Lecture 7.1
Epipolar geometry

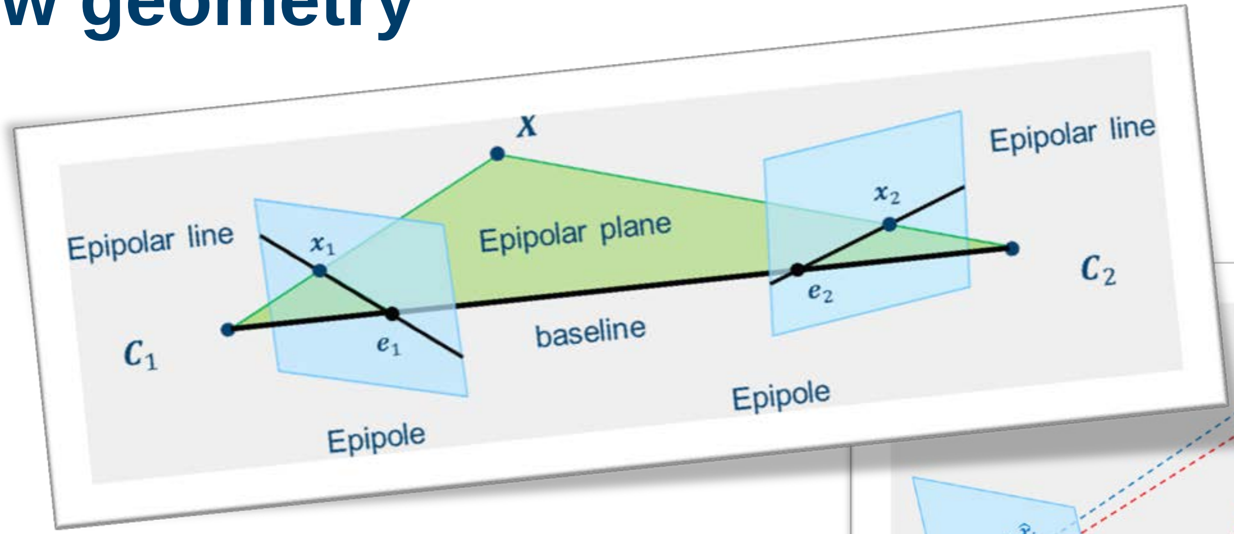
Thomas Opsahl

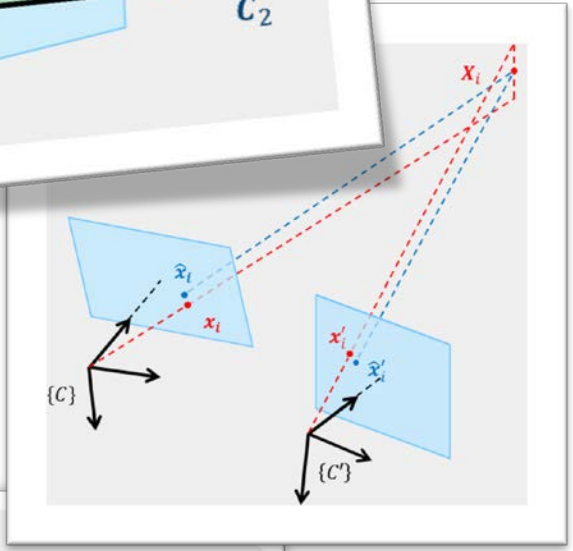
Weekly overview – Two-view geometry

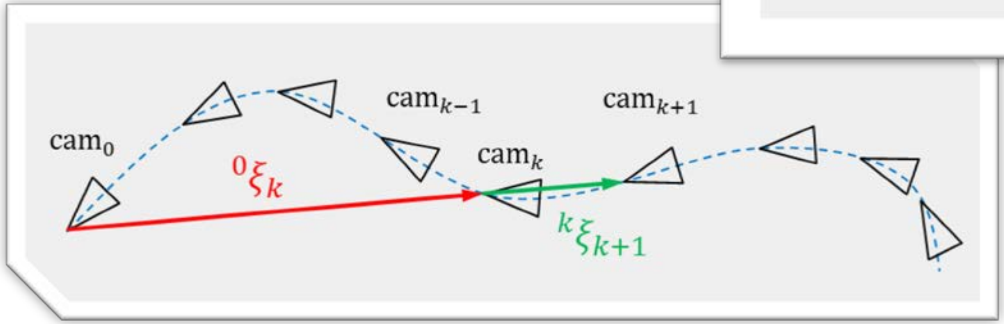
- Epipolar geometry
  - Algebraic representation & estimation
  - The essential matrix
  - The fundamental matrix

- Triangulation
  - Sparse 3D scene reconstruction from 2D correspondences

- Relative pose from epipolar geometry
  - Estimating the relative pose from the essential matrix
  - Visual odometry



