UNIT - I

Random Variables

Introduction

Consider an experiment of throwing a coin twice. The outcomes {HH, HT, TH, TT} consider the sample space. Each of these outcome can be associated with a number by specifying a rule of association with a number by specifying a rule of association (eg. The number of heads). Such a rule of association is called a random variable. We denote a random variable by the capital letter (X, Y, etc) and any particular value of the random variable by x and y.

Thus a random variable X can be considered as a function that maps all elements in

the sample space S into points on the real line. The notation X(S)=x means that x

is the value associated with the outcomes S by the Random variable X.

PROBABILITY

Probability (or) Chance: Probably, Chances, Likely, Possible - The terms convey the same meaning.

Example:

1. Probably your method is correct
2. The chances of getting ranks Ram and Babu are equal.
3. It is likely that Ram may not come for taking his classes today.
4. It is possible to reach the college by 8.30am.

Ordinary Language: The word probability means uncertainty about happening.

Mathematics or Statistics: A numerical measure of uncertainty is practiced by the important branch of statistics is called the Theory of Probability.

Day to Day Life:

- **Certainty** - Every day the sun rises in the east
- **Impossibility** - It is possible to live without water
- **Uncertainty** - Probably Raman gets that job. In the theory of probability, we represent certainty by 1, impossibility by 0 and uncertainty by a positive fraction which lies between 0 and 1.

Applications: There is no area in social, physical (or) natural sciences where the probability theory is not used.

- It is the base of the fundamental laws of statistics.
- It gives solutions to betting of games.

• It is extensively used in business situations characterized by uncertainty.

• It is essential tool in statistical inference and forms the basis of the Decision Theory.

Random Experiment (or) Trial and Event (or) Cases: Consider an experiment of throwing a coin. When tossing a coin, we may get a head or tail. Here tossing of a coin is a trial and getting a head or tail is an event.

Throwing of a die is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

Favorable Events: The number of outcomes favorable to an event in an experiment is the number of outcomes which entail the happening of the event

Example: In tossing 2 coins the cases favorable to the event of getting a head are HT, TH, and HH.

Exhaustive Events: The total number of possible outcomes in any trial is known as exhaustive events.

Example: In tossing a coin the possible outcomes are getting a head or tail. Hence we have 2 exhaustive events in throwing a coin.

Mutually Exclusive Event: Two events are said to be mutually exclusive when the occurrence of one affects the occurrence of the other. In other words, if A & B are mutually exclusive events and if A happens then B will not happen and vice versa.

Example: In tossing a coin the events head or tail are mutually exclusive, since both tail & head cannot appear in the same time

Equally Likely Events: Two events are said to be equally likely if one of them cannot be expected in preference to the other.

Example: In tossing a coin, head or tail are equally likely event

Independent Event : Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other.

Example : In tossing a coin, the event of getting a head in the 1st toss is independent of getting a head in the 2nd toss, 3rd toss, etc.

Mathematical Definition of Probability: If P is the notation for probability of happening of the event, then

$$P(A) = \frac{number\ of\ favourable\ case\ to\ A}{Total\ number\ of\ out\ comes} = \frac{m}{n}$$

Statistical Definition of Probability: If n trials, an event E happens m times, then $P(E) = \lim_{n\to\infty} \frac{m}{n}$

<u>Axiomatic Definition of Probability:</u>

1. For any event A, $P(A) \geq 0$.
2. $P(S) = 1$
3. If $A_1, A_2, A_3, \dots, A_n$ are finite number of disjoint events of S, then
$P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \cdots = \sum P(A_i)$

ADDITION LAW OF PROBABILITY

Case (i): When events are mutually exclusive

If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

Case (ii): When events are not mutually exclusive

If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

THEOREM : ADDITION LAW OF PROBABILITY

If A and B are any two events and are not disjoint, then $\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

THEOREM : MULTIPLICATION LAW OF PROBABILITY

For two events A and B, $\mathbf{P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B)}$, $\mathbf{P(A) > 0, P(B) > 0}$ ***where*** $\mathbf{P(B/A)}$ ***represents the conditional probability of occurrence of B when the event A has already happened and*** $\mathbf{P(A/B)}$ ***is the conditional probability of happening A, given that B has already happened.***

RANDOM VARIABLE

In this sample space each of these outcomes can be associated with a number by specifying a rule of association. Such a rule of association is called a random variables.

