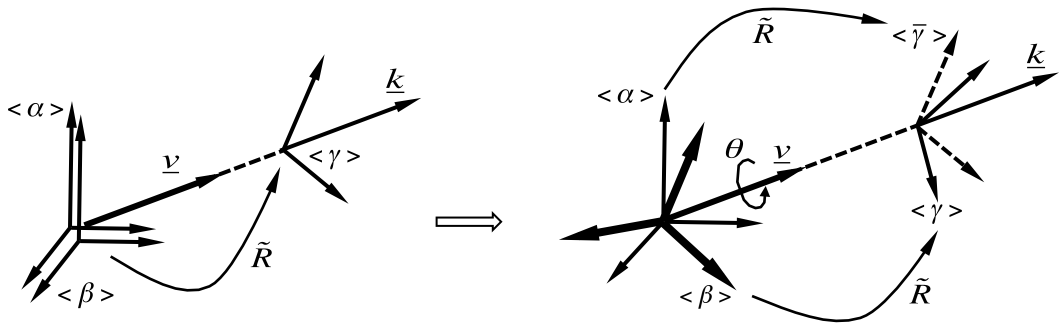


Then by now executing the θ rotation around $\underline{v}$, by maintaining $<\beta>$ and $<\gamma>$ rigidly connected (i.e. $<\gamma>$ is assumed belonging to the reference system of $<\beta>$) we can consequently see how, at the *end of the rotation*, the resulting $R(v,\theta)$ can also be written as (see the second figure, in the right, above and follow the path $<\alpha>,<\bar{\gamma}>,<\gamma>,<\beta>$, with $<\bar{\gamma}>$ marking the position of $<\gamma>$ before rotation)

$$R(v,\theta) = \tilde{R}\, e^{[k\wedge]\theta}\tilde{R}^T$$

Where, by now series expanding the matrix exponential we have

$$R(v,\theta) = \tilde{R}\left[\sum_{n=0}^{+\infty}\frac{[k\wedge]^n}{n!}\theta^n\right]\tilde{R}^T = \left[\sum_{n=0}^{+\infty}\frac{\tilde{R}[k\wedge]^n\tilde{R}^T}{n!}\theta^n\right] = \left[\sum_{n=0}^{+\infty}\frac{(\tilde{R}[k\wedge]\tilde{R}^T)^n}{n!}\theta^n\right]$$

Where the last r.h.s. is easily shown to be equal to its antecedent one, in force of the fact that $\tilde{R}\tilde{R}^T = I$

Then, by formerly noting that the term $\tilde{R}[k\wedge]\tilde{R}^T$ is just the projection of the vector product operator $[k\wedge]$ on frame $<\alpha>$; and by also keeping into account that the projection of k on $<\alpha>$ is just v; this necessarily implies

$$\tilde{R}[k\wedge]\tilde{R}^T = [v\wedge]$$

That, once substituted into the above, directly leads to the first two equalities in the equivalent angle-axis formulae; that is the following ones

$$R(v,\theta) = e^{[v\wedge]\theta} = e^{[\rho\wedge]}$$

At this point, in order to also prove the last one (i.e. the Rodrigues formula) it is sufficient re-expand in series the above result; that is

$$e^{[v\wedge]\theta} = \sum_{n=0}^{+\infty}\frac{[v\wedge]^n}{n!}\theta^n$$

And then making appearing, within the right-hand-side, the composing series expansions of $sen\theta$ and $cos\theta$ (the completion of the proof is however left as exercise to the reader)

5.1.1 The Inverse equivalent angle-axis problem. The unit vectors lemma

The previous subsection has provided the analytical expressions characterizing any orientation matrix constructed as $R(v,\theta)$.
In this subsection we instead consider the inverse problem; that is, given an orientation matrix R find, if any, at least an angle-axis couple (v,θ) (or equivalently its associated rotation vector $\rho \doteq v\theta$) realizing it via the associated $R(v,\theta)$.
To this respect an existence theorem, due to Euler, simply state that actually

◊ *Euler theorem*

Any orientation matrix R admits an equivalent angle-axis representation. ◊

Thus guaranteeing that the inverse equivalent angle-axis problem always admits a solution.
Meanwhile, just due to the Euler theorem, from now onward any orientation matrix can consequently be denoted also as a *rotation matrix*, with an identical meaning.
The proof of the Euler theorem is however postponed to the end of this subsection.

