LECTURE - 01
DATE - 07.03.2022 DAY - MONDAY

UNIT - 1 PROBABILITY AND RANDOM VARIABLE

Pre-Requisite Terms :

- RANDOM EXPERIMENT : It is an experiment which can be repeated any number of times under the same conditions, but does not give unique result. A random experiment is also called trial and the outcomes are known as events.

  Example : Rolling a die is a trial and getting 5 (or) 6 is an event (outcome).

- SAMPLE SPACE : The collection of all possible outcomes of an experiment is called sample space.

  Example : Tossing a coin
  $S = \{H, T\} \Rightarrow n(S) = 2$

  When a die is thrown
  $S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$

- MUTUALLY EXCLUSIVE EVENTS (OR DISJOINT) : Two events A and B are said to be mutually exclusive if $A \cap B = \phi$.

- EQUALLY LIKELY EVENTS : Two events are said are said to be equally likely if either of them cannot occur in preference to other.

  Example : When tossing a die all 6 sample points are equally likely event.

- **FAVOURABLE EVENTS**: The number of outcomes favourable in an event / trial.

- **EXHAUSTED EVENTS**: Outcomes are said to be exhaustive when they include all possible outcomes.

- **INDEPENDENT EVENTS**: Two events A & B are said to be independent when outcomes do not depend on each other.

Let A be an event and P is the probability of getting an event A, then its mathematical definition is

$$P(A) = \frac{\text{Total no. of favourable outcomes}}{\text{Total no. of possible outcomes}}$$

* **Axioms of Probability**:

(i) $0 \leq P(A) \leq 1$

(ii) $P(S) = 1$ (or) Total Probability $= 1$

* **RESULT**:

(i) P – probability of success
q – probability of failure
Then $\boxed{p+q=1}$

(ii) $P[\text{Impossible event}] = \phi$ (or) 0

(iii) **ADDITION LAW OF PROBABILITY**:

Let A & B be 2 events

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \longrightarrow$ either A or B or both

(iv) MULTIPLICATION LAW :

$$P(A \cap B) = \begin{cases} 0 \rightarrow A \,\&\, B \text{ are mutually exclusive.} \\ P(A)\,P(B) \rightarrow A \,\&\, B \text{ are independent events.} \end{cases}$$

must, and

(v) CONDITIONAL PROBABILITY :

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Event in denominator should be happened first.

A given B

Example : B → Enter the exam hall
A → To take the test

* Examples :-

Q1) From a bag containing 10 black and 12 white balls, a random ball is drawn at random. What is the probability that it is black?

Solution : Total number of balls = 10 + 12 = 22

$$P[1 \text{ black ball}] = \frac{\text{Total number of fav. outcomes}}{\text{Total number of possible outcomes}}$$

$$= \frac{10C_1}{22C_1} = \frac{10}{22} = \underline{\underline{0.4545}}$$

$$\because \boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$$

Q2) From a well shuffled deck of 52 playing cards 4 cards are selected at random. Find the probability of that selected card is :-

(i) 3 Spades, 1 Heart

(ii) 2 Kings, 1 Ace, 1 Queen

(iii) All are diamonds.

(iv) There is only 1 card from each suit.

Solution: In a deck of 52 playing cards there are: 13 diamonds, 13 spades, 13 hearts, 13 clubs, 4 aces, 4 Kings and 4 Queens.

(i) $P[\text{3 spades, 1 heart}] = \dfrac{{}^{13}C_3 \times {}^{13}C_1}{{}^{52}C_4} = \underline{\underline{0.0137}}$

(ii) $P[\text{2 Kings, 1 Ace, 1 Queen}] = \dfrac{{}^{4}C_2 \times {}^{4}C_1 \times {}^{4}C_1}{{}^{52}C_4} = \underline{\underline{0.0004}}$

(iii) $P[\text{All are diamonds}] = \dfrac{{}^{13}C_4}{{}^{52}C_4} = \underline{\underline{0.0026}}$

(iv) $P[\text{There is only 1 card from each suit}] = \dfrac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4}$

$= \underline{\underline{0.1055}}$

LECTURE – 02
DATE – 08.03.22 DAY – TUESDAY

* RANDOM VARIABLE :-

Definition: A random variable X whose value is determined by the outcome of random experiment.

Example :- Tosing a coin 3 times then the expectation of Head.

$X = \{0, 1, 2, 3\}$

Classification of Random Variable

Discrete Random Variable → Countable → $X = 0, 1, 2, ----$

Continuous Random Variable → Uncountable → $0 < X < 1$

* DISCRETE RANDOM VARIABLE :

If the random variable X, takes countable number of real values then X is said to be Discrete Random Varia-ble.

Example → Number of telephone calls per unit time.

Results :- (i) $P(x_i) \geq 0 \; \forall i$

(ii) Probability Mass Function

* PROBABILITY MASS FUNCTION (Proofs)

For a discrete random variable X,

(i) $P(x_i) \geq 0, \; \forall i$

(ii) $\sum_{i=0}^{\infty} P(x_i) = 1$, then X is probability mass function is it is represented by

x	1	2	-	-	-
$P(x)$	1/2	1/3	-	-	-

(iii) Expected Value of X (or) Mean (or) Average (or) $E(x)$

$$\boxed{E(x) = \sum_{x=0}^{\infty} x\,P(x)}$$

(iv) Variance : $\boxed{E(x^2) - [E(x)]^2 = Var(x)}$ (OR)

$$E(x^2) - (Mean)^2 = Var(x)$$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 P(x)$$

$\sqrt{var(x)} = \sigma$ [Standard Deviation]

(v) $E(ax+b) = aE(x) + b$

(vi) $var(ax+b) = a^2\, var(x)$

(vii) $var(cons) = 0$

* PROBLEMS FROM DISCRETE RANDOM VARIABLE

Q1) A random variable X has the following probability mass function,

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

then find (i) k (ii) $P(x<6)$ (iii) $P(x \geq 6)$ (iv) $P(0<x<5)$
(v) Distribution function

Solution : (i) $\because$ The given data is probability mass function

$$\sum P(x_i) = 1$$

$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow 10k^2 + 9k = 1$

$\Rightarrow 10k^2 + 9k - 1 = 0$

$\Rightarrow 10k^2 + 10k - k - 1 = 0$

$\Rightarrow 10k(k+1) - 1(k+1) = 0$

$\Rightarrow (k+1)(10k-1) = 0$

$\Rightarrow k+1 = 0$ $\qquad\qquad \Rightarrow 10k - 1 = 0$

$\boxed{k = -1}$ $\qquad\qquad \Rightarrow 10k = 1$

cannot be possible $\qquad\qquad \Rightarrow \boxed{k = 1/10}$

x	0	1	2	3	4	5	6	7
$P(x)$	0	$1/10$	$2/10$	$2/10$	$3/10$	$1/100$	$2/100$	$17/100$

(ii) $P(x<6) = 1 - P(x \geq 6)$

$= 1 - [P(6) + P(7)]$

$= 1 - \left[\frac{2}{100} + \frac{17}{100}\right] = \frac{81}{100}$

(iii) $P(x \geq 6) = P(6) + P(7) = \frac{2}{100} + \frac{17}{100} = \frac{19}{100}$