Eg : Number of heads

We denote random variable by the letter (X, Y, etc) and any particular value of the random variable by x or y.

S = {HH, HT, TH, TT} X(S) = {2, 1, 1, 0}

Thus a random X can be the considered as a fun. That maps all elements in the sample space S into points on the real line. The notation X(S) = x means that x is the value associated with outcome s by the R.V.X.

Example

In the experiment of throwing a coin twice the sample space S is S = {HH,HT,TH,TT}. Let X be a random variable chosen such that X(S) = x (the number of heads).

Note:

Any random variable whose only possible values are 0 and 1 is called a Bernoulli random variable.

Types of Random variables : (i) Discrete random variable (ii) Continuous random variable

DISCRETE RANDOM VARIABLE

Definition : A discrete random variable is a R.V.X whose possible values consitute finite set of values or countably infinite set of values.

Examples

All the R.V.'s from Example : 1 are discrete R.V's

Remark

The meaning of $P(X \leq a)$ is simply the probability of the set of outcomes 'S' in the sample space for which $X(s) \leq a$. Or $P(X \leq a) = P\{S : X(S) \leq a\}$

In the above example: 1 we should write

$P(X \leq 1) = P(HH, HT, TH) = ¾$

Here $P(X \leq 1) = ¾$ means the probability of the R.V.X (the number of heads) is less than or equal to 1 is ¾ .

Distribution function of the random variable X or cumulative distribution of the random variable X

Def :

The distribution function of a random variable X defined in $(-\infty, \infty)$ is given by

$F(x) = P(X \leq x) = P\{s : X(s) \leq x\}$

Note:

Let the random variable X takes values x1, x2, ….., xn with probabilities P1, P2, ….., Pn and let x1< x2< ….. <xn

Then we have

$F(x) = P(X < x1) = 0, -\infty < x < x,$

$F(x) = P(X < x1) = 0, P(X < x1) + P(X = x1) = 0 + p1 = p1$

$F(x) = P(X < x2) = 0, P(X < x1) + P(X = x1) + P(X = x2) = p1 + p2$

$F(x) = P(X < xn) = P(X < x1) + P(X = x1) + ….. + P(X = xn)$

$= p1 + p2+ ………. + pn = 1$

PROPERTIES OF DISTRIBUTION FUNCTIONS

Property : 1 $P(a < X \leq b) = F(b) – F(a)$, where $F(x) = P(X \leq x)$

Property : 2 $P(a \leq X \leq b) = P(X = a) + F(b) – F(a)$

Property : 3 $P(a < X < b) = P(a < X \leq b) - P(X = b)$

$= F(b) – F(a) – P(X = b)$ by prob (1)

PROBABILITY MASS FUNCTION (OR) PROBABILITY FUNCTION

Let X be a one dimenstional discrete R.V. which takes the values x1, x2, …… To each possible outcome 'x_i' we can associate a number p_i. i.e.,

$P(X (x_i) = pi$ called the probability of x_i. The number $p_i = P(x_i)$ satisfies the following conditions.

(i) $p(x_i) \geq 0, \forall_i$ (ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

The function p(x) satisfying the above two conditions is called the probability mass function (or) probability distribution of the R.V.X. The probability distribution {xi, pi} can be displayed in the form of table as shown below.

$X = x_i$	x_1	x_2	....	x_i
$P(X = x_i) = p_i$	p_1	p_2	....	p_i

Notation

Let 'S' be a sample space. The set of all outcomes 'S' in S such that X(S) = x is denoted by writing X = x.

$P(X = x) = P\{S : X(s) = x\}$

|||ly $P(x \leq a) = P\{S : X() \in (-\infty, a)\}$

and $P(a < x \leq b) = P\{s : X(s) \in (a, b)\}$

$P(X = a \text{ or } X = b) = P\{(X = a) \cup (X = b)\}$

$P(X = a \text{ and } X = b) = P\{(X = a) \cap (X = b)\}$ and so on.

Theorem :1 If X1 and X2 are random variable and K is a constant then KX_1, $X_1 + X_2$, X_1X_2, $K_1X_1 + K_2X_2$, X_1-X_2 are also random variables.

Theorem :2

If 'X' is a random variable and f(•) is a continuous function, then f(X) is a random variable.

