Probability and Random variables

Definition of Probability

Probability is the study of randomness and uncertainty of an experiment. It is a numerical measure of the likelihood that an event will occur, which is expressed as a number between 0 and 1.

Experiment

By experiment we mean an act of conducting a controlled test or investigation. The experiment results in some outcome.

The possible results of an experiment may be one or more.

Based on the number of possible results in an experiment, we classify the experiments into two as

DETERMINISTIC and PROBABILISTIC

Deterministic Experiment

The Experiments which have only one possible result or outcome i.e. whose result is certain or unique are called **deterministic** or **predictable** experiments. The result of these experiments is predictable with certainty and is known prior to its conduct.

E.g. An Experiment conducted to verify the Newton's Laws of Motion.

Probabilistic Experiment

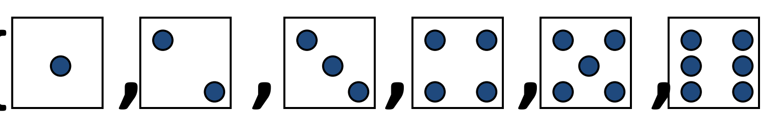
Experiments whose result is uncertain are called **nondeterministic** or **unpredictable** or **Probabilistic** experiments. There are more than one possible results or outcomes.

E.g. An Experiment of tossing a coin.

Random Experiment

An Experiment whose result is uncertain i.e. a random experiment is a probabilistic experiment. The experiment results in two or more outcomes. It cannot be predicted prior to its conduct.

E.g. Whenever a fair dice is thrown, it is known that any of the 6 possible outcomes will occur, but it cannot be predicted what exactly the outcome will be.

S ={  }

Sample Space

The set of all possible outcomes of the experiment, which are assumed equally likely.

E.g. In rolling a die, outcomes are

S = {⚀ , ⚁ , ⚂ , ⚃ , ⚄ , ⚅ }

Event

A sub-set of Sample space of Random Experiment

Probability of an Event

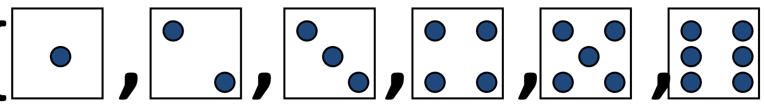
Suppose that the sample space $S = \{o_1, o_2, o_3, \ldots o_N\}$ has a finite number, N, of outcomes. Also each of the outcomes is equally likely.

Then for any event E

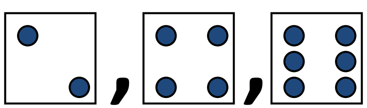
$$P[E] = \frac{n(E)}{n(S)} = \frac{n(E)}{N} = \frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

- Find the probability of getting an even number when a die is thrown.

When a die is thrown the sample space is

S ={  }

The total number of possible outcomes is 6

The favourable number of outcomes is 3, that is {  }

$\therefore$ The required probability is= $\frac{3}{6} = \frac{1}{2}$

Probability Set Function

Let S be the sample space and if A subset of S, then P(A) is the probability that the outcome of the random experiment is an element of the set A. Then the probability of the event A, P(A) is defined as a real number satisfying the following axioms.

