Matrix Representation of Graph :=

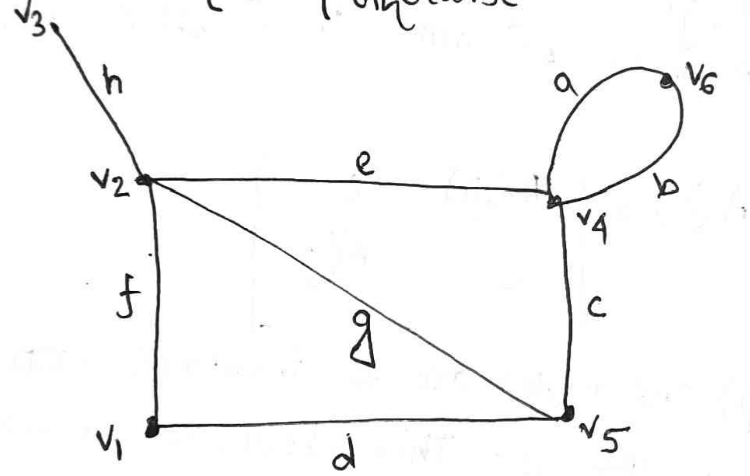
- Incidence Matrix.
- Adjacency Matrix.

Incidence Matrix

Let G be a graph with n vertices, e edges, and no self loops. Define an n by e matrix $A = [a_{ij}]$, whose n row corrospond to the n vertices and the e columns corresponds to the e edges, as follows:

The matrix element

$$a_{ij} = \begin{cases} 1 & \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i \\ 0 & \text{otherwise} \end{cases}$$



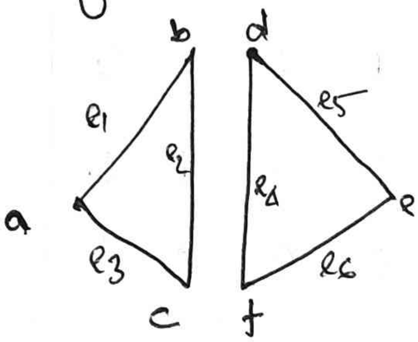
$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The following observation about the incident matrix A can be made.

1. Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.
2. The number of 1's in each row equals the degree of the corresponding vertex.
3. A row with all 0's, therefore, represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix, for example column 1 and 2.
5. If a graph G is disconnected and consist of two components g_1 and g_2, the incident matrix A(G) of graph G can be written in a block-diagonal form as

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]$$

where $A(g_1)$ and $A(g_2)$ are the incidence matrixes of components g_1 and g_2. This observation results from the fact that no edge in g_1 is incident on vertices of g_2, and vice-versa. Obviously, this remark is also true for a disconnected graph with any number of components.

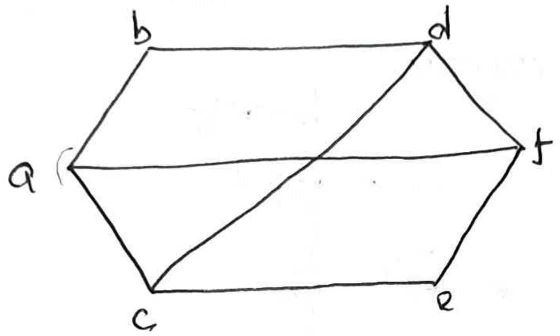


$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}
\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\left[\begin{array}{ccc|ccc}
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array}\right] \end{array}$$

Adjacency Matrix :=

Let G be a graph with n vertices, e edges and no parallel edges ; is a $(n \times n)$ symmetric matrix .

$$M[G] = [m_{ij}] = \begin{cases} 1 & \text{if } i^{th} \text{ vertex is adjacent to } j^{th} \text{ vertex} \\ 0 & \text{otherwise.} \end{cases}$$



* two vertices are adjacent – common edge
* two edges are adjacent – common vertex .

$$M(G) = m_{ij} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

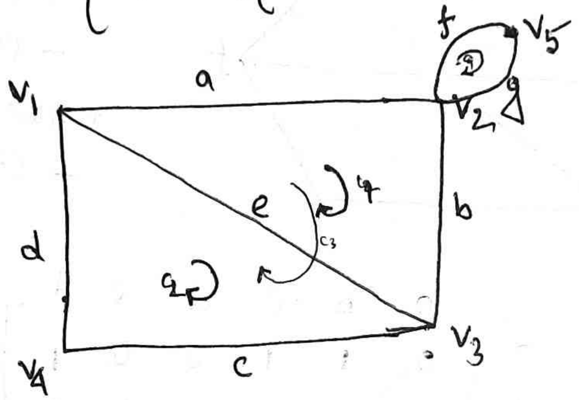
Observation :

