On the Dynamical Hierarchy in Gathering Protocols with Circulant Topologies

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May 22, 2024

In this article we investigate the convergence behavior of gathering protocols with fixed circulant topologies using tools form dynamical systems. Given a fixed number of mobile entities moving in the Euclidean plane, we model a (linear) gathering protocol as a system of (linear) ordinary differential equations whose equilibria are exactly all possible gathering points. Then, for a circulant topology we derive a decomposition of the state space into stable invariant subspaces with different convergence rates by utilizing tools form dynamical systems theory. It turns out, that decomposition is identical for every (linear) circulant gathering protocol, whereas only the convergence rates depend on the weights in interaction graph itself. We end this article with a brief outlook on robots with limited viewing range.
Keywords: Multi-agent Systems, Consensus Of Multi-agent Systems, Consensus Protocol, Leaderless, Network Topology, Communication Protocol Control Of Multi-agent Systems, Formation Control

1. Introduction

This article applies dynamical systems theory to the design and analysis of collective behavior of swarms of mobile entities, called robots from now on, which are widely studied in the computer science community under the headline of *distributed computing* (e.g. [Ham18, FPS19]). The problem of interest, is the *gathering problem* in which the robots are supposed to converge to a single, not predefined, point. The only capabilities the robots have are observing other robots' positions, performing computations in their local memory, and moving. The strategy that each robot pursues is called a *protocol*. While most studies of the gathering problem employ discrete time (e.g. [DKL+11, CFH+20, CP05, Flo19, CHJ+23]), some protocols using continuous time have been proposed as well ([DKKM15, KMadH19]). The latter framework is the focus of this manuscript.

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In mathematics and physics literature, comparable multiagent systems have been extensively studied in the past decades. Gathering is an instance of famous problems known under terms such as *synchronization, agreement problems, rendezvous problems, consensus algorithms*, and likely many more. We mention only some prominent results that are closely related to this paper. For example, in synchronization literature one commonly studies Laplacian (or Laplacian like) dynamics of coupled oscillators. Very general internal dynamics and coupling functions are possible and synchronization is not restricted to equilibria (see e.g. [HCP94, PRK03, DB14] and references therein). Very powerful synchronization results have also been derived in the control theory literature under the headline of *distributed consensus control*, *distributed consensus algorithms*, or *asymptotic agreement problems*. Asymptotic consensus was investigated for abstract variables, time varying interaction topologies, communication delay, and even defective agents (e.g. [OSM03b, OSM03a, OSM04, Mor04, CLYH13]). We reference specific results in Section 2 below to characterize gathering protocols in our setting.

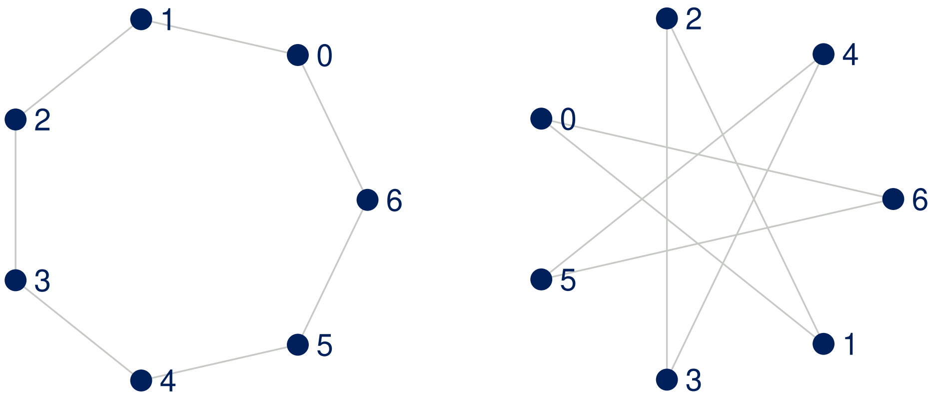
The key contribution of this article is to show that applying the mathematical theory of dynamical systems to the gathering problem allows for a profound understanding of the collective dynamics in terms of fine grained gathering rates. To that end, we assume the interaction structure to be *circulant* and to remain unchanged throughout the dynamics. For the most part, we restrict to linear protocols, however, we outline options to generalize to nonlinear protocols as well. We unveil a foliation of the state space of all robots' positions into dynamically invariant subspaces in which the configurations gather with different speeds. This foliation is independent of the precise protocol. It allows to identify gathering rates for different initial configurations (cf. Theorem 4.3) and to decompose initial configurations into components with different gathering rates (cf. Corollary 4.5).

Comparable results are known in the synchronization literature for coupled oscillators. There, dynamical hierarchy corresponds to so-called *normal modes* or *eigenmodes* which can unveil communities that synchronize faster than others (see for example [ADGPV06, MM08]). Furthermore, overall convergence speed to synchrony is commonly related to spectral properties of the Laplacian or a more general system matrix which has in particular been done in the context of circulant interaction topologies (e.g. [OSM04, IA16, Rub16, ILN18]). However, to the best of our knowledge a similarly detailed exploitation of the full spectral decomposition of phase space and the interpretation in terms of a dynamical hierarchy of configurations in positional variables has not been done before.

Before we delve into the details of modeling the gathering problem as a dynamical system/distributed control system and analyzing its collective dynamics, we illustrate the main results at the hand of the following example.

Example 1.1. *Consider $N = 7$ mobile robots $\{0, \ldots, 6\}$ running the* Go-To-The-Middle *protocol, i.e., if $z_i(t) \in \mathbb{R}^2$ is the position of robot i at time t the i-th robot will move towards the midpoint between two fixed neighbors. Theorem 4.3 shows that the configuration in Fig. 1 (a) has the slowest gathering rate, while the configuration in Fig. 1 (b) has the fastest gathering rate. Further intermediate configurations are unveiled as well.*

The article is organized as follows: In Section 2 we gather some necessary preliminary notations in order to model the gathering problem as a dynamical system governed by an ordinary differential equation and we state the convergence result for general linear protocols. In this context, we argue why it is reasonable to focus on linear systems (see



(a) Slowest gathering configuration (b) Fastest gathering configuration

Figure 1: Illustration of the gathering hierarchy exhibited by $N = 7$ robots running the Go-To-The-Middle protocol. Blue dots indicate positions of each robot. Gray lines symbolize the communication relation.

Section 2.1). In the following Section 3 we formalize the underlying interaction structure by a weighted graph and deduce a convergence result for linear circulant systems. The main dynamical analysis of such systems is then carried out in Section 4, where a hierarchy of different gathering rates is proven. Finally, in Section 5 we end this article with a brief outlook on robots with limited viewing range. In the appendix Appendix A, we provide some additional technical background for the convergence results discussed in Section 2. This is for the sake of completeness only and does not contain novel results.

2. Introductory Considerations

We begin by setting up a dynamical systems model for the collective dynamics of a collection of $N \in \mathbb{N}$ interacting mobile robots. The position of robot $i \in \{0, \dots, N-1\}$ at time $t \in \mathbb{R}$ is denoted by $z_i(t) = (x_i(t), y_i(t)) \in \mathbb{R}^2$ – we typically omit the time argument for notational brevity. The collection of all robots' positions is called a *configuration*. Each robot observes the positions of (a subset of) the other robots in relative coordinates and adapts its own direction and speed of movement instantaneously. However, we take the stance of an external observer describing the dynamics in global coordinates. The adaptation strategies (or *distributed control strategies*) are given by the following ordinary differential equations (ODEs)

$$\dot{z}_i = -z_i + f_i(z_0, \dots, z_{N-1}), \quad i = 0, \dots, N-1. \tag{1}$$

The equation can be interpreted as each robot computing a *target point* using $f_i \colon \mathbb{R}^{2N} \to \mathbb{R}^2$ and pointing its velocity vector towards it. In general, not every neighboring position z_j will influence the dynamics of the i-th robot. The collection of all equations (1) for $i \in \{0, \dots, N-1\}$ models the evolution of the configuration.

Definition 2.1. *A protocol modeled by* (1) *is called* gathering*, if the solution with any initial configuration converges to a* gathering point $z_i = z^* = (x^*, y^*) \in \mathbb{R}^2$ *for all* i *for* $t \to \infty$ *and if the system is in equilibrium in any such gathering point, i.e.,* $-z^* + f_i(z^*, \ldots, z^*) = 0$ *for all* i*. The subspace of all gathering configurations is denoted by*

$$V_0 = \left\{ Z^* = (z^*, \ldots, z^*) \,\middle|\, z^* \in \mathbb{R}^2 \right\} \subseteq \mathbb{R}^{2N}.$$

In the remainder of this article, we typically consider *linear* strategies, i.e., the target point is a linear combination of the neighboring positions and (1) has the form

$$\dot{z}_i = -z_i + \sum_{j=0}^{N-1} w_{i,j} z_j, \tag{2}$$

where $w_{i,j} \in \mathbb{R}$ is a constant *weight* which we collect in the *weight matrix* $W = (w_{i,j})_{i,j=0}^{N-1} \in \mathbb{R}^{N\times N}$. It can readily be seen, that a necessary condition for a linear strategy to be gathering is $z^* = \sum_{j=0}^{N-1} w_{i,j} z^*$, which in turn is satisfied if and only if the weights satisfy

$$\sum_{j=0}^{N-1} w_{i,j} = 1 \text{ for all } i \in \{ 0, \ldots, N-1 \}. \tag{3}$$

In this situation, we say that the weights, resp. the weight matrix, and also the linear strategy are *consistent*.

Remark 2.2. *A negative weight forces a robot to compute the position of a neighbor* reflected at the origin *and forces it to intentionally* move away *from that neighbor which we deem unnatural to achieve gathering. We will restrict to non-negative weights below. Furthermore,* $w_{i,j} = 0$ *if and only if* i *does not use the position of* j *to compute its target point.*

Collecting all robots' positions in a single vector $Z = (z_0, \ldots, z_{N-1}) \in \mathbb{R}^{2N}$, allows us to rewrite (2) as

$$\dot{Z} = (-\mathbf{I}_{2N} + \mathbf{W})\, Z, \tag{4}$$

where $\mathbf{I}_{2N} \in \mathbb{R}^{2N\times 2N}$ denotes the identity matrix and $\mathbf{W} \in \mathbb{R}^{2N\times 2N}$ is given by the Kronecker product

$$\mathbf{W} := W \otimes \mathbf{I}_2 = \left(\begin{array}{cc|c|cc} w_{0,0} & 0 & \cdots & w_{0,N-1} & 0 \\ 0 & w_{0,0} & \cdots & 0 & w_{0,N-1} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline w_{N-1,0} & 0 & \cdots & w_{N-1,N-1} & 0 \\ 0 & w_{N-1,0} & \cdots & 0 & w_{N-1,N-1} \end{array}\right).$$

In some cases, it turns out to be helpful to arrange the configuration vector as $\widetilde{Z} := (X, Y) \in \mathbb{R}^{2N}$, where $X = (x_0, \ldots, x_{N-1}) \in \mathbb{R}^N$ denotes the x-coordinates of the N robots and $Y = (y_0, \ldots, y_{N-1}) \in \mathbb{R}^N$ the y-coordinates, respectively. In these coordinates, the linear system (4) takes the form

$$\dot{\widetilde{Z}} = \left(-\mathbf{I}_{2N} + \widetilde{\mathbf{W}}\right) \widetilde{Z} \tag{5}$$

with

$$\widetilde{\mathbf{W}} := \mathbf{I}_2 \otimes W = \left(\begin{array}{c|c} W & \mathbf{0} \\ \mathbf{0} & W \end{array}\right) \in \mathbb{R}^{2N\times 2N}.$$

The behavior of linear systems of ordinary differential equations is well understood. Applying the closed form of solutions (e.g. [Tes12]) to the two decoupled systems in (5), we immediately obtain the following result characterizing linear gathering protocols (compare [KMadH11] for time-discrete systems).

