Graph Coloring:-

The assign of colors to the vertices of G, one color to each vertex, so that adjacent vertex assigned different colors is called proper coloring of G or simple vertex coloring.

Chromatic Number:- A graph G is called m chromatic if it required exactly m colors to paint it properly & the number m is called chromatic number of graph G.

i.e $\chi(G) = m$

Some observation that follows directly from the defination :-

1) A graph consist of only isolated vertices is - 1 chromatic.

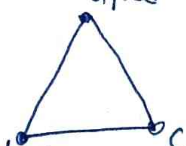
$n = 1$
$m = 1$

a/yellow

2) A graph with one or more edges (not a self loop) is at least 2 chromatic (also called bichromatic).

$n = 2$
$m = 2$

3) A complete graph of n vertices is n-chromatic, as all its vertices are adjacent. Hence a graph containing a complete graph of p vertices is at least n chromatic. For instance, every graph having a triangle is at least 3 chromatic



$n = 3$
$m = 3$

4) a graph having a maximum degree Δ. so the chromatic no. is

$$\boxed{\chi(G) \leq 1 + \Delta(G)}$$

5) a graph having minimum degree δ. so the chromatic no. is

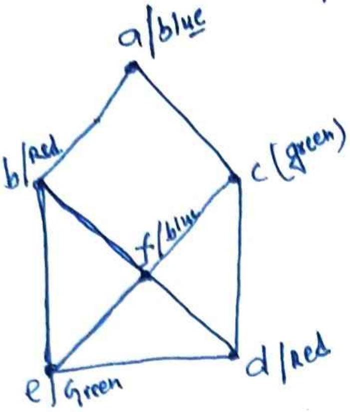
$$\boxed{\chi(G) \geq \frac{n}{n - \delta(G)}}$$

6) If G has no complete graph of $(\Delta(G) + 1)$ vertex. The chromatic no. of this graph is

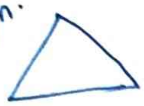
$$\boxed{\chi(G) \geq \Delta(G)}$$

Problem:-

Find the chromatic number of the following graph G.



Complete graph.



$n = 6$

max degree $(\Delta) = 4$.

minimum degree $(\delta) = 2$

Firstly we calculate the range of no. of required color

$\chi(G) \leq 1 + \Delta(G)$

$\leq 1 + 4$

≤ 5

$\chi(G) \geq \frac{n}{n - \delta(G)} \geq \frac{6}{6-2} \geq \frac{6}{4} \geq \frac{3}{2}$

$1.5 \cong 2$

range $\frac{3}{2} \leq \chi(G) \leq 5$

Graph contain a triangle means

$\chi(G) = 3$

So, the range $\boxed{3 \leq \chi(G) \leq 5}$

So, consider

$\Delta(b, e, f) \rightarrow$ (Red, green, blue)

vertex d adjacent to - e and f

d $\rightarrow$ Red.

c $\xrightarrow{\text{adjacent}}$ f and d

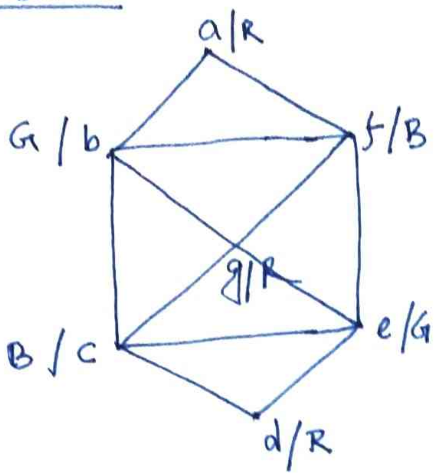
c $\rightarrow$ green.

a $\xrightarrow{\text{adjacent}}$ b and c

a $\rightarrow$ blue.

So, the chromatic no of this Graph $\chi(G) = 3$

Problem-2 : Find chromatic for the following graph :-



$n = 7$

graph contain triangle.

$m = 3$.

