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On Combinatorial Models for Affine Demazure Crystals of Levels Zero and One

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Abstract. The non-symmetric Macdonald polynomials are a family of orthogonal polynomials with parameters q and t. There are two well-known combinatorial models for computing these polynomials: a tableau model in type A, due to Haglund, Haiman and Loehr, and the type-independent model due to Ram and Yip, based on alcove walks. We establish a bijection between these two models in the case $t = 0$ (in type A). Furthermore, we construct a crystal structure on alcove walks in arbitrary type. In type A, we prove that this crystal structure is compatible with Assaf's Demazure crystal on semistandard key tabloids via our bijection; these crystals are colored directed graphs encoding the structure of certain affine Demazure-type submodules.

Keywords: Kirillov-Reshetinkhin crystals, Macdonald polynomials, alcove model.

1 Introduction

Kashiwara's crystals [8] are related to the representation theory of quantum groups $U_q(\mathfrak{g})$ as q goes to 0. Certain representations (including highest weight modules, their Demazure submodules, and Kirillov-Reshetikhin modules) possess *crystal bases*, whose structure is encoded in a colored directed graph, called a *crystal graph*. The vertices of this graph are the elements of the crystal basis, and the directed i-edges $b \to f_i(b)$ give the action of the *Kashiwara operators* (analogues of the Chevalley generators) as $q \to 0$.

For the highest weight representations of a complex semisimple Lie algebra, there exist several combinatorial models for the corresponding crystal graph. One example is a uniform model in all Lie types called the *alcove model* [13], which is based on enumerating saturated chains in the Bruhat graph of the Weyl group. The enumeration is determined by an alcove path. In fact, the alcove model was defined in a more general setting for symmetrizable Kac-Moody algebras.

More recently the alcove model was generalized to the "quantum setting" [10]. The term quantum refers to the use of paths in the quantum Bruhat graph. An application of the *quantum alcove model* is in the theory of *Kirillov–Reshetinkhin (KR) modules* [9], which are finite-dimensional modules $W^{r,s}$ for affine Lie algebras, not of highest weight, indexed by $r \times s$ rectangles. In many cases KR modules admit crystal bases, which are denoted $B^{r,s}$, and the quantum alcove model was shown to uniformly describe certain KR crystals for untwisted affine types [12].

Macdonald [14] introduced a remarkable family of symmetric polynomials $P_\lambda(x, q, t)$ with parameters q and t and indexed by dominant nonaffine weights λ of an irreducible affine root system. Shortly after, Opdam and Cherednik provided a non-symmetric version of Macdonald polynomials $E_\mu(x, q, t)$. Here μ is an arbitrary nonaffine weight.

There is a close connection between KR crystals and the Macdonald polynomials. In [12] the quantum alcove model was used to establish the equality of $P_\lambda(x; q, 0)$ with the graded character of a tensor product of "single-column" Kirillov-Reshetikhin (KR) modules for untwisted affine Lie algebras. As in the symmetric case, the same authors used the quantum alcove model to show that $E_\mu(x, q, 0)$ is equal to a graded character of a Demazure-type submodule of a tensor product of KR modules [11].

Due to their connections to many different subjects (such as representation theory of quantum groups), explicit formulas for computing $P_\lambda(x, q, t)$ and $E_\mu(x; q, t)$ are desired. The simplest formulas for the Macdonald polynomials are in type A. Haglund, Haiman, and Loehr (HHL) give formulas for both the symmetric [5] and non-symmetric [6] Macdonald polynomials for general q and t, in terms of non-attacking fillings of shape λ or μ respectively. Recently Assaf defined objects called *semi-standard key tabloids* [1]. They give the monomial terms in the HHL formula for the type A non-symmetric Macdonald polynomials at $t = 0$. Semi-standard key tabloids were endowed with an affine crystal structure [3]. As a consequence, a crystal-theoretic proof that the type A nonsymmetric Macdonald polynomials at $t = 0$ are a positive q-graded sum of finite type Demazure characters is obtained. [3]

For an arbitrary Lie type, there is another formula due to Ram and Yip [16] based on the alcove model. While having the benefit of being type-independent, the alcove formula has many more terms than the tableau formulas in type A. However, the alcove path formula carries more information. In particular it allows for computing the global energy function and expressing the combinatorial R-matrix (see 2.2).