Theorem 2.3. *A linear protocol modeled by (2) is gathering if and only if the weight matrix $W \in \mathbb{R}^{N\times N}$ has a simple eigenvalue 1 with eigenvector $(1,\dots,1)^T \in \mathbb{R}^N$ and all other eigenvalues $\lambda \in \mathbb{C}$ satisfy $\Re(\lambda) < 1$. In this case, the gathering point is the average of the initial positions $\frac{1}{N}\sum_{i=0}^{N-1} z_i(0)$.*

Remark 2.4. *Seminal works such as [OSM03b, OSM03a, OSM04, Mor04, BS03] and many more thereafter have classified gathering protocols (or consensus control strategies in their context) in much more general contexts. Their results typically discuss connectivity properties (cf. Section 3 below).*

2.1. A Note on Nonlinear Strategies

The linear model (2) is idealized, as it does not prevent unbounded velocity and realizes gathering only in the limit $t \to \infty$. Both restrictions are at least problematic for the application. However, in commonly investigated protocols robots compute their *direction* as a linear combination of the (relative) positions of their neighbors and move with bounded speed until they are gathered. Thus, the velocity vector in (2) is normalized by a *nonlinear* function $\mathcal{N} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathcal{N}(0) = 0$ as

$$\dot{z}_i = \mathcal{N}\left(-z_i + \sum_{j=0}^{N-1} w_{i,j} z_j\right) \tag{6}$$

(cf. [KMadH19]). Since we are interested in gathering protocols and their dynamics, we have to investigate gathering points which we assume to be equilibrium points of (6) and their stability properties. These can at least locally be deduced from a linear approximation of the nonlinear system. To that end, arrange all right hand sides of (6) into a single vector $F_{\mathcal{N}}(Z)$ and rewrite the system in vector notation $\dot{Z} = F_{\mathcal{N}}(Z)$, as before. Then locally in a neighborhood of a gathering point $Z^* \in V_0$ the dynamics of the nonlinear system (6) is topologically conjugate to that of the linear system

$$\dot{\zeta} = DF_{\mathcal{N}}(Z^*)\zeta, \tag{7}$$

where $\zeta = Z - Z^*$ and $DF_{\mathcal{N}}(Z^*)$ is the Jacobian at the gathering point – this requires $\mathcal{N}$ to be continuously differentiable. In particular, the local stability properties of Z^* are are the same as the stability properties of the equilibrium $0 \in \mathbb{R}^{2N}$ of the linear system (7) which are determined by the spectral properties of $DF_{\mathcal{N}}(Z^*)$. For a more detailed exposition of this principle consider for example [HSD13, Section 8], and the famous Hartman-Grobman-Theorem (e.g. [Tes12]).

Using the chain rule we can write the Jacobian $DF_{\mathcal{N}}(Z^*)$ as

$$DF_{\mathcal{N}}(Z^*) = (-\mathbf{I}_N + W) \otimes D\mathcal{N} \left(-z^* + \sum_{j=0}^{N-1} w_{i,j} z^* \right).$$

By the consistency condition (3) it is is reasonable to simplify this equation to

$$DF_{\mathcal{N}}(Z^*) = (-\mathbf{I}_N + W) \otimes D\mathcal{N}(0).$$

which is a 'Kronecker-scaled' version of (4) by the Jacobian $D\mathcal{N}(0) \in \mathbb{R}^{2\times 2}$. Thus, to preserve the structure in (4) it suffices to require

$$D\mathcal{N}(0) = c\mathbf{I}_2, \tag{8}$$

for some $c \in \mathbb{R}$. In this case we have

$$DF_{\mathcal{N}}(Z^*) = -\mathbf{I}_{2N} + ((1-c)\mathbf{I}_N + cW) \otimes \mathbf{I}_2$$

and it follows that the linearized system (7) is precisely of the form (4) with weight matrix $W' = (1-c)\mathbf{I}_N + cW \in \mathbb{R}^{N\times N}$. In particular, for $c = 1$ we have $W' = W$ and all the results on linear systems that we present, can be directly applied locally to nonlinear systems of the form (6) as well. For instance, Theorem 2.3 tells us that (6) is (locally) gathering if and only if the corresponding non-normalized system (2) with weight matrix $W \in \mathbb{R}^{N\times N}$ is gathering.

Remark 2.5.

(a) *For instance, the requirement in (8) is satisfied if $\mathcal{N}(x, y) = (n(x, y), n(y, x))^T$ for some smooth function $n : \mathbb{R}^2 \to \mathbb{R}$ such that $n(x, -y) = n(x, y)$ for all $x, y \in \mathbb{R}$. In this case, we have $c = \frac{\partial}{\partial x} n(0, 0) = \frac{\partial}{\partial y} n(0, 0)$.*

(b) *A common approach in distributed computing is to have robots move with constant speed one which requires $\mathcal{N}(x, y) = \frac{1}{\|(x,y)\|}(x, y)^T$. However, this is not differentiable in gathering points and cannot be used for our analysis. We instead propose a 'smooth' version*

$$\mathcal{N}_\varepsilon(x, y) = \frac{1}{\|(x, y)\| + \exp\left(-\frac{\|(x,y)\|^2}{\varepsilon}\right)}(x, y)^T, \tag{9}$$

with $\varepsilon > 0$. Note that $\mathcal{N}_\varepsilon$ is of the form discussed in (a) with c = 1. For large $\|(x, y)\|$, the term $\exp\left(-\frac{\|(x,y)\|^2}{\varepsilon}\right)$ can be neglected, such that $\mathcal{N}_\varepsilon$ behaves like $\frac{1}{\|(x,y)\|}$. In the limit $\|(x, y)\| \to 0$ the exponential term becomes one, and hence so does the entire prefactor. Thus, close to the gathering point the nonlinear system (6) is almost linear.

(c) *In general, any smoothing like (9) is a modification of the non-smooth (nonlinear) protocol. Hence, one has to analyze whether the modified strategy is still gathering or how it has to be further modified. In particular, the effects of the limit $\varepsilon \to 0$ have to be studied. We leave this analysis for further research.*

3. Circulant Interaction Topology

We focus on the situation that the interaction structure is fixed for all times independent of the robots' positions. Pairwise interactions between robots are encoded by a (directed) *graph* $G = (V, E)$. The *vertices* are the robots $V = \{0, \dots, N-1\}$ and an *edge* $e = (j, i) \in E \subseteq V \times V$ represents the fact that robot i adapts its movement using the position of robot j. In other terms, robot j influences robot i. We refer to this graph as the *interaction graph.* In the context of linear strategies we further assume the interaction graph to be *weighted*: each edge $e = (j, i) \in E$ is assigned the *weight* $w_{i,j}$ from (2). A commonly considered special case is when the interaction graph is *symmetric* or *undirected*, i.e, the weight matrix $W \in \mathbb{R}^{N \times N}$ is symmetric. The interaction graph is said to be *connected* (or *weakly connected*) if for any two vertices $j, i \in \{0, \dots, N-1\}$ there is an *undirected path* from j to i. If there is even always an *directed path* the graph is said to be *strongly connected.*

The focus of this article lies in the dynamical analysis of (2) with a *circulant* interaction topology. The interaction graph is circulant if and only if there are integers $0 \le s_1 < \dots < s_k \le N-1$ – sometimes referred to as *jumps* – such that

$$(j, i) \in E \iff j = i + s_r \bmod N \text{ for an } r \in \{1, \dots, k\}.$$[1]

Circulant interaction structure implies that the weight matrix $W \in \mathbb{R}^{N \times N}$ is a *circulant matrix* given by

$$\begin{aligned} W &= \operatorname{circ}(w_0, w_1, \dots, w_{N-1}) \\ &:= \begin{pmatrix} w_0 & w_1 & \cdots & w_{N-2} & w_{N-1} \\ w_{N-1} & w_0 & w_1 & & w_{N-2} \\ \vdots & w_{N-1} & w_0 & \ddots & \vdots \\ w_2 & & \ddots & \ddots & w_1 \\ w_1 & w_2 & \cdots & w_{N-1} & w_0 \end{pmatrix} \in \mathbb{R}^{N \times N}. \end{aligned}$$

We say the vector $w = (w_0, \dots, w_{N-1})$ *generates* the circulant matrix W as $W_{i,j} = w_{(j-i) \bmod N}$ and therefore the underlying graph and topology. A circulant symmetric matrix W has fewer degrees of freedom. In fact, in this case we have the extra condition

$$w_{N-i} = w_i, \quad i = 1, \dots, N-1,$$

and W is determined by only $\lfloor \frac{N}{2} \rfloor + 1$ elements $w_0, \dots, w_{\lfloor \frac{N}{2} \rfloor}$.

Example 3.1.

(a) As a first running example we consider the N-BUG problem [WK69]. In this protocol each robot i is only influenced by its first right neighbor, which yields a circulant interaction structure of the form $W = \operatorname{circ}(0, w_1, 0 \dots, 0)$ with $w_1 \neq 0$. Here, we only have one jump $s_1 = 1$ and in Fig. 2 we illustrate its topology.

(b) As a second running example we consider the GO-TO-THE-MIDDLE *protocol (cf. [KMadH11]). In contrast to the N-BUG problem, the underlying interaction graph is symmetric. Its weight matrix is given by $W = \operatorname{circ}(0, \frac{1}{2}, 0, \dots, \frac{1}{2})$, i.e., robot i is influenced by its first left and right neighbor.*

[1] $i + s \bmod N = i + s,$ if $i + s < N; i + s - N,$ if $i + s \ge N.$

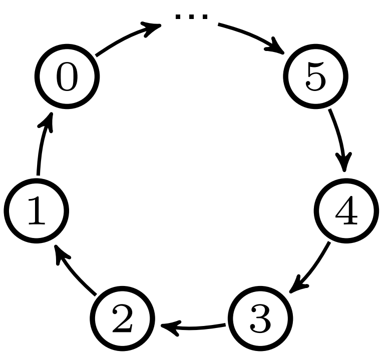
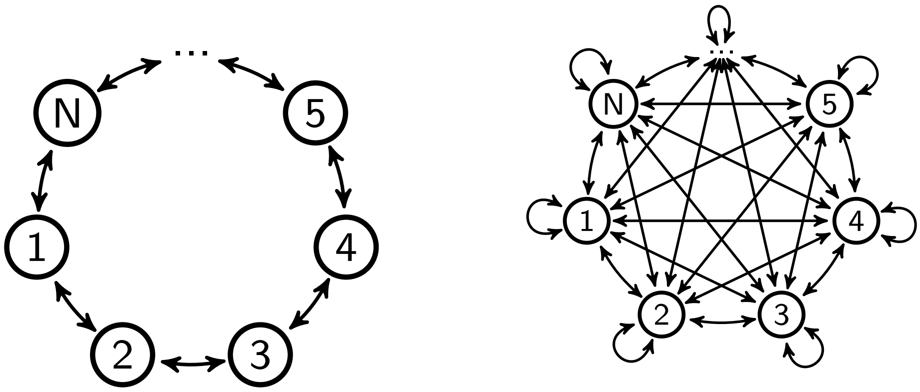


Figure 2: Illustration of the circulant interaction graph of the N-BUG problem. The arrangement of the vertices does not reflect the physical location of robots.

(c) *Finally as a third example, we take a look at the (global)* GO-TO-THE-AVERAGE *protocol (also called* GO-TO-THE-CENTER-OF-GRAVITY *in [CP04]). In this model every robot has global vision and is therefore influenced by all robots (including itself). Hence, the corresponding interaction graph is complete and its weight matrix can be written as* $W = \mathrm{circ}(\frac{1}{N}, \ldots, \frac{1}{N})$. *In Fig. 3b its complete interaction graph is illustrated.*



(a) Interaction graph of the GO-TO-THE-MIDDLE protocol

(b) Complete interaction graph of the GO-TO-THE-AVERAGE protocol

Figure 3: Illustration of circulant and symmetric interaction graphs. Note that the arrow tips may have been omitted due to its symmetric nature.

