Brachistochrone Problem as Teaching Material – Application of KeTCindy with Maxima –

Setsuo Takato

Toho University, Japan
takato@phar.toho-u.ac.jp,
ketpic.com

Abstract. KeTCindy, which we have been developing in collaboration with Cinderella, can produce mathematics and physics teaching materials of various types such as printed materials, screen presentations, and interactive materials. These materials are more effective when mutually combined and arranged properly. As described in this paper, we use the brachistochrone problem as an example to show what materials are produced and how they might be used in classes at the collegiate level.

Keywords: KeTCindy, Cinderella, LaTeX, Maxima, teaching materials

1 Introduction

Johann Bernoulli postulated the well-known brachistochrone problem at the end of 17th century[5]. It can be stated simply as follows:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.

In fact, the problem attracted the interest of many contemporary mathematicians, who gave the solution in their own ways. Nowadays, educators often use it as a teaching material for calculus of variations in their mathematics and physics classes. As described herein, we present a series of teaching materials using KETCindy which we developed and which anyone can download freely[1]. KETCindy is a macro package of Cinderella[2], which is dynamic geometry system, and Scilab[3], a numerically oriented programming language, to produce and to insert finely detailed figures into LaTeX documents not only easily but also interactively. Moreover, KETCindy has extended functionalities to call other mathematical software packages, for example, Maxima, to use its results in KETCindy. The following Table 1 shows what KETCindy can do for now.

Table 1 Can-do table of KETCindy

s1	Geometric Figure	s6	Animation	s11	Calling Asir
s2	Graph of Function	s7	Presentation	s12	Calling Fricas
s3	Making Table	s8	Calling R	s13	Calling Mesthlab
s4	Bézier Curve	s9	Surface	s14	Data Processing
s5	3D Figure	s10	Calling Maxima	s15	TeX Style Files

We present an example to produce figures with KETCindy. These figures will be used in our materials.

For simplicity, we put point A on origin O. Then we set the coordinates of point B as $(5, -5)$. Figure 1 presents screens of Cinderella, Cindy Screen at the left, Script Editor at the upper right and Console at the lower right.

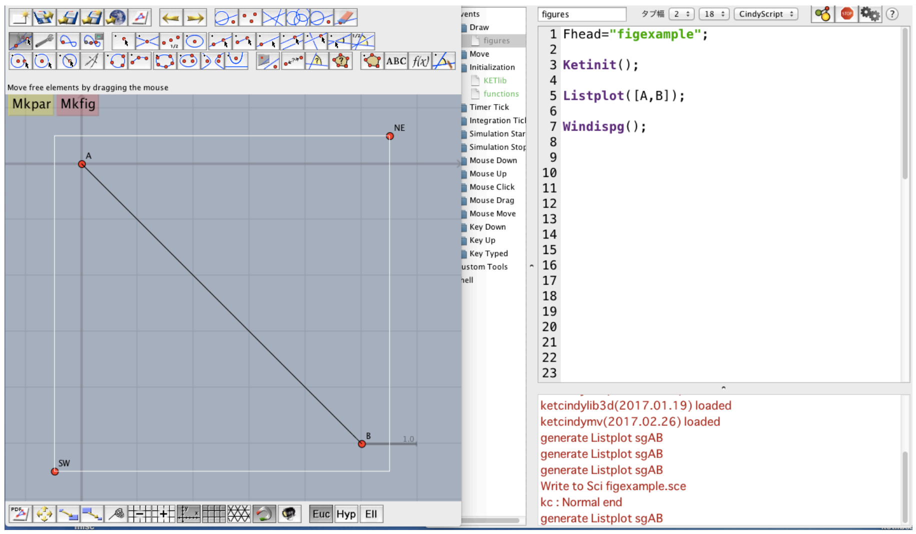


Figure 1 Cinderella screens

The process to produce a figure is the following.

