Combinatorial models in the representation theory of quantum affine Lie algebras

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Background

Certain affine crystals, known as Kirillov-Reshetikhin (KR) crystals, were realized in a uniform way (for all untwisted affine types) in terms of the quantum alcove model. Their graded characters were shown to coincide with the symmetric Macdonald polynomials at $t = 0$. We generalize these results to certain (level zero) Demazure-type subcrystals of KR crystals and non-symmetric Macdonald polynomials, based on a "non-symmetric version" of the quantum alcove model.

In type A, we also construct an affine crystal isomorphism between the new model and the crystal of Assaf and González on so-called semistandard key tabloids. The latter was initially developed as a model for certain level one affine Demazure crystals of type A.

Young Tableaux

A fundamental combinatorial object in representation theory is a Young tableau, which is a collection of boxes arranged in a staircase shape with numbers in each box.

$\begin{array}{|c|c|c|c|}\hline 9 \\ \cline{1-1} 7 & 7 & 7 \\ \cline{1-3} 1 & 2 & 5 & 3 \\ \hline\end{array}$ is a Young tableau of shape $(4, 3, 1)$.

Young Tableaux encode information about the representations of the Lie Group/Algebra $SL_n/\mathfrak{sl}_n$ (volume-preserving linear transformations)

- dimension
- number of irreducible representations
- character

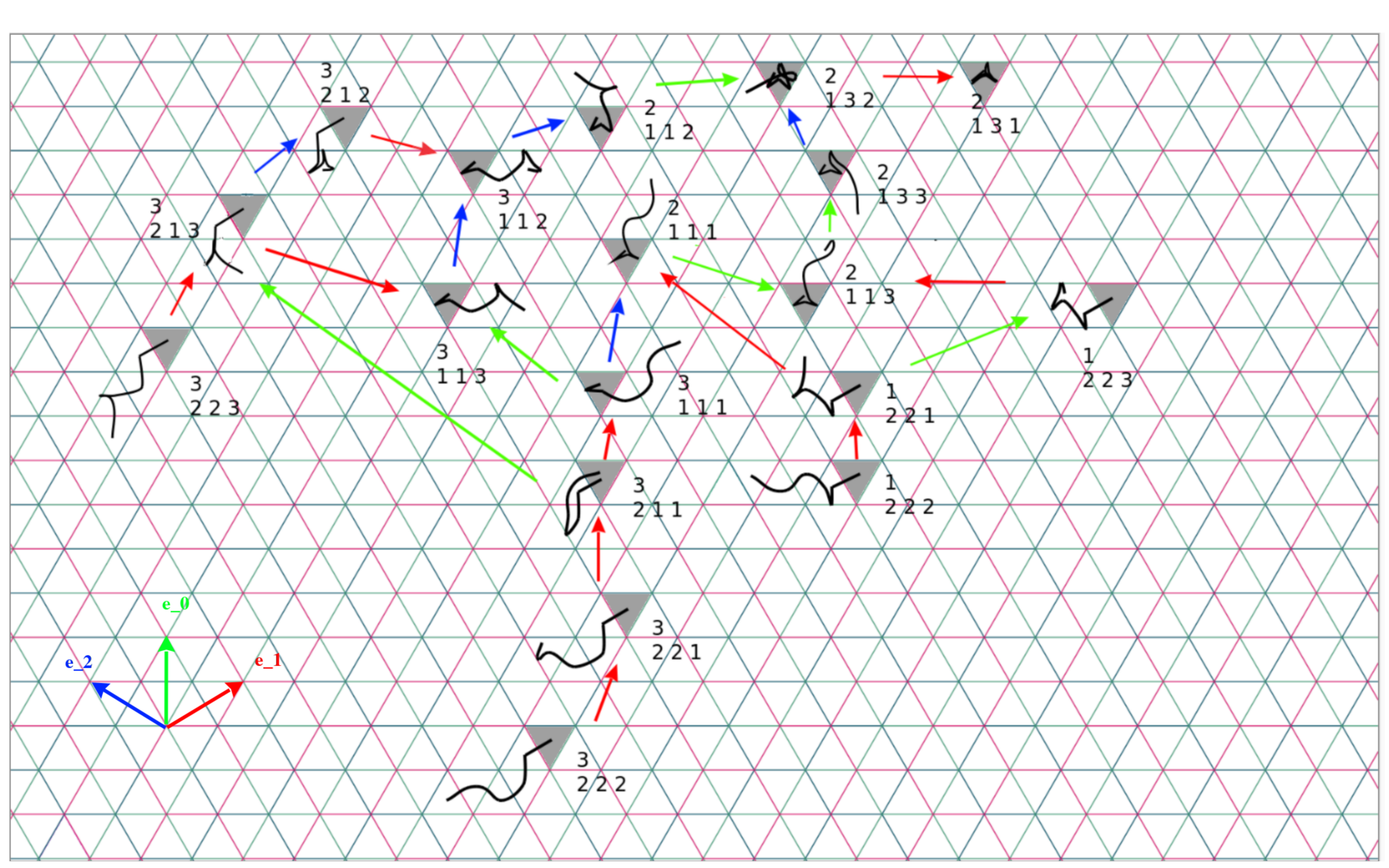
Crystal Bases

Main idea: use colored directed graphs to encode certain representations of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$ ($\mathfrak{g}$ complex semisimple or affine Lie algebra).
Crystal graph: directed graph with colored edges e_i, f_i.

affine Demazure crystals $\widetilde{B}_v(\lambda)$

Correspond to certain $finite$-dimensional representations (not highest weight) of affine Lie algebras $\mathfrak{g}$.

