Robotics, Geometry and Control - Nonoholonomic constraints

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The material for these slides is largely taken from the *text*

- Nonholonomic Mechanics and Control - A. Bloch *et al*, Springer, 2003

Revisit the coin equations

- Matrix representation of the constraint conveys the fact that permissible motions of the coin are such that allowable velocity directions at any point on the manifold $(x_p, y_p, \theta_p, \phi_p)$- described by the vector

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}_{(x_p, y_p, \theta_p, \phi_p)}$$

are annihilated by the matrix

$$\begin{bmatrix} \sin(\theta_p) & -\cos(\theta_p) & 0 & 0 \\ \cos\theta_p & \sin\theta_p & 0 & -r \end{bmatrix}$$

Integrability

- To explore integrability, we ask the question: Can

$$\Omega(\boldsymbol{q}) = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 & 0 \\ \cos\theta & \sin\theta & 0 & -r \end{bmatrix}$$

be expressed as the gradient of two functions λ_1 and λ_2 with $\lambda_i(\cdot) : \boldsymbol{q} \rightarrow R^1$ such that

$$\Omega \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{d(\lambda_1)}{dt} \\ \frac{d(\lambda_2)}{dt} \end{bmatrix} = 0$$

- If yes, then

$$\lambda_1(\boldsymbol{q}) = c_1 \quad \lambda_2(\boldsymbol{q}) = c_2 \quad \text{where } c_1 \text{ and } c_2 \text{ are constants}$$

Frobenius' theorem

- This implies that the system evolves on a manifold of reduced dimension (in this case reduced by 2) and the constraints are integrable (or holonomic).
- However, in this particular case, it is not possible to find such functions λ_is and the constraints are hence non-integrable.
- We arrive at this conclusive negative answer by employing a result from differential geometry called **Frobenius' theorem**.

Hamilton's principle

- Hamilton's principle - The trajectory of a mechanical system moving from a fixed configuration q_a at time a to a fixed configuration q_b at time b is the stationary point of the functional

$$\int_a^b \mathcal{L}(q^i, \dot{q}_i)dt$$

where $\mathcal{L}(q^i, \dot{q}_i)$ is the Lagrangian of the system

- The equations of motion are thus obtained by solving

$$\delta(\int_a^b \mathcal{L}(q^i, \dot{q}_i)dt) = 0$$

- The stationary point condition yields the familiar Euler-Lagrange equations

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}) - \frac{\partial \mathcal{L}}{\partial q^i} = 0 \quad i = 1, \ldots, n$$

Variational calculus

- Denote the vector space of admissible trajectories as V. Each trajectory is differentiable in $[a, b]$ and must satisfy $q(a) = q_a$ and $q(b) = q_b$
- The functional
$$\int_a^b \mathcal{L}(q^i, \dot{q}_i)dt$$
is a mapping from V to R.
- Hamilton's principle says that the trajectory that the system takes is an extremum (stationary point) of the functional amongst the set of admissible trajectories.
- Characterize (parametrize) admissible trajectories with the variable ϵ ($\epsilon \in [0, 1]$) as
$$q(t, \epsilon) \quad \text{is differentiable and} \quad q(a, \epsilon) = q_a, \;\; q(b, \epsilon) = q_b$$
- The extremal trajectory corresponds to $\epsilon = 0$

Variational calculus

- Variation at time t

$$\delta q(t) = \frac{d}{d\epsilon} q(t,\epsilon)|_{\epsilon=0}$$

- The first order condition for an extremum is

$$\delta\left(\int_a^b \mathcal{L}(q^i, \dot{q}_i)dt\right) = 0 \Rightarrow \frac{d}{d\epsilon}\left(\int_a^b \mathcal{L}(q^i(t,\epsilon), \dot{q}_i(t,\epsilon))dt\right)|_{\epsilon=0} = 0$$

Constraints

- Holonomic constraints - reduce the dimension of the configuration space
- Nonholonomic constraints - do not reduce the dimension of the configuration space
- We shall now incorporate nonholonomic constraints into the mechanical system and derive the constrained equations of motion
- Our attention is restricted to Pfaffian constraints of the form

$$A(q)\dot{q} = 0 \qquad A(q) \in R^{m\times n}, \dot{q} \in R^n$$

Nonholonomic systems and constraint forces

- The constraint forces do no work under virtual displacements
- Thus if there are m independent constraints we have m forces that enforce these constraints
- These can be viewed as being enforced by m independent vectors in a vector space that is dual to the space of all virtual displacements (this space of virtual displacements can also be interpreted as the space of all velocities)
- Thus

$$F = \sum_{j=1}^{m} \lambda_j a^j \quad \text{where} \quad a^j \in R^n, \lambda_j \in R$$

- The condition of zero work done over a trajectory is

$$\int_a^b \left(\sum_{i=1}^{n} F_i \delta q_i\right) dt = 0$$

The Constraint forces as annihilators

- In a vector space framework, the forces annihilate the virtual displacements (or velocities)
- Thus if the space of displacements (velocities) to start with was of dimension n (same as the number of configuration variables), then the m constraints reduce the dimension to $n - m$. Thus velocities are restricted to an $n - m$ dimensional subspace
- The forces belong to an annihilator subspace of dimension m (m - linearly independent constraints)

Hamilton's principle for nonholonomic systems

- The equations of motion are thus obtained by solving

$$\delta(\int_a^b \mathcal{L}(q^i, \dot{q}_i)dt) + \int_a^b (\sum_{i=1}^n F_i \delta q_i)dt = 0$$

- This yields the **Lagrange/D'Alembert** equations

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}) - \frac{\partial \mathcal{L}}{\partial q^i} = \sum_{j=1}^m \lambda_j a_i^j \quad i = 1, \ldots, n$$

where $F_i = \sum_{j=1}^m \lambda_j a_i^j$

Some examples

- The rolling coin

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_v\dot{\theta}^2 + \frac{1}{2}I_h\dot{\phi}^2$$

Constraints

$$\dot{x} - r\dot{\phi}\cos(\theta) = 0 \quad \dot{y} - r\dot{\phi}\sin(\theta) = 0$$

- The knife edge

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_v\dot{\theta}^2 + mgy\sin(\alpha)$$

Constraint

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$$

More examples

- The Chaplygin sleigh

$$\mathcal{L} = \frac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2}I_v\dot{\theta}^2$$

Constraint

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$$

- The Heisenberg system

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Constraint

$$\dot{z} = y\dot{x} - x\dot{y}$$