Then, supported by the Euler existence theorem, we can now proceed toward finding a suitable inverse equivalent angle-axis algorithm.
This can be done on the basis of a fundamental lemma, the so-called *unit vectors lemma for frames* that by the way will also reveal of *paramount importance*, not only for devising the inverse equivalent angle-axis algorithm, but also, as we shall see at due time, for the future development of robot control techniques.
We shall now state and comment such lemma, before passing to use it for providing the searched inverse equivalent angle axis algorithm.
The proof of such lemma is however, it also, postponed at the end of the present subsection

◊ *Unit vectors lemma for frames: geometric form*

The following free from coordinates equalities hold true for any couple of frames $<a>$, $<b>$

$$\begin{cases} (\underline{i}_a \wedge \underline{i}_b) + (\underline{j}_a \wedge \underline{j}_b) + (\underline{k}_a \wedge \underline{k}_b) = 2\underline{v}\, sen\theta \\ (\underline{i}_a \bullet \underline{i}_b) + (\underline{j}_a \bullet \underline{j}_b) + (\underline{k}_a \bullet \underline{k}_b) = 1 + 2\cos\theta \end{cases}$$

With $(\underline{v},\theta)$ the equivalent angle-axis of frame $<b>$ with respect to $<a>$, once represented in free from coordinate notations. ◊

As regard the above lemma, it is of a certain importance the fact that it is expressed in free-from coordinate notation, thus directly allowing its application on a whatever projecting frame; even different from the involved ones $<a>$, $<b>$.
However, by projecting the above free from coordinates formulas, indifferently on one of the involved frames $<a>$, $<b>$ (and *not* on others) the following algebraic form of the same lemma is readily obtained

◊ *Unit vectors lemma for frames: algebraic form*

The following equalities hold true for both frames $<a>$, $<b>$

$$\begin{cases} (R - R^{T}) = 2[v\wedge]\, sen\theta \\ Tr(R) = 1 + 2\cos\theta \end{cases}$$

With R the orientation matrix of frame $<b>$ with respect to $<a>$; and (v,θ) the associated equivalent-angle-axis, indifferently projected on frame $<a>$ or $<b>$. ◊

The proof of the above algebraic form is left for exercise to the reader.

On the basis above lemma, the inverse equivalent angle-axis algorithm can be now devised, as hereafter constructively explained.

Inverse equivalent angle-axis algorithm

From the second equation of the unit vectors lemma in its geometric form, evaluate the right side and then let

$$\delta \doteq \cos\theta$$

- In case $|\delta| < 1$ (i.e. $\theta \neq \{2k\pi ; (2k+1)\pi\}$)

  From the first equation of the lemma in geometric form, evaluate its resulting right-hand-side and formerly let

  $$\underline{\sigma} \doteq \underline{v}\, sen\theta \neq 0$$

  Where the non-zero condition is trivially the consequence of being $|\delta| < 1$; thus allowing for

  $$(\underline{v}, \theta) = \pm\left[\frac{\underline{\sigma}}{|\underline{\sigma}|}, Atan(|\underline{\sigma}|, \delta) + 2k\pi\right]$$

  Where the well-defined ratio within the brackets simply corresponds to the choice for $\underline{v}$ leading to the positive value $|\underline{\sigma}|$ for $sen\theta$; and whose corresponding angle θ then follows immediately