The classification of circulant gathering protocols follows as a consequence of Theorem 3.2 in [BS03], Theorem 4 in [OSM04], and the observation that the consistency condition (3) is necessary for a protocol to be gathering as outlined below (2). In fact, under the assumption of consistency, the system matrix $-\mathbf{I}_{2N} + \mathbf{W}$ in (4) equals $L \otimes \mathbf{I}_2$, where L is the graph Laplacian matrix of the weighted interaction graph so that these results apply.

Theorem 3.2. *For a circulant linear protocol modeled by* (2) *with weight matrix* $W = \mathrm{circ}(w_0, \dots, w_{N-1}) \in \mathbb{R}^{N\times N}$*, where* $w_i \geq 0$ *for* $i = 0, \dots, N-1$*, the following are equivalent:*

(i) The protocol is gathering.

(ii) The interaction graph is connected and the weight matrix is consistent $\sum_{i=0}^{N-1} w_i = 1$*.*

We would like to point out that weak and strong connectivity of a circulant graph are equivalent and thus we drop the distinction. Connectivity of a circulant graph can be determined using a purely combinatorial condition: G is connected if and only if $\gcd(N, s_1, \dots, s_k) = 1$ (cf. [vD86, Corollary 1]).

Example 3.3. *Consider again the protocols of our running Example 3.1 whose interaction graphs are connected due to the previous remark.*

(a) Applying the results of Theorem 3.2 to the N-BUG *problem yields that it is gathering if and only if its generating weight vector* $w \in \mathbb{R}^N$ *has the form* $w = (0, 1, 0, \dots, 0)^T \in \mathbb{R}^N$*.*

(b) Theorem 3.2 implies that the only gathering linear protocol on the undirected next neighbor graph Fig. 3a is the GO-TO-THE-MIDDLE *protocol.*

(c) Similarly, Theorem 3.2 implies that the GO-TO-THE-AVERAGE *protocol is gathering. It is, in fact, the only gathering linear protocol on the complete undirected graph for which all weights are equal.*

4. A hierarchy of convergence rates in circulant gathering systems

In the previous parts of this article, we have studied conditions for linear (circulant) protocols to be gathering. For a circulant interaction graph, i.e., a circulant weight matrix $W \in \mathbb{R}^{N\times N}$, we can actually exploit its structure even more to describe the dynamics of such protocols more precisely. As indicated in Theorem 2.3, the longtime behavior of (2) mainly depends on the spectral properties of $W \in \mathbb{R}^{N\times N}$. Fortunately, all eigenvalues and eigenvectors can be analytically derived for such a circulant matrix (see e.g. [Gra05]).

Proposition 4.1. *Let* $W \in \mathbb{R}^{N\times N}$ *be a circulant matrix generated by a vector* $w = (w_0, \dots, w_{N-1})^T \in \mathbb{R}^N$*. Then its eigenvectors* $v_j \in \mathbb{C}$ $(j = 0, \dots, N-1)$ *are given by*

$$v_j = (1, \omega^j, \omega^{2j}, \dots, \omega^{(N-1)j})^T \in \mathbb{C}^N, \tag{10}$$

where $\omega = \exp\left(\frac{2\pi \mathbf{i}}{N}\right)$ *is a primitive* N*-th root of unity. The corresponding eigenvalues are given by*

$$\lambda_j = \sum_{i=0}^{N-1} w_i \omega^{ij} \in \mathbb{C}. \tag{11}$$

Remark 4.2.

(a) *For $j = 0$, we obtain the vector $v_0 = (1, \dots, 1)^T \in \mathbb{R}^N$ as an eigenvector with corresponding (simple) eigenvalue $\lambda_0 = \sum_{i=0}^{N-1} w_i$, which has to be one for the protocol to be gathering according to Theorem 2.3.*

(b) *Since $W \in \mathbb{R}^{N \times N}$ has real-valued entries, we may immediately compute*

$$\lambda_j = \overline{\lambda_{N-j}} \text{ for } j = 1, \dots, N-1, \tag{12}$$

which implies for even $N = 2k$ that $\lambda_k \in \mathbb{R}$ is real-valued. Moreover, if $W \in \mathbb{R}^{N \times N}$ is symmetric, (12) reduces to

$$\lambda_j = \lambda_{N-j} \text{ for } j = 1, \dots, N-1,$$

and the eigenvalues $\lambda_1, \dots, \lambda_{\lfloor \frac{N-1}{2} \rfloor} \in \mathbb{R}$ are duplicated.

(c) *By definition (10), all eigenvectors $v_j \in \mathbb{C}^N$ are independent of the generating vector $w \in \mathbb{R}^N$, i.e., the given circulant protocol. In Fig. 4 we visualize the eigenvectors $v_j \in \mathbb{C}$, $j \neq 0$, in the complex plane for some $N \in \mathbb{N}$.*

For the upcoming analysis we consider the form (5) of the protocol as its block-diagonal structure is most useful to derive solutions in closed form. In general, solutions of (5) are linear combinations of the so-called fundamental solutions

$$\begin{aligned} \widetilde{Z}^x_{j,i}(t) &= e^{(\lambda_j - 1)t} \sum_{\ell=0}^{i} \frac{t^\ell}{\ell!} (\xi_{j,\ell}, 0), \\ \widetilde{Z}^y_{j,i}(t) &= e^{(\lambda_j - 1)t} \sum_{\ell=0}^{i} \frac{t^\ell}{\ell!} (0, \xi_{j,\ell}) \end{aligned} \tag{13}$$

where $\lambda_0, \dots, \lambda_k$ are the eigenvalues of W counting geometric multiplicities and

$$\xi_{0,1}, \dots, \xi_{0,m_0}, \dots, \xi_{k,1}, \dots, \xi_{k,m_k} \in \mathbb{C}^N$$

are the corresponding eigenvectors and generalized eigenvectors: $(W - \lambda_j \mathbf{I}_N)\xi_{j,0} = 0$ and $(W - \lambda_j \mathbf{I}_N)\xi_{j,i} = \xi_{j,i-1}$ for $j = 0, \dots, k$ and $i = 2, \dots, m_j$.

By Proposition 4.1 a circulant matrix W is diagonalizable as the eigenvectors $v_j \in \mathbb{C}^N$ are linearly independent. Thus, these fundamental solutions reduce to

$$\widetilde{Z}^x_j(t) = e^{(\lambda_j - 1)t} \begin{pmatrix} v_j \\ 0 \end{pmatrix} \text{ and } \widetilde{Z}^y_j(t) = e^{(\lambda_j - 1)t} \begin{pmatrix} 0 \\ v_j \end{pmatrix},$$

i.e, $m_j = 1$ and $\xi_{j,0} = v_j$ for $j = 0, \dots, N-1$. However, for $j \neq 0$, the eigenvector v_j and its corresponding eigenvalue λ_j is, in general, complex-valued, which gives little to no possibility for interpretation in terms of configurations of N robots moving in the two-dimensional Euclidean plane.

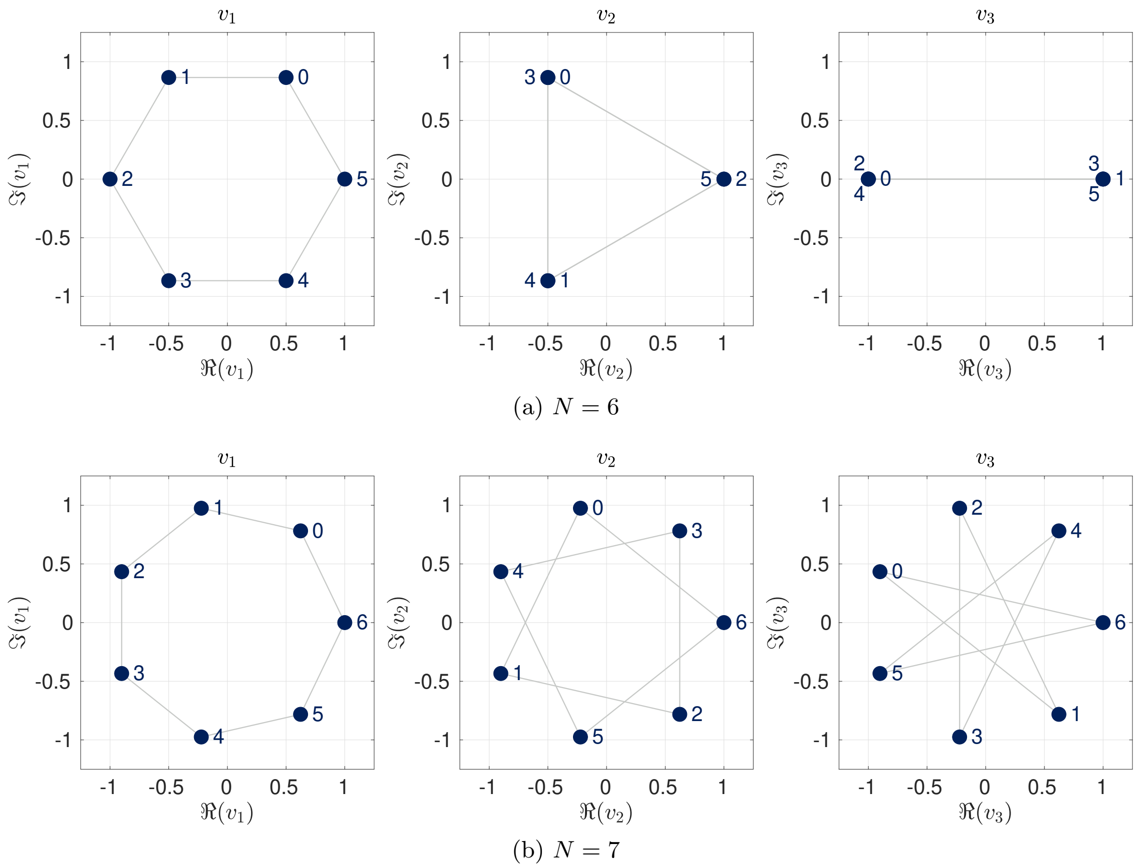


Figure 4: Visualization of the complex-valued eigenvectors $v_j \in \mathbb{C}^N$, $j \neq 0$, of a circulant matrix in the complex plane for (a) $N = 6$ and (b) $N = 7$. Here, the number i next to the blue dots correspond to i-th entry of $v_j \in \mathbb{C}$. The connections in gray between the points illustrate the (cyclically) ascending index $i = 0, 1, \ldots, N-1$. In particular, it does not indicate any dynamical influences according to an underlying interaction graph, i.e., the generating vector $w \in \mathbb{R}^N$. Note that, by definition all points are on the unit circle and thus form a regular N-polygon (see (10)). For increasing j, if $N = 2k$ is even, the N-polygon 'degenerates' since some entries are the same, while for $N = 2k+1$ only the ordering changes. Later on, we assign to every entry of $v_j \in \mathbb{C}$ its corresponding robot such that the dots also represent the position $x_i = \Re((v_j)_i))$ and $y_i = \Im((v_j)_i)$ of the i-th robot in the Euclidean plane (by changing the axis labels accordingly). Hence, the *generating* configuration of the stable invariant subspace $V_j \subseteq \mathbb{R}^{2N}$ in the Euclidean plane is also visualized (cf. Theorem 4.3.