We introduce a conjectured model, based on the quantum alcove model, for Demazure-type subcrystals of (tensor products of) column-shape KR crystals for untwisted types. Our model differs from the model used in [11] as it drops a condition known as the "end" condition used to select the subcrystal from the full crystal. Specializing to type A, we define an explicit crystal isomorphism from our model to the Demazure crystal on semi-standard key tabloids, which proves our conjecture in type A and provides a bijection between HHL terms and Ram-Yip terms for $t = 0$.

2 Background

2.1 Root systems

Let $\mathfrak{g}$ be a complex simple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. Let $\Phi \subset \mathfrak{h}_\mathbb{R}^*$ be the corresponding irreducible root system, and Φ^+ the set of positive roots. Let $\alpha_i \in \Phi^+$

be the *simple roots* for i in some indexing set I. We denote by $\langle \cdot, \cdot \rangle$ the nondegenerate scalar product on $\mathfrak{h}_{\mathbb{R}}^*$ induced by the Killing form. Given a root α, we consider the corresponding coroot $\alpha^\vee := 2\alpha / \langle \alpha, \alpha \rangle$ and reflection s_α. The *weight lattice* P is generated by the *fundamental weights* ω_i for $i \in I$, which satisfy $\langle \omega_j, \alpha_i^\vee \rangle = \delta_{i,j}$. The set of dominant weights is denoted P^+. Define $\rho := \sum_{i \in I} \omega_i$, and for a root α, set $|\alpha| := \operatorname{sgn}(\alpha)\alpha$.

Let W be the corresponding finite Weyl group with Coxeter generators $s_i := s_{\alpha_i}$. With respect to these generators, W comes with a length function $\ell(\cdot)$. Let $w_\circ$ be the longest element of W. The Bruhat order on W is defined as the transitive closure of its covers $w \lessdot ws_\alpha$ if $\ell(w) + 1 = \ell(ws_\alpha)$.

For a root $\alpha \in \Phi$ and an integer $k \in \mathbb{Z}$, consider the reflection $s_{\alpha,k}$ in the affine hyperplane $H_{\alpha,k} := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* | \langle \lambda, \alpha \rangle = k\}$. The *affine Weyl group* W_{aff} is generated by such reflections. The hyperplanes $H_{\alpha,k}$ divide $\mathfrak{h}_{\mathbb{R}}^*$ into connected components called *alcoves*. The *fundamental alcove* is denoted $A_\circ$. An important object in defining the quantum alcove model is a certain directed graph on W.

Definition 1. *The* quantum Bruhat graph *of W is the directed graph with vertex set W and edges $w \xrightarrow{\beta} ws_\beta$ labeled by $\beta \in \Phi^+$ if:*

(1) $w \lessdot ws_\beta$, or

(2) $\ell(ws_\beta) = \ell(w) - 2\langle \beta^\vee, \rho \rangle + 1$.

2.2 Kirillov–Reshetikhin crystals

Given a simple or affine Lie algebra $\mathfrak{g}$ with simple roots α_i for $i \in I$, a $\mathfrak{g}$-*crystal* is a nonempty set B together with maps $e_i, f_i : B \to B \sqcup \{0\}$ for $i \in I$ and $\mathrm{wt} : B \to P$ satisfying $f_i(b) = b'$ if and only if $e_i(b') = b$ for $b \in B$. The maps e_i, f_i are called crystal operators and they give B the structure of a colored directed graph with edges $b \xrightarrow{i} f_i(b)$. For $\mathfrak{g}$-crystals B_1 and B_2, the tensor product $B_1 \otimes B_2$ is defined on the set $B_1 \times B_2$ with a rule that determines on which factor e_i and f_i act [7].