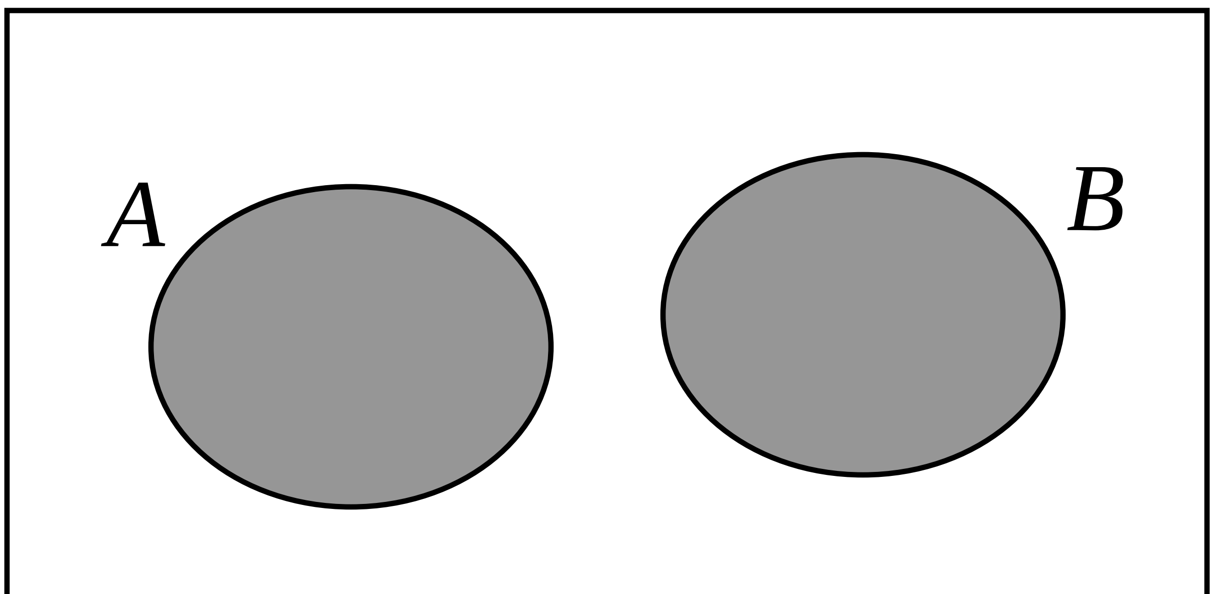
1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. If $A_1, A_2, A_3, \ldots, A_n, \ldots$ are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n, \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n) \ldots$$

If two events *A* and *B* are **mutually exclusive** then:

They have no outcomes in common.
They can't occur at the same time. The outcome of the random experiment can not belong to both *A* and *B.*

$$A \cap B = \phi$$



Some Important Formulas

1. If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This rule is known as additive rule on probability.

For three events A, B and C, we have,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) \\ - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

2. If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

In general, if A_1, A_2, … , A_n are mutually exclusive, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

3. If A and A^c are complementary events, then

$$P(A) + P(A^c) = 1$$

4. $P(S) = 1$

5. $P(\Phi) = 0$

- A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?

Let A be the event of drawing a spade

B be the event of drawing a ace

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}$$

- In tossing a coin what is the probability of getting Head or tail?

Let A be the event of getting Head

B be the event of getting Tail

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Conditional Probability

The conditional probability of an event B, assuming that the event A has happened, is denoted by P(B/A) and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{provided } P(A) \neq 0$$

Product theorem of probability

Rewriting the definition of conditional probability, We get

$$P(A \cap B) = P(A)P(B/A)$$

The product theorem can be extended to 3 events, A, B and C as follows:

$$P(A \cap B \cap C) = P(A)P(B/A)P(C/A \cap B)$$

- The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane

  (a) arrives on time, given that it departed on time, and

  (b) departed on time, given that it has arrived on time.

  (c) arrives on time, given that it did not departed on time.

(a) The probability that a plane arrives on time, given that it departed on time, is

$$P(A/D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94$$

(b) The probability that a plane departed on time, given that it has arrived on
time, is

$$P(D/A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95$$

(c) The probability that a plane arrives on time, given that it is not departed on time, is

$$P(D') = 1 - P(D) = 1 - 0.83 = 0.17$$

$$P(A/D') = \frac{P(D' \cap A)}{P(D')}, \qquad \because (D' \cap A) \cup (D \cap A) = A$$

$$= \frac{P(A) - P(D \cap A)}{P(D')}$$

$$= \frac{0.82 - 0.78}{0.17}$$

$$= 0.24$$

Independent Events

A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

If the two events A and B are independent, the product theorem takes the form

$$P(A \cap B) = P(A) \times P(B),$$

Conversely, if $P(A \cap B) = P(A) \times P(B)$, the events are said to be independent (pair wise independent).

