Combinatorial Models for Type A Specialized Non-symmetric Macdonald Polynomials and Affine Demazure Crystals

Sam Spellman

March 7, 2023

Motivation

Many known connections between:

- representation theory of semisimple Lie algebras
- Schubert calculus and geometry of flag manifolds

Motivation

Many known connections between:

- representation theory of semisimple Lie algebras
- Schubert calculus and geometry of flag manifolds

- We want to develop a quantum/affine version of these connections.

Motivation

Many known connections between:

- representation theory of semisimple Lie algebras
- Schubert calculus and geometry of flag manifolds

- We want to develop a quantum/affine version of these connections.

Part of this story involves nonsymmetric Macdonald polynomials, affine Demazure crystals, and combinatorics.

Macdonald Polynomials

Polynomials in $(x_1, x_2, \ldots, x_n)$ associated to an irreducible rank n affine root system, with coefficients a rational function in q and t.

Macdonald Polynomials

Polynomials in $(x_1, x_2, \ldots, x_n)$ associated to an irreducible rank n affine root system, with coefficients a rational function in q and t.

There are two versions:

- symmetric (under the Weyl group action) $P_\lambda(x, q, t)$, for a dominant weight λ.

Macdonald Polynomials

Polynomials in $(x_1, x_2, \dots, x_n)$ associated to an irreducible rank n affine root system, with coefficients a rational function in q and t.

There are two versions:

- symmetric (under the Weyl group action) $P_\lambda(x, q, t)$, for a dominant weight λ.
- non-symmetric $E_\mu(x, q, t)$, for an arbitrary weight μ.

Specialized Macdonald Polynomials

Various specializations of these polynomials give characters of representations:

- $P_\lambda(x, 0, 0)$ are the irreducible characters of simple Lie algebras.
- $P_\lambda(x, q, 0)$ are graded charaters of KR modules for affine Lie algebras.

Specialized Macdonald Polynomials

Various specializations of these polynomials give characters of representations:

- $P_\lambda(x, 0, 0)$ are the irreducible characters of simple Lie algebras.
- $P_\lambda(x, q, 0)$ are graded charaters of KR modules for affine Lie algebras.

- $E_\mu(x, 0, 0)$ are characters of Demazure modules of simple Lie agebras.
- $E_\mu(x, q, 0)$ are graded characters of Demazure submodules of KR modules.

Crystals

- A crystal is a nonempty set B along with operators $e_i : B \to B \cup \{0\}$ and $f_i : B \to B \cup \{0\}$ subject to some conditions.
- Can be thought of as a finite, colored, directed graph.

Crystals

- A crystal is a nonempty set B along with operators $e_i : B \to B \cup \{0\}$ and $f_i : B \to B \cup \{0\}$ subject to some conditions.
- Can be thought of as a finite, colored, directed graph.
- All the representations in the last slide have a crystal structure.

Combinatorial Models for Macdonald Polynomials

There are two main combinatorial formulas:

1. (2008) Ram-Yip formula, sum over alcove walks. Valid in all Lie types.

Combinatorial Models for Macdonald Polynomials

There are two main combinatorial formulas:

1. (2008) Ram-Yip formula, sum over alcove walks. Valid in all Lie types.
    - For $P_\lambda(x, q, 0)$, $E_\mu(x, q, 0)$ the quantum alcove model of [LNSSS].
    - $P_\lambda(x, q, 0)$ has crystal structure due to Lenart, Lubovsky.

Combinatorial Models for Macdonald Polynomials

There are two main combinatorial formulas:

1. (2008) Ram-Yip formula, sum over alcove walks. Valid in all Lie types.
   - For $P_\lambda(x, q, 0)$, $E_\mu(x, q, 0)$ the quantum alcove model of [LNSSS].
   - $P_\lambda(x, q, 0)$ has crystal structure due to Lenart, Lubovsky.

2. (2005) Haglund-Haiman-Loehr formula, sum over fillings of Young diagrams subject to certain conditions. Valid in type A only.

Combinatorial Models for Macdonald Polynomials

There are two main combinatorial formulas:

1. (2008) Ram-Yip formula, sum over alcove walks. Valid in all Lie types.
   - For $P_\lambda(x, q, 0)$, $E_\mu(x, q, 0)$ the quantum alcove model of [LNSSS].
   - $P_\lambda(x, q, 0)$ has crystal structure due to Lenart, Lubovsky.
2. (2005) Haglund-Haiman-Loehr formula, sum over fillings of Young diagrams subject to certain conditions. Valid in type A only.
   - For $E_\mu(x, q, 0)$ the relevant fillings are the semistandard key tabloids (SSKT) defined by Assaf and Gonzalez.
   - SSKT come with a crystal structure.

We want to establish a connection between the non-symmetric type A Ram-Yip formula to the HHL formula when $t = 0$.