Note

If F(x) is the distribution function of one dimensional random variable then

I. $0 \leq F(x) \leq 1$

II. If $x < y$, then $F(x) \leq F(y)$

III. $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

IV. $F(\infty) = \lim_{x \to \infty} F(x) = 1$

V. If 'X' is a discrete R.V. taking values x_1, x_2, x_3
Where $x_1 < x_2 < x_{i-1}\ x_i$ .... then
$P(X = x_i) = F(x_i) - F(x_{i-1})$

Example:

A random variable X has the following probability function

Values of X	0	1	2	3	4	5	6	7	8
Probability P(X)	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Determine the value of 'a'

(ii) Find P(X<3), P(X≥3), P(0<X<5)

(iii) Find the distribution function of X.

Solution

Table 1

Values of X	0	1	2	3	4	5	6	7	8
p(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) We know that if p(x) is the probability of mass function then

$$\sum_{i=0}^{8} p(x_i) = 1$$

p(0) + p(1) + p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + p(8) = 1

a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1

81 a = 1

a = 1/81

put a = 1/81 in table 1, e get table 2

Table 2

X = x	0	1	2	3	4	5	6	7	8
P(x)	1/81	3/81	5/81	7/81	9/81	11/81	13/81	15/81	17/81

(ii) $P(X < 3)$ = p(0) + p(1) + p(2)

= 1/81+ 3/81 + 5/81 = 9/81

(ii) $P(X \geq 3)$ = 1 - p(X < 3)

= 1 - 9/81 = 72/81

(iii) $P(0 < x < 5)$ = p(1) + p(2) + p(3) + p(4) here 0 & 5 are not include

= 3/81 + 5/81 + 7/81 + 9/81

$$= \frac{3+5+7+8+9}{81} = \frac{24}{81}$$

(iv) To find the distribution function of X using table 2, we get

$X = x$	$F(X) = P(x \leq x)$
0	$F(0) = p(0) = 1/81$
1	$F(1) = P(X \leq 1) = p(0) + p(1)$ $= 1/81 + 3/81 = 4/81$
2	$F(2) = P(X \leq 2) = p(0) + p(1) + p(2)$ $= 4/81 + 5/81 = 9/81$
3	$F(3) = P(X \leq 3) = p(0) + p(1) + p(2) + p(3)$ $= 9/81 + 7/81 = 16/81$
4	$F(4) = P(X \leq 4) = p(0) + p(1) + \ldots. + p(4)$ $= 16/81 + 9/81 = 25/81$
5	$F(5) = P(X \leq 5) = p(0) + p(1) + \ldots.. + p(4) + p(5)$ $= 2/81 + 11/81 = 36/81$
6	$F(6) = P(X \leq 6) = p(0) + p(1) + \ldots.. + p(6)$ $= 36/81 + 13/81 = 49/81$
7	$F(7) = P(X \leq 7) = p(0) + p(1) + \ldots. + p(6) + p(7)$ $= 49/81 + 15/81 = 64/81$
8	$F(8) = P(X \leq 8) = p(0) + p(1) + \ldots.. + p(6) + p(7) + p(8)$ $= 64/81 + 17/81 = 81/81 = 1$

CONTINUOUS RANDOM VARIABLE

Def : A R.V.'X' which takes all possible values in a given internal is called a continuous random variable.

Example : Age, height, weight are continuous R.V.'s.

PROBABILITY DENSITY FUNCTION

Consider a continuous R.V. 'X' specified on a certain interval (a, b) (which can also be a infinite interval $(-\infty, \infty)$).

If there is a function y = f(x) such that

$$\lim_{\Delta x \to 0} \frac{P(x < X < x + \Delta x)}{\Delta x} = f(x)$$

Then this function f(x) is termed as the probability density function (or) simply density function of the R.V. 'X'.

It is also called the frequency function, distribution density or the probability density function.

The curve y = f(x) is called the probability curve of the distribution curve.