- The principal of digonal matrix are all contain value 0. i.e, it do not contain self loop.
- Suppose $\deg(a) = 3$; i.e summation of each row or each column that return the degree of this particular vertex.
- $$M[G] = \left[\begin{array}{c|c} M[G_1] & 0 \\ \hline 0 & M[G_2] \end{array}\right]$$ • it is the adjacency matrix for disconnected graph.
- adjacency matrix is a symmetric matrix because $x_{ij} = x_{ji}$

Circuit matrix :-

Let the number of different circuit in a graph G be q and the number of edges in G be e. Then a circuit matrix B = $[b_{ij}]$ of G is a q by e, (0, 1) – matrix defined as follows :

$$b_{ij} = \begin{cases} 1 & \text{if } i^{th} \text{ circuit includes } j^{th} \text{ edges.} \\ 0 & \text{otherwise.} \end{cases}$$



Circuit are present here,

$c_1 = \{a, b, e\}$
$c_2 = \{d, c, e\}$
$c_3 = \{f, g\}$
$c_4 = \{a, b, c, d\}$.

* m by e
m ↓ no. of circuit
e ↓ no. of edges.

$$[B] = [b_{ij}] = \begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline c_1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ c_2 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ c_3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ c_4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}$$

Observation :-

- A column of all zeros corresponds to a noncircuit edges. i.e, an edge that does not belong to any ~~circuit~~ circuit.
- Each row of B(G) is a circuit vector.

- A circuit matrix is capable of representing a self-loop — the corresponding row will have a single 1.
- The number of 1s in a row is equal to the number of edges in the corresponding circuit.
- If a graph G is disconnected and consist of two blocks (or components) g_1 and g_2, the circuit matrix B(G) can be written as a block diagonal form as

$$B(G) = \left[\begin{array}{c|c} B(g_1) & 0 \\ \hline 0 & B(g_2) \end{array}\right]$$

where $B(g_1)$ and $B(g_2)$ are the circuit matrices of g_1 and g_2. That means circuit in g_1 have no edges blonging to g_2, and vice-versa.

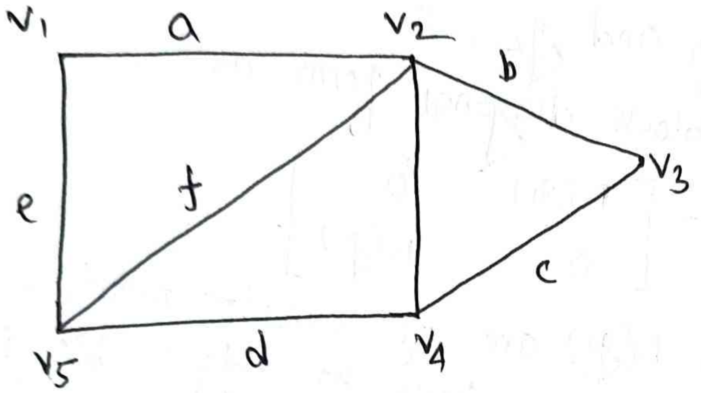
- Permutation of any two rows or columns in circuit matrix simply corresponds to relabeling the circuit and edges.
- Two graphs G_1 and G_2 will have the same circuit matrix if and only if G_1 and G_2 are isomorphic.

Fundamental circuit matrix and Rank of B :-

- A set of fundamental circuits with respect to any spanning tree in a connected graph, are the only independent circuits in a graph.
- The rest of the circuits can be obtained as ring sums of these circuits.
- Thus, in circuit matrix, if we retain only those rows that corresponds to a set of fundamental circuits and remove all other rows, we would not lose any information. The remaining rows can be reconstructed from the rows corresponding to the set of fundamental circuit.

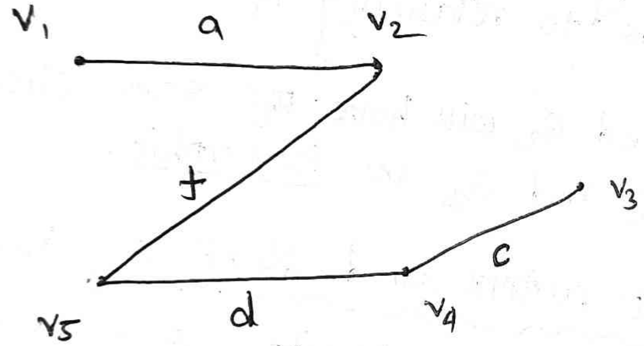
Hence,

A submatrix $B_f(G)$ of circuit matrix $B(G)$ in which all rows corresponds to fundamental circuit with respect to spanning tree in G of graph G.



