

Homogeneous coordinates in \mathcal{R}^3

- A point \mathbf{X} in \mathcal{R}^3 is presented by a 4-vector. The homogeneous vector $\mathbf{X} = (X_1, X_2, X_3, X_4)^\top$ with $X_4 \neq 0$ corresponds to the Cartesian point $(X, Y, Z)^\top$ in \mathcal{R}^3 with

$$X = X_1/X_4, Y = X_2/X_4, Z = X_3/X_4.$$

- The 3D point $(X, Y, Z)^\top$ has homogeneous representation $\mathbf{X} = (X, Y, Z, 1)^\top$.
- Homogeneous points with $X_4 = 0$ correspond to ideal points.
- A projective transformation on \mathcal{P}^3 is a linear transformation operating on homogeneous 4-vectors and is represented by a non-singular 4×4 matrix

$$\mathbf{X}' = \mathbf{H}\mathbf{X}.$$

- The matrix \mathbf{H} has 16 elements and 15 degrees of freedom.
- A homography in \mathcal{P}^3 maps lines onto lines and preserves intersections between e.g. lines and planes.

3D homographies – p. 1

Three points define a plane

- Assume we have three distinct points \mathbf{X}_i in the plane π . Each point \mathbf{X}_i has to satisfy $\pi^\top \mathbf{X}_i = 0, i = 1, \dots, 3$ or

$$\begin{bmatrix} \mathbf{X}_1^\top \\ \mathbf{X}_2^\top \\ \mathbf{X}_3^\top \end{bmatrix} \pi = \mathbf{0}.$$

- The plane π is calculated from the null-space of the matrix.
- Example: $\mathbf{X}_1 = (0, 0, 0, 1)^\top$, $\mathbf{X}_2 = (1, 0, 0, 1)^\top$, $\mathbf{X}_3 = (0, 1, 0, 1)^\top$. The plane through these points are given by

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \pi = \mathbf{0} \Rightarrow \pi = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

i.e. the plane $Z = 0$.

3D homographies – p. 3

Planes in \mathcal{P}^3

- A plane in \mathcal{R}^3 may be written as

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

with homogeneous representation

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)^\top$$

in \mathcal{P}^3 .

- The incidence expression $\pi^\top \mathbf{X} = 0$ means that the point \mathbf{X} is on the plane π and that the plane π intersects the point \mathbf{X} .

3D homographies – p. 2

Three plane defines a point

- Similarly three distinct planes π_i define an intersection point as the solution of

$$\begin{bmatrix} \pi_1^\top \\ \pi_2^\top \\ \pi_3^\top \end{bmatrix} \mathbf{X} = \mathbf{0}.$$

- Example: $\pi_1 = (1, 0, 0, -3)^\top$, $\pi_2 = (1, 1, 0, -2)^\top$, $\pi_3 = (0, 1, 1, 0)^\top$. The intersection between the planes is given by

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \mathbf{X} = \mathbf{0} \Rightarrow \mathbf{X} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

i.e. the points $(X, Y, Z)^\top = (3, -1, 1)^\top$.

- Planes are dual to points in \mathcal{P}^3 .

3D homographies – p. 3

Transformations of a plane

- Under a point transformation $\mathbf{X}' = \mathbf{H}\mathbf{X}$ planes are transformed as

$$\pi' = \mathbf{H}^{-\top} \pi.$$

Parameterization of points in a plane

- All points in a plane π may be written as

$$\mathbf{X} = \mathbf{M}\mathbf{x},$$

where the columns of the 4×3 matrix \mathbf{M} spans the 3-dimensional null-space of π^\top , i.e. $\pi^\top \mathbf{M} = \mathbf{0}$.

