Hypernetworks: cluster synchronisation is a higher-order effect

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Abstract

Many networked systems are governed by non-pairwise interactions between nodes. The resulting higher-order interaction structure can then be encoded by means of a hypernetwork. In this paper we consider dynamical systems on hypernetworks by defining a class of admissible maps for every such hypernetwork. We explain how to classify robust cluster synchronisation patterns on hypernetworks by finding balanced partitions, and we generalize the concept of a graph fibration to the hypernetwork context. We also show that robust synchronisation patterns are only fully determined by polynomial admissible maps of high order. This means that, unlike in dyadic networks, cluster synchronisation on hypernetworks is a higher-order, i.e., nonlinear effect. We give a formula, in terms of the order of the hypernetwork, for the degree of the polynomial admissible maps that determine robust synchronisation patterns. We also demonstrate that this degree is optimal by investigating a class of examples. We conclude by displaying how this effect may cause remarkable synchrony breaking bifurcations that occur at high polynomial degree.

1 Introduction

Summary of the main results Recent advances in applications ranging from physics (coupled oscillator networks) over ecology (species interaction models) to social sciences (social interaction models) have indicated that, instead of by pairwise interactions, ensemble dynamics of networked real-world systems are frequently driven by simultaneous interactions of groups of network agents, so-called *higher-order interactions* [1, 2, 3]. While examples of these structural aspects have been exploited in theoretical (mathematical) studies as well, a unifying framework that defines coupled dynamical systems corresponding to a higher-order hypergraph structure is still largely lacking. Consequently, this paper

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- generalises the concepts of coupled cell networks (Definition 2.1), admissible maps and vector fields (Definition 2.4), graph fibrations (Definition 3.1) and quotient networks (Definition 3.5), and studies the properties of these generalisations to define, manipulate, and analyse coupled dynamical systems on hypergraphs;

- demonstrates that, unlike in classical (dyadic) networks, cluster synchronisation on hypernetworks is determined by higher-order terms in the equations of motion, and is thus a purely nonlinear effect (Section 4 and Section 5). We provide a precise expression for the polynomial degree at which cluster synchronisation is determined (Theorem 4.1) and show by means of examples that a lower polynomial degree is in general not sufficient (Theorem 5.4).

The relation between this paper and the existing concepts mentioned under the first bullet point, shall be made more precise in the background section below. Here, we would like to point out that the main result mentioned under the second bullet point distinguishes hypernetwork dynamical systems from classical (dyadic) network dynamical systems, where cluster synchronisation is known to be completely determined by the linear terms in the equations of motion [4, 5]. As far as we know, this is one of the first examples of a dynamical phenomenon that is fundamentally different in hypernetworks than in classical networks. Furthermore, we show numerically in Section 5 that this phenomenon leads to a remarkable new type of bifurcations. A systematic analytical investigation of this type of bifurcation will be presented in a separate paper.

We elaborate on the background and relations to classical network dynamical systems in the remainder of this introduction, but we first present an example.

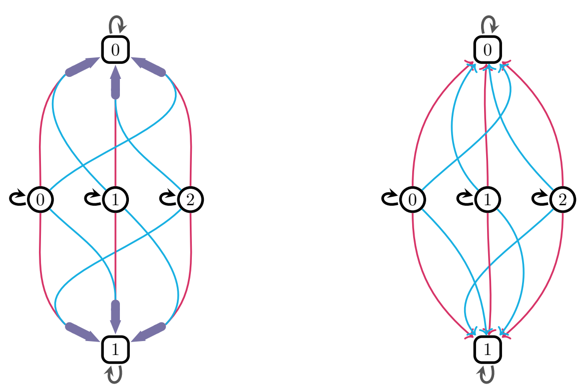


Figure 1: The left figure depicts the hypernetwork that yields equations as in (1.1). The coloring encodes the distinct inputs of each hyperedge, while the arrows specify their targets. Replacing each hyperedge by two edges yields the network on the right, with governing equations as in (1.5). The classical network and the hypernetwork have different robust synchrony spaces.

Example 1.1. Consider the differential equations

$$\begin{aligned}
\dot{x}_0 &= G(x_0),\\
\dot{x}_1 &= G(x_1),\\
\dot{x}_2 &= G(x_2),\\
\dot{y}_0 &= F(y_0, (x_0, x_1), (x_1, x_2), (x_2, x_0)),\\
\dot{y}_1 &= F(y_1, (x_0, x_2), (x_1, x_0), (x_2, x_1)),
\end{aligned} \tag{1.1}$$

for $x_0, x_1, x_2, y_0, y_1 \in \mathbb{R}$. We require that the function $F: \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the invariance equations

$$\begin{aligned}
&F(Y, (X_0, X_1), (X_2, X_3), (X_4, X_5)) =\\
&F(Y, (X_2, X_3), (X_0, X_1), (X_4, X_5)) =\\
&F(Y, (X_0, X_1), (X_4, X_5), (X_2, X_3)).
\end{aligned} \tag{1.2}$$

In other words, the three pairs of X-inputs of F can be exchanged without changing the value of F (this is why we emphasize pairs of variables among the arguments of F using additional brackets). We make no assumptions on the function $G: \mathbb{R} \to \mathbb{R}$. As a result of (1.2), we may think of the cells with states y_0 and y_1 as being targeted by three identical *hyperedges* of order two. The hypernetwork that encodes the structure of equations (1.1) is depicted in the left panel of Figure 1. In Section 2, we shall define, in general, *admissible maps and vector fields* for a broad class of hypernetworks.

Synchronisation occurs when groups of cells in the system evolve synchronously. Note for example that substituting $x_0 = x_1 = x_2$ and $y_0 = y_1$ in (1.1) yields that $\dot{x}_0 = \dot{x}_1 = \dot{x}_2$ and $\dot{y}_0 = \dot{y}_1$. This implies that the synchrony space

$$\{x_0 = x_1 = x_2 \text{ and } y_0 = y_1\}$$

is invariant under the flow of any ODE of the form (1.1). The (larger) subspace $\{y_0 = y_1\}$, on the other hand, is not. Choosing for instance

$$F(Y, (X_0, X_1), (X_2, X_3), (X_4, X_5)) = X_0 X_1^2 + X_2 X_3^2 + X_4 X_5^2$$

—which satisfies (1.2)— we find that

$$\dot{y}_0 = x_0 {x_1}^2 + x_1 {x_2}^2 + x_2 {x_0}^2 \quad \text{while} \quad \dot{y}_1 = x_0 {x_2}^2 + x_2 {x_1}^2 + x_1 {x_0}^2 .$$

Generically, we will thus have that $\dot{y}_0 \neq \dot{y}_1$ when $y_0 = y_1$, i.e., the synchrony space $\{y_0 = y_1\}$ is not flow-invariant for this F.

Note that the function F that we chose here is *nonlinear*. To see why this is important, note that any *linear* F satisfying (1.2) is of the form

$$\begin{aligned}
&F(Y, (X_0, X_1), (X_2, X_3), (X_4, X_5))\\
&= aY + bX_0 + cX_1 + bX_2 + cX_3 + bX_4 + cX_5 .
\end{aligned} \tag{1.3}$$

For such F, we see from (1.1) that

$$\begin{aligned}
\dot{y}_0 &= ay_0 + (bx_0 + cx_1) + (bx_1 + cx_2) + (bx_2 + cx_0),\\
\dot{y}_1 &= ay_1 + (bx_0 + cx_2) + (bx_1 + cx_0) + (bx_2 + cx_1).
\end{aligned}$$

It is easy to see that the right hand sides of these ODEs are equal when $y_0 = y_1$, and therefore the synchrony space $\{y_0 = y_1\}$ is flow-invariant whenever F is linear. Perhaps surprisingly, we conclude that linear systems of the form (1.1) have more flow-invariant synchrony spaces than general nonlinear ones.

One way to understand this phenomenon is to observe from (1.3) that any linear F satisfying (1.2) automatically satisfies the stronger invariance equations

$$\begin{aligned}
&F(Y, X_0, X_1, X_2, X_3, X_4, X_5) = \\
&F(Y, X_2, X_1, X_0, X_3, X_4, X_5) = \\
&F(Y, X_0, X_1, X_4, X_3, X_2, X_5) = \\
&F(Y, X_0, X_3, X_2, X_1, X_4, X_5) = \\
&F(Y, X_0, X_1, X_2, X_5, X_4, X_3)\,.
\end{aligned} \tag{1.4}$$

This in turn means that any linear system of the form (1.1) is automatically an admissible system for the (classical) coupled cell network shown in the right panel in Figure 1. This network has been constructed by replacing each hyperedge of the original hypernetwork by two edges. The admissible ODEs of this classical network are of the form

$$\begin{aligned}
\dot{x}_0 &= G(x_0), \\
\dot{x}_1 &= G(x_1), \\
\dot{x}_2 &= G(x_2), \\
\dot{y}_0 &= F(y_0, x_0, x_1, x_1, x_2, x_2, x_0)\,, \\
\dot{y}_1 &= F(y_1, x_0, x_2, x_1, x_0, x_2, x_1)\,,
\end{aligned} \tag{1.5}$$

with F satisfying (1.4). One quickly checks that the synchrony space $\{y_0 = y_1\}$ is invariant under the flow of all systems of the form (1.5).

It is known that the patterns of robust synchrony of a classical (dyadic) coupled cell network are completely determined by its linear admissible maps, see [4, 5]. Example 1.1 shows that this is not true for hypernetwork systems. We shall investigate this phenomenon in detail in this paper.

Background Systems of interacting dynamical units are prevalent in nature, whether it is the coordinated activity of neurons in the brain, interacting species in ecology, or opinion building in social networks. To investigate interconnected systems mathematically, one studies *network dynamical systems*—coupled (nonlinear) maps or differential equations that describe individual units and their interactions. These systems behave vastly different from systems without an underlying connection structure. Among the most striking phenomena observed in network dynamical systems are *(cluster) synchronisation*—some or all cells evolve identically—and unusual bifurcation behaviour.

A prominent method to define dynamical systems that respect network structure is the *groupoid formalism* developed by Golubitsky, Stewart, and collaborators [6, 7], and Field [8]. It allows to translate structural features of the network into dynamical properties using algebraic tools that can be summarized using the language of *graph fibrations* [9] and *quiver representations* [10]. Specifically, network dynamical systems are modelled by coupled cell systems as exemplified by the system in (1.5). Key results in this field include the classification of *robust* patterns of synchrony—i.e., dynamically invariant, independent of the governing functions; compare to Example 1.1—and the observation that these are determined already by linear systems (e.g. [11, 4, 12, 13, 5, 14]), the classification of generic bifurcations in terms of the network structure (e.g. [15, 16, 17, 18, 19]), as well as insights into real world problems (e.g. [20, 21, 22]) with no claim of this list being complete.