Remark

If f(x) is p.d.f of the R.V.X then the probability that a value of the R.V. X will fall in some interval (a, b) is equal to the definite integral of the function f(x) a to b.

$$P(a < x < b) = \int_a^b f(x)dx \quad \text{(or)}$$

$$P(a \le X \le b) = \int_a^b f(x)dx$$

PROPERTIES OF P.D.F

The p.d.f f(x) of a R.V.X has the following properties

1. In the case of discrete R.V. the probability at a point say at x = c is not zero. But in the case of a continuous R.V.X the probability at a point is always zero.

$$P(X = c) = \int_{-\infty}^{\infty} f(x)dx = [x]_c^C = C - C = 0$$

2. If x is a continuous R.V. then we have p(a ≤ X ≤ b) = p(a ≤ X < b)= p(a < X V b)

IMPORTANT DEFINITIONS INTERMS OF P.D.F

If f(x) is the p.d.f of a random variable 'X' which is defined in the interval (a, b) then

i	Arithmetic mean	$\int_a^b x\, f(x)dx$
ii	Harmonic mean	$\int_a^b \frac{1}{x} f(x)dx$
iii	Geometric mean 'G' log G	$\int_a^b \log x\, f(x)dx$
iv	Moments about origin	$\int_a^b x^r\, f(x)dx$
v	Moments about any point A	$\int_a^b (x-A)^r\, f(x)dx$
vi	Moment about mean μ_r	$\int_a^b (x-\text{mean})^r\, f(x)dx$
vii	Variance μ_2	$\int_a^b (x-\text{mean})^2\, f(x)dx$
viii	Mean deviation about the mean is M.D.	$\int_a^b \lvert x-\text{mean}\rvert\, f(x)dx$

Mathematical Expectations

Def : Let 'X' be a continuous random variable with probability density function f(x). Then the mathematical expectation of 'X' is denoted by E(X) and is given by

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

It is denoted by

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x) dx$$

Thus

$$\mu_1^{'} = E(X) \qquad (\mu_1^{'} \text{ about origin})$$

$$\mu_2^{'} = E(X^2) \qquad (\mu_2^{'} \text{ about origin})$$

$$\therefore \text{Mean} = \overline{X} = \mu_1^{'} = E(X)$$

And

$$\text{Variance} = \mu_2^{'} - \mu_2^{'2}$$

$$\text{Variance} = E(X^2) - [E(X)]^2 \qquad \text{(a)}$$

* r^{th} moment (abut mean)

Now

$$E\{X - E(X)\}^r = \int_{-\infty}^{\infty} \{x - E(X)\}^r f(x) dx$$

$$= \int_{-\infty}^{\infty} \{x - \overline{X}\}^r f(x) dx$$

Thus

$$\mu_r = \int_{-\infty}^{\infty} \{x - \overline{X}\}^r f(x) dx \qquad \text{(b)}$$

$$\text{Where} \quad \mu_r = E[X - E(X)^r]$$

This gives the r^{th} moment about mean and it is denoted by μ_r

Put r = 1 in (B) we get

$$\mu_r \quad = \quad \int_{-\infty}^{\infty} \{x - \overline{X}\} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x\,f(x)\,dx - \int_{-\infty}^{\infty} \overline{x}\,f(x)\,dx$$

$$= \quad \overline{X} - \overline{X}\int_{-\infty}^{\infty} f(x)\,dx \qquad \left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

$$= \quad \overline{X} - \overline{X}$$

$$\mu_1 \quad = \quad 0$$

Put r = 2 in (B), we get

$$\mu_2 \quad = \quad \int_{-\infty}^{\infty} (x - \overline{X})^2 f(x)\,dx$$

$$\text{Variance} = \mu_2 \quad = \quad E[X - E(X)]^2$$

Which gives the variance interms of expectations.

Note

Let g(x) = K (Constant), then

$$E\left[g(X)\right] = E(K) \quad = \quad \int_{-\infty}^{\infty} K\,f(x)\,dx$$

$$= \quad K\int_{-\infty}^{\infty} f(x)\,dx \qquad \left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

$$= \quad K \,.\, 1 \quad = \quad K$$

Thus $E(K) = K \Rightarrow E[\text{a constant}] = \text{constant}$.

EXPECTATIONS (Discrete R.V.'s)

Let 'X' be a discrete random variable with P.M.F p(x)

Then

$$E(X) = \sum_x x\, p(x)$$

For discrete random variables 'X'

$$E(X^r) = \sum_x x^r\, p(x) \qquad \text{(by def)}$$

If we denote

$$E(X^r) = \mu_r'$$

Then

$$\mu_r' = E[X^r] = \sum_x x^r p(x)$$

Put r = 1, we get

$$\text{Mean } \mu_r' = \sum x\, p(x)$$

Put r = 2, we get

$$\mu_2' = E[X^2] = \sum_x x^2 p(x)$$

$$\therefore \mu_2 = \mu_2' - \mu_1'^2 = E(X^2) - \{E(X)\}^2$$

The r^{th} moment about mean

$$\mu_r' = E[\{X - E(X)\}^r]$$

$$= \sum_x (x - \overline{X})^r p(x), \qquad E(X) = \overline{X}$$

Put r = 2, we get

$$\text{Variance} = \mu_2 = \sum_x (x - \overline{X})^2 p(x)$$

ADDITION THEOREM (EXPECTATION)

