Robotics, Geometry and Control - Differential Geometry

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The material for these slides is largely taken from the *texts*

- Calculus on manifolds - Vol. 1, by M. Spivak, published by W. A. Benjamin, Inc., New York, 1965.
- Topology, geometry and gauge fields - Vol. 1 - by G. L. Naber, Springer, New York, 1998.
- Ordinary Differential Equations - V. I. Arnold, Springer
- Mechanics and Symmetry - J. E. Marsden and T. Ratiu, Springer, 1994

Introduction

- The study of differential geometry in our context is motivated by the need to study dynamical systems that evolve on spaces other than the usual Euclidean space.
- Single pendulum. double pendulum

- **Definition**

  A manifold is a topological space M with the following property. For any $x \in M$, there exists a neighbourhood B of x which is homeomorphic to R^n (for some fixed $n \geq 0$). (We shall need more - "smooth" manifolds)

Charts and atlases

- We start with a topological space X and a positive integer n
- An **n-dimensional chart** on X is a pair (U, ϕ) where U is open and ϕ is a homeomorphism
- X is said to be **locally Euclidean** if there exists a positive integer n such that for each $x \in X$, there is an n-dimensional chart (U, ϕ) on X with $x \in U$
- Two charts (U_1, ϕ_1) and (U_2, ϕ_2) and $U_1 \bigcap U_2 \neq 0$ then $\phi_1 \circ \phi_2^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ are homeomorphisms
- If both are C^∞ then the two charts are $\mathbf{C}^\infty$ -related
- An **atlas** of **dimension** n on X is a collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ of n-dimensional charts on X, any two of which are C^∞-related and such that $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X$

- A maximal atlas contains every chart on X that is admissible to it - or whenever (U, ϕ) is a chart on X that is C^∞-related to every $(U_\alpha, \phi_\alpha), \alpha \in \mathcal{A}$, then there exists an $\alpha_0 \in \mathcal{A}$ such that $U = U_{\alpha 0}$ and $\phi = \phi_{\alpha 0}$
- **Example**: The charts (U_S, ϕ_S) and (U_N, ϕ_N) together form an atlas on S^1.
- Consider the standard atlas on **R**. Now consider the chart $((-\pi/2, \pi/2), tan)$. Is this **admissible** ? How about (R, ϕ) where $\phi(x) = x^3$?
- A maximal n-dimensional atlas for a topological manifold X is called a **differentiable structure** on X and a topological manifold together with some differentiable structure is called a **differentiable** or **smooth** or $\mathbf{C}^\infty$ manifold.

A circle

- A circle S^1 - subset of R^2
- Consider $U_S = S^1 - \{N\}$ and $U_N = S^1 - \{S\}$
- Map $\phi_s : U_S \to R$ (homeomorphism ?)

$$\phi_s(x^1, x^2) = \frac{x^1}{1 - x^2} \qquad \phi_s^{-1}(y) = (\frac{2y}{y^2+1}, \frac{y^2-1}{y^2+1})$$

- $\phi_N : U_N \to R$

$$\phi_N(x^1, x^2) = \frac{x^1}{1 + x^2} \qquad \phi_N^{-1}(y) = (\frac{2y}{y^2+1}, \frac{-y^2+1}{y^2+1})$$

- $U_N \bigcap U_S = S^1 - \{N, S\}$
- $\phi_S \circ \phi_N^{-1}(y) = y^{-1} = \phi_N \circ \phi_S^{-1}(y)$

A sphere

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- A sphere S^2 - subset of R^3
- Consider $U_S = S^2 - \{N\}$ and $U_N = S^2 - \{S\}$
- Map $\phi_s : U_S \to R$ (homeomorphism ?)

$$\phi_s(x^1, x^2, x^3) = (\frac{x^1}{1 - x^3}, \frac{x^2}{1 - x^3}) \qquad \phi_N(x^1, x^2, x^3) = (\frac{x^1}{1 + x^3}, \frac{x^2}{1 + x^3})$$