In recent years, there has been growing interest in the effect of simultaneous nonlinear interactions between three or more units—commonly referred to as *higher-order interactions*—on the network dynamics. This has been ignited by developments in various disciplines: For example in neuroscience, one observes that the signal of one neuron activates or inhibits the communication channel between two other ones (cf. [1]). In ecology, the simultaneous competition for resources of multiple species leads to nonstationary fluctuations of species abundancies typically observed in ecological networks of competing species (cf. [2]). In social sciences, multi-agent interactions can lead to a change of the average opinion in consensus dynamics (cf. [3]). Moreover, recent results show that higher-order interactions can emerge from data-driven model reconstruction, even when the original system is a pairwise coupled network (cf. [23]). These advances suggest that also in the mathematical investigation of network dynamics the underlying structures be sharpened from networks represented by graphs to *hypernetworks* represented by *hypergraphs* (Figure 1) in order to keep precise track of group interactions. Significant progress has been made in that regard. However, most investigations have studied individual examples or specific physical systems (see for example the excellent surveys [24, 25, 26, 27] and references therein).

From a theoretical perspective, a major obstacle to gauging the impact of higher-order interactions on dynamical phenomena stems from the fact that the existing approaches to define network dynamical systems are either not well suited to incorporate or simply do not contain higher-order interactions at all. Generic systems according to the groupoid formalism are unspecific to the precise structural nature of the interactions (e.g. a generic function F in (1.5) allows for arbitrary group-interactions influencing y_0 and y_1). On the other hand, application inspired systems rely on pairwise interactions only, for example by imposing additive input structure (e.g. the governing function F in (1.5) is chosen to have the form $F(Y_0, X_0, \ldots, X_5) = H(Y_0) + \sum_j a_j J(Y_0, X_j)$ with suitable functions H and J and weights a_j) (e.g. [28]).

Additionally, in the existing literature it is not always clear how the higher-order interactions, that is, the hypernetwork structure, enters or shapes the equations governing the dynamics. In particular, different authors use different conventions, which makes it hard to compare results qualitatively. A comprehensive, unifying formalism to define admissible dynamical systems that respect the structure of a given hypernetwork is necessary. First approaches to address this issue have been made only very recently and allowed for intriguing results. We want to highlight two main lines of work. One approach has been to investigate hypernetworks with an *additive input structure* (cf. [29, 30, 31, 32, 33, 34]). On the other hand, the investigation of *simplicial complexes*—i.e., hypergraphs with additional structural properties—allows for analytic results (cf. [35, 36]). In both cases, there are tools to determine how the higher-order network structure shapes dynamics, e.g. in the form of robust cluster synchrony as well as in structure-dependent stability properties.

The goal of this paper is to take a more general stance in the sense that we consider directed hypergraphs or hypernetworks (which are more general than simplicial complexes) and define admissible maps and vector fields without the restriction to additive input structure. In particular, we generalise the groupoid formalism to hypernetworks and exploit this generalisation to characterise and understand synchronisation in the hypernetwork context. We foresee that this formalism can eventually be used to model equations of motion or evolution in the applications outlined above. It would follow that the theoretical results can explain or even predict unexpected behaviour in these systems. For example, modelling an opinion formation process according to this formalism allows to determine all robust patterns of synchrony. These, in turn, might explain the newly observed average opinion or even the occurrence of multiple opposing opinions that are shared by groups of agents.

Structure of the article This article is structured as follows. In Section 2, we introduce hypernetworks and their admissible maps as well as balanced partitions and robust synchrony subspaces. In Section 3, we relate balanced partitions to hypergraph fibrations and quotient hypernetworks. In Section 4, we characterise robust cluster synchrony in terms of balanced partitions, and we give a polynomial degree at which cluster synchronisation is determined. Finally, Section 5 presents a class of examples that show that the polynomial degree at which cluster synchronisation is determined, found in Section 4, is optimal. We conclude Section 5 with an example of highly unusual bifurcation behavior in a hypernetwork system, in which steady-state branches break synchrony only up to high order in the bifurcation parameter.

2 Hypernetworks and their admissible maps

In this section, we formalise the idea that hypernetworks can encode the structure of the interactions between dynamical variables. Before introducing dynamics on hypernetworks, we first define hypernetworks as a type of directed hypergraph. Of course, the concept of the structure of a hypergraph is not new, see [37] for a recent survey.

Definition 2.1. A *hypernetwork* is a tuple $\mathbf{N} = (V, H, s, t)$ in which V is a finite set of vertices and H is a finite set of hyperedges. The map s assigns to each hyperedge a finite ordered list of source vertices $s(h) = (s_1(h), \ldots, s_k(h)) \in V^k$. The length k of $s(h)$ is called the *order* of h, and the *order* of the hypernetwork is the maximum of the order of its hyperedges. The map $t : H \to V$ assigns to each hyperedge a unique target vertex.

In addition, all vertices and hyperedges are assigned a type (chosen from some finite set), such that

1. if two hyperedges $h_1, h_2 \in H$ have the same type, then they have equal order. Moreover, their sources $s_i(h_1)$ and $s_i(h_2)$ have the same type for each $i = 1, \dots, k$ (where k is the order of h_1 and h_2), and their targets $t(h_1)$ and $t(h_2)$ have the same type;

2. if two vertices $v_1, v_2 \in V$ have the same type, then there is a type-preserving bijection $\alpha : t^{-1}(v_1) \to t^{-1}(v_2)$ between the hyperedges that target v_1 and v_2.

A subset of vertices $V' \subset V$ such that $s(h) \in (V')^k$ for all $h \in H$ with $t(h) \in V'$ together with hyperedges $H' = \{h \in H \mid t(h) \in V'\}$ defines a *sub-hypernetwork of* $\mathbf{N}$, $\mathbf{N}' = (V', H', s|_{H'}, t|_{H'})$. We write $\mathbf{N}' \sqsubset \mathbf{N}$.

Remark that a single vertex could act multiple times as a source of a hyperedge h, while it could also act as a source of multiple hyperedges. Definition 2.1 generalises the definition of a coupled cell network as in the groupoid formalism. In fact, a coupled cell network is simply a hypernetwork in which every hyperedge has order one. Our generalisation formalises the idea that, when the hyperedges h_1 and h_2 have the same type, then the source vertices $s(h_1) = (s_1(h_1), \dots, s_k(h_1))$ together influence the target vertex $t(h_1)$ through the hyperedge h_1, in exactly the same way as the source vertices $s(h_2)$ impact the target vertex $t(h_2)$ through the hyperedge h_2.

Example 2.2. The left panel of Figure 1 depicts a hypergraph with 5 vertices of two types. It contains 5 (i.e. hyperedges of order 1) of two types that each form a self-loop on one of the vertices. Additionally, it contains 6 equal-type hyperedges of order two, with two cells of the first type as sources, and a cell of the second type as target. This example (and its generalisations) will appear at various places in this paper.

Example 2.3. Figure 2 displays a hypernetwork with 5 vertices. Vertices v_0, v_1 and v_2 are of the same type and form a classical first-order network: they are targeted only by edges. Vertices w_0 and w_1 are also of the same type and each receive one edge (from themselves; not depicted) and three equal-type hyperedges of order 2. Both inputs of these hyperedges are taken from v_0, v_1 and v_2.

We are now ready to define hypernetwork dynamical systems in terms of the admissible maps for the hypernetwork.

Definition 2.4. Let $\mathbf{N} = (V, H, s, t)$ be a hypernetwork. Assume that for every $v \in V$ an *internal phase space* $\mathbb{R}^{n_v}$ is given in such a way that $n_{v_1} = n_{v_2}$ whenever v_1 and v_2 are of the same vertex-type. That is, vertices of the same type have identical internal phase spaces. A map or vector field

$$f : \bigoplus_{v \in V} \mathbb{R}^{n_v} \to \bigoplus_{v \in V} \mathbb{R}^{n_v}$$

defined on the *total phase space* $\bigoplus_{v\in V}\mathbb{R}^{n_v}$ is called **N**-*admissible* if it is of the form

$$f_v(x) = F_v\left(\bigoplus_{h\,:\,t(h)=v}\mathbf{x}_{s(h)}\right)$$

for some *response function* F_v. Here we write

$$\mathbf{x}_{s(h)} = (x_{s_1(h)},\ldots,x_{s_k(h)}) \in \mathbb{R}^{n_{s_1(h)}}\oplus\ldots\oplus\mathbb{R}^{n_{s_k(h)}}$$

for the ordered list of source variables of h, and

$$\bigoplus_{h\,:\,t(h)=v}\mathbf{x}_{s(h)} \in \bigoplus_{h\,:\,t(h)=v}\left(\mathbb{R}^{n_{s_1(h)}}\oplus\ldots\oplus\mathbb{R}^{n_{s_k(h)}}\right)$$

for the list of all the input variables of F_v. We furthermore require that the F_v satisfy the following invariance condition: if $\alpha : t^{-1}(v_1)\to t^{-1}(v_2)$ is any hyperedge-type-preserving bijection between the targeting hyperedges of two vertices of the same type, then

$$F_{v_2}\left(\bigoplus_{t(h_2)=v_2}\mathbf{x}_{s(h_2)}\right) = F_{v_1}\left(\bigoplus_{t(h_1)=v_1}\mathbf{x}_{s(\alpha(h_1))}\right). \tag{2.1}$$

Remark 1. The invariance condition (2.1) states that the evolution of two vertices of the same type depends "in the same way" on the state of the sources of any equal-type hyperedges targeting them. The condition can also be seen as a *local symmetry* property for the admissible vector field. For example, consider the situation where $v\in V$ is targeted by two hyperedges of the same type h_1 and h_2. This implies that there is a bijection between the targeting hyperedges of v that exchanges h_1 and h_2 (and keeps all other hyperedges fixed). Equation (2.1) with $v_1=v_2=v$ then states that F_v is invariant under the exchange of $\mathbf{x}_{s(h_1)}$ and $\mathbf{x}_{s(h_2)}$.

Example 2.5. The general admissible ODE for the hypernetwork in Figure 2 (also described in Example 2.3) is of the form

$$\begin{aligned}
\dot{x}_0 &= G(x_0, x_0, x_0)\,,\\
\dot{x}_1 &= G(x_1, x_1, x_0)\,,\\
\dot{x}_2 &= G(x_2, x_1, x_2)\,,\\
\dot{y}_0 &= F(y_0, (x_0, x_1), (x_1, x_2), (x_2, x_0))\,,\\
\dot{y}_1 &= F(y_1, (x_0, x_2), (x_1, x_0), (x_2, x_1))\,.
\end{aligned} \tag{2.2}$$

Here, x_0, x_1, x_2 are the variables describing the state of the cells of the first type (circular) while y_0, y_1 determine the states of cells of the second type (square). The ODE is admissible precisely when

$$\begin{aligned}
&F(Y_0, (X_0, X_1), (X_2, X_3), (X_4, X_5)) =\\
&F(Y_0, (X_2, X_3), (X_0, X_1), (X_4, X_5)) =\\
&F(Y_0, (X_2, X_3), (X_4, X_5), (X_0, X_1))\,.
\end{aligned} \tag{2.3}$$

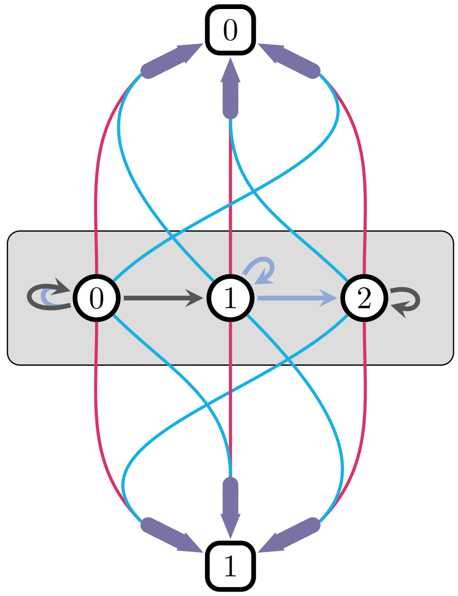


Figure 2: A hypernetwork that contains 3 cells of the same type that form a classical first-order subnetwork. The 2 additional square cells are targeted by 3 hyperedges of order 2 each, which only have cells of the subnetwork as sources. In contrast to Figure 1, we have left out the self-loop edges corresponding to the first entries of the governing functions in (2.2).