By Conditional Probability, we have

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{provided P(A)} \neq 0$$

that is $P(A \cap B) = P(A)P(B/A)$

If $P(B/A) = P(B)$, then the events are said to be independent.

- In the Flight arrival and departure time experiment, we note that $P(A/D) = 0.94$, where as $P(A) = 0.82$.

  i.e., $P(A/D) \neq P(A)$

  indicating that A depends on D.

- Two cards are drawn in succession from an ordinary deck, with replacement. The events are defined as

  A: the first card is an ace,

  B: the second card is a spade.

Since the first card is replaced, our sample space for both the first and the second draw consists of 52 cards, containing 4 aces and 13 spades. Hence,

$P(B/A) = 13/52$

$P(B) = 13/52$

i.e., $P(B/A) = P(B)$

when this is true, the events A and B are said to be independent.

- A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Let A and B represent the respective events that the fire engine and the ambulance are available. Then

$$P(A \cap B) = P(A)\,P(B) = (0.98)\,(0.92) = 0.9016$$

Random Variable

A random variable is a function that associates a real number with each element in the sample space. Normally a capital letter, say X, is used to denote a random variable and its corresponding small letter, x in this case, for one of its values.

Discrete Random Variable

If the random variable taken the values only on the set {0, 1, 2, 3, ….. n} is called a Discrete random variable.

E.g. The number of printing mistakes in each page of a book, the number of telephone calls received by the telephone operator.

- Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. The possible outcomes and the values x of the random variable X, where X is the number of red balls, are

Sample Space	x
RR	2
RB	1
BR	1
BB	0

- Suppose that our experiment consists of tossing 3 fair coins. If we let X denote the number of heads appearing, then X is a random variable taking on one of the values 0, 1, 2, 3 with respective probabilities

S = {(H, H, H), (H, H, T), (H, T, H), (T, H, H), (T, T, H), (T, H, T), (H, T, T), (T, T, T)}

P{X = 0} = P{(T, T, T)} = 1/8

P{X = 1} = P{(T, T, H), (T, H, T), (H, T, T)} = 3/8

P{X = 2} = P{(T, H, H), (H, T, H),(H, H, T) }= 3/8

P{X = 3} = P{(H, H, H)} = 1/8

Discrete Probability Distributions

A discrete random variable assumes each of its values with a certain probability.

In the case of tossing a coin three times, the variable X, representing the number of heads, assumes the value 2 with probability 3/8, since 3 of the 8 equally likely sample points result in two heads and one tail. The possible values x of X and their probabilities are

X	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

Probability Mass Function

Let X be a one dimensional discrete random variable which takes the values x_1 ,x_2 ,x_3,.... Then P(X = x_i) = P(x_i) satisfies the following conditions

1. $P(x_i) \geq 0$
2. $\sum_{i=1}^{\infty} P(x_i) = 1$

Cumulative Distribution Function of Discrete Random Variable X

The distribution function of a discrete random variable X defined in $(-\infty,\infty)$ is given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad -\infty < x < \infty$$

Properties of the Distribution function

1. $P(a < X \leq b) = F(b) - F(a)$
2. $P(a \leq X \leq b) = P(X = a) + F(b) - F(a)$
3. $P(a < X < b) = F(b) - F(a) - P(X = b)$
4. $P(a \leq X < b) = F(b) - F(a) - P(X = b) + P(X = a)$

- A random variable X has the following probability function

Value of X, x_i	0	1	2	3	4	5	6	7	8
Probability P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

1. Determine the value of '*a*'.
2. Find P(X < 3), P(X $\geq$ 3). P(0 < X < 5).
3. Find the distribution function of X.

1. Since $\sum_{i=1}^{\infty} P(x_i) = 1$

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$a = \frac{1}{81}$$

2. $P(X < 3) = P(0) + P(1) + P(2) = a + 3a + 5a = 9a = 1/9$

$P(X \geq 3) = 1 - P(X < 3) = 8/9$

$P(0 < X < 5) = P(1) + P(2) + P(3) + P(4)$

$= 3a + 5a + 7a + 9a = 24a = 24/81$

3.

x	0	1	2	3	4	5	6	7	8
P(x)	*a*	*3a*	*5a*	*7a*	*9a*	*11a*	*13a*	*15a*	*17a*
F(x)	*a*	*4a*	*9a*	*16a*	*25a*	*36a*	*49a*	*64a*	*81a*

Alternate method for sub-division 2, using the cumulative distribution function F(x).