(a, b, f) - (Red, green, blue).

	adjacent vertex	
g -	(b, f)	- Red.
e -	(g, f)	- green.
c -	(b, g)	- blue.
d -	(c, e)	- Red.

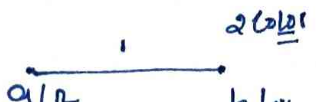
so, $\boxed{\chi(G) = 3 \text{ (chromatic no.)}}$

Chromatic polynomial :-

- A graph G of n vertices can be colored properly in different ways by a sufficiently large no. of colors.
- This property of a graph is expressed by a means of polynomial is called chromatic polynomial.
- It is denoted by $P_n(\lambda)$ of a graph of n vertices

$$P_n(\lambda) = \sum_{i=1}^{n} c_i \binom{\lambda}{i}$$

where c_i - different ways of properly coloring of graph G using i different colors.



$c_1 = 0$

$c_n = n!$

n vertices $= n!$

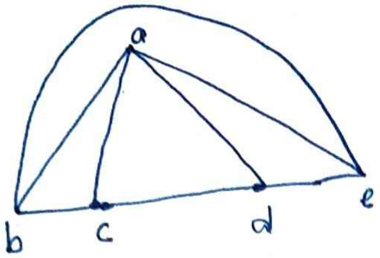
- 1) ~~Each~~ each c_i has to be evaluated individually for the given graph. For example, any graph with even one edge requires at least two color for proper coloring and therefore $c_1 = 0$.

- A graph with n vertices and using n different colors can be properly colored in $n!$ ways; that is

$c_n = n!$

Problem:-

Find chromatic polynomial for following graph.



$n = 5$

so, formula is

$$P_n(\lambda) = \sum_{i=1}^{n} c_i \binom{\lambda}{i}$$

$$P_5(\lambda) = \sum_{i=1}^{5} c_i \binom{\lambda}{i}.$$

so, we calculate c_1, c_2, c_3, c_4, c_5

- Graph G has a triangle so we require atleast 3 colors to paint (a, b, c) triangle.

$$C_1 = C_2 = 0$$

- Now 3 colors can be assigned to 3 vertices a, b, c in $3! = 6$ different ways.

$$C_3 = 6.$$

- C_4 ⇒ out of 4 colors 3 can be selected and assigned to a, b, c in ${}^4P_3 = 24$ ways.

- Now d will have two choices: forth colors / color as b

$$C_4 = \overset{4!}{24} \cdot 2 = 48 \text{ ways.}$$

- $C_5 = 5! = 120.$

$$P_5(\lambda) = c_1 \frac{\lambda(\lambda-1)}{1!} + c_2 \frac{\lambda(\lambda-1)(\lambda-2)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{4!} + c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{5!}$$

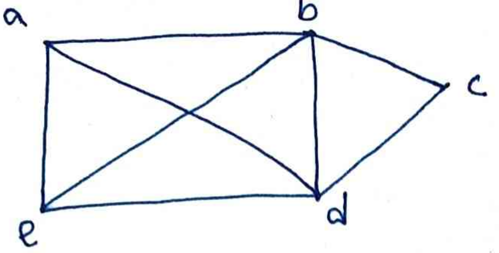
$$= 0 + 0 + \frac{6\lambda(\lambda-1)(\lambda-2)}{3!} + 48\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 120\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)}{5!}$$

$$= \boxed{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^2-5\lambda+7)}$$

This is the choromatic polynomial of following graph.

Prob

Find the chromatic polynomial of following graph.



$$P_n(\lambda) = \sum_{i=1}^{n} c_i \binom{\lambda}{i}$$

$$P_5(\lambda) = \sum_{i=1}^{5} c_i \binom{\lambda}{i}$$ So, we calculate c_1, c_2, c_3, c_4, c_5

Graph contain a complete graph (a, b, d, e) of 4 vertices. So it requires at least 4 color to color it

So, $\boxed{c_1 = c_2 = c_3 = 0}$

c_4 – 4 color be c_1, c_2, c_3, c_4 assigned vertex a, b, d, e in

$4! = 24.$

There are two color for vertex c.