To address this problem, we proceed as follows: Let $N = 2k$ be even or $N = 2k + 1$ be odd. First, for any $j = 1, \ldots, k$, we consider the pair $v_j \in \mathbb{C}^N$ and its complex conjugate $\overline{v_j} = v_{N-j} \in \mathbb{C}^N$ and define the subspace $\widetilde{V}_j \subseteq \mathbb{R}^N$ spanned by the real- and imaginary parts of v_j and v_{N-j}, which is

$$\widetilde{V}_j = \operatorname{span}\left(\Re(v_j), \Im(v_j)\right) \subseteq \mathbb{R}^N.$$

For $N = 2k$ even, $\widetilde{V}_k$ simply becomes $\widetilde{V}_k = \operatorname{span}(v_k) \subseteq \mathbb{R}^N$, since v_k has in fact real-valued entries (cf. Remark 4.2 (b)).

By abusing the notation and considering $\widetilde{V}_j$ to be also the matrix containing the basis vectors of $\widetilde{V}_j$ we immediately have

$$W\widetilde{V}_j = \widetilde{V}_j\Lambda_j \text{ with } \Lambda_j = \begin{pmatrix} \Re(\lambda_j) & -\Im(\lambda_j) \\ \Im(\lambda_j) & \Re(\lambda_j) \end{pmatrix} \tag{14}$$

since the pair of eigenvalues $\lambda_j \in \mathbb{C}$ and $\overline{\lambda_j} = \lambda_{N-j}$ act as a matrix $\Lambda_j \in \mathbb{R}^{2\times 2}$ in the subspace $\widetilde{V}_j \subseteq \mathbb{R}^N$. For brevity we will use this double meaning from now on. For $N = 2k$ even, (14) reduces to the eigenvalue equation $Wv_k = \Lambda_k v_k$ for $\Lambda_k = \lambda_k \in \mathbb{R}$. Finally, we set $\widetilde{V}_0 = \operatorname{span}(v_0)$ and $\Lambda_0 = \lambda_0 = 1$.

As the eigenvectors $v_j \in \mathbb{C}^N$ are linearly independent, we obtain a decomposition $\mathbb{R}^N = \bigoplus_{j=0}^{k} \widetilde{V}_j$, in which the weight matrix $W \in \mathbb{R}^{N\times N}$ becomes a block-diagonal of the form

$$\widetilde{V}^{-1}W\widetilde{V} = \begin{pmatrix} \Lambda_0 & & & \\ & \Lambda_1 & & \\ & & \ddots & \\ & & & \Lambda_k \end{pmatrix} \in \mathbb{R}^{N\times N} \tag{15}$$

where $\widetilde{V} = (\widetilde{V}_0 \cdots \widetilde{V}_k) \in \mathbb{R}^{N\times N}$.

Now, recall the block-diagonal structure in $\widetilde{\mathbf{W}} \in \mathbb{R}^{2N\times 2N}$, which corresponds to the disconnected dynamical behavior of both coordinates of all robots. Hence, the subspace $\widetilde{V}_j \subseteq \mathbb{R}^N$ can be considered as either x- or y-coordinates and we set

$$\begin{aligned} V_j^x &= \operatorname{span}\left(\begin{pmatrix} \Re(v_j) \\ 0 \end{pmatrix}, \begin{pmatrix} \Im(v_j) \\ 0 \end{pmatrix}\right) \subseteq \mathbb{R}^{2N}, \\ V_j^y &= \operatorname{span}\left(\begin{pmatrix} 0 \\ \Re(v_j) \end{pmatrix}, \begin{pmatrix} 0 \\ \Im(v_j) \end{pmatrix}\right) \subseteq \mathbb{R}^{2N}. \end{aligned}$$

As both $V_j^x \subseteq \mathbb{R}^N$ and $V_j^x \subseteq \mathbb{R}^N$ correspond to the same eigenvalue, we build their direct sum and define

$$V_j = V_j^x \oplus V_j^y = \operatorname{span}\left(\begin{pmatrix} \Re(v_j) \\ 0 \end{pmatrix}, \begin{pmatrix} \Im(v_j) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \Re(v_j) \end{pmatrix}, \begin{pmatrix} 0 \\ \Im(v_j) \end{pmatrix}\right). \tag{16}$$

which by construction is an invariant subspace of the model (5). For this decomposition

$\mathbb{R}^{2N} = \bigoplus_{j=0}^{k} V_j$ the matrix $\widetilde{\mathbf{W}} \in \mathbb{R}^{2N\times 2N}$ in (5) becomes

$$V^{-1}\widetilde{W}V = \begin{pmatrix} \Lambda_0 & & & & \\ & \Lambda_0 & & & \\ & & \ddots & & \\ & & & \Lambda_k & \\ & & & & \Lambda_k \end{pmatrix} \in \mathbb{R}^{2N\times 2N} \tag{17}$$

where $V = (V_0 \cdots V_k) \in \mathbb{R}^{2N\times 2N}$.

For $j = 0$, we recover the 2-dimensional subspace $V_0 \subseteq \mathbb{R}^{2N}$ of all gathering points (in $\widetilde{Z}$-coordinates). If $j \neq 0$, then there is a pair of eigenvalues $\lambda_j \in \mathbb{C}$ and $\overline{\lambda_j} = \lambda_{N-j} \in \mathbb{C}$ (or simply $\lambda_k \in \mathbb{R}$ for $N = 2k$ even) belonging to $V_j \subseteq \mathbb{R}^{2N}$ and we call the real part $\Re(\lambda_j) \in \mathbb{R}$ *convergence rate* in $V_j \subseteq \mathbb{R}^{2N}$. Note that $\Re(\lambda_j) < 1$ for $j \neq 0$ as we assume the underlying protocol to be gathering. Hence, $V_j \subseteq \mathbb{R}^{2N}$ is a *stable invariant subspace* of (5) meaning solutions starting in one of the $V_j \subseteq \mathbb{R}^{2N}$ remain in $V_j \subseteq \mathbb{R}^{2N}$ for all times and converge with convergence rate $\Re(\lambda_j)$ to 0 exponentially fast. The smaller $\Re(\lambda_j) < 1$ is, the faster the solution converges.

As an arbitrary initial configuration $\widetilde{Z} \in \mathbb{R}^{2N}$ can be written as a linear combination of the basis vectors of all $V_j \subseteq \mathbb{R}^{2N}$, we conclude that for $j \neq 0$ every part in $V_j \subseteq \mathbb{R}^{2N}$ vanishes as time proceeds and only the gathering point in $V_0 \subseteq \mathbb{R}^{2N}$ remains (cf. Corollary 4.5).

Moreover, if there is a convergence rate $\Re(\lambda_s)$ such that

$$\Re(\lambda_s) < \Re(\lambda_j) \quad \forall j \neq s,$$

the subspace $V_s \subseteq \mathbb{R}^{2N}$ is called *strong stable invariant subspace*. In this case, the convergence rate of any configuration in $V_s \subseteq \mathbb{R}^{2N}$ is faster than any configuration in any other $V_j \subseteq \mathbb{R}^{2N}$ for $0 \neq j \neq s$.

For odd $N \in \mathbb{N}$, every subspace $V_j \subseteq \mathbb{R}^{2N}$, $j \neq 0$, is 4-dimensional. However, if $N = 2k$ is even, the subspace $V_k \subseteq \mathbb{R}^{2N}$ for $j = k$ is only 2-dimensional which is due to the fact that $v_k \in \mathbb{R}^N$ has only real-valued entries.

By definition of $V_j \subseteq \mathbb{R}^{2N}$ in (16) the spanning vectors correspond to configurations, where all robots have only non-zero components in one coordinate (x or y). Thus, in particular for visualization purposes of the configurations contained in $V_j \subseteq \mathbb{R}^{2N}$ in the Euclidean plane, we propose the following basis instead

$$V_j = \operatorname{span}\left(\begin{pmatrix} \Re(v_j) \\ \Im(v_j) \end{pmatrix}, \begin{pmatrix} \Im(v_j) \\ \Re(v_j) \end{pmatrix}, \begin{pmatrix} \Re(v_j) \\ -\Im(v_j) \end{pmatrix}, \begin{pmatrix} -\Im(v_j) \\ \Re(v_j) \end{pmatrix} \right).$$

Here, the later three basis vectors (considered as points/robots in the Euclidean plane) are reflections of the first one. In this sense, we say $v_j \in \mathbb{C}^N$, respectively the configuration $(\Re(v_j), \Im(v_j))^T \in \mathbb{R}^{2N}$, *generates* the subspace $V_j \subseteq \mathbb{R}^{2N}$. In particular, the x-coordinate of the i-th robot is given by the real part of the i-the entry of $v_j \in \mathbb{C}^N$, while its y-coordinate is the corresponding imaginary part. Hence, the generating configuration $(\Re(v_j), \Im(v_j))^T \in \mathbb{R}^{2N}$ of $V_j \subseteq \mathbb{R}^{2N}$ can be visualized as in Fig. 4 by simply changing the axis labels to x and y. However, note that by choosing this adapted basis, the block structure into blocks Λ_j in (17) is slightly changed as we will get $\dim V_j \times \dim V_j$-dimensional blocks instead.

We summarize the results found above in the following theorem and state that the $2N$-dimensional state space of a linear circulant gathering protocol can decomposed as follows.

Theorem 4.3. *For a linear circulant gathering protocol of $N \in \mathbb{R}$ robots there exist a family of stable invariant subspaces $(V_j)_{j=1}^{k} \subseteq \mathbb{R}^{2N}$ with convergence rates $\Re(\lambda_j) < 1$ and a 2-dimensional subspace $V_0 \subseteq \mathbb{R}^{2N}$ of gathering points such that*

$$\mathbb{R}^{2N} = V_0 \oplus \left(\bigoplus_{j=1}^{k} V_j \right), \text{ where} \tag{18}$$

(i) if $N = 2k+1$ is odd, then every subspace $V_j \subseteq \mathbb{R}^{2N}$ is 4-dimensional.

(ii) if $N = 2k$ is even, then the subspace $V_k \subseteq \mathbb{R}^{2N}$ is 2-dimensional, whereas the remaining $V_j \subseteq \mathbb{R}^{2N}$, $j \neq k$, are 4-dimensional.

Moreover, each subspace $V_j \subseteq \mathbb{R}^{2N}$ is spanned by a generating eigenvector $v_j \in \mathbb{C}^N$ of the weight matrix $W \in \mathbb{R}^{N\times N}$ in the sense that

$$V_j = \operatorname{span}\left(\begin{pmatrix} \Re(v_j) \\ \Im(v_j) \end{pmatrix}, \begin{pmatrix} \Im(v_j) \\ \Re(v_j) \end{pmatrix}, \begin{pmatrix} \Re(v_j) \\ -\Im(v_j) \end{pmatrix}, \begin{pmatrix} -\Im(v_j) \\ \Re(v_j) \end{pmatrix} \right).$$

As indicated in Remark 4.2 (c) this decomposition in (18) is the identical for every linear circulant gathering protocol. Only the explicit values of the convergence rates $\Re(\lambda_j)$ depend on the protocol itself.

