Non-linear estimation

- Given a non-linear model $g(x)$ depending on a parameter vector x and a number of observations b, non-linear estimation is about finding the parameter vector x_* that explains the observations the best.
- The model $g(x)$ should be twice continuously differentiable
- If we choose to minimize the geometric distance, the problem is a non-linear least squares problem

$$\min_x f(x) = \frac{1}{2} r(x)^T r(x) = \frac{1}{2}\|r(x)\|^2 = \frac{1}{2}\|g(x) - b\|^2,$$

  where $f(x)$ is the objective function and $r(x)$ is known as the residual function.
- For the Mahalanobis distance the problem becomes a weighted non-linear least squares problem

$$\min_x \frac{1}{2} r(x)^T W r(x) = \frac{1}{2}\|L r(x)\|^2 = \frac{1}{2}\|L(g(x) - b)\|^2,$$

  where $W = \Sigma^{-1}$ and $L^T L = W$.

The Gauss-Newton method

- An algorithm to solve unconstrained least squares problem is the Gauss-Newton method.
- The method requires implementations of a *model function* $g(x)$, *residual function* $r(x)$ and *jacobian* of the residual function $J(x)$.
- For each iteration k, the search direction vector p_k is calculated as the solution of

$$\min_{p_k} \frac{1}{2}\|L\left(J_k p_k + r_k\right)\|^2,$$

  or

$$J_k^T W J_k p_k = -J_k^T W r_k,$$

  where $J_k = J(x_k)$, $r_k = r(x_k)$. The Jacobian $J(x)$ thus have to have full rank in every point we pass.

Gauss-Newton with line-search

- Given the search direction p_k, the *step length* α_k is calculated such that

$$x_{k+1} = x_k + \alpha_k p_k$$

  is a “sufficiently better” point that x_k.
- “Sufficiently better” is judged by the objective function and the step length is accepted if the new point satisfies the *Armijo condition*

$$f(x_k + \alpha_k p_k) \le f(x_k) + \mu \alpha_k \nabla_x f(x_k)^T p_k$$

  for a given constant $0 < \mu < 1$. The gradient $\nabla_x f(x_k) = J_k^T W r_k$.
- To avoid taking too short steps, a *back-tracking* technique is often employed where the first value in the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \ldots$$

  that satisfies the Armijo condition is taken as the step length.

Homography, error in one image only

- Let p_i be known points in $\mathcal{P}^2$ and let q_i be measured, normalized points in $\mathcal{R}^2$.
- Parameters (unknowns):

$$x = [h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}]^T.$$

- Model:

$$g(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{bmatrix}, g_i(x) = \begin{bmatrix} \frac{r_{i1}}{r_{i3}} \\ \frac{r_{i2}}{r_{i3}} \end{bmatrix} = \begin{bmatrix} \frac{h_{11}p_{i1}+h_{12}p_{i2}+h_{13}p_{i3}}{h_{31}p_{i1}+h_{32}p_{i2}+p_{i3}} \\ \frac{h_{21}p_{i1}+h_{22}p_{i2}+h_{23}p_{i3}}{h_{31}p_{i1}+h_{32}p_{i2}+p_{i3}} \end{bmatrix},$$

$$J(x) = \begin{bmatrix} J_1(x) \\ \vdots \\ J_n(x) \end{bmatrix}, J_i(x) = \begin{bmatrix} \frac{p_{i1}}{r_{i3}} & \frac{p_{i2}}{r_{i3}} & \frac{p_{i3}}{r_{i3}} & 0 & 0 & 0 & \frac{-p_{i1}r_{i1}}{r_{i3}^2} & \frac{-p_{i2}r_{i1}}{r_{i3}^2} \\ 0 & 0 & 0 & \frac{p_{i1}}{r_{i3}} & \frac{p_{i2}}{r_{i3}} & \frac{p_{i3}}{r_{i3}} & \frac{-p_{i1}r_{i2}}{r_{i3}^2} & \frac{-p_{i2}r_{i2}}{r_{i3}^2} \end{bmatrix}$$

$$b = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}.$$

- If the points are measured with different variance use $W = \Sigma^{-1}$, otherwise $W = I$.

Constrained problems

- The corresponding formulation with constraints is

$$\min_x \frac{1}{2} r(x)^T W r(x)$$
$$\text{s.t. } c(x) = 0$$

- The constraint function $c(x)$ should be twice continuously differentiable.

