**SRM Institute of Science and Technology- Ramapuram campus**
**Department of Mathematics**
**18MAB204T- Probability and Queuing Theory**

**Year/Sem: II/IV**

**Branch: CSE, IT**

## Unit I - PROBABILITY AND RANDOM VARIABLES

### PART-B

1. A lot consists of 10 good articles, 4 with minor defects and 2 with major defective. Two articles are chosen from the lot at random (without replaced). Find the probability that both are good.
   (a) 3/8 (b) 7/8 (c) 5/8 (d) 1/8
   Answer: a
   Solution:
   P (both are good)= $\frac{No.of\ ways\ drawing\ 2\ goods\ articles}{total\ no.of\ ways\ of\ drawning\ 2\ articles}$
   $$= \frac{10C_2}{16C_2} = \frac{3}{8}$$

2.

If X and Y are independent r. v's with variance 2 and 3. Find the Variance of 3X+4Y

2

(a) 46 (b) 64 (c) 66 (d) 66.5
Answer: c
Solution:
We knoe that $var(aX + bY) = a^2var(X) + b^2var(Y)$
$$var(3X + 4Y) = 3^2var(X) + 4^2var(Y)$$
$$= 9 * 2 + 16 * 3 = 66$$

3.

Obtain the probability function or probability distribution from the following Distribution function

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| f(x) | 0.1 | 0.4 | 0.9 | 1 |

(a)

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(x) | 0.1 | 0.3 | 0.5 | 0.1 |

(b)

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(x) | 0.1 | 0.4 | 0.5 | 0.1 |

(c)

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(x) | 0.1 | 0.3 | 0.9 | 0.1 |

(d)

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(x) | 0.1 | 0.3 | 0.5 | 1 |

Answer: a

$$P(x = 0) = f(0) = 0.1$$

$P(x = 1) = f(1) - f(0) = 0.4 - 0.1 = 0.3$
$P(x = 2) = f(2) - f(1) = 0.9 - 0.4 = 0.5$
$P(x = 3) = f(3) - f(2) = 1 - 0.9 = 0.1$

4.

Let x be a discrete random variable whose cumulative distribution is

$$F(x)= \begin{cases} 0, x < -3 \\ \frac{1}{6}, -3 < x < 6 \\ \frac{1}{3}, 6 < x < 10 \\ 1, x > 10 \end{cases}$$

i) find P(x>4) , p(-5<x<4)

(a) $p(x > 4) = \frac{1}{6}, p(-5 < x < 4) = \frac{1}{8}$
(b) $p(x > 4) = \frac{1}{6}, p(-5 < x < 4) = \frac{1}{6}$
(c) $p(x > 4) = \frac{1}{8}, p(-5 < x < 4) = \frac{1}{8}$
(d) $p(x > 4) = \frac{1}{6}, p(-5 < x < 4) = \frac{1}{7}$

Answer : b

Solution:

$$P(x < 4) = P(x = 0) + P(x = 3) = 0 + \frac{1}{6} = \frac{1}{6}$$

$$P(-5 < x < 4) = F(4) - F(-5) = \frac{1}{6} - 0 = \frac{1}{6}$$

5.

If Var (x) = 4 Find var (3X+8), where X is a random variable.

(a) 33 (b) 44 (c) 66 (d) 36

Answer: d

We know that $var(aX) = a^2 var(X)$

$$var(3X + 8) = 3^2 var(X) + 0 = 9 * 4 = 36$$

6. The first four moments of a distribution about A = 4 are 1, 4 , 10 , and 45 respectively.

Find the value of $\mu_1$ ,$\mu_2$ ,$\mu_3$ $and$ $\mu_4$.

(a) $\mu_{1=5}$ ,$\mu_{2=3}$ ,$\mu_{3=0}$ $and$ $\mu_{4=26}$

(b) $\mu_{1=3}$ ,$\mu_{2=5}$ ,$\mu_{3=0}$ $and$ $\mu_{4=26}$

(c) $\mu_{1=5}$ ,$\mu_{2=3}$ ,$\mu_{3=26}$ $and$ $\mu_{4=6}$

(d)$\mu_{1=5}$ ,$\mu_{2=3}$ ,$\mu_{3=0}$ $and$ $\mu_{4=6}$

Answer: a

**Solution:**

Given $\mu_1' = 1, \mu_2' = 4, \mu_3' = 100$ & $\mu_4' = 25$ the point A=5

Mean = A + $\mu_1'$ =4+1=5

Variance = $\mu_2 = \mu_2' - \mu_1'^2 = 4 - 1 = 3$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$
$$= 10 - 3(4)(1) + 2(1)^3 = 0$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' - 3\mu_1'^4$$
$$= 45 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4$$
$$= 45 - 40 + 24 - 3$$

7.

