PROBABILITY: It is a measure of uncertainity about the happening of an event.

RANDOM EXPERIMENT: An experiment in which the Outcomes can be predicted in advance.

SAMPLE SPACE (S): The collection of all possible Outcomes of random experiment.

Ex: Tossing a coin $S = \{H, T\}$

Throwing a die $S = \{1, 2, 3, 4, 5, 6\}$

Sample space is also called as "exhaustive events."

SURE EVENT (Or) CERTAIN EVENT: The sample space 's' is sure event. $P(S) = 1$

IMPOSIBLE (OR) NULL EVENT: The event can't happen in an Experiment. $P(\phi) = 0$

MUTUALLY EXCLUSIVE EVENT: Two (or) more said to be ME Event if they can't happen together

$P(A \cap B) = 0$

INDEPENDENT EVENT: Two (or) more events said to be independent if occurence of event don't effect the Occurrence of the other

$P(A \cap B) = P(A) \cdot P(B)$

CONDITIONAL PROBABILITY: [Dependent Event]

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

↳ given

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

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* for Independent

$$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)\cdot P(B)}{P(B)}$$

$$= P(A)$$

$$P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A)\cdot P(B)}{P(A)}$$

$$= P(B)$$

MATHEMATICAL DEFINATION OF PROBABILITY

If A is any object, S is the sample space

$$P(A) = \frac{\text{No of favourable Outcome}}{\text{No of Exhaustive events}}$$

$$P(A) = \frac{n(A)}{n(S)}$$

PROPERTIES:

(i) $P(A) \geq 0$

(ii) $0 \leq P(A) \leq 1$ [Range]

Addition theorem:

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

If A, B are MEE

$$P(A\cup B) = P(A) + P(B)$$

RANDOM VARIABLE: (X)

Random Variable 'X' is a function from Space to Real number.

It can be represented as ⇒ $X: S \rightarrow R$

(or)

A function which assign some real value to every outcome of an experiment called RANDOM VARIABLE

Random Variable

Discrete RV [outcomes takes integer values]

* A RV which takes only integer values called DRV

Ex: No of boys in a queue for a particular period

Continous RV [outcomes takes any value in Real line] $(-\infty, \infty)$

* A RV which takes any value in Real line $(-\infty, \infty)$

Ex: Temperature
Amount of Rainfall
Height & weight of individuals

RESULTS OF PROBABILITY:

1. $P(\phi) = 0$
2. $P(\bar{A}) = 1 - P(A)$
3. $P(\bar{A}) + P(A) = 1$
4. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
5. $P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$ → if A & B are independent

PROBABILITY MASS FUNCTION [PMF]:

If X is a DRV with probabilities of $P_1, P_2 \ldots P_n$ then P is called a PMF Provided

(i) $P_i \geq 0 \quad \forall i$

(ii) $\sum_{i=1}^{n} P_i = 1$

PROBABILITY DENSITY FUNCTION: [PDF]

If X is a CRV with probabilities of $f(x_1), f(x_2) \ldots$ then $f(x)$ is said to be PDF provided

$f(x) \geq 0 \quad \forall i$

$\int_{-\infty}^{\infty} f(x)\, dx$

PROBABILITY DISTRIBUTION FUNCTION (OR) CUMMULATIVE [CDF]

$$F(x) = P(X \le x)$$

$$F(x) = \begin{cases} \sum_x x\,p(x) & \text{if } X \text{ is DRV} \\ \int_{-\infty}^{\infty} f(x)\,dx & \text{if } X \text{ is CRV} \end{cases}$$

Properties of CDF :

(1) $F(-\infty) = 0$

$F(\infty) = 1$

(2) $f(x) = F'(x)$

$f(x) = \frac{d}{dx} F(x)$ → Relation b/w CDF & PDF

(3) $P(a \le x \le b) = F(b) - F(a)$

PROBLEMS ON DRV :

(1) A RV X has the following probability function

X	0	1	2	3	4	5	6	7	8
P(X)	a	3a	5a	7a	9a	11a	13a	15a	17a

(a) Find 'a'

(b) Find $P(X<3)$, $P(X \ge 3)$, $P(0<X<5)$

(c) CDF of x (probability distribution)

(a) To find 'a'

since $\sum_i P_i = 1$

$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$81a = 1$

$a = 1/81$

b) $P(X<3) = P(X=0)+P(X=1)+P(X=2)$

$= a+3a+5a$

$= 9a$

$= 9\times{}^{1}/_{81}$

$= {}^{1}/_{9}$

$P(X<3) = {}^{1}/_{9}$

$P(X\geq 3) = 1-P(X<3)$

$\cancel{= P(X=3)+P(X>}$

$= 1-{}^{1}/_{9}$

$= {}^{8}/_{9}$

$P(0<X<5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$= 9a+3a+5a+7a$

$= 24a$

$= 24\times\frac{1}{81}$

$= {}^{8}/_{27}$

(c) <u>CDF</u>

$F(x) = P(X\leq x)$

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$
$F(x)=P(X\leq x)$	a	$4a$	$9a$	$16a$	$25a$	$36a$	$49a$	$64a$	$81a$
$F(x)$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1

