Lie Groups and Algebras for optimisation and motion representation

AVL/MRG Reading Group

Tuesday 6th May 2008

- Chapter 2, *An invitation to 3D vision*, Ma & al
- Chapter 5, *PhD* Mei 2007
- *Computing MAP trajectories by representing, propagating and combining PDFs over groups*, Smith & al, ICCV 2003

Why use Lie Groups ?

Some uses...

- Interpolation
- Motion representation
- General theory for the minimal representation of geometric objects
- Representation of PDFs over groups

Outline

1. Definitions
2. Representing motion and geometric objects
3. Interpolation
4. Minimisation
5. Uncertainty

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Matrix Lie group (1/2)

Properties of a group $(G, \circ)$:

- closed : $(g_1, g_2) \in G^2 \Rightarrow g_1 \circ g_2 \in G$,
- associative :
  $\forall (g_1, g_2, g_3) \in G^3, (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$,
- has a neutral (unit) element e : $\forall g \in G^3, e \circ g = g \circ e = g$,
- $\circ$ is invertible : $\forall g \in G, \exists g^{-1} \in G | g \circ g^{-1} = g^{-1} \circ g = e$

Matrix Lie group (2/2)

Lie group $(G, \circ)$

- $(G, \circ)$ is a group,
- G is a smooth manifold, ie has the topology of $\mathbb{R}^n$, (the inverse function is differentiable everywhere)

All closed subgroups of the general linear group $\mathbb{GL}(n)$ (group of all invertible matrices) are Lie groups.

Matrix exponential (1/2)

$e^{\mathbf{A}}$

$$e^{\mathbf{A}} = \mathbf{I}_n + \sum_{p \geq 1} \frac{\mathbf{A}^p}{p!} = \sum_{p \geq 0} \frac{\mathbf{A}^p}{p!}, \qquad \text{beware : } e^{\mathbf{X}} e^{\mathbf{Y}} \neq e^{\mathbf{X}+\mathbf{Y}}$$

This series is absolutely convergent and thus well-defined.

$\log \mathbf{A}$

- Under the condition $\|\mathbf{A} - \mathbf{I}\| < 1$, the logarithm of **A** is defined as :

$$\log \mathbf{A} = \sum_{p \geq 0} (-1)^{p+1} \frac{(\mathbf{A} - \mathbf{I})^p}{p}$$

Matrix exponential (2/2)

Calculating $e^{\mathbf{A}}$ in practice

- explicit formulas (eg. Rodrigues' formula for $\mathbb{SO}(3)$). A general way of finding explicit formulas is to use the Cayley–Hamilton theorem.
- diagonalisation (not generally a good idea),
- *Nineteen dubious ways to compute the exponential of a matrix*, Moler and Loan, 1978 (or 2003)
- *The scaling and squaring method for the matrix exponential revisited*, N. Higham, 2005 (`expm` in Matlab)

Lie algebra

Lie algebra $\mathfrak{g}$ of the Lie group G

The set of all matrices $\mathbf{X}$ such that $e^{t\mathbf{X}}$ is in G for all real numbers t.

$\mathfrak{g}$ is an algebra (vector space+ ring)

- Real vector space
  - $\forall t, t\mathbf{X} \in G$
  - $\mathbf{X} + \mathbf{Y} \in G$
- $[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX} \in G$ (Lie bracket)

Lie groups and algebras

Exponential map

If G is a matrix Lie group with Lie algebra $\mathfrak{g}$, then the exponential mapping for G is the map :

$$\exp : \mathfrak{g} \longrightarrow G$$

In general the mapping is neither one-to-one nor onto but provides the *link* between the group and the Lie algebra.

There exists a neighborhood v about zero in $\mathfrak{g}$ and a neighborhood V of $\mathbf{I}$ in G such that $\exp : v \longrightarrow V$ is smooth and one-to-one onto with smooth inverse.

Paths

Path-connectedness

G is **path-connected** if given any two matrices **A** and **B** in G, there exists a continuous path $\mathbf{A}(t)$, $a \leq t \leq b$, lying in G with $\mathbf{A}(a) = \mathbf{A}$ and $\mathbf{A}(b) = \mathbf{B}$.

$\mathbb{SO}(n)$, $\mathbb{SL}(n)$ and $\mathbb{SE}(n)$ are connected ($\mathbb{O}(n)$ is not).

