
Geometric Model of Camera

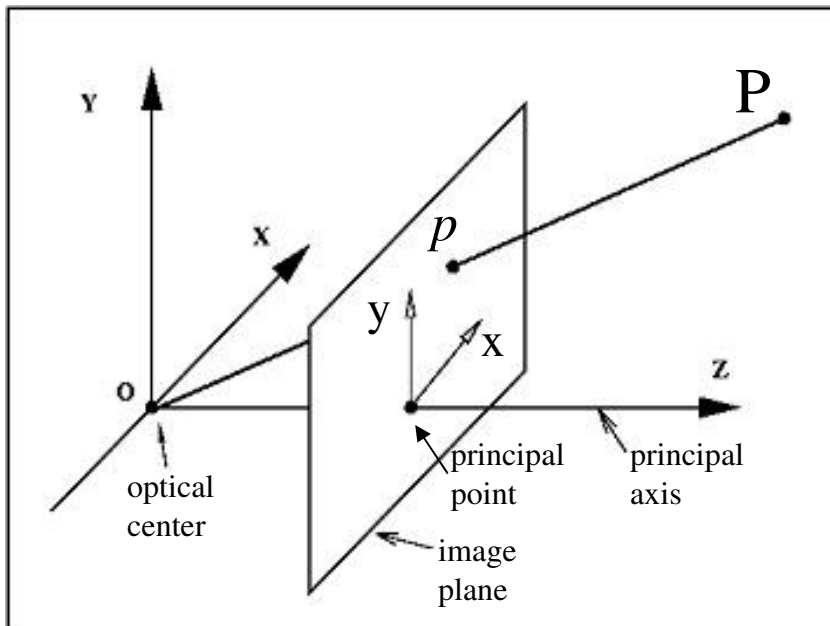
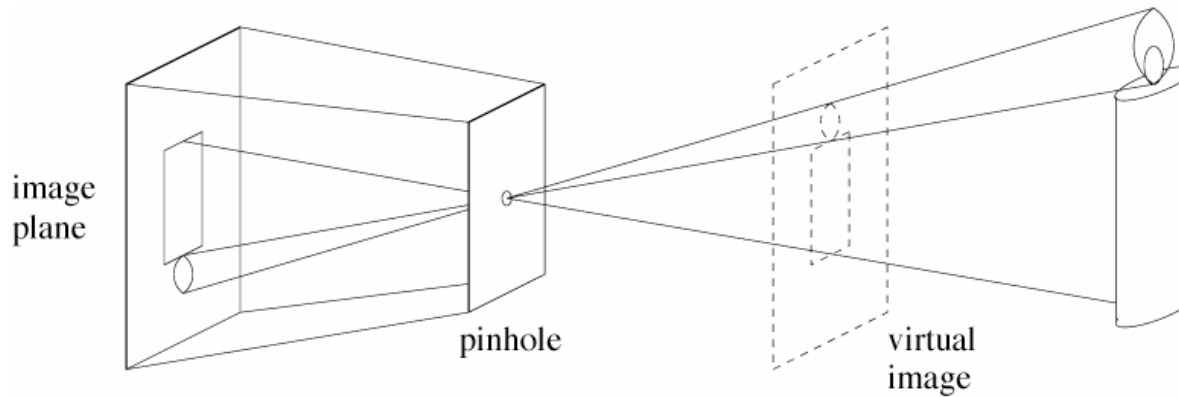
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Geometric Model of Camera