1. Add geometric points such as C and D on Cindy Screen, which are movable at any position by dragging.
2. Write scripts of KETCindy on Script Editor. We remark here that Cinderella has the “Cindy Script” programming language, which is easy to use and which distinguishes Cinderella from other dynamic geometry software. For example, these scripts are

```
Ketinit();  // Initialization of KeTCindy.
Bezier("1",[A,B],[C,D],["Num=50"]); //Draw a Bezier curve.
PutonCurve("P","bz1");  //Put a point P on the curve.
    //The name of the figure is appeared on the Console.
Pointdata("1",P,["Size=4"]); //Display P also as TeX figure.
Letter([P,"ne","P",B,"e","B"]);
   //Display letters also as TeX figure.
Windispg(); //Display on Cindy Screen.
```

3. Move points C, D, and P if necessary.
4. Press `Mkfig` button, then one can obtain data from the figure. A pdf will be displayed for checking.
5. Modify the scripts or geometric points. Then press the `Mkfig` button until one is satisfied.

6. Insert the figure into TEX documents using the `\input` command.

We show the Cindy Screen and the figure to be input in the TEX document.

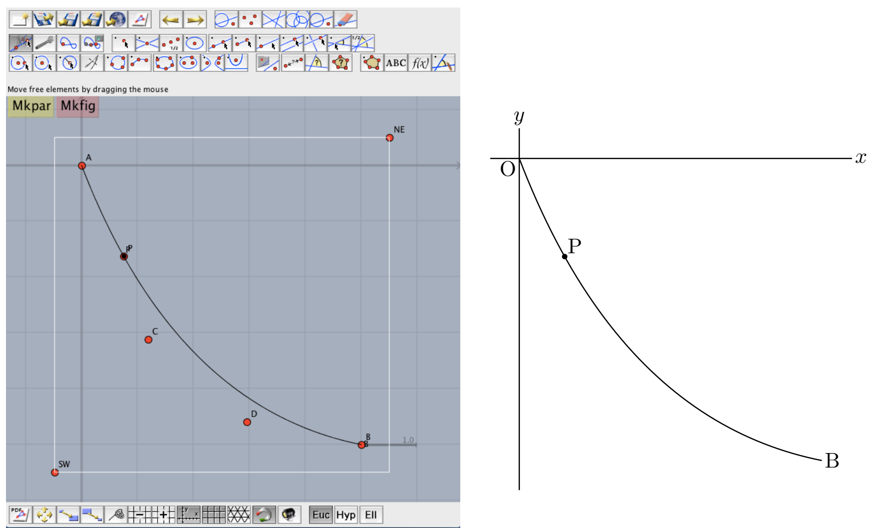


Figure 2 Cindy screen and the TEX figure

From section 2, we describe a series of teaching materials.

2 Finding Equations of Motion and Total Time

The contents of this section will be used to produce printed material to be distributed during the first class session. It is noteworthy that these figures are produced with KETCindy.

Let a curve C from O to B be represented by $\boldsymbol{r} = \big(x(u),\ y(u)\big)$ $(0 \leq u \leq U)$, a point P start O with the velocity $v = 0$, and the value u at time t be $u(t)$. Assume that the motion is frictionless, so the energy conservation law is satisfied. Let g represent the acceleration of gravity.

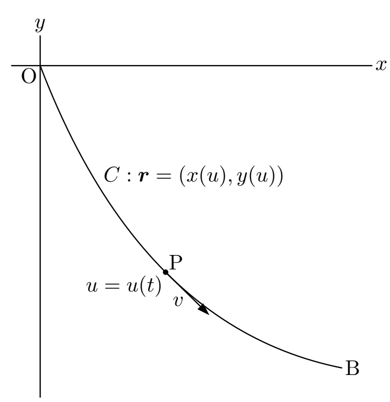
Then, from $\dfrac{1}{2}mv^2 = mgh = -mgy$,

$$v = \sqrt{-2gy}. \tag{1}$$

Let s be the length of the curve C from O to P, then

$$\frac{ds}{du} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\text{where } \dot{x} = \frac{dx}{du},\ \ \dot{y} = \frac{dy}{du}.$$



Using $v = \dfrac{ds}{dt}$ and (1),

$$v = \frac{ds}{du}\frac{du}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2}\,\frac{du}{dt}. \tag{2}$$

Thereby, we get the differential equation of the motion

$$\frac{du}{dt} = \sqrt{\frac{-2gy}{\dot{x}^2 + \dot{y}^2}},\ u(0) = 0, \tag{3}$$

and the formula for calculating total time T as

$$T = \int_0^U \sqrt{\frac{\dot{x}^2 + \dot{y}^2}{-2gy}}du. \tag{4}$$

Here, (3) has a singular solution $u = 0$. Also, (4) is an improper integral.