If a random variable X takes the value 2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)
Find the probability distribution of X

(a) $p(x = 1) = \frac{15}{60}$ ,$p(x = 2) = \frac{9}{60}$,$p(x = 3) = \frac{30}{60}$ ,$p(x = 4) = \frac{6}{60}$

(b) $p(x = 1) = \frac{20}{66}$ ,$p(x = 2) = \frac{10}{66}$,$p(x = 3) = \frac{30}{66}$ ,$p(x = 4) = \frac{6}{66}$

(c) $p(x = 1) = \frac{15}{62}$ ,$p(x = 2) = \frac{10}{62}$,$p(x = 3) = \frac{30}{62}$ ,$p(x = 4) = \frac{7}{62}$

(d) $p(x = 1) = \frac{15}{61}$ ,$p(x = 2) = \frac{10}{61}$,$p(x = 3) = \frac{30}{61}$ ,$p(x = 4) = \frac{6}{61}$

Answer: d

**Solution:**

2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)=K

P(X=1)=k/2
P(X=2)=k/3
P(X=3)=k
P(X=2)=k/5

$$\therefore \frac{k}{2}+\frac{k}{3}+k+\frac{k}{5}=1$$

$$\frac{61k}{30}=1$$

$$K=\frac{30}{61}$$

| X | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| P(X) | 15/61 | 10/61 | 30/61 | 5/61 |

8.

Find the Cumulative distribution function F(X) corresponding to the p.d.f of

$f(x)=\frac{1}{\Pi}\left(\frac{1}{1+x^2}\right), -\infty < x < \infty$

(a) $\frac{1}{\pi}\left(\tan^{-1} x+\frac{\pi}{2}\right)$
(b) $\frac{1}{\pi}\left(\tan^{-1} x-\frac{\pi}{2}\right)$
(c) $\frac{1}{\pi}\left(\tan^{-1} x+\frac{\pi}{4}\right)$
(d) $\frac{1}{\pi}\left(\tan^{-1} x+\frac{\pi}{6}\right)$

Answer: a

**Solution:** W K T

f(X)=P(X≤x)

$$= \int_{-\infty}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\Pi}\left(\frac{1}{1+x^2}\right)dx$$

$$= \frac{1}{\Pi}[\tan^{-1} x]_{-\infty}^{x}$$

$$F(x) = \begin{matrix} \frac{1}{\Pi}[\tan^{-1} x - \tan^{-1}(-\infty)] \\ \frac{1}{\Pi}[\tan^{-1} x + \frac{\Pi}{2} \end{matrix}$$

9. A random variable X has $\mu = 12$ and $\sigma^2 = 9$ and unknown probability distribution.

Find $(6 < X < 18)$ .

(a) $P(6 < X < 18) \geq \frac{3}{4}$(b) $P(6 < X < 18) \leq \frac{3}{4}$ (c)$P(6 < X < 18) \geq \frac{1}{4}$ (d) $P(6 < X < 18) \geq \frac{6}{8}$

Answer: a

Solution:

Lower bound for the probability by using Tcebycheffs inequality

$P\{|X - \mu| \leq c\} \geq 1 - \dfrac{\sigma^2}{c^2}$

$P\{\mu - c \leq X \leq \mu + c\} \geq 1 - \frac{\sigma^2}{c^2}$taking$\mu = 12$ *and* $\sigma = 9$ *, we get*

$P\{12 - c \leq X \leq 12 + c\} \geq 1 - \dfrac{9}{c^2}$

Putting $c$=6 $P\{6 \leq X \leq 18\} \geq 1 - \frac{9}{36} = \frac{3}{4}$

10.

A random variable X has p d f f(X)=Cx$e^{-x}$, x>0. fine the value of C and C d f of X

(a)$1 - e^{-x}(x - 1)$

(b) $1 + e^{-x}(x + 1)$

(c) $1 - e^{x}(x + 1)$

(d) $1 + e^{-x}(-x - 1)$

Answer: d

11.

The C.d.f of a random variable X is F(X) =1-(1+x) $e^{-x}$ , x>0. Find the density Function of X

**Solution:**

$$WKT$$

$$f(x)=\frac{d(F(x))}{dx}$$

$$=\frac{d}{dx}[1-(1+x)e^{-x}]$$

$$=0-\frac{d}{dx}((1+x)e^{-x})$$

$$=-[(1+x)(e^{-x})+e^{-x}(1)]$$

$$=-[-e^{-x}-xe^{-x}+e^{-x}]$$

$$=xe^{-x}, x>0$$

12. There are 20 boys and 15 girls in a class of 35 students. A students is chosen at random find the probability that the chosen students is a (i) boy (ii) girl

