Basel 問題と関連して $+\alpha$（2020.12.3 大島利雄）

$$
\begin{aligned}
A_m^{(k)} &:= \sum_{n=0}^{m} \frac{1}{n^2} + \sum_{j=0}^{k} \frac{((2j-1)!!)^2}{4^j(2j+1)(m^2-\frac{1^2}{4})(m^2-\frac{3^2}{4})\cdots(m^2-\frac{(2j-1)^2}{4})(m+\frac{2j+1}{2})} \\
&= \sum_{n=0}^{m} \frac{1}{n^2} + \frac{1}{m+\frac{1}{2}} - \frac{1}{12(m^2-\frac{1}{4})(m+\frac{3}{2})} + \frac{9}{80(m^2-\frac{1}{4})(m^2-\frac{9}{4})(m+\frac{5}{2})} \\
&\quad + \cdots + (-1)^k \frac{((2k-1)!!)^2}{4^k(2k+1)(m^2-\frac{1^2}{4})\cdots(m^2-\frac{(2k-1)^2}{4})(m+\frac{2k+1}{2})},
\end{aligned}
$$

$$
0 < (-1)^{k+1}\Bigl(\sum_{n=1}^{\infty} \frac{1}{n^2} - A_m^{(k)}\Bigr) < \frac{((2k+1)!!)^2}{4^{k+1}(2k+3)(m-k-\frac{1}{2})^{2k+3}} \qquad (0 < k < m).
$$