Gauss-Newton for constrained problem

- There is a Gauss-Newton-based algorithm for constrained problem.
- Beside the residual function $r(x)$ and its Jacobian $J(x)$, the constraint function $c(x)$ and its jacobian $K(x)$ are needed.
- At every iteration, the linearized problem is formulated

$$\min_{p_k} \frac{1}{2} \| L \left(J_k p_k + r_k \right) \|^2,$$
$$\text{s.t. } c_k + K_k p_k = 0,$$

corresponding to the linear equation system

$$\begin{bmatrix} -J_k^T W J_k & K_k^T \\ K_k & 0 \end{bmatrix} \begin{bmatrix} p_k \\ \lambda_k \end{bmatrix} = \begin{bmatrix} J_k^T W r_k \\ -c_k \end{bmatrix},$$

where $r_k = r(x_k)$, $J_k = J(x_k)$, $c_k = c(x_k)$, $K_k = K(x_k)$. The vector λ_k is an estimate of the Lagrange multipliers for the constraints.

Line-search for constrained problems

- As in the unconstrained case, a *step length* α_k is calculated such that $x_{k+1} = x_k + \alpha_k p_k$ is a “sufficiently better” point that x_k.
- However, “sufficiently better” is more complicated than in the unconstrained case, since it is necessary to balance reductions of the objective function with violations of the constraints.
- A possible solution is to use a quadratic *merit function*

$$\psi(x, \nu_k) = f(x) + \frac{1}{2} \nu_k \|c(x)\|^2 = f(x) + \frac{1}{2} \nu_k c(x)^T c(x),$$

where the parameter ν_k determines the penalty for violating the constraint.

- The Armijo condition applied to the merit function is

$$\psi(x_k + \alpha_k p_k, \nu_k) \le \psi(x_k) + \mu \alpha_k \nabla_x \psi(x_k, \nu_k)^T p_k,$$

where the gradient $\nabla_x \psi(x_k, \nu) = J_k^T W r_k + \nu K_k^T c_k$.

- The penalty parameter ν_k is initially small and is increased as the iteration progresses, forcing the iterates to stay closer to the constraint.

Reprojection error

- Let p_i be measured points in $\mathcal{P}^2$ in image 1 and let q_i be measured points in $\mathcal{P}^2$ in image 2. Let $\hat{p}_i$ be the estimated point in image 1 and $\hat{q}_i$ the estimated point in image 2.
- Parameters

$$x = [h_{11}, \ldots, h_{33}, \hat{p}_i, \hat{q}_i]^T.$$

- Model:

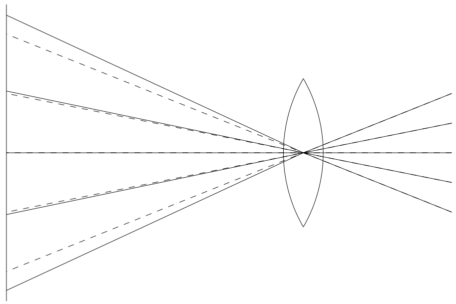
$$g_i(x) = \begin{bmatrix} \frac{\hat{p}_{i1}}{\hat{p}_{i3}} \\ \frac{\hat{p}_{i2}}{\hat{p}_{i3}} \\ \frac{\hat{q}_{i1}}{\hat{q}_{i3}} \\ \frac{\hat{q}_{i2}}{\hat{q}_{i3}} \end{bmatrix}, J_i(x) = \begin{bmatrix} 0 \ldots 0 & \frac{1}{\hat{p}_{i3}} & 0 & \frac{-\hat{p}_{i1}}{\hat{p}_{i3}^2} & 0 & 0 & 0 & 0 \ldots 0 \\ 0 \ldots 0 & 0 & \frac{1}{\hat{p}_{i3}} & \frac{-\hat{p}_{i1}}{\hat{p}_{i3}^2} & 0 & 0 & 0 & 0 \ldots 0 \\ 0 \ldots 0 & 0 & 0 & 0 & \frac{1}{\hat{q}_{i3}} & 0 & \frac{-\hat{q}_{i1}}{\hat{q}_{i3}^2} & 0 \ldots 0 \\ 0 \ldots 0 & 0 & 0 & 0 & 0 & \frac{1}{\hat{q}_{i3}} & \frac{-\hat{q}_{i1}}{\hat{q}_{i3}^2} & 0 \ldots 0 \end{bmatrix}$$

$$c_j(x) = \begin{bmatrix} \hat{p}_j^T \hat{p}_j - 1 \\ \hat{q}_j^T \hat{q}_j - 1 \\ H \hat{p}_j - \hat{q}_j \end{bmatrix}, K_j(x) = \begin{bmatrix} \hat{p}_j^T & 0 & 0 & 0 \ldots 0 & & 0 \ldots 0 & & 0 \ldots 0 \\ 0 & \hat{p}_j^T & 0 & 0 \ldots 0 & H & 0 \ldots 0 & -I & 0 \ldots 0 \\ 0 & 0 & \hat{p}_j^T & 0 \ldots 0 & & 0 \ldots 0 & & 0 \ldots 0 \end{bmatrix},$$

$$c(x) = \begin{bmatrix} c_1(x) \\ \vdots \\ c_n(x) \\ h_{11}^2 + \cdots + h_{33}^2 - 1 \end{bmatrix}, K(x) = \begin{bmatrix} K_1(x) \\ \vdots \\ K_n(x) \\ 2h^T \; 0 \ldots 0 \end{bmatrix}.$$

Lens distortion

- A lens is designed to bend rays of light to construct a focused image.
- A side effect is that the collinearity between incoming and outgoing rays is destroyed.



