



NORMALISING CONSTANT

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► **Problem:** MCMC can't measure the normalising constant Z

► **Solution:** Introduce a reference distribution η (e.g gaussian) that dominates target

- By the Radon-Nikodym theorem

$$Z = \int_{\mathbb{X}} \gamma(x) dx = \int_{\mathbb{X}} w(x) \eta(x) dx = \eta[w], \quad w = \frac{\gamma(x)}{\eta(x)}$$

- Can estimate Z using the Monte Carlo estimator

$$X_1, \dots, X_N \sim \eta \qquad \hat{Z} = \frac{1}{N} \sum_{n=1}^N w(X_n)$$

▶ In-normalised density can only compare between two states

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► **Unbiased:**

$$\mathbb{E}[\hat{Z}] = \frac{1}{N} \sum_{n=1}^N \mathbb{E}[w(X_n)] = Z$$

► **Consistent:** convergence is a.s.

$$\lim_{N \rightarrow \infty} \hat{Z} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N w(X_n) = Z$$

► **Variance:**

$$\mathbb{V} \left[\frac{\hat{Z}}{Z} \right] = \frac{1}{N} \mathbb{V}_{X \sim \eta} \left[\frac{\pi(X)}{\eta(X)} \right] = \frac{e^{D_2(\pi \parallel \eta)} - 1}{N}$$

- The variance grows exponentially as η deviates from the target

$$D_2(\pi \parallel \eta) = \log \left(1 + \mathbb{V}_{X \sim \eta} \left[\frac{\pi(X)}{\eta(X)} \right] \right) \geq \text{KL}(\eta \parallel \pi)$$