Generators

Generators

- Let $g(t_i) = \exp(t_i \mathbf{A}_i)$ define a subgroup of G, then $\mathbf{A}_i = \left.\frac{\partial g(t_i)}{\partial t_i}\right|_{t_i=0}$ is a generator of $\mathfrak{g}$.
- The set of generators form a basis and any element $\mathbf{x} \in \mathfrak{g}$ can be written :

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{n} x_i \mathbf{A}_i$$

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Special Orthogonal Group

$$\left[\det(e^{\mathbf{A}}) = e^{\mathrm{trace}(\mathbf{A})}\right]$$

$$\mathbb{SO}(3) = \{\mathbf{R} \in \mathbb{GL}(3) \mid \mathbf{R}\mathbf{R}^\top = \mathbf{I}, \det(\mathbf{R}) = +1\}$$

- preserves orientation (not a reflexion)

Associated Lie algebra :

$$\mathfrak{so}(3) = \{[\omega]_\times \in \mathbb{R}^{3\times 3} | \omega \in \mathbb{R}^3\}$$

Lie algebra representation and Euler angles

The Lie algebra representation :

$$(x_1, x_2, x_3) \longmapsto \exp(x_1 \, [e_1]_\times + x_2 \, [e_2]_\times + x_3 \, [e_3]_\times)$$

Euler angles :

$$(x_1, x_2, x_3) \longmapsto \exp(x_1 \, [e_1]_\times) \exp(x_2 \, [e_2]_\times) \exp(x_3 \, [e_3]_\times)$$

Special Euclidean Group

$$\mathbb{SE}(3) = \left\{ \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{GL}(4) \mid \mathbf{R} \in \mathbb{SO}(3), \mathbf{t} \in \mathbb{R}^3 \right\}$$

- preserves distances
- preserves orientation (not a reflexion)

Associated Lie algebra *twist* :

$$\mathfrak{se}(3) = \{ \begin{bmatrix} [\omega]_\times & \mathbf{v} \\ 0 & 0 \end{bmatrix} | \omega, \mathbf{v} \in \mathbb{R}^3 \}$$

- $\mathbf{v}$ is the linear velocity
- ω is the angular velocity

Expressing velocity

Velocity of a point in homogeneous coordinates ($\mathbf{x}(t) \in \mathfrak{se}(3)$) :

$$\dot{\mathbf{X}}(t) = \mathbf{x}(t)\mathbf{X}(t)$$

If $\mathbf{Y}(t) = \mathbf{T}\mathbf{X}(t)$ with $\mathbf{T} \in \mathbb{SE}(3)$ (change of coordinates) :

$$\dot{\mathbf{Y}}(t) = \mathbf{T}\mathbf{x}(t)\mathbf{T}^{-1}\mathbf{Y}(t)$$

Adjoint map on $\mathfrak{se}(3)$:

$$\begin{array}{rccc} Ad_{\mathbf{T}} : & \mathfrak{se}(3) & \longrightarrow & \mathfrak{se}(3) \\ & \mathbf{x} & \longmapsto & \mathbf{T}\mathbf{x}\mathbf{T}^{-1} \end{array}$$

Adjoint representation of $\mathfrak{se}(3)$ ($e^{ad_{\mathbf{x}}} = Ad_{e^{\mathbf{x}}}$) :

$$\begin{array}{rccc} ad_{\mathbf{X}} : & \mathfrak{se}(3) & \longrightarrow & \mathfrak{se}(3) \\ & \mathbf{Y} & \longmapsto & [\mathbf{X}, \mathbf{Y}] \end{array}$$

Another example : Special Linear Group

$$\mathbb{SL}(3) = \{\mathbf{H} \in \mathbb{GL}(3) \mid \det(\mathbf{H}) = +1\}$$

- ensures an invertible matrix with a minimal amount of parameters,
- subgroups include *affine transforms* or *translations* that are directly obtained by choosing the correct generators

This representation for a homography leads to “better” results than the “standard” minimal representation :

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & 1 \end{bmatrix}$$

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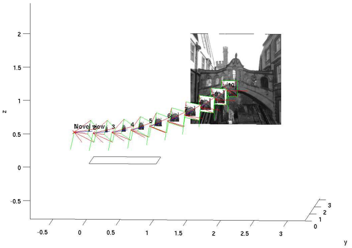
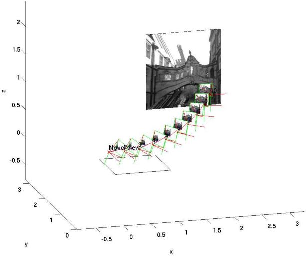
Interpolation