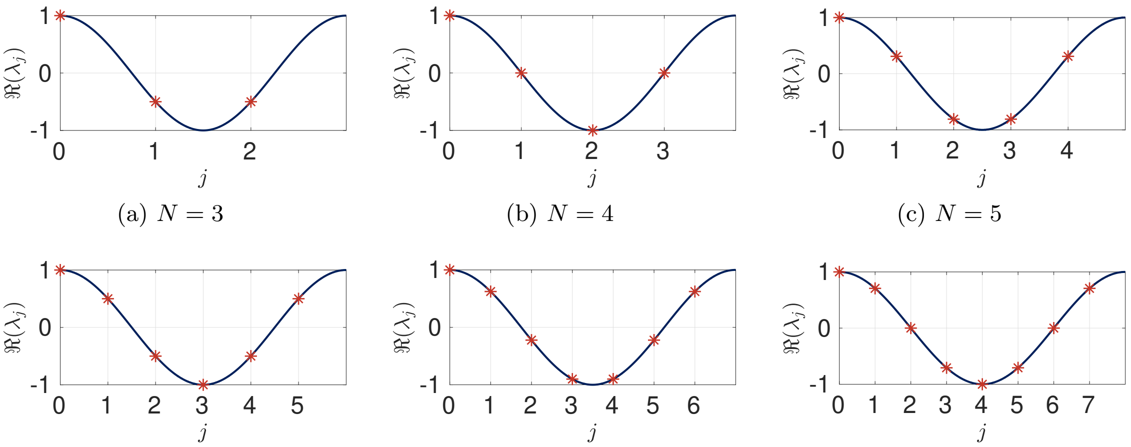
Example 4.4. *We apply Theorem 4.3 to the running Example 3.1 in order to illustrate the dynamical hierarchy of the decomposition (18). To this end, we explicitly compute the eigenvalues $\lambda_j \in \mathbb{C}$, the corresponding convergence rates $\Re(\lambda_j) \in \mathbb{R}$ and visualize the generating configurations $(\Re(v_j), \Im(v_j))^T \in \mathbb{R}^{2N}$.*

(a) For the N-BUG problem we have $w = (0,1,0,\ldots,0)^T \in \mathbb{R}^N$ (cf. Example 3.3 (a)) which by using the eigenvalue formula (11) gives us

$$\lambda_j = \omega^j = \cos\left(\frac{2\pi j}{N}\right) + \mathbf{i}\sin\left(\frac{2\pi j}{N}\right)$$

and thus $\Re(\lambda_j) = \cos\left(\frac{2\pi j}{N}\right)$ for $j = 0,1,\ldots,N-1$. For some choices of $N \in \mathbb{N}$ we illustrate the convergence rates in Fig. 5.

Note that they are strictly decreasing in j (up to k), i.e., $\Re(\lambda_j) > \Re(\lambda_{j+1})$ for $j < k$. Hence, the stable subspaces $V_j \subseteq \mathbb{R}^{2N}$ are hierarchically ordered such that any configuration $Z \in V_j$ converges faster to a gathering point than any other configuration $\widetilde{Z} \in V_i$ for $i < j$. For $N = 6$ and $N = 7$ their generating configurations are visualized in Fig. 4. In particular, this model has a strong stable subspace $V_k \subseteq \mathbb{R}^{2N}$ with corresponding convergence rate $\Re(\lambda_k) = \cos\left(\frac{2\pi k}{N}\right)$. For $N = 2k$ even, this strong stable subspace is 2-dimensional and $\Re(\lambda_k) = -1$. Its generating configuration is illustrated in the right panel of Fig. 4a. On the other hand, for $N = 2k+1$ odd, it is 4-dimensional with $\Re(\lambda_k) = \cos\left(\frac{2\pi k}{2k+1}\right)$.



(a) $N = 3$ (b) $N = 4$ (c) $N = 5$

(d) $N = 6$ (e) $N = 7$ (f) $N = 8$

Figure 5: Illustration of the convergence rates $\Re(\lambda_j)$ of the N-bug problem and the Go-To-The-Middle protocol for $N \in \{3, \dots, 8\}$ by red stars. For reference the cosine curve is also plotted in blue. As observed in Remark 4.2 (b) the convergence rates $\Re(\lambda_j)$ are symmetrically distributed (for $j \neq 0$).

(b) *Since the* Go-To-The-Middle *protocol is symmetric, we have real-valued eigenvalues $\lambda_j \in \mathbb{R}$ (cf. Remark 4.2 (b)), i.e., they coincide with the convergence rates $\Re(\lambda_j)$. Using the eigenvalue formula* (11) *with $w = (0, \frac{1}{2}, 0, \dots, \frac{1}{2})^T \in \mathbb{R}^N$ we compute*

$$\lambda_j = \Re(\lambda_j) = \cos\left(\frac{2\pi j}{N}\right) \quad \textit{for } j = 0, \dots, N-1.$$

*Observe that the convergence rates for this protocol are the same as for the N-*bug *problem discussed in (a) whose rates are illustrated in Fig. 5. In particular, we obtain the same hierarchical decomposition. Note that, from the dynamical systems perspective both models gather at the same speed. As the decomposition* (18) *is independent of the generating vector $w \in \mathbb{R}^N$, the generating configurations are also visualized in Fig. 4.*

(c) *The generating vector $w = (\frac{1}{N}, \dots, \frac{1}{N})^T \in \mathbb{R}^N$ of the* Go-To-The-Average *protocol yields the eigenvalues*

$$\lambda_j = \frac{1}{N} \sum_{i=0}^{N-1} \omega^{ij} = \begin{cases} 1, & \textit{if } j = 0, \\ 0, & \textit{if } j \neq 0, \end{cases}$$

since the average of the N-th roots of unity vanishes for $j \neq 0$. Again, by the symmetry of the protocol all eigenvalues λ_j are indeed real and thus coincide with the convergence rates. In particular, for this protocol we have $\Re(\lambda_j) = 0$ for all $j \neq 0$ and every configuration converges with the same speed. Even though Theorem 4.3 decomposes the state space into stable subspaces $V_j \subseteq \mathbb{R}^{2N}$ (also shown in Fig. 4), they cannot be distinguished by their convergence rates $\Re(\lambda_j) \in \mathbb{R}$.

For the dynamics of an given arbitrary initial configuration $\widetilde{Z} \in \mathbb{R}^{2N}$ an immediate consequence of Theorem 4.3 is the following.

Corollary 4.5. *Let $\widetilde{Z}(0) = (X(0), Y(0)) \in \mathbb{R}^{2N}$ be an initial configuration and the state space $\mathbb{R}^{2N}$ be decomposed as in (18). Then the solution $\widetilde{Z}(t) \in \mathbb{R}^{2N}$ of the linear gathering protocol (5) with initial condition $\widetilde{Z}(0) \in \mathbb{R}^{2N}$ can be written as*

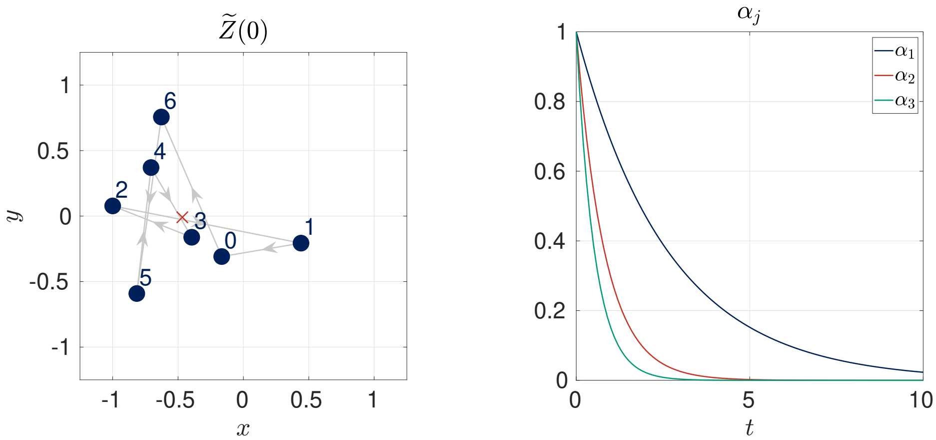
$$\widetilde{Z}(t) = \widetilde{Z}^* + \sum_{j=1}^{k} \alpha_j(t)\ \Xi_j(t), \ \textit{where} \tag{19}$$

(i) $\widetilde{Z}^* = (X^*, Y^*) = (x^*, \dots, x^*, y^*, \dots, y^*)$ is the final gathering point of $\widetilde{Z} \in \mathbb{R}^{2N}$. In particular, $x^* = \frac{1}{N}\sum_{i=0}^{N-1} X_i(0)$ and $y^* = \frac{1}{N}\sum_{i=0}^{N-1} Y_i(0)$.

(ii) $\alpha_j(t) = \exp((-1 + \Re(\lambda_j))t) \in \mathbb{R}$ is the exponentially decaying coefficient corresponding to $\Xi_j(t) \in V_j$.

(iii) by abusing notation, $\Xi_j(t) = V_j \beta_j(t) \in V_j$ for some coefficient vector $\beta_j(t) \in \mathbb{R}^{\dim V_j}$ with constant norm, i.e., $\|\beta_j(t)\|_2 = \|\beta_j(0)\|_2$ for all $t \geq 0$.

Example 4.6. *We illustrate the consequences of Corollary 4.5 for the N-BUG problem. For $N = 7$, we show in Fig. 6a a random initial configuration $\widetilde{Z} \in \mathbb{R}^{2N}$ as well as its final gathering point $\widetilde{Z}^* \in \mathbb{R}^{2N}$. The corresponding exponentially decaying coefficients $\alpha_j(t) \in \mathbb{R}$ are plotted in Fig. 6b. Moreover, the initial decomposition into $\Xi_j(0) \in V_j$ is*

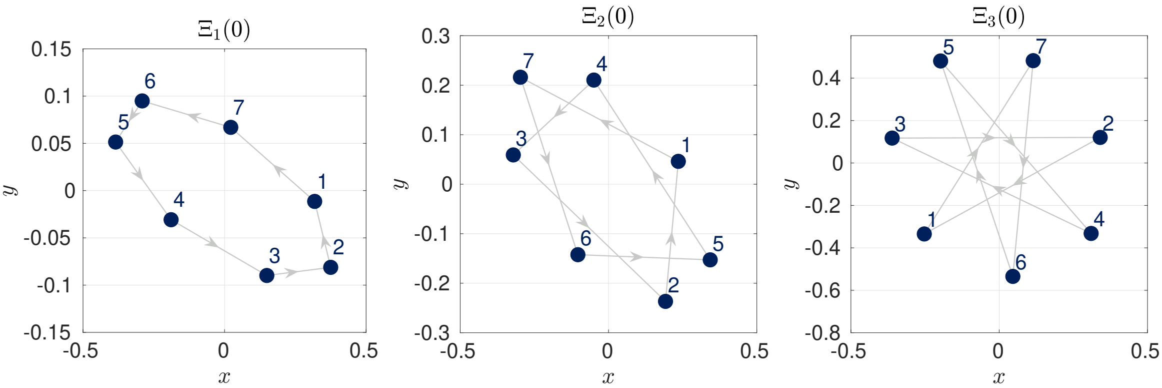


(a) $\widetilde{Z}(0) \in \mathbb{R}^{2N}$ and $\widetilde{Z}^* \in \mathbb{R}^{2N}$

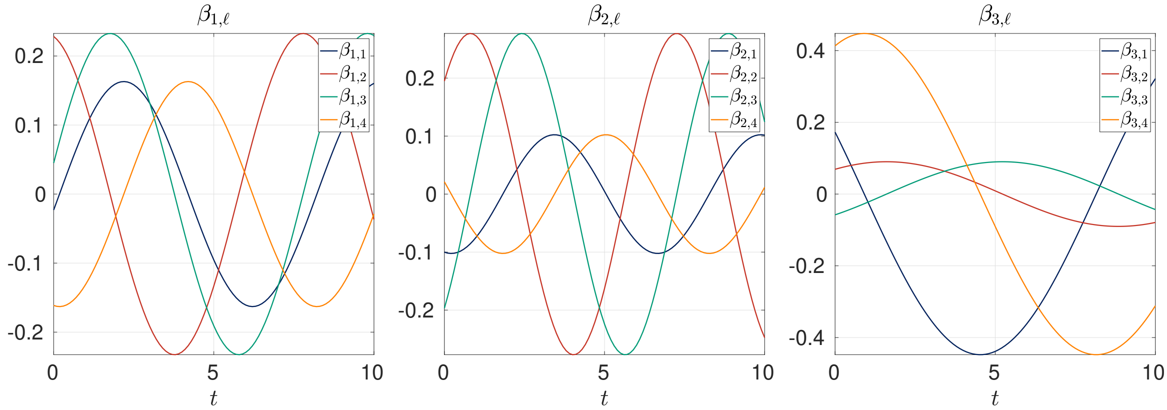
(b) Trajectory of $\alpha_j(t) \in \mathbb{R}$

Figure 6: Visualization of a random initial configuration $\widetilde{Z}(0) \in \mathbb{R}^{2N}$ and its gathering point $Z^* \in \mathbb{R}^{2N}$ (red cross) in the Euclidean plane for $N = 7$ in (a). Its corresponding decaying coefficients $\alpha_j(t) \in \mathbb{R}$ are shown in (b). Note that $\alpha_j(t) \in \mathbb{R}$ decreases faster for increasing j, which illustrates the dynamical hierarchy discussed in Example 4.4 (a).

visualized in Fig. 7a, whereas the coefficients $\beta_j(t) \in \mathbb{R}^{\dim V_j}$ are shown in Fig. 7b. A related short movie of the simulation of $\widetilde{Z}(t) \in \mathbb{R}^{2N}$ and $\Xi_j(t) \in V_j$ can be found online at this page: https://math.uni-paderborn.de/en/ag/chair-of-applied-mathematics/research/research-projects/swarmdynamics



(a) Initial decomposition of $\widetilde{Z}(0)$ into $\Xi_j(0) \in V_j$.