$c_4 = 24 \cdot 2 = 48$ ways.

$c_5 - 5! = 120.$

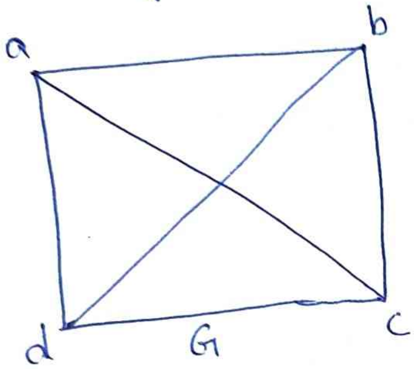
So, $$P_5(\lambda) = c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$= 0 + 0 + 0 + 48 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 120 \cdot \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$= \boxed{\lambda^5 + 8\lambda^4 - 23\lambda^3 - 28\lambda^2 + 20\lambda}$$ so, this is the chromatic polynomial of the given graph.

Covering

- In a graph G, a set of edges h is said to be cover G if every vertex in G is incident on atleast one edge in h.
- A set of edges that cover a graph G is called an edge covering or a covering subgraph or simply a covering of G.



$G = \{a, b, c, d\}$

$$\boxed{\begin{array}{l} h = \{ab, cd\} \\ h = \{ac, bd\} \end{array}}$$

So, both set of h edges are the cover of graph G.

Observation

i) A graph G is its own covering.

ii) A spanning tree in a connected graph G is also a cover of G.

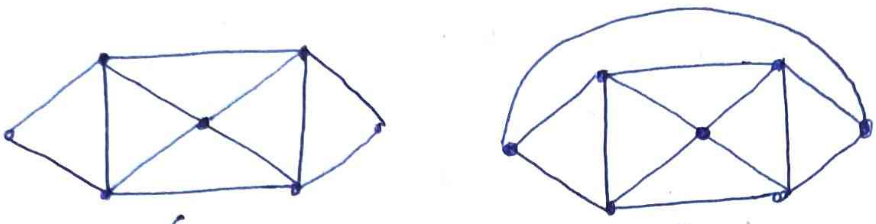
iii) Hamiltonian circuit is G is also a covering in G.

iv) The covering exist in a graph G if G has no isolated vertex.

Four color problem:-

- The four color Theorem: The chromatic number of a plannar graph is no greater than 4.

  example:- G_1 chromatic number = 3,
  G_2 chromatic number = 4.



G_1 G_2

If the regions of a plannar graph are colored so that adjacent regions have different colors, then no more than 4 colors are required.

i.e, $\chi(G) \leq 4$.

Algorithm for graph coloring :=

Backtracking : Graph coloring Algorithm.

```
Algorithm m Coloring (k)
// This algorithm was formed using recursive backtracking.
// schema, The graph is represented by its boolean adjacency.
// matrix G[1:n, 1:n]. All assignments of 1,2,... m to the
// vertices of the graph such that adjacent vertices are assigned.
// distinct integers are printed k is the index of the next vertex.
// to color.
{
   repeat
   {
      Next value (k)
      if (x[k] = = 0) then return.
      if (k = = n) then write (x[1:n])
      else
         mcoloring (k+1)
   } until (false).
}
```

Finding all m coloring of a graph

Algorithm Next Value (k).

// x[1] ---- x[k-1] have been assigned integer values in the range 1 to m such that adjacent vertices have distinct integers A value for x[k] is determine in the range 0 to m x[k] is assigned next highest number color ~~assigned~~ while maintaining distinctness from the adjacent vertices of vertex k if no such color exists, then x[k] is 0.

```
{
    repeat
    {
        x[k] = (x[k]+1) mod (m+1)          // next highest node.
        if (x[k] = = 0) then return        // color exhausted
        for j = 1 to n do
        {
            if ((G[k, j] ≠ 0) and (x[k] = = x[j]))
            then break                     // adjacent with same Color
        }
        if (j = = (n+1)) then return       // new color found.
    } until (false)                        // otherwise find other
}
```

Generating the next Color.