We give a simple example where C is a line.

Example 1 Line $\boldsymbol{r} = (au,\ -bu)\ \ (0 \le u \le U)$

Because $\dot{x}^2 + \dot{y}^2 = a^2 + b^2$, we have the equation from (3),

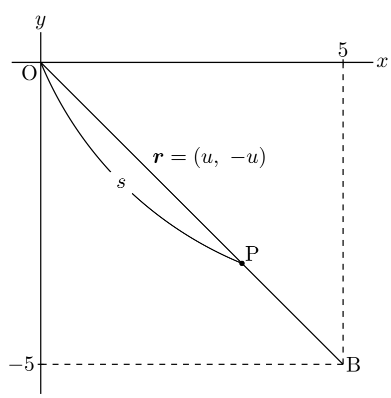
$$\frac{du}{dt} = \sqrt{\frac{2gbu}{a^2 + b^2}},\ u(0) = 0.$$

Solving this equation, we obtain

$$u = \frac{bgt^2}{2(a^2 + b^2)},$$

and from (4),

$$T = \sqrt{\frac{2(a^2 + b^2)U}{bg}}.$$



Especially, line OB is represented as $\boldsymbol{r} = (u,\ -u)\ \ (0 \le u \le 5)$. Therefore,

$$T = 1.42857 \ \text{ where } g = 9.8 \text{ is used.}$$

3 Use of KETCindy with Maxima

Finding the equation for curves other than a line (example 1) is not so easy. Therefore, we use Maxima in KETCindy. Contents of this section might be used for preparing teaching materials, not for materials itself.

Here we explain how to use KETCindy with Maxima when curve C is an inverted cycloid, a circle, and a parabola.

3.1 The case of an inverted cycloid

An inverted cycloid from O is represented as

$$\boldsymbol{r} = \bigl(a(u - \sin u),\ -a(1 - \cos u)\bigr) \quad (0 \le u \le U).$$

To simplify equation (3), we need only write the following scripts on Scripts Editor of Cinderella.

```
cmdL=[
 "assume",["g>0"],
 "fxy:[a*(u-sin(u)),-a*(1-cos(u))]",[],
 "d2:diff(fxy[1],u)^2+diff(fxy[2],u)^2",[],
 "d2:trigsimp",["d2"],
 "n2:2*g*(-fxy[2])",[],
 "so1:ratsimp",["n2/d2"],
 "so2:ratsimp(sqrt(so1))",[],
 "so2",[]
];
CalcbyM("cyc",cmdL);
```

Here `cmdL` is the list of Maxima commands, where each command is given as a pair of the name and arguments, and where `CalcbyM` is the command of KETCindy to call Maxima. The results are assigned to a variable `cyc` as a string or a list of strings. Actually

`cyc="sqrt(g)/sqrt(a)"`

Therefore, equation (3) has a simple form. Moreover, it is easily solvable as

$$\frac{du}{dt} = \sqrt{\frac{g}{a}},\ u = \sqrt{\frac{g}{a}}\,t.$$

Scripts to decide the inverted cycloid through B$(5, -5)$ are shown below.

```
cmdL=[
  "assume",["a>0"],
  "fxy:[a*(u-sin(u)),-a*(1-cos(u))]",[],
  "eq:ratsimp",["(fxy[1]+fxy[2])/a"],
  "so1:find_root",["eq","u","%pi/2","%pi"],
  "eqr:ev",["fxy[1]-5","u=so1"],
  "so2:find_root",["eqr","a",0,5],
  "so1::so2",[]
 ];
 CalcbyM("coeff",cmdL);println(coeff);
```

The results are $U = 2.412011143913525,\ a = 2.864585187658752$. From these, the total time is obtained.

$$T = \sqrt{\frac{a}{g}}\,U = 1.30406$$

3.2 The case of a circle

In the same way as an inverted cycloid, the equation for a circle

$$\boldsymbol{r} = \bigl(5(1-\cos u),\ -5\sin u\bigr) \quad (0 \le u \le \frac{\pi}{2})$$

can be obtained.

```
cir="(sqrt(2)*sqrt(g)*sqrt(sin(u)))/sqrt(5)"
```

Therefore,

$$\frac{du}{dt} = \sqrt{\frac{2g}{5}}\sqrt{\sin u}.$$

However, we cannot find the exact solution of the equation above. The total time (4) is obtainable with

