A.2.1 Unit vector lemma

Given two frames $\langle a \rangle$, $\langle b \rangle$, the following equivalence holds:

$$\begin{cases} (\boldsymbol{i}_a \wedge \boldsymbol{i}_b) + (\boldsymbol{j}_a \wedge \boldsymbol{j}_b) + (\boldsymbol{k}_a \wedge \boldsymbol{k}_b) = 2\boldsymbol{v}\sin\theta \\ (\boldsymbol{i}_a \cdot \boldsymbol{i}_b) + (\boldsymbol{j}_a \cdot \boldsymbol{j}_b) + (\boldsymbol{k}_a \cdot \boldsymbol{k}_b) = 1 + 2\cos\theta \end{cases} \tag{A.3}$$

These equations allow us to found out the inverse problem, i.e. from the knowledge of ${}^a_bR \rightarrow \boldsymbol{v}, \theta$. Knowing that

$${}^a\boldsymbol{v} = {}^b\boldsymbol{v} \tag{A.4}$$

because it is the axis of rotation (i.e. it is an eigenvector of $\boldsymbol{R}$)

Suppose to project (A.3) in a frame we like, say $\langle s \rangle$, it becomes

$$\begin{cases} [{}^s\boldsymbol{i}_a \wedge]{}^s\boldsymbol{i}_b + [{}^s\boldsymbol{j}_a \wedge]{}^s\boldsymbol{j}_b + [{}^s\boldsymbol{k}_a \wedge]{}^s\boldsymbol{k}_b = 2{}^s\boldsymbol{v}\sin\theta \\ ({}^s\boldsymbol{i}_a^T\,{}^s\boldsymbol{i}_b) + ({}^s\boldsymbol{j}_a^T\,{}^s\boldsymbol{j}_b) + ({}^s\boldsymbol{k}_a^T\,{}^s\boldsymbol{k}_b) = 1 + 2\cos\theta \end{cases} \tag{A.5}$$

where we have defined the skew matrix as

$$[{}^s\boldsymbol{i}_a \wedge] = \begin{bmatrix} 0 & -{}^s\boldsymbol{i}_{a_z} & {}^s\boldsymbol{i}_{a_y} \\ {}^s\boldsymbol{i}_{a_z} & 0 & -{}^s\boldsymbol{i}_{a_x} \\ -{}^s\boldsymbol{i}_{a_y} & {}^s\boldsymbol{i}_{a_x} & 0 \end{bmatrix} \tag{A.6}$$

Solving (A.5) gives us ${}^s\boldsymbol{v}, \theta$

Note that:

$${}^s_a\boldsymbol{R} = [{}^s\boldsymbol{i}_a|...] \qquad {}^s_b\boldsymbol{R} = [{}^s\boldsymbol{i}_b|...] \tag{A.7}$$

This means that ${}^s_a\boldsymbol{R}$ and ${}^s_b\boldsymbol{R}$ contain all the vectors needed in the projected versor lemma equation. From their knowledge it is possible to compute the angle-axis representation, which will be projected on the common frame $\langle s \rangle$.

What is the use of all these different representations?

- the rotation matrix is used to compute a sequence of rotations;
- the Euler angles are human-friendly; also they can be estimated by onboard sensors with the use of Kalman filters (or any other estimation technique such as complementary filters);
- angle-axis representation is useful for computing the reference rate to drive a particular frame towards another one asymptotically.