Lens distortion

- The effect of lens distortion is that the projected point is moved toward or away from a point of symmetry.
- One formulation of lens distortion is as a symmetric distortion around a distortion center.

$$\begin{bmatrix} x_d \\ y_d \end{bmatrix} = \mathbf{p}_d = f(\tilde{\mathbf{p}}) = L(\tilde{x}, \tilde{y}) \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix},$$

where $(\tilde{x}, \tilde{y})^\top$ are the coordinates without distortion

$$L(\tilde{x}, \tilde{y}) = 1 + \kappa_1 r + \kappa_2 r^2 + \ldots, \quad r = \sqrt{(\tilde{x} - x_c)^2 + (\tilde{y} - y_c)},$$

is the distortion as a function of the distance from the distortion center $(x_c, y_c)^\top$.

- Some features of this formulation:
  - The distorted coordinates are expressed as a function of the perfect coordinates.
  - The distortion center is undefined if the distortion is zero.
  - The function $L(\tilde{x}, \tilde{y})$ is not differentiable in $(\tilde{x}, \tilde{y}) = (x_c, y_c)$.

Lens distortion

- The fotogrammetric formulation of lens distortion is a separation into a symmetric (radial) and asymmetric (tangential) about the principal point.

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \tilde{\mathbf{p}} = g(\mathbf{p}_d) = \begin{bmatrix} x_d \\ y_d \end{bmatrix} - \left(\begin{bmatrix} x_r \\ y_r \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix} \right),$$

where

$$\begin{bmatrix} x_r \\ y_r \end{bmatrix} = (K_1 r^2 + K_2 r^4 + \ldots) \begin{bmatrix} x_d \\ y_d \end{bmatrix},$$

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 2P_1 \Delta x \Delta y + P_2(r^2 + 2\Delta x^2) \\ 2P_2 \Delta x \Delta y + P_1(r^2 + 2\Delta y^2) \end{bmatrix},$$

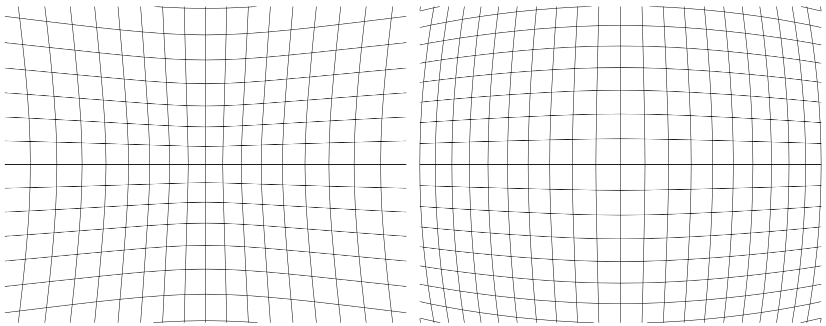
is the radial and tangential distortion, respectively, as a function of the distance to the principal point

$$r^2 = \Delta x^2 + \Delta y^2, \quad \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} x_d - x_p \\ y_d - y_p \end{bmatrix}$$

Lens distortion

- Some properties for this formulation:
  - More stable formulation for small distortions.
  - Differentiable in all points.
  - Describes the *correction* of the lens distortion, i.e. the perfect coordinates $\tilde{\mathbf{p}}$ are described as a function of the distorted coordinates $\mathbf{p}_d$.
- The radial distortion follows from that the lens bends rays of light. It is neglectable only for large focal lengths.
- Any tangential distortion is due to de-centering of the optical axis for the various lens components. It is neglectable except for high precision measurements.
- The lens distortion varies with the focal length. To use a calibrated camera, the focal length (and hence any zoom) must be the same as during calibration.

Lens distortion



Positive radial distortion (*pin-cushion*) | Negative radial distortion (*barrel*)

Calibration

- Calibration of the lens distortion is usually performed in one of two ways:
  - During camera calibration where the other internal camera parameters are determined.
  - By constructing a measure for "deviation from straight lines".