(b) Trajectory of the corresponding coefficients $\beta_j(t) \in \mathbb{R}^{\dim V_j}$ such that $\Xi_j(t) = V_j\beta_j(t) \in V_j$.

Figure 7: Illustration of the decomposition (19) and its individual dynamics. Note that $\beta_j(t) \in \mathbb{R}^{\dim V_j}$ only consists of sinusoidal curves since the decaying part is contained in $\alpha_j(t) \in \mathbb{R}$. In particular, we have $\|\beta_j(t)\| = \|\beta_j(0)\|$ for all $t \geq 0$.

5. Robots with Limited Vision

One of the key issues in current distributed computing research is to develop and analyze protocols which use only local information: realistically, the robots have a limited viewing range $\mathcal{C} > 0$ within which they can perceive the positions of other robots. While synchronization results under similar constraints are available (e.g. [OSM03a]), it remains largely unclear if visibility of different agents with a limited vision radius can be preserved under the dynamics. In fact, if two communicating robots lose sight of each other a local deterministic protocol cannot guarantee that they will ever regain sight of each other [KMadH19]. We conclude with a brief outlook on this question.

A matrix $W \in \mathbb{R}^{N\times N}$ is called *non-defective* if all its eigenvalues are real and their algebraic and geometric multiplicities agree. For a non-defective matrix all fundamental solutions (13) are real and of the form exponential multiplied with a constant vector. In original coordinates, the same can be seen for individual robots $z_i(t)$ and by linearity also for differences of two robots. For a gathering protocol, all exponentials are strictly decreasing in t or constant. In particular, the distance between two arbitrary robots cannot increase.

Proposition 5.1. *Consider a linear gathering protocol with weight matrix $W \in \mathbb{R}^{N\times N}$ and a valid initial configuration $z_0(0), \dots, z_{N-1}(0) \in \mathbb{R}^2$, that is,*

$$\|z_i(0) - z_j(0)\| \le \mathcal{C} \quad \text{for all } (j,i) \in E.$$

If $W \in \mathbb{R}^{N\times N}$ is non-defective then visibility is preserved under the dynamics, i.e.,

$$\|z_i(t) - z_j(t)\| \le \mathcal{C} \quad \text{for all } t \ge 0 \text{ and } (j,i) \in E.$$

Circulant communication strategies are not necessarily non-defective. For instance, $W = \operatorname{circ}(0, 5, -4)$ describes a circulant linear gathering strategy according to Theorem 2.3 but has complex eigenvalues. The initial condition $z_0(0) = (0,0)$, $z_1(0) = \frac{1}{\sqrt{10}}(-\mathcal{C}, 3\mathcal{C})$, $z_2(0) = \frac{1}{\sqrt{10}}(-2\mathcal{C}, 2\mathcal{C})$ satisfies $\|z_0(0) - z_1(0)\| = \mathcal{C}$ and $\|z_i(0) - z_j(0)\| \le \mathcal{C}$ for all i, j. However, one readily observes that the distance between robots 0 and 1 increases for small $t > 0$ and therefore these robots lose sight of each other.

This example depends crucially on the fact that there are negative weights. If we restrict to non-negative weights we may prove that visibility is preserved by gathering protocols. The proof uses the consistency of the weight matrix of a gathering circulant protocol to show that the two maximally distant robots cannot move further away from each other.

Proposition 5.2. *Consider a circulant linear gathering protocol with weight matrix $W = \operatorname{circ}(w_0, \dots, w_{N-1}) \in \mathbb{R}^{N\times N}$ and assume all weights are non-negative: $w_i \ge 0$ for all $i \in \{0, \dots, N-1\}$. Let $z_0(0), \dots, z_{N-1}(0) \in \mathbb{R}^2$ be a valid initial configuration, that is,*

$$\|z_i(0) - z_j(0)\| \le \mathcal{C} \quad \text{for all } (j,i) \in E.$$

Then, visibility is preserved under the dynamics, i.e.,

$$\|z_i(t) - z_j(t)\| \le \mathcal{C} \quad \text{for all } t \ge 0 \text{ and } (j,i) \in E.$$

Proof. Throughout this proof, we use the convention that robot indices are counted $\bmod N$, as outlined in Section 3. In particular, the equations of motion (2) become

$$\dot{z}_i = -z_i + \sum_{j=0}^{N-1} w_j z_{i+j}. \tag{20}$$

Fix $(j,i) \in E$ such that

$$\|z_i(0) - z_j(0)\| = \max_{(l,k)\in E} \|z_k(0) - z_l(0)\|.$$

That is, robots i and j are furthest apart initially among those that do communicate. We drop the time-dependence in notation and compute and estimate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|z_i - z_j\|^2 &= 2\left(-\|z_i - z_j\|^2 + \sum_{k=0}^{N-1} w_k \langle z_i - z_j, z_{i+k} - z_{j+k}\rangle\right) \\
&\le 2\left(-\|z_i - z_j\|^2 + \sum_{k=0}^{N-1} w_k \|z_i - z_j\| \cdot \|z_{i+k} - z_{j+k}\|\right) \\
&\le 2\|z_i - z_j\|^2 \left(-1 + \sum_{k=0}^{N-1} w_k\right) \le 0.
\end{aligned}$$

Therein, the first estimate is due to the Cauchy-Schwarz inequality. For the second estimate, note that the sum in the third expression contains only terms $\|z_{i+k} - z_{j+k}\|$ for $(j+k, i+k) \in E$ which satisfy $\|z_{i+k} - z_{j+k}\| \leq \|z_i - z_j\|$ by assumption. In fact, since $W = (w_{i,j})_{i,j=1}^{N-1}$ is circulant, we have $w_{i+k,j+k} = w_{i,j} = w_l \neq 0$ for $j = i+l \mod N$ and any $k \in \{0, \dots, N-1\}$. Thus, also $(j+k, i+k) \in E$. The final estimate is due to the fact that W is consistent.

Hence, the distance of the two maximally distant robots cannot increase initially and by using $z_0(t), \dots, z_{N-1}(t)$ as a new initial configuration, the same is true for arbitrary $t > 0$.

Note that we may not rule out that two robots k, l with $(l, k) \in E$ that are not maximally distant increase their distance. However, assume $\|z_k(t^*) - z_l(t^*)\| > \mathcal{C}$ for some $t^* > 0$. Since $\|z_i(t) - z_j(t)\|$ is non-increasing, there must be some $0 < t < t^*$ at which $\|z_k(t) - z_l(t)\| = \|z_i(t) - z_j(t)\| \leq \mathcal{C}$ by continuity of the solution $z(t)$. Then we may again use $z_0(t), \dots, z_{N-1}(t)$ as a new initial configuration and the argument above with the roles of (j, i) and (l, k) switched yields that $\|z_k(t) - z_l(t)\|$ cannot increase any further contradicting the assumption. In particular, k and l cannot lose sight of each other. This completes the proof. $\square$

Acknowledgements

The authors would like to express their special gratitude to Friedhelm Meyer auf der Heide, Jannik Castenow, and Jonas Harbig who provided invaluable inspiration and counsel for the distributed computing side of the project and took part in the development of some of the research ideas presented in this manuscript in numerous spirited and productive discussions. This work is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project number 453112019.

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A. Technical Background

We end this manuscript with an appendix on some extended technical details concerning the convergence results in Sections 2 and 3. These are included for the sake of completeness without claiming novelty of the presented results.

A.1. Gathering in General Linear Systems

We apply the theory of linear systems of ordinary differential equations to systems of the form (2) for arbitrary weight matrices W to discuss the convergence result in Theorem 2.3 in more detail. In particular, we may deduce several basic properties that W has to satisfy in order for system (2) to model a gathering protocol. Again, the form (5) is most useful for these considerations.

First, we investigate when every gathering point is invariant under the dynamics induced by the protocol. By definition this requires the system (2) to be in equilibrium for any fully synchronous configuration $Z^* = (z^*, \dots, z^*) \in \mathbb{R}^{2N}$. From the transformed system (5) we can immediately read off that this means that any point $X^* = (x^*, \dots, x^*) \in \mathbb{R}^N$ is a zero of $-(\mathbf{I}_N + W)$. It can readily be seen that this is case if and only if X^* is an eigenvector to the eigenvalue 1 of W, which in turn requires all rows of the weight matrix to sum to 1. This shows the following proposition.

Proposition A.1. *The following are equivalent:*

(i) Any gathering point $Z^* = (z^*, \dots, z^*) \in \mathbb{R}^{2N}$ is an equilibrium point of the protocol (2).

(ii) The weight matrix $W \in \mathbb{R}^{N\times N}$ has an eigenvalue 1 with corresponding eigenvector $(1, \dots, 1)^T \in \mathbb{R}^N$.

(iii) The weight matrix $W \in \mathbb{R}^{N\times N}$ is consistent.

Proof. *(i)* $\implies$ *(ii)*: Let us first assume that $Z^* \in \mathbb{R}^{2N}$ is an equilibrium point of (2). In particular, $f(Z^*) = z^*$. In transformed coordinates, the equilibrium point is given by $\widetilde{Z}^* = (X^*, Y^*) = (x^*, \dots, x^*, y^*, \dots, y^*) \in \mathbb{R}^{2N}$ and the right hand side of (5) vanishes in $\widetilde{Z}^* \in \mathbb{R}^{2N}$. We obtain,

$$\widetilde{\mathbf{W}}\widetilde{Z}^* = \mathbf{I}_{2N}\widetilde{Z}^*,$$

which is equivalent to

$$WX^* = X^* \quad \text{and} \quad WY^* = Y^*$$

Since $X^* = x^* \cdot (1, \dots, 1)^T \in \mathbb{R}^N$, this completes the proof of the first direction.

(ii) $\implies$ *(iii)*: Let $(1, \dots, 1)^T \in \mathbb{R}^N$ be an eigenvector of $W \in \mathbb{R}^{N\times N}$ to the eigenvalue 1. Then

$$\sum_{j=0}^{N-1} w_{i,j} = (W(1, \dots, 1)^T)_i = ((1, \dots, 1)^T)_i = 1$$

for all $i \in \{0, \dots, N-1\}$, which proves *(iii)*.