Solution

$$n(s)=35, n(B)=20, n(G)=15$$

Probability of choosing a boy is $P(B)=\frac{n(B)}{n(S)}=\frac{20}{35}$

Probability of choosing a girl is $P(G)=\frac{n(G)}{n(s)}=\frac{15}{35}$

**13.**

If a random variable X has the m.g.f $M_x(t)=\frac{2}{2-t}$, determine the variance of X

**Solution:**

$$\mu_1'=\frac{d}{dx}[M_x(t)]_{t=0}$$

$$=[\frac{(2-t).0-2(-1)}{(2-t)^2}]_{t=0}$$

$$=\frac{2}{(2-0)^2}=\frac{1}{2}$$

$$\mu_2'=\frac{d^2}{dt^2}[M_x(t)]_{t=0}$$

$$=\frac{d}{dt}[\frac{2}{(2-t)^2}]_{t=0}$$

$$=[\frac{(2-t)^2.0-2(2-t).2)(-1)}{(2-t)^4}]_{t=0}$$

$$=\frac{4}{8}=\frac{1}{2}$$

$$\text{var } \mu_2=\mu_2'-(\mu_1')^2=\frac{1}{2}-(\frac{1}{2})^2$$

$$=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

**14.** A continuous random variable X has the distribution function $F(x)=\begin{cases} 1 & x<1 \\ k(x-1)^4, & 1<x<3 \\ 0\ , & otherwise \end{cases}$ Determine k and the pdf.

**Solution:**

$$Let\ F(x)=\begin{cases} 1 & x<1 \\ k(x-1)^4, & 1<x<3 \\ 0, & otherwise \end{cases}\ be\ the\ distribution\ function\ of\ the\ R.V\ X$$

$$Then\ the\ p.d.f\ of\ X\ is\ f(x)=\frac{d}{dx}F(x)$$

$$f(x)=\begin{cases} 4k(x-1)^3 & for\ 0<x<3 \\ 0 & otherwise \end{cases}$$

$$Since\ f(x)\ is\ the\ p.d.f,\ we\ have\ \int_{-\infty}^{\infty} f(x)dx=1$$

$$\int_1^3 4k(x-1)^3\,dx=1$$

$$\left\{4k\frac{(x-1)^4}{4}\right\}_1^3=1 \Rightarrow k\{16-0\}=1 \Rightarrow k=\frac{1}{16}$$

**15.**The diameter of an electric cable X is a continuous R.V with PDF

$f(x)=kx(1-x),\ 0<x<1$ .Find the (i) value of ‘ k ‘

**Solution**

$$Let\ f(x)=\begin{cases} kx(1-x), & 0\le x\le 1 \\ 0 & otherwise \end{cases}\ be\ the\ p.d.f\ of\ the\ R.V\ X$$

$$Then\ \int_{-\infty}^{\infty} f(x)dx=1 \Rightarrow \int_0^1 k\,x(1-x)dx=1 \Rightarrow k\int_0^1 (x-x^2)\,dx=1 \Rightarrow k\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_0^1=1$$

$$\therefore k\left(\frac{1}{2}-\frac{1}{3}\right)=1 \Rightarrow k=6$$

**16.** Find the m.g.f and the r th moment for the distribution whose p.d.fis f(x) = k $e^{-x}$, 0 < x <∞. Find also the standard deviation.

$$Let\ f(x)=k\,e^{-x}\ .x>0\ be\ the\ p.d.f\ of\ the\ R.V\ X$$

$$Then\ \int_{-\infty}^{\infty} f(x)dx=1 \Rightarrow \int_0^{\infty} k\,e^{-x}\,dx=1 \Rightarrow k\left[\frac{e^{-x}}{-1}\right]_0^{\infty}=1 \Rightarrow k(0-(-1))=1 \Rightarrow k=1$$

$$The\ M_X(t)=E\left[e^{tx}\right]=\int_0^{\infty} k\,e^{-x}\,e^{tx}dx=\int_0^{\infty} e^{-x(1-t)}\,dx=\left\{\frac{e^{-x(1-t)}}{-(1-t)}\right\}_0^{\infty}=0+\frac{1}{1-t}$$

$$Hence\ M_X(t)=\frac{1}{1-t}\ [OR]\ M_X(t)=(1-t)^{-1}$$

**17** Let X be a random variable with p.m.f $p(x)=\frac{1}{2^x},\ x=0,1,2,\ldots$ Find the m.g.f of X.

*Let* $p(x)=\frac{1}{2^x},\ x=0,1,2,\ldots$ *be the p.m.f of the R.V X.*

*Then the m.g.f of X is* $M_X(t)=E\left(e^{tx}\right)$

$$=\sum_x e^{tx}\ p(x)$$

$$=\sum_{x=0}^{\infty} e^{tx}\ \frac{1}{2^x}=\sum_{x0}^{\infty}\left(\frac{e^t}{2}\right)^x=1+\left(\frac{e^t}{2}\right)+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\ldots$$

$$=\left(1-\frac{e^t}{2}\right)^{-1}$$

$$M_X(t)=\left(\frac{2-e^t}{2}\right)^{-1}\qquad (OR)\quad M_X(t)=\frac{2}{2-e^t}$$

18. (i) Find the moment generating function of the continuous probability distribution whose density is $2e^{-2x}, x\geq 0$ also find first four raw and central moments.

