Geometry of Vectors

Carlo Tomasi

This note explores the geometric meaning of norm, inner product, orthogonality, and projection for vectors. For vectors in three-dimensional space, we also examine the geometry of cross product and triple product. All these concepts are often introduced algebraically in textbooks, but understanding their geometry is particularly useful for computer vision.

1 Cartesian Coordinates

The *law of cosines*[1] states that if a, b, c are the sides of a triangle and the angle between a and b is θ, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta \; .$$

The special case for $\theta = \pi/2$ radians is known as Pythagoras' theorem.

The definitions that follow focus on three-dimensional space. Two-dimensional geometry can be derived as a special case when the third coordinate of every point is set to zero.

A *Cartesian reference system* for three-dimensional space is a point in space called the *origin* and three mutually perpendicular, directed lines though the origin called the *axes*. The order in which the axes are listed is fixed, and is part of the definition of the reference system. The plane that contains the second and third axis is the *first reference plane*. The plane that contains the third and firs axis is the *second reference plane*. The plane that contains the first and second axis is the *third reference plane*.

It is customary to mark the axis directions by specifying a point on each axis and at unit distance from the origin. These points are called the *unit points* of the system, and the *positive direction* of an axis is from the origin towards the axis' unit point. A Cartesian reference system is *right-handed* if the smallest rotation that brings the first unit point to the second is counterclockwise when viewed from the third unit point. The system is *left-handed* otherwise.

The *Cartesian coordinates* of a point in three-dimensional space are the signed distances of the point from the first, second, and third reference plane, in this order, and collected into a vector. The sign for coordinate i is positive if the point is in the half-space (delimited by the i-th reference plane) that contains the positive half of the i-th reference axis. It follows that the Cartesian coordinates of the origin are $\mathbf{o} = (0, 0, 0)$, those of the unit points are the vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$, and the vector $\mathbf{p} = (x, y, z)$ of coordinates of an arbitrary point in space can also be written as follows:

$$\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \; .$$

The point $\mathbf{p}$ can be reached from the origin by the following polygonal path:

$$\mathbf{o} \, , \; x\mathbf{e}_1 \, , \; x\mathbf{e}_1 + y\mathbf{e}_2 \, , \; \mathbf{p} \; .$$

[1] A proof of this law based on trigonometry is straightforward but tedious, and a useful exercise.

Each segment of the path is followed by a right-angle turn, so Pythagoras' theorem can be applied twice to yield the distance of $\mathbf{p}$ from the origin:

$$d(\mathbf{o}, \mathbf{p}) = \sqrt{x^2 + y^2 + z^2} \ .$$

From the definition of norm of a vector we see that

$$d(\mathbf{o}, \mathbf{p}) = \|\mathbf{p}\| \ .$$

So the norm of the vector of coordinates of a point is the distance of the point from the origin. A vector is often drawn as an arrow pointing from the origin to the point whose coordinates are the components of the vector. Then, the result above shows that the *length* of that arrow is the norm of the vector. Because of this, the words "length" and "norm" are often used interchangeably.

2 Inner Product and Orthogonality

The law of cosines yields a geometric interpretation of the inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a}^T\mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \cos\theta \tag{1}$$

where θ is the acute angle between the two arrows that represent $\mathbf{a}$ and $\mathbf{b}$ geometrically. So the inner product of two vectors is the product of the lengths of the two arrows that represent them and of the cosine of the angle between them.

To prove this result, consider a triangle with sides

$$a = \|\mathbf{a}\| \quad , \quad b = \|\mathbf{b}\| \quad , \quad c = \|\mathbf{b} - \mathbf{a}\|$$

and with an angle θ between a and b. Then the law of cosines yields

$$\|\mathbf{b} - \mathbf{a}\|^2 \ = \ \|\mathbf{a}\|^2 \ + \ \|\mathbf{b}\|^2 \ - \ 2\|\mathbf{a}\| \, \|\mathbf{b}\| \, \cos\theta \ .$$

From the definition of norm we then obtain

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b} \ = \ \|\mathbf{a}\|^2 \ + \ \|\mathbf{b}\|^2 \ - \ 2\|\mathbf{a}\| \, \|\mathbf{b}\| \, \cos\theta \ .$$

Canceling equal terms and dividing by -2 yields the desired result.