We want to establish a connection between the non-symmetric type A Ram-Yip formula to the HHL formula when $t = 0$.

Main Results

- In type A for $t = 0$ we define an explicit bijection with inverse map from alcove walks to SSKT.
- We construct an affine Demazure crystal structure on alcove walks (colored directed graph).
- Conjecture: the bijection is compatible with our crystal operators and the operators between semistandard key tabloids by Assaf, Gonzalez.

Type A_{n-1} notation

- $V = \mathbb{R}^n / span(1, 1, \ldots, 1)$
- The finite Weyl group is S_n
- The root system is $\Phi := \{\epsilon_i - \epsilon_j | i, j \leq n\}$

Type A_{n-1} notation

- $V = \mathbb{R}^n / span(1, 1, \ldots, 1)$
- The finite Weyl group is S_n
- The root system is $\Phi := \{\epsilon_i - \epsilon_j | i, j \leq n\}$
- Denote by (i, j) the root $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ and the transposition t_{ij};

Type A_{n-1} notation

- $V = \mathbb{R}^n / span(1, 1, \dots, 1)$
- The finite Weyl group is S_n
- The root system is $\Phi := \{\epsilon_i - \epsilon_j | i, j \leq n\}$
- Denote by (i, j) the root $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ and the transposition t_{ij};
- Identify a dominant weight λ with the partition $\lambda = (\lambda_1 \geq \dots \lambda_{n-1} \geq \lambda_n = 0)$

Type A_{n-1} notation

- $V = \mathbb{R}^n / span(1, 1, \ldots, 1)$
- The finite Weyl group is S_n
- The root system is $\Phi := \{\epsilon_i - \epsilon_j | i, j \leq n\}$
- Denote by (i, j) the root $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ and the transposition t_{ij};
- Identify a dominant weight λ with the partition $\lambda = (\lambda_1 \geq \ldots \lambda_{n-1} \geq \lambda_n = 0)$
- an arbitrary weight μ with the weak composition $(\mu_1, \mu_2, \cdots, \mu_n) \in \mathbb{Z}^n$

The Quantum Alcove Model

For a positive root α Let $H_{\alpha,k}$ be the hyperplane defined by

$$H_{\alpha,k} = \{\lambda \in V | \langle \lambda, \alpha \rangle = k \text{ for } k \in \mathbb{Z}\}$$

The Quantum Alcove Model

For a positive root α Let $H_{\alpha,k}$ be the hyperplane defined by

$$H_{\alpha,k} = \{\lambda \in V | \langle \lambda, \alpha \rangle = k \text{ for } k \in \mathbb{Z}\}$$

Alcoves: connected components of $V \setminus \left(\bigcup H_{\alpha,k}\right)$.

The Quantum Alcove Model

For a positive root α Let $H_{\alpha,k}$ be the hyperplane defined by

$$H_{\alpha,k} = \{\lambda \in V | \langle \lambda, \alpha \rangle = k \text{ for } k \in \mathbb{Z}\}$$

Alcoves: connected components of $V \setminus \left(\bigcup H_{\alpha,k} \right)$.

Fundamental alcove $A_\circ$, with a vertex at 0.

The Quantum Alcove Model

For a positive root α Let $H_{\alpha,k}$ be the hyperplane defined by

$$H_{\alpha,k} = \{\lambda \in V | \langle \lambda, \alpha \rangle = k \text{ for } k \in \mathbb{Z}\}$$

Alcoves: connected components of $V \setminus \left(\bigcup H_{\alpha,k}\right)$.

Fundamental alcove $A_\circ$, with a vertex at 0.

An Alcove path is a shortest sequence of adjacent alcoves

$$\Gamma = A_\circ \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} A_2 \xrightarrow{-\gamma_3} \dots \xrightarrow{-\gamma_m} A_m$$

The Quantum Alcove Model

For a positive root α Let $H_{\alpha,k}$ be the hyperplane defined by

$$H_{\alpha,k} = \{\lambda \in V | \langle \lambda, \alpha \rangle = k \text{ for } k \in \mathbb{Z}\}$$

Alcoves: connected components of $V \setminus \left(\bigcup H_{\alpha,k}\right)$.

Fundamental alcove $A_\circ$, with a vertex at 0.