This means that the three pairs of input states of F can be arbitrarily permuted. In the literature on coupled cell networks this is often indicated by a bar over the inputs. We have decided not to use this notation here, as it might mistakenly suggest all variables may be permuted individually, instead of only the pairs.

Remark 2. In our definition, hyperedges are directed and target precisely one cell. However, hypergraphs that do not satisfy these conventions can often be represented using this formalism as well. For instance, a hyperedge with multiple targets can be considered as multiple hyperedges with one target and all having the same source vertices. Similarly, an undirected hyperedge can be considered as the collection of all possible directed hyperedges connecting the involved vertices. For example, an undirected hyperedge with three vertices may be interpreted as the collection of 27 directed hyperedges of degree 2, as in Figure 3. As a matter of fact, an admissible ODE for one such undirected hyperedge is of the form

$$\begin{aligned}
\dot{x}_1 &= F(x_1, x_2, x_3)\\
\dot{x}_2 &= F(x_1, x_2, x_3)\\
\dot{x}_3 &= F(x_1, x_2, x_3),
\end{aligned}$$

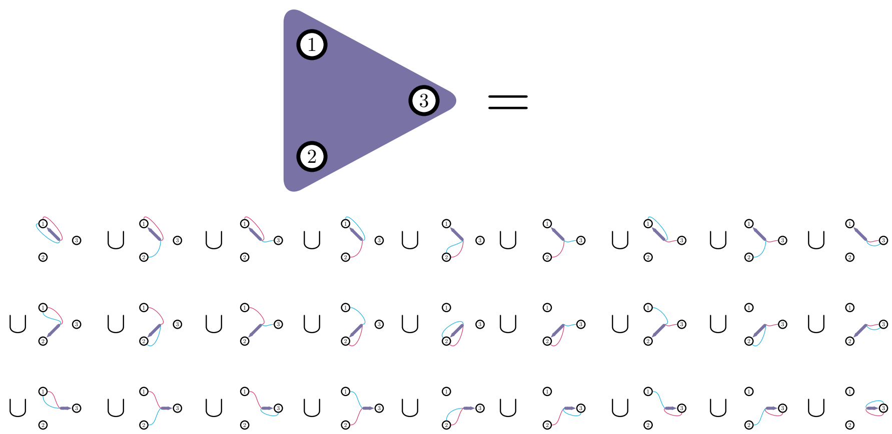


Figure 3: An undirected hyperedge between three vertices can be regarded as the union of 27 directed hyperedges of degree 2. Inspired by Figure 1 in [34].

where F is invariant under arbitrary permutations of its final three inputs. On the other hand, a system that is admissible with respect to the 27 hyperedges will be of the form

$$\dot{x}_1 = G((x_1, x_1), (x_1, x_2), (x_1, x_3), (x_2, x_1), (x_2, x_2), (x_2, x_3), (x_3, x_1), (x_3, x_2), (x_3, x_3)),$$
$$\dot{x}_2 = G((x_1, x_1), (x_1, x_2), (x_1, x_3), (x_2, x_1), (x_2, x_2), (x_2, x_3), (x_3, x_1), (x_3, x_2), (x_3, x_3)),$$
$$\dot{x}_3 = G((x_1, x_1), (x_1, x_2), (x_1, x_3), (x_2, x_1), (x_2, x_2), (x_2, x_3), (x_3, x_1), (x_3, x_2), (x_3, x_3)),$$

where G is invariant under arbitrary permutations of the tuple inputs (though not of the entries of the tuples). Setting

$$\begin{aligned} F(X_1, X_2, X_3) = G((&X_1, X_1), (X_1, X_2), (X_1, X_3), \\ &(X_2, X_1), (X_2, X_2), (X_2, X_3), \\ &(X_3, X_1), (X_3, X_2), (X_3, X_3)), \end{aligned}$$

then yields an admissible governing function for the undirected system, i.e., F is invariant under permutations of its final three inputs. This convention is similar to the one in the classical network literature where an undirected edge between two vertices is equivalent to two directed edges, one in each direction.

In the remainder of this section, we describe balanced partitions and synchrony subspaces in hypernetwork systems. Consider a partition $P = \{V_1, \ldots, V_C\}$ (also called *colouring*) of the vertices V of a hypernetwork $\mathbf{N}$ with the property that whenever $v_1, v_2 \in V_c$ are in the same element of the partition, then they are of the same vertex-type. We can then define the polysynchrony space

$$\mathrm{Syn}_P := \{x_{v_1} = x_{v_2} \text{ when } v_1, v_2 \text{ are in the same element of } P\}.$$

This is the subspace of the total phase space of the hypernetwork dynamical system on which the states of two cells are synchronised when these cells belong to the same element of the partition P. When the polysynchrony space is dynamically invariant for any **N**-admissible map, we say it is *robust*.

In Definition 2.6 we define what it means for a partition to be *balanced*. This notion can be read as the partition being compatible with the hypernetwork structure, and it is the obvious generalisation of the corresponding notion for coupled cell networks. Theorem 4.1 in Section 4 below states that a partition of the cells is balanced if and only if Syn_P is robust.

Definition 2.6. A partition $P = \{V_1, \ldots, V_C\}$ of the vertices in a hypernetwork **N** is *balanced* if for all $v_1, v_2 \in V_c$ in the same element of the partition we have

1. v_1 and v_2 have the same vertex-type, i.e., P is a refinement of the partition into vertex-types;
2. there is a hyperedge-type-preserving bijection $\alpha : t^{-1}(v_1) \to t^{-1}(v_2)$ such that for every hyperedge $h_1 \in t^{-1}(v_1)$ of order k and every source index $1 \leq i \leq k$, the sources $s_i(h_1), s_i(\alpha(h_1)) \in V_d$ are also in the same element of the partition.

Example 2.7. In Example 2.3, the partition $P = \{v_0, v_1, v_2\} \cup \{w_0, w_1\}$ of the cells in the hypernetwork is balanced. As a matter of fact, all cells v_0, v_1, v_2 receive (hyper-)edges from some v_i, which are all in the same element of the partition. Similarly, the cells w_0, w_1 receive one edge from themselves, which are in the same element of the partition, and three hyperedges of order 2 with source cells v_i, v_j, which are also all in the same element of the partition.

In accordance with Theorem 4.1 in Section 4 below, one observes from the equations presented in Example 2.5 that the cluster synchrony space

$$\mathrm{Syn}_P = \{x_0 = x_1 = x_2 \text{ and } y_0 = y_1\}$$

for this partition is flow-invariant for any admissible map, i.e., it is robust.

In contrast, the (larger) space $\{y_0 = y_1\}$ is not invariant under every admissible map. In fact, the partition $\{v_0\} \cup \{v_1\} \cup \{v_2\} \cup \{w_0, w_1\}$ is not balanced, as w_0 and w_1 are targeted by hyperedges whose ordered inputs come from different elements in the partition.

3 Hypergraph fibrations

In the context of classical networks—i.e., hypernetworks of order 1—, it is well known that the dynamics restricted to a robust synchrony subspace is that of a network as well, which is called the *quotient network* (see [11, 8, 4, 7]). It arises by collapsing synchronous vertices to a single one and attaching arrows consistent with the original network—a construction for which it is essential that the partition is balanced. It was shown more recently, that this result is an instance of so-called *graph fibrations*, which were introduced in [38]. These are morphisms of the underlying graphs that induce linear maps sending solutions of one network dynamical system to solutions of another network dynamical system [9]. The goal of this section is to generalize this concept to hypernetworks.

We begin by defining hypergraph fibrations. The definition is almost identical to the one for classical networks.

Definition 3.1 (Hypergraph fibration)**.** Consider two hypernetworks $\mathbf{N} = (V, H, s, t)$ and $\mathbf{N}' = (V', H', s', t')$ and let $\phi\colon \mathbf{N} \to \mathbf{N}'$ be a map such that

1. ϕ sends vertices to vertices, i.e., $\phi(v) \in V'$ for all $v \in V$;

2. ϕ sends hyperedges to hyperedges, i.e., $\phi(h) \in H'$ for all $h \in H$;

3. ϕ preserves the types of vertices and hyperedges, i.e, $\phi(v)$ and v as well as $\phi(h)$ and h are of the same type respectively for all $v \in V$ and $h \in H$;

4. ϕ sends the source vertices $s(h) \in V^k$ of a hyperedge $h \in H$ to the source vertices $s'(\phi(h)) \in (V')^k$ of its image $\phi(h) \in H'$ and respects their order, i.e., $s'(\phi(h)) = (s'_1(\phi(h)), \dots, s'_k(\phi(h))) = (\phi(s_1(h)), \dots, \phi(s_k(h)))$;

5. ϕ sends the unique target vertex of a hyperedge to the unique target vertex of its image, i.e., $t'(\phi(h)) = \phi(t(h))$ for all $h \in H$;

6. and for every vertex $v \in V$, the restriction $\phi|_{t^{-1}(v)}\colon t^{-1}(v) \to (t')^{-1}(\phi(v))$ is a type-preserving bijection of hyperedges.

Then ϕ is called a *hypergraph fibration* or a *fibration of hypernetworks*.

Remark 3.

1. We chose the name hyper*graph* fibration in order to highlight that these maps are a generalization of *graph* fibrations even though the objects that are mapped are called hyper*networks*.

2. Note that **4.** is well-defined since hyperedges of the same type have the same order. In particular, the existence of a hypergraph fibration $\phi\colon \mathbf{N} \to \mathbf{N}'$ implies that the order of $\mathbf{N}'$ is equal to or greater than the order of $\mathbf{N}$.

3. Point **6.** is also referred to as the *fibration property* of the map ϕ.

Example 3.2. Let $\mathbf{N}$ be the hypernetwork in Figure 2 from our running example (Example 2.3). Furthermore, consider the second hypernetwork $\mathbf{N}'$ as depicted in Figure 4. Any map $\phi\colon \mathbf{N} \to \mathbf{N}'$ that sends the three circular vertices v_0, v_1 and v_2 to the circular vertex v_0, the two square vertices w_0 and w_1 to the square vertex w_0, all **light grey** and **grey** hyperedges of order 1 to the **light grey** and **grey** hyperedges of order 1 respectively, the 3 **purple** hyperedges of order 2 that target the square vertex w_0 bijectively to the 3 **purple** hyperedges of order 2, and the 3 **purple** hyperedges of order 2 that target the square vertex w_1 bijectively to the 3 **purple** hyperedges of order 2, is a hypergraph fibration.

The key result of [9] relating graph fibrations to dynamical systems translates to the hypergraph context almost immediately.