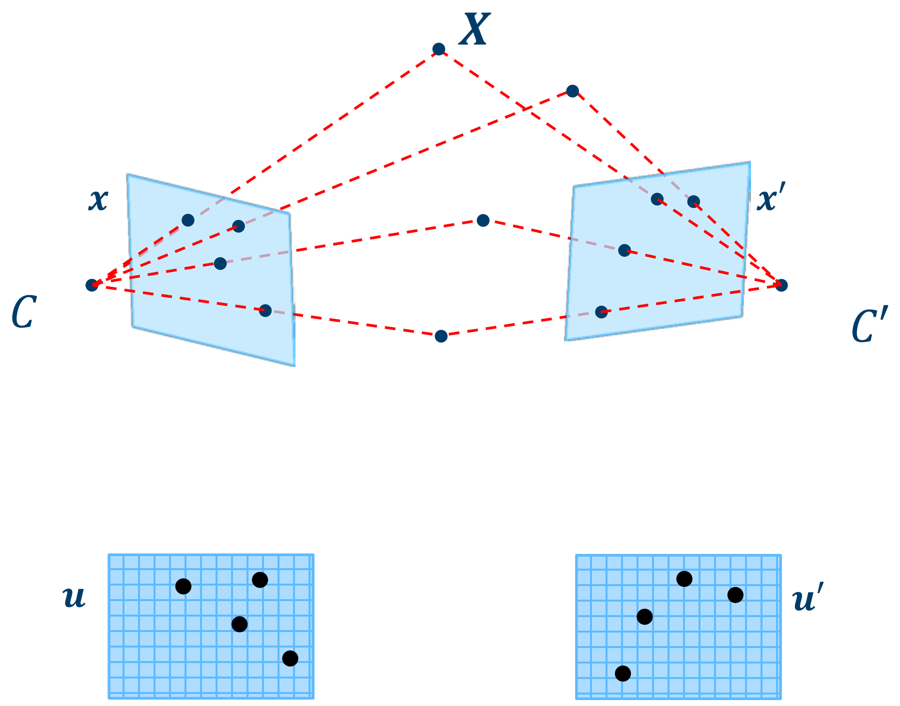




Introduction

- Observing the same points in two views puts a strong geometrical constraint on the cameras

- Algebraically this epipolar constraint is usually represented by two related 3×3 matrices



Introduction

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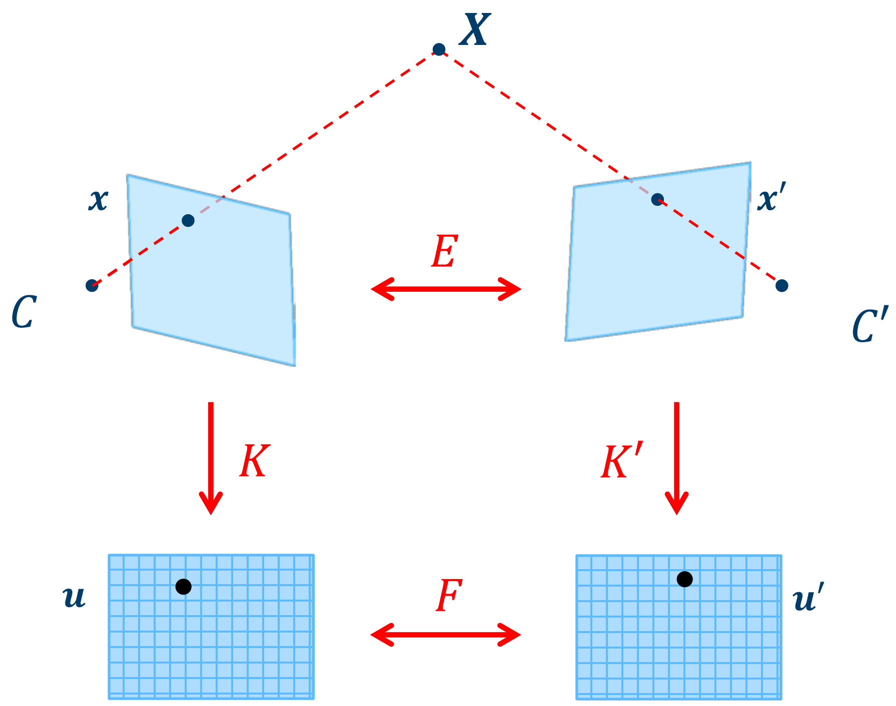
- Algebraically this epipolar constraint is usually represented by two related 3×3 matrices

- The fundamental matrix F
$$\tilde{\boldsymbol{u}}'^T F \tilde{\boldsymbol{u}}$$