- $\phi_s^{-1} : R^2 \to U_S$

$$\phi_s^{-1}(y) = \frac{1}{(1 + \|y\|^2)}(2y_1, 2y_2, \|y\|^2 - 1)$$

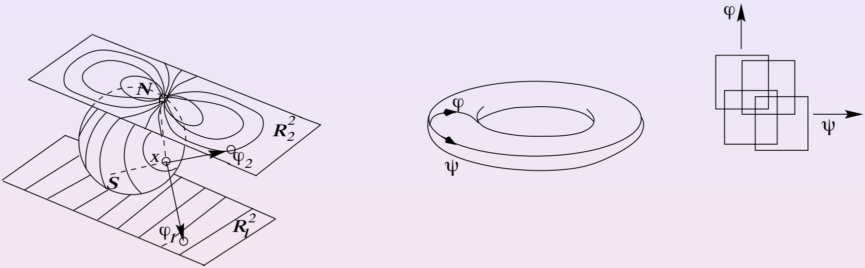

Figure: An atlas for a sphere and a torus

Other examples of smooth manifolds

- Any nonempty open subset of R^n
- $M(n) = \{n \times n \text{ real matrices}\} = R^{n^2}$
- $GL(n) = \{n \times n \text{ real, invertible matrices}\}$
- S^n - a sphere in $n+1$ dimensional Euclidean space

Coordinates

- Charts assign an n-tuple of reals to an element on the manifold
- Choosing a basis for R^n, one could assign coordinates to these n-tuples
- So we introduce the notion of coordinate functions $x \triangleq (x^1, \ldots, x^n)$.
- The ith coordinate of a point p on the manifold is $x^i(\phi(p)) : X \rightarrow R$
- Coordinate function $x^i \circ \phi : U \rightarrow R$
- We could choose another set of coordinates as well - say y
- Sometimes we skip ϕ and talk directly of x (the coordinate function) as a map from U to R^n.

Change of coordinates - chain rule

- How do we move from one coordinate system to another ?
- Consider a smooth function $f(\cdot) : X \rightarrow R$ and consider two coordinate systems x and y.

$$f \circ x^{-1} = f \circ y^{-1} \circ y \circ x^{-1}$$

- Then the partial derivative of $f(\cdot)$ with respect to x^j (the jth partial in the x coordinate system) is given by

$$D_j(f \circ x^{-1}) = D_j(f \circ y^{-1} \circ y \circ x^{-1}) = D_k(f \circ y^{-1}) \cdot D_j^k(y \circ x^{-1})$$

- Note

$$f \circ y^{-1} : R^n \rightarrow R \qquad\qquad y \circ x^{-1} : R^n \rightarrow R^n$$

Change of coordinates - example

- Think of the Cartesian plane and two coordinate systems (x, y) and (r, θ).
- Now $x = r\cos\theta$ and $y = r\sin\theta$
- Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ with the chain rule mentioned in the last slide.

A submanifold

Definition

A subset X' of an n-dimensional smooth manifold X will be called a **k-dimensional submanifold** of X if for for each $p \in X'$ there exists a chart (U, ϕ) of X at p such that $\phi|U \bigcap X'$ onto an open set is some copy of R^k in R^n.

- Is the unit circle a submanifold of R^2 ?
- Is the set of real-symmetric $n \times n$ matrices a submanifold of $R^{n\times n}$?

Differentiable functions

- $F : X \rightarrow R$ is $\mathbf{C}^\infty$ on X, if, for every chart (U, ϕ) in the differentiable structure for X, the coordinate expression $F \circ \phi^{-1}$ is a C^∞ real-valued function on the open subset $\phi(U)$ of R^n.
- A bijection $F : X \rightarrow Y$ for which both F and $F^{-1} : Y \rightarrow X$ are $\mathbf{C}^\infty$ is called a **diffeomorphism**

Tangent vectors

- Two curves $c_1 : R \supset (-a, a) \to X$ and $c_2 : (-a, a) \supset R \to X$ in X are called *equivalent* at p if

$$c_1(0) = c_2(0) = p \qquad\qquad (\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$$

  in some chart (U, ϕ).
- An *equivalence* class $[c(\cdot)]$ where $c(0) = p$ is called a tangent vector to X at p.
- The set of all tangent vectors at p is called the tangent space at p and is denoted by T_pX.
- Identify T_pX with $T_{\phi(p)}R^n$.

Tangent bundle

- The disjoint union of all tangent spaces to a manifold forms the tangent bundle.

$$TX = \bigcup_{p \in X} T_p X$$

- TX has a smooth manifold structure (an atlas for X induces an atlas for TX.)
- Call $\pi : TX \to X$ the canonical projection defined by

$$v_p \to p \qquad \forall v_p \in T_p X$$

- If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for X, then $\{(\pi^{-1}(U_\alpha), D\phi_\alpha)\}$is an atlas for TX.