(iv) $P(0<x<5) = P(1) + P(2) + P(3) + P(4)$

$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10}$

(v) <u>Distribution</u> <u>Function</u>:

x	$P(x)$	$F(x)$
0	0	0
1	$1/10$	$1/10$
2	$2/10$	$3/10$
3	$2/10$	$5/10$
4	$3/10$	$8/10$
5	$1/100$	$81/100$
6	$2/100$	$83/100$
7	$17/100$	$100/100 \rightarrow 1$

LECTURE – 03

DATE – 09.03.2022 DAY – WEDNESDAY

Q2) The probability mass function is defined by :

x	-2	-1	0	1	2
$p(x)$	a	a	$2a$	a	a

Find (i) a (ii) $P\left[\frac{1}{2} < x < \frac{5}{2} / x > 1\right]$

(iii) Mean and Variance (iv) Distribution function

Solution :(i): It is probability mass function

$$\sum p(x) = 1$$

$\Rightarrow a + a + 2a + a + a = 1$

$\Rightarrow 6a = 1$ $\Rightarrow \boxed{a = \frac{1}{6}}$

x	-2	-1	0	1	2
$P(x)$	$1/6$	$1/6$	$2/6$	$1/6$	$1/6$

(ii) $P\left[\frac{1}{2} < x < \frac{5}{2} / x > 1\right] = \dfrac{P\left[\left(\frac{1}{2} < x < \frac{5}{2}\right) \cap x > 1\right]}{P(x > 1)}$

$\boxed{P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}}$ $= \dfrac{P\left[(x = 1, 2) \cap (x = 2)\right]}{P(x = 2)}$

$$= \frac{P(x = 2)}{P(x = 2)} = 1$$

(iii) Mean :

$E(x) = \sum x p(x) = \left(-\cancel{2} \times \frac{1}{6}\right) + \left(-\cancel{1} \times \frac{1}{6}\right) + 0 + \left(\cancel{1} \times \frac{1}{6}\right) + \left(\cancel{2} \times \frac{1}{6}\right)$

$= 0$

$\boxed{E(x) = 0}$

Variance : Var(x) : $E(x^2) = \sum x^2 p(x)$

$= (-2)^2\left(\frac{1}{6}\right) + (-1)^2\left(\frac{1}{6}\right) + 0 + \left(1^2 \times \frac{1}{6}\right) + \left(2^2 \times \frac{1}{6}\right)$

$= \frac{4}{6} + \frac{1}{6} + \frac{1}{6} + \frac{4}{6} = \frac{10}{6}.$

Var (x) : $E(x)^2 - [E(x)]^2$

$= \frac{10}{6} - 0 = \frac{10}{6}$

$\boxed{\therefore \text{Var}(x) = \frac{10}{6} = \sigma^2}$

Standard Deviation $= \sigma = \sqrt{\frac{10}{6}}$

(iv) Distribution Function :

x	$P(x)$	$F(x)$
-2	$1/6$	$1/6$
-1	$1/6$	$2/6$
0	$2/6$	$4/6$
1	$1/6$	$5/6$
2	$1/6$	$6/6 \to 1$

Q3) A Random variable X has the following probab-ility Mass function,

x	0	1	2	3	4
$P(x)$	k	3k	5k	7k	9k

(i) Find k (ii) $P(x \geq 3)$ (iii) $P[0 < x < 4)$
(iv) Mean & Variance (v) Distribution Function

Solution: (i) $\sum P(x) = 1$

$\Rightarrow k + 3k + 7k + 5k + 9k = 1$ $\Rightarrow 25k = 1$

$\Rightarrow \boxed{k = 1/25}$

x	0	1	2	3	4
P(x)	1/25	3/25	5/25	7/25	9/25

(ii) $P(x \geq 3) = P(3) + P(4)$

$$= \frac{7}{25} + \frac{9}{25} = \frac{16}{25}$$

(iii) $P[0 < x < 4] = P(1) + P(2) + P(3)$

$$= \frac{3}{25} + \frac{5}{25} + \frac{7}{25} = \frac{15}{25} = \frac{3}{5}$$

(iv) Mean : $E(x) = \sum x P(x)$

$$= \left(0 \times \frac{1}{25}\right) + \left(1 \times \frac{3}{25}\right) + \left(2 \times \frac{5}{25}\right) + \left(3 \times \frac{7}{25}\right) + \left(4 \times \frac{9}{25}\right)$$

$$= \frac{3}{25} + \frac{10}{25} + \frac{21}{25} + \frac{36}{25}$$

$$= \frac{70}{25} = \underline{\underline{2.8}}$$

$$\boxed{E(x) = 2.8}$$

$Var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \sum x^2 P(x)$

$$= 0 + \left(1^2 \times \frac{3}{25}\right) + \left(2^2 \times \frac{5}{25}\right) + \left(3^2 \times \frac{7}{25}\right) + \left(4^2 \times \frac{9}{25}\right)$$

$$= \frac{3}{25} + \frac{20}{25} + \frac{63}{25} + \frac{144}{25} = \frac{230}{25} = \underline{\underline{9.2}}$$

$$\boxed{E(x^2) = 9.2}$$

$Var(x) = E(x^2) - [E(x)]^2 = 9.2 - (2.8)^2$

$$= 9.2 - 7.84 = \underline{\underline{1.36}}$$

$$\boxed{Var(x) = 1.36}$$

(v) Distribution Function :

x	$P(x)$	$F(x)$
0	1/25	1/25
1	3/25	4/25
2	5/25	9/25
3	7/25	16/25
4	9/25	25/25 $\longrightarrow$ 1

LECTURE — 04

DATE – 10.03.2022 DAY – THURSDAY

Q4) If X is the random variable that takes the value 1, 2, 3, 4 such that,

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$$

Find the probability distribution, continuous distribution function, Mean and Variance.

Solution : Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

$2P(X=1) = k$	$3P(X=2) = k$	$P(X=3) = k$	$5P(X=4) = k$
$P(X=1) = \frac{k}{2}$	$P(X=2) = \frac{k}{3}$		$P(X=4) = k/5$

X	1	2	3	4
P(X)	k/2	k/3	k	k/5

$$\Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\Rightarrow \frac{15k + 10k + 30k + 6k}{30} = 1$$

$$\Rightarrow \frac{40k + 21k}{30} = 1 \qquad \Rightarrow \frac{61k}{30} = 1$$

$$\Rightarrow 61k = 30 \qquad \boxed{k = \frac{30}{61}}$$

x	1	2	3	4
$P(x)$	$15/61$	$10/61$	$30/61$	$6/61$

Distribution Function :-

Continuous distribution function :

x	$P(x)$	$F(x)$
1	$15/61$	$15/61$
2	$10/61$	$25/61$
3	$30/61$	$55/61$
4	$6/61$	$61/61 \longrightarrow 1$

Mean : $E(x) = \sum xP(x)$

$$= 1\left(\frac{15}{61}\right) + 2\left(\frac{10}{61}\right) + 3\left(\frac{30}{61}\right) + 4\left(\frac{6}{61}\right)$$

$$= \frac{15}{61} + \frac{20}{61} + \frac{90}{61} + \frac{24}{61}$$

$$= \frac{35+114}{61} = \frac{149}{61}$$

$$\boxed{E(x) = 2.442}$$

Var(x) : $E(x^2) = \sum x^2 p(x)$

$$= \frac{15}{61} + \frac{40}{61} + \frac{270}{61} + \frac{96}{61}$$

$$= \frac{421}{61} = \underline{6.90}$$

$$\boxed{E(x^2) = 6.90}$$

$$\boxed{Var(x) = 0.9366}$$

$$Var(x) = 6.90 - (2.442)^2 = \underline{0.9366}$$

Discrete Random Variable $\sum$ $\longrightarrow$ $\int$ Continuous Random Variable

* CONTINUOUS RANDOM VARIABLE

If a random variable takes finitely or infinitely uncountable values then it is continuous random variable and the probability function is $f(x)$.