G

first construct a spanning tree.



So, branches are = $\{a, f, d, c\}$

Chords = $\{e, g, b\}$

So, In here the fundamental circuit are

(i) $\{a, e, f\}$

(ii) $\{f, d, g\}$

(iii) $\{f, d, c, b\}$

$$B_f(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccccccc} e & g & b & a & f & d & c \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Observation

- If n is the number of vertices and e is the number of edges in a connected graph, then B_f is an $(e-n+1)$ by e matrix, because the number of fundamental circuit being produced by one chord.

- A matrix B_f thus arranged can be written as

$$B_f = [I_\mu \,\vdots\, B_t]$$

where I_μ is an identity matrix of order $\mu = e-n+1$, and B_t is the remaining μ by $(n-1)$ submatrix, corresponding to the branches of the spanning tree.

So, the rank of $B_f = \mu = e-n+1$.

Since B_f is a submatrix of the circuit matrix B, the rank of $\boxed{B \geq e-n+1}$

~~Proof~~ Theorem :- If B is a circuit matrix of a connected graph G with e edges and n vertices,

$$\text{rank of } B = e-n+1.$$

Proof:- If A is an incident matrix of G, then

$$A \cdot B^T = 0 \pmod 2.$$

Therefore, according to sylvester's theorem

$$\text{rank of } A + \text{rank of } B \leq e;$$

i.e,

$$\text{rank of } B \leq e - \text{rank of } A.$$

Since rank of $A = n-1$.

We have rank of $B \leq e-n+1$.

But rank of $B \geq e-n+1$.

Therefore, we must have

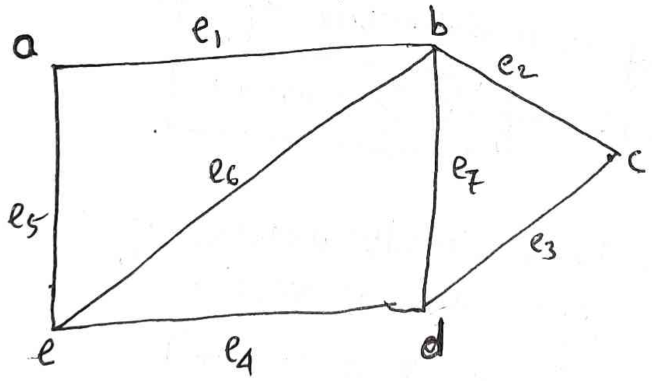
$$\text{rank of } B = e-n+1$$

Proof

Path Matrix / Reachability Matrix :=

A path matrix is defined for a specific pair of vertices in a graph, say (x, y) and is written as $P(x, y)$. The rows $P(x, y)$ Corresponds to different paths between the vertices x and y, and the Column Correspond to the edges in G. That is, the path matrix for (x, y) vertices is $P(x, y) = [P_{ij}]$, where

$$P_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ edge line in } i^{th} \text{ path.} \\ 0 & \text{otherwise.} \end{cases}$$



Possible path from vertice a to d.

① (e_1, e_7)

② (e_1, e_6, e_7)

③ (e_5, e_4)

④ (e_5, e_6, e_7)

⑤ (e_1, e_2, e_3)

$$P(a,d) = \begin{array}{c c} & \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

observation :-

- A column of all 0's corresponds to an edge that does not lie in any path between x and y.
- A column of all 1's corresponds to an edges that lies in every path between x and y.
- There is no row with all 0's.
- The ring sum of any two rows in $P(x,y)$ corresponds to a circuit or an edge disjoint union of circuits.

Relationship among A_f, B_f' and C_f :=

In here the relationship ~~between~~ among the reduced incidence matrix A_f, the fundamental circuit matrix B_f and fundamental cutset matrix C_f of a connected graph.

we know that,

step 1>

$$B_f = [I_\mu \mid B_t]$$

$$C_f = [C_c \mid I_{n-1}]$$

- where t denotes the submatrix corresponding to the branches of a spanning tree.
- and c denotes the submatrix corresponding to the chords.

step2)

let spanning tree for B_f and C_f be the same and let the order of the edges in both equation be the same.

step3) Reduce incidence matrix A_f of size $(n-1)$ by e.

step4) let the edges be arranged in the same order as in B_f and C_f.

step5) Partition A_f into the Submatrix $A_f = [A_c \mid A_t]$ —— (3)

step6) since Column of A_f and B_f are same

$$A \cdot B^T = B \cdot A^T = 0 \quad (\text{mod 2 arthimatic}).$$

$$A_f \cdot B_f^T = 0$$

$$[A_c \mid A_t] \cdot [I_\mu \mid B_t]^T = 0.$$

$$[A_c \mid A_t] \cdot \begin{bmatrix} I_\mu \\ \hdashline B_t^T \end{bmatrix} = 0$$

$$A_c + A_t B_t^T = 0 \quad \text{——— (4)}$$

since A_t is non singular, A_t^{-1} exist.