$P(X < 3) = P(X \leq 2) = F(2) = 9a = 1/9$

$P(X \geq 3) = 1 - P(X < 3) = 1 - (1/9) = 8/9$

$P(0 < X < 5) = F(5) - F(0) - P(X = 5)$

$= 36a - a - 11a$

$= 24a$

$= 24/81$

- A random variable X has the following probability function

Value of X, x_i	0	1	2	3	4	5	6	7
Probability P(x)	0	a	$2a$	$2a$	$3a$	a^2	$2a^2$	$7a^2 + a$

1. Determine the value of 'a'.
2. Find P(1.5 < X < 4.5 / X > 2).
3. Find the smallest value of λ for which P(X $\leq \lambda$) > 1/2.

1. Since $\sum_{i=1}^{\infty} \mathrm{P}(\mathrm{x_i}) = 1$

$10\,a^2 + 9a = 1$

a = 1/10 or -1. As a= -1 is meaningless, a = 1/10

2.

$$\mathrm{P}(1.5 < \mathrm{X} < 4.5 \,/\, \mathrm{X} > 2) = \frac{\mathrm{P}[(1.5 < \mathrm{X} < 4.5) \cap (\mathrm{X} > 2)]}{\mathrm{P}(\mathrm{X} > 2)}$$

$$= \frac{P(X=3) + P(X=4)}{1 - [P(X=0) + P(X=1) + P(X=2)} = \frac{5}{7}$$

3. P(X $\leq$ 0) = 0; P(X $\leq$ 1) = 0.1; P(X $\leq$ 2) = 0.3;
P(X $\leq$ 3) = 0.5 and P(X $\leq$ 4) = 0.8

$\therefore$ λ = 4 for which P(X $\leq$ λ) > 1/2 .

Continuous Random Variable

If a random variable takes on all values within a certain interval, then the random variable is called Continuous random variable.

E.g., The height, age and weight of individuals, the amount of rainfall on a rainy day.

Probability Density Function

If X is a continuous random variable then $f(x)$ is called the probability density function of X provided $f(x)$ satisfies the following conditions;

1. $f(x) \geq 0, \forall x$

2. $$\int_{-\infty}^{\infty} f(x)dx = 1$$

Cumulative Probability Distribution of Continuous Random Variable X

The **cumulative distribution function** F(x) of a continuous random variable X with density function f(x) is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt, \ -\infty < x < \infty$$

Results:

a)1. $P(a \le x \le b) = \int_a^b f(x)dx = F(b) - F(a)$

2. When X is continuous r.v.

$$P(X = a) = P(a \le X \le a) = \int_a^a f(x)dx = 0$$

$\therefore$ P(a < X $\le b$) = P($a \le$ X $\le b$) = P(a < X < b) = P($a \le$ X < b).

b)If F(x) is the distribution function of one dimensional random variables, then

1. $0 \le F(x) \le 1$
2. If $x < y$, then $F(x) \le F(y)$
3. $F(-\infty) = 0$, $F(\infty) = 1$.
4. If X is discrete r.v. taking values $x_1, x_2, x_3, \ldots$ where $x_1 < x_2 < x_3 < \ldots$ then $P(X = x_i) = F(x_i) - F(x_{i-1})$.
5. If X is continuous r.v., then . $\frac{dF(x)}{dx} = f(x)$

- If the density function of a continuous r.v. X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$$

1. Find the value of *a*
2. Find the cumulative distribution function of X
3. Find P(1.5 < X ≤ 3)
4. Find P(X > 1.5)

1. Since $f(x)$ is a p.d.f.

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{0}^{3} f(x)dx = \int_{0}^{1} axdx + \int_{1}^{2} adx + \int_{2}^{3} (3a - ax)dx = 1$$