* PROBABILITY DENSITY FUNCTION (PDF)

If $\int_{-\infty}^{\infty} f(x)\,dx = 1$, then $f(x)$ is probability density function (pdf).

RESULTS :

(i) Mean $E(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx$

(ii) Variance $= E(x^2) - [E(x)]^2$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

(iii) cdf $F(x) = \int_{-\infty}^{x} f(x)\,dx$ (x → (always less), $-\infty$ → (available))

Q5) Given that pdf of the random variable X is $f(x) = kx$, $0 < x < 1$. Find

(i) k (ii) $P(x > 0.5)$ (iii) cdf. (iv) Means and Variance

Solution : (i) ∵ $f(x)$ is a probability density functions

$$\int_0^1 kx\,dx = 1 \quad \Rightarrow k\left(\frac{x^2}{2}\right)_0^1 = 1$$

$$\Rightarrow k\left(\tfrac{1}{2}\right) = 1 \quad \Rightarrow \boxed{k = 2}$$

Hence, $f(x) = 2x$, $0 < x < 1$.

(ii) $P(x > 0.5) = \int_{0.5}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{0.5}^{1}$

$= \left(x^2\right)_{0.5}^{1} = 1 - (0.5)^2 = \boxed{0.75}$.

(iii) <u>cdf</u>: $F(x) = \int_{0}^{x} 2x\,dx = \cancel{2}\left(\frac{x^2}{\cancel{2}}\right)_{0}^{x}$

$\boxed{F(x) = x^2,\ 0 < x < 1}$

(iv) <u>Mean</u>: $E(x) = \int_{0}^{1} x\, f(x)\,dx$

$= \int_{0}^{1} x(2x)\,dx = 2\int_{0}^{1} x^2\,dx$

$= 2\left(\frac{x^3}{3}\right)_{0}^{1} = \frac{2}{3}\left(x^3\right)_{0}^{1}$

$\boxed{E(x) = \frac{2}{3}}$

$E(x^2) = \int_{0}^{1} x^2 f(x)\,dx = \int_{0}^{1} x^2 (2x)\,dx$

$= 2\left[\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2}(1)$

$\boxed{E(x^2) = \frac{1}{2}}$

$Var(x) = E(x^2) - [E(x)]^2$

$= \frac{1}{2} - \left(\frac{2}{3}\right)^2$

$= \underline{0.055}$

$\boxed{Var(x) = 0.055}$

LECTURE - 05

DATE - 14.03.2022 DAY - MONDAY

* HOMEWORK :

Q) If $f(x) = \begin{cases} kxe^{-x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$ Find :

(i) k (ii) Mean and Variance (iii) cdf.

Solution : (i) ∵ f(x) is pdf $= \int_0^\infty f(x)dx = 1$

$$\int_0^\infty kxe^{-x}dx = 1$$

$$\Rightarrow k\int_0^\infty \underset{u}{x}\,\underset{dv}{e^{-x}\,dx} = 1$$

$$\boxed{\int u\,dv = uv - u'v_1 + u''v_2 - \cdots}$$

$u = x$, $u' = 1$, $u'' = 0$

$dv = e^{-x}dx$

$v = \frac{e^{-x}}{-1} \Rightarrow v = -e^{-x}$

$v_1 = \frac{e^{-x}}{(-1)^x} \Rightarrow v_1 = e^{-x}$

$$\Rightarrow k\left[-xe^{-x} - e^{-x}\right]_0^\infty = 1 \qquad \boxed{e^{-\infty} = 0}$$

$$\Rightarrow k\left[0 - (0-1)\right] = 1 \Rightarrow k(1) = 1 \Rightarrow \boxed{k=1}$$

$$\boxed{f(x) = xe^{-x}, \; x>0}$$

(ii) Mean : $E(x) = \int_0^\infty x f(x)dx$

$$= \int_0^\infty x\,xe^{-x}\,dx = \int_0^\infty \underset{u}{x^2}\,\underset{dv}{e^{-x}dx}$$

$$= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^\infty$$

$$= \{(0) - (0-2)\} = 2 \qquad \boxed{E(x) = 2}$$

$$E(x^2) = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty x^2 x e^{-x}\,dx$$

$$= \int_0^\infty \underbrace{x^3}_{u}\ \underbrace{e^{-x}}_{dv}\,dx = \left[-x^3e^{-3x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}\right]_0^\infty$$

$$= [(0) - (-6)] = 6 \qquad \boxed{E(x^2) = 6}$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2$$

$$= 6 - (2)^2 = 6 - 4 = 2$$

$$\boxed{\text{Var}(x) = 2}$$

(iii) cdf :- $F(x) = \int_0^x f(x)\,dx = \int_0^x xe^{-x}\,dx$

$$F(x) = \left[-xe^{-x} - e^{-x}\right]_0^x$$

$$\boxed{F(x) = \left\{-xe^{-x} - e^{-x} + 1\right\}} \quad \boxed{x > 0}$$

* PREVIOUS YEAR UNIVERSITY QUESTION FROM DISCRETE & CONTINUOUS RANDOM VARIABLES

Q1) If X has the probability distribution,

x	-1	0	1	2
$P(x)$	0.3	0.1	0.4	0.2

Find the Mean, Variance and cdf. (Part-B, 4 marks)

Solution: Mean $(E(x)) = \sum x P(x)$

$$= -0.3 + 0.0 + 0.4 + 0.4$$

$$= 0.8 - 0.3$$

$$= 0.5$$

$$\boxed{E(x) = 0.5}$$

$Var(x) = E(x^2) = \sum x^2 p(x)$

$= 0.3 + 0 + 0.4 + 0.8$

$= \underline{1.5}$

$\boxed{Var(x) = 1.5}$

$Var(x) : E(x)^2 - [E(x)]^2$

$= 1.5 - (0.5)^2$

$= 1.5 - 0.25 = \underline{1.25}$

$\boxed{Var(x) = 1.25}$

Standard Deviation $(\sigma) = \sqrt{Var} = \underline{1.118}$.

cdf

x	$P(x)$	$F(x)$
-1	0.3	0.3
0	0.1	0.4
1	0.4	0.8
2	0.2	1.0

Q2) What is the expectation on when a die is thrown?
(Part-B, 4 marks)

Solution: Random Variables $= X = \{1, 2, 3, 4, 5, 6\}$

X	1	2	3	4	5	6
P(x)	1/6	1/6	1/6	1/6	1/6	1/6

Expectation: $E(x) = \sum x p(x)$

$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$

$= \frac{1}{6}[1+2+3+4+5+6] = \frac{21}{6} = \frac{7}{2}$

$\boxed{E(x) = \frac{7}{2}}$

Q3) What is the value of $var(4x+8)$ when $var(x)$ is 6?
(Part A, 1 mark)

Solution: $\boxed{var(ax+b) = a^2\, var(x)}$

$var(4x+8) = 4^2 var(x)$

$= 16 \times 6 = \underline{\underline{96}}.$

Q4) If $E(x^2) = 8$, $E(x) = 2$, then find $var(x)$? (Part A, 1 mark)