(iii) $\implies$ *(i)*: Assume that all rows of the weight matrix $W \in \mathbb{R}^{N\times N}$ sum to 1 and let $Z^* \in \mathbb{R}^{2N}$ be an arbitrary gathering point, i.e., $\widetilde{Z}^* = (X^*, Y^*) = (x^*, \dots, x^*, y^*, \dots, y^*) \in \mathbb{R}^{2N}$ in transformed coordinates. Then

$$(WX^*)_i = \sum_{j=0}^{N-1} w_{i,j} \cdot x^* = x^*$$

for all $i \in \{0, \dots, N-1\}$ and by the same argument also $(W^*Y^*)_i = y^*$. In particular, we obtain

$$\widetilde{\mathbf{W}}\widetilde{Z}^* = (x^*, \dots, x^*, y^*, \dots, y^*)^T = \mathbf{I}_{2N}\widetilde{Z}^*.$$

This implies that $\widetilde{Z}^* \in \mathbb{R}^{2N}$ is a zero of the right hand side of (5) and in turn that $Z^* \in \mathbb{R}^{2N}$ is an equilibrium of (2). □

By a similar argument we may prove a slightly stronger statement. In fact, (5) shows that any equilibrium point of the transformed system is necessarily an eigenvector of the weight matrix corresponding to the eigenvalue 1. Together with Proposition A.1 this shows:

Proposition A.2. *The following are equivalent:*

(i) The subspace $V_0 = \{ Z^* = (z^*, \dots, z^*) \mid z^* \in \mathbb{R}^2 \} \subseteq \mathbb{R}^{2N}$ consists of all equilibrium points of (2).

(ii) The weight matrix W has an eigenvalue 1 with eigenvector $(1, \dots, 1)^T \in \mathbb{R}^N$ whose geometric multiplicity is 1.

Proof. Using Proposition A.1 it suffices to prove that no point $Z \notin V_0$ is an equilibrium point of (2) if and only if the geometric multiplicity of the eigenvalue 1 is 1. This can be seen readily from the transformed system (5). In fact, any equilibrium satisfies

$$\left(-\mathbf{I}_{2N} + \widetilde{\mathbf{W}}\right)\widetilde{Z} = 0 \iff \widetilde{\mathbf{W}}\widetilde{Z} = \widetilde{Z} \iff \begin{cases} WX & = X \\ WY & = Y \end{cases}$$

in transformed coordinates $\widetilde{Z} = (X, Y) \in \mathbb{R}^{2N}$. In particular, $\widetilde{Z} \in \mathbb{R}^{2N}$ is an equilibrium outside of $V_0 \subseteq \mathbb{R}^{2N}$ if and only if $X \in \mathbb{R}^N$ or $Y \in \mathbb{R}^N$ is an eigenvector of $W \in \mathbb{R}^{N\times N}$ to the eigenvalue 1 that is linearly independent of $(1, \dots, 1)^T \in \mathbb{R}^N$. Under the given assumptions, this exists if and only if the geometric multiplicity of the eigenvalue 1 is greater than 1. □

Next, we investigate when linear protocols guarantee that each initial configuration converges to an equilibrium. This property can essentially be read off from the fundamental solutions (13) which form any solution of the system via linear combinations. It can readily be seen that whenever an eigenvalue satisfies $\Re(\lambda) \neq 1$ the corresponding fundamental solutions are dominated by the exponential for $t \to \infty$. In particular, they converge to 0 if $\Re(\lambda) < 1$ and they 'diverge' if $\Re(\lambda) > 1$. For an eigenvalue with $\Re(\lambda) = 1$ the exponential is taken of a purely imaginary number if $\lambda \neq 1$. Hence, real and imaginary parts of the corresponding fundamental solution are given by a (periodic) trigonometric function multiplied with a polynomial expression, which do not converge. Finally, the eigenvalue $\lambda = 1$ causes the exponential to be constant and the corresponding fundamental solution to be given by the polynomial expressions. These converge if and only if they are in fact constant, which can only happen if the eigenvalue has only true eigenvectors. These considerations summarize to the following result.

Proposition A.3. *The following are equivalent:*

(i) For any initial configuration $Z(0) = (z_1(0), \ldots, z_N(0)) \in \mathbb{R}^{2N}$ the solution of (2) converges to an equilibrium point $\overline{Z} \in \mathbb{R}^{2N}$.

(ii) All eigenvalues $\lambda \in \mathbb{C}$ of $W \in \mathbb{R}^{N\times N}$ satisfy $\Re(\lambda) < 1$ or $\lambda = 1$, and if 1 is an eigenvalue of $W \in \mathbb{R}^{N\times N}$ then its algebraic and geometric multiplicities agree.

Proof. (i) $\implies$ (ii): Assume that any solution of (2) converges to an equilibrium point $\overline{Z} \in \mathbb{R}^{2N}$. We first consider the case that $W \in \mathbb{R}^{N\times N}$ has an eigenvalue $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 1$. As a simplification, we begin with $\lambda \in \mathbb{R}$. Then $\lambda > 1$ and there is a corresponding real eigenvector $\xi \in \mathbb{R}^N$. Note that $\widetilde{Z}(t) = \exp((\lambda-1)t)(\xi, 0)$ is the corresponding fundamental solution for $j = 1$ in (13) of the transformed system (5). In fact, one readily computes $\dot{\widetilde{Z}}(t) = (\lambda - 1)\widetilde{Z}(t)$ and

$$\begin{aligned}\left(-\mathbf{I}_{2N} + \widetilde{\mathbf{W}}\right)\widetilde{Z}(t) &= \exp((\lambda-1)t)\left(-\mathbf{I}_{2N} + \widetilde{\mathbf{W}}\right)(\xi, 0)\\ &= \exp((\lambda-1)t)(-1+\lambda)(\xi, 0)\\ &= (\lambda-1)\widetilde{Z}(t).\end{aligned}$$

Since $\lambda - 1 > 0$, we have $\|\widetilde{Z}(t)\| \to \infty$ for $t \to \infty$, which contradicts the assumption. If the eigenvalue λ is complex the argument remains the same, however the computations become slightly more complicated. In fact, replacing $\widetilde{Z}(t)$ with $\Re(\exp((\lambda-1)t)(\xi, 0))$ yields the same result.

Next, consider an eigenvalue $\lambda \neq 1$ with $\Re(\lambda) = 1$. That is, the eigenvalue is of the form $\lambda = 1 + \mathbf{i}\omega$ for some $\omega \in \mathbb{R}$. Again omitting the details, $\widetilde{Z}(t) = \Re(\exp((\lambda-1)t)(\xi, 0)) = \cos(\omega t)\Re(\xi, 0))$ for a complex valued eigenvector $\xi \in \mathbb{C}^N$ is a fundamental solution of the system (5) that does not converge to any equilibrium. In fact, it is a periodic solution.

Finally, consider the case that $\lambda = 1$ is an eigenvalue of $W \in \mathbb{R}^{N\times m}$ whose geometric multiplicity is less than its algebraic multiplicity. Then there exist two linearly independent vectors $\xi, \zeta \in \mathbb{R}^N$ such that ξ is an eigenvector and ζ a corresponding generalized eigenvector, i.e., $(W - \mathbf{I}_N)\zeta = \xi$. Then, $\widetilde{Z}(t) = (\zeta, 0) + t(\xi, 0)$ is a fundamental solution of (5). Once again, this solution does not converge to an equilibrium point, as $\|(\zeta, 0) + t(\xi, 0)\| \to \infty$ for $t \to \infty$.

(ii) $\implies$ (i): Assume that all eigenvalues $\lambda \in \mathbb{C}$ of $W \in \mathbb{R}^{N\times N}$ satisfy $\Re(\lambda) < 1$ or $\lambda = 1$ and that if $\lambda = 1$ is an eigenvalue then its algebraic and geometric multiplicities agree. Note that for all fundamental solutions in (13) in the limit $t \to \infty$ all terms are dominated by the exponential as long as $\Re(\lambda_i - 1) \neq 0$.

Under our assumptions, the only situation in which $\Re(\lambda_i - 1) = 0$ is given by the eigenvalue $\lambda_1 = \cdots = \lambda_\ell = 1$, where ℓ is the algebraic and geometric multiplicity. For these eigenvalues, the two fundamental solutions in (13) are precisely of the form $Z_i^x(t) = (\xi_i, 0)$ and $Z_i^y(t) = (0, \xi_i)$, where $\xi_i \in \mathbb{R}^N$ are the corresponding eigenvectors of $W \in \mathbb{R}^{N\times N}$ for $i = 1, \ldots, \ell$ – note that no generalised eigenvectors exist. For all other eigenvalues $\lambda_{\ell+1}, \ldots, \lambda_k$ we have $\Re(\lambda_i - 1) < 0$. As a result, any linear combination of the fundamental solutions converges to a linear combination of $(\xi_1, 0), \ldots, (\xi_k, 0), (0, \xi_1), \ldots, (0, \xi_k)$. As any linear combination of eigenvectors to the eigenvalue 1 constitutes another eigenvector, this linear combination is a zero of the right hand side of (5) and therefore an equilibrium point. □

The combination of Propositions A.2 and A.3 proves the convergence result in Theorem 2.3. While the necessary and sufficient condition in that theorem is purely algebraic, we can also use Proposition A.2 to prove a necessary condition on the interaction structure of the robots to allow for a linear protocol to be gathering. In fact, if the interaction graph is not (weakly) connected, there are multiple groups of robots that do not communicate with each other. Then these groups might gather individually, but the protocol cannot guarantee that all groups chose the same gathering point. This is the statement of the following.

Proposition A.4. *Consider a linear protocol modeled by (2) with weight matrix $W \in \mathbb{R}^{N\times N}$. If the protocol is gathering then the interaction graph is (weakly) connected.*

Proof. For any two vertices $i, j \in \{0, \dots, N-1\}$ of the interaction graph, define $i \sim j$ if and only if there is an undirected path from j to i (if and only if there is an undirected path from i to j). It can readily be seen that this constitutes an equivalence relation on $\{0, \dots, N-1\}$. Thus, it generates a partition $P_1, \dots, P_k \subset V$ such that $P_r \neq \emptyset$, $P_r \cap P_s = \emptyset$ if $r \neq s$, and $P_1 \cup \dots \cup P_k = \{0, \dots, N-1\}$, where $i, j \in P_r$ if and only if $i \sim j$. These partitions are called the *connected components* of the interaction graph. By definition, the interaction graph is weakly connected if and only if all vertices are connected via undirected paths, i.e., if and only if $k = 1$.

Assume that the interaction graph is not weakly connected. Then $k \geq 2$. Define $P = P_1$ and $Q = P_2 \cup \dots \cup P_k$. By construction, there are no paths between any vertices in P and Q. In particular, there are no edges from any vertex in P to any vertex in Q and vice versa, i.e., $(j, i) \in E$ implies $i, j \in P$ or $i, j \in Q$. Since the interaction structure is reflected in the weight matrix $W = (w_{i,j})_{i,j=0}^{N-1}$, this in particular implies that $w_{i,j} = 0$ whenever $i \in P$ and $j \in Q$ or $i \in Q$ and $j \in P$.