Theorem 1

If X and Y are two continuous random variable with pdf fx(x) and fy(y) then

E(X+Y) = E(X) + E(Y)

MULTIPLICATION THEOREM OF EXPECTATION

Theorem 2

If X and Y are independent random variables,

Then E(XY) = E(X) . E(Y)

Note :

If X1, X2, ……, Xn are 'n' independent random variables, then

$E[X_1, X_2, \ldots, X_n] = E(X_1), E(X_2), \ldots, E(X_n)$

Theorem 3

If 'X' is a random variable with pdf f(x) and 'a' is a constant, then

(i) $E[a\, G(x)] = a\, E[G(x)]$

(ii) $E[G(x)+a] = E[G(x)+a]$

Where G(X) is a function of 'X' which is also a random variable.

Theorem 4

If 'X' is a random variable with p.d.f. f(x) and 'a' and 'b' are constants, then $E[ax + b] = a\, E(X) + b$

Cor 1:

If we take $a = 1$ and $b = -E(X) = -X$, then we get

$E(X - X) = E(X) - E(X) = 0$

Note

$$E\left(\frac{1}{X}\right) \neq \frac{1}{E(X)}$$

$$E[\log (x)] \neq \log E(X)$$

$$E(X^2) \neq [E(X)]^2$$

EXPECTATION OF A LINEAR COMBINATION OF RANDOM VARIABLES

Let $X_1, X_2, \ldots, X_n$ be any 'n' random variable and if $a_1, a_2, \ldots, a_n$ are constants, then

$E[a_1X_1 + a_2X_2 + \ldots + a_nX_n] = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$

Result

If X is a random variable, then

$Var(aX + b) = a^2 Var(X)$ 'a' and 'b' are constants.

Covariance :

If X and Y are random variables, then covariance between them is defined as $Cov(X, Y) = E\{[X - E(X)]\,[Y - E(Y)]\}$

$Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$ (A)

If X and Y are independent, then

$E(XY) = E(X)\, E(Y)$

Sub (B) in (A), we get $Cov(X, Y) = 0$

$\therefore$ If X and Y are independent, then

Cov (X, Y) = 0

Note

(i) Cov(aX, bY) = ab Cov(X, Y)

(ii) Cov(X+a, Y+b) = Cov(X, Y)

(iii) Cov(aX+b, cY+d) = ac Cov(X, Y)

(iv) Var (X1 + X2) = Var(X1) + Var(X2) + 2 Cov(X1, X2)

If X1, X2 are independent

Var (X1+ X2) = Var(X1) + Var(X2)

EXPECTATION TABLE

Discrete R.V's	**Continuous R.V's**
1. $E(X) = \sum x\, p(x)$	1. $E(X) = \int_{-\infty}^{\infty} x f(x) dx$
2. $E(X^r) = \mu_r' = \sum_x x^r p(x)$	2. $E(X^r) = \mu_r' = \int_{-\infty}^{\infty} x^r\ f(x) dx$
3. Mean $= \mu_r' = \sum x\, p(x)$	3. Mean $= \mu_r' = \int_{-\infty}^{\infty} x\ f(x) dx$
4. $\mu_2' = \sum x^2\ p(x)$	4. $\mu_2' = \int_{-\infty}^{\infty} x^2\ f(x) dx$
5. Variance $= \mu_2' - \mu_1'^2 = E(X^2) - \{E(X)\}^2$	5. Variance $= \mu_2' - \mu_1'^2 = E(X^2) - \{E(X)\}^2$

SOLVED PROBLEMS ON DISCRETE R.V'S

Example :1

When die is thrown, 'X' denotes the number that turns up. Find E(X), E(X^2) and Var (X).

