f-DIVERGENCES

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- Let $f : \mathbb{R}_+ \to \mathbb{R}$ be an convex function such that $f(1) = 0$, define the f-divergence:

$$D_f(\mu' \| \mu) = \mu\left[f \circ \frac{\mathrm{d}\mu'}{\mathrm{d}\mu}\right] = \int_{\mathbb{X}} f\left(\frac{\mathrm{d}\mu'}{\mathrm{d}\mu}(x)\right)\mu(x)\mathrm{d}x$$

- The likelihood ratio or Radon-Nikodym derivative $\mu(\mathrm{d}x) = \mu(x)\mathrm{d}x$ and $\mu'(\mathrm{d}x) = \mu'(x)\mathrm{d}x$

$$\frac{\mathrm{d}\mu'}{\mathrm{d}\mu} : \mathbb{X} \to \mathbb{R}, \qquad \frac{\mathrm{d}\mu'}{\mathrm{d}\mu}(x) = \frac{\mu'(x)}{\mu(x)}$$

$$f(r) = \frac{1}{2}\,|\,1 - r\,| \qquad \mathrm{TV}(\mu, \mu') = \frac{1}{2}\int_{\mathbb{X}} |\,\mu(x) - \mu'(x)\,|\,\mathrm{d}x = 1 - \int_{\mathbb{X}} \mu(x) \wedge \mu'(x)\mathrm{d}x$$

$$f(r) = (1 - r)^2 \qquad \chi^2(\mu' \| \mu) = \mathbb{V}_\mu \left[\frac{\mathrm{d}\mu'}{\mathrm{d}\mu} \right] = \int_{\mathbb{X}} \left(\frac{\mu'(x)}{\mu(x)} - 1 \right)^2 \mu(x) \mathrm{d}x$$

$$f(r) = r \log r \qquad\qquad \mathrm{KL}(\mu' \| \mu) = \int_{\mathbb{X}} \mu'(x) \log \frac{\mu'(x)}{\mu(x)} \mathrm{d}x$$

- We have $D_f(\mu'||\mu) \geq 0$ and with equality if and only if $\mu = \mu'$

Examples:

f-DIVERGENCES

- The likelihood ratio or Radon-Nikodym derivative $\mu(\mathrm{d}x) = \mu(x)\mathrm{d}x$ and $\mu'(\mathrm{d}x) = \mu'(x)\mathrm{d}x$

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- Let $f : \mathbb{R}_+ \to \mathbb{R}$ be an convex function such that $f(1) = 0$, define the f-divergence:

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- We have $D_f(\mu' \| \mu) \geq 0$ and with equality if and only if $\mu = \mu'$

- **Examples:**

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CONVERGENCE