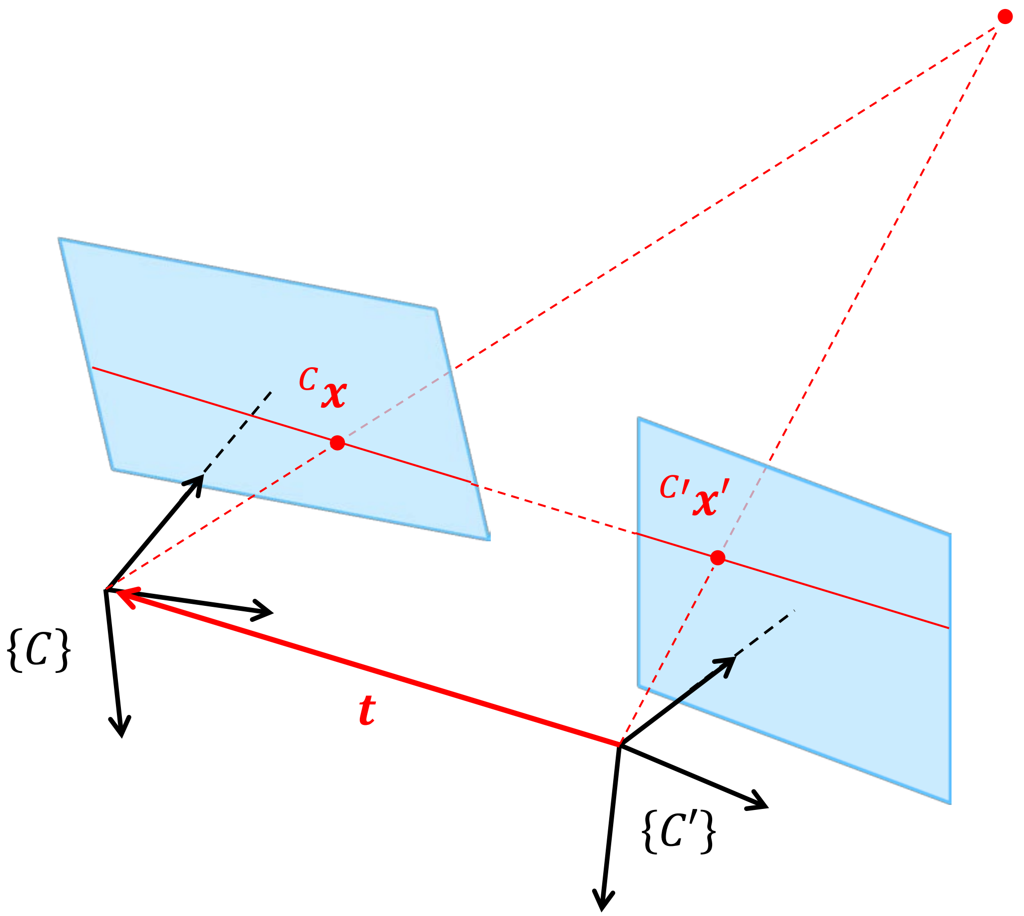
- The essential matrix E
$$\tilde{\boldsymbol{x}}'^T E \tilde{\boldsymbol{x}}$$

- These are related through the two calibration matrices K and K'



The essential matrix E

- Let ${}^{C}\boldsymbol{x} \leftrightarrow {}^{C'}\boldsymbol{x}'$ be corresponding points in the normalized image planes and let the pose of $\{C\}$ relative to $\{C'\}$ be
$$^{C'}\xi_C = \begin{bmatrix} R & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}$$



The essential matrix E

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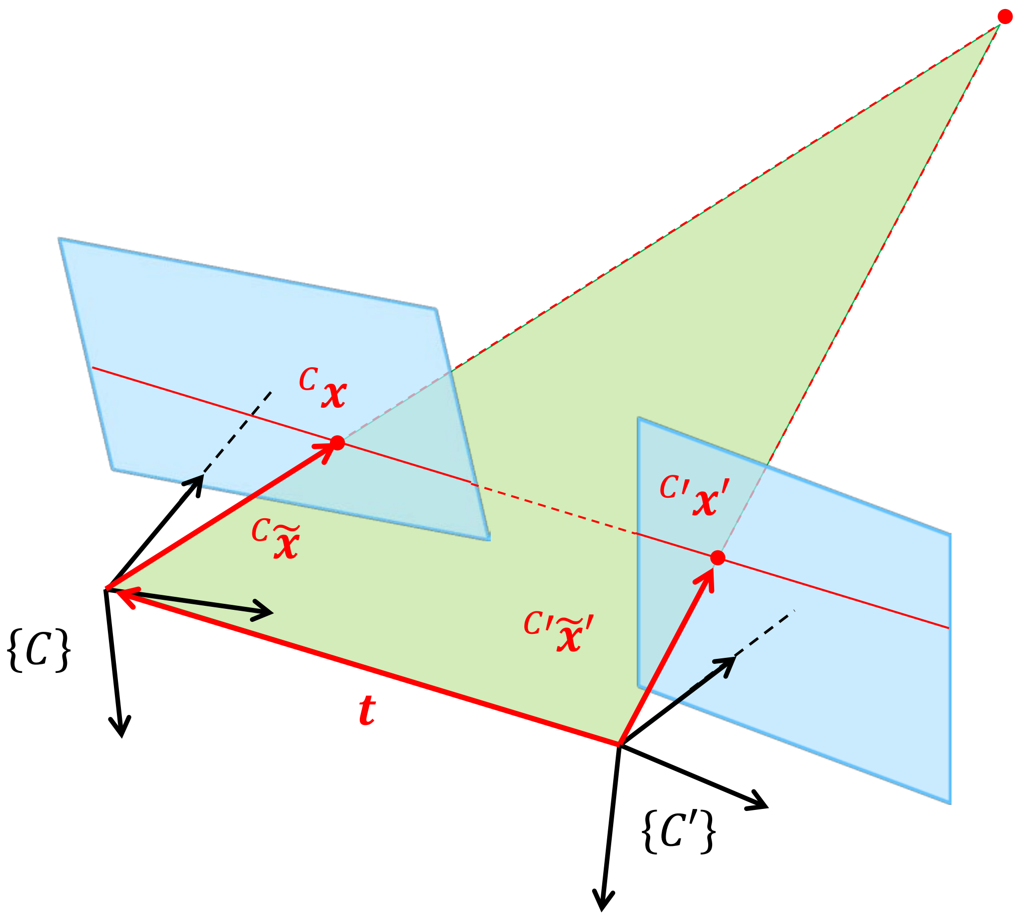
$$^{C'}\xi_C = \begin{bmatrix} R & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