Vector fields

- A **vector field** on a smooth manifold X is a map $\mathbf{V}$ that assigns to each $p \in X$ a tangent vector $\mathbf{V}(p)$ in $T_p(X)$.
- If this assignment is smooth, the vector field is called **smooth** or $\mathbf{C}^\infty$.
- The collection of all C^∞ vector fields on a manifold X denoted by $\mathcal{X}(X)$ is endowed with an algebraic structure as follows: Let $\mathbf{V}, W \in \mathcal{X}(X)$, $a \in R$ and $f \in C^\infty(X)$
  - $\mathbf{V} + \mathbf{W}(p) = \mathbf{V}(p) + \mathbf{W}(p)$ $(\in \mathcal{X}(X))$
  - $a(\mathbf{V})(p) = a\mathbf{V}(p)$ $(\in \mathcal{X}(X))$
  - $(f\mathbf{V})(p) = f(p)\mathbf{V}(p)$ $(\in \mathcal{X}(X))$

An alternate viewpoint of tangent vectors

- A tangent vector **v** to a point p on a surface will assign to every smooth real-valued function f on the surface a "directional derivative" $\mathbf{v}(f) = \bigtriangledown f(p) \cdot \mathbf{v}$.
- Draw a curve $\alpha(.) : M = \mathbf{R} \rightarrow \mathbf{R}$ on the plane. Let $\alpha \in \mathbf{C}^\infty$. Consider $\frac{d\alpha(t)}{dt}|_{t=p}$. What is this expression ? It takes a curve $\alpha(\in \mathbf{C}^\infty)$ to the reals at a point p.
- A **tangent vector (derivation)** to a differentiable manifold X at a point p is a real-valued function **v**: $C^\infty \rightarrow R$ that satisfies
  - (**Linearity**) $\mathbf{v}(af + bg)$ = a **v**(f) + b **v**(g)
  - (**Leibnitz Product Rule**) $\mathbf{v}(fg) = f(p)\mathbf{v}(g) + \mathbf{v}(f)g(p)$ for all $f, g \in \mathbf{C}^\infty(X)$ and all $a, b \in R$
- The set of all tangent vectors to X at p is called the **tangent space** to X at p and denoted $T_p(X)$.

An alternate viewpoint of vector fields

- A **vector field** $\mathcal{X}$ on a differentiable manifold X is a linear operator that maps smooth functions to smooth functions. Mathematically

$$\mathcal{X} : \mathbf{C}^{\infty}(X) \to \mathbf{C}^{\infty}(X)$$

Integral curve

Definition

Let $\mathcal{V}$ be a smooth vector field on the manifold X. A smooth curve $\alpha : \mathcal{I} \to X$ in X is an *integral curve* for $\mathcal{V}$ if its velocity vector at each point coincides with the vector assigned to that point

$$\frac{d\alpha}{dt} = \mathcal{V}(\alpha(t))$$

Maximal integral curve

Theorem

Let $\mathcal{V}$ be a smooth vector field on the manifold X and p a point in X. Then there exists an interval $(a(p), b(p)) \in R$ and a smooth curve $\alpha_p : (a(p), b(p)) \to X$ such that

- $0 \in (a(p), b(p))$ *and* $\alpha_p(0) = p$.
- α_p *is an integral curve for* $\mathcal{V}$.
- *If* (c, d) *is an interval containing* 0 *and* $\beta : (c, d) \to X$ *is an integral curve for* $\mathcal{V}$ *with* $\beta(0) = p$*, then* $(c, d) \subset (a(p), b(p))$ *and* $\beta = \alpha_p|(c, d)$. *Thus,* α_p *is called the maximal integral curve of* $\mathcal{V}$ *through* p *at* $t = 0$.

Maps between manifolds and ”the linear approximation”

Consider a smooth map $f : X \to Y$. At each $p \in X$ we define a linear transformation

$$f_{*p} : T_p(X) \to T_{f(p)}(Y)$$

called the **derivative** of f at p, which is intended to serve as a ”linear approximation to f near p,” defined as

- For each $\mathbf{v} \in T_p(X)$ we define $f_{*p}(\mathbf{v})$ to be the operator on $\mathbf{C}^\infty(f(p))$ defined by $(f_{*p}(\mathbf{v}))(g) = \mathbf{v}(g \circ f)$ for all $g \in \mathbf{C}^\infty(f(p))$.