Type A_4 Crystal Graph of $\widetilde{B}_v(\lambda)$, $\lambda = (3, 1, 0)$, $v = 231$



The Quantum Alcove Model in Type A_{n-1}

The main ingredient is the Weyl group $\mathbf{W} = \langle s_\alpha : \alpha \in \Phi \rangle$.

Definition. For a weight $\mu = v\lambda$, we associate an alcove path called a $\mu - chain$ (many choices possible):

$$\Gamma' = (\beta_1, \beta_2, \ldots, \beta_m).$$

Let $r_i := s_{\beta_i}$. We consider subsets of positions in Γ';

$$J = (j_1 < j_2 < \ldots < j_s) \subseteq \{1, \ldots, m\}.$$

Definition. A subset $J = \{j_1 < j_2 < \ldots < j_s\}$ is *v-admissible* if we have a path in the quantum Bruhat graph

$$v \xleftarrow{\beta_{j_1}} vr_{j_1} \xleftarrow{\beta_{j_2}} vr_{j_1}r_{j_2} \ldots \xleftarrow{\beta_{j_s}} vr_{j_1} \ldots r_{j_s}.$$

v-admissibility can easily be checked by arranging the numbers 1 to n on a clock.
We say $a \prec b \prec c$ if b lies between a and c going clockwise.

Theorem: The v-admissible subsets of Γ' form a model for the crystal $\widetilde{B}_v(\lambda)$.
The crystal operators add and remove a position from J:

$$e_i(J) = J \setminus \{m\} \cup \{k\} \qquad f_i(J) = J \setminus \{k\} \cup \{m\}$$

can be thought of as adding a removing "folds" from the path Γ'

The Type A_{n-1} Tableau Model

Assaf and Gonzalez show that the crystal $\widetilde{B}_v(\lambda)$ can be modeled by objects similar to Young tableaux, called semi-standard key tabloids.

no coinversions: $\begin{array}{|c|c|}\hline 2 \\ \cline{1-1} \\ \hline 4 & 1 \\ \hline\end{array}$ or $\begin{array}{|c|c|}\hline 5 & 1 \\ \hline \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$ **non-attacking:** $\begin{array}{|c|}\hline a \\ \hline \\ \hline b \\ \hline\end{array} \implies a \neq b$

The crystal operators e_i, f_i, swap entries i with $i + 1$, or swap n with 1.