Solution

Let 'X' be the R.V. denoting the number that turns up in a die. 'X' takes values 1, 2, 3, 4, 5, 6 and with probability 1/6 for each

X = x	1	2	3	4	5	6
p(x)	1/6	1/6	1/6	1/6	1/6	1/6
	$p(x_1)$	$p(x_2)$	$p(x_3)$	$p(x_4)$	$p(x_5)$	$p(x_6)$

Now

$$E(X) = \sum_{i=1}^{6} x_i\, p(x_i)$$

$$= x_1p(x_1) + x_2p(x_2) + x_3p(x_3) + x_4p(x_4) + x_5p(x_5) + x_6p(x_6)$$

$$= 1 \text{ x } (1/6) + 1 \text{ x } (1/6) + 3 \text{ x } (1/6) + 4 \text{ x } (1/6) + 5 \text{ x } (1/6) + 6 \text{ x } (1/6)$$

$$= 21/6 \quad = \quad 7/2 \qquad (1)$$

$$E(X) = \sum_{i=1}^{6} x_i\, p(x_p)$$

$$= x_1^2p(x_1)+x_2^2p(x_2)+x_3^2p(x_3)+x_4^2p(x_4)+x_5^2p(x_5)+x_6p(x_6)$$

$$= 1(1/6) + 4(1/6) + 9(1/6) + 16(1/6) + 25(1/6) + 36(1/6)$$

$$= \frac{1+4+9+16+25+36}{6} \quad = \quad \frac{91}{6} \qquad (2)$$

Variance (X) $= \text{Var (X)} = E(X^2) - [E(X)]^2$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} \quad = \frac{35}{12}$$

Example :2

Find the value of (i) C (ii) mean of the following distribution given

$$f(x)=\begin{cases} C(x-x^2), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

$$\text{Given } f(x)=\begin{cases} C(x-x^2), & 0<x<1 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^1 C(x-x^2)dx = 1 \qquad [\text{using (1)}]\ [\therefore\ 0<x<1]$$

$$C\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$C\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

$$C\left[\frac{3-2}{6}\right] = 1$$

$$\frac{C}{6} = 1 \qquad C = 6 \qquad (2)$$

Sub (2) in (1), $f(x) = 6(x - x^2)$, $0 < x < 1$ (3)

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_0^1 x\,6(x - x^2)\,dx \qquad \text{[from (3)]} \qquad [\therefore 0 < x < 1]$$

$$= \int_0^1 (6x^2 - x^3)\,dx$$

$$= \left[\frac{6x^3}{3} - \frac{6x^4}{4}\right]_0^1$$

$\therefore$ Mean = ½

Mean	C
½	6

CONTINUOUS DISTRIBUTION FUNCTION

Def :

If f(x) is a p.d.f. of a continuous random variable 'X', then the function

$$F_X(x) = F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)\,dx, \quad -\infty < x < \infty$$

is called the distribution function or cumulative distribution function of the random variable.

* PROPERTIES OF CDF OF A R.V. 'X'

(i) $0 \le F(x) \le 1, \; -\infty < x < \infty$

(ii) $\underset{x \to -\infty}{\mathrm{Lt}} F(x) = 0, \qquad \underset{x \to -\infty}{\mathrm{Lt}} F(x) = 1$

(iii) $P(a \le X \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$

(iv) $F'(x) = \dfrac{dF(x)}{dx} = f(x) \ge 0$

(v) $P(X = x_i) = F(x_i) - F(x_i - 1)$

Example :1.4.1

Given the p.d.f. of a continuous random variable 'X' follows

$$f(x)=\begin{cases}6x(1-x), & 0<x<1\\ 0 & \text{otherwise}\end{cases}$$, find c.d.f. for 'X'

Solution

Given $f(x)=\begin{cases}6x(1-x), & 0<x<1\\ 0 & \text{otherwise}\end{cases}$

The c.d.f is $F(x)=\int_{-\infty}^{x} f(x)dx, -\infty<x<\infty$

(i) When $x<0$, then

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_{-\infty}^{x} 0\, dx \quad = 0$$

(ii) When $0<x<1$, then

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$$

$$= 0 + \int_{0}^{x} 6x(1-x)dx \quad = 6\int_{0}^{x} x(1-x)dx = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^x$$

$$= 3x^2 - 2x^3$$

(iii) When $x>1$, then

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 6x(1-x)dx + \int_{0}^{x} 0\, dx$$

$$= 6\int_{0}^{1}(x-x^2)dx \quad = 1$$

Using (1), (2) & (3) we get

$$F(x)=\begin{cases}0, & x<0\\ 3x^2-2x^3, & 0<x<1\\ 1, & x>1\end{cases}$$

Example:1.4.2

(i) If $f(x)=\begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ defined as follows a density function ?