Kirillov–Reshetikhin (KR) modules, denoted $W^{r,s}$, are finite-dimensional modules, not of highest weight, of affine Lie algebras. They are indexed by positive integer multiples of a fundamental weight $s\omega_r$. In classical types $s\omega_r$ corresponds to a rectangle of height r and length s. In the case $s = 1$, the modules are called "single-column" KR modules. In most cases, $W^{r,s}$ was shown to have a crystal basis, and the corresponding *Kirillov–Reshetikhin crystal* is denoted $B^{r,s}$. For a composition $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, tensor products of column-shape KR crystals for untwisted types are defined as follows:

$$B = B^{\mathbf{a}} := \bigotimes_{i=1}^{k} B^{a_i,1} \,. \tag{2.1}$$

The crystal B is known to be connected as an affine crystal, but disconnected as a classical crystal. If $\mathbf{a'}$ is a permutation of $\mathbf{a}$, then there is a unique crystal isomorphism between $B^{\mathbf{a}}$ and $B^{\mathbf{a}'}$, called the *combinatorial R-matrix* .

Let $\mathfrak{g}_{\mathrm{af}}$ be an affine Lie algebra and let $\lambda \in P^+$ be a dominant weight for $\mathfrak{g}$. Consider the level zero extremal weight module $V(\lambda)$ over the quantum group $U_q(\mathfrak{g}_{\mathrm{af}})$, generated by a vector v_λ (of weight λ). For a Weyl group element w, the *Demazure submodule* $V_w(\lambda) \subset V(\lambda)$ is given by $V_w(\lambda) := U_q^+(\mathfrak{g}) \cdot wv_\lambda$.

Beck and Nakajima introduced a subtle finite-dimensional quotient of $V_{w_\circ}^+(\lambda)$, denoted $U_{w_\circ}^+(\lambda)$ (see [11, §3.3]). As a $U_q(\mathfrak{g})$-module, $U_{w_\circ}^+(\lambda)$ is isomorphic to a tensor product of KR modules $W^{i,1}$. The image of the level zero Demazure module $V_w^+(\lambda)$ under the projection to $U_{w_\circ}^+(\lambda)$ is denoted $U_w^+(\lambda)$ and is a *"Demazure-type"* submodule.

2.3 The quantum alcove model

Fix a dominant weight λ. The quantum alcove model depends on a sequence of roots $\Gamma := (\beta_1, \beta_2, \cdots, \beta_m)$ called a λ-chain [10]. This is equivalent to a shortest sequence of adjacent alcoves from the fundamental alcove $A_\circ$ to the translate $A_\circ - \lambda$. Define the reflection $r_i := s_{\beta_i}$.

Definition 2. *A subset $J = \{j_1, j_2, \ldots, j_s\} \subset [m]$ (possibly empty) is called* admissible *if we have a path in the quantum Bruhat graph*

$$1 \to r_{j_1} \to r_{j_1} r_{j_2} \to \cdots \to r_{j_1} r_{j_2} \cdots r_{j_s} := end(J).$$

Let $\mathcal{A}(\Gamma)$ be the set of admissible subsets of J.

Theorem 1. *[10, 12] Let $\lambda = \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_k}$ be a dominant weight of a untwisted affine $\mathfrak{g}$. Then $\mathcal{A}(\Gamma)$ with the proper crystal operators e_i, f_i gives a combinatorial model for the tensor product of KR crystals $B^\lambda := B^{i_1,1} \otimes B^{i_2,1} \otimes \cdots \otimes B^{i_k,1}$.*

Let λ be a dominant weight, whose stabilizer is denoted W_λ. Given a Weyl group element u, we denote by $\lfloor u \rfloor$ the lowest coset representative of uW_λ.