(2) If X be a DRV with following distribution

X	-2	-1	0	1	2	3
$P(X)$	0.1	k	0.2	$2k$	0.3	$3k$

(i) find 'k'

$$\sum_i p_i = 1$$

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$0.6 + 6k = 1$$

$$6k = 1 - 0.6$$

$$= 0.4$$

$$k = 0.4/6$$

$$k = 4/60 = 1/15$$

$$\boxed{k = \frac{1}{15}}$$

(ii) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) + P(X=-2) + P(X=-1)$

$$= 0.2 + 2k + 0.3 + 0.1 + k$$

$$= 0.5 + 2/15 + 0.1 + 1/15$$

$$= 0.6 + \frac{3}{15}$$

$$= 0.6 + 0.2$$

$$= 0.8$$

$$P(X > 2) = P(X = 3)$$

$$\Rightarrow 3k = 3/15 = 0.2$$

(iii) $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$P(x > 0 / x < 3) = \frac{P(X=1) + P(X=2)}{0.8} = \frac{2k + 0.3}{0.8} = \frac{0.54}{0.8}$$

$$= 0.54$$

CDF : $F(x) = F(X \le x)$

$F(X \le x)$	0.1	0.166	0.366	0.498	0.798	1

$F(x)$

(iv) $P(X \le a) > 1/2 = 0.5$

Min value of a = ?

$\boxed{a = 2}$

Expectation (Mean (or) Average)

$$E(X) = \begin{cases} \sum_x x p(x), & X \text{ is DRV} \\ \int_{-\infty}^{\infty} x f(x)\,dx, & \text{if } X \text{ is CRV} \end{cases}$$

MEAN SQUARE VALUE:

$$E(X^2) = \sum x^2 p(x)$$

VARIANCE:

$$\sigma^2 = Var(X)$$

STAN

$$= E(X^2) - [E(X)]^2$$

STANDARD DEVIATION:

$$\sigma = \sqrt{V(X)}$$

PROPERTIES:

1. $E(a) = a$, a is const
2. $E(aX+b) = aE(X)+b$
3. $Var(aX+b) = a^2 Var(X)$
4. $Var(a) = 0$, a is const
5. $Var(aX) = a^2 Var(X)$

(3) Find probability mass function

(ii) Find Mean and Variance of X where X is a RV with

$P(X=-2) = P(X=-1) = P(X=1)$ and $P(X<0) = P(X=0) = P(X>0)$

$P(X=x) = \frac{1}{2^x}$

(ii) $P(X \text{ is even})$

$= P(X=2)+P(X=4)+P(X=6)\dots$

$= \frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\dots$

$= \frac{1}{2^2}\left[1+\frac{1}{2^2}+\frac{1}{2^4}+\dots\right]$

$= \frac{1}{4}\left[1-\frac{1}{2^2}\right]^{-1}$

$\Rightarrow \frac{1}{4}\left[1-\frac{1}{4}\right]^{-1} = 1/3$

(iii) $P(X \text{ is divisible by } 3)$

$P(X=3)+P(X=6)+P(X=9)+\dots$

$= \frac{1}{2^3}+\frac{1}{2^6}+\frac{1}{2^9}+\dots$

$= \frac{1}{2^3}\left[1+\frac{1}{2^3}+\frac{1}{(2^3)^2}+\dots\right]$

$\Rightarrow \frac{1}{8}\left[1-\frac{1}{2^3}\right]^{-1} = \frac{1}{8}\left[\frac{7}{8}\right]^{-1}$

$= 1/7$

(iv) $P(X>5)$

$= P(X=6)+P(X=7)+P(X=8)+\dots$

$= \frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^8}+\dots$

$= \frac{1}{2^6}\left[1+\frac{1}{2}+\frac{1}{2^2}+\dots\right]$

$\Rightarrow \frac{1}{2^6}\left[1-1/2\right]^{-1} = \frac{1}{64}\times 2 = \frac{1}{32} = 1/32$

(v) Find Mean & Variance of X

Mean: $E(X) = \sum_{x=1}^{\infty} x p(x)$

$= \sum_{x=1}^{\infty} x \cdot \frac{1}{2^x}$

$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + \cdots$

$= \frac{1}{2}\left[1 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2^2}\right]$