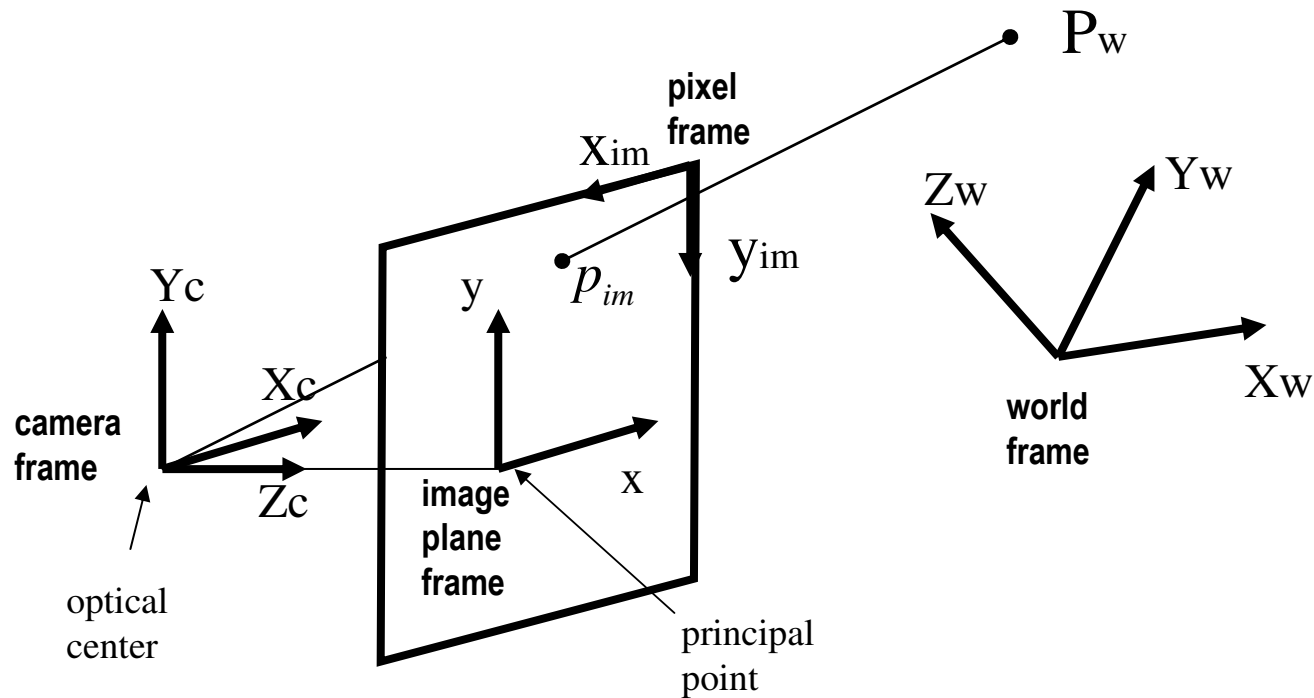
Perspective projection



$$P(X,Y,Z) \rightarrow p(x,y)$$

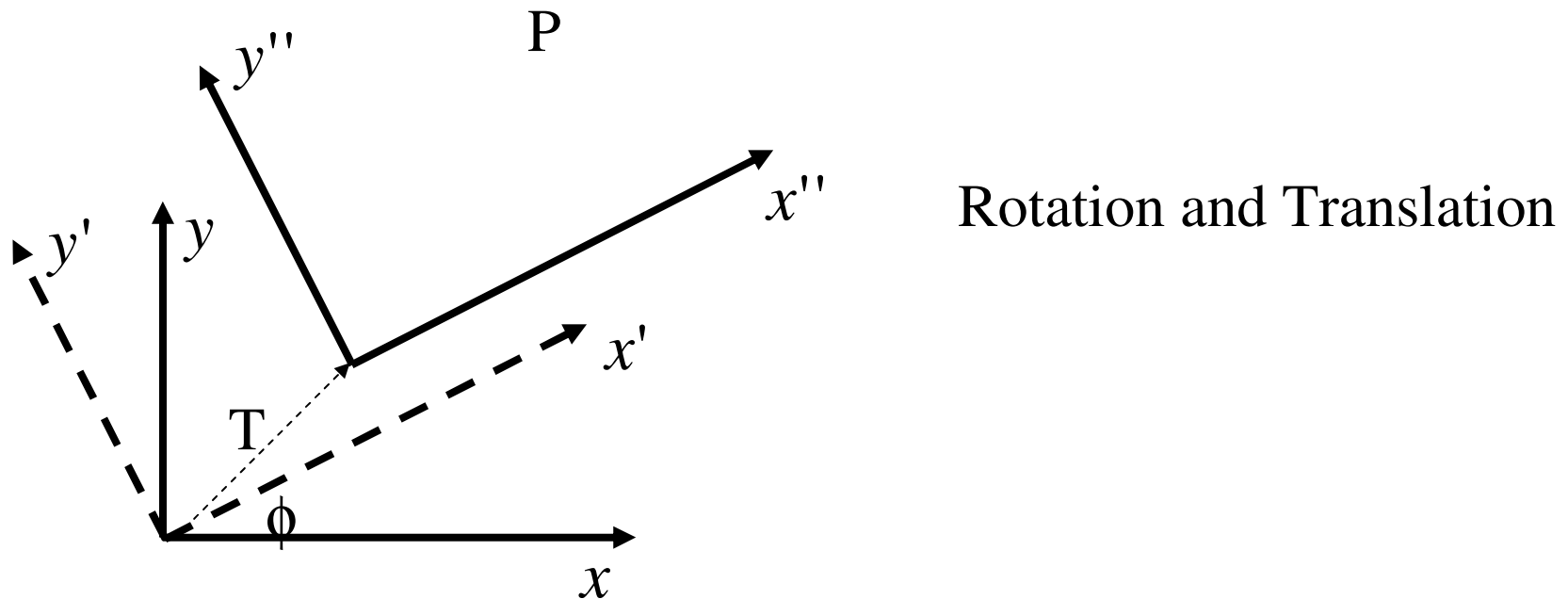
$$x = f \frac{X}{Z} \quad y = f \frac{Y}{Z}$$

Four Coordinate Frames



Camera model:
$$p_{im} = \begin{bmatrix} \text{transformation} \\ \text{matrix} \end{bmatrix} P_w$$

Coordinate Transformation – 2D



$$p' = \begin{bmatrix} p_x' \\ p_y' \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

$$p'' = \begin{bmatrix} p_x' \\ p_y' \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

Homogeneous Coordinates

Go one dimensional higher:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

w is an arbitrary non-zero scalar, usually we choose 1.

From homogeneous coordinates to Cartesian coordinates:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 / x_3 \\ x_2 / x_3 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 / x_4 \\ x_2 / x_4 \\ x_3 / x_4 \end{bmatrix}$$