Solution: $var(x) = E(x^2) - [E(x)]^2$

$= 8 - 2^2 = \underline{4}.$

$\boxed{var(x) = 4}$

Q5) If $f(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$ is pdf then find k and cdf. (Part B, 4 mark)

Solution: $\because f(x)$ is pdf. $\Rightarrow \int_{-\infty}^{\infty} k\left(\frac{1}{1+x^2}\right) dx = 1$

$k\left[\tan^{-1}(x)\right]_{-\infty}^{\infty} = 1$

$\Rightarrow k\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right] = 1$

$\Rightarrow k\left[\tan^{-1}(\infty) + \tan^{-1}(\infty)\right] = 1$

$\Rightarrow k'\left[2\tan^{-1}(\infty)\right] = 1$

$\Rightarrow k\left[\frac{2\pi}{2}\right] = 1 \quad \Rightarrow \boxed{k = \frac{1}{\pi}}$

$\boxed{f(x) = \frac{1}{\pi} \times \frac{1}{1+x^2}, -\infty < x < \infty}$

cdf: $F(x) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+x^2} dx$

$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{\infty}$

$= \frac{1}{\pi}\left\{\tan^{-1} x - \tan^{-1}(-\infty)\right\}$

$$= \frac{1}{\pi}\left\{ \tan^{-1}x + \tan^{-1}\infty \right\}$$

$$\boxed{f(x) = \frac{1}{\pi}\left\{ \tan^{-1}x + \frac{\pi}{2} \right\} ; -\infty < x < \infty}$$

LECTURE – 06

DATE – 15.03.2022 DAY – TUESDAY

Q6) If X is a continuous random variable with pdf $f(x) = Ax$, $0 < x < 1$, then find

(i) A (ii) Mean, Variance and cdf. (iii) $P(x \leq 0.4)$

(iv) $P(x > 3/4)$ (v) $P(x > 1/2)$ (vi) $P(1/2 < x < 3/4)$

(vii) $P(x > 3/4 \mid x > 1/2)$ (Part C, 12 Marks)

Solution : $\because f(x)$ is a pdf $\Rightarrow \int_0^1 Ax\,dx = 1$

(i) $A\left(\frac{x^2}{2}\right)_0^1 = 1 \Rightarrow A(1/2) = 1$ $\boxed{A = 2}$

$$\boxed{f(x) = 2x , \quad 0 < x < 1}$$

(ii) Mean : $E(x) = \int_0^1 x\, f(x)\,dx = \int_0^1 2x^2\,dx$

$$= 2\left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3} \qquad \boxed{E(x) = \frac{2}{3}}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\, 2x\,dx = \int_0^1 2x^3\,dx$$

$$= 2\left(\frac{x^4}{4}\right)_0^1 = \frac{1}{2} \qquad \boxed{E(x^2) = \frac{1}{2}}$$

Variance : $Var(x) = E(x^2) - [E(x)]^2$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18} \qquad \boxed{Var(x) = \frac{1}{18}}$$

(iii) $P(x \leq 0.4) = \int_0^{0.4} 2x\,dx = \cancel{2}\left(\frac{x^2}{\cancel{2}}\right)_0^{0.4}$

$= (x^2)_0^{0.4} = \underline{\underline{0.16}}$

$$\boxed{P(x \leq 0.4) = 0.16}$$

(iv) $P(x > 3/4) = \int_{3/4}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{3/4}^{1} = 1 - \left(\frac{3}{4}\right)^2 = \underline{\underline{0.4375}}$

$$\boxed{P(x > 3/4) = 0.4375}$$

(v) $P(x > 1/2) = \int_{1/2}^{1} 2x\,dx = (x^2)_{1/2}^{1} = \underline{\underline{0.75}}$

$$\boxed{P(x > 1/2) = 0.75}$$

(vi) $P(1/2 < x < 3/4) = \int_{1/3}^{3/4} 2x\,dx = (x^2)_{1/2}^{3/4} = \underline{\underline{0.315}}$

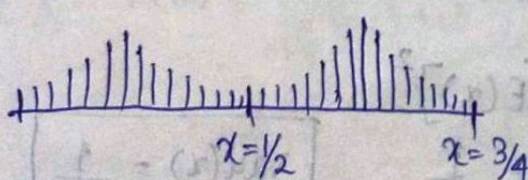
$$\boxed{P(1/2 < x < 3/4) = 0.315}$$

(vii) $P\left(\underbrace{x > 3/4}_{A} \Big/ \underbrace{x > 1/2}_{B}\right)$

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

$= \dfrac{P(x > 3/4 \cap x > 1/2)}{P(x > 1/2)}$

$= \dfrac{P(x > 3/4)}{P(x > 1/2)} = \dfrac{0.4375}{0.75} = \underline{0.5833}$



$$\boxed{P\left(x > 3/4 \Big/ x > \frac{1}{2}\right) = 0.5833}$$

* MOMENTS AND MOMENT GENERATING FUNCTION (MGF)

MOMENTS

RAW MOMENTS (Moments about origin)

$$\mu_r' = \begin{cases} \sum x^r p(x) \\ x^r f(x)\,dx \end{cases}$$

where $r = 1, 2, 3, 4$

CENTRAL MOMENTS (Moments about mean)

$$\mu_r = \begin{cases} \sum (x-\bar{x})^r p(x) \\ \int (x-\bar{x})^r f(x) \end{cases}$$

* RELATION BETWEEN RAW & CENTRAL MOMENTS

$\mu_1 = 0$ (always)

$\mu_2 = \mu_2' - (\mu_1')^2$ (Variance)

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$$\boxed{\mu_1' = \text{Mean}}$$

Q1) If X has the pdf $f(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$. Find first 4 central moments.

Solution: $f(x) = \dfrac{x+1}{2}, \quad -1 < x < 1$

$\mu_r' = \int x^r f(x)\,dx$ where $r = 1$, $\Rightarrow \mu_1' = \int x\, f(x)\,dx$

$$\mu_1' = \int_{-1}^{1} x\left(\frac{x+1}{2}\right)dx = \frac{1}{2}\int_{-1}^{1}(x^2 + x)\,dx$$

$$= \frac{1}{2}\left\{\frac{x^3}{3} + \frac{x^2}{2}\right\}_{-1}^{1} = \frac{1}{2}\left\{\frac{1}{3} + \cancel{\frac{1}{2}} + \frac{1}{3} - \cancel{\frac{1}{2}}\right\}$$

$$= \frac{1}{2}\left[2\left(\frac{1}{3}\right)\right] \Rightarrow \boxed{\mu_1' = \frac{1}{3}}$$

$\boxed{r=2}$ $$\mu_2' = \int x^2 f(x)\,dx = \int_{-1}^{1} x^2\left(\frac{x+1}{2}\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1}(x^3+x^2)\,dx = \frac{1}{2}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left\{\left(\frac{1}{4}+\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{3}\right)\right\}$$

$$= \frac{1}{2}\left(\cancel{\frac{1}{4}}+\frac{1}{3}-\cancel{\frac{1}{4}}+\frac{1}{3}\right) = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3} = \frac{1}{3}$$

$$\boxed{\mu_2' = \frac{1}{3}}$$

$$\mu_3' = \int_{-1}^{1} x^3 f(x)\,dx = \int_{-1}^{1} x^3\left(\frac{x+1}{2}\right)dx \qquad \boxed{\mu_3' = \frac{1}{5}}$$

$$\mu_4' = \int_{-1}^{1} x^4 f(x)\,dx = \int_{-1}^{1} x^4\left(\frac{x+1}{2}\right)dx \qquad \boxed{\mu_4' = \frac{1}{5}}$$

* CENTRAL MOMENTS

$$\boxed{\mu_1 = 0}$$

$$\mu_0 = \mu_2' = (\mu_1')^2 = \frac{2}{9} \qquad \boxed{\mu_2 = \frac{2}{9}}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 \qquad \boxed{\mu_3 = \frac{-8}{135}}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1)^4$$

$$= \left(\frac{1}{5}\right) - 4\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + 6\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right)^4$$

$$= \frac{48}{405} \qquad \boxed{\mu_4 = \frac{48}{405}}$$

LECTURE – 07

DATE – 16.03.2022 DAY – WEDNESDAY

* PROBLEMS BASED ON MGF

Q1) For the probability mass function $P(X=x) = \frac{1}{2^x}$, $X = 1, 2, 3, \ldots\ldots$. Find MGF, Mean and variance.