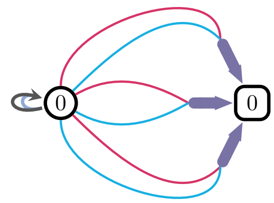


Figure 4: A hypernetwork $\mathbf{N}'$ that contains 2 vertices of different types, 2 hyperedges of order 1 (classical edges) and 3 hyperedges of order 2. There is a (surjective) hypergraph fibration mapping from the hypernetwork $\mathbf{N}$ in Figure 2 to $\mathbf{N}'$. Therefore, we call $\mathbf{N}'$ a *quotient* of $\mathbf{N}$. As before, we have left out self-loops corresponding to self-influence of each cell.

Theorem 3.3. *Let $\mathbf{N} = (V, H, s, t)$ and $\mathbf{N}' = (V', H', s', t')$ be two hypernetworks and $\phi\colon \mathbf{N} \to \mathbf{N}'$ a hypergraph fibration. Furthermore, let $f^{\mathbf{N}}\colon \bigoplus_{v\in V} \mathbb{R}^{n_v} \to \bigoplus_{v\in V} \mathbb{R}^{n_v}$ and $f^{\mathbf{N}'}\colon \bigoplus_{v'\in V'} \mathbb{R}^{n_{v'}} \to \bigoplus_{v'\in V'} \mathbb{R}^{n_{v'}}$ be admissible maps for $\mathbf{N}$ and $\mathbf{N}'$ respectively. That is, they are defined by*

$$f_v^{\mathbf{N}}\left(\bigoplus_{w\in V} x_w\right) = F_v\left(\bigoplus_{h\,:\,t(h)=v} \mathbf{x}_{s(h)}\right) \quad \textit{and} \tag{3.1}$$

$$f_{v'}^{\mathbf{N}'}\left(\bigoplus_{w'\in V'} x_{w'}\right) = F'_{v'}\left(\bigoplus_{h'\,:\,t'(h')=v'} \mathbf{x}_{s'(h')}\right), \tag{3.2}$$

where

$$\mathbf{x}_{s(h)} = \left(x_{s_1(h)}, \ldots, x_{s_k(h)}\right) \quad \textit{and} \quad \mathbf{x}_{s'(h')} = \left(x_{s'_1(h')}, \ldots, x_{s'_{k'}(h')}\right).$$

Assume that the internal dynamics of vertices $v \in V$ and $v' \in V'$ of the same type are governed by the same function, i.e., for every hyperedge-type-preserving bijection $\alpha\colon t^{-1}(v) \to (t')^{-1}(v')$ it holds that

$$F_v\left(\bigoplus_{h\,:\,t(h)=v} \mathbf{x}_{s'(\alpha(h))}\right) = F'_{v'}\left(\bigoplus_{h'\,:\,t'(h')=v'} \mathbf{x}_{s'(h')}\right). \tag{3.3}$$

Then the linear map $R_\phi\colon \bigoplus_{v'\in V'} \mathbb{R}^{n_{v'}} \to \bigoplus_{v\in V} \mathbb{R}^{n_v}$ defined by

$$R_\phi\left(\bigoplus_{v'\in V'} x_{v'}\right) = \bigoplus_{v\in V} x_{\phi(v)} \tag{3.4}$$

is a semiconjugacy between the admissible maps, that is,

$$R_\phi \circ f^{\mathbf{N}'} = f^{\mathbf{N}} \circ R_\phi.$$

In particular, R_ϕ sends solution curves of $\dot{y} = f^{\mathbf{N}'}(y)$ on $\bigoplus_{v' \in V'} \mathbb{R}^{n_{v'}}$ to solution curves of $\dot{x} = f^{\mathbf{N}}(x)$ on $\bigoplus_{v \in V} \mathbb{R}^{n_v}$.

Proof. The proof is almost identical to that in [9] and follows by filling in the suitable definitions and assumptions from the theorem. One computes

$$
\begin{aligned}
\left(f^{\mathbf{N}} \circ R_\phi\right)\left(\bigoplus_{w' \in V'} x_{w'}\right) &= f^{\mathbf{N}}\left(\bigoplus_{w \in V} x_{\phi(w)}\right) && (3.5)\\
&= \bigoplus_{v \in V} f^{\mathbf{N}}_v \left(\bigoplus_{w \in V} x_{\phi(w)}\right) && (3.6)\\
&= \bigoplus_{v \in V} F_v \left(\bigoplus_{h\,:\,t(h)=v} \left(\bigoplus_{w\,:\,w \in s(h)} x_{\phi(w)}\right)\right) && (3.7)\\
&= \bigoplus_{v \in V} F_v \left(\bigoplus_{h\,:\,t(h)=v} \left(\bigoplus_{w'\,:\,w' \in s'(\phi(h))} x_{w'}\right)\right) && (3.8)\\
&= \bigoplus_{v \in V} F_v \left(\bigoplus_{h\,:\,t(h)=v} \mathbf{x}_{s'(\phi(h))}\right) && (3.9)\\
&= \bigoplus_{v \in V} F'_{\phi(v)} \left(\bigoplus_{h'\,:\,t'(h')=\phi(v)} \mathbf{x}_{s'(h')}\right) && (3.10)\\
&= R_\phi \left(\bigoplus_{v' \in V'} F'_{v'} \left(\bigoplus_{h'\,:\,t'(h')=v'} \mathbf{x}_{s'(h')}\right)\right) && (3.11)\\
&= \left(R_\phi \circ f^{\mathbf{N}'}\right)\left(\bigoplus_{w' \in V'} x_{w'}\right) && (3.12)
\end{aligned}
$$

We list the reasoning behind each equality:

(3.5): by definition of R_ϕ (3.4),
(3.6): by definition of admissibility,
(3.7): by (3.1),
(3.8): by **4**. of Def. 3.1,
(3.9): by definition of $\mathbf{x}_{s'(h')}$,
(3.10): by (3.3), since v and $\phi(v)$ are of the same type and $\phi|_{t^{-1}(v)} \colon t^{-1}(v) \to (t')^{-1}(v')$ is a hyperedge-type-preserving bijection,
(3.11): by definition of R_ϕ (3.4),
(3.12): by (3.2).

$\square$

Remark 4.

1. Condition (3.3) is well-defined. Vertices of the same type have the same internal phase space. Furthermore, hyperedge-type-preserving bijections

$\alpha\colon t^{-1}(v) \to (t')^{-1}(v')$ for v and v' of the same type exist due to the local symmetry property (2.1) and the fibration property (**6.**) in Definition 3.1 of ϕ.

2. We remark without further explanation that Theorem 3.3 allows to extend the methods in [10]—to encode network structure by means of quiver representations—to hypernetworks.

Example 3.4. We illustrate Theorem 3.3 with the hypergraph fibration in Example 3.2. The admissible vector field for $\mathbf{N}$ is given by the right hand side of (2.2). Condition (3.3) implies that the $\mathbf{N}'$-admissible vector field is

$$f^{\mathbf{N}'}(x_0, y_0) = \begin{pmatrix} G(x_0, x_0, x_0) \\ F(y_0, (x_0, x_0), (x_0, x_0), (x_0, x_0)) \end{pmatrix}.$$

The linear map R_ϕ (for any choice of ϕ) is defined as

$$R_\phi(x_0, y_0) = (x_0, x_0, x_0, y_0, y_0)^T$$

and it indeed semi-conjugates $f^{\mathbf{N}}$ and $f^{\mathbf{N}'}$.

Hypergraph fibrations can often be used to encode structural properties of a hypernetwork (for details in the context of classical networks consult [10]). For example, it can readily be checked that $\mathbf{N}' = (V', H', s', t')$ is (more precisely, can be identified with) a sub-hypernetwork of $\mathbf{N} = (V, H, s, t)$ if and only if the inclusion $\iota\colon V' \to V$ extends to an *injective* hypergraph fibration $\iota\colon \mathbf{N}' \to \mathbf{N}$ (in particular, ι identifies s' and t' with s and t, respectively). The linear map $R_\iota\colon \bigoplus_{v\in V} \mathbb{R}^{n_v} \to \bigoplus_{v\in V'} \mathbb{R}^{n_v}$ is then the projection

$$R_\iota\left(\bigoplus_{v\in V} x_v\right) = \bigoplus_{v\in V'} x_{\iota(v)} = \bigoplus_{v\in V'} x_v.$$

Surjective hypergraph fibrations are more interesting for our considerations. As in the context of classical networks, these correspond to balanced partitions.

Definition 3.5 (Quotient hypernetwork)**.** Let $\mathbf{N}, \mathbf{N}'$ be two hypernetworks. If there exists a surjective hypergraph fibration $\phi\colon \mathbf{N} \to \mathbf{N}'$, then we call $\mathbf{N}'$ a *quotient hypernetwork* (or simply a *quotient*) of $\mathbf{N}$.

Proposition 3.6. *Let $\mathbf{N}$ be a hypernetwork and $\mathbf{N}'$ a quotient of $\mathbf{N}$ corresponding to the surjective hypergraph fibration $\phi\colon \mathbf{N} \to \mathbf{N}'$. Then the polydiagonal*

$$\mathrm{Syn}_\phi = \{x_{v_1} = x_{v_2} \text{ when } \phi(v_1) = \phi(v_2)\}$$

is a robust synchrony subspace of $\mathbf{N}$. Furthermore, any robust synchrony subspace arises in this way.

Remark 5. The second part of the proposition depends on the more involved Theorem 4.1 in Section 4 below from which we only use the result that robust synchrony implies that the corresponding partition is balanced.

Proof. The first statement follows as a corollary from Theorem 3.3: the linear map R_ϕ as defined in (3.4) is an embedding of the total phase space of $\mathbf{N}'$ into the total phase space of $\mathbf{N}$ whose image is the synchrony subspace Syn_ϕ. Due to the semiconjugacy we have for any admissible map $f^{\mathbf{N}}$ that

$$
\begin{aligned}
f^{\mathbf{N}}\left(\mathrm{Syn}_\phi\right) &= f^{\mathbf{N}}\left(R_\phi\left(\bigoplus_{v'\in V'}\mathbb{R}^{n_{v'}}\right)\right)\\
&= R_\phi\left(f^{\mathbf{N}'}\left(\bigoplus_{v'\in V'}\mathbb{R}^{n_{v'}}\right)\right) \subset R_\phi\left(\bigoplus_{v'\in V'}\mathbb{R}^{n_{v'}}\right) \subset \mathrm{Syn}_\phi\,.
\end{aligned}
$$

This shows that Syn_ϕ is robust.