Theorem 2. *[11] The set of admissible subsets $J \in \mathcal{A}(\Gamma)$ satisfying the condition $end(J) \leq \lfloor w \rfloor$ is a model for the crystal corresponding to the Demazure-type module $U_w^+(\lambda)$.*

Based on the quantum alcove model, the following is derived in [12, 11]:

Theorem 3. *Let $\mathfrak{g}_{af}$ be an untwisted affine Lie algebra and λ a dominant weight of $\mathfrak{g}$.*

1. *The symmetric Macdonald polynomial $P_\lambda(x, q, 0)$ is a graded character of the tensor product of column-shape KR crystals B^λ, so it is expressed by $\mathcal{A}(\Gamma)$.*

2. *The nonsymmetric Macdonald polynomial $E_{w\lambda}(x, q, 0)$ is a graded character of the Demazure-type crystal corresponding to $U_w^+(\lambda)$, so it is expressed by the subset of $\mathcal{A}(\Gamma)$ defined in Theorem 2.*

In each case, the grading is by the energy function.

2.4 Key tabloids and their crystal structure

In this section we introduce a tableau model for the level one Demazure crystal in type A defined in [1]. The objects of the model are fillings of diagrams subject to conditions on entries as we describe below.

A *diagram* dg(μ) of the weak composition $\mu = (\mu_1, \mu_2, \ldots \mu_n)$ consists of μ_i boxes left-justified in row i. We label the box in row i column j as (i, j).

For example, if $\mu = (3, 0, 2)$ then $dg(\mu) = \begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array} \\ \\ \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \end{array}$.

Definition 3. *A* filling *is an assignment of a positive integer $m_{(i,j)} \leq n$ to each box (i, j).*

Definition 4. *Given a diagram dg(μ), we say two boxes are* attacking *if they are in the same column or they are in adjacent columns with the leftmost box strictly higher than the rightmost. A filling is* non-attacking *if every pair of attacking boxes have distinct entries and no entry in the first column exceeds its row index.*

Definition 5. *A* triple *is a collection of three boxes with two row adjacent and either (Type I) the third cell is above the left and the lower row is strictly longer, or (Type II) the third cell is below the right and the higher row is weakly longer. The orientation of a triple is determined by reading the entries of the boxes from smallest to largest. A* coinversion triple *is a Type I triple oriented counterclockwise or a Type II triple oriented clockwise.*

Definition 6. *The* semi-standard key tabloids *of shape μ, denoted* SSKD(μ)*, are the non-attacking fillings of dg(μ) with no coinversion triples.*

Example 1. $\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array} \\ \\ \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline \end{array}$ *is a semi-standard key tabloid of shape $\mu = (3, 0, 2)$.*

The *weight* of a tabloid T, denoted wt(T) is defined as the weak composition with ith part equal to the number of entries i in T. For example wt$(T) = (2, 1, 2)$ in the example above. Assaf defined raising and lowering operators e_i, f_i for $i = 0, 1, \cdots, n-1$, which act on T by swapping certain entries i and $i + 1$ (1 and n for $i = 0$). The affected entries are determined by a pairing rule [2, defs 3.1-3.6]. The operators e_i, f_i, and the map wt define a crystal structure on SSKD(μ).

Theorem 4. *[1] The above crystal structure on semi-standard key tabloids of shape μ is isomorphic to a certain level one Demazure crystal of $\widehat{\mathfrak{sl}}_n$.*

Theorem 5. *[6] The specialized non-symmetric Macdonald polynomials are given by*

$$E_\mu(x, q, 0) = \sum_{T \in SSKD(\mu)} q^{\mathrm{maj(T)}} x^{\mathrm{wt}(T)} ; \tag{2.2}$$

where $\mathrm{maj}(T)$ is the sum of all legs of boxes (i, j) of T with $m_{(i,j)} < m_{(i,j+1)}$.

Remark 1. *[4] Level one and level zero Demazure crystals are known to be isomorphic, up to removal of some 0-edges of the latter.*

3 Modified quantum alcove model

We introduce an alternative model that is conjectured to describe "Demazure-type" modules $U_w^+(\lambda)$. The objects are analogues of admissible subsets as in Definition 2. The model has the benefit of removing the condition $\mathrm{end}(J) \leq \lfloor w \rfloor$ in Theorem 2.