(obviously, within a $2k\pi$ modularity) as indicated, via the use of the so called "four quadrants inverse tangent" $Atan\,[sen(\cdot), cos(\cdot)]$ operator.
In the above, by choosing the sign + a minimal *positive* angle within $(0,\pi)$ is assigned, by the *Atan* operator, to the corresponding choice for $\underline{v}$; while by choosing the sign – a minimal module *negative* angle within $(0,-\pi)$ is assigned by the associated *Atan* operator to the performed opposite choice for $\underline{v}$
In any case note how the associated minimal norm rotation vector $\underline{\rho}=\underline{v}\theta$ (i.e. free from any nonzero $2k\pi$ modularity) is always the *same* for both choices.

- In case $\delta = 1$ (i.e. $\theta = 2k\pi$)

  In this occurrence the first equation of the lemma cannot be of any help for devising $\underline{v}$, since its output will necessarily be the null vector $\underline{\sigma} = 0$.
  Nevertheless, since $\theta = 2k\pi$ implies and is implied by the fact that the two frames are actually parallel (i.e. ${}^a_bR = I$); we have that $<b>$ can therefore be interpreted as rotated around any unit vector of any angle multiple of 2π; thus allowing stating that

  $$(\underline{v}, \theta) = (\forall, 2k\pi)$$

- In case $\delta = -1$ (i.e. $\theta = (2k+1)\pi$)

  Here also the first equation of the lemma cannot be of any help for devising $\underline{v}$, since its output will necessarily be again the null vector $\underline{\sigma} = 0$.
  Nevertheless, since for rotations of type $(2k+1)\pi$ two possible choices (and no others) actually exist for $\underline{v}$ (as hereafter clarified) such two choices can be consequently devised as follows

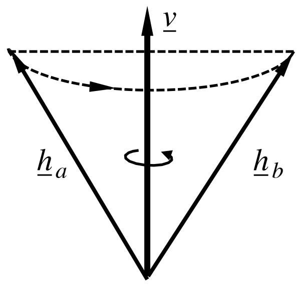
  Formerly evaluate the unit vector

  $$\underline{v}_0 \doteq \frac{\underline{h}_a + \underline{h}_b}{|\underline{h}_a + \underline{h}_b|}$$

  With the couple $(\underline{h}_a, \underline{h}_b)$ represented by any choice of one among the three couples of homologous unit vectors $(\underline{i}_a, \underline{i}_b)$, $(\underline{j}_a, \underline{j}_b)$, $(\underline{k}_a, \underline{k}_b)$ exhibiting a non-zero resultant; and then let

  $$(\underline{v}, \theta) = \left[\pm\underline{v}_0 \,, (2k+1)\pi\right]$$

  The reason for the above specific procedure simply relies on the trivial fact that, for $\theta = (2k+1)\pi$ and for two homologous unit vectors not exhibiting a zero resultant, their relative positioning necessarily results as reported in the following figure

  Whose normalized resultant $\underline{v}_0$ obviously constitutes one between two possible choices for $\underline{v}$.
  Finally, for completely commenting the above particular procedure, we can note that in case a couple of homologous unit vectors instead exhibits a null resultant, this corresponds to a rotation of $\theta = (2k+1)\pi$ just performed around an axis which is orthogonal to both the unit vectors of the null-resultant couple of homologous unit-vectors; a fact that however, as it can be easily verified, cannot ever impose a null resultant to any of the remaining two homologous couples.
  Thus, just due to this property, instead of the above formula, the following one can also be used

  $$\underline{v}_0 \doteq \frac{(\underline{i}_a + \underline{i}_b) + (\underline{j}_a + \underline{j}_b) + (\underline{k}_a + \underline{k}_b)}{\left|(\underline{i}_a + \underline{i}_b) + (\underline{j}_a + \underline{j}_b) + (\underline{k}_a + \underline{k}_b)\right|}$$

Not requiring any preliminary check about the possible existence of a null-resultant couple of homologous unit vectors.