2D Transformation with Homogeneous Coordinates

2D coordinate transformation:

$$p'' = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

2D coordinate transformation using homogeneous coordinates:

$$\begin{bmatrix} p_x'' \\ p_y'' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & T_x \\ -\sin \phi & \cos \phi & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

3D Rotation Matrix

Rotate around each coordinate axis:

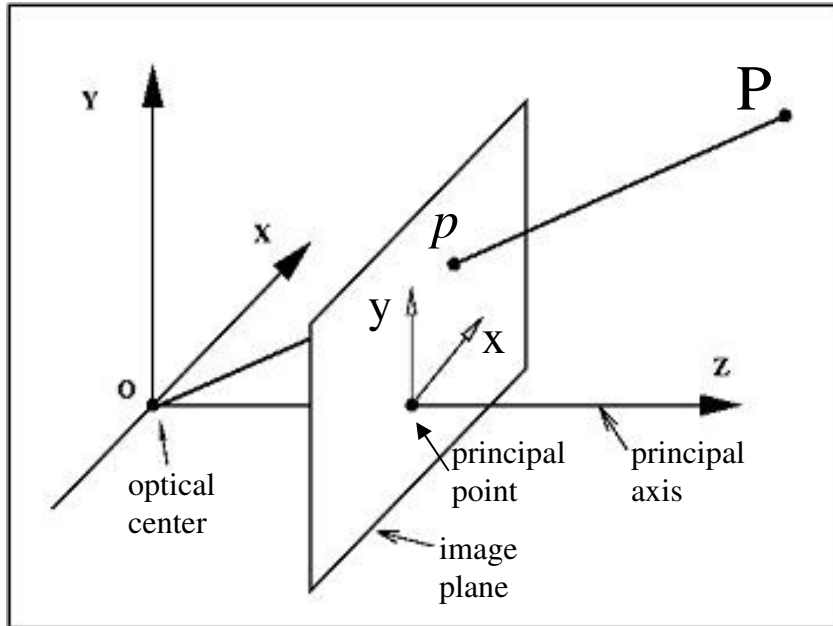
$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \quad R_2(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \quad R_3(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combine the three rotations:

$$R = R_1 R_2 R_3$$

3D rotation matrix has three parameters.