Solution: $\boxed{\because X \text{ takes known values it is discrete}}$

MGF: $M_X(t) = \sum_{x=1}^{\infty} e^{tx} P(x)$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \sum_{x=1}^{\infty} \frac{(e^t)^x}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^t + \ldots\ldots$$

$$= \frac{e^t}{2}\left\{1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \ldots\ldots\right\}$$

$$\boxed{(1-x)^{-1} = 1 + x + x^2 + \ldots\ldots}$$

$$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2}\left(\frac{2-e^t}{2}\right)^{-1} = \frac{e^t}{\cancel{2}}\left(\frac{\cancel{2}}{2-e^t}\right)$$

$$= \frac{e^t}{2-e^t} \qquad \boxed{M_X(t) = \frac{e^t}{2-e^t}}$$

Mean: $M_X(t) = \frac{e^t}{2-e^t}$

$M_X'(t) = \frac{vu' - uv'}{v^2}$ $\qquad u = e^t,\ v = 2-e^t$; $u' = e^t,\ v' = -e^t$

$$M_X'(t) = \frac{(2-e^t)e^t + (e^t)^2}{(2-e^t)^2} = \frac{2e^t - \cancel{(e^t)^2} + \cancel{(e^t)^2}}{(2-e^t)^2} = \frac{2e^t}{(2-e^t)^2}$$

$$M_X'(t) = \frac{2e^t}{(2-e^t)^2} \xrightarrow{t=0} \frac{2e^0}{(2-e^0)^2} = 2/1 = 2.$$

$$\boxed{\text{Mean} = E(x) = 2}$$

$$M_x''(t) = \frac{vu' - uv'}{v^2}, \quad u = 2e^t, \; v = (2-e^t)^2$$
$$u' = 2e^t, \; v' = 2(2-e^t)(-e^t)$$

$$M_x''(t) = \frac{2e^t(2-e^t)^2 + 2e^t\, 2e^t(2-e^t)}{(2-e^t)^4}$$

$$\xrightarrow{t=0} = \frac{2(1) + 2(2)(1)}{1} = 2+4 = \underline{\underline{6}}.$$

$$\boxed{E(x^2) = 6}$$

Variance : $Var(x) = E(x^2) - [E(x)]^2$
$= 6-4 = \underline{\underline{2}}.$

$$\boxed{Var(x) = 2}$$

* ANOTHER METHOD TO CALCULATE

Mean and Variance :

From $M_x(t)$,

Co-efficient of $t/1! = E(x)$

Co-efficient of $t^2/2! = E(x^2)$

---

* HOMEWORK :

Q2) For a triangular distribution, $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & x \ge 2 \end{cases}$

Calculate Mean, Variance, cdf & MGF.

Solution : $\because$ x take unknown values between 0 & 1, 1 & 2 etc, it is continuous

$$MGF = M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$\begin{array}{ccccccc} & x & & 2-x & & 0 & \\ 0 & & 1 & & 2 & & \infty \end{array}$$

$$M_x(t) = \int_0^1 \underset{u}{x} \, \underset{dv}{e^{tx}} \, dx + \int_1^2 \underset{u}{(2-x)} \, \underset{dv}{e^{tx}} dx + \int_2^{\infty} 0$$

LECTURE – 08

DATE – 17.03.2022 DAY – THURSDAY

Q3) For a triangular distribution $f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1 \le x < 2 \\ 0, & x \geq 2 \end{cases}$.

Find the MGF, Mean and cdf. (Part C, 12 Marks)

Solution: $M_x(t) = \int_{-\infty}^{\infty} f(x) \, e^{tx} dx$

$$\begin{array}{ccccccc} & x & & 2-x & & 0 & \\ 0 & & 1 & & 2 & & \infty \end{array}$$

$$= \int_0^1 \underset{u}{x} \, \underset{dv}{e^{tx}} \, dx + \int_1^2 \underset{u}{(e-x)} \, \underset{dv}{e^{tx}} \, dx$$

$$= \left[x\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2} \right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} + \frac{e^{tx}}{t^2} \right]_1^2$$

$$= \left[\frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} \right] + \left[\left(0 + \frac{e^{2t}}{t^2} \right) - \left(\frac{e^t}{t} + \frac{e^t}{t^2} \right) \right]$$

$$= \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{(e^t)^2}{t^2} - \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2}$$

$$= \frac{1}{t^2}\left[(e^t)^2 - 2e^t + 1 \right] \qquad \boxed{M_x(t) = \frac{1}{t^2}\left[e^t - 1 \right]^{-2}}$$

$$\text{Mean} = E(x) = \left[M_x'(t) \right]_{t=0} = \frac{vu' - uv'}{v^2}$$

$u = (e^t - 1)^2$ $\qquad v = t^2$

$u' = 2(e^t - 1)(e^t)$ $\qquad v' = 2t$

$$M_x'(t) = \frac{t^2 \, 2(e^t - 1) e^t - 2t(e^t - 1)^2}{t^4} \xrightarrow{t=0} \boxed{\text{Mean} = 0}$$

cdf :- $\boxed{F(x) = \int_0^x f(x)\,dx}$

0<x<1 :

$$F(x) = \int_0^x x\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{x^2}{2}.$$

$1 \leq x < 2$:

$$F(x) = \int_0^1 x\,dx + \int_1^x (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= 2x - \frac{x^2}{2} - 1$$

$$F(x) = 2x - \frac{x^2}{2} - 1$$

$$F(x) = \begin{cases} \frac{x^2}{2}, & 0<x<1 \\ 2x - \frac{x^2}{2}, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

Q4) If $f(x) = \frac{1}{2} e^{-x/2}$, $x>0$. Find mean, cdf, MGF.