Comments

1) The algorithm has been formally stated in free from coordinate notations for sake of generality; but its implementation necessarily requires the preliminary projection of all the involved unit vectors on some suitably chosen frame, with output results they also projected on such frame. On the other hand, since such projecting frame can be arbitrarily chosen, this allows for its most convenient choice, depending from the scopes the algorithm has to be used.

2) Also, the algorithm has been formally stated in its complete form; that is in such a way to provide its entire set of solutions in terms of separated values for $\underline{v}$ and θ. However since in the robotic practice the main interest generally is in evaluating the associated rotation vector $\underline{\rho} = \underline{v}\theta$ of *minimal norm* (i.e. $|\underline{\rho}| \le \pi$); then, for the first two above listed cases (i.e. $/\delta/ < 1$ and $\delta = 1$) this is trivially provided by executing the product without any 2π modularity in the angle θ; and this always provides a *unique* minimal norm rotation vector $\underline{\rho}$.

3) However, within these cases (i.e. $/\delta/ < 1$ and $\delta = 1$) the algorithm evidences a discontinuity only regarding the evaluation of the unit vector $\underline{v}$ (i.e. *not* for the relevant rotation vector $\underline{\rho} = \underline{v}\theta$) which occurs in the vicinities of $\delta = 1$ (obviously from the left); where the solution set for $\underline{v}$ jumps from a couple of values (for $\delta < 1$) to an arbitrary continuous infinity (for $\delta = 1$). The practical occurrence of such discontinuity is obviously regulated by the processing machine precision, which establishes the threshold $1 - \varepsilon < 1$, within which δ has to be considered unitary. However keep clearly in mind how such discontinuity is instead *absent* within the evaluation of the associated unique rotation vector $\underline{\rho} = \underline{v}\theta$ of *minimal norm*.

4) For the third case $\delta = -1$, there is instead no possibility of resulting with a unique minimum norm solution for the rotation vector $\underline{\rho}$, as it instead is for the previous cases. More precisely, as it can be verified from the corresponding general formula, $\delta = -1$ is actually the *sole case* where the number of minimal norm solutions for $\underline{\rho}$ cannot be less than the resulting two $\underline{\rho} = \pm \underline{v}_0 \pi$; each one of them however representing the convergence value of a unique minimum norm rotation vectors coming from $/\delta/ < 1$, but such that $\delta \to -1$.
The following simple example may serves for clarifying this point
----

◊ *Unit vectors lemma for couples of unit vectors*

For any couple of unit vectors $\underline{v}_a$ and $\underline{v}_b$ the following relationships actually hold by definition

$$\begin{cases} \underline{v}_a \wedge \underline{v}_b = \underline{v}\, sen\theta \\ \underline{v}_a \bullet \underline{v}_b = cos\theta \end{cases}$$

In the order, respectively corresponding to the vector and scalar products of the two unit vectors; where θ is the minimum angle for rotating $\underline{v}_a$ on $\underline{v}_b$ around the unit vector $\underline{v}$ (orthogonal to the plane $(\underline{v}_a, \underline{v}_b)$ and then to both $\underline{v}_a, \underline{v}_b$), in turn constructed via the well-known right-hand-rule from $\underline{v}_a$ toward $\underline{v}_b$. ◊

Then, provided the rotation vector of $\underline{v}_b$ with respect to $\underline{v}_a$ is accordingly defined just as $\underline{\rho} \doteq \underline{v}\theta$, we can see in the above relationships the "lemma" that (actually with an overemphasizing terminology) allows us to separately evaluate $(\underline{v}, \theta)$, and consequently $\underline{\rho}$, just via an algorithm strictly similar to the one to used for frames (deduce it for exercise).

Also this lemma will reveal of *paramount importance*, not only for finding the rotation vector of a couple of unit vectors; but also, as we shall again see at due time, for the future development of robot control techniques.

Proof of the Euler theorem
