

# Robotics, Geometry and Control - Conventional Robotics

Ravi Banavar<sup>1</sup>

<sup>1</sup>Systems and Control Engineering  
IIT Bombay

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The material for these slides is largely taken from the *text*

- ▶ A Mathematical Introduction to Robotic Manipulation - R. Murray, Z. Li and S. Sastry, CRC Press 1994.

# Conventional Robotics

- ▶ Products of exponential formula
- ▶ Manipulator Jacobian
- ▶ Singularities
- ▶ Dynamics of open chain manipulators
- ▶ Christoffel symbols
- ▶ Passivity and skew-symmetric structure
- ▶ Position control and computed torque control

# A robotic manipulator

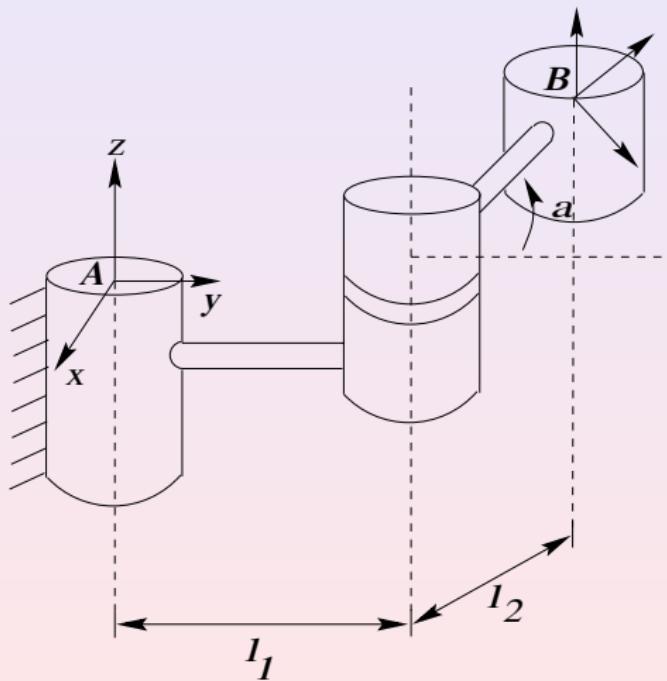


Figure: A conventional manipulator

# Forward kinematics

- ▶ Forward kinematics map  $g_{bt} : Q \rightarrow SE(3)$ .
- ▶  $\xi_1$  and  $\xi_2$  are the twists associated with the first and second joint respectively.
- ▶  $g_{bt}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{bt}(0)$
- ▶ For  $n$  joints - product of exponentials

$$g_{bt}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} g_{bt}(0)$$

- ▶ Recall geometric interpretation - exponential map takes the Lie algebra to the Lie group

# Inverse kinematics

- ▶ Given  $g_d \in SE(3)$ , find  $\theta = (\theta_1, \dots, \theta_n) \in Q$  such that

$$g_d = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} g_{bt}(0)$$

# Manipulator Jacobian

- ▶ Say  $\theta(\cdot) : R \supset (a, b) \rightarrow Q$  and  $g_{bt}(\theta(t)) : Q \rightarrow SE(3)$
- ▶ Recall the spatial representation of velocity (in  $se(3)$ )

$$\hat{V}_{bt}^s = \dot{g}_{bt}(\theta)g_{bt}^{-1}(\theta) = \sum_{i=1}^n \left( \frac{\partial g_{bt}}{\partial \theta_i} \dot{\theta}_i \right) g_{bt}^{-1}(\theta) = \sum_{i=1}^n \left[ \frac{\partial g_{bt}}{\partial \theta_i} g_{bt}^{-1}(\theta) \right] \dot{\theta}_i$$

- ▶ Vectorial notation

$$V_{bt}^s = J_{bt}^s(\theta)\dot{\theta} \quad V_{bt}^s \in R^6, J_{bt}^s \in R^{6 \times n} \quad J_{bt}^s - \text{spatial manipulator Jacobian}$$

- ▶ Geometric interpretation - the Jacobian is the tangent map from  $T_q Q \rightarrow T_e SE(3)$ .
- ▶ Another representation

$$J_{bt}^s(\theta) = \begin{bmatrix} \xi_1 & \xi_2' & \dots & \xi_n' \end{bmatrix}$$

$$\xi'_i = Ad_{e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_{i-1} \theta_{i-1}}} \xi_i$$