(Part B, 4 Marks)

Solution: $M_x(t) = \int_0^\infty e^{tx} f(x)\,dx$

$$= \int_0^\infty e^{tx} \frac{1}{2} e^{-x/2}\,dx = \frac{1}{2}\int_0^\infty e^{tx} e^{-x/2}\,dx$$

$$= \frac{1}{2}\int_0^\infty e^{tx - x/2}\,dx = \frac{1}{2}\int_0^\infty e^{-x(1/2 - t)}\,dx$$

$$= \frac{1}{2}\left\{\frac{e^{-x(1/2-t)}}{-(1/2-t)}\right\}_0^{\infty} = \frac{1}{2}\left[0 - \frac{1}{-(1/2-t)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\frac{1-2t}{2}}\right] = \frac{1}{1-2t} = M_x(t)$$

$$\boxed{M_x(t) = \frac{1}{1-2t}}$$

$$M_x(t) = (1-2t)^{-1} = 1 + (2t) + (2t)^2 + \ldots\ldots$$

$$= 1 + 2\frac{t}{1!} + 4t^2 + \ldots\ldots$$

Mean $E(x)$ = Co-efficient of $\frac{t}{1!}$ $\boxed{E(x) = 2}$

cdf: $F(x) = \int_1^x \frac{1}{2} e^{-x/2} dx = \frac{1}{2}\left(\frac{e^{-x/2}}{-1/2}\right)_0^x = -\left[e^{-x/2}\right]_0^x$

$$= -\left[e^{-x/2} - 1\right] = F(x) \quad x > 0$$

$$\boxed{F(x) = -\left[e^{-x/2} - 1\right]}$$

Q5) If $M_x(t) = \frac{3}{3-t}$, find mean, variance, standard deviation. (Part B, 4 Marks) (University Question)

Solution: $M_x(t) = \frac{3}{3-t} = 3(3-t)^{-1}$

$$\boxed{(1-x)^{-1} = 1 + x + x^2 + \ldots\ldots}$$

$$M_x(t) = 3 \cdot 3^{-1}\left(1 - t/3\right)^{-1} = \cancel{3} \cdot \frac{1}{\cancel{3}}\left(1 - t/3\right)^{-1}$$

$$M_x(t) = 1 + \frac{1}{3} + \frac{1^2}{3^2} + \ldots\ldots$$

$$= 1 + \frac{1}{3}\frac{t}{1!} + \frac{1}{3^2}\frac{t^2}{2!}2! + \ldots\ldots$$

$$= 1 + \frac{1}{3}\left(\frac{t}{1!}\right) + \frac{2}{9}\left(\frac{t^2}{2!}\right) + \ldots\ldots$$

$E(x)$ = coefficient of $\frac{t}{1!} = \frac{1}{3} = E(x)$ $\boxed{E(x) = \frac{1}{3}}$

$E(x^2)$ = coefficient of $\frac{t^2}{2!} = \frac{2}{9} = E(x^2)$ $\boxed{E(x^2) = \frac{2}{9}}$

$Var(x) = E(x^2) - [E(x)]^2$

$= \frac{2}{9} - \frac{1}{9} = \frac{1}{9} = Var(x)$ $\boxed{Var(x) = \frac{1}{9}}$

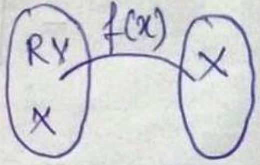
Standard deviation $= \sqrt{Var}$

$= \sqrt{1/9} = \frac{1}{3} = \sigma$ $\boxed{\sigma = \frac{1}{3}}$

* FUNCTIONS OF RANDOM VARIABLE

Let X be the random variable with pdf $f(x)$, then the transfered random variable Y is its pdf $f(y)$ is given by

$$\boxed{f(y) = f(x)\left|\frac{dx}{dy}\right|}$$



* PROBLEMS :

Q1) Let X be a random variable with density function $f(x) = 2x$, $0 < x < 1$ & $y = 3x + 6$. Find pdf of y.

Solution: $y = 3x + 6$

$\Rightarrow y - 6 = 3x$

$\Rightarrow x = \frac{1}{3}(y-6)$

$\frac{dx}{dy} = \frac{1}{3}(1)$

$\left|\frac{dx}{dy}\right| = \frac{1}{3}$

Limits:

x	0	1
$y = 3x+6$	6	9

$f(y) = f(x)\left|\frac{dx}{dy}\right|$

$= 2x\left(\frac{1}{3}\right) = 2\,\frac{1}{3}(y-6)\,\frac{1}{2}$

$$\boxed{f(y) = \frac{2}{9}(y-6),\ 6 < y < 9}$$

LECTURE – 09
DATE – 21.03.22 DAY – MONDAY

* TRANSFORMATION OF RANDOM VARIABLE $(x \to y)$

$$f(y) = f(x)\left|\frac{dx}{dy}\right|$$

Q2) Let X be a random variable with $f(x) = \frac{1}{4} x e^{-x/2}, x>0$ if $y = \frac{-x}{2}+2$. Find pdf of y. (Part-C, 6 marks)

Solution: Given that $y = -x/2 + 2$.

$$\Rightarrow y-2 = \frac{-x}{2} \quad \Rightarrow x = -2(y-2)$$

$$\frac{dx}{dy} = -2(1) \quad \Rightarrow \left|\frac{dx}{dy}\right| = 2$$

$$f(y) = f(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{1}{4} x e^{-x/2}(2) = \frac{1}{2} x e^{-x/2}$$

$$\Rightarrow \frac{1}{2}\{-2(y-2)\} e^{2(y-2/2)} \Rightarrow -(y-2)e^{y-2}$$

$$f(y) = (2-y)e^{y-2}, \; y>2$$

Limits:

$x>0$. $\Rightarrow -2(y-2) > 0$

$\Rightarrow y-2>0 \quad \Rightarrow \boxed{y>2}$

Q3) Let X be an exponential distribution with parameters '1'. Find pdf of $y=\sqrt{x}$.

Solution: General form of exponential

$$\boxed{\begin{array}{c} f(x) = \lambda e^{-\lambda x}, \; \lambda>0, \; x>0 \\ \lambda \to \text{parameter} \end{array}}$$

Here, $\lambda = 1$, $\boxed{f(x) = e^{-x}, \; x>0}$

$y = \sqrt{x}$ (given)

$y^2 = x \Rightarrow 2y = \frac{dx}{dy} \quad \Rightarrow \left|\frac{dx}{dy}\right| = 2y$

$f(y) = f(x)\left|\frac{dx}{dy}\right|$

$= e^{-x} 2y = e^{-y^2}(2y), \quad y>0$

<u>Limits</u>:

$x>0, \quad \Rightarrow x = y^2$

$\Rightarrow y^2 > 0 \quad \Rightarrow \boxed{y>0}$

Q4) Given random variable X with density function

$f(x) = \begin{cases} 4x, & 0<x<2 \\ 0, & \text{Otherwise} \end{cases}$. Find pdf of $y = 2x^3$.