NOW equation (4) $\times A_t^{-1}$

$$A_t^{-1} \cdot A_c + B_t^T = 0$$

$$A_t^{-1} \cdot A_c = -B_t^T$$

since in mod 2 arthmatic $-1 = 1$.

$$\boxed{A_t^{-1} \cdot A_c = B_t^T} \quad \text{——— (A)}$$

Similarly, since the column in B_f and C_f are same

$$C_f \cdot B_f^T = 0$$

$$\left[C_c \,\vdots\, I_{n-1} \right] \cdot \left[I_u \,\vdots\, B_t \right]^T = 0$$

$$\left[C_c \,\vdots\, I_{n-1} \right] \cdot \left[\frac{I_u}{B_t^T} \right] = 0$$

$$C_c + B_t^T = 0$$

$$C_c = -B_t^T$$

Since in mod 2 arithmatic $-1 = 1$.

$$\boxed{C_c = B_t^T} \text{ ——— (B)}$$

$$\boxed{A_t^{-1} \cdot A_c = B_t^T} \text{ ——— (A)}$$

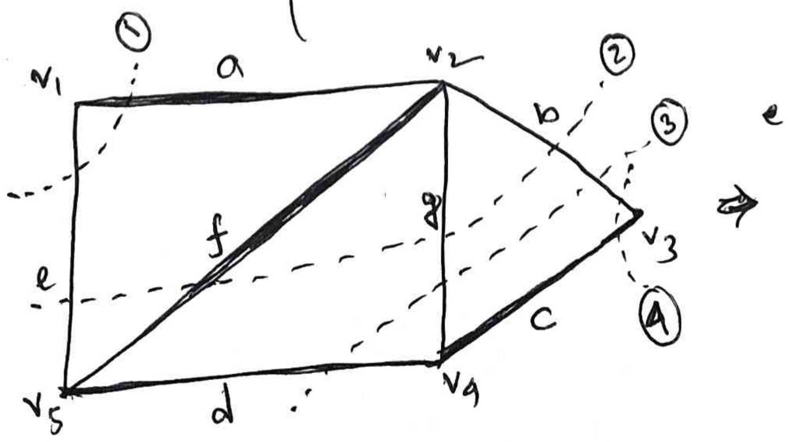
Now we can written,

$$\boxed{A_t^{-1} \cdot A_c = B_t^T = C_c}$$ Relationship.

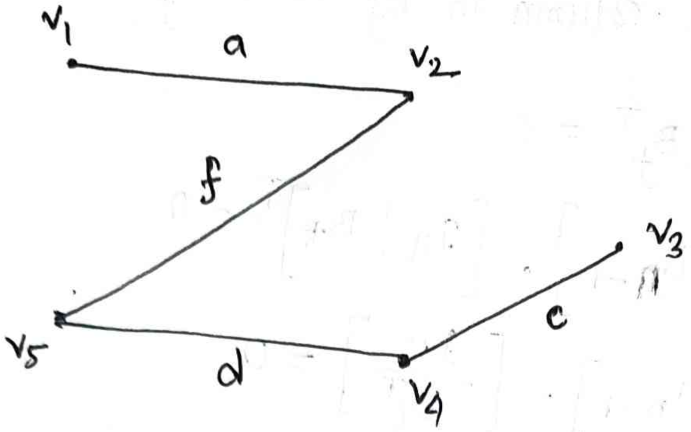
Cut set matrix (C_f) :=

We can define a cut set matrix $C = [c_{ij}]$ in which the rows correspond to the cut sets and the column to the edges of the graph, as follows

$$c_{ij} = \begin{cases} 1 & \text{if } i^{th} \text{ cut-set contain } j^{th} \text{ edge.} \\ 0 & \text{otherwise.} \end{cases}$$



⇒



Branches = $\{a, f, d, c\}$

Chond = $\{e, g, b\}$.

So, in here ~~the~~ fundamental cut-set are.

① $\{e, \acute{a}\}$

② $\{e, \acute{f}, g, b\}$

③ $\{\acute{d}, g, b\}$

④ $\{\acute{c}, b\}$.

⇒ [In here, we are selected as a one brach and remanning are chord]

So, Cut-set matrix is represented as,

$$
C = C_{ij} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccc:cccc} e & g & b & a & f & d & c \end{array} \\
\left[\begin{array}{ccc:cccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & & 1
\end{array}\right]
\end{array}
$$