3.1 v-admissible subsets

Given a weight μ, let $v \in W$ be the element of maximal length such that $\mu = v\lambda$ for a dominant weight λ, and define $w := vw_\circ$. Consider a reduced alcove path Γ from $A_\circ$ to $wA_\circ + \mu$:

$$\Gamma := (A_\circ \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} A_2 \xrightarrow{-\gamma_3} \cdots \xrightarrow{-\gamma_m} A_m = wA_\circ + \mu). \tag{3.1}$$

The sequence of crossed hyperplanes is given by $H_{|\gamma_k|, m_k}$ where m_k is defined by this relation. To describe the combinatorial objects of the model, we need another alcove path Γ' obtained from Γ by applying $-v^{-1}$ on the left:

$$\Gamma' = -v^{-1}\Gamma = (-v^{-1}A_\circ \xrightarrow{-\beta_1} A_1' \xrightarrow{-\beta_2} A_2' \xrightarrow{-\beta_3} \cdots \xrightarrow{-\beta_m} A_m' = A_\circ - \lambda). \tag{3.2}$$

Note that in this case all the roots β_i are positive. Let $J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$. We can interpret J geometrically as a "folding" of the alcove path Γ along the hyperplanes $H_{|\gamma_{j_i}|, m_{j_i}}$ for $1 \leq i \leq s$ as follows: Let $i = i(k)$ be the largest index for which $j_i < k$. Then after folding Γ, the k-th hyperplane crossed in the folded path is given by:

$$H_{|\delta_k|, l_k^J} := \widetilde{r}_{j_1} \cdots \widetilde{r}_{j_i} H_{|\gamma_k|, m_k} , \tag{3.3}$$

where $\widetilde{r}_k := s_{\gamma_k, m_k}$. Note that $\delta_k = r_{j_1} r_{j_2} \cdots r_{j_i}(\gamma_k)$ where r_{j_1} is the projection of $\widetilde{r}_{j_1}$ onto the finite Weyl group, etc. The objects of our model are subsets $J \subset [m]$ satisfying the following condition:

Definition 7. *We say $J = \{j_1 < j_2 < \cdots < j_s\} \subset [m]$ is v-*admissible *and write $J \in \mathcal{A}(\Gamma')$ if there is a path ending at v in the quantum Bruhat graph:*

$$vs_{\beta_{j_1}}s_{\beta_{j_2}}\cdots s_{\beta_{j_s}} \to \cdots \to vs_{\beta_{j_1}}s_{\beta_{j_2}} \to vs_{\beta_{j_1}} \to v\ .$$

For J v-admissible, *define* $\mathrm{wt}(J) := \widetilde{r}_{j_1}\widetilde{r}_{j_2}\cdots\widetilde{r}_{j_s}(\mu)$ *and* $\mathrm{ht(J)} := \sum_{\mathrm{i}} \mathrm{l}^{\mathrm{J}}_{\mathrm{j_i}}$ *where the sum is over all i such that the edge corresponding to β_{j_i} is a quantum edge, i.e an edge of type (2) in Definition 1.*

Theorem 6. *[16, consequence of Ram-Yip formula]*

$$E_\mu(x,q,0) = \sum_{J\in\mathcal{A}(\Gamma)} q^{ht(J)}x^{\mathrm{wt}(J)}\ . \tag{3.4}$$

3.2 Crystal operators on v-admissible subsets

We now define crystal operators e_i, f_i on $\mathcal{A}(\Gamma')$. For $J \in \mathcal{A}(\Gamma')$ and $\alpha \in \Phi^+$ a positive root, define:

$$I_\alpha = I_\alpha(J) := \{i \in [m] \mid \delta_i = \pm\alpha\}\ , \qquad \widehat{I}_\alpha = \widehat{I}_\alpha(J) := I_\alpha \cup \{\infty\}\ ,$$

$$l^\infty_\alpha := \langle \mathrm{wt}(J), \mathrm{sgn}(\alpha)\alpha^\vee\rangle, \qquad \delta_\infty := r_{j_1}r_{j_2}\cdots r_{j_s}v\rho\ .$$