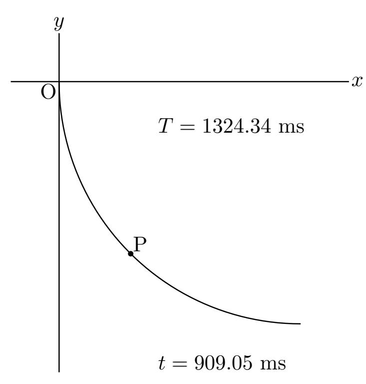
```
Mxfun("tt","integrate",["sqrt(5/(2*g*sin(u))"]);
```

`Mxfun` is a command of KETCindy to execute a single command of Maxima. The results are

$$\frac{\sqrt{5}\,\beta\left(\frac{1}{4},\frac{1}{2}\right)}{2\sqrt{2}\sqrt{g}} = 1.32434.$$

To obtain the numerical solution of the equation, we use `deqplot`, a command KETCindy provided to calculate the numerical solution of a differential equation using Runge-Kutta method. Then we display the integral curve. Here, because the equation has a singular solution $u = 0$, the process requires a little ingenuity as shown below.

1. Take u_0 close to zero, for example, $u_0 = 0.001$, and approximate the solution from 0 to u_0 by a straight line. Using example 1, we have the time at which u becomes u_0
$$t_0 = \sqrt{\frac{2(a^2+b^2)u_0}{bg}} \qquad (5)$$
where $(a,\ b) = \bigl(x(u_0),\ y(u_0)\bigr)$.
2. Use `deqplot` from t_0 to $T = 1.32434$.
```
equ="u‘=(sqrt(2*9.8/5*sin(u)))";
range="t="+textformat([t0,T],6);
Deqplot("cir",equ,range,t0,u0);
```
Then $u = u(t)$ is obtained numerically.



Here we consider the case of a circle which has the same length of the inverted cycloid. KETCindy with Maxima can find the length exactly or numerically.

$$L = 7.3705338507$$

Let C be a circle through points O and B with the equation

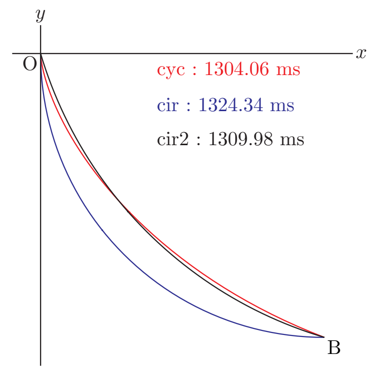
$$\boldsymbol{r} = (5 + s - r\cos u,\ s - r\sin u).$$

Using KETCindy with Maxima again, we can determine s and r so that the arc length equals to L

$$s = 2.11125,\ r = 7.41803,$$

and obtain the total time

$$T = 1.30998.$$



3.3 The case of a parabola

Let a parabola be represented as

$$\boldsymbol{r} = \left(u,\ \frac{1}{5}(u^2 - 10u)\right) \ \ (0 \leq u \leq 5).$$

We can neither solve equation (3) nor obtain the total time (4) exactly. Therefore, the process should be numerical as follows.

1. Take a slightly large value t_1 such as $t_1 = 1.5$, and use `deqplot` from t_0 to t_1.
2. Let `dtpara` be the returned list of plotdata
   `dtpara=[[t0,u0],...,[t1,u1]];`
   Execute the following scripts:

```
n=max(select(1..(length(dtpara)),dtpara_#_2<5));
p1=dtpara_(n); p2=dtpara_(n+1);
T=p1_1+(5-p1_2)*(p2_1-p1_1)/(p2_2-p1_2);
```

   Then we have total time $T = 1.33099$.

4 Examples of interactive materials

In this section, we introduce two examples that are expected to be useful immediately after the first lecture, of which the contents are of section 1.

4.1 Using a Bézier curve

The Bézier curve from O(0, 0) to B(5, −5) with control points C(c_1, c_2) and D(d_1, d_2) is defined as

$$\boldsymbol{r} = 3C(1-u)^2u + 3D(1-u)u^2 + Bu^3 \ \ (0 \leq u \leq 1).$$

We get the equation (3) with Maxima, although it is extremely complicated. Actually, the string returned from Maxima is

`sqrt(((6*g*d2-6*g*c2+10*g)*u^3+(12*g*c2-6*g*d2)*u^2-6*g*c2*u)/..,`

which has 370 characters. Also the total time is obtainable in the same way as the case of a parabola.