Then, we define $Z^* = (z_1^*, \dots, z_N^*)^T \in \mathbb{R}^{2N}$ as

$$z_i^* = \begin{cases} (1,1)^T & \text{if } i \in P, \\ 0 & \text{if } i \in Q. \end{cases}$$

Considering this point as an initial configuration, we determine the robots' behavior by applying the right hand side of (2) to it. For $i \in P$ we obtain

$$\begin{aligned} \dot{z}_i^* &= -z_i^* + \sum_{j=0}^{N-1} w_{i,j} z_j^* = -z_i^* + \sum_{j\in P} w_{i,j} z_j^* \\ &= \left(-1 + \sum_{j\in P} w_{i,j}\right)(1,1)^T = \left(-1 + \sum_{j=0}^{N-1} w_{i,j}\right)(1,1)^T. \end{aligned}$$

As the protocol is assumed to be gathering, we have $\sum_{j=0}^{N-1} w_{i,j} = 1$ for all $i \in \{0, \dots, N-1\}$ by Proposition A.1 and we obtain $\dot{z}_i^* = 0$. Similarly, for $i \in Q$ we compute

$$\dot{z}_i^* = -z_i^* + \sum_{j\in Q} w_{i,j} z_j^* = \left(-1 + \sum_{j\in Q} w_{i,j}\right) 0 = 0.$$

Combining these two computations, we see that $Z^* \in \mathbb{R}^{2N}$ is an equilibrium point of (2). However, as $Z^* \notin V_0$, the protocol cannot be gathering due to Proposition A.2. □

Remark A.5. *More precisely it is already the property that any initial configuration converges to a gathering point that implies (weak) connectivity.*

On the other hand, if we restrict ourselves to matrices with non-negative weights as suggested in Remark 2.2, we can also formulate a sufficient condition for a protocol to be gathering. If the interaction graph is strongly connected we can apply the Perron-Frobenius theorem for non-negative matrices (e.g. [Gan09, Theorem 2]). It tells us that the weight matrix has a simple real eigenvalue – called the Perron-Frobenius eigenvalue – that is in between the minimal and the maximal row sums and all other eigenvalues have real parts that are strictly smaller than the Perron-Frobenius eigenvalue. Hence, if the weight matrix is also consistent the Perron-Frobenius eigenvalue is 1 and the weight matrix satisfies the conditions in Theorem 2.3.

Proposition A.6. *Consider a linear protocol modeled by* (2) *with non-negative weights* $w_{i,j} \in \mathbb{R}$. *If the interaction graph is strongly connected and the weight matrix is consistent – i.e.,* $\sum_{j=0}^{N-1} w_{i,j} = 1$ *for all* $i \in \{0, \ldots, N-1\}$ *– then the protocol is gathering.*

Proof. Recall that the weight matrix $W \in \mathbb{R}^{N\times N}$ is irreducible, if the underlying interaction graph is strongly connected. As furthermore all entries of $W \in \mathbb{R}^{N\times N}$ are non-negative, we may apply the Perron-Frobenius theorem for non-negative matrices (e.g. [Gan09, Theorem 2]). It tells us that the spectral radius $\rho(W) = \max\{|\lambda_1|, \ldots, |\lambda_k|\}$, where $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of $W \in \mathbb{R}^{N\times N}$, is itself an eigenvalue $\lambda = \rho(W)$ of $W \in \mathbb{R}^{N\times N}$. It is called the *Perron-Frobenius eigenvalue*, which is real by definition and all eigenvalues satisfy $|\lambda_i| \leq \lambda$. Furthermore, according to the Perron-Frobenius theorem, the Perron-Frobenius eigenvalue is simple. In particular, this implies $\Re(\mu) < \lambda$ for all eigenvalues $\mu \neq \lambda$, since λ is real. Finally, note that for irreducible, non-negative matrices the Perron-Frobenius eigenvalue is also bounded by the minimal and maximal row sums of $W \in \mathbb{R}^{N\times N}$ [Gan09, Remark 2], i.e.,

$$\min_i \sum_{j=0}^{N-1} w_{i,j} \leq \lambda \leq \max_i \sum_{j=0}^{N-1} w_{i,j}.$$

By assumption, all row sums satisfy $\sum_{j=0}^{N-1} w_{i,j} = 1$ so that we have $\lambda = 1$. To summarize, $W \in \mathbb{R}^{N\times N}$ has a simple eigenvalue $\lambda = 1$ and all other eigenvalues μ satisfy $\Re(\mu) < 1$. Furthermore, due to Proposition A.1, the eigenvector to the eigenvalue $\lambda = 1$ is $(1, \ldots, 1)^T \in \mathbb{R}^N$. Hence, by Theorem 2.3, the protocol is gathering. □

Next, we provide a proof that for any linear gathering protocol the gathering point is always the average of the initial positions (cf. Theorem 2.3). As we have discussed above any solution to (5) is a linear combination of the fundamental solutions (13) and for a gathering protocol all of them converge to 0 except for the fundamental solutions corresponding to the simple eigenvalue 1 with eigenvector $\xi = (1, \ldots, 1)^T \in \mathbb{R}^N$. The coefficients of this linear combination are constant and determined by the initial configuration $\widetilde{Z}(0) = (X(0), Y(0)) \in \mathbb{R}^2$. In fact the coefficients of $X(0)$ and $Y(0)$ corresponding to this basis vector are given by

$$a = \frac{\langle X(0), \xi\rangle}{\langle \xi, \xi\rangle} = \frac{1}{N}\sum_{i=1}^{N} x_i(0), \qquad b = \frac{\langle Y(0), \xi\rangle}{\langle \xi, \xi\rangle} = \frac{1}{N}\sum_{i=1}^{N} y_i(0).$$

Thus, the protocol converges to $a(\xi,0)^T + b(0,\xi)^T$ which has the average x initial values in its first N entries and the average y initial values in its last N entries.

Proposition A.7. *Consider a linear gathering protocol with weight matrix $W \in \mathbb{R}^{N\times N}$ and an initial configuration of the robots $z_0(0),\dots,z_{N-1}(0) \in \mathbb{R}^2$. Then, the position $z_i(t)$ of any robot $i \in \{0,\dots,N-1\}$ converges to the gathering point $z^* = \frac{1}{N}\sum_{i=0}^{N-1} z_i(0) \in \mathbb{R}^2$, i.e., the average of the initial positions, for $t \to \infty$.*

Proof. First, note that $W \in \mathbb{R}^{N\times N}$ has a simple eigenvalue 1 with corresponding eigenvector $(1,\dots,1)^T \in \mathbb{R}^N$ with all other eigenvalues having real part less than 1, since it models a gathering protocol. As stated before, the solution $\widetilde{Z}(t)$ with initial condition $\widetilde{Z}(0)$ of the linear system of ordinary differential equations in transformed coordinates (5) is given by a linear combination of the fundamental solutions in (13). The coefficients of this linear combination can be determined by setting $t = 0$ and requiring the linear combination to be equal to $Z(0)$. Setting $t = 0$ in (13), one immediately sees that all fundamental solutions are reduced to precisely one of the eigenvectors and generalized eigenvectors that has been copied into the X- or the Y-coordinates respectively. In particular, solving for the linear coefficients boils down to solving

$$X(0) = \sum_{i=1}^{k}\sum_{j=1}^{m_i} a_{i,j}\xi_{i,j} \quad \text{and} \quad Y(0) = \sum_{i=1}^{k}\sum_{j=1}^{m_i} b_{i,j}\xi_{i,j} \tag{21}$$

for the coefficients $a_{i,j}, b_{i,j} \in \mathbb{R}$, where $\widetilde{Z}(0) = (X(0), Y(0)) \in \mathbb{R}^{2N}$ is the initial condition $Z(0)$ in transformed coordinates. Since the eigenvectors and generalized eigenvectors of $W \in \mathbb{R}^{N\times N}$ constitute a basis of $\mathbb{R}^N$, both equations can uniquely be solved. It remains to observe that all fundamental solutions satisfy

$$\|Z^{x,y}_{i,j}(t)\| \to 0 \quad \text{for} \quad t \to \infty,$$

for every eigenvalue $\lambda \neq 1$. For the simple eigenvalue 1 the corresponding fundamental solutions are constant. Without loss of generality, we set $\lambda_1 = 1$ and $\xi_{1,1} = (1,\dots,1)^T \in \mathbb{R}^N$ for the corresponding eigenvector (cf. Proposition A.1). Then we have

$$\widetilde{Z}(t) \to a_{1,1}((1,\dots,1)^T, 0) + b_{1,1}(0,(1,\dots,1)^T) = (a_{1,1},\dots,a_{1,1}, b_{1,1},\dots,b_{1,1})^T$$

for $t \to \infty$. In particular, the position of every robot converges to $(a_{1,1}, b_{1,1})$.

It remains to show that indeed $(a_{1,1}, b_{1,1}) = \left(\frac{1}{N}\sum_{i=1}^{N} x_i(0), \frac{1}{N}\sum_{i=1}^{N} y_i(0)\right)$. To that end, consider (21), which describes the decomposition of $X(0)$ and $Y(0)$ into the eigenbasis of $W \in \mathbb{R}^{N\times N}$. Since the eigenvalue $\lambda_1 = 1$ is simple, the corresponding eigenspace is one-dimensional. This allows us to compute the corresponding coefficients to be

$$a_{1,1} = \frac{\langle X(0), \xi_{1,1}\rangle}{\langle \xi_{1,1}, \xi_{1,1}\rangle} = \frac{1}{N}\sum_{i=1}^{N} x_i(0),$$
$$b_{1,1} = \frac{\langle Y(0), \xi_{1,1}\rangle}{\langle \xi_{1,1}, \xi_{1,1}\rangle} = \frac{1}{N}\sum_{i=1}^{N} y_i(0).$$

□

A.2. Linear Circulant Gathering Models

To obtain the gatherint result for circulant interaction structures, one readily specifies the findings of Section 2 for this class. That is, we assume the weight matrix $W = \mathrm{circ}(w_0, \dots, w_{N-1}) \in \mathbb{R}^{N\times N}$ to be a circulant matrix generated by a weight vector $w = (w_0, \dots, w_{N-1})^T \in \mathbb{R}^N$ from now on. It remains to show the following necessary condition.

Proposition A.8. *Consider a circulant linear protocol modeled by (2) with weight matrix $W = \mathrm{circ}(w_0, \dots, w_{N-1}) \in \mathbb{R}^{N\times N}$. If the protocol is gathering, then the interaction graph is strongly connected and $W \in \mathbb{R}^{N\times N}$ is consistent, i.e., $\sum_{j=0}^{N-1} w_{i,j} = \sum_{j=0}^{N-1} w_j = 1$ for all $i \in \{0, \dots, N-1\}$.*

Proof. Assume the protocol is gathering. Then Proposition A.4 implies that the interaction graph is weakly connected. As it is also circulant, it is therefore also strongly connected as mentioned above. Furthermore, due to Proposition A.1, all rows of the weight matrix sum to 1 and for all $i \in \{0, \dots, N-1\}$ we obtain

$$1 = \sum_{j=0}^{N-1} w_{i,j} = \sum_{j=0}^{N-1} w_j,$$

since all rows of W contain the same set of entries. □

Together with Theorem 2.3 this proves Theorem 3.2.

Remark A.9. *The consistency condition $\sum_{i=0}^{N-1} w_i = 1$ for non-negative weights implies $0 \le w_i \le 1$ for all $i \in \{0, \dots, N-1\}$. In this case $W \in \mathbb{R}^{N\times N}$ is a* stochastic *matrix. As row and column sums coincide for circulant matrices, $W \in \mathbb{R}^{N\times \mathbb{N}}$ is in fact* doubly stochastic *under these assumptions.*

The characterization of gathering protocols in Theorem 3.2 distinguishes circulant protocols from arbitrary linear ones. For example, consider the non-circulant weight matrix

$$W = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

All its rows sum to 1 and its eigenvalues are $1, \frac{1}{2}, -\frac{1}{2}$. Thus, according to Proposition A.1 and Theorem 2.3 it is gathering. However, the underlying interaction graph is $G = (V, E)$ with

$$V = \{0, 1, 2\} \text{ and } E = \{(0,1), (1,0), (2,0), (2,1), (2,2)\}.$$

There is no directed path from vertices 0 or 1 to vertex 2 so that the interaction graph is not strongly connected – it is weakly connected, however.