Lemma

*(**Chain Rule**) Let $f : X \to Y$ and $g : Y \to Z$ be smooth maps between differentiable manifolds. Then $g \circ f : X \to Z$ is smooth and for every $p \in X$*

$$(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$$

An alternate viewpoint of maps between manifolds

The derivative of f at p could be constructed as follows. Once again $f(\cdot) : X \to Y$. (The notation f_{*p} is now changed to $T_p f$.)

- Choose a parametrized curve $c(\cdot) : (-\epsilon, \epsilon) \to X$ with $c(0) = p$ and $\dot{c}(0) = v_p$.
- Construct the curve $f \circ c$. Then define
$$T_p f \cdot v_p = \frac{d}{dt}|_{t=0}(f \circ c)(t)$$
- With coordinates, we have
$$T_{x(p)}(y \circ f \circ x^{-1}) = \frac{\partial}{\partial x^j}(y \circ f \circ x^{-1})|_{x(p)}$$
The Jacobian at $x(p)$
- The rank of f at p is the rank of the Jacobian matrix at $x(p)$ and *this is independent of the choice of the charts*

Immersions, submersions

- If $f : X \to Y$ is smooth, then if
  1. If $T_p f$ is *onto* for all $p \in X$, then f is called a submersion
  2. If $T_p f$ is *one-to-one* for all $p \in X$, then f is called a immersion
  3. If f is an immersion and one-to-one, then $f(X)$ is an *immersed submanifold*
- Examples

Critical points and regular points

Consider a smooth map $f : X \to Y$

- A point $p \in X$ is called a critical point of f if $T_p f$ is not onto.
- A point $p \in X$ is called a regular point of f if $T_p f$ is onto.
- A point $y \in Y$ is called a *critical value* if $f^{-1}(y)$ contains a critical point. Otherwise, y is called a regular value of f.

Submersion theorem

Theorem

If $f : X^n \to Y^k$ is a smooth map, and $y \in f(X) \subset Y$ is a regular value of f, then $f^{-1}(y)$ is a regular submanifold of X of dimension $n - k$.

- Example: $f(x, y) = x^2 + y^2 - 1$. Take $f^{-1}(0)$.
- Show that $O(n) = \{A \in M_n(R) | A^T A = I\}$ is a submanifold of $M_n(R)$. What is its dimension ?
  Hint: Consider a map $f : R^{n\times n} \to S^{n\times n}$ (symmetric matrices). Let $f(A) = A^T A$ and examine the value I.

The cotangent space

Definition

For any $f \in \mathbf{C}^{\infty}(p)$ we define an operator $df(p) = df_p : T_p(X) \to R$ called the **differential** of f at p by

$$df(p)(\mathbf{v}) = df_p(\mathbf{v}) = \mathbf{v}(f)$$

for every $\mathbf{v} \in T_p(X)$. Since df_p is linear, it is an element of the dual space of $T_p(X)$ called the **cotangent space** of X at p and denoted by $T_p^*(X)$

- basis - $\{\frac{\partial}{\partial x^i}\}$ and dual basis $\{dx^i\}$

Cotangent bundle

- The disjoint union of all cotangent spaces to a manifold forms the cotangent bundle.
$$T^*X = \bigcup_{p \in X} T_p^* X$$
- T^*X has a smooth manifold structure (an atlas for X induces an atlas for T^*X.)
- Call $\pi : T^*X \to X$ the canonical projection defined by
$$v_p \to p \qquad \forall v_p \in T_p^* X$$
- If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for X, then $\{(\pi^{-1}(U_\alpha),)\}$is an atlas for T^*X.