$= \frac{1}{2}\left[1 - \frac{1}{2}\right]^{-2}$ $\quad \left[\because 1 + 2x + 3x^2 + 4x^3 + \cdots = (1-x)^{-2}\right]$

$= \frac{1}{2}\left[\frac{1}{2}\right]^{-2}$

$= 2$

$\boxed{E(x) = 2}$

Variance:

$E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 p(x)$

$= \sum_{x=1}^{\infty} 1 \cdot \frac{1}{2}$

$= 1 \cdot \frac{1}{2} + 2^2 \times \frac{1}{2^2} + 3^2 \cdot \frac{1}{2^3} + \cdots$

$= \sum_{x=1}^{\infty} [x^2 + x - x] \cdot \frac{1}{2^x}$

$\sum_{x=1}^{\infty} x(x+1)\frac{1}{2^x} - \sum_{x=1}^{\infty} x \frac{1}{2^x}$

$= 1(2)\frac{1}{2} + 2(3)\frac{1}{2^2} + 3(4)\frac{1}{2^3} - E(X)$

$= 1 + \frac{3}{2} + 6\left[\frac{3}{2^2}\right] - E(X)$

$= \left[1 + 3\left[\tfrac{1}{2}\right] + 6\left(\tfrac{1}{2^2}\right) + \cdots\right] - 2$

$$= \left(1 - \frac{1}{2}\right)^{-3} - 2$$

$$= \left(\frac{1}{2}\right)^{-3} - 2$$

$$= 2^3 - 2$$

$$E(x^2) = 6$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 6 - (2)^2$$

$$= 6 - 4$$

$$= 2$$

CONTINUOUS RANDOM VARIABLE:

Probability density function $f(x)$:

conditions:

$$f(x) \geq 0 \quad \forall x$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Cummulative function:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

$$\boxed{F'(x) = f(x)}$$

Mean:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$V(X) = E(x^2) - [E(X)^2]$$

(1) Let X be a CRV with pdf

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0 & x > 3 \end{cases}$$

(i) <u>To find 'a'</u>:

Since f(x) is pdf,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx + \int_3^{\infty} 0\,dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$a\left[\tfrac{1}{2}\right] + a(2-1) + \left[9a - \frac{9a}{2} - \left(6a - \frac{4a}{2}\right)\right] = 1$$

$$\frac{a}{2} + a + \left[3a - \frac{5a}{2}\right] = 1$$

$$-\frac{4a}{2} + 4a = 1$$

$$a\left[4 - \frac{4}{2}\right] = 1$$

$$a[2] = 1$$

$$a = 1/2$$

$$f(x) = \begin{cases} x/2 & 0 \le x \le 1 \\ 1/2 & 1 \le x \le 2 \\ \left(3/2 - \frac{x}{2}\right) & 2 \le x \le 3 \\ 0 & x > 3 \end{cases}$$

(ii) $P(X < 1.5) = \int_{\infty}^{1.5} f(x)\,dx$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{1.5} \frac{1}{2}\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \frac{1}{2}\left[x\right]_1^{1.5}$$

$$= \frac{1}{4} + \frac{1}{2}\left[1.5 - 1\right]$$

$$= \frac{1}{4} + \frac{1}{2}\left[\frac{1}{2}\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2}\right] \quad \frac{1}{2}$$

$$\Rightarrow \frac{1}{2}\left[\frac{3}{2}\right] = \frac{3}{4}$$

(iii) $P(X < 1.5 / X < 2)$

$$P(X<1.5/X<2) = \frac{P(X<1.5) \cap P(X<2)}{P(X<2)}$$

$$\left[\because P(A/B) = \frac{P(A \cap B)}{P(B)}\right]$$

$$= \frac{P(X<1.5)}{P(X<2)}$$

$$P(X<2) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{1}{2}x\right]_1^2$$

$$= \frac{1}{4} + \left[\frac{2}{2} - \frac{1}{2}\right]$$

$$\Rightarrow \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$$= \frac{P(X<1.5)}{P(X<2)}$$

$$= \frac{1/2}{3/4}$$

$$= \frac{1}{2} \times 4/3$$

$$= 2/3$$

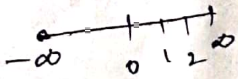
(iv) CDF

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$$f(x) = \begin{cases} \frac{x}{2} & (0,1) \\ \frac{1}{2} & (1,2) \\ \frac{1}{2}(3-x) & (2,3) \\ 0 & x>3 \end{cases}$$

When $x<0$,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = 0$$



When $0 \le x \le 1$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{x} x/2\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^x = \frac{x^2}{4}$$