- In terms of vectors, the equation for the epipolar plane can be written like
$$({}^{C'}\tilde{\boldsymbol{x}}' \times \boldsymbol{t}) \cdot (R\,{}^{C}\tilde{\boldsymbol{x}}) = 0$$

- Rewritten in terms of matrices this takes the form
$${}^{C'}\tilde{\boldsymbol{x}}'^{T}[\boldsymbol{t}]_{\times} R\,{}^{C}\tilde{\boldsymbol{x}} = 0$$

- This relationship defines the essential matrix
$E = [\boldsymbol{t}]_{\times} R$

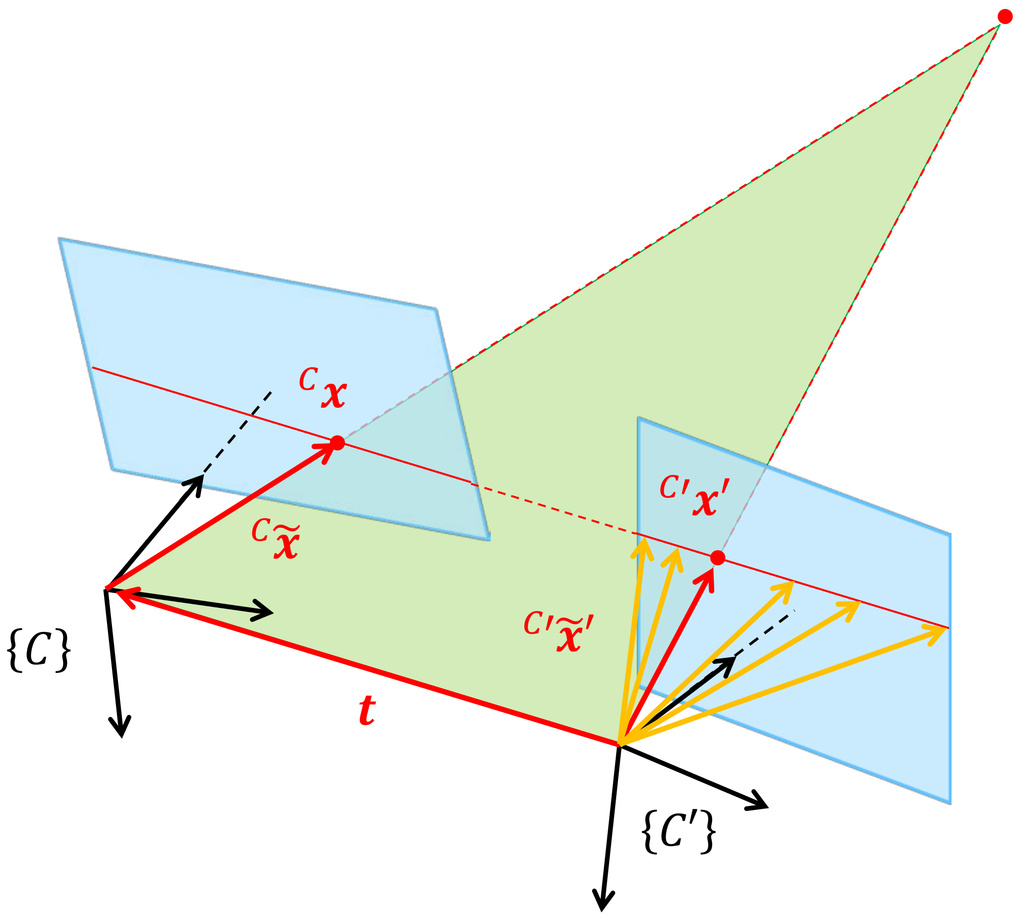
$$\boxed{\tilde{\boldsymbol{x}}'^{T} E \tilde{\boldsymbol{x}} = 0}$$



The essential matrix E

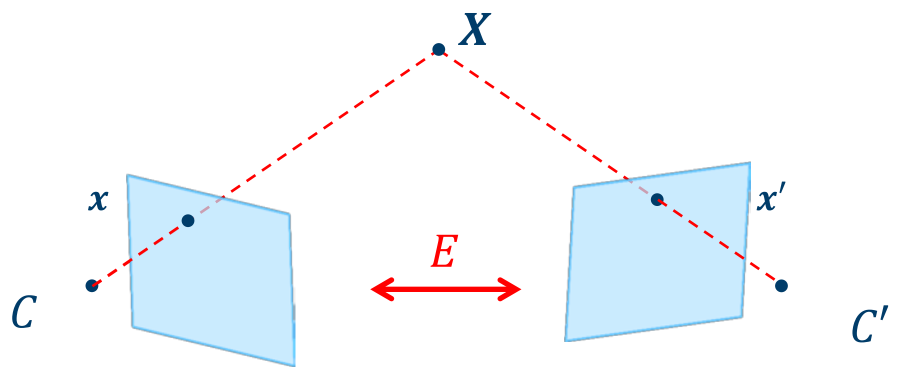
- The essential matrix E represents the epipolar constraint on corresponding normalized points