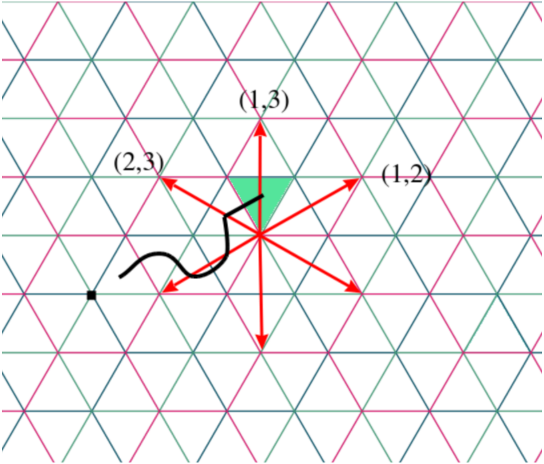
An Alcove path is a shortest sequence of adjacent alcoves

$$\Gamma = A_\circ \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} A_2 \xrightarrow{-\gamma_3} \dots \xrightarrow{-\gamma_m} A_m$$

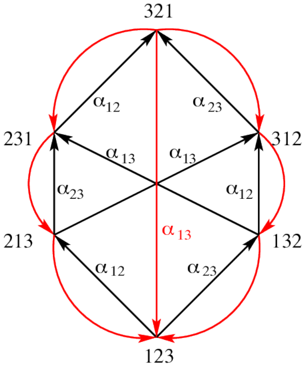
Γ is encoded by the sequence of roots $(\gamma_1, \gamma_2, \dots, \gamma_m)$

Alcove picture in type A_2 for $\mu = (0, 3, 1)$

$$\Gamma = ((12)(13)|(12)|(32)(12))$$



Quantum Bruhat Graph



Directed graph with labeled edges

$$w \xrightarrow{\alpha} ws_\alpha, \quad \text{if :}$$

$\ell(ws_\alpha) = \ell(w) + 1$ (covers of Bruhat order),

or

$$\ell(ws_\alpha) = \ell(w) - 2ht(\alpha^\vee) + 1$$

Special alcove paths (μ-chains)

Fix an arbitrary weight $\mu = (\mu_1, \cdots, \mu_n)$ with $\mu = v\lambda$, $v \in S_n$ of maximal length, and λ a dominant weight (partition).

Special alcove paths (μ-chains)

Fix an arbitrary weight $\mu = (\mu_1, \cdots, \mu_n)$ with $\mu = v\lambda$, $v \in S_n$ of maximal length, and λ a dominant weight (partition).

Construct an alcove path

$$\Gamma := \left(A_0 = A_\circ \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} \ldots \xrightarrow{-\gamma_m} A_m = wA_\circ + \mu\right).$$

where w depends on μ.

Special alcove paths (μ-chains)

Fix an arbitrary weight $\mu = (\mu_1, \cdots, \mu_n)$ with $\mu = v\lambda$, $v \in S_n$ of maximal length, and λ a dominant weight (partition).

Construct an alcove path

$$\Gamma := (A_0 = A_\circ \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} \ldots \xrightarrow{-\gamma_m} A_m = wA_\circ + \mu) .$$

where w depends on μ.

We will need a closely related path to Γ:

$$\Gamma' := -v^{-1}\Gamma = (\beta_1, \beta_2, \ldots \beta_m)$$

The β roots are all positive

v-admissible Subsets

From the path $\Gamma' = (\beta_1, \beta_2, \dots, \beta_m)$, select a subset of positions

$$J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$$

v-admissible Subsets

From the path $\Gamma' = (\beta_1, \beta_2, \ldots, \beta_m)$, select a subset of positions

$$J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$$

We say J is v-admissible if we have a path in the quantum Bruhat graph:

$$v \leftarrow vs_{\beta_{j_1}} \leftarrow vs_{\beta_{j_1}} s_{\beta_{j_2}} \leftarrow \cdots \leftarrow vs_{\beta_{j_1}} s_{\beta_{j_2}} \ldots s_{\beta_{j_s}}$$

where s_{β_i} is the transposition (i_1, i_2) if $\beta_i = \epsilon_{i_1} - \epsilon_{i_2}$.

v-admissible Subsets

From the path $\Gamma' = (\beta_1, \beta_2, \dots, \beta_m)$, select a subset of positions

$$J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$$

We say J is v-admissible if we have a path in the quantum Bruhat graph:

$$v \leftarrow vs_{\beta_{j_1}} \leftarrow vs_{\beta_{j_1}} s_{\beta_{j_2}} \leftarrow \cdots \leftarrow vs_{\beta_{j_1}} s_{\beta_{j_2}} \dots s_{\beta_{j_s}}$$

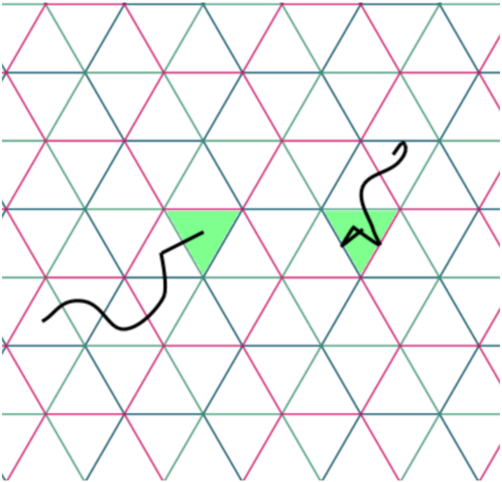
where s_{β_i} is the transposition (i_1, i_2) if $\beta_i = \epsilon_{i_1} - \epsilon_{i_2}$.