Setting $\theta = \pi/2$ in the result above yields another important corollary:

The arrows that represent two vectors $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular if an only if the two vectors are orthogonal:

$$\mathbf{a}^T\mathbf{b} = 0 \ .$$

Because of this result, the words "perpendicular" and "orthogonal" are often used interchangeably.

3 Vector Projection

Given two vectors $\mathbf{a}$ and $\mathbf{b}$, the *projection*[2] of $\mathbf{a}$ onto $\mathbf{b}$ is the vector $\mathbf{p}$ that represents the point p on the line through $\mathbf{b}$ that is nearest to the endpoint of $\mathbf{a}$. See figure 1.

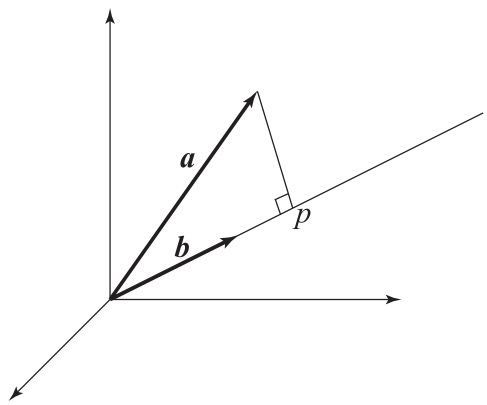

Figure 1: The vector from the origin to point p is the projection of $\mathbf{a}$ onto $\mathbf{b}$. The line from the endpoint of $\mathbf{a}$ to p is orthogonal to $\mathbf{b}$.

The projection of $\mathbf{a}$ onto $\mathbf{b}$ is the vector

$$\mathbf{p} = P_{\mathbf{b}}\mathbf{a}$$

where $P_{\mathbf{b}}$ is the following square, symmetric matrix:

$$P_{\mathbf{b}} = \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}} \; .$$

The magnitude of the projection is

$$\|\mathbf{p}\| = \frac{|\mathbf{b}^T\mathbf{a}|}{\|\mathbf{b}\|} \; . \qquad (2)$$

To prove this, observe that since by definition point p is on the line through $\mathbf{b}$, the projection vector $\mathbf{p}$ has the form $\mathbf{p} = x\mathbf{b}$, where x is some real number. From elementary geometry, the line between p and the endpoint of $\mathbf{a}$ is shortest when it is perpendicular to $\mathbf{b}$:

$$\mathbf{b}^T(\mathbf{a} - x\mathbf{b}) = 0$$

which yields

$$x = \frac{\mathbf{b}^T\mathbf{a}}{\mathbf{b}^T\mathbf{b}}$$

so that

$$\mathbf{p} = x\mathbf{b} = \mathbf{b}\,x = \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}}\,\mathbf{a}$$

[2] The term “vector projection” may be used to avoid confusion with the very different notion of camera projection.

as advertised. The magnitude of $\mathbf{p}$ can be computed as follows. First, observe that

$$P_{\mathbf{b}}^2 = \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}} \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}} = \frac{\mathbf{b}\mathbf{b}^T\mathbf{b}\mathbf{b}^T}{(\mathbf{b}^T\mathbf{b})^2} = \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}} = P_{\mathbf{b}}$$

so that the projection matrix $P_{\mathbf{b}}$ is *idempotent*:

$$P_{\mathbf{b}}^2 = P_{\mathbf{b}} \; .$$

This means that applying the matrix once or multiple times has the same effect. Then,

$$\|\mathbf{p}\|^2 = \mathbf{p}^T\mathbf{p} = \mathbf{a}^T P_{\mathbf{b}}^T P_{\mathbf{b}}\mathbf{a} = \mathbf{a}^T P_{\mathbf{b}} P_{\mathbf{b}}\mathbf{a} = \mathbf{a}^T P_{\mathbf{b}}\mathbf{a} = \mathbf{a}^T \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}}\mathbf{a} = \frac{(\mathbf{b}^T\mathbf{a})^2}{\mathbf{b}^T\mathbf{b}}$$

which yields equation (2).