- Note that although $\tilde{\boldsymbol{x}}'^T E \tilde{\boldsymbol{x}} = 0$ is a necessary requirement for the correspondence $\boldsymbol{x} \leftrightarrow \boldsymbol{x}'$ to be geometrically possible, it does not guarantee its correctness



The essential matrix E

- Properties of E
  - $E = [\boldsymbol{t}]_\times R$
  - Homogeneous
  - $rank = 2$
  - $\det = 0$
  - 5 degrees of freedom
  - E can be estimated from a minimum of 5 point correspondences
  - If $\boldsymbol{x}$ and $\boldsymbol{x}'$ are corresponding normalized image points, then $\tilde{\boldsymbol{x}}'^T E \tilde{\boldsymbol{x}} = 0$
  - E has 2 singular values that are equal and a third that is zero



- It is possible to decompose $E = [\boldsymbol{t}]_\times R$ to determine the relative pose between cameras
  - Translation only up to scale
  - Topic of another lecture

The fundamental matrix F

- The epipolar constraint on image points is naturally connected to the essential matrix by the calibration matrices K and K'

$$K\,{}^{C}\tilde{\mathbf{x}} = \tilde{\mathbf{u}} \Rightarrow {}^{C}\tilde{\mathbf{x}} = K^{-1}\tilde{\mathbf{u}}$$

$$K'\,{}^{C'}\tilde{\mathbf{x}}' = \tilde{\mathbf{u}}' \Rightarrow {}^{C'}\tilde{\mathbf{x}}' = K'^{-1}\tilde{\mathbf{u}}' \Rightarrow {}^{C'}\tilde{\mathbf{x}}'^{T} = \tilde{\mathbf{u}}'^{T} K'^{-T}$$

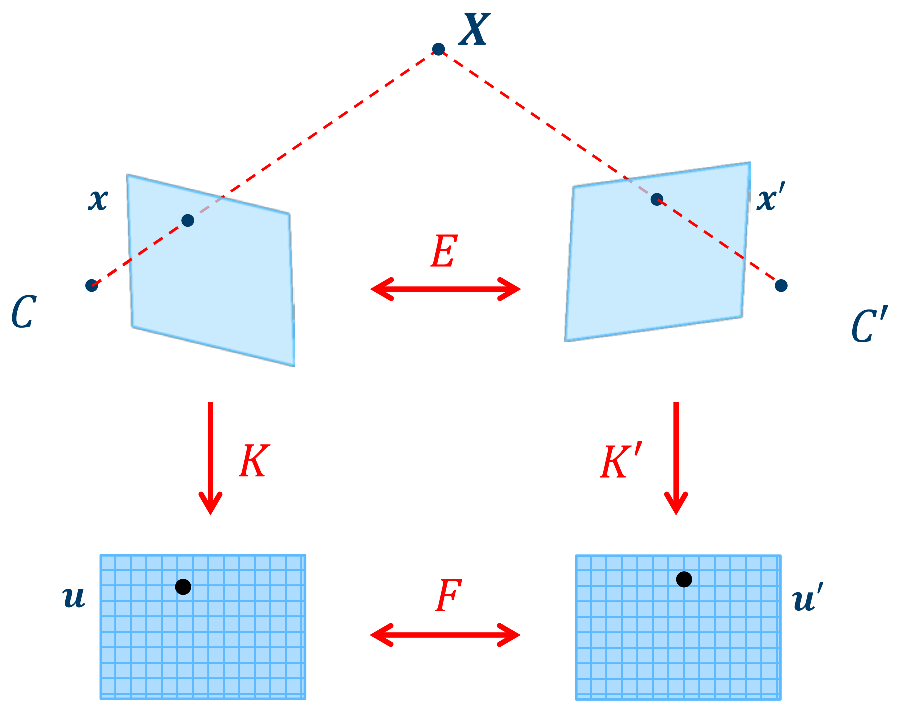
- Combined with the epipolar constraint for normalized image points we get