Solution :

**(i)** Given f(x) = $2e^{-2x}, x\geq 0$ we know that $e^{-\infty}=0$ & $e^0=1$ $\quad M_X(t)=E[e^{tX}]=$ $\int_0^{\infty} e^{tx}\,2e^{-2x}dx=2\int_0^{\infty} e^{-x(2-t)}dx=2\left\{\frac{e^{-x(2-t)}}{t-2}\right\}_0^{\infty}$

$M_X(t)=\left(\frac{2}{2-t}\right)=\left(1-\frac{t}{2}\right)^{-1}$

$=\left(1+\frac{t}{2}+\left(\frac{t}{2}\right)^2+\left(\frac{t}{2}\right)^3+\left(\frac{t}{2}\right)^4+\left(\frac{t}{2}\right)^5\ldots\right)$ -------- (1)

$M_X(t)=\left(1+\frac{t}{1!}\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+\frac{t^4}{4!}\mu_4'+\frac{t^5}{5!}\mu_5'+\cdots\right)$ -------- (2) comparing (1) & (2) we get the first four raw moments $\mu_1'=\frac{1}{2}$ , $\mu_2'=\frac{1}{2}$, $\mu_3'=\frac{3}{4}$ & $\mu_4'=\frac{24}{16}$ and the first four central moments $\quad \mu_1=0$ , $\mu_2=\mu_2'-(\mu_1')^2=\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^2=\frac{1}{4}$ ,

$$\mu_3=\mu_3'-3\mu_2'\mu_1'+2(\mu_1')^3=\frac{3}{4}-3\frac{1}{4}+2\left(\frac{1}{4}\right)=\frac{1}{2}$$

$$\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'(\mu_1')^2-3(\mu_1')^4=\frac{24}{16}-4\left(\frac{3}{8}\right)+6\left(\frac{1}{8}\right)-3\left(\frac{1}{16}\right)=\frac{9}{16}$$

19.A discrete random variable X has M.g.f = $M_X(t)=\left(\frac{1}{4}+\frac{3}{4}\right)^5$ find E(X) , Var (X) and P(X=2)

The given m.g.f $M_X(t)=\left(\frac{1}{4}+\frac{3}{4}\right)^5$ is of the form $M_X(t)=(q+pe^t)^n$ which is the m.g.f of the Binomial distribution whose p.m.f is P(X=x)= $nC_x p^x q^{n-x}$ , mean E(X) = np and Variance = npq

Here q = $\frac{1}{4}$ , p = $\frac{3}{4}$ and n = 5

=> P(X=2)= $5C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3=10\left(\frac{9}{16}\right)\left(\frac{1}{64}\right)=\frac{45}{512}$

Mean $\mu_1'=np=\frac{15}{4}$, Var(X) = $\frac{15}{16}$

20. Find the cdf for the following pdf f(x) = $\begin{matrix} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0 & else\,where \end{matrix}$

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow a = 1/2$$

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$x \in (0,1) \qquad F(x) = \frac{x^4}{4}$$

$$x \in (1,2) \qquad F(x) = \frac{x}{2} - \frac{1}{4}$$

$$x \in (2,3) \qquad F(x) = \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

$$x > 3 \qquad F(x) = 1$$

**Solution:**

$$\int_{0}^{\infty} f(x)dx = 1$$

$$\int_{0}^{\infty} Cxe^{-x}dx = 1$$

$$C[x(-e^{-x}) - (1)(e^{-x})] = 1$$

$$C[(0-0) - (0-1)] = 1$$

$$\therefore C = 1$$

$$F(x) = P(X \le x)$$

$$= \int_{-\infty}^{x} Cxe^{-x}dx$$

$$= \int_{-\infty}^{0} xe^{-x}dx + \int_{0}^{x} Cxe^{-x}dx$$

$$= 0 + [x(-e^{-x}) - ((1)(e^{-x}))]_0^x$$

$$= [(-xe^{-x} - e^{-x}) - (0-1)]$$

$$= -xe^{-x} - e^{-x} + 1$$

$$F(x) = 1 + e^{-x}(-x-1)$$