The following graphical representation of the levels l^J_i for $i \in l^\infty_\alpha$ is useful for defining the crystal operators. Set:

$$\widehat{I}_\alpha = \{i_1 < i_2 < \cdots < i_n < i_{n+1} = \infty\} \text{ and } \varepsilon_i := \begin{cases} 1 & \text{if } i \notin J \\ -1 & \text{if } i \in J \end{cases}.$$

If $\alpha > 0$, we define the continuous piece-wise linear function $g_\alpha : [0, n+\frac{1}{2}] \to \mathbb{R}$ by

$$g_\alpha(0) = -\frac{1}{2}, \quad g'_\alpha(x) = \begin{cases} \mathrm{sgn}(\delta_{i_k}) & \text{if } x \in (k-1, k-\frac{1}{2}),\ k = 1,\ldots,n \\ \varepsilon_{i_k}\mathrm{sgn}(\delta_{i_k}) & \text{if } x \in (k-\frac{1}{2}, k),\ k = 1,\ldots,n \\ \mathrm{sgn}(\langle\delta_\infty, -\alpha^\vee\rangle) & \text{if } x \in (n, n+\frac{1}{2}). \end{cases} \tag{3.5}$$

If $\alpha < 0$, we define g_α to be the graph obtained by reflecting $g_{-\alpha}$ across the x-axis. For any α we have

$$\mathrm{sgn}(\alpha)l^J_{i_k} = -g_\alpha\left(k-\frac{1}{2}\right) \text{ for } k = 1,\ldots,n \tag{3.6}$$

$$\mathrm{sgn}(\alpha)l^\infty_\alpha := \langle \mathrm{wt}(J), -\alpha^\vee\rangle = -g_\alpha\left(n+\frac{1}{2}\right). \tag{3.7}$$

Let J be a v-admissible subset. Fix $i \in \{0,\ldots,n-1\}$, so α_i is a simple root if $i > 0$, or

$-\alpha_0$ if $i = 0$. Let $M := \max(g_{\alpha_i})$. Assuming that $M > \langle \mathrm{wt}(J), \alpha_i^\vee \rangle$, let k be the maximum index j in I_{α_i} for which we have $\mathrm{sgn}(\alpha_i) l_j^J = -M$, and let m be the successor of k in $\widehat{I}_{\alpha_i}$. Assuming also that $M \geq \delta_{i,0}$, we have $k \in J$, and either $m \notin J$ or $m = \infty$. Define:

$$f_i(J) := \begin{cases} (J \setminus \{k\}) \cup \{m\} & \text{if } M > \langle \mu(J), \alpha_i^\vee \rangle \text{ and } M \geq \delta_{i,0} \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{3.8}$$

The operators e_i are defined similarly. We use the convention that $J \setminus \{\infty\} = J \cup \{\infty\} = J$.

The following theorem shows that $\mathcal{A}(\Gamma')$ is closed under the crystal operators f_i, e_i, and so is a well-defined model.

Theorem 7. *If J is a v-admissible subset and if $f_i(J) \neq \mathbf{0}$, then $f_i(J)$ is also a v-admissible subset. Similarly for $e_i(J)$. Moreover, $f_i(J) = J'$ if and only if $e_i(J') = J$.*

Conjecture 1. *The set of v-admissible subsets $\mathcal{A}(\Gamma')$ together with maps e_i, f_i, and* wt *defined above is a model for level zero "Demazure-type" crystals of tensor products of KR crystals as in [11, §3.3].*

The entire crystal is constructed by repeated application of the operators e_i, f_i, starting from the lowest element $J = \emptyset$ ($f_i(J) = \mathbf{0}$ for all i). By comparison, the Demazure-type crystal from Theorem 2 is constructed by repeated application of the operators f_i, starting from the highest element $J = \emptyset$ ($e_i(J) = \mathbf{0}$ for all i) and checking the condition $\mathrm{end}(J) \leq \lfloor w \rfloor$ at each step.