$a = 1/2$

2.

$$F(x) = P(X \le x) = 0, \quad x < 0$$

$$F(x) = \int_0^x ax\,dx = \left[\frac{ax^2}{2}\right]_0^x = \frac{ax^2}{2} = \frac{x^2}{4}, \quad 0 \le x \le 1$$

$$F(x) = \int_0^1 a\,x dx + \int_1^x a\,dx$$

$$= \left[\frac{ax^2}{2}\right]_0^1 + [ax]_1^x$$

$$= \left[\frac{a}{2}\right] + [ax - a] = ax - \frac{a}{2} = \frac{x}{2} - \frac{1}{4}, \; 1 \le x \le 2$$

2.

$$F(x)=\int_0^1 a\,x dx+\int_1^2 a\,dx+\int_2^3 (3a-ax)\,dx$$

$$=\left[\frac{ax^2}{2}\right]_0^1+[ax]_1^2+\left[3ax-\frac{ax^2}{2}\right]_2^x$$

$$=\left[\frac{a}{2}\right]+a+\left[\left(3ax-\frac{ax^2}{2}\right)-\left(6a-2a\right)\right]$$

$$=3ax-\frac{ax^2}{2}-\frac{5a}{2}$$

$$=\frac{1}{4}\left(6x-x^2-5\right),\ \ 2\le x\le 3$$

$$F(x)=P(X\le x)=1,\ \ x>3$$

2. Cumulative Distribution function

$$F(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ \dfrac{x^2}{4}, & 0 \le x \le 1 \\ \dfrac{1}{4} + \dfrac{x-1}{2}, & 1 \le x \le 2 \\ \dfrac{-5}{4} + \dfrac{6x - x^2}{4}, & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

3. $F(1.5)=\frac{1}{4}+\frac{x-1}{2}=\frac{1}{4}+\frac{1.5-1}{2}=\frac{1}{2},\quad 1\le x\le 2$

$F(3)=\frac{-5}{4}+\frac{6x-x^2}{4}=-\frac{5}{4}+\frac{6(3)-(3)^2}{4}=1,\ \ 2\le x\le 3$

$\therefore P(1.5<x<3)=F(3)-F(1.5)=1-\frac{1}{2}=\frac{1}{2}$

4. $$P(X > 1.5) = \int_{1.5}^{3} f(x)dx$$

$$= \int_{1.5}^{2} \frac{1}{2}dx + \int_{2}^{3} \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \frac{1}{2}$$

Mathematical Expectation

Let X be a random variable having a probability distribution function f(x). Expected value of the random variable is the arithmetic mean of the random variable. The mean or expected value of the random variable u(X), Then

If X is discrete type of random variable

$$\mu_{u(X)} = E[u(x)] = \sum_{x} u(x) f(x)$$

If X is continuous type of random variable

$$\mu_{u(X)} = E[u(x)] = \int_{-\infty}^{\infty} u(x) f(x)\, dx$$

Special Mathematical Expectation

Let $u(X) = X$, where X is a random variable of the discrete type having a p.d.f. $f(x)$. Then

1. Mean of the random variable is $E[X] = \mu_X$
2. If a is constant, $E[a] = a$
3. If a and b are constants, $E[aX \pm b] = a\,E[X] \pm b$
4. $E[f(X) \pm g(X)] = E[f(X)] \pm E[g(X)]$
5. Variance of the random variable

   $\sigma_X^2 = Var[X] = E[(X - \mu_X)^2] = E[X^2] - E[X]^2$
6. If a is constant, $Var[a] = 0$
7. $Var[aX \pm b] = a^2\, Var[X]$

- In a gambling game a man is paid Rs. 5 if he gets all heads or all tails when three coins are tossed, and he will pay out Rs. 3 if either one or two heads show,. What is his expected gain?

The sample space for the possible outcomes when three coins are tossed simultaneously, or equivalently if 1 coin is tossed three times, is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

The amount the gambler can:

win is Rs.5; if event

E_1 = {HHH, TTT} occurs and

lose Rs.3; if event

E_2 = {HHT, HTH, THH, HTT, THT, TTH} occurs.