Perspective Projection



$$x = f \frac{X}{Z} \quad y = f \frac{Y}{Z}$$

These are *nonlinear*.

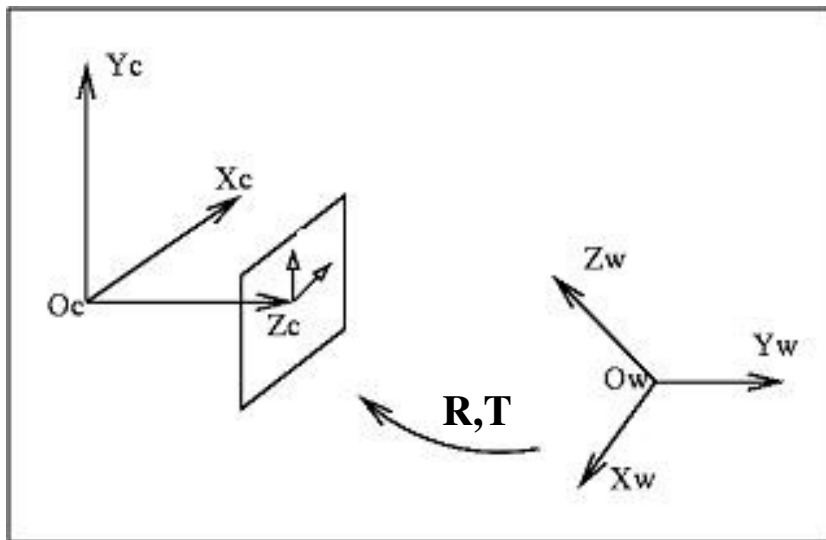
Using homogenous coordinate, we have a *linear* relation:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x = u / w \quad y = v / w$$

World to Camera Coordinate

Transformation between the camera and world coordinates:



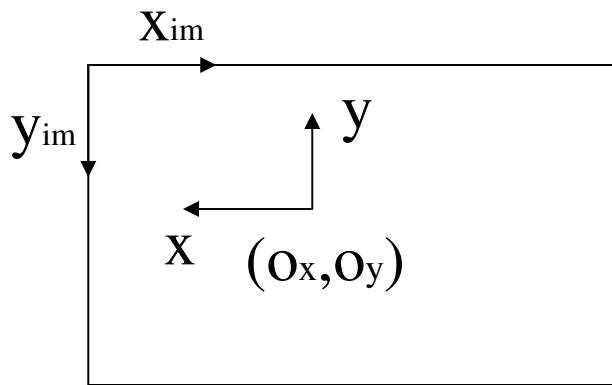
$$\mathbf{X}_c = \mathbf{R}\mathbf{X}_w + \mathbf{T}$$

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Image Coordinates to Pixel Coordinates

$$x = (o_x - x_{im})s_x \quad y = (o_y - y_{im})s_y$$

s_x, s_y : pixel sizes



$$\begin{bmatrix} x_{im} \\ y_{im} \\ 1 \end{bmatrix} = \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Put All Together – World to Pixel

$$\begin{aligned}
 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\
 &= \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = K \begin{bmatrix} R & T \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}
 \end{aligned}$$

$$x_{im} = x_1 / x_3 \quad y_{im} = x_2 / x_3$$

Camera Intrinsic Parameters

$$K = \begin{bmatrix} -f / s_x & 0 & o_x \\ 0 & -f / s_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

K is a 3x3 upper triangular matrix, called the **Camera Calibration Matrix**.