Conversely, assume Syn is a robust synchrony subspace of $\mathbf{N}$—i.e., it is defined by equality of the coordinates corresponding to certain vertices—and let $P = \{V_1, \ldots, V_C\}$ be the corresponding partition of V, which is balanced due to Theorem 4.1 below. In particular, we have that v_1 and v_2 are in the same element of the partition if and only if $x_{v_1} = x_{v_2}$ holds throughout Syn, and we have that $\mathrm{Syn} = \mathrm{Syn}_P$. We construct a quotient $\mathbf{N}'$ and a surjective hypergraph fibration $\phi\colon \mathbf{N} \to \mathbf{N}'$. The set of vertices is given by the elements of the partition $V' = P$ and we assign the type of v to V_i for any $v \in V_i$—this is well-defined, since P refines the partition into vertex types. Then we set $\phi(v) = V_i$ whenever $v \in V_i$. Next, let $v_1, \ldots, v_C$ be a set of representatives of P. For each v_i and each $h \in t^{-1}(v_i)$ we add a hyperedge h' of the same type to H' such that $t'(h') = V_i$ and the inputs $s'(h') = (s'_1(h'), \ldots, s'_k(h'))$ are defined by $s'_j(h') = V_l$, when $s_j(h) \in V_l$. Finally, we define ϕ on hyperedges. To that end, let $v \in V_i$ and $\alpha\colon t^{-1}(v) \to t^{-1}(v_i)$ be the hyperedge-type-preserving bijection from Definition 2.6. For any $h \in t^{-1}(v)$, we define $\phi(h)$ to be the hyperedge h' constructed above from $\alpha(h)$, which targets the representative v_i. This construction is well-defined: $s_j(h)$ and $s_j(\alpha(h))$ are in the same element of P for all $1 \le j \le k$, since the partition is balanced. Furthermore, one may readily check that ϕ is indeed a hypergraph fibration which is surjective by construction.

It remains to check that $\mathbf{N}'$ realizes the synchrony space, i.e., that $\mathrm{Syn}_\phi = \mathrm{Syn}_P$. This follows immediately from the fact that, by construction, $\phi(v_1) = \phi(v_2)$ if and only if v_1 and v_2 are in the same element of the partition P. □

In combination with Theorem 3.3, we immediately obtain

Corollary 3.7. *Let $\mathbf{N}'$ be a quotient of $\mathbf{N}$ under the surjective hypergraph fibration $\phi : \mathbf{N} \to \mathbf{N}'$. The dynamics of an $\mathbf{N}$-admissible vector field restricted to the robust synchrony subspace Syn_ϕ is given by an $\mathbf{N}'$-admissible vector field.*

Example 3.8. Any hypergraph fibration in Example 3.2 is surjective. Hence, $\mathbf{N}'$ as depicted in Figure 4 is a quotient of $\mathbf{N}$ as depicted in Figure 2. It corresponds to the synchrony subspace $\{x_0 = x_1 = x_2 \text{ and } y_0 = y_1\}$ as in Example 2.7.

4 Balanced partitions and robust synchrony

In the previous section we have shown that robust synchrony uniquely corresponds to surjective graph fibrations. In this section, we provide another characterization. Theorem 4.1 is the main result regarding robust synchrony in hypernetworks, generalising the well-known result for network dynamical systems which states that balanced partitions correspond to robust synchrony. However, the result for hypernetworks is more subtle than the result for networks: which synchrony spaces are robust is not determined by the linear admissible maps, but by higher order polynomial admissible maps. For networks (i.e. hypernetworks of order $k = 1$) Theorem 4.1 reduces to the aforementioned well-known result. In what follows, we fix internal phase spaces $\mathbb{R}^{n_v}$ for each of the nodes.

Theorem 4.1. *Let $\mathbf{N} = \{V, H, s, t\}$ be a hypernetwork and $P = \{V_1, \ldots, V_C\}$ a partition of V that refines the partition into vertex-types. As above, define*

$$\mathrm{Syn}_P := \{x_{v_1} = x_{v_2} \text{ when } v_1, v_2 \text{ are in the same element of } P\ \}\,.$$

The following are equivalent:

i) The partition P is balanced.

ii) Syn_P is invariant under any $\mathbf{N}$-admissible map, i.e., it is robust.

iii) Syn_P is invariant under any polynomial $\mathbf{N}$-admissible map of degree at most $\frac{k(k+1)}{2}$, where k is the order of the hypernetwork.

We split the proof of Theorem 4.1 into two main parts. The first part deals with the implication $i) \implies ii)$ and is relatively straightforward and similar to the proof for dyadic networks. Note that the implication $ii) \implies iii)$ is trivial. The second part of the proof concerns the implication $iii) \implies i)$. This implication is considerably harder to prove than the corresponding implication for dyactic networks, and requires that we first develop some combinatorial machinery.

Proof of Theorem 4.1, $i) \implies ii)$. Assume that the partition P is balanced and that v_1 and v_2 are in the same element of P. Then v_1 and v_2 are of the same vertex-type and there is a hyperedge-type-preserving bijection $\alpha : t^{-1}(v_1) \to t^{-1}(v_2)$ so that for each hyperedge $h_1 \in t^{-1}(v_1)$ and every source index i, the sources $s_i(h_1)$ and $s_i(\alpha(h_1))$ are also in the same element of the partition P. For $x \in \mathrm{Syn}_P$, we thus have that $x_{s_i(h_1)} = x_{s_i(\alpha(h_1))}$ for every i, which we may write as $\mathbf{x}_{s(h_1)} = \mathbf{x}_{s(\alpha(h_1))}$. It follows that for any admissible map and any $x \in \mathrm{Syn}_P$, we have that

$$\begin{aligned} f_{v_1}(x) =& F_{v_1}\left(\bigoplus_{t(h_1)=v_1} \mathbf{x}_{s(h_1)}\right) = F_{v_1}\left(\bigoplus_{t(h_1)=v_1} \mathbf{x}_{s(\alpha(h_1))}\right) = \\ & F_{v_2}\left(\bigoplus_{t(h_2)=v_2} \mathbf{x}_{s(h_2)}\right) = f_{v_2}(x)\,. \end{aligned}$$

The third equality uses the property of an admissible map. This proves that $f(\mathrm{Syn}_P) \subset \mathrm{Syn}_P$, so the synchrony space is invariant under any admissible map. $\square$

Proving the implication *iii*) $\implies$ *i*) involves some combinatorics of finite number sequences and polynomials that we need to develop first. To this end, let us denote by $\mathcal{C} = \{1, \dots, C\}$ a finite set of symbols, and by $\mathcal{C}^m$ the set of ordered sequences of length m with entries in $\mathcal{C}$. The reason for introducing sequences is because, given a partition $P = (V_1, \dots, V_C)$, we want to keep track of the elements in the partition that each hyperedge receives its inputs from. More precisely, to a given hyperedge h of order m we associate the *signature* of h, $\mathcal{S}(h)$, as the ordered sequence $(c_1, \dots, c_m) \in \mathcal{C}^m$ where $s_i(h) \in V_{c_i}$ for all $i \in \{1, \dots, m\}$. We wish to understand how different monomials in a response function change when we restrict to Syn_P, which is ultimately determined by the signature of each hyperedge involved.

We begin by putting a strict partial ordering $\succ$ on $\mathcal{C}^m$ as follows. Given sequences $a, b \in \mathcal{C}^m$, we set $a \succ b$ if

- the number of Cs appearing in a is greater than the number of Cs appearing in b, or;
- the number of Cs appearing is the same for a and b, but the number of $(C-1)$s appearing in a is greater than the number of $(C-1)$s appearing in b, or;

$\vdots$

- the number of Cs appearing is the same for a and b, as is the number of $(C-1)$s, $(C-2)$s and so forth, up to the number of 3s, but the number of 2s appearing in a is greater than the number of 2s appearing in b.

Note that we never have to consider the number of 1s, as an equal number of 2s up to Cs means an equal number of 1s as well (both sequences have equal length m). It follows that two sequences are only incomparable to each other if they have all symbols appearing an equal number of times. It is not hard to see that $\succ$ indeed defines a strict partial ordering. We also point out that the sequence $a_C := (C, \dots, C)$ satisfies $a_C \succ b$ for all $b \neq a_C$.

Next, let S_m denote the symmetric group on m elements. Given $a = (c_1, \dots, c_m) \in \mathcal{C}^m$ and $\sigma \in S_m$, we write $\mathcal{M}_a^\sigma \in \mathbb{Z}[Z_1, \dots, Z_C]$ for the monomial given by

$$\mathcal{M}_a^\sigma(Z) = Z_{c_1}^{\sigma(1)} Z_{c_2}^{\sigma(2)} \cdots Z_{c_m}^{\sigma(m)} .$$

Note that the total degree of $\mathcal{M}_a^\sigma$ is always $1 + \cdots + m = \frac{m(m+1)}{2}$. The monomials $\mathcal{M}_a^\sigma$ will show up as the restriction to Syn_P of the terms in some conveniently chosen response functions, where a will be the signature of an edge determined by P.

Example 4.2. Suppose $m = 3$ and $C \geq 2$, and let $a = (2, 2, 1)$ and $b = (1, 1, 2)$. We see that $a \succ b$, as the number of 2s appearing in a is larger than the number of 2s appearing in b. Let $\mathrm{Id} \in S_3$ denote the identity permutation. We have

$$\mathcal{M}_a^{\mathrm{Id}}(Z) = Z_2^{\mathrm{Id}(1)} Z_2^{\mathrm{Id}(2)} Z_1^{\mathrm{Id}(3)} = Z_2^1 Z_2^2 Z_1^3 = Z_1^3 Z_2^3$$

and likewise

$$\mathcal{M}_b^{\mathrm{Id}}(Z) = Z_1^1 Z_1^2 Z_2^3 = Z_1^3 Z_2^3 \,.$$

Hence, we see that in this case $\mathcal{M}_a^{\mathrm{Id}} = \mathcal{M}_b^{\mathrm{Id}}$.

Finally, we need the notion of a permutation $\tau \in S_m$ that is *attuned* to a sequence $a \in \mathcal{C}^m$. To this end, suppose a has the symbol c appearing on the positions $I_c \subset \{1, \ldots, m\}$, for all $c \in \mathcal{C}$. The permutation τ is attuned to a if τ takes on its $\#I_C$ largest values on I_C, its next $\#I_{C-1}$ largest values on I_{C-1} and so forth.

Example 4.3. Given $a = (3, 3, 1, 2, 1) \in \mathcal{C}^5$, the permutation $\sigma \in S_5$ given by

$$\sigma(1) = 4, \sigma(2) = 5, \sigma(3) = 1, \sigma(4) = 3, \sigma(5) = 2$$

is attuned to a. The same holds true when we switch the values of $\sigma(1)$ and $\sigma(2)$ or those of $\sigma(3)$ and $\sigma(5)$. The identity permutation is, for instance, not attuned to a.

The result we need regarding these notions is the following:

Lemma 4.4. *Let $a \in \mathcal{C}^m$ be a sequence and suppose the permutation $\tau \in S_m$ is attuned to a. If $b \neq a$ is another sequence such that $\mathcal{M}_b^\tau = \mathcal{M}_a^\tau$, then necessarily $b \succ a$.*

Proof. Let us denote by $I_c^a, I_c^b \subset \{1, \ldots, m\}$ the positions on which a and b have the symbol $c \in \mathcal{C}$, respectively. Note that we may write

$$\mathcal{M}_a^\tau = \prod_{c=1}^{C} Z_c^{\sum_{i \in I_c^a} \tau(i)} \,. \tag{4.1}$$

By assumption, (4.1) equals

$$\mathcal{M}_b^\tau = \prod_{c=1}^{C} Z_c^{\sum_{i \in I_c^b} \tau(i)}$$

so that

$$\sum_{i \in I_c^a} \tau(i) = \sum_{i \in I_c^b} \tau(i) \ \text{ for all } \ c \in \mathcal{C} \,. \tag{4.2}$$

We start by looking at $c = C$. As τ is attuned to a, it holds that I_C^a consists of the $\#I_C^a$ distinct values $i \in \{1, \ldots, m\}$ for which $\tau(i)$ is largest. Hence, the only way Equation (4.2) can hold for $c = C$ is if either $I_C^a = I_C^b$ or $\#I_C^b > \#I_C^a$.

