



第2.6章 图像恢复(Restoration)

- 数字图象恢复与增强的目的类似，也是旨在改善图象的质量。
- 但恢复是力求保持图象的本来面目，即以保真原则为其前提，这是区别于增强的基本不同点。因而恢复时要了解图象质量下降的物理过程，找出或估计其物理模型。恢复的过程就是沿着质量下降的逆过程来重现原始图象。

图象降质的模型

□ 线性位移不变系统，加性噪声

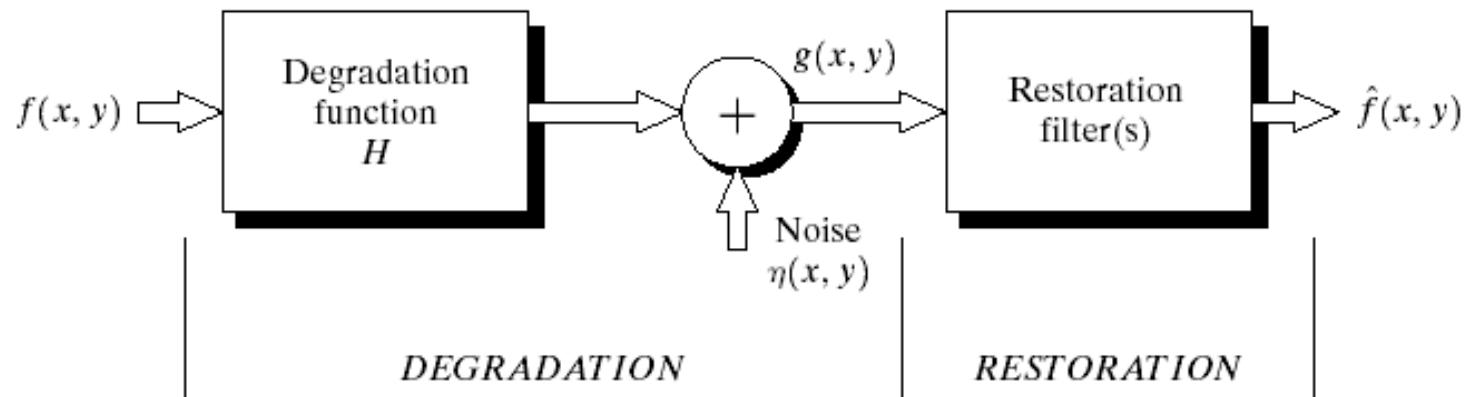


FIGURE 5.1 A model of the image degradation/restoration process.

$$g(x, y) = h(x, y) * f(x, y) + \eta(x, y)$$

$$G(u, v) = H(u, v)F(u, v) + N(u, v)$$



噪声模型

□ 噪声的产生

- 图像获取: CCD相机
- 图像传输: 无线传输

□ 噪声的空域特性和频域特性

- 空域特性: 噪声的直方图分布
- 频域特性: 噪声在Fourier频谱上的分布

□ 几种典型的噪声模型

- 高斯噪声 (Gaussian noise), 瑞利噪声(Rayleigh noise),
厄兰噪声(Erlang noise), 指数噪声(Exponential noise),
均匀噪声(Uniform noise), 椒盐噪声(salt-and-pepper),



几种典型的噪声模型

$$p(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-\mu)^2/2\sigma^2} \text{ (高斯噪声)}$$

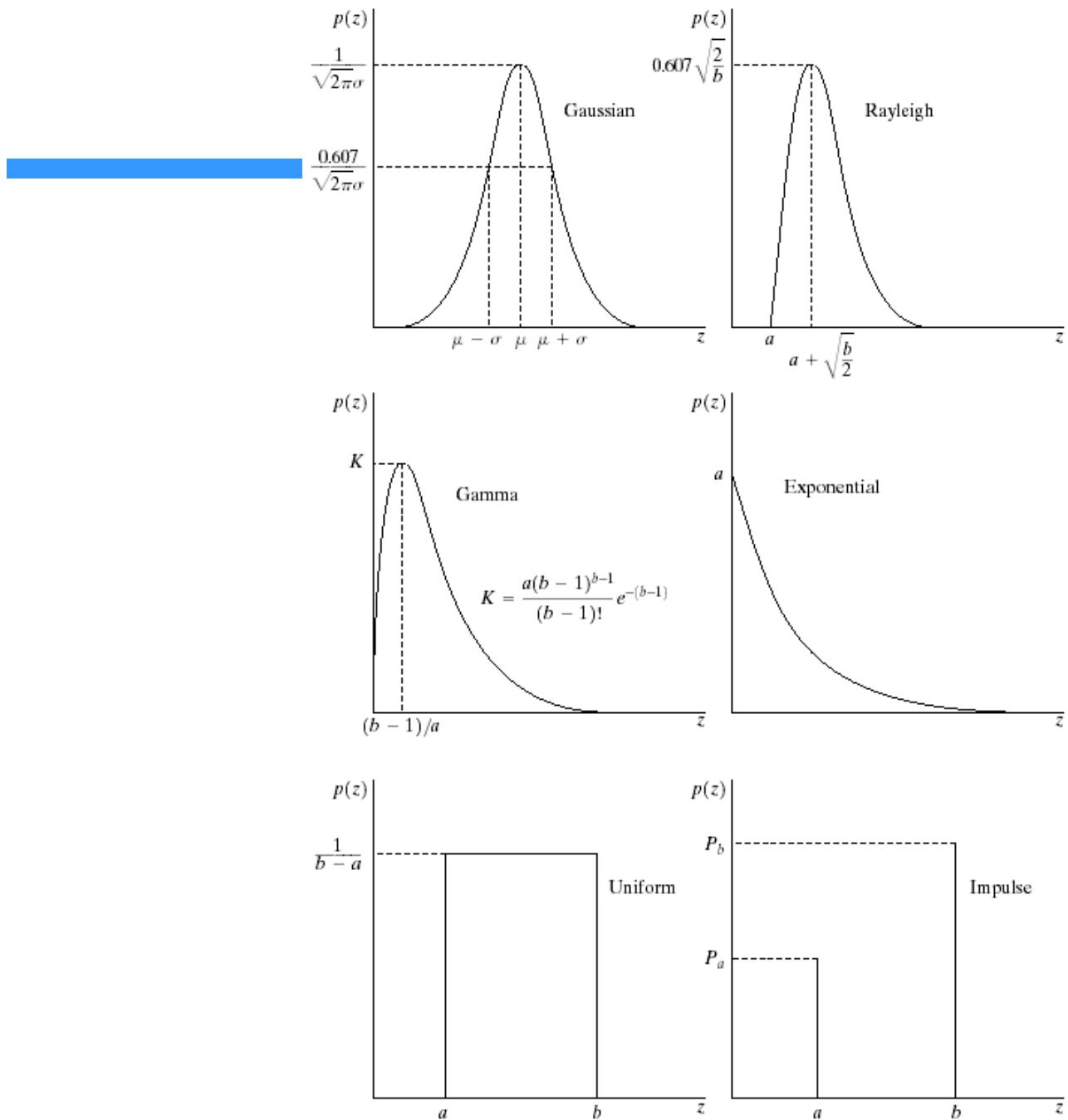
$$p(z) = \begin{cases} \frac{2}{b} (z-a) e^{-(z-a)^2/b} & \text{for } z \geq a \\ 0 & \text{for } z < a \end{cases} \text{ (