$${}^{C'}\tilde{\mathbf{x}}'^{T} E\, {}^{C}\tilde{\mathbf{x}} = 0$$

$$\tilde{\mathbf{u}}'^{T} K'^{-T} E K^{-1} \tilde{\mathbf{u}} = 0$$

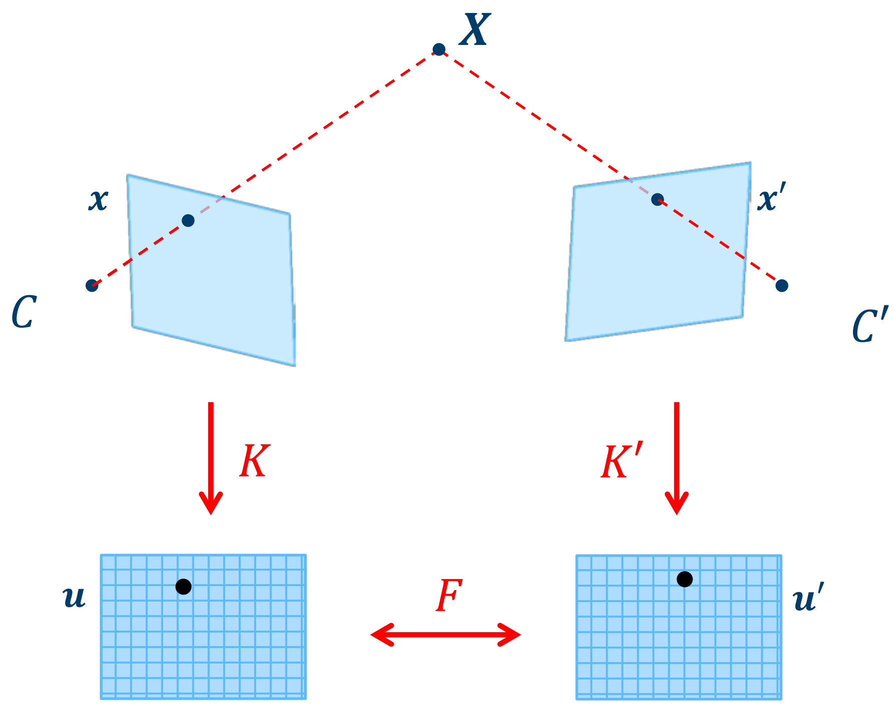
- This defines the fundamental matrix $F = K'^{-T} E K^{-1}$

$$\boxed{\tilde{\mathbf{u}}'^{T} F \tilde{\mathbf{u}} = 0}$$



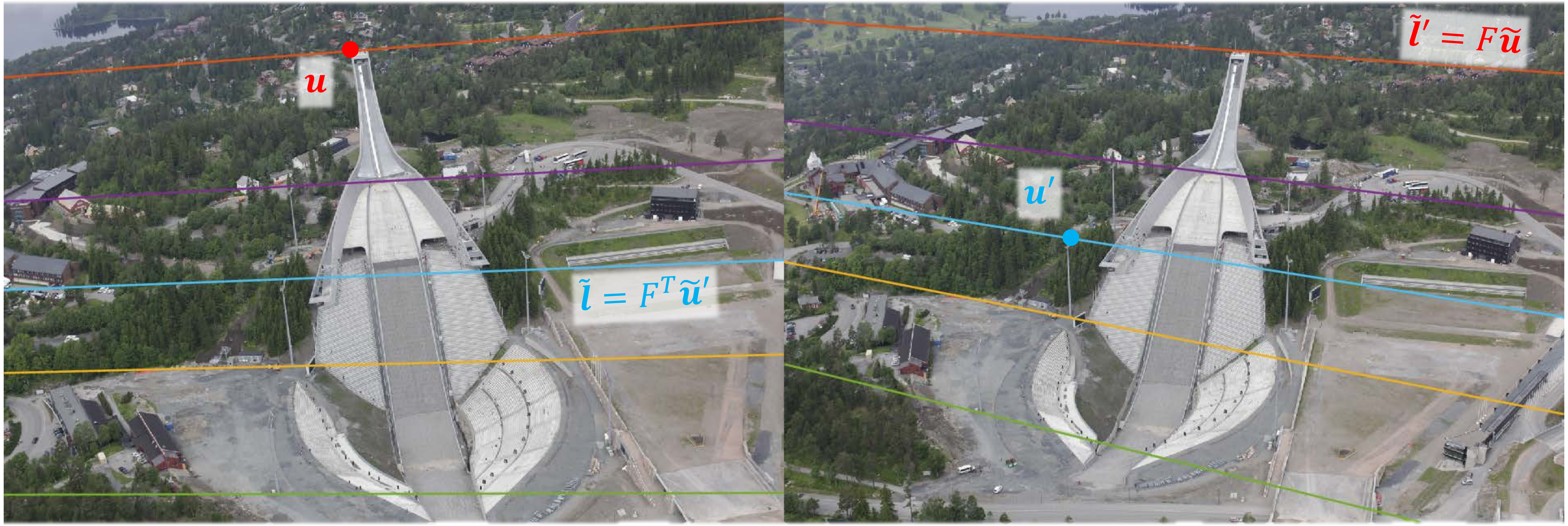
The fundamental matrix F

- Properties of F
  - $F = K'^{-T} E K^{-1}$
  - Homogeneous
  - $rank = 2$
  - $\det = 0$
  - 7 degrees of freedom
  - F can be estimated from a minimum of 7 point correspondences
  - If $\boldsymbol{u}$ and $\boldsymbol{u}'$ are corresponding image points, then $\tilde{\boldsymbol{u}}'^T F \tilde{\boldsymbol{u}} = 0$
  - For any point $\boldsymbol{u}$ in image 1, the corresponding epipolar line $\boldsymbol{l}'$ in image 2 is given by
$$\tilde{\boldsymbol{l}}' = F\tilde{\boldsymbol{u}}$$
  - For any point $\boldsymbol{u}'$ in image 2, the corresponding epipolar line $\boldsymbol{l}$ in image 1 is given by
$$\tilde{\boldsymbol{l}} = F^T\tilde{\boldsymbol{u}}'$$