Put C and D as geometric points on Cindy Screen, then, moving these points, we can change the Bézier curve shape. The initial position might be on the line OB as the left side of Figure 3. Here, the total time T(1428 ms in this case), almost equal to that of example 1, is displayed. In accordance with the teacher's instructions, students try to minimize the total time T by moving points C and D. They will learn where these points should be put. When value T satisfies them sufficiently (the left side of Figure 4), instruct them to display the inverted cycloid as the right side of Figure 4. Then they will understand clearly that the inverted cycloid is just the brachistochrone curve.

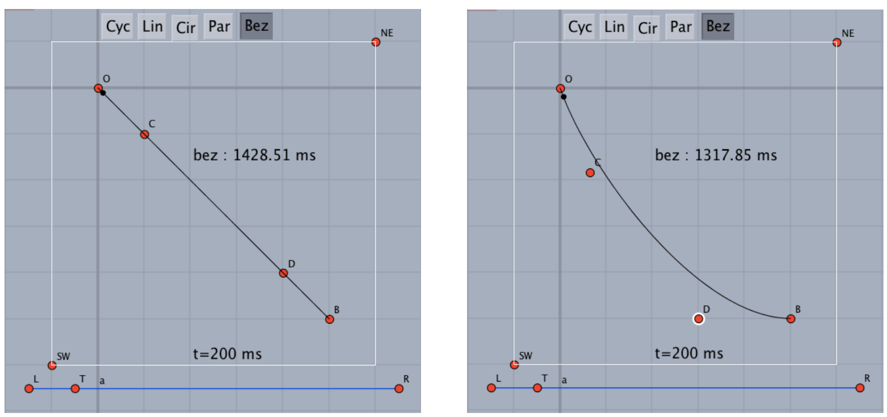


Figure 3 Bézier curves and control points

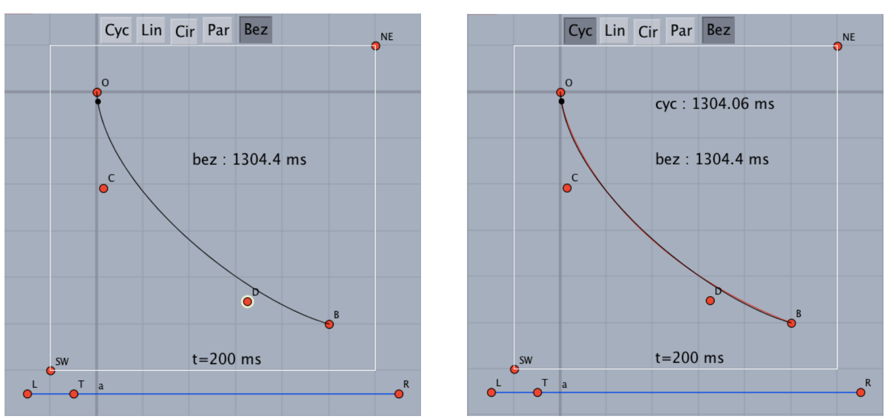


Figure 4 Fitting the Bézier curve to the inverted cycolid

4.2 Moving the point with a slider

We can make a slider on Cindy Screen, putting two points L, R, and T on the segment LR. Let $t = \mathrm{T} - \mathrm{L}$ represent the time from the initial time. Because the motion of the point is solvable exactly or numerically, we can display the position of the point at time t as Figure 5.