Interpolation

Let $\mathbf{T}_1 = e^{\mathbf{x}_1} \in \mathbb{SE}(3)$ and $\mathbf{T}_2 = e^{\mathbf{x}_2} \in \mathbb{SE}(3)$, a smooth trajectory can be obtained as $\mathbf{T}(x) = e^{\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2}$ with $\lambda = 0..1$.





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A generic minimisation problem...

Let :

$$f: \begin{array}{ccc} G & \longrightarrow & \mathbb{R} \\ \mathbf{g} & \longmapsto & f(\mathbf{g}) \end{array}$$

We want to solve, with $\bar{\mathbf{f}} \in \mathbb{R}$:

$$\bar{\mathbf{g}} = \min_{\mathbf{g}} d(f(\mathbf{g}), \bar{\mathbf{f}})$$

Gradient descent update :

$$\widehat{\mathbf{g}} \leftarrow \widehat{\mathbf{g}} + \mathbf{g}_k$$

$\widehat{\mathbf{g}}$ has no reason to still belong to G !!! (eg. rotation)

Using Lie algebras...

$$
\begin{array}{rcccc}
h: & \mathbb{R}^n & \longrightarrow & \mathfrak{g} & \longrightarrow \mathbb{R} \\
 & \mathbf{x} & \longmapsto & G(\mathbf{x}) & \longmapsto f(\widehat{\mathbf{g}} \circ e^{G(\mathbf{x})})
\end{array}
$$

The parameterisation only needs to be valid locally.
New update :

$$\widehat{\mathbf{g}} \leftarrow \widehat{\mathbf{g}} \circ e^{G(\mathbf{x}_k)}$$

$\widehat{\mathbf{g}}$ is guaranteed to still belong to the group.
Important condition : the initial value and optimal value have to be path-connected (in the case of $\mathbb{O}$, there are two components...).

Example...

Pose estimation (Lu *et al.*) :

$$\min_{\mathbf{x},\mathbf{t}_x} \sum_{i=1}^{n} \|(\mathbf{I} - \mathbf{Q}_i)\left(\mathbf{R}(\mathbf{x})\mathbf{R}\mathbf{p}_i + \mathbf{t} + \mathbf{t}_x\right)\|^2$$

Jacobians :

$$\begin{aligned} \nabla_x f_i &= (\mathbf{I} - \mathbf{Q}_i)\left[\ [\mathbf{e}_1]_\times \quad [\mathbf{e}_2]_\times \quad [\mathbf{e}_3]_\times\ \right]_{3\times3\times3} \mathbf{R}\mathbf{p}_i \\ \nabla_{\mathbf{t}_x} f_i &= (\mathbf{I} - \mathbf{Q}_i) \end{aligned}$$

$$\begin{aligned} \mathbf{R}_{k+1} &\leftarrow \mathbf{R}(\mathbf{x})\mathbf{R}_k \\ \mathbf{t}_{k+1} &\leftarrow \mathbf{t}_k + \mathbf{t}_x \end{aligned}$$

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BCF

Baker-Campbell-Hausdorff formula

Solution to $\mathbf{Z} = \log\left(e^{\mathbf{X}}e^{\mathbf{Y}}\right)$:

$$\mathbf{Z} = \mathbf{X} + \mathbf{Y} + \frac{1}{2}[\mathbf{X}, \mathbf{Y}] + \frac{1}{12}[\mathbf{X}, [\mathbf{X}, \mathbf{Y}]] - \frac{1}{12}[\mathbf{Y}, [\mathbf{X}, \mathbf{Y}]] + \dots$$

Further reading

- *Geometric Means in a Novel Vector Space Structure on Symmetric Positive-Definite Matrices*, V. Arsigny *et al.*, SIAM Journal on Matrix Analysis and Applications, 2006.
- *Processing Data in Lie Groups : An Algebraic Approach. Application to Non-Linear Registration and Difusion Tensor MRI.*, V. Arsigny, PhD, 2006.
- *An Elementary Introduction to Groups and Representations*, Brian C. Hall.