<u>Solution</u>: $f(x) = 4x$

$y = 2x^3 \Rightarrow x^3 = y/2 \Rightarrow \boxed{\left(\frac{y}{2}\right)^{1/3} = x}$

$\Rightarrow x = \frac{y^{1/3}}{2^{1/3}} \quad \Rightarrow \frac{dx}{dy} = \frac{1}{2^{1/3}} \times \frac{1}{3} y^{1/3 - 1}$

$= \frac{1}{2^{1/3}} \times \frac{1}{3} y^{-2/3}$

$\frac{dx}{dy} = \frac{1}{3(2^{1/3})} y^{-2/3} \quad \Rightarrow \left|\frac{dx}{dy}\right| = \frac{1}{3(2)^{1/3}} y^{-2/3}$

$f(y) = f(x)\left|\frac{dx}{dy}\right| = 4x \frac{1}{3(2^{1/3})} y^{-2/3}$

$= 4\left(\frac{y}{2}\right)^{1/3} \cdot \frac{1}{3(2^{1/3})} y^{-2/3}$

<u>Limits</u>: $0<x<2$ when $x=0 \Rightarrow \frac{y^{1/3}}{2^{1/3}} = 0$

$x = \frac{y^{1/3}}{2^{1/3}}$

$\boxed{y=0}$

when $x = 2$ $\Rightarrow \frac{y^{1/3}}{2^{1/3}} = 2$ $\Rightarrow y^{1/3} = 2 \cdot 2^{1/3}$

$\Rightarrow y^{1/3} = 2^{4/3}$

$\Rightarrow y = 2^{\frac{4}{\cancel{3}} \times \frac{\cancel{3}}{1}}$

$\Rightarrow y = 2^4 = 16$ $\boxed{y = 16}$

x	0	2
y	0	16

$$\boxed{f(y) = 4\left(\frac{y}{2}\right)^{1/3} \cdot \frac{1}{3(2^{1/3})} y \quad , \quad 0 < y < 16}$$

* RECAPITULATION

Unit - 1 : Probability and Random Variable

Topics : * Introduction to Random Variable

↓

Assign numerical values to our expectations

Discrete → Countable Real values

x	
$P(x)$	

If $\sum P(x) = 1$ then given table is probability mass function.

Continuous → Infinite / Uncountable real values

$f(x) = x^2 \, , \; 0 < x < 2$

$f(x) \, , \; a < x < b$

If $\int_a^b f(x)\,dx = 1$ then $f(x)$ is probability density function (pdf)

* Expected (or) Mean (or) Average :-

$E(x) = \sum_{x=0}^{\infty} x P(x)$

Var : $E(x^2) - [E(x)]^2$

$= E(x^2) - \sum x^2 P(x)$

* **Discrete :-**

Moments

about origin → Real moments

$$\mu_r' = \sum x^r P(x)$$

$r = 1, 2, 3, 4$

about mean → Central moments

$$\mu_r = \sum (x-\bar{x})^r P(x)$$

$r = 1, 2, 3, 4$

* **MGF :-**

$$M_x(t) = \sum_{x=0}^{\infty} e^{tx} P(x)$$

$[M_x'(t)]_{t=0} \Rightarrow$ Mean, $[M_x''(t)]_{t=0} = E(x^2)$

* **Continuous :**

$$E(x) = \int x f(x)\,dx$$

$$E(x^2) = \int x^2 f(x)\,dx$$

$$\mu_r' = \int x^r f(x)\,dx$$

$$\mu_r = \int (x-\bar{x})^r f(x)\,dx$$

$$M_x(t) = \int e^{tx} f(x)\,dx$$

* NOTE :- cdf

$$F(x) = \int_{-\infty}^{x} f(x)\,dx \qquad \frac{d\,F(x)}{d(x)} = f(x)$$

23/3/22

LECTURE – 10

DATE – 22.03.22 DAY – TUESDAY

Q1) A random variable X has the following cdf. (Part-C, 12 Marks)

$$F(x) = \begin{cases} x^2, & 0 < x < 1/2 \\ 1 - \frac{3}{25}(3-x)^2, & 1/2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

Find pdf, $P[|x| \le 1]$, $P[1/2 \le x < 4]$.

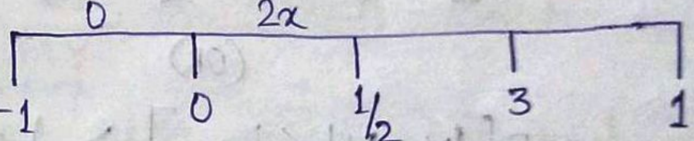
(i) Solution :
$$\boxed{\underset{\text{cdf}}{\underbrace{F(x)}} = \int_{-\infty}^{x} \underset{\text{pdf}}{\underbrace{f(x)}}\,dx \;\Rightarrow f(x) = \frac{d}{dx}(F(x))}$$

$$f(x) = \begin{cases} 2x, & 0 \le x < 1/2 \\ 0 - \frac{3}{25}\cdot 2(3-x)(-1), & 1/2 \le x < 3 \\ 0, & x \ge 3 \end{cases}$$

$$\boxed{\underset{\text{pdf}}{\underbrace{f(x)}} = \begin{cases} 2x, & 0 \le x < 1/2 \\ \frac{6}{25}(3-x), & 1/2 \le x < 3 \\ 0, & x \ge 3 \end{cases}}$$

(ii) $P[|x| \le 1]$ $\quad |x| \le a \quad -a \le x \le a$

$\hookrightarrow P(-1 \le x \le 1) =$



$$= \int_0^{1/2} 2x\,dx + \int_{1/2}^{1} \frac{6}{25}(3-x)\,dx$$

$$= 2\left(\frac{x^2}{8}\right)_0^{1/2} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{1/2}^{1}$$

$$\boxed{P[|x| \le 1] = \frac{13}{25}}$$

$$= \frac{1}{4} + \frac{6}{25}\left\{\left(3 - \frac{1}{2}\right) - \left(\frac{3}{2} - \frac{1}{8}\right)\right\}$$

$$= \frac{1}{4} + \frac{\cancel{6}^{\,3}}{25}\left(\frac{9}{\cancel{8}_{4}}\right) = \frac{1}{4} + \frac{27}{100} = \boxed{\frac{13}{25}}$$

(iii) $$P\left(\frac{1}{2} \le x < 4\right) = \int_{1/3}^{1/2} 2x + \int_{1/2}^{3} \frac{6}{25}(3-x)\,dx + \int_3^4 0$$

$$= 2\left[\frac{x^2}{2}\right]_{1/3}^{1/2} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{1/2}^{3}$$

$$= \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \frac{6}{25}\left\{\left(9 - \frac{9}{2}\right) - \left(\frac{3}{2} - \frac{1}{8}\right)\right\}$$

$= \frac{5}{36} + \frac{\cancel{6}^{3}}{\cancel{25}}\left(\frac{\cancel{25}}{\cancel{8}_{4}}\right) = \frac{5}{36} + \frac{3}{4} = \boxed{\frac{8}{9}}$ $\boxed{P[1/2 \leq x < 4] = \frac{8}{9}}$

* CHEBYSHEV'S INEQUALITY

Chebyshev's Inequality is a probability theory that guarantees only a definite fraction of values will be found within a specific distance from mean of the distribution.

- **Definition:** If X is a random variable with mean (μ) and variance (σ^2). Then,

$$P\left[|X-\mu| \geq k\sigma\right] \leq \frac{1}{k^2} \text{ (Upper bound)}$$

(OR)

$$P\left[|X-\mu| \leq k\sigma\right] \geq 1-\frac{1}{k^2} \text{ (Lower bound)}$$

$k \to$ Number of standard deviations from Mean

- **PROBLEMS:**

Q1) Let X be a random variable with mean 12, variance 9. If the probability distribution is not known. Find $P(6<x<18)$.