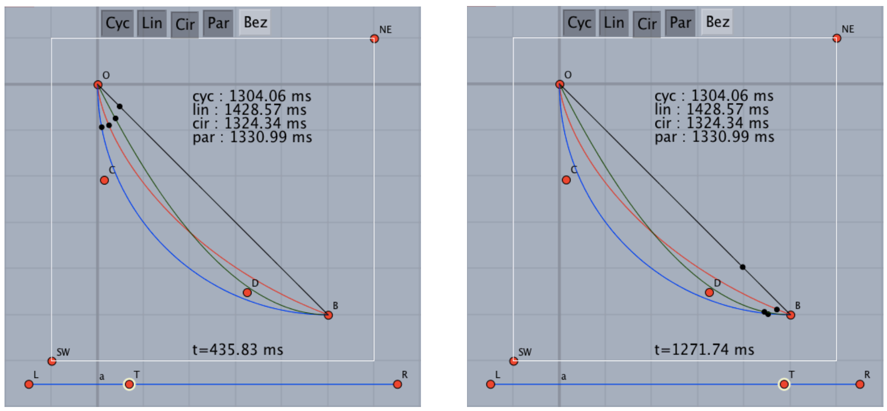


Figure 6 Cindy screens with a slider

5 Derivation of the brachistochrone curve

After using interactive materials of previous section, it would be better for teachers to explain why an inverted cycloid is the brachistochrone curve. We describe the derivation briefly, although one can find many reports in the literature about it. The problem is to find the function which minimizes the left-hand side of (4). Here, we set $-y$ to y and

$$f = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{y}}.$$

The Euler equations in variational calculus in this case are

$$\begin{cases} \dfrac{\partial f}{\partial x} - \dfrac{d}{du}\left(\dfrac{\partial f}{\partial \dot{x}}\right) = 0 \\ \dfrac{\partial f}{\partial y} - \dfrac{d}{du}\left(\dfrac{\partial f}{\partial \dot{y}}\right) = 0 \end{cases}.$$

Using the first equation, we have

$$\frac{\partial f}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{y(\dot{x}^2 + \dot{y}^2)}} = \sqrt{c}.$$

Therefore, we have

$$\dot{x}^2 = cy(\dot{x}^2 + \dot{y}^2)$$

$$\frac{\dot{x}}{\dot{y}} = \sqrt{\frac{cy}{1 - cy}} = \sqrt{\frac{y}{2r - y}} \quad \left(r = \frac{1}{2c}\right).$$

Put $y = 2r\sin^2\frac{u}{2} = r(1-\cos u)$. From

$$2r - y = 2r\cos^2\frac{u}{2},\ \dot{y} = 2r\sin\frac{u}{2}\cos\frac{u}{2},$$

we have the following:

$$\dot{x} = \frac{\sin\frac{u}{2}}{\cos\frac{u}{2}} \cdot 2r\sin\frac{u}{2}\cos\frac{u}{2} = 2r\sin^2\frac{u}{2} = r(1-\cos u).$$

Solving the equation above, we get

$$\begin{cases} x = r(u - \sin u) \\ y = r(1 - \cos u) \end{cases},$$

which is the equation of a cycloid.

6 Conclusions

We have been developing KETCindy in collaboration with Cinderella to produce teaching materials of various types. We showed an example, the brachistochrone problem, as a series of materials, It would be more effective when we mutually associate these materials and arrange them properly. Teachers will give other applicable examples, similar to the brachistochrone problem and which are expected to enhance a learning effect in their class.

Acknowledgments

This work was supported by JSPS KAKENHI Grant Numbers 15K01037, 15K00944, and 16K01152.

References

1. KeTpic.com, `http://ketpic.com`
2. Cinderella, `http://www.cinderella.de/tiki-index.php`
3. Scilab, `http://www.scilab.org`
4. Maxima, `http://maxima.sourceforge.net`
5. MacTutor History of Mathematics archive, `http://www-history.mcs.st-and.ac.uk/HistTopics/Brachistochrone.html`
6. Takato S., McAndrew A., Vallejo J.A., Kaneko M., Collaborative Use of KeTCindy and Free Computer Algebra Systems, Mathematics in Computer Science, 1–12, 2017, doi:10.1007/s11786-017-0303-7
7. Takato S., What is and how to Use KeTCindy – Linkage Between Dynamic Geometry Software and Collaborative Use of KETCindy and Free Computer Algebra Systems and LATEX Graphics Capabilities –, Mathematical Software –ICMS 2016, LNCS **9725**, 371–379, Springer, 2016.

8. Kobayashi S., Takato S., Cooperation of KeTCindy and Computer Algebra System, Mathematical Software – ICMS 2016, LNCS **9725**, 351–358, Springer, 2016.
9. Kaneko M., Yamashita S., Kitahara K., Maeda Y., Nakamura Y., Kortenkamp U, Takato S., KETCindy – Collaboration of Cinderella and KETpic, Reports on CADGME 2014 Conference Working Group, The International Journal for Technology in Mathematics Education, 22(4), 179-185, 2015.