When $1 \le x < 2$

$$F(x) = \int_{-\infty}^{0} 0 + \int_{0}^{1} \frac{x}{2}dx + \int_{1}^{x} \frac{1}{2}dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{1}{2}x\right]_1^x = \frac{1}{4} + \frac{1}{2}x - \frac{1}{2}$$

$$= 1/2\,x - 1/4$$

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When $2 \le x \le 3$

$$= \int_{-\infty}^{0} 0 + \int_0^1 \frac{x}{2} dx + \int_1^2 \frac{1}{2} dx + \int_2^x \frac{1}{2}(3-x)\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{1}{2}x\right]_1^2 + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^x$$

$$= \frac{1}{4} + \left[1 - \frac{1}{2}\right] + \left[\frac{3x}{2} - \frac{x^2}{4} - (3-1)\right]$$

$$= \frac{1}{4} + \left[1/2\right] + \left[\frac{3x}{2} - \frac{x^2}{4} - 2\right]$$

$$= \frac{3}{4} - 2 + \frac{3x}{2} - \frac{x^2}{4}$$

$$= \frac{3-8}{4} + \frac{6x - x^2}{4}$$

$$= \frac{6x - x^2 - 5}{4}$$

When $x > 3$

$$F(x) = \int_{-\infty}^{0} 0 + \int_0^1 \frac{x}{2} dx + \int_1^2 \frac{1}{2} x + \int_2^3 \frac{1}{2}(3-x)dx + \int_3^x 0$$

$$= 1$$

CDF:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{4} & 1 \le x \le 2 \\ \frac{6x - x^2 - 5}{4} & 1 \le x \le 3 \\ 1 & x > 3 \end{cases}$$

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(2) If $f(x) = \begin{cases} xe^{-x^2/2}, & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$, (i) Check whether $f(x)$ is a pdf (ii) Mean

(i) PT $f(x)$ is a pdf :-

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{\infty} xe^{-x^2/2} dx$$

Take $\frac{x^2}{2} = t$

$\frac{\cancel{2}x\,dx}{\cancel{2}} = dt$

$x\,dx = dt$

$$= \int_{0}^{\infty} xe^{-x^2/2} dx$$

$$= \int_{0}^{\infty} e^{-t} dt$$

$$= \left(\frac{e^{-t}}{-1}\right)_0^{\infty} = \left(\frac{e^{-\infty} - e^{-0}}{1}\right)$$

$$= 1$$

$f(x)$ is a Pdf //

(ii) Mean:

$$E(X) = \int_{-\infty}^{\infty} f(x)dx$$

$$= \int_{0}^{\infty} xe^{-x^2/2} dx$$

$x^2 = 2t$

$x = \sqrt{2t}$

$$\int_{0}^{\infty} \sqrt{2t}\, e^{-t} dt$$

$$\sqrt{2} \int_{0}^{\infty} t^{1/2} e^{-t} dt$$ [Gama Integral]

$$\sqrt{2}\, \sqrt{\tfrac{1}{2}}$$

$E(X) = 1$

(1) If $F(x) = 1-(1+x)e^{-x}, x \geq 0$. find (i) Pdf,

(ii) Mean & Variance of x

<u>CDF</u>:

$$F(x) = 1-(1+x)e^{-x}$$

<u>Pdf</u>:

$$F'(x) = f(x)$$

$$f(x) = \frac{d}{dx}[F(x)]$$

$$= \frac{d}{dx}\left[1-(1+x)e^{-x}\right]$$

$$= \left[0-(-(1+x)e^{-x} + (1)e^{-x})\right]$$

$$= -\left[-e^{-x} - xe^{-x} + e^{-x}\right]$$

$$f(x) = xe^{-x}, x \geq 0$$

(ii) <u>Mean</u>:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} x(xe^{-x})\,dx$$

$$= \int_0^{\infty} \underset{u}{x^2}\, \underset{v}{e^{-x}}\,dx$$

$u = x^2 \quad V = e^{-x}$

$u' = 2x \quad V_1 = -e^{-x}$

$u'' = 2 \quad V_2 = e^{-x}$

$u''' = 0 \quad V_3 = -e^{-x}$

$$uV_1 - u'V_2 + u''V_3$$

$$= \left[-x^2e^{-x} - 2xe^{-x} + 2(-e^{-x})\right]$$

$$= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty}$$

$$= \left[0 - (0 - 0 - 2)\right]$$

$$\boxed{E(x) = 2}$$

Variance: $V(x) = E(x^2) - [E(x)]^2$

$$E(x^2) = \int_0^{\infty} x^3 e^{-x}\, dx$$

$$u = x^3 \qquad v = e^{-x}$$

$$E(x^2) = \left[-x^3e^{-x} + (3x^2e^{-x}) - 6xe^{-x} - 6e^{-x}\right]_0^{\infty}$$

$$= 6$$

$$V(x) = 6 - 4$$

$$\boxed{V(x) = 2}$$

Moments:

Raw moments (Moments about Origin)

$$\mu_r' = E(x^r)$$

$$\mu_0' = 1, \ \mu_1' = \text{Mean}, \ \mu_2' = E(x^2)$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

Central Moments: (Moments about mean)

$$\mu_r = E(x - \bar{x})^r$$

Moments about A:

$$\Gamma(r) = E(x - A)^2$$

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Central Moments: $\mu_r = E(x-\bar{x})^r$

$$\boxed{\mu_0 = 1,\ \mu_1 = 0}$$

Variance $\Rightarrow \mu_2 = \mu_2' - \mu_1'^2$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

relation b/w raw & central moments

$\vdots$

$\mu_0 = E(x-\bar{x})^0$

$E(1) = 1$

$\mu_1 = E(x-\bar{x})$

$= E(x) - E(\bar{x})$

$= \bar{x} - \bar{x}$

$= 0$

Moment Generating Function (MGF):

$$\boxed{M_x(t) = E(e^{tx})}$$

$$M_x(t) = E(e^{tx}) = \begin{cases} \sum_x e^{tx} p(x), & x \text{ is DRV} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & x \text{ is CRV} \end{cases}$$

Properties:

1) $\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]$

2) Additive property of MGF

MCQ

$M_{x_1} + M_{x_2} + \cdots x_n(t) = M_{x_1}(t) \cdot M_{x_2}(t) \ldots M_{x_n}(t)$

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(1) If $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

Find MGF of X

$$M_X(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^1 x e^{tx} dx + \int_1^2 e^{tx}(2-x)\,dx$$

$u = x \qquad v = e^{tx}$

$u' = 1 \qquad v_1 = \frac{e^{tx}}{t}$

$u'' = 0 \qquad v_2 = \frac{e^{tx}}{t^2}$

$$= \left[x\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\left(\frac{e^{tx}}{t}\right) - (-1)\left(\frac{e^{tx}}{t^2}\right)\right]_1^2$$

$$= \left[\cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} - 0 + \frac{1}{t^2}\right] + \left[0 + \frac{e^{2t}}{t^2} - \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2}\right]$$

$$M_X(t) = \frac{1}{t^2}\left[-e^t + e^{2t} - e^t\right]$$

$$= \frac{1}{t^2}\left[e^{2t} - 2e^t + 1\right]$$

$$= \left(\frac{e^t - 1}{t}\right)^2$$

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2) If $f(x) = \begin{cases} ke^{-x}, & x \geq 0 \\ 0, & o/w \end{cases}$, find (i) k

(ii) MGF

(iii) Mean, Variance

(iv) first 3 central Moments

$k = ?$

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{0}^{\infty} ke^{-x}\,dx = 1$

$k\left(\frac{e^{-x}}{-1}\right)_0^{\infty} = 1$

$k[-e^{-\infty} + e^{-0}] = 1$

$k[1] = 1$

$\boxed{k = 1}$

MGF: $M_x(t) = E(e^{tx})$

$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$= \int_{0}^{\infty} e^{tx} e^{-x}\,dx$

$= \int_{0}^{\infty} e^{x(t-1)}\,dx$

$= \left[\frac{e^{-(1-t)x}}{-(1-t)}\right]_0^{\infty}$

$= \frac{e^{-\infty} + e^{0}}{1-t}$

$$\boxed{M_x(t) = \frac{1}{1-t}}$$

Mean: $E(x) = \mu_r'$

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$$

$$\mu_1' = \left[\frac{d}{dt}\frac{1}{1-t}\right]_{t=0}$$

$$\left[\frac{d}{dt}(1-t)^{-1}\right]_{t=0}$$

$$\left[\cancel{1}(1-t)^{-2}\right]_{t=0} \qquad \boxed{E(x) = 1}$$

$$\boxed{\mu_1' = 1}$$

Variance:

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$\Rightarrow \mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \left\{\frac{d}{dt}\left(\frac{d}{dt}M_x(t)\right)\right\}_{t=0}$$

$$= \left[\frac{d}{dt}(1-t)^{-2}\right]_{t=0}$$

$$= \left[2(1-t)^{-3}\right]_{t=0}$$

$$= 2(1-0)^{-3}$$

$$\boxed{\mu_2' = 2}$$

$$\mu_2 = 2 - (1)^2$$

$$= 2 - 1$$

$$= \cancel{2}$$

$$\boxed{V(X) = 1}$$

(IV) First three central moments.