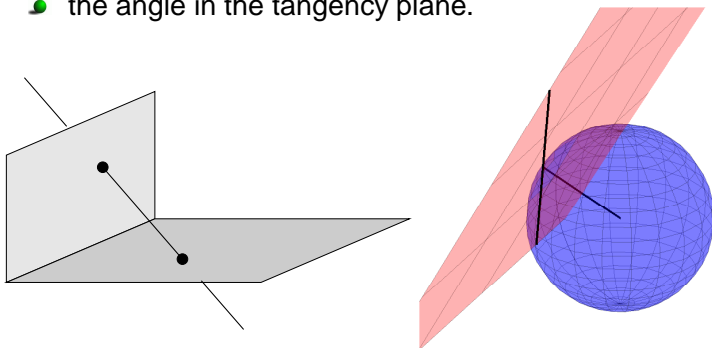
- The 3-vector \mathbf{x} is a homogeneous point in \mathbb{P}^2 .

3D homographies – p. 5

3D homographies – p. 5

Lines in \mathcal{P}^3

- A line is defined as the intersection of two planes or the *join* of two points.
- A line in \mathcal{P}^3 has 4 degrees of freedom. These may be interpreted as the intersection points of two orthogonal planes or
 - the shortest distance to the origin (sphere with radius),
 - latitude and longitude for a tangent point, and
 - the angle in the tangency plane.



3D homographies – p. 7

3D homographies – p. 7

Line representations

- A line may be represented as the range space of two vectors. Let \mathbf{A} and \mathbf{B} be two distinct points in \mathcal{P}^3 . The intersecting line is the linear combination of the two. Let

$$\mathbf{W} = \begin{bmatrix} \mathbf{A}^\top \\ \mathbf{B}^\top \end{bmatrix}.$$

- The line may also be represented as the intersection of two planes. Let \mathbf{P} and \mathbf{Q} be two distinct planes and

$$\mathbf{W}^* = \begin{bmatrix} \mathbf{P}^\top \\ \mathbf{Q}^\top \end{bmatrix}.$$

Line representations

Then

- the range space of \mathbf{W}^\top contains all points $\lambda\mathbf{A} + \mu\mathbf{B}$ on the line,
- the null-space of \mathbf{W} is the bundle of planes having the line as its axis,
- the range space of $\mathbf{W}^{*\top}$ is the bundle of planes $\lambda'\mathbf{P} + \mu'\mathbf{Q}$ with the line as its axis,
- the null-space of \mathbf{W}^* contains all points on the line,
- the matrix pairs $\mathbf{W}, \mathbf{W}^{*\top}$ and $\mathbf{W}^*, \mathbf{W}^\top$ spans each other's null-space, $\mathbf{W}\mathbf{W}^{*\top} = \mathbf{W}^*\mathbf{W}^\top = \mathbf{0}_{2 \times 2}$.

The line representations \mathbf{W} and \mathbf{W}^* are dual.

3D homographies – p. 9

Range space and intersection

- The plane π spanned by the line \mathbf{W} and the point \mathbf{X} is given by the null-space of

$$\mathbf{M} = \begin{bmatrix} \mathbf{W} \\ \mathbf{X}^\top \end{bmatrix}.$$

- If the null-space is 2-dimensional the point \mathbf{X} is on the line \mathbf{W} and the plane is not unique.
- The point \mathbf{X} being the intersection between the line \mathbf{W} and the plane π is given by the null-space of

$$\mathbf{M}^* = \begin{bmatrix} \mathbf{W}^* \\ \pi^\top \end{bmatrix}.$$

- If the null-space is 2-dimensional the line \mathbf{W} is contained in the plane π and the point is not unique.

3D homographies – p. 11

Example

- The X -axis may be represented by

$$\mathbf{W} = \begin{bmatrix} \mathbf{A}^\top \\ \mathbf{B}^\top \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{W}^* = \begin{bmatrix} \mathbf{P}^\top \\ \mathbf{Q}^\top \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

- All points on the X axis are described by

$$\mathbf{X} = \lambda\mathbf{A} + \mu\mathbf{B} = \begin{bmatrix} \lambda \\ 0 \\ 0 \\ \lambda + \mu \end{bmatrix}$$

since all points in the plane satisfy $\mathbf{P} : Z = 0$ and $\mathbf{Q} : Y = 0$.