4 The model in type A

4.1 A special choice of μ-chain

Recall that the objects of the quantum alcove model depend on a choice of alcove path (different choices with yield isomorphic models). In order to obtain explicit maps from the quantum alcove model to the tabloid model, a particular alcove path Γ' is needed.

To this end, consider the setup as in Section 3, now specialized to type A_{n-1}. Let λ' be the conjugate partition to λ and let $\mathrm{dg}(\mu)$ be an empty diagram of shape μ. Γ' will be constructed by concatenating smaller sequences of roots $\Gamma'(j)$, one for each column of $\mathrm{dg}(\mu)$, i.e,

$$\Gamma' = \Gamma'(1)\Gamma'(2)\cdots\Gamma'(\lambda_1).$$

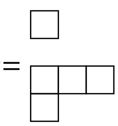
Suppose $\lambda'_j = k$. Rename and reorder $v(1), v(2), \cdots, v(k)$ as $i_1, i_2, \cdots, i_k$ with

$$i_1 < i_2 < \cdots < i_k\,. \tag{4.1}$$

For $1 \leq s \leq k$, let $m_s := \min(\{a \in \{k+1, \cdots, \lambda'_{j-1}\} | i_s > v(a)\} \cup \{\lambda'_{j-1} + 1\})$ (by convention we set $\lambda'_0 = n$). Then $\Gamma'(j)$ is given by the following sequence of roots:

$$\Gamma'(j) = \begin{pmatrix} (v^{-1}(i_1), m_1) & (v^{-1}(i_1), m_1 + 1) & \dots & (v^{-1}(i_1), n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (v^{-1}(i_k), m_k) & (v^{-1}(i_k), m_k + 1) & \dots & (v^{-1}(i_k), n) \end{pmatrix}. \tag{4.2}$$

Proposition 1. *[15]* Γ'*, defined as the above concatenation, is the sequence of roots* $(\beta_1, \beta_2, \beta_3, \dots)$ *encoding an alcove path of the form (3.2), with which it is identified.*

Example 2. *Let* $\mu = (1, 0, 3, 1)$*. Then* $v = v^{-1} = 3412$*,* *dg(*μ*)=* 

The first column has 3 *boxes, so* $i_1 = 1, m_1 = 5$*;* $i_2 = 3, m_2 = 4$*; and* $i_3 = 4, m_3 = 4$*.*
The second column has 1 *box, so* $i_1 = 3, m_1 = 3$*.*
The third column has 1 *box, so* $i_1 = 3, m_1 = 2$*.*

The corresponding sequences of roots $\Gamma'(j)$ *are:*

$$\Gamma'(1) = ((1,4)(2,4)); \quad \Gamma'(2) = ((1,3)(1,4)); \quad \Gamma'(3) = ((1,2)(1,3)(1,4))$$

Concatenating gives the alcove path

$$\Gamma' = ((1,4)(2,4)|(1,3)(1,4)|(1,2)(1,3)(1,4))$$

4.2 The filling map

We now describe a map from v-admissible subsets $\mathcal{A}(\Gamma')$ to semi-standard key tabloids $SSKD(\mu)$. It is a forgetful map that produces a filling from a subset of roots J.

Recall that a v-admissible subset $J \in \mathcal{A}(\Gamma')$ selects a set of positions within the alcove path Γ'. Let T_j be the sequence of transpositions corresponding to the positions in $\Gamma'(j)$ that are selected, noting that T_j could possibly be empty. Given a v-admissible subset J and a column j of $\mathrm{dg}(\mu)$, define:

$$u^j := vT_1T_2 \cdots T_j$$

where the right hand side is the permutation obtained from v by consecutively multiplying by each transposition present in $T_1, T_2, \dots, T_j$ from left to right.