The action of f_2: $\begin{array}{cccccccc} 5 & 3 & 1 \\ 3 & 2 & 2 & 2 & 5 & 5 & 5 & 2 \\ \\ 2 & 1 & 4 & 4 & 3 & 1 & 1 \end{array} \longrightarrow \begin{array}{cccccccc} 5 & 2 & 1 \\ 3 & 3 & 3 & 2 & 5 & 5 & 5 & 2 \\ \\ 2 & 1 & 4 & 4 & 3 & 1 & 1 \end{array}$

The action of e_2: $\begin{array}{cccccccc} 5 & 2 & 1 \\ 3 & 3 & 3 & 2 & 5 & 5 & 5 & 3 \\ \\ 2 & 1 & 4 & 4 & 3 & 1 & 1 \end{array} \longrightarrow \begin{array}{cccccccc} 5 & 2 & 1 \\ 3 & 3 & 3 & 2 & 5 & 5 & 5 & 2 \\ \\ 2 & 1 & 4 & 4 & 3 & 1 & 1 \end{array}$

The Two Realizations

- The tableaux model is simpler and has less structure.
- The quantum alcove model has extra structure which makes it easier to do several computations (energy function, combinatorial R-Matrix, charge statistic...)

Relating the Two Models

We build a map $fill : \mathcal{A}(\Gamma') \to SSKT(\mu)$ where $\mu = v\lambda$ for a dominant weight λ.

Definition: For each column j, we construct a chain of roots $\Gamma'(j)$

Definition: A μ-*chain* is given as a concatenation of the above subchains:
$\Gamma' := \Gamma'(1)\Gamma'(2) \ldots \Gamma'(\lambda'_1)$.

Example Consider $n = 4$ and $\mu = (1, 2, 0, 3)$. Then $v = 4213$ and a μ-chain is

$$\Gamma' = \Gamma'(1)\Gamma'(2)\Gamma'(3) = ((1,4)|(2,4),(1,3),(1,4)|(1,2),(1,3),(1,4)).$$

Example $J = \{1, 5, 6\} \in \mathcal{A}(\Gamma')$.

$$(\underline{(1,4)}|(2,4),(1,3),(1,4)|\underline{(1,2)},\underline{(1,3)},(1,4))$$

We get the corresponding path in the Bruhat order/quantum Bruhat graph

$$v = \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline\end{array} \xrightarrow{(1,4)} \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \mid \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \mid \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \xrightarrow{(1,2)} \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline 3 \\ \hline 2 \\ \hline\end{array} \xrightarrow{(1,3)} \begin{array}{|c|}\hline 4 \\ \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array} = end(J).$$

Keeping track of the blue entries gives us a filling: $\begin{array}{|c|c|c|}\hline 1 \\ \cline{1-1} 2 & 2 \\ \cline{1-2} 3 & 3 & 1 \\ \hline\end{array}$

Permuting the rows by v gives $fill(J) = \begin{array}{|c|c|c|}\hline 3 & 3 & 1 \\ \hline \\ \cline{1-2} 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$

The Reverse Map in Type A_{n-1}

Consider the semi-standard key tabloid from the previous example

$$T = \begin{array}{|c|c|c|}\hline 3 & 3 & 1 \\ \hline \\ \cline{1-2} 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

Use entries of columns i and $i - 1$ viewed as sets to build the desired sub-list of $\Gamma'(i)$ where column 0 is the size n column with entries v.
This is done with a "greedy" algorithm: Path-A(u,j,C,M)

Theorem: The resulting bijection is a crystal isomorphism.

The Greedy Algorithm

We rebuild the desired sublist of $\Gamma'(i)$ by going through $\Gamma'(i)$ root by root.

If $C'(v(j_1)) \prec u(j_2) \prec u(v(j_1))$, then we include the transposition (j_1, j_2). Otherwise skip. Continue.

So for our example, we have $\Gamma'(3) = ((1,2),(1,3),(1,4))$.

Comparing $C = \begin{array}{|c|}\hline 3 \\ \hline \\ \hline 2 \\ \hline\end{array}$ and $C' = \boxed{1}$ of T:

Starting with $(1,2)$, we see $C'(v(1)) = C'(4) = 1 \prec u(2) = 2 \prec u(v(1)) = u(4) = 4$.
So we include the root $(1,2)$ and set $u := u(1,2) = 2314$

We continue with the root $(1,3)$ and have $C'(4) = 1 = u(3) = 1$.
So we include $(1,3)$ and stop since we've reached our target.