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 1$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_3' = \left[\frac{d^3}{dt^3} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt} 2(1-t)^{-3}\right]_{t=0}$$

$$= \left[6(1-t)^{-4}\right]_{t=0}$$

$$= 6$$

$$\mu_3 = 6 - 3(2)(1) + 2(1)^3$$

$$= 6 + 2(1) - 6$$

$$= 2$$

$$\boxed{\mu_3 = 2}$$

(2) If $f(x) = kx(2-x)$, $0 \le x \le 2$

(i) find k

$$\int_0^2 kx(2-x)\,dx = 1$$

$$k\left[\frac{x^2}{2}\times 2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[4 - \frac{8}{3}\right] = 1$$

$$k\left[\frac{4}{3}\right] = 1$$

$$\boxed{k = 3/4}$$

(ii) r^{th} moment about ~~reg~~ igion

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx = E(x^r)$$

$$= \frac{3}{4}\int_0^2 (2x-x^2)x^r\,dx$$

$$= \frac{3}{4}\int_0^2 \left[2x^{r+1}-x^{2+r}\right]dx$$

$$= \frac{3}{4}\left[\frac{2x^{r+2}}{r+2}-\frac{x^{3+r}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{2\cdot 2^{r+2}}{r+2}-\frac{2^{3+r}}{r+3}\right]$$

$$= \frac{3}{4}\cdot 2^{r+3}\left[\frac{1}{r+2}-\frac{1}{r+3}\right]$$

$$= \frac{3}{4}\cdot 2^2\cdot 2^{r+1}\left[\frac{\cancel{r}+3-\cancel{r}-2}{(r+2)(r+3)}\right]$$

$$= 3\,2^{r+1}\left[\frac{1}{(r+2)(r+3)}\right]$$

$$\boxed{\mu_r' = 3\left[\frac{2^{r+1}}{(r+2)(r+3)}\right]}$$

put $r=1$

$$\mu_1' = \frac{3(4)}{12} = \frac{12}{12} = 1$$

$$\mu_2' = \frac{3(8)}{20} = \frac{24}{20} = 6/5$$

$$\mu_3' = \frac{3(16)}{5\cdot 6} = 8/5$$

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = \frac{6}{5}-1^2 = \frac{6-5}{5} = 1/5$$

$$\mu_3 = \frac{8}{5} - 3(6/5)(1) + 2(1)^3$$

$$\Rightarrow 8/5 - \frac{18}{5} + 2 = \frac{8-18+10}{5} = 0$$

(4) If $P(X=x) = \frac{1}{2^x}$, $x = 1, 2, 3 \dots$ find MGF, Mean, Variance

MGF :

$$M_x(t) = E(e^{tx})$$

$$= \int_{x=-\infty}^{\infty} e^{tx} p(x)\,dx \quad \left[= \sum_{x=1}^{\infty} e^{tx} p(x) \right.$$

$$= \int_{1}^{\infty} e^{tx} \cdot \frac{1}{2^x}\,dx \quad = \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{2^x}$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \dots\right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2-e^t}{2}\right]^{-1}$$

$$\boxed{M_x(t) = \frac{e^t}{2-e^t}}$$

Mean : $\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$

$$= \left[\frac{d}{dt} \frac{e^t}{2-e^t}\right]_{t=0}$$

$$= \left[\frac{(2-e^t)(e^t) + (e^t)(e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t - \cancel{e^{2t}} + \cancel{e^{2t}}}{(2-e^t)^2}\right]_{t=0}$$

$$\boxed{\mu_1' = 2}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$\frac{d}{dt}\left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{(2-e^t)^2 2e^t + 2e^t 2(2-e^t)e^t}{(2-e^t)^4}\right]_{t=0}$$

$$\mu_2' = \left[\frac{2+4(1)}{1}\right]$$

$$\boxed{\mu_2' = 6}$$

$$\boxed{E(X) = 2 \quad , \quad V(X) = 6 - 4 = 2}$$ //

Function Of RV :

If X is a RV with pdf $f(x)$ and $Y = g(X)$ is a function of RV X, then the pdf of Y is