One-forms

A **one -form** on a smooth manifold X is a smooth map $\alpha(\cdot)$ that assigns to each $p \in X$ a cotangent vector $\alpha(p)$ in $T_p^* X$

Pull-back and push-forward of vector fields

- Suppose $f : X \to Y$ is a diffeomorphism, then $f_* : \mathcal{X}(X) \to \mathcal{X}(Y)$ *the push-forward* of a vector field on X is given by

$$(f_* V)(q) = T_{f^{-1}(q)} f \cdot V(f^{-1}(q)) \qquad \forall q \in Y, V \in \mathcal{X}(X)$$

- Suppose $f : X \to Y$ is a diffeomorphism, then $f^* : \mathcal{X}(Y) \to \mathcal{X}(X)$ *the pull-back* of a vector field on Y is given by

$$(f^* V)(p) = T_{f(p)} f^{-1} \cdot V(f(p)) \qquad \forall p \in X, V \in \mathcal{X}(Y)$$

The Lie bracket

Definition

The Lie bracket of two vector fields on a smooth manifold X is defined as

$$[\cdot,\cdot] : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z} \qquad [P, Q] \triangleq PQ - QP$$

where for $f \in \mathbf{C}^{\infty}(X)$

$$[P, Q](f) = P(Q(f)) - Q(P(f))$$

Properties of the Lie bracket

- $[X, Y] = -[Y, X]$
- $[X + Y, Z] = [X, Z] + [Y, Z]$
- $[fX, gY] = f \cdot (X(g))Y - g(Y(f)) + f.g.[X, Y]$
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

A Lie group

Definition

A smooth manifold G together with a group structure is called a Lie group if the group operation is smooth.

$$(g, h) \to g \cdot h \qquad \text{is smooth}$$

- The identity element of the Lie group is usually denoted by e.
- *Left translation* of a group

$$L_g : G \to G \qquad h \to gh$$

- *Right translation* of a group

$$R_g : G \to G \qquad h \to hg$$

Examples of Lie groups

- R or multiple copies of R (as R^n.).
- The unit circle S^1 or multiple copies of S^1 (as $S^1 \times \ldots \times S^1$).
- The set of $n \times n$ matrices with real entries.
- The set of $n \times n$ real-orthogonal matrices $O(n)$.
- The set of $n \times n$ real-rotation matrices $SO(n)$.

The Lie algebra

- The Lie bracket on T_eG is given by

$$[\xi, \eta] = [\mathcal{X}_\xi, \mathcal{X}_\eta] \quad \text{bracket of vector fields}$$

- The vector space T_eG with the $[\cdot, \cdot]$ is called the *Lie algebra* of G and is denoted by $\mathfrak{g}$.

Invariant vector fields

Definition

A vector field V on a Lie group G is said to be left invariant if $\forall g \in G$

$$T_h L_g \cdot X(h) = X(gh) \qquad \text{OR} \qquad L_{g*} V = V$$

Some properties

- For $X, Y \in \mathcal{X}_L(G) \Rightarrow [X, Y] \in \mathcal{X}_L(G)$.
- Identify $\mathcal{X}_L(G)$ with $T_e G$.

Exponential maps

- The "exponential map" maps a Lie algebra to the Lie group

$$\exp : \mathfrak{g} \to G \qquad \eta \to \exp(\eta) = \gamma_\eta(1)$$

where $\gamma_\eta(t)$ is the integral curve of $X_\eta \in \mathcal{X}_L(G)$ with $\gamma_\eta(0) = e$.

- The exponential map is a diffeomorphism from a neighbourhood of 0 in $\mathfrak{g}$ onto a nbhd of e in G.
- Recall linear systems and the solution of $\dot{x} = Ax$. We have $\gamma_A(t) = e^{At}$. Can you make a connection ?

Group actions

A (left) action of a Lie group G on a manifold M is a smooth map

$$\Phi : G \times M \to M$$

such that $\Phi(e, x) = x \forall x \in M \Phi(g, \Phi(h, x)) = \Phi(gh, x) \forall g, h \in G, x \in M$

- Examples: 1. Action of $SO(3)$ on R^3. 2. Action of a group on itself

Adjoint and co-adjoint actions

- Define

$$i_g : G \to G \qquad i_g = R_{g^{-1}} \circ L_g$$

- Adjoint action of G on $\mathfrak{g}$

$$Ad_g : \mathfrak{g} \to \mathfrak{g} \qquad Ad_g(\eta) = T_e i_g(\eta)$$

- Co-adjoint action

$$Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^* \qquad [Ad_g^*(\alpha), \eta] = [\alpha, Ad_g \eta]$$

Orbits

- Say G acts on M. The orbit of $x \in M$ is defined by

$$G \cdot x = \{g \cdot x \in M | g \in G\}$$