The *scalar projection* is defined as

$$\frac{\mathbf{b}^T\mathbf{a}}{\|\mathbf{b}\|}$$

and is positive if and only if the angle between $\mathbf{a}$ and $\mathbf{b}$ has a magnitude smaller than $\pi/2$.

From the definition of projection we also see the following fact.

> The coordinates of a point in space are the projections of the vector of coordinates of the point onto the three unit vectors that define the coordinate axes.

This result is trivial in the basic Cartesian reference frame with unit points $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$. If $\mathbf{p} = (x, y, z)$, then obviously

$$\mathbf{e}_1\mathbf{p} = x \quad , \quad \mathbf{e}_2\mathbf{p} = y \quad , \quad \mathbf{e}_3\mathbf{p} = z \; .$$

The result becomes less trivial in Cartesian reference systems where the axes have different orientations, as we will see soon.

4 Cross Product and Triple Product

The *cross product* of two 3-dimensional vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is the 3-dimensional vector

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2 \, , \; a_3b_1 - a_1b_3 \, , \; a_1b_2 - a_2b_1) \; .$$

The cross product $\mathbf{c}$ of $\mathbf{a}$ and $\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned} \mathbf{c}^T\mathbf{a} &= (a_2b_3 - a_3b_2)a_1 \; + \; (a_3b_1 - a_1b_3)a_2 \; + \; (a_1b_2 - a_2b_1)a_3 = 0 \\ \mathbf{c}^T\mathbf{b} &= (a_2b_3 - a_3b_2)b_1 \; + \; (a_3b_1 - a_1b_3)b_2 \; + \; (a_1b_2 - a_2b_1)b_3 = 0 \end{aligned}$$

(check that all terms do indeed cancel).

The vector $\mathbf{c}$ is oriented so that the triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is right-handed. This fact is not proven here.

If θ is the acute angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$(\mathbf{a}^T\mathbf{b})^2 \; + \; \|\mathbf{a} \times \mathbf{b}\|^2 \; = \; \|\mathbf{a}\|^2 \, \|\mathbf{b}\|^2$$

as can be shown by straightforward manipulation:

$$
\begin{aligned}
(\mathbf{a}^T\mathbf{b})^2 &= (a_1b_1 + a_2b_2 + a_3b_3)\,(a_1b_1 + a_2b_2 + a_3b_3) \\
&= \;\; a_1^2b_1^2 + a_1b_1a_2b_2 + a_1b_1a_3b_3 \\
&\quad + a_2^2b_2^2 + a_1b_1a_2b_2 + a_2b_2a_3b_3 \\
&\quad + a_3^2b_3^2 + a_1b_1a_3b_3 + a_2b_2a_3b_3 \\
&= a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1b_1a_2b_2 + 2a_2b_2a_3b_3 + 2a_1b_1a_3b_3
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathbf{a}\times\mathbf{b}\|^2 &= (a_2b_3 - a_3b_2)^2 \;+\; (a_3b_1 - a_1b_3)^2 \;+\; (a_1b_2 - a_2b_1)^2 \\
&= \;\; a_2^2b_3^2 + a_3^2b_2^2 - 2a_2b_2a_3b_3 \\
&\quad + a_1^2b_3^2 + a_3^2b_1^2 - 2a_1b_1a_3b_3 \\
&\quad + a_1^2b_2^2 + a_2^2b_1^2 - 2a_1b_1a_2b_2 \\
&= \;\; a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_2^2 + a_1^2b_3^2 + a_3^2b_1^2 \\
&\quad - 2a_1b_1a_2b_2 - 2a_2b_3a_2b_2 - 2a_1b_1a_3b_3
\end{aligned}
$$

so that

$$
(\mathbf{a}^T\mathbf{b})^2 \;+\; \|\mathbf{a}\times\mathbf{b}\|^2 \;=\; a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2
$$

but also

$$
\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 \;=\; a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2
$$

so that

$$
(\mathbf{a}^T\mathbf{b})^2 \;+\; \|\mathbf{a}\times\mathbf{b}\|^2 \;=\; \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 \tag{3}
$$

as desired. The following geometric interpretation of the cross product then follows:

> The cross product of two three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector $\mathbf{c}$ orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, oriented so that the triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is right-handed, and with magnitude
>
> $$\|\mathbf{c}\| \;=\; \|\mathbf{a}\times\mathbf{b}\| \;=\; \|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta \tag{4}$$
>
> where θ is the acute angle between $\mathbf{a}$ and $\mathbf{b}$.

The orthogonality of $\mathbf{c}$ to both $\mathbf{a}$ and $\mathbf{b}$ and right-handedness of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ has already been shown. The result on the magnitude is a consequence of equation (3). From this equation we obtain

$$
\|\mathbf{a}\times\mathbf{b}\|^2 \;=\; \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 \,-\, (\mathbf{a}^T\mathbf{b})^2 \;=\; \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 \,-\, \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2\,\cos^2\theta \;=\; \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2\,\sin^2\theta
$$

or

$$
\|\mathbf{a}\times\mathbf{b}\| \;=\; \pm\|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta\,.
$$

Since the angle θ is acute (from equation (1)), all quantities in the last equation are nonnegative, so that the $-$ sign yields an impossible equation. This results in equation (4).

From its expression, we see that the magnitude of $\mathbf{a} \times \mathbf{b}$ is the area of a rectangle with sides $\mathbf{a}$ and $\mathbf{b}$.

Suppose that we need to compute cross products of the form $\mathbf{a} \times \mathbf{p}$ where $\mathbf{a}$ is a fixed vector but $\mathbf{p}$ changes. It is then convenient to write the cross product as the product of a matrix $\mathbf{a}_\times$ that depends on $\mathbf{a}$ and of $\mathbf{p}$. Spelling out the definition of the cross product yields the following anti-symmetric matrix:

$$\mathbf{a}_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} .$$

The *triple product* of three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is defined as follows:

$$\mathbf{a}^T(\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) .$$

It is immediate to verify that

$$\mathbf{a}^T(\mathbf{b} \times \mathbf{c}) = \mathbf{b}^T(\mathbf{c} \times \mathbf{a}) = \mathbf{c}^T(\mathbf{a} \times \mathbf{b}) = -\mathbf{a}^T(\mathbf{c} \times \mathbf{b}) = -\mathbf{c}^T(\mathbf{b} \times \mathbf{a}) = -\mathbf{b}^T(\mathbf{a} \times \mathbf{c}) .$$

Again, from its expression, we see that the triple product of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is the volume of a parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: the cross product $\mathbf{p} = \mathbf{b} \times \mathbf{c}$ is a vector orthogonal to the plane of $\mathbf{b}$ and $\mathbf{c}$, and with magnitude equal to the base of the parallelepiped. The inner product of $\mathbf{p}$ and $\mathbf{a}$ is the magnitude of $\mathbf{p}$ times that of $\mathbf{a}$ times the cosine of the angle between them, that is, the base area of the parallelepiped times its height. This gives the volume of the solid. See Figure 2.

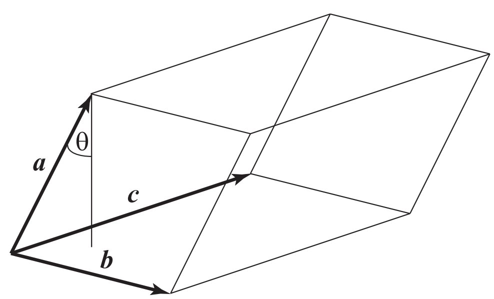

Figure 2: The triple product of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is the volume of the parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.