(ii) If so determine the probability that the variate having this density will fall in the interval (1, 2).

Solution

Given $f(x)=\begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) In $(0, \infty)$, e^{-x} is +ve

$\therefore f(x) \geq 0$ in $(0, \infty)$

(b) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx$

$= \left[-e^{-x}\right]_0^{\infty} \qquad = -e^{-\infty} + 1$

$= 1$

Hence f(x) is a p.d.f

(ii) We know that

$P(a \leq X \leq b) \qquad = \int_a^b f(x)\,dx$

$P(1 \leq X \leq 2) \qquad = \int_1^2 f(x)\,dx \qquad = \int_1^2 e^{-x}\,dx = [-e^{-x}]_{+1}^{2}$

$= \int_1^2 e^{-x}\,dx = [-e^{-x}]_{+1}^{2}$

$= -e^{-2} + e^{-1} \qquad = -0.135 + 0.368 \qquad = 0.233$

Example:1.4..3

A probability curve $y = f(x)$ has a range from 0 to ∞. If $f(x) = e^{-x}$, find the mean and variance and the third moment about mean.

Solution

$$\text{Mean} = \int_0^\infty x\, f(x)\, dx$$

$$= \int_0^\infty x\, e^{-x}\, dx \qquad = \left[x[-e^{-x}] - [e^{-x}]\right]_0^\infty$$

Mean = 1

$$\text{Variance} \quad \mu_2 = \int_0^\infty (x - \text{Mean})^2 f(x) dx$$

$$= \int_0^\infty (x-1)^2 e^{-x}\, dx$$

$$\mu_2 = 1$$

Third moment about mean

$$\mu_3 = \int_a^b (x - \text{Mean})^3 f(x) dx$$

Here a = 0, b = ∞

$$\mu_3 = \int_a^b (x-1)^3 e^{-x}\, dx$$

$$= \left\{(x-1)^3(-e^{-x}) - 3(x-1)^2(e^{-x}) + 6(x-1)(-e^{-x}) - 6(e^{-x})\right\}_0^\infty$$

$$= -1 + 3 - 6 + 6 = 2$$

$$\mu_3 = 2$$

MOMENT GENERATING FUNCTION

Def : The moment generating function (MGF) of a random variable 'X' (about origin) whose probability function f(x) is given by

$$M_X(t) = E[e^{tX}]$$

$$= \begin{cases} \displaystyle\int_{x=-\infty}^{\infty} e^{tx} f(x)dx, & \text{for a continuous probably function} \\ \displaystyle\sum_{x=-\infty}^{\infty} e^{tx} p(x), & \text{for a discrete probably function} \end{cases}$$

Where t is real parameter and the integration or summation being extended to the entire range of x.

Example :1.5.1

Prove that the r^{th} moment of the R.V. 'X' about origin is $M_X(t) = \int_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$

Proof

$$\text{WKT } M_X(t) = E(e^{tX})$$

$$= E\left[1+\frac{tX}{1!}+\frac{(tX)^2}{2!}+\frac{(tX)^3}{3!}+....++\frac{(tX)^r}{r!}+....\right]$$

$$= E[1]+tE(X)+\frac{t^2}{2!}E(X^2)+.....+\frac{t^r}{r!}E(X^r)+........$$

$$M_X(t) = 1+t\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+.....+\frac{t^r}{r!}\mu_r'+........$$

[using $\mu_r' = E(X^r)$]

Thus r^{th} moment = coefficient of $\frac{t^r}{r!}$

Note

1. The above results gives MGF interms of moments.
2. Since $M_X(t)$ generates moments, it is known as moment generating function.

Example:1.5.2

Find μ_1' and μ_2' from $M_X(t)$

Proof

$$\text{WKT } M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'$$

$$M_X(t) = \mu_0'+\frac{t}{1!}\mu_1'+\frac{t^2}{2!}\mu_2'+.....+\frac{t^r}{r!}\mu_r' \qquad \text{(A)}$$

Differenting (A) W.R.T 't', we get

$$M_X'(t) = \mu_1'+\frac{2t}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+..... \qquad \text{(B)}$$

Put $t = 0$ in (B), we get

$$M_X'(0) = \mu_1' = \text{Mean}$$

$$\text{Mean} = M_1'(0) \quad \text{(or)} \quad \left[\frac{d}{dt}(M_X(t))\right]_{t=0}$$

$$M_X''(t) = \mu_2'+t\,\mu_3'+.......$$

Put $t = 0$ in (B)

$$M_X''(0) = \mu_2' \quad \text{(or)} \quad \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0}$$

In general $\mu_r' = \left[\frac{d^r}{dt^r}(M_X(t))\right]_{t=0}$

Example :1.5.3

Obtain the MGF of X about the point X = a.