J can be viewed geometrically as a folding of Γ along hyperplanes.

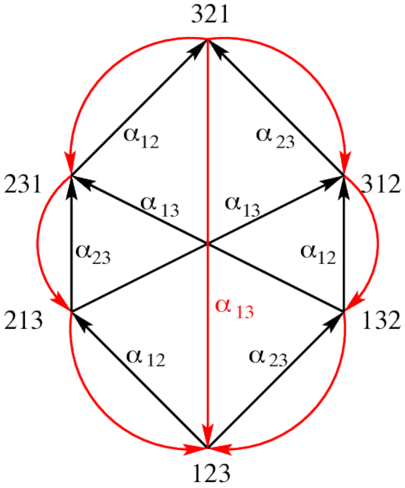
Example for $\mu = (0,3,1)$, $v = 231$, $J = \{1,2,5\}$

$\Gamma = ((12)(13)|(12)|(32)(12))$

$\Gamma' = ((13)(23)|(13)|(12)(13))$



Quantum Bruhat graph for $W = S_3$

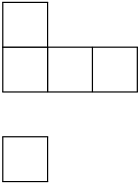


Semistandard Key Tabloids (SSKT(μ))

Semistandard Key Tabloids (SSKT(μ))

A diagram $\mathrm{dg}(\mu)$ of the weak composition μ consists of μ_i boxes left-justified in row i.

For example, $\mathrm{dg}(\mu)$ is shown below for $\mu = (1, 0, 3, 1)$:



Semistandard Key Tabloids (SSKT(μ))

A semistandard key tabloid of shape μ is a filling of dg(μ) with positive integers from 1 to n, subject to some conditions on entries in pairs and triples of boxes.

$$\mu = (0, 2, 1, 2)$$

$$\begin{array}{|c|c|}\hline 4 & 4 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \hline 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \hline 1 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \hline 1 & 1 \\ \hline \end{array}$$

Filling map from v-admissible subsets to SSKT(μ)

From an alcove path Γ' and a v-admissible subset J:

- Start from the permutation v.
- v^j is the permutation obtained by applying to v all transpositions in J up to the j-th segment of Γ'.
- The entry in row i, column j is

$$\mathit{fill}(J)(i,j) := v^j v^{-1}(i)$$

Filling map from v-admissible subsets to SSKT(μ)

From an alcove path Γ' and a v-admissible subset J:

- Start from the permutation v.
- v^j is the permutation obtained by applying to v all transpositions in J up to the j-th segment of Γ'.
- The entry in row i, column j is

$$\mathit{fill}(J)(i,j) := v^j v^{-1}(i)$$

Recall $\Gamma' = ((13)(23)|(13)|(12)(13))$, $J = \{1,2,5\}$, $v = 231$

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array} \xrightarrow{(1,3)} \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array} \xrightarrow{(2,3)} \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \;\Big|\; \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \;\Big|\; \xrightarrow{(1,3)} \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

Filling map from v-admissible subsets to SSKT(μ)

From an alcove path Γ' and a v-admissible subset J:

- Start from the permutation v.
- v^j is the permutation obtained by applying to v all transpositions in J up to the j-th segment of Γ'.
- The entry in row i, column j is

$$fill(J)(i,j) := v^j v^{-1}(i)$$

Recall $\Gamma' = ((13)(23)|(13)|(12)(13))$, $J = \{1,2,5\}$, $v = 231$

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array} \xrightarrow{(1,3)} \begin{array}{|c|}\hline 2\\\hline 3\\\hline 1\\\hline\end{array} \xrightarrow{(2,3)} \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \;\Big|\; \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \;\Big|\; \xrightarrow{(1,3)} \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

$$fill(J) = \begin{array}{|c|cc}\hline 2 & \multicolumn{2}{c}{} \\\hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\\hline\end{array}$$

Theorem

Given a v-admissible subset J, fill(J) is a semistandard key tabloid of shape μ with inverse map given by a "greedy" algorithm.

Theorem

The set of v-admissible subsets of Γ', together with operators $e_0, e_1, \ldots, e_{n-1}$ and $f_0, f_1, \ldots, f_{n-1}$ defined by

$$e_i(J) = J \setminus \{m\} \cup \{k\}$$

$$f_i(J) = J \setminus \{k\} \cup \{m\}$$

form an affine Demazure crystal, where k and m are positions in the alcove path Γ (or possibly ∞), and are determined by J and i.

Affine Demazure Crystal for $\mu = (0, 3, 1)$