$$\boxed{f(y) = f(x)\left|\frac{dx}{dy}\right|}$$

(1) let X be a RV with pdf $f(x) = 2x$, $0 < x < 1$, find pdf of

(i) $Y = 3X + 6$

(ii) $Y = 8X^3$

iii) $Y = \tan^{-1} x$

pdf of x is $f(x) = 2x$, $0 < x < 1$

(i) pdf of $Y = 3X + 6$

$$f(y) = f(x)\left|\frac{dx}{dy}\right|$$

$$\frac{dy}{dx} = 3 \Rightarrow \frac{dx}{dy} = 1/3$$

$$\left|\frac{dx}{dy}\right| = \left|\frac{1}{3}\right| = \frac{1}{3}$$

$$y = 3x + 6$$

$$\frac{y-6}{3} = x$$

$$f(y) = \left|\frac{1}{3}\right| 2x$$

$$= \frac{2}{3}\left(\frac{y-6}{3}\right)$$

$$f(y) = \frac{2}{9}(y-6)$$

$$0 < x < 1$$

$$0 < \frac{y-6}{3} < 1$$

$$0 < y - 6 < 3$$

$$6 < y < 9$$

$$\boxed{f(y) = \frac{2}{9}(y-6),\ 6<y<9}$$

(ii) $\underline{y = 8x^3}$:

$$\frac{dy}{dx} = 3(8x^2)$$

$$x = \frac{y^{1/3}}{8}$$

$$\frac{dx}{dy} = \frac{1}{24x^2}$$

$$\left|\frac{dx}{dy}\right| = \left|\frac{1}{24x^2}\right| = \frac{1}{24x^2}$$

$$f(y) = f(x)\left|\frac{dx}{dy}\right|$$

$$= 2x \cdot \frac{1}{24x^2}$$

$$= \frac{1}{12x} = \frac{2}{12 \cdot y^{1/3}} = \frac{y^{-1/3}}{6}$$

$$\boxed{f(y) = \frac{y^{-1/3}}{6},\ 0<y<8}$$

$$0 < x < 1$$

$$0 < \frac{y^{-1/3}}{2} < 1$$

$$0 < y^{-1/3} < 2$$

(ii) $Y = \tan^{-1}x$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

$X = \tan y.$

$\left|\frac{dx}{dy}\right| = \left|1+x^2\right| = 1+x^2$

$f(y) = (2x)(1+x^2)$

$= 2x + 2x^3$

$= 2\tan y\,(1+x^2)$

$= 2\tan y \sec^2 y.$

$0 < x < 1$

$0 < \tan^{-1} y < 1$

$0 < y = \tan^{-1}(1)$

$$\boxed{f(y) = 2\tan y \sec^2 y,\ 0 < y < \frac{\pi}{4}}$$

Tchebycheff's Inequality:

If X is a RV with mean and Variance σ^2, then

$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$ (upper bound)

(or)

$P(|X - \mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$ (lower bound)

(1) Let X be a RV with mean 12 & Variance 9. If probability function is not known find $P(6 < X < 18)$

Given, $\mu = 12$, $\sigma^2 = 9$

$\sigma = 3$

By Tchebycheff's Inequality,

$P(|X - \mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$

$|x| \leq a$

$-a < x < a$

$P(|X - 12| \leq 3k) \geq 1 - \frac{1}{k^2}$

$P(-3k \leq X - 12 \leq 3k) \geq 1 - 1/k^2$

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$P(-3k+12 \le x \le 3k+12) \ge 1-\frac{1}{k^2}$

$P(6 < x < 18) = ?$

$-3k+12 = 6$

$-3k = -6$

$\boxed{k=2}$

$P(6 < x < 18) \ge 1 - 1/4$

$P(6 < x < 18) \ge 3/4$

→ 2 dice are rolled simultaneously at X represents sum of numbers occurs ~~odd~~ on top of the dice

$P(|x-7| \ge 3] \le \frac{35}{54}$ show that

X	2	3	4	5	6	7	8	9	10	11	12
P(x)	1/36	2/36	3/36	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	1/36

Mean μ :

$E(x) = \sum_{x=2}^{12} x\,p(x)$

$= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{22}{36} + \frac{12}{36}$

$= \frac{252}{36} = 7$

$\boxed{\mu = 7}$

Variance (σ^2)

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=2}^{12} x^2 p(x)$$

$$= \frac{4}{36} + \frac{18}{36} + \frac{48}{36} + \frac{100}{36} + \frac{180}{36} + \frac{294}{36} + \frac{320}{36} + \frac{324}{36} + \frac{300}{36} + \frac{242}{36} + \frac{144}{36}$$

$$E(X^2) = \frac{1974}{36}$$

$$V(X) = \frac{1974}{36} - (7)^2$$

$$\sigma^2 = \frac{105}{18}$$

$$\boxed{\sigma = \sqrt{\frac{105}{18}}}$$

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

To prove:

$$P(|X - 7| \geq 3] \leq \frac{35}{54}$$

$$P\left(|X - 7| \geq \sqrt{\frac{105}{18}}\, k\right) \leq \frac{1}{k^2}$$

$$\frac{105}{18} k^2 = 9$$

$$k^2 = \frac{18 \times 9}{105} = \frac{54}{35}$$

$$\frac{1}{k^2} = \frac{35}{54}$$

$$\therefore P(|X - 7| \geq 3) \leq \frac{35}{54}.$$

//

(3) If $f(x)=e^{-x}$, $x\geq 0$, PT $P[|X-1|\geq 2]\leq \frac{1}{4}$

S.T, the actual probability is e^{-3}

Mean :