3D homographies – p. 11

Plücker matrices

- A *Plücker matrix* \mathbf{L} is a skew symmetric line representation of the join between two points

$$\mathbf{L} = \mathbf{A}\mathbf{B}^\top - \mathbf{B}\mathbf{A}^\top.$$

- The matrix \mathbf{L} has rank 2. The null-space spans the pencil of planes with \mathbf{L} as their axis.
- The matrix \mathbf{L} has 4 degrees of freedom — skew symmetric 4×4 matrix with $|\mathbf{L}| = 0$.
- The matrix \mathbf{L} is independent of the points \mathbf{A}, \mathbf{B} used to define it. If another point $\mathbf{C} = \mathbf{A} + \mu\mathbf{B}$ was used, the resulting matrix would be

$$\hat{\mathbf{L}} = \mathbf{A}\mathbf{C}^\top - \mathbf{C}\mathbf{A}^\top = \mathbf{A}(\mathbf{A}^\top + \mu\mathbf{B}^\top) - (\mathbf{A} + \mu\mathbf{B})\mathbf{A}^\top = \mu(\mathbf{A}\mathbf{B}^\top - \mathbf{B}\mathbf{A}^\top) = \mu\mathbf{L}$$

- Under a point transformation $\mathbf{X}' = \mathbf{H}\mathbf{X}$, the matrix transforms as

$$\mathbf{L}' = \mathbf{H}\mathbf{L}\mathbf{H}^\top.$$

3D homographies – p. 11

Dual Plücker matrices

- A dual Plücker matrix L^* is a skew symmetric line representation of the intersection between two planes

$$L^* = PQ^\top - QP^\top.$$

- Under a point transformation $X' = HX$, the matrix transforms as

$$L^{*'} = H^{-\top} L^* H^{-1}.$$

- The matrix L^* can be calculated directly from L

$$\begin{aligned}\ell_{12} &= \ell_{34}^* \\ \ell_{13} &= \ell_{42}^* \\ \ell_{14} &= \ell_{23}^* \\ \ell_{23} &= \ell_{14}^* \\ \ell_{42} &= \ell_{13}^* \\ \ell_{34} &= \ell_{12}^*\end{aligned}$$

3D homographies – p. 13

Plücker line coordinates

- Plücker line coordinates are the 6 non-zeros elements of the Plücker matrix L

$$\mathcal{L}(L) = \begin{bmatrix} \ell_{12} \\ \ell_{13} \\ \ell_{14} \\ \ell_{23} \\ \ell_{42} \\ \ell_{34} \end{bmatrix} \Leftrightarrow L(\mathcal{L}) = \begin{bmatrix} 0 & \ell_1 & \ell_2 & \ell_3 \\ -\ell_1 & 0 & \ell_4 & -\ell_5 \\ -\ell_2 & -\ell_4 & 0 & \ell_6 \\ -\ell_3 & \ell_5 & -\ell_6 & 0 \end{bmatrix}$$

- The constraint $|L| = 0$ translates to

$$\ell_{12}\ell_{34} + \ell_{13}\ell_{42} + \ell_{14}\ell_{23} = 0$$

or

$$\mathcal{L}^\top \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathcal{L} = 0$$

3D homographies – p. 13

Plücker line coordinates as a matrix-vector product

- Given two points A and B , the Plücker line coordinates may be calculated as

$$\mathcal{L} = M(A)B = \begin{bmatrix} -A_2 & A_1 & 0 & 0 \\ -A_3 & 0 & A_1 & 0 \\ -A_4 & 0 & 0 & A_1 \\ 0 & -A_3 & A_2 & 0 \\ 0 & A_4 & 0 & -A_2 \\ 0 & 0 & -A_4 & A_3 \end{bmatrix} B$$