Since E_1 and E_2 occur with probabilities 1/4 and 3/4 respectively, it follows that

$$\mu = E[X] = \sum_x x\, p(x) = (5)\left(\frac{1}{4}\right) + (-3)\left(\frac{3}{4}\right) = -1$$

In this game, the gambler will on an average lose Rs. 1 per toss of the three coins.

- Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \dfrac{20{,}000}{x^3}, & \mathrm{x} > 100 \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected life of this type of device.

We have,

$$\mu = E[X] = \int_{100}^{\infty} x \frac{20000}{x^3} dx = \int_{100}^{\infty} \frac{20000}{x^2} dx = 200$$

Therefore, we can expect this type of device to last, on average 200 hours.

- Suppose that the number of cars X that pass through a car wash between 4.00p.m and 9.00p.m. on any day has the following distribution:

x	4	5	6	7	8	9
P(X = x)	1/12	1/12	1/4	1/4	1/6	1/6

Let g(X) = 2X – 1 represent the amount of money in rupees, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

The attendant can expect to receive

$$
\begin{aligned}
E[g(X)] &= E[2X-1] \\
&= \sum_{x=4}^{9}(2x-1)\,p(x) \\
&= (7)\left(\frac{1}{12}\right)+(9)\left(\frac{1}{12}\right)+(11)\left(\frac{1}{4}\right)+(13)\left(\frac{1}{4}\right)+(15)\left(\frac{1}{6}\right)+(17)\left(\frac{1}{6}\right) \\
&= 12.67
\end{aligned}
$$

- Let X be a random variable with density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected value of g(X) = 4X + 3.

We have

$$E[g(X)] = E[4X+3]$$

$$= \int_{-1}^{2} \frac{(4x+3)x^2}{3}\,dx = \frac{1}{3}\int_{-1}^{2}(4x^3+3x^2)\,dx = 8$$

- The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X *having the* probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the mean and variance of X.

- Calculating *E(X) and E(X²), we have*

$$\mu = E[X] = 2\int_1^2 x(x-1)\,dx = \frac{5}{3}$$

$$E[X^2] = 2\int_1^2 x^2(x-1)\,dx = \frac{17}{6}$$

$$\therefore Var[X] = \frac{17}{6} - \left[\frac{5}{3}\right]^2 = \frac{1}{18}$$

Moment Generating Function

Moment generating function (MGF) of a r.v. X (discrete or continuous) is defined as $E[e^{tX}]$, where t is a real variable and denoted as M(t).

i.e., $M(t) = E[e^{tX}]$

If X is discrete type of random variable

$$M(t) = \sum_{x} e^{tx} f(x)$$

If X is continuous type of random variable

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Also, $M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$ we have, $E(X^n) = \left[\frac{d^n}{dt^n} M(t) \right]_{t=0}$

- If X represents the outcome, when a fair die is tossed, find the MGF of X and hence find E(X) and Var(X).

The probability distribution of X is given by

P(X = *i*) = 1/6, *i* = 1,2,3,4,5,6

$$M(t) = \sum_x e^{tx} p_x = \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$$

$$E(X) = [M'(t)]_{t=0}$$

$$= \left[\frac{1}{6}(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t})\right]_{t=0} = \frac{7}{2}$$

$$E(X^2) = [M''(t)]_{t=0}$$

$$= \frac{1}{6}(e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t})_{t=0} = \frac{91}{6}$$

$$Var(X) = E(X^2) - [E(X)]^2 == \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Chebyshev's Inequality

If X is a random variable with mean μ and variance σ^2, then for any positive number *K*, we have

$$P\{|X-\mu| \geq K\sigma\} \leq \frac{1}{K^2} \quad \text{or} \quad P\{|X-\mu| < K\sigma\} \geq 1-\frac{1}{K^2}$$