There are five intrinsic parameters:

- (a) The pixel sizes in x and y directions s_x, s_y
- (b) The focal length f
- (c) The principal point (o_x, o_y) , which is the point where the optic axis intersects the image plane.

Extrinsic Parameters

$$p_{im} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K \begin{bmatrix} R & T \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = M \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$[R|T]$ defines the **extrinsic parameters**.

The 3x4 matrix $M = K[R|T]$ is called the **projection matrix**.

Weak Perspective Model

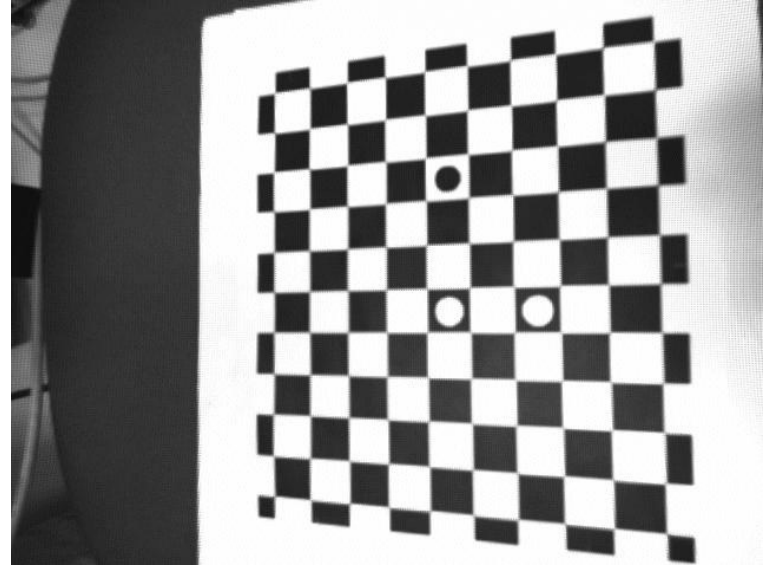
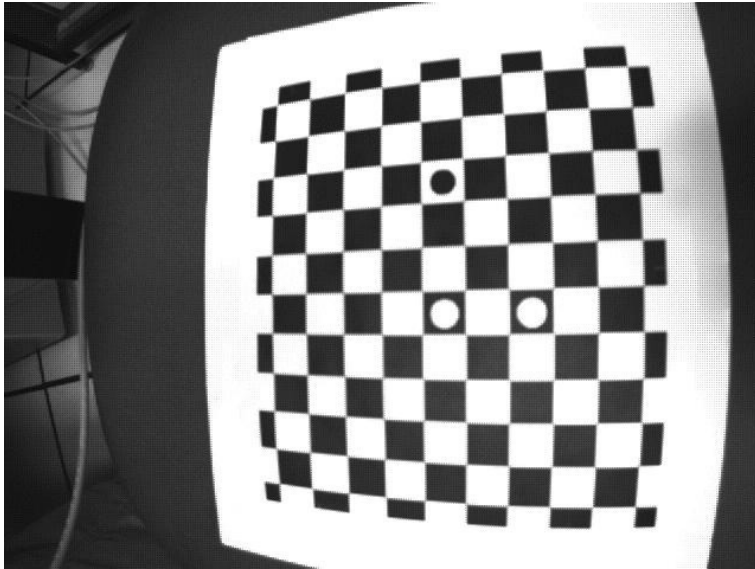
If the relative distance between any two points along the principal axis is much smaller than the average distance \bar{z}
The camera projection can be approximated as:

$$x = f \frac{X}{Z} \approx \frac{f}{\bar{Z}} X$$

$$y = f \frac{Y}{Z} \approx \frac{f}{\bar{Z}} Y$$

This is the **weak-perspective** camera model.

Radial Distortions



$$x = x_d (1 + k_1 r^2 + k_2 r^4)$$

$$y = y_d (1 + k_1 r^2 + k_2 r^4)$$

(x_d, y_d) are the coordinates of the distorted points, and

$$r^2 = x_d^2 + y_d^2$$