- From two planes P and Q , the dual Plücker line coordinates may be calculated as

$$\mathcal{L}^* = M(P)Q = \begin{bmatrix} -P_2 & P_1 & 0 & 0 \\ -P_3 & 0 & P_1 & 0 \\ -P_4 & 0 & 0 & P_1 \\ 0 & -P_3 & P_2 & 0 \\ 0 & P_4 & 0 & -P_2 \\ 0 & 0 & -P_4 & P_3 \end{bmatrix} Q$$

3D homographies – p. 15

Plücker line coordinates as a matrix-vector product

- The primal and dual Plücker line coordinates are related by

$$\mathcal{L}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathcal{L} = D\mathcal{L}, \quad \mathcal{L} = D\mathcal{L}^*.$$

- Thus, the dual Plücker line coordinates formed from two points A and B may be written as

$$\mathcal{L}^* = DM(A)B.$$

and the primal Plücker line coordinates formed from two planes P and Q as

$$\mathcal{L} = DM(P)Q.$$

3D homographies – p. 15

Plane from line and one point

- The plane π defined by the join of a point X and a line L is given as

$$\pi = L^*X.$$

- The point X is on L iff $L^*X = 0$.
- If the line is constructed from two points A and B

$$\mathcal{L} = M(A)B$$

the plane becomes

$$\pi = L(\mathcal{L}^*)X = L(DM(A)B)X,$$

which is a point-only expression.

3D homographies – p. 17

Example

- Take the points $X_1 = (0, 0, 0, 1)^T$, $X_2 = (1, 0, 0, 1)^T$, $X_3 = (0, 1, 0, 1)^T$.
- Construct the dual Plücker line
- Construct a Plücker line between the first two points

$$\mathcal{L} = M(X_1)X_2 =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$L(\mathcal{L}) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$\mathcal{L}^* = D\mathcal{L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad L(\mathcal{L}^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Calculate the plane from the dual Plücker matrix and the third point.

$$\pi = L(\mathcal{L}^*)X_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

i.e. the plane $Z = 0$.

3D homographies – p. 17

Point from line and one plane

- Similarly, the point X defined by the intersection of the line L with the plane π is given by

$$X = L\pi.$$

- The line \mathcal{L} is on the plane π iff $L\pi = 0$.
- If the line is constructed from two planes P and Q

$$\mathcal{L}^* = M(P)Q$$

the point becomes

$$X = L(\mathcal{L})\pi = L(DM(P)Q)\pi,$$

which is a plane-only expression.

3D homographies – p. 19

Second order surfaces

- A point *quadric* in \mathcal{P}^3 is defined by the equation

$$X^T Q X = 0,$$

where Q is a symmetric 4×4 matrix with 9 degrees of freedom (homogeneous symmetric 4×4 matrix).

- The intersection between a quadric Q and a plane π is a conic C .
- If M is a null-space matrix to π all point in the plane may be written as $X = Mx$. Points on π are on Q if

$$X^T Q X = x^T \underbrace{M^T Q M}_C x = x^T C x = 0$$

i.e. on the conic $C = M^T Q M$ in the plane π .

3D homographies – p. 19

Second order surfaces

- The dual of a point quadric is a plane quadric. Dual quadrics are equations for planes: The tangent plane π of the point quadric Q satisfies $\pi^\top Q^* \pi = 0$, where Q^* is the adjoint matrix to Q ($Q^* = Q^{-1}$ if Q is invertible).

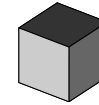
- Under a point transformation $X' = HX$, quadrics are transformed as

$$Q' = H^{-\top} Q H^{-1} \text{ and } Q^{*'} = H Q^* H^\top.$$

Transformation hierarchies for \mathcal{P}^3

Euclidean
6 d.o.f.

$$\begin{bmatrix} R & t \\ 0^\top & 1 \end{bmatrix}$$



volume

Similar
7 d.o.f.

$$\begin{bmatrix} sR & t \\ 0^\top & 1 \end{bmatrix}$$



the absolute conic Ω_∞

Affine
12 d.o.f.

$$\begin{bmatrix} A & t \\ 0^\top & 1 \end{bmatrix}$$



parallelity, the plane at infinity π_∞

Projective
15 d.o.f.

$$\begin{bmatrix} A & t \\ v^\top & v \end{bmatrix}$$



contact points

where the matrix A is an invertible 3×3 matrix, R is a 3D orthogonal matrix, t is a translation 3-vector, v is a 3-vector, v and s are scalars, and $0 = (0, 0, 0)^\top$.