We know that

$$\sigma^2 = E[X-\mu]^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

$$= \int_{-\infty}^{\mu-K\sigma} (x-\mu)^2 f(x)dx + \int_{\mu-K\sigma}^{\mu+K\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+K\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

$$\geq \int_{-\infty}^{\mu-K\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+K\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

From the first Integral	From the second Integral
$x \leq \mu - K\sigma$ $-(x-\mu) \geq K\sigma$ $(x-\mu)^2 \geq K^2\sigma^2$	$x \geq \mu + K\sigma$ $(x-\mu) \geq K\sigma$ $(x-\mu)^2 \geq K^2\sigma^2$

$$\sigma^2 \geq \int_{-\infty}^{\mu-K\sigma} K^2\sigma^2 f(x)dx + \int_{\mu+K\sigma}^{\infty} K^2\sigma^2 f(x)dx$$

$$1 \geq \int_{-\infty}^{\mu-K\sigma} K^2 f(x)dx + \int_{\mu+K\sigma}^{\infty} K^2 f(x)dx$$

$$1 \geq K^2\{P[X \leq \mu - K\sigma] + P[X \geq \mu + K\sigma]$$

$$1 \geq K^2\{P[X - \mu \leq -K\sigma] + P[X - \mu \geq K\sigma]$$

$$1 \geq K^2\{P[| X - \mu | \geq K\sigma]$$

Hence

$$P\{| X - \mu | \geq K\sigma\} \leq \frac{1}{K^2}$$

Also we know,

$$P[|X-\mu| \geq K\sigma] + P[|X-\mu| < K\sigma] = 1, \quad \because P(A) + P(A^C) = 1$$

$$P[|X-\mu| < K\sigma] = 1 - P[|X-\mu| \geq K\sigma]$$

$$\therefore P[|X-\mu| < K\sigma] \geq 1 - \frac{1}{K^2}$$

- A discrete RV X takes the values $-1,0,1$ with probabilities 1/8, 3/4, 1/8 respectively. Evaluate $P[|X - \mu| \geq 2\sigma]$ and compare it with the upper bound given by chebyshev's inequality.

We have,

$$E[X] = -1\times\frac{1}{8} + 0\times\frac{3}{4} + 1\times\frac{1}{8} = 0$$

$$E[X^2] = 1\times\frac{1}{8} + 0\times\frac{3}{4} + 1\times\frac{1}{8} = \frac{1}{4}$$

$$Var[X] = E[X^2] - E[X]^2 = \frac{1}{4}$$

$$\sigma = \sqrt{Var[X]} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\therefore\ P[|X-\mu| \geq 2\sigma] = P[|X| \geq 1] = P[X = -1] + P[X = 1] = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Using Chebyshev's inequality,

$$P[|X-\mu| \geq 2\sigma] \leq \frac{1}{2^2} = \frac{1}{4}, here\, K = 2$$

- Let X be a continuous random variable whose probability density function given by $f(x)=e^{-x}$, $0 \le x \le \infty$. Using Chebyshev's inequality verify

$$P[|X-\mu|>2] \le \frac{1}{4}$$

and show that actual probability is e^{-3}.

- We have,

$$E[X] = \int_0^\infty xe^{-x}dx$$

$$= \left[-xe^{-x} - e^{-x}\right]_0^\infty = 1$$

$$E[X^2] = \int_0^\infty x^2e^{-x}dx$$

$$= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^\infty = 2$$

$$Var[X] = E[X^2] - E[X]^2 = 2 - 1 = 1$$

$$\sigma = 1$$

Therefore,

$$
\begin{aligned}
P[|X-1|>2] &= P(-\infty \le X-1 \le -2) + P(2 \le X-1 \le \infty) \\
&= P(-\infty \le X \le -1) + P(3 \le X \le \infty) \\
&= 0 + \int_3^\infty f(x)dx \\
&= e^{-3}
\end{aligned}
$$

Using the Chebyshev's inequality, $P\{|X-\mu| \ge K\sigma\} \le \dfrac{1}{K^2}$

$$K\sigma = 2;\ \ K = 2, \quad \because \sigma = 1$$

$$P[|X-\mu|>2] \le \frac{1}{4}$$