In the latter case we indeed have $b \succ a$, whereas the former requires we look at $c = C - 1$ to determine if and how a and b are related under $\succ$.

Suppose therefore that $I^a_c = I^b_c$ for all $c > d$, for some fixed $d \in \mathcal{C}$. Again, because τ is attuned to a, we see that I^a_d consists of the $\#I^a_d$ distinct values

$$i \in \{1, \dots, m\} \setminus \left(I^a_C \sqcup \dots \sqcup I^a_{d+1}\right) = \{1, \dots, m\} \setminus \left(I^b_C \sqcup \dots \sqcup I^b_{d+1}\right)$$

for which $\tau(i)$ is largest. The only way Equation (4.2) can hold for $c = d$ is therefore when $I^a_d = I^b_d$ or $\#I^b_d > \#I^a_d$. Again, the latter case means $b \succ a$, whereas the former means we look at $c = d - 1$ next. If at no point in this procedure we conclude that $b \succ a$, then eventually we arrive at $I^a_c = I^b_c$ for all $c > 1$, which implies that also $I^a_1 = I^b_1$. This means $a = b$ though, which we excluded by assumption. Hence we indeed have $b \succ a$. $\square$

Proof of Theorem 4.1, iii) $\implies$ i). To keep notation as simple as possible, we write $h_1 \sim h_2$ to indicate that the hyperedges h_1 and h_2 are of the same hyperedge-type, and likewise use $v_1 \sim v_2$ to denote equal vertex-type for the nodes v_1 and v_2. We will write $v_1 \sim_P v_2$ to indicate that the nodes v_1 and v_2 are in the same class of the partition P. The set of classes or "colors" of P will be indexed by $\mathcal{C} = \{1, \dots, C\}$.

We will prove that the partition P is balanced by constructing appropriate admissible polynomial maps that allow us to count hyperedges with certain properties. We claim that P is balanced if the following holds: for any hyperedge h_0 and sequence $a \in \mathcal{C}^m$, with m the order of h_0, the cardinality of the set

$$\mathcal{N}^v_{h_0,a} := \{h \sim h_0 \mid t(h) = v \text{ and } \mathcal{S}(h) = a\} \tag{4.3}$$

is the same for all nodes v in the same class of P. Here $\mathcal{S}(h)$ denotes the signature of h determined by P, as defined above. After all, if these cardinalities match, then for any two nodes $v_1 \sim_P v_2$ we may build a bijection $\alpha : t^{-1}(v_1) \to t^{-1}(v_2)$ that maps $\mathcal{N}^{v_1}_{h_0,a}$ into $\mathcal{N}^{v_2}_{h_0,a}$ for all a and h_0. This bijection then satisfies the second condition of Definition 2.6. As P refines the partition into vertex-types by assumption, this shows P is indeed balanced.

We therefore fix a node v_0 and a hyperedge h_0 of order $1 \leq m \leq k$. The response functions we will use to determine the cardinality of the sets $\mathcal{N}^v_{h_0,a}$ are as follows. Given $\sigma \in S_m$, when $v \sim v_0$ we set

$$f^\sigma_v(x) = F^\sigma_v\left(\bigoplus_{t(h)=v} \mathbf{x}_{s(h)}\right) = \sum_{\substack{t(h) = v \\ h \sim h_0}} Q^\sigma\left(\mathbf{x}_{s(h)}\right), \tag{4.4}$$

in which

$$Q^\sigma\left(\mathbf{x}_{s(h)}\right) := \prod_{i=1}^{m} \left(x_{s_i(h)}\right)_1^{\sigma(i)} \cdot e^{n_v}_1 . \tag{4.5}$$

Here $e^{n_v}_1$ denotes the first unit vector $(1, 0, \dots, 0)$ in $\mathbb{R}^{n_v}$ and $\left(x_{s_i(h)}\right)_1^{\sigma(i)} \in \mathbb{R}$ is the first component of $x_{s_i(h)} \in \mathbb{R}^{n_{s_i(h)}}$ raised to the power $\sigma(i)$. Note that for one-dimensional internal dynamics the function (4.6) is just given by $Q^\sigma\left(\mathbf{x}_{s(h)}\right) = x_{s_1(h)}^{\sigma(1)} \dots x_{s_m(h)}^{\sigma(m)}$. We also point out that each F_v^σ is polynomial of degree $1 + \dots + m = \frac{m(m+1)}{2} \leq \frac{k(k+1)}{2}$ and satisfies the symmetry-properties imposed on response functions. When v is not vertex-equivalent to v_0 we set $f_v^\sigma(x) = 0$. In particular, we then have that $f_{v_1}^\sigma$ and $f_{v_2}^\sigma$ agree on Syn_P whenever $v_1 \sim_P v_2$ by assumption *iii)*.

We now parametrize the synchrony space Syn_P by a variable $Y = (Y_c)_{c\in\mathcal{C}}$ via the map $Y \mapsto x = (x_v)_{v\in V}$ defined by $x_v := Y_c$ if $v \in V_c$. It follows that for $x \in \mathrm{Syn}_P$ we may write $\mathbf{x}_{s(h)} = (Y_{c_1}, \dots, Y_{c_m})$ where $s_i(h) \in V_{c_i}$. Note that the indices of this latter vector are precisely the signature $\mathcal{S}(h) = (c_1, \dots, c_m)$. Additionally, if we write $Z_c := (Y_c)_1$ for all $c \in \mathcal{C}$ and $Z = (Z_c)_{c\in\mathcal{C}}$, then the first component of the vector valued function (4.4), evaluated on Syn_P, is precisely given by

$$(f_v^\sigma(x))_1 = \sum_{\substack{t(h) = v \\ h \sim h_0}} \mathcal{M}_{\mathcal{S}(h)}^\sigma(Z)\,. \tag{4.6}$$

Note that each term h in the summation of (4.6) yields a term $\mathcal{M}_a^\sigma$ if $\mathcal{S}(h) = a$. Hence, we may also write

$$(f_v^\sigma(x))_1 = \sum_{a\in\mathcal{C}^m} \left(\#\mathcal{N}_{h_0,a}^v\right) \mathcal{M}_a^\sigma(Z)\,, \tag{4.7}$$

for $x \in \mathrm{Syn}_P$. It may happen that $\mathcal{M}_a^\sigma = \mathcal{M}_b^\sigma$ for distinct sequences $a, b \in \mathcal{C}^m$, as Example 4.2 shows. Hence, we may not directly read off $\#\mathcal{N}_{h_0,a}^v$ as the number of monomials $\mathcal{M}_a^\sigma$ appearing in $(f_v^\sigma)_1|_{\mathrm{Syn}_P}$.

We will therefore use the strict partial ordering $\succ$ defined before. Our claim is: given a sequence $a \in \mathcal{C}^m$, if the values of $\#\mathcal{N}_{h_0,b}^v$ are known for all b such that $b \succ a$, then we can retrieve $\#\mathcal{N}_{h_0,a}^v$ from $(f_v^\sigma)_1|_{\mathrm{Syn}_P}$ for some appropriately chosen permutation σ. To show that this indeed holds, we choose $a \in \mathcal{C}^m$ and fix a permutation $\tau \in S_m$ that is attuned to a. It follows from Equation (4.7) that the polynomial $(f_v^\tau)_1|_{\mathrm{Syn}_P}$ will involve the monomial $\mathcal{M}_a^\tau$ exactly $M_{v,a}^\tau$ times, where

$$M_{v,a}^\tau = \sum_{\substack{b \in \mathcal{C}^m \\ \mathcal{M}_b^\tau = \mathcal{M}_a^\tau}} \#\mathcal{N}_{h_0,b}^v\,. \tag{4.8}$$

However, Lemma 4.4 tells us that the sum in (4.8) goes only over sequences b satisfying $b \succ a$, apart from a itself. Hence, we may indeed determine $\#\mathcal{N}_{h_0,a}^v$ from the given information. Note that the largest sequence $a_C = (C, \dots, C)$ satisfies $\mathcal{M}_{a_C}^\sigma(Z) = Z_C^{m(m+1)/2}$ for any permutation $\sigma \in S_m$, whereas for all other sequences b we have $\mathcal{M}_b^\sigma(Z) \neq Z_C^{m(m+1)/2}$ for all permutations $\sigma \in S_m$. Hence, the number $\#\mathcal{N}_{h_0,a_C}^v$ equals the coefficient in front of the monomial $\mathcal{M}_{a_C}^\sigma$ in $(f_v^\sigma)_1|_{\mathrm{Syn}_P}$ for any permutation σ. As any sequence $b \neq a_C$ satisfies $a_C \succ b$, we see that we may iteratively find all numbers $\#\mathcal{N}_{h_0,a}^v$ from the maps $(f_v^\sigma)_1|_{\mathrm{Syn}_P}$ in this way.

Finally, as $f^\sigma_{v_1}$ and $f^\sigma_{v_2}$ agree on Syn_P for any $\sigma \in S_m$ whenever $v_1 \sim_P v_2$, we see that $\#\mathcal{N}^{v_1}_{h_0,a} = \#\mathcal{N}^{v_2}_{h_0,a}$ for all such nodes v_1, v_2, all hyperedges h_0 and all signatures a. This shows that P is indeed balanced. $\square$

In Section 5 we present a family of examples to show that the number $\frac{k(k+1)}{2}$ is optimal and for general hypernetworks cannot be reduced. This implies in particular that *only* in classical networks (hypernetworks of order one) robust synchrony is determined by the linear admissible maps.

Remark 6. If the internal phase spaces $\mathbb{R}^{n_v}$ also agree for some nodes that are not of the same vertex-type, then we may define the space

$$\mathrm{Syn}_P := \{x_{v_1} = x_{v_2} \text{ when } v_1, v_2 \text{ are in the same element of } P\}.$$

for some partition P that does not refine the partition into vertex-types. However, such spaces are not even invariant under all constant $\mathbf{N}$-admissible maps, as can be seen by setting $f_v(x) = C_v$ for some vectors $C_v \in \mathbb{R}^{n_v}$ such that $C_v = C_w$ if and only the nodes v and w have the same vertex-type.

Example 4.5. Recall our running example (see Examples 2.3 and 2.5) which consists of a hypernetwork of order two of which the admissible maps are defined in terms of a response function F satisfying the invariance equations in (2.3). In Example 2.7 we concluded that the partition $P = \{v_0\} \cup \{v_1\} \cup \{v_2\} \cup \{w_0, w_1\}$ is not balanced. One can check that the corresponding synchrony space $\{y_0 = y_1\}$ is not robust: it is not invariant under all admissible maps, and in particular not under all polynomial admissible maps of order $\frac{2(2+1)}{2} = 3$ and higher. However, one may also verify that $\{y_0 = y_1\}$ is actually invariant under all linear and quadratic admissible maps.