Definition 8. *The map* fill *is a map from subsets* J *(not necessarily v-admissible) to fillings of shape μ with entry in row i_a, column j given by*

$$\mathrm{fill}(\mathrm{J})(i_a, j) := u^j v^{-1}(i_a).$$

Theorem 8. *If* J *is v-admissible, then* fill(J) *is non-attacking and has no coinversion triples. Moreover,* fill *is a weight preserving bijection between v-admissible subsets and semi-standard key tabloids of shape μ.*

Example 3. *Continuing Example 2, let $J = \{2, 5, 6, 7\}$. One can check that J is v-admissible.*

$$T_1 = (2,4); \quad T_2 = \emptyset; \quad T_3 = (1,2)(1,3)(1,4)$$

Starting from v, we apply in order the transpositions above:

$$v = \begin{array}{|c|}\hline 2\\\hline 1\\\hline 4\\\hline 3\\\hline\end{array} \xrightarrow{(2,4)} \begin{array}{|c|}\hline 4\\\hline \boldsymbol{1}\\\hline \mathbf{2}\\\hline \boldsymbol{3}\\\hline\end{array} \;\Big|\; \begin{array}{|c|}\hline 4\\\hline 1\\\hline 2\\\hline \boldsymbol{3}\\\hline\end{array} \;\Big|\; \xrightarrow{(1,2)} \begin{array}{|c|}\hline 4\\\hline 1\\\hline 3\\\hline 2\\\hline\end{array} \xrightarrow{(1,3)} \begin{array}{|c|}\hline 4\\\hline 2\\\hline 3\\\hline 1\\\hline\end{array} \xrightarrow{(1,4)} \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline \boldsymbol{4}\\\hline\end{array}$$

$$u^1 = 3412(2,4) = \boldsymbol{321}4; \quad u^2 = \boldsymbol{3}214; \quad u^3 = 3214(1,3)(1,4) = \boldsymbol{4}231$$

Recall that $v^{-1} = 3412$. The entry in box (i_s, j) is $u^j(v^{-1}(i_s))$, so $\mathrm{fill}(J) =$ $\begin{array}{lll} \boxed{2} & & \\ \boxed{3} & \boxed{3} & \boxed{4} \\ & & \\ \boxed{1} & & \end{array}$

and one can check that fill(J) *is non-attacking with no coinversion triples.*

4.3 The inverse Map

The existence of a map from semi-standard key tabloids to v-admissible subsets hinges on the following order that is closely related to the quantum bruhat graph for $W = S_n$:

Definition 9. *Define an order $\prec_i$ on $[n]$ by*

$$i \prec_i i+1 \prec_i i+2 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i-1$$

It is convenient to think of this order as arranging the numbers 1 to n as on a clock.

To reconstruct a v-admissible subset from a semi-standard key tabloid T, a "greedy" algorithm is run once for each box of T, working up the columns then across the rows. The initial input is the permutation v. If the target box is in column j, row i_a and has entry b, we select every root of $\Gamma'(J)$ (4.2) of the form $(v^{-1}(i_a), m_a + p)$ (viewed as a transposition) that brings us closer to our target entry (with respect to the order $\prec_b$) and consecutively multiply the current permutation by each selected transposition.

Proposition 2. *Application of the proccess described above terminates, produces a unique path in the quantum Bruhat graph, and is the inverse of the map* fill*.*

5 Main result

Here we recap the main constructions and state our main result. For a dominant weight λ and a Weyl group element w with $w = vw_\circ$, we construct a crystal structure on v-admissible subsets that is conjectured to be isomorphic to the level zero "Demazure-type" crystal corresponding to $U_w^+(\lambda)$. In type A, Assaf constructs a crystal on semi-standard key tabloids that gives a certain level one affine Demazure crystal. Through our bijection fill, we show that the crystal operators on tabloids are compatible with our crystal operators on v-admissible subsets, proving our main result:

Theorem 9. *The map* fill *is a crystal isomorphism from v-admissible subsets to semi-standard key tabloids. This proves Conjecture 1 in type A.*

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