Solution: Here, Mean $= \mu = 12$

Variance $= \sigma^2 = 9 \Rightarrow \sigma = 3$

$P(6<x<18)$

By Chebyshev's Inequality,

$$\boxed{P\left[|x-\mu| \leq k\sigma\right] \geq 1-1/k^2}$$

$|x| \leq a$
$-a \leq x \leq a$

$\Rightarrow P\left[|x-12| \leq 3k\right] \geq 1 - 1/k^2$

$\Rightarrow P\left\{-3k \leq (x-12) \leq 3k\right\} \geq 1 - 1/k^2$

$\Rightarrow P\left\{-3k+12 \leq x \leq 3k+12\right\} \geq 1 - 1/k^2$

$\Rightarrow P\left\{-3k+12 \leq x \leq 3k+12\right\} \geq 1 - 1/k^2$

$\Rightarrow P\left\{6 < x < 18\right\} \geq 1 - 1/2^2$

$\Rightarrow -3k+12 = 6$ $\quad \Rightarrow \boxed{P\left(6 < x < 18\right) \geq \frac{3}{4}}$

$\boxed{k=2}$

LECTURE - 11

DATE - 23.03.2022 DAY - WEDNESDAY

* CHEBYSHEV'S INEQUALITY

$P\left[|x-\mu| \geq k\sigma\right] \leq \frac{1}{k^2}$ (OR) $P\left[|x-\mu| \leq k\sigma\right] \geq 1-\frac{1}{k^2}$

Q2) 2 dice are rolled simultaneously and X represents sum of the numbers occur on top of the dice.

S.T. $P\left[|x-7| \geq 3\right] \leq \frac{35}{54}$.

Solution : sample space for 2 dice thrown simultaneously $(S) =$ {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

$n(S) = 36$

$X \rightarrow$ Sum of the numbers occur on top of the dice.

(i.e.) Minimum value $(1,1) = 1+1 = 2$

Maximum value $(6,6) = 6+6 = 12$

X	2	3	4	5	6	7	8	9	10	11	12	0
p(x)	1/36	2/36	3/36	4/36	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0

Mean $= \sum x P(x) = \left(2\times\frac{1}{36}\right)+\left(3\times\frac{2}{36}\right)+ \cdots\cdots + \left(12\times\frac{1}{36}\right)$

$= \underline{\underline{7}}$. $\boxed{\sum x P(x) = 7 = \mu}$

$E(x^2) = \sum x^2 P(x) = \left(2^2\times\frac{1}{36}\right)+\left(3^2\times\frac{2}{36}\right)+ \cdots\cdots + \left(12^2\times\frac{1}{36}\right)$

$= \underline{\underline{\frac{1974}{36}}}$ $\boxed{\sum x^2 P(x) = \frac{1974}{36}}$

$Var = E(x^2) - [E(x)]^2 = \frac{1974}{6} - 7^2 = \frac{1974 - 49}{6}$

$= \frac{105}{18} = \sigma^2$ $\boxed{Var(x) = \sigma^2 = \frac{105}{18}}$

$\boxed{\sigma = \sqrt{\frac{105}{18}}}$

To calculate, $P\left[|x-7| \geq 3\right] \leq \frac{1}{k^2}$

$k\sigma = 3$

$\Rightarrow k\sqrt{\frac{105}{18}} = 3$

$\Rightarrow k = 3\sqrt{\frac{18}{105}}$

$\Rightarrow k^2 = 3^2 \times \frac{18}{105} = \frac{54}{35} = k^2$

$k\sigma \leq \frac{1}{\frac{54}{35}} \leq \frac{35}{54}$

$\boxed{k^2 = \frac{54}{35}}$

Hence, Proved.

Q3. If $f(x) = e^{-x}$, $x \geq 0$. Prove that $P[|x-\underset{\mu}{1}| \geq 2] \leq 1/4$.

Solution: Here, Mean $\mu = 1$.

$$E(x^2) = \int_0^\infty x^2 f(x)dx = \int_0^\infty \underset{u}{x^2} \underset{dv}{e^{-x}} dx$$

$$= \left[-x^2e^{-x} \cdot -2xe^{-x} - 2\right]_0^\infty = [0-(-2)] = 2.$$

$$\boxed{E(x^2) = 2}$$

$$Var = E(x^2) - [E(x)]^2 = 2 - 1 = 1 \qquad \sigma^2 = 1 \quad \Rightarrow \boxed{\sigma = 1}$$

To calculate, $P[|x-1| \geq 2| \leq 1/k^2$

$$\boxed{P[|x-\mu| \geq k\sigma] \leq 1/k^2}$$

$\mu = 1$, $k\sigma = 2$

$$k = \frac{2}{\sigma} = 2 \leq \frac{1}{2^2}$$

$$\boxed{k = 2} \leq 1/4$$

Hence, proved.

Unit-1 Topics

- Discrete & Continuous Random Variable →
- Mean, Var of Discrete RV. → Part B
- Mean, Var of Continuous RV → Part C
- Cdf of Continuous RV → Part C
- MGF of continuous & discrete RV → Part C
- Chebyshev's Inequality → Part C
- cdf of Discrete RV → Part B
- Functions of several variables → Part B (4 marks)

* IMPORTANT RESULTS AND FORMULAS FOR UNIT-1

1. $P[\text{getting an event}] = \dfrac{\text{Total no. of favourable outcomes}}{\text{Total no. of possible outcomes}}$

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = 0$ → Mutually exclusive

$P(A \cap B) = P(A)P(B)$ → Independent.

3. Conditional Probability :

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \qquad P(B/A) = \frac{P(A \cap B)}{P(A)}$$

4. $0 \leq P(A) \leq 1$ & $\sum P(A_i) = 1$

5. Discrete Random Variable (RV) → Countable

x	
$P(x)$	

Table 1

(i) If $\sum P(x) = 1$ then table 1 is probability mass function (pmf).

(ii) Mean [expected value of x (or) Average]

$$E(x) = \sum x P(x)$$

(iii) $E(x^2) = \sum x^2 P(x)$

(iv) $Var(\sigma^2) = E(x^2) - [E(x)]^2$

(v) $SD = \sqrt{var} = \sigma$

6. (vi) **Moments**

about origin — about mean

Raw moments

$\mu_r' = \sum x^r P(x)$

$r = 1, 2, 3, 4$

Central moments

$\mu_r = \sum (x - \bar{x}) P(x)$

$r = 1, 2, 3, 4$

Always $\boxed{\mu_1 = 0}$

(vii) Relation between μ_r' & μ_r.

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1'^2 + 6\mu_2'(\mu_1')^2 - 3(\mu_1)^4$$

(viii) Mean $\longrightarrow M_x(t) = \sum e^{tx} P(x)$

(ix) Continuous Random Variable $f(x)$

→ If $\int_{-\infty}^{\infty} f(x)\,dx = 1$, then $f(x)$ is probability density function (pdf).

(x) Mean $= E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

(xi) $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$.

(xii) $\text{Var} = E(x^2) - \text{Mean}^2$

(xiii) Moments:

↓ (about Origin)

Raw Moments $\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx$

$r = 1, 2, 3, 4$

→ (About Mean) Central Moments

$$\mu_r = \int_{-\infty}^{\infty} (x-\bar{x})^r f(x)dx$$

$r = 1, 2, 3, 4$

Always $\boxed{\mu_1 = 0}$

(xiv) MGF: $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$

(xv) Mean $= \left[\mu_x'(t)\right]_{t=0}$, $E(x^2) = \left[\mu_x''(t)\right]_{t=0}$

(xvi) Transformation: $f(y) = f(x)\left|\frac{dx}{dy}\right|$

(xvii) Chebyshev's Inequality:

$$P\left[|x-\mu| \geq k\sigma\right] \leq \frac{1}{k^2} \quad \text{(Upper Bound)}$$

(OR)

$$P\left[|x-\mu| \leq k\sigma\right] \geq 1 - \frac{1}{k^2} \quad \text{(Lower Bound)}$$