5 An interesting class of examples

In this section, we construct a class of hypernetworks that have interesting "near-synchrony" properties. More precisely, given $k \geq 2$ and a hypernetwork of order at most k and with $k+1$ cells of identical type, we construct a new hypernetwork of order k with a synchrony space that is not robust, but that is nevertheless invariant under every polynomial admissible map of degree strictly less than $\frac{k(k+1)}{2}$. These examples therefore show that the bound of $\frac{k(k+1)}{2}$ in Theorem 4.1 cannot in general be decreased. We also present a brief exploration of a remarkable synchrony breaking bifurcation in one of the hypernetworks constructed in this way.

To introduce our construction, let S_n denote the symmetric group on n elements (i.e., the group of permutations of n elements) and write $S^0_n, S^1_n \subseteq S_n$ for the set of even and odd permutations, respectively.

Definition 5.1. Let $\mathbf{N}$ be a given hypernetwork with $k+1 \geq 3$ nodes $v_0, \dots, v_k$, all of identical type. The *augmented hypernetwork* $\mathbf{N}^\diamond$ is obtained from $\mathbf{N}$ by adding two additional nodes w_0 and w_1, their self-loops and $(k+1)!$ hyperedges of order k. The two additional nodes are of the same type, which is different from that of the v_i, and the additional hyperedges are likewise of a same, new type. These new hyperedges are labelled by the elements of the symmetric group on $k+1$ elements, S_{k+1}. Given $\sigma \in S_{k+1}$, the hyperedge h_σ satisfies

$$s(h_\sigma) = (v_{\sigma(1)}, \dots, v_{\sigma(k)}) , \qquad t(h_\sigma) = w_{\mathrm{sgn}(\sigma)} , \tag{5.1}$$

where S_{k+1} is understood to act on the ordered set $\{0, \dots, k\}$ and $v_{\sigma(0)}$ is therefore the only node in $\mathbf{N}$ that is not a source of h_σ. We will refer to $\mathbf{N}$ as the *core* of the augmented hypernetwork $\mathbf{N}^\diamondsuit$.

Note that in Definition 5.1 the order of the augmented hypernetwork is k, provided the core has order k or less. In particular, this holds when the core is a classical (dyadic) network.

Example 5.2. Let $\mathbf{N}$ be the classical network consisting of three disconnected nodes, with only a single self-loop for each of them. It follows that the augmented hypernetwork $\mathbf{N}^\diamondsuit$ is the one shown in the left panel of Figure 1. Here the circular nodes belong to $\mathbf{N}$, whereas the square ones are the newly added w_0 and w_1. Note that the core $\mathbf{N}$ is not required to be connected in Definition 5.1.

Example 5.3. The hypernetwork from Example 2.3 is the augmented hypernetwork with core the classical three-node network shown within the box in Figure 2 (i.e. consisting of the circular nodes and the arrows in between).

Given an augmented network $\mathbf{N}^\diamondsuit$ with core $\mathbf{N}$, we shall denote by $x_0, \dots, x_k$ the dynamical variables of the cells $v_0, \dots, v_k$ and by y_0, y_1 the variables of the cells w_0, w_1. For convenience, we assume from here on out that all cells have a one-dimensional internal phase space. It follows that the equations of motion for the y-variables are given by

$$\dot{y}_0 = F\left(y_0, \textstyle\bigoplus_{\sigma \in S^0_{k+1}} \mathbf{x}_\sigma\right) , \; \dot{y}_1 = F\left(y_1, \textstyle\bigoplus_{\sigma \in S^1_{k+1}} \mathbf{x}_\sigma\right) . \tag{5.2}$$

Here we used the notation

$$\mathbf{x}_\sigma = (x_{\sigma(1)}, \dots x_{\sigma(k)}) \in \mathbb{R}^k$$

for the source variables of the hyperedge h_σ. The response function

$$F = F\left(Y, \bigoplus_{\sigma \in S^0_{k+1}} \mathbf{X}_\sigma\right) \text{ from } \mathbb{R} \oplus \bigoplus_{\sigma \in S^0_{k+1}} \mathbb{R}^k \text{ to } \mathbb{R}$$

is assumed to be invariant under any permutation of the $\frac{(k+1)!}{2}$ entries $\mathbf{X}_\sigma \in \mathbb{R}^k$, which implies in particular that the notation $\bigoplus_{\sigma \in S^0_{k+1}} \mathbf{X}_\sigma$ for the arguments of F is unambiguous: the $\mathbf{X}_\sigma$ can be substituted into F in arbitrary order. As S^0_{k+1} and S^1_{k+1} have the same cardinality, the invariance of F likewise means that the expression for $\dot{y}_1$ in Equation (5.2) is well-defined. Note that each $\dot{x}_i$

depends only on the x-variables, according to the hypernetwork structure of the core $\mathbf{N}$. This is because the core is a sub-hypernetwork of the augmented hypernetwork.

Our main result about these augmented hypernetworks is the following.

Theorem 5.4. *Let $\mathbf{N}^\diamond$ be an augmented hypernetwork whose core consists of $k+1$ nodes. Assume one-dimensional internal dynamics for each of the nodes, and write F for the response function of the y-nodes as in Equation (5.2). The space $\{y_0 = y_1\}$ is invariant for all admissible systems for $\mathbf{N}^\diamond$ with F polynomial of total degree strictly less than $\frac{k(k+1)}{2}$, but not for all polynomials F of total degree $\frac{k(k+1)}{2}$. In particular, $\{y_0 = y_1\}$ is not a robust synchrony space.*

As the dynamics of the cells in the core does not depend on the y-variables, we see that invariance of the space $\{y_0 = y_1\}$ is equivalent to the condition

$$F\left(y, \bigoplus_{\sigma \in S^0_{k+1}} \mathbf{x}_\sigma\right) = F\left(y, \bigoplus_{\sigma \in S^1_{k+1}} \mathbf{x}_\sigma\right) \tag{5.3}$$

for all $y = y_0 = y_1 \in \mathbb{R}$ and $x = (x_0, \ldots, x_k) \in \mathbb{R}^{k+1}$. This explains why the result of Theorem 5.4 does not depend on the core $\mathbf{N}$. At the end of this section we will investigate a phenomenon in augmented hypernetworks that does depend on specifics of the core. Equation (5.3) also suggests that in order to prove Theorem 5.4, we first need to gather results on functions with the symmetry properties of F. To this end we have the following lemmas.

Lemma 5.5. *Let*

$$Q : \bigoplus_{\sigma \in S^0_{k+1}} \mathbb{R}^k \to \mathbb{R}$$

be a function that is invariant under all permutations of its $\#S^0_{k+1}$ entries in $\mathbb{R}^k$. We have

$$Q\left(\bigoplus_{\sigma \in S^0_{k+1}} \mathbf{x}_\sigma\right) = Q\left(\bigoplus_{\sigma \in S^1_{k+1}} \mathbf{x}_\sigma\right)$$

for $x = (x_0, \ldots, x_k) \in \mathbb{R}^{k+1}$ whenever $x_i = x_j$ for some distinct $i, j \in \{0, \ldots, k\}$.

Proof. Let i and j be as in the lemma and let $\kappa \in S^1_{k+1}$ denote the transposition that interchanges i and j, while leaving the other indices fixed. By assumption we have $x_{\kappa(\ell)} = x_\ell$ for all $\ell \in \{0, \ldots, k\}$. It follows that for all $\sigma \in S_{k+1}$ and $m \in \{1, \ldots, k\}$ it holds that

$$(\mathbf{x}_{\kappa\sigma})_m = x_{\kappa(\sigma(m))} = x_{\kappa(\ell)} = x_\ell = x_{\sigma(m)} = (\mathbf{x}_\sigma)_m\,,$$

where we have set $\ell = \sigma(m)$. Hence, we see that $\mathbf{x}_{\kappa\sigma} = \mathbf{x}_\sigma$ for all $\sigma \in S_{k+1}$. From the fact that $\kappa S^0_{k+1} = S^1_{k+1}$ as sets, together with the symmetry properties of Q, we indeed find

$$Q\left(\bigoplus_{\sigma\in S^0_{k+1}} \mathbf{x}_\sigma\right) = Q\left(\bigoplus_{\sigma\in S^0_{k+1}} \mathbf{x}_{\kappa\sigma}\right) = Q\left(\bigoplus_{\sigma\in S^1_{k+1}} \mathbf{x}_\sigma\right),$$

which completes the proof. □

Lemma 5.6. *Let Q be as in Lemma 5.5 and assume in addition that this function is polynomial. There exists a polynomial $S\colon \mathbb{R}^{k+1} \to \mathbb{R}$ such that*

$$Q\left(\bigoplus_{\sigma\in S^0_{k+1}} \mathbf{x}_\sigma\right) - Q\left(\bigoplus_{\sigma\in S^1_{k+1}} \mathbf{x}_\sigma\right) = S(x) \prod_{\substack{i,j=0\\ i>j}}^{k} (x_i - x_j) \tag{5.4}$$

for all $x = (x_0, \ldots, x_k) \in \mathbb{R}^{k+1}$.

Proof. It follows from Lemma 5.5 that the left hand side of Equation (5.4) vanishes whenever $x_i = x_j$ for some distinct $i, j \in \{0, \ldots, k\}$. Let us for the moment fix two such indices $i \neq j$. Any polynomial P in the variables $x = (x_0, \ldots, x_k)$ may be written as

$$P(x) = (x_i - x_j)T(x) + R(x_0, \ldots, \hat{x}_i, \ldots, x_k),$$

for some polynomials T and R, and where $\hat{x}_i$ means that R does not depend on x_i. This can be seen by setting $x_i = (x_i - x_j) + x_j$ and separating out multiples of $x_i - x_j$. If P vanishes when we set $x_i = x_j$ then necessarily $R = 0$, so that $x_i - x_j$ divides P.

Returning to Equation (5.4), we conclude that the left hand side is divisible by $x_i - x_j$ for all pairs of distinct indices (i, j). If we impose $i > j$ then the factors $x_i - x_j$ are all different irreducible polynomials (i.e., not differing by a unit). Using that the polynomial ring $\mathbb{R}[x_0, \ldots, x_k]$ is a unique factorization domain (a UFD), we conclude that the left hand side of Equation (5.4) is indeed divisible by

$$\prod_{\substack{i,j=0\\ i>j}}^{k} (x_i - x_j),$$

from which Equation (5.4) follows. □

Corollary 5.7. *Let Q be a polynomial as in Lemmas 5.5 and 5.6. If Q is of total degree less than $\frac{k(k+1)}{2}$, then*

$$Q\left(\bigoplus_{\sigma\in S^0_{k+1}} \mathbf{x}_\sigma\right) = Q\left(\bigoplus_{\sigma\in S^1_{k+1}} \mathbf{x}_\sigma\right) \tag{5.5}$$

for all $x = (x_0, \ldots, x_k) \in \mathbb{R}^{k+1}$.