## Body manipulator Jacobian

- ▶ Another representation

$$J_{bt}^b(\theta) = \begin{bmatrix} \xi_1^\dagger & \xi_2^\dagger & \dots & \xi_n^\dagger \end{bmatrix}$$

$$\xi_i^\dagger = Ad_{e^{\hat{\xi}_i \theta_i} \dots e^{\hat{\xi}_n \theta_n} g_{bt}(0)} \xi_i$$

- ▶ The body Jacobian is related to the spatial Jacobian as

$$J_{bt}^s(\theta) = Ad_{g_{bt}(\theta)} J_{bt}^b(\theta)$$

- ▶ Recall the geometric interpretation

## Dynamics of rigid body manipulators

- ▶ Kinetic energy = sum of the kinetic energy of each link . For the  $i$ th link

$$T_i(\theta, \dot{\theta}) = \frac{1}{2}(V_{bl_i}^b)^T \mathcal{M}_i(V_{bl_i}^b) = \frac{1}{2}\dot{\theta}^T [J_{bl_i}^b(\theta)^T \mathcal{M}_i J_{bl_i}^b(\theta)]\dot{\theta}$$

- ▶ Total kinetic energy

$$T(\theta, \dot{\theta}) = \sum_{i=1}^n \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}$$

- ▶ Manipulator inertia matrix

$$M(\theta) = \sum_{i=1}^n J_{bl_i}^b(\theta)^T \mathcal{M}_i J_{bl_i}^b(\theta)$$

# The Lagrangian

- ▶ Potential energy

$$V(\theta) = \sum_{i=1}^n m_i g h_i(\theta)$$

- ▶ Lagrangian -  $\mathcal{L}(\theta, \dot{\theta}) =$

$$\frac{1}{2} \sum_{i,j=1}^n M_{ij}(\theta) \dot{\theta}_i \dot{\theta}_j - V(\theta)$$

- ▶ Equations of motion for each link

$$\sum_{j=1}^n M_{ij}(\theta) \ddot{\theta}_j + \sum_{j,k=1}^n \Gamma_{ijk}(\theta) \dot{\theta}_j \dot{\theta}_k + \frac{\partial V}{\partial \theta_i}(\theta) = - i \quad i = 1, \dots, n$$

## Interpretation of the equations of motion

- ▶ Christoffel symbols corresponding to the inertia matrix  $M(\theta)$ .

$$\Gamma_{ijk} = \frac{1}{2} \left[ \frac{\partial M_{ij}(\theta)}{\partial \theta_k} + \frac{\partial M_{ik}(\theta)}{\partial \theta_j} - \frac{\partial M_{kj}(\theta)}{\partial \theta_i} \right]$$

- ▶ Coriolis matrix

$$C_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^n \Gamma_{ijk} \dot{\theta}_k$$

- ▶ Coriolis and centrifugal forces

$$C(\theta, \dot{\theta})\dot{\theta}$$

- ▶ Equations of motion are often expressed as

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + N(\theta, \dot{\theta}) = \tau$$

## Geometric viewpoint

- ▶ The dynamic equations are described on the manifold  $TQ$ .
- ▶ An element of  $TQ$  is a two tuple  $(\theta, \dot{\theta})$ .
- ▶ The vector field on  $TQ$  is given by

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -M^{-1}(\theta)[C(\theta, \dot{\theta})\dot{\theta} + N(\theta, \dot{\theta})] \end{bmatrix}$$

- ▶ The Lagrangian is a map  $\mathcal{L} : TQ \rightarrow \mathbb{R}$ .
- ▶ The KE is a metric on the tangent space  $T_q Q$  induced by the inertia matrix  $M(q)$  as

$$\langle \dot{\theta}_i, \dot{\theta}_j \rangle \triangleq \frac{1}{2} \sum_{i,j=1}^n M_{ij}(\theta) \dot{\theta}_i \dot{\theta}_j$$

- ▶ The inertia matrix  $M(q)$  is symmetric and positive definite and  $\dot{M} - 2C$  is skew-symmetric

## Computed-torque control of open-chain manipulators

- ▶ Assuming  $\theta$  and  $\dot{\theta}$  are measurable, for a desired trajectory  $\theta(t)_d$ , the control law is

$$\tau_{ff} = M(\theta)\ddot{\theta}_d + C(\theta, \dot{\theta})\dot{\theta} + N(\theta, \dot{\theta})$$

- ▶ To account for modelling uncertainties we modify the control law as

$$\tau_{fc} = \tau_{oc} - \underbrace{M(\theta)[K_p(\theta - \theta_d) - K_v(\dot{\theta} - \dot{\theta}_d)]}_{\text{feedback - Proportional and damping terms}}$$

- ▶ The error equation is

$$\ddot{e} + K_v \dot{e} + K_p e = 0$$