The plane at infinity

- The plane at infinity π_∞ has canonical form

$$\pi_\infty = (0, 0, 0, 1)^\top$$

and contains all directions

$$D = (X_1, X_2, X_3, 0)^\top$$

in \mathcal{P}^3 .

- Two planes are parallel iff their intersection line is on π_∞ .
- A line is parallel with a plane or a line iff their intersection is on π_∞ .
- The plane at infinity is fixed under an affine transformation.

The absolute conic Ω_∞

- The absolute conic Ω_∞ is a point conic on π_∞ . Points on Ω_∞ satisfies

$$\left. \begin{array}{l} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{array} \right\} = 0$$

and describes a relation between all directions in \mathcal{P}^3 , i.e. points with $X_4 = 0$:

$$(X_1, X_2, X_3)I(X_1, X_2, X_3)^\top = 0.$$

- All sphere intersect π_∞ on Ω_∞ .

Metric properties for Ω_∞

- If π_∞ and Ω_∞ are known it is possible to calculate angles between lines.
- Let $(\mathbf{d}_1^\top, 0)^\top$ and $(\mathbf{d}_2^\top, 0)^\top$ be directions in \mathcal{P}^3 . The angle θ between \mathbf{d}_1 and \mathbf{d}_2 is given by

$$\cos \theta = \frac{\mathbf{d}_1^\top \mathbf{d}_2}{\sqrt{(\mathbf{d}_1^\top \mathbf{d}_1)(\mathbf{d}_2^\top \mathbf{d}_2)}}$$

that may be re-written as

$$\cos \theta = \frac{\mathbf{d}_1^\top \Omega_\infty \mathbf{d}_2}{\sqrt{(\mathbf{d}_1^\top \Omega_\infty \mathbf{d}_1)(\mathbf{d}_2^\top \Omega_\infty \mathbf{d}_2)}}$$

- Thus, if the projection Ω'_∞ of Ω_∞ is known we can decide if two projected lines \mathbf{d}'_1 and \mathbf{d}'_2 are orthogonal.

3D homographies – p. 25

The absolute dual quadric Q_∞^*

- The absolute dual quadric Q_∞^* is represented by a 4×4 homogeneous matrix of rank 3

$$Q_\infty^* = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}.$$

- The angle θ between two planes π_1 and π_2 is given by

$$\cos \theta = \frac{\pi_1^\top Q_\infty^* \pi_2}{\sqrt{(\pi_1^\top Q_\infty^* \pi_1)(\pi_2^\top Q_\infty^* \pi_2)}}.$$

- Thus, if the projection Q_∞^{*f} is known we may decide if two projected planes π'_1 and π'_2 are orthogonal.

3D homographies – p. 26

Metric properties of Q_∞^*

- Q_∞^* is preserved by a similarity transformation only.
- Under a general homography

$$H = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & k \end{bmatrix}$$

Q_∞^* is mapped onto $Q_\infty^{*f} = H Q_\infty^* H^\top$, i.e.

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & k \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}^\top & \mathbf{v} \\ \mathbf{t}^\top & k \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{A}^\top & \mathbf{A}\mathbf{v} \\ \mathbf{v}^\top \mathbf{A}^\top & \mathbf{v}^\top \mathbf{v} \end{bmatrix}.$$

- This equality (up to scale) is only true if $\mathbf{v} = \mathbf{0}$ and \mathbf{A} is a scaled orthogonal matrix, i.e. a similarity.
- The plane at infinity π_∞ is a null-vector of Q_∞^* .

3D homographies – p. 27