Proof. From Lemma 5.6 we get Equation (5.4) for some polynomial S. The left hand side of Equation (5.4) has degree less than $\frac{k(k+1)}{2}$, whereas the factor $\prod_{\substack{i,j=0\\ i>j}}^{k}(x_i - x_j)$ on the right hand side has degree $\frac{k(k+1)}{2}$. So Equation (5.4) can only hold when $S = 0$, which proves Equation (5.5). $\square$

Finally, we introduce a symmetric function that will be helpful in the proof of Theorem 5.4. Given $k \in \mathbb{N}$, we define the *power sum symmetric polynomial*

$$
\begin{aligned}
&P_{(k)} \colon \bigoplus_{\sigma \in S^0_{k+1}} \mathbb{R}^k \to \mathbb{R} \\
&P_{(k)}\left(\bigoplus_{\sigma \in S^0_{k+1}} \mathbf{X}_\sigma\right) = \sum_{\sigma \in S^0_{k+1}} X^1_{\sigma,1} X^2_{\sigma,2} \cdots X^k_{\sigma,k}\,,
\end{aligned}
\tag{5.6}
$$

where we write $\mathbf{X}_\sigma = (X_{\sigma,1}, \dots, X_{\sigma,k}) \in \mathbb{R}^k$ for $\sigma \in S^0_{k+1}$. Note that $P_{(k)}$ is symmetric under all permutations of its $\#S^0_{k+1}$ entries $\mathbf{X}_\sigma$ in $\mathbb{R}^k$, and has total degree $1 + \cdots + k = \frac{k(k+1)}{2}$.

Proof of Theorem 5.4. We first show that the space $\{y_0 = y_1\}$ is not in general invariant when the response function F in Equation (5.2) is a polynomial of degree $\frac{k(k+1)}{2}$. To this end, we set

$$
F\left(Y, \bigoplus_{\sigma \in S^0_{k+1}} \mathbf{X}_\sigma\right) = P_{(k)}\left(\bigoplus_{\sigma \in S^0_{k+1}} \mathbf{X}_\sigma\right)\,,
$$

where $P_{(k)}$ is defined by Equation (5.6). It follows that

$$
\dot{y}_0 = F\left(y_0, \bigoplus_{\sigma \in S^0_{k+1}} \mathbf{x}_\sigma\right) = \sum_{\sigma \in S^0_{k+1}} x^1_{\sigma(1)} \cdots x^k_{\sigma(k)}\,, \tag{5.7}
$$

and similarly

$$
\dot{y}_1 = F\left(y_1, \bigoplus_{\sigma \in S^1_{k+1}} \mathbf{x}_\sigma\right) = \sum_{\sigma \in S^1_{k+1}} x^1_{\sigma(1)} \cdots x^k_{\sigma(k)}\,. \tag{5.8}
$$

It is not hard to see that the right hand sides of Equations (5.7) and (5.8) are not equal. For example, we can only have

$$
x^1_{\sigma(1)} \cdots x^k_{\sigma(k)} = x^0_0 x^1_1 \cdots x^k_k
$$

if $\sigma \in S_{k+1}$ is the identity. Hence, this particular monomial appears in Equation (5.7) but not in Equation (5.8). This shows that the polynomials on the right hand sides of both equations are indeed different, and hence that $\{y_0 = y_1\}$ is not robust.

On the other hand, suppose now that F is a polynomial of degree $d < \frac{k(k+1)}{2}$ satisfying the required symmetry conditions. It follows that we may write

$$F\left(Y, \bigoplus_{\sigma \in S^0_{k+1}} \mathbf{X}_\sigma\right) = \sum_{\ell=0}^{d} Y^\ell\, Q_\ell \left(\bigoplus_{\sigma \in S^0_{k+1}} \mathbf{X}_\sigma\right),$$

where each polynomial Q_ℓ is invariant under all permutations of its $\frac{(k+1)!}{2}$ entries $\mathbf{X}_\sigma$, and of degree strictly less than $\frac{k(k+1)}{2}$. By Corollary 5.7 we have

$$Q_\ell \left(\bigoplus_{\sigma \in S^0_{k+1}} \mathbf{x}_\sigma\right) = Q_\ell \left(\bigoplus_{\sigma \in S^1_{k+1}} \mathbf{x}_\sigma\right) \tag{5.9}$$

for all $\ell \in \{0, \ldots, d\}$. This in turn implies that

$$\dot{y}_0 = F\left(y_0, \bigoplus_{\sigma \in S^0_{k+1}} \mathbf{x}_\sigma\right) = F\left(y_1, \bigoplus_{\sigma \in S^1_{k+1}} \mathbf{x}_\sigma\right) = \dot{y}_1$$

whenever $y_0 = y_1$, which proves that the space $\{y_0 = y_1\}$ is dynamically invariant. This completes the proof. $\square$

Remark 7. The synchrony space $\{y_0 = y_1\}$ is the fixed point space of the map

$$S : (x_0, \ldots, x_k, y_0, y_1) \mapsto (x_0, \ldots, x_k, y_1, y_0)\ .$$

From the proof of Theorem 5.4 it is clear that S is a symmetry of the admissible vector field for the augmented hypernetwork whenever F is polynomial of degree less than $\frac{k(k+1)}{2}$, but not in general. As fixed point spaces are dynamically invariant for equivariant systems, this offers an alternative interpretation of Theorem 5.4.

We conclude this section by describing a type of bifurcation that appears to occur abundantly in augmented hypernetworks. The idea is that, even though $\{y_0 = y_1\}$ is not a robust synchrony space, its invariance under polynomial response functions of sufficiently low degree, makes it act as a "ghost" synchrony space that can still influence bifurcations. In particular, in various examples we found steady-state bifurcation branches $(x(\lambda), y(\lambda))$ in which $y_0(\lambda)$ and $y_1(\lambda)$ are not exactly equal, but do agree up to unusually high degree in the bifurcation parameter λ. We have dubbed this phenomenon "reluctant synchrony breaking" and we investigate it in detail in a companion paper. Here we only illustrate it in one numerical example.

Example 5.8. We revisit the augmented hypernetwork from Examples 2.3 and 5.3, of which the admissible ODEs are given by Equation (2.2) in Example 2.5. Instead of a single admissible vector field, we consider a family of them by defining for each $\lambda \in \mathbb{R}$ the response functions

$$G_\lambda(X_0, X_1, X_2) = -X_0 + X_1 - X_2 + 8\lambda X_0 + 4X_0^2 \text{ and} \tag{5.10}$$

$$F_\lambda(Y, (X_0, X_1), (X_2, X_3), (X_4, X_5)) = -5Y + 14\lambda \tag{5.11}$$
$$- h(10X_0 - 12X_1) - h(10X_2 - 12X_3) - h(10X_4 - 12X_5)\,,$$

in which

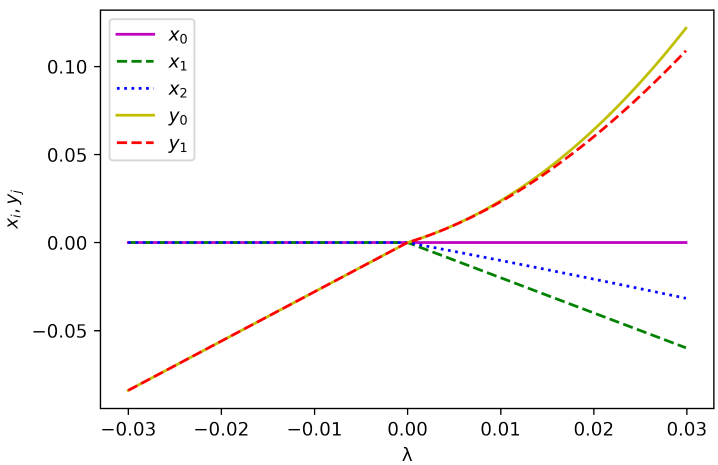
$$h(x) = \sin(x) + \cos(x) - 1 = \sqrt{2}\sin(x + \frac{\pi}{4}) - 1\,. \tag{5.12}$$

For every value of λ, the function F_λ satisfies the symmetry conditions required in Equation (2.3). As $G_0(0,0,0) = F_0(0,(0,0),(0,0),(0,0)) = 0$, the resulting admissible vector field has a fixed point at the origin for $\lambda = 0$. Moreover, by construction the Jacobian of the admissible vector field around the origin is singular, so that we may expect a steady-state bifurcation to occur as λ is varied near 0.

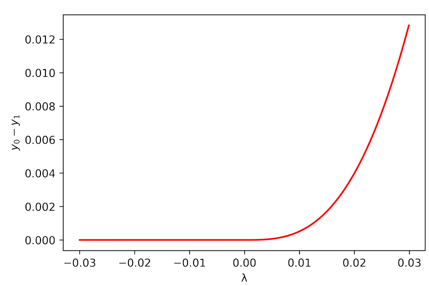
Figure 5 shows the results of a numerical bifurcation analysis of the problem, in which we found a stable branch of steady states for small $\lambda < 0$ and a stable branch of steady states for small $\lambda > 0$. Figure 5a is a bifurcation diagram showing the values of the different components $x_i(\lambda)$ and $y_i(\lambda)$ on these stable branches. The branch for $\lambda < 0$ appears to lie in the robust synchrony space $\{x_0 = x_1 = x_2 \text{ and } y_0 = y_1\}$. For $\lambda > 0$ the core (corresponding to the x-values) becomes fully non-synchronous. For small values of λ, the y-values on this branch seem to remain equal, but at higher values of λ it becomes clear that a small separation occurs. Figure 5b corroborates this observation, showing a non-linear departure from the space $\{y_0 = y_1\}$, with Figure 5c indicating that in fact $y_0(\lambda) - y_1(\lambda) \sim \lambda^3$ for $\lambda > 0$.

Such "reluctant synchrony breaking" is highly anomalous in general vector fields (with generic observables y_0, y_1). However, we claim that it happens for generic one-parameter families of admissible vector fields for this augmented hypernetwork —assuming certain bifurcations happen in the core— and is therefore not an artifact of our particular choice of response functions. Rather, our response functions are merely chosen to guarantee stability of each branch for the correct sign of λ, to produce clear pictures, and to illustrate that the reluctant behavior $y_0(\lambda) - y_1(\lambda) \sim \lambda^3$ is not just a consequence of (low-order) polynomial response functions.

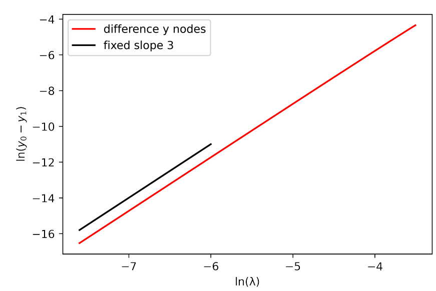
Figure 5 was obtained by forward integrating the equations of motion for each of 600 equidistributed values of $\lambda \in [-0.03, 0.03]$, using Euler's method with time steps of 0.1. For each value of λ integration was performed up to $t = 2000$ and starting from the point $(x_0, x_1, x_2, y_0, y_1) = (0.1, -0.2, 0.3, 0.4, 0.5)$ in phase space. Note that only stable branches can be visualised in this way.



(a) The stable branches of a synchrony breaking bifurcation.



(b) The difference between the y-nodes along the stable branches.



(c) A log–log plot of the difference between the y-nodes, for $\lambda > 0$.

Figure 5: Numerically obtained bifurcation diagram for a family of systems corresponding to the augmented hypernetwork shown in Figure 2. For comparison, the black line segment in the log-log plot has slope 3, indicating that $y_0(\lambda) - y_1(\lambda) \sim \lambda^3$.

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