- The epipole $\boldsymbol{e}'$ in image 2 is F's left singular vector corresponding to the zero singular value
- The epipole $\boldsymbol{e}$ in image 1 is F's right singular vector corresponding to the zero singular value

Example



- These fundamental lines were determined using the fundamental matrix between images
- Recall that points and lines are dual in $\mathbb{P}^2$

$$\tilde{\boldsymbol{l}}^T \tilde{\boldsymbol{u}} = 0 \Leftrightarrow \begin{bmatrix} l_0 & l_1 & l_2 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0 \Leftrightarrow l_0 u + l_1 v + l_2 = 0$$

Estimating F

- Several algorithms
  - Non-iterative: 7-pt, 8-pt
  - Iterative: Minimize epipolar error
  - Robust: RANSAC with 7-pt

- From the definition it follows that each point correspondence $\boldsymbol{u}_i \leftrightarrow \boldsymbol{u}_i{}'$ contributes with 1 equation

$$\boldsymbol{u}_i'^T F \boldsymbol{u}_i = 0$$

$$\begin{bmatrix} u_i' & v_i' & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} u_i u_i' & u_i' v_i & u_i' & u_i v_i' & v_i v_i' & v_i' & u_i & v_i & 1 \end{bmatrix} \boldsymbol{f} = 0$$

- So given several correspondences we get a homogeneous system of linear equations that we can solve by SVD

$$\begin{bmatrix} u_1 u_1' & u_1' v_1 & u_1' & u_1 v_1' & v_1 v_1' & v_1' & u_1 & v_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n u_n' & u_n' v_n & u_n' & u_n v_n' & v_n v_n' & v_n' & u_n & v_n & 1 \end{bmatrix} \boldsymbol{f} = 0$$

$$A\boldsymbol{f} = 0$$

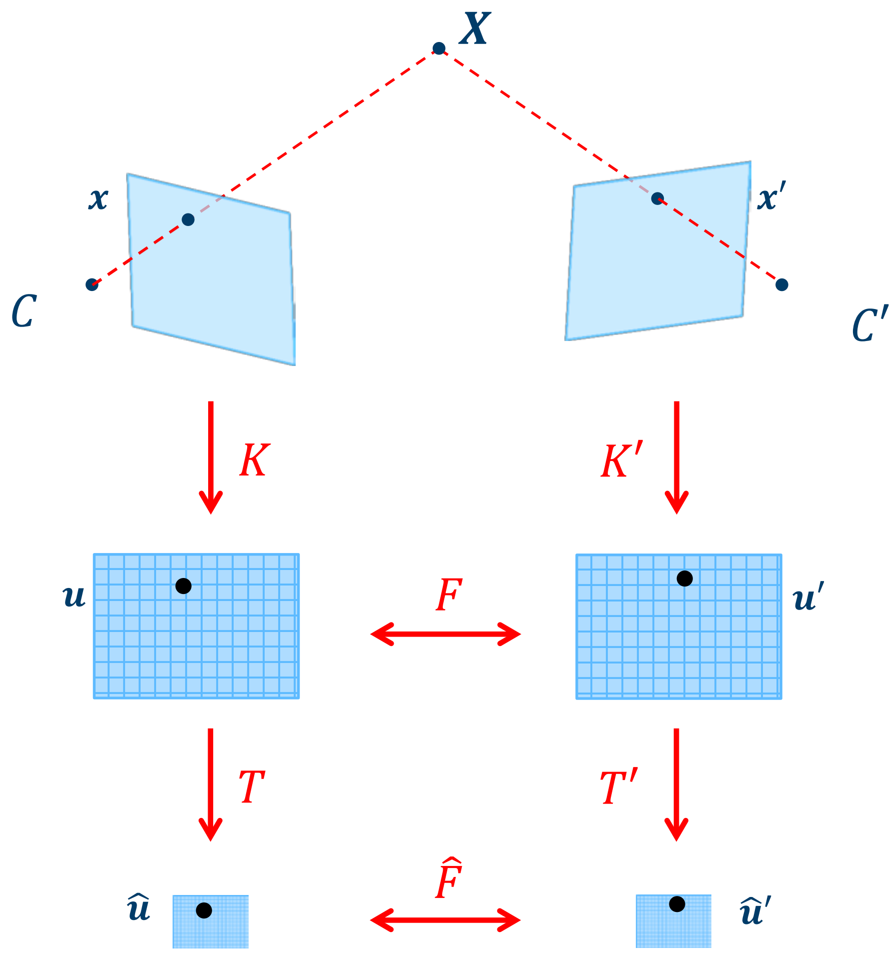
- As before, we see that the matrix A contains terms that can be very different in scale, so point sets $\{\boldsymbol{u}_i\}$ and $\{\boldsymbol{u}_i{}'\}$ should be normalized in advance
  - Centroid → origin
  - Mean distance from origin should be $\sqrt{2}$

Estimating F

The normalized 8-point algorithm

Given $n \geq 8$ correspondences $\boldsymbol{u}_i \leftrightarrow \boldsymbol{u}_i{}'$, do the following