$$E(X)=\int_0^\infty x f(x)\,dx$$

$$=\int_0^\infty x e^{-x}\,dx$$

$$=\left[-xe^{-x}-e^{-x}\right]_0^\infty$$

$$=\left[0+e^{-0}\right]=1$$

$$E(X^2)=\int_0^\infty x^2 f(x)\,dx=\int_0^\infty x^2 e^{-x}\,dx$$

$$=\left[-x^2e^{-x}-(2x)(e^{-x})+2(-e^{-x})\right]_0^\infty$$

$$=0-(-2)=2$$

$$\sigma^2=V(X)=2-1^2$$
$$=1$$

$$\boxed{\sigma=1}$$

$$P(|X-\mu|\geq k\sigma)\leq\frac{1}{k^2}$$

$$P(|X-1|\geq k)\leq\frac{1}{k^2}$$

comparing,

$$P(|X-1|\geq 2)\leq\frac{1}{4}.$$

$$1-P[|X-1|<2]\ \cancel{\geq}\ 1-\left(\frac{1}{4}\right)$$

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$= 1 - P(-2 < X - 1 < 2)$

$= 1 - P(-1 < X < 3)$

$= 1 - \int_{-1}^{3} f(x)\,dx$

$= 1 - \int_{-1}^{3} e^{-x}\,dx \qquad x \geq 0$

$= 1 - \int_{0}^{3} e^{-x}\,dx$

$= 1 - \left[-e^{-x}\right]_0^3$

$= 1 - \left[-e^{-3} + e^{-0}\right]$

$= \cancel{1} + e^{-3} - \cancel{1}$

$= e^{-3}$ //

(4) A fair die is tossed 720 times use Tchebychell Inequality to find a lower bound for probability of getting 100 to 140 sixes

$P(100 < X < 140) \geq 1 - \frac{1}{k^2}$

for Binomial distribution,

$\mu(\text{mean}) = np$

$\text{Variance} = npq$

$p + q = 1$, $n \to$ no. of trails.

$\mu = np = 720 \times \frac{1}{6} = 120$

$\sigma^2 = V = (720)(1/6)(5/6) = 120 \times \frac{5}{6}$

$= 100$

$\boxed{\sigma = 10}$

$P(|X - 120| \leq k(10)) \geq 1 - \frac{1}{k^2}$

$P(-10k < X - 120 < 10k) \geq ? \, 1 - 1/k^2$

$P(-10k + 120 < X < 10k + 120) \geq ? \, 1 - 1/k^2$

$-10k + 120 = 100$

$-10k = -20$

$\boxed{k = 2}$

$P(100 < X < 140) \geq \frac{3}{4}$

$\boxed{\text{lower bound is } 3/4}$

(1) A RV X has the following . pmf

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

find i) k

ii) $P(1.5 < X < 4.5 / X > 2)$

iii) find smallest value of λ for which $P(X \leq \lambda) \geq 1/2$

(i) Since $\sum_i p_i = 1$

$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k = 1$

$10k^2 + 9k - 1 = 0$

$k = \frac{-9 \pm \sqrt{81+40}}{20}$

$= \frac{-9 \pm 11}{20}$

$k = \frac{2}{20}, -1$

Since, $k \neq -1$ $\quad p_i \geq 0$

$$\boxed{k = 1/10}$$

Pmf :

X	0	1	2	3	4	5	6	7
P(X)	0	$\frac{1}{10}$	$2/10$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{7}{100}+\frac{1}{10}$
F(X)	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	1

(ii) $P(1.5 < x < 4.5)$

$$= \frac{P(1.5 < x < 4.5)}{P(X>2)}$$

$$= \frac{P(3)+P(4)}{1-P(X\leq 2)} = \frac{5/10}{1-(4/10)}$$

$$= \frac{5}{10} \times \frac{10}{6} = 5/6.$$

(iii) $P(X \leq \lambda) \geq \frac{1}{2}$

$P(X\leq 0) = 0 \ngeq \frac{1}{2}$

$P(X\leq 1) = \frac{1}{10} = 0.1 \ngeq 1/2$

$P(X\leq 2) = \frac{3}{10} = 0.3 \ngeq 1/2$

$P(X\leq 3) = 5/10 = 0.5 > 1/2$

$P(X\leq 4) = 8/10 = 0.8 > 1/2$

$$\boxed{\lambda = 4}$$

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