Proof

The moment generating function of X about the point X = a is $M_X(t) \quad = E[e^{t(X-a)}]$

$$= E\left[1 + t(X-a) + \frac{t^2}{2!}(X-a)^2 + ... + \frac{t^r}{r!}(X-a)^r + ...\right]$$

$$\begin{bmatrix} \text{Formula} \\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... \end{bmatrix}$$

$$= E(1) + E[t(X-a)] + E[\frac{t^2}{2!}(X-a)^2] + ... + E[\frac{t^r}{r!}(X-a)^r] + ...$$

$$= 1 + t\,E(X-a) + \frac{t^2}{2!}E(X-a)^2 + ... + \frac{t^r}{r!}E(X-a)^r + ...$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + ... + \frac{t^r}{r!}\mu_r' + ... \qquad \text{Where } \mu_r' = E[(X-a)^r]$$

$$[M_X(t)]_{x=a} \quad = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + ... + \frac{t^r}{r!}\mu_r' + ...$$

Result:

$M_{CX}(t) = E[e^{tcx}]$ (1)

$M_X(t) = E[e^{ctx}]$ (2)

From (1) & (2) we get

$M_{CX}(t) = M_X(ct)$

Example :1.5.4

If X_1, X_2, ….., X_n are independent variables, then prove that

$$M_{X_1+X_2+....+X_n}(t) = E[e^{t(X_1+X_2+....+X_n)}]$$

$$= E[e^{tX_1}.e^{tX_2}.....e^{tX_n}]$$

$$= E(e^{tX_1}).E(e^{tX_2}).....E(e^{tX_n})$$

[$\therefore$ X_1, X_2, ….., X_n are independent]

$$= M_{X_1}(t).M_{X_2}(t)..........M_{X_n}(t)$$

Example:1.5.5

Prove that if $\cup = \dfrac{X-a}{h}$, then $M_{\cup}(t) = e^{\frac{-at}{h}}.M_X^{\left(\frac{t}{h}\right)}$, where a, h are constants.

Proof

By definition

$$M_{\cup}(t) = E\left[e^{tu}\right] \qquad \because \left[M_X(t) = E[e^{tx}]\right]$$

$$= E\left[e^{t\left(\frac{X-a}{h}\right)}\right]$$

$$= E\left[e^{\frac{tX}{n}-\frac{ta}{n}}\right]$$

$$= E[e^{\frac{tX}{h}}]\ \ E[e^{\frac{-ta}{h}}]$$

$$= e^{\frac{-ta}{h}}\ E[e^{\frac{tX}{h}}] \qquad \text{[by def]}$$

$$= e^{\frac{-ta}{h}}\ .\ M_X^{\left(\frac{t}{h}\right)}$$

$\therefore M_{\cup}(t) = e^{\frac{-at}{h}}.M_X\left(\dfrac{t}{h}\right)$, where $\cup = \dfrac{X-a}{h}$ and $M_X(t)$ is the MGF about origin.

Example:1.5.6

Find the MGF for the distribution where

$$f(x) = \begin{cases} \dfrac{2}{3} & \text{at } x = 1 \\ \dfrac{1}{3} & \text{at } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution

Given $\quad f(1) = \dfrac{2}{3}$

$f(2) = \dfrac{1}{3}$

$f(3) = f(4) = \ldots\ldots = 0$

MGF of a R.V. 'X' is given by

$$M_X(t) \quad = E[e^{tx}]$$

$$= \sum_{x=0}^{\infty} e^{tx} f(x)$$

$= e^0 f(0) + e^t f(1) + e^{2t} f(2) + \ldots\ldots$

$= 0 + e^t f(2/3) + e^{2t} f(1/3) + 0$

$= 2/3e^t + 1/3e^{2t}$

$\therefore$ MGF is $M_X(t) = \dfrac{e^t}{3}[2 + e^t]$