1. Normalize points $\{\boldsymbol{u}_i\}$ and $\{\boldsymbol{u}_i{}'\}$ using similarity transforms T and T'
2. Build matrix A from point-correspondences and compute its SVD
3. Extract the "solution" $\bar{F}$ from the right singular vector corresponding to the smallest singular value
4. Compute the SVD of $\bar{F}$: $\bar{F} = USV^T$
5. Enforce zero determinant by setting $s_{33} = 0$ and compute a proper fundamental matrix
$$\hat{F} = USV^T$$
6. Denormalize $F = T'^T \hat{F} T$



$$\hat{\boldsymbol{u}}'^T \hat{F} \hat{\boldsymbol{u}} = 0$$

$$(T'\boldsymbol{u}')^T \hat{F} (T\boldsymbol{u}) = 0 \quad \Rightarrow \quad F = T'^T \hat{F} T$$

$$\boldsymbol{u}'^T T'^T \hat{F} T \boldsymbol{u} = 0$$

Estimating F

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The 7-point algorithm

- Given 7 correspondences, A will be a 7×9 matrix which in general will be of rank 7

- So the null space of A is 2-dimensional and the fundamental matrix must be a linear combination of the two right null vectors of A
$$\boldsymbol{f}(\alpha) = \alpha \boldsymbol{f}_1 + (1 - \alpha)\boldsymbol{f}_2$$
$$F(\alpha) = \alpha F_1 + (1 - \alpha)F_2$$

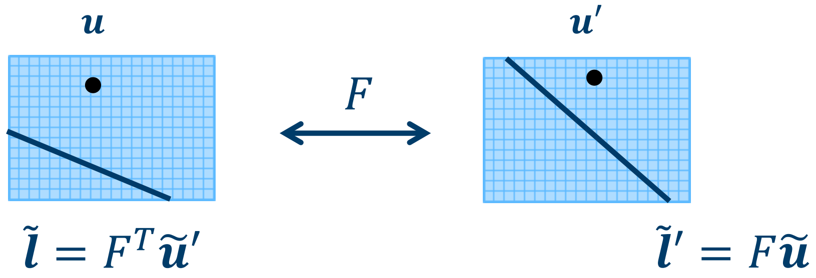
- The additional constraint $det(F) = 0$ leads to a cubic polynomial equation in α which has 1 or 3 solutions α_i which in turn yields 1 or 3 F’s

- This algorithm is to prefer in a RANSAC scheme, since it is minimal and since for a single sampling of 7 correspondences one might get 3 fundamental matrices to test for inliers

Estimating F

- Improved estimates of F can be obtained using iterative methods like Levenberg-Marquardt to minimize the epipolar error

$$\epsilon = \sum d(\tilde{\boldsymbol{u}}, F^T\tilde{\boldsymbol{u}}') + d(\tilde{\boldsymbol{u}}', F\tilde{\boldsymbol{u}})$$



- OpenCV
  - cv::Mat cv::findFundamentalMat
  - Arguments are
    InputArray points1
    InputArray points2
    int method – {CV_FM_7POINT, CV_FM_8POINT, CV_FM_RANSAC, CV_FM_LMEDS}
    double param1
    double param2
    OutputArray mask

- Matlab
  - estimateFundamentalMatrix

Estimating F

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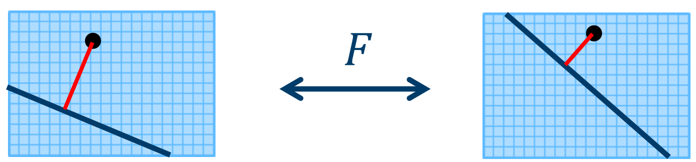

- Distance between homogeneous point $\boldsymbol{u}$ and line $\boldsymbol{l}$, with the homogeneous representation $\tilde{\boldsymbol{l}} = [\tilde{l}_1, \tilde{l}_2, \tilde{l}_3]^T$

$$d\left(\tilde{\mathbf{u}}, \tilde{\boldsymbol{l}}\right) = \frac{\tilde{\mathbf{u}}^T \tilde{\boldsymbol{l}}}{\sqrt{\tilde{l}_1^{\,2} + \tilde{l}_2^{\,2}}}$$

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- Matlab
  - estimateFundamentalMatrix

Estimating E

- For calibrated cameras (K and K' are known), we can first estimate F and then compute $E = K'^T F K$
- It is also possible to estimate E directly from 5 normalized point correspondences $\boldsymbol{x}_i \leftrightarrow \boldsymbol{x}_i'$
  - Algorithm proposed by David Nistér in 2004
  - Involves finding the roots of a 10^{th} degree polynomial
- In RANSAC schemes, the 5-point algorithm is the fastest alternative
  - To acieve 99% confidence with 50% outliers, the 5-point algorithm only requires 145 tests while the 8-point algorithm requires 1177 tests

- OpenCV
  - cv::Mat cv::findEssentialMat
  - 5-pt algorithm
- Matlab
  - Currently not available as a built in function in Matlab
  - MexOpenCV
- OpenGV: http://laurentkneip.github.io/opengv/ contains several interesting functions
  - 5-pt algorithm
  - 2-pt algorithm based on known relative rotation

Summary

- Algebraic representation of epipolar geometry
  - The essential matrix
  - The fundamental matrix

- Estimating the epipolar geometry
  - Estimate F: 7pt, 8pt, RANSAC
  - Estimate E: 5pt

- Additional reading:
  - Szeliski: 7.2