RESEARCH STATEMENT

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My research interests span the areas of algebraic combinatorics and representation theory. A theme in my research is in developing combinatorial models to facilitate computations, particularly related to the representation theory of Lie algebras and quantum groups. My main approach is to encode complex algebraic or geometric objects as combinatorial objects, then use combinatorial methods to prove statements about the objects or to run explicit computations. In this regard my work has both theoretical and practical applications. Computer software such as SAGE plays a role in this process; from helping test the models to executing the algorithms/computations.

Background

Crystals

Crystal bases have origins in the representation theory of quantum groups as well as in the theory of exactly solvable lattice models in statistical physics. They were introduced by Kashiwara [10], and provide a way to view the representation theory of quantum groups $U_q(\mathfrak{g})$ (q-deformations of the universal enveloping algebra) at $q = 0$. Quantum groups were first introduced independently by Drinfeld [5] and Jimbo [9] in the context of two-dimensional statistical mechanics. In this setting the parameter q represents temperature, so the limit $q \to 0$ corresponds to behavior near "absolute zero". In this limit, crystal bases come with a simple combinatorial structure in the form of a colored directed graph called a *crystal*. This graph contains the essential information for solving classical problems in representation theory such as computing characters, decomposing tensor products, and branching rules. In order to reap the above benefits, effective realizations of crystals are needed.

In the literature, there has been many developments in combinatorial models for crystals. Geometric models, as well as another big class of models, based on lattice points in polytopes (BZ patterns) [6], are some examples. In the case of highest weight crystals, Kashiwara and Nakashima [11] constructed type-dependent tableau models for non-exceptional classical types. The tableaux are built out of columns called *KN columns*. For symmetrizable Kac-Moody algebras, a type-independent model is due to Littleman [16] and is based on paths in the weight lattice. Lenart and Postnikov [15] developed another type-independent model for highest weight crystals called the *alcove model*, which is based on the affine Weyl group and enumerating saturated chains in Bruhat order. The alcove model has advantages over the Littleman path model due to its simplicity, ease of computations, and widespread applications.

Kirillov-Reshetikhin (KR) modules [12] are finite-dimensional modules $W^{r,s}$ (not of highest weight) for affine Lie algebras, indexed by positive integers r, s. In most cases these modules admit crystal bases, whose crystal graphs are denoted $B^{r,s}$. A generalization of the alcove model, called the *quantum alcove model*, [13] describes tensor products of "column-shape" KR crystals $B^{r,1}$ in untwisted types. It is based on enumerating chains in the so-called quantum Bruhat graph of the corresponding finite Weyl group, which is obtained from the Hasse diagram of the Bruhat order by adding extra downward edges.

Macdonald polynomials

The symmetric Macdonald polynomials $P_\lambda(x; q, t)$, indexed by dominant weights λ, are Weyl group invariant polynomials with rational function coefficients in q, t [17]. They generalize the irreducible characters of the corresponding simple Lie algebras, which are recovered for $q = t = 0$. Due to their connections to subjects such as: double affine Hecke algebras, affine Lie algebras, Hilbert schemes, statistical physics, etc, explicit formulas are desired. A uniform combinatorial formula for $P_\lambda(x;q,t)$ was given by Ram and Yip [18] in terms of *alcove walks*, and in type A there is also a formula based on fillings of skyline-shaped diagrams due to Haglund-Haiman-Loehr (HHL) [7]. There are also non-symmetric versions of Macdonald polynomials $E_\mu(x;q,t)$, where μ is an arbitrary weight [17], as well as Ram-Yip and HHL formulas for them.

Current Work

In [14] it was shown that non-symmetric Macdonald polynomials coincide with graded characters of Demazure-type submodules of tensor products of single-column KR modules. As discussed in the mentioned paper, the latter can be thought of as Demazure submodules of certain quotients of *level zero extremal weight modules.* The corresponding crystals are called *DARK crystals* (**K**irillov-**R**eshetikhin **A**ffine **D**emazure) in [3]. Thus, there is expected to be a combinatorial model for the mentioned DARK crystals based on the alcove walks in the non-symmetric Ram-Yip formula specialized at $t = 0$.

My current work is the development of a "non-symmetric" version of the quantum alcove model to describe the DARK crystals mentioned above. This model has the benefit over the model described in [14] in that it eliminates a condition on the objects known as the *end condition.*

Specializing to type A, the non-symmetric Macdonald polynomials are also expressed in terms of *semistandard key tabloids.* These were introduced in [1] as special cases of the *non-attacking skyline fillings* in the HHL formula for non-symmetric Macdonald polynomials [8]. They were given an affine crystal structure in [2], which realizes the crystals of certain level one Demazure modules for the affine Lie algebra $\widehat{\mathfrak{sl}_n}$.

Another aspect of my work is the construction of an affine crystal isomorphism between the "non-symmetric" quantum alcove model in type A and the corresponding semistandard key tabloids, i.e., an alcoves-to-fillings map in the type A Demazure case. While the tabloid model is simpler, it has less easily accessible information, so it is hard to use for computations, for instance of the energy function and the combinatorial R-matrix. As these computations are much simpler in the quantum alcove model, an alternative is to relate them to the tabloid model, via an affine crystal isomorphism from the former model to the latter one.

Future Work

Moving forward, the above results are expected to generalize to types B,C, and D. In the near future I would like to construct analogous alcove-to-fillings correspondences in the non-symmetic case, based on previous work in [4] (types B and D) and [13] (type C). The goal is to derive types B, C, and D analogues of semistandard key tabloids (i.e. Demazure-type analogues of the corresponding concatenations of KN columns) which would give tableau formulas for the type B, C, and D Macdonald polynomials.

I am also interested in other areas of combinatorics, including symmetric function theory, graph theory, and Schubert calculus, as well as further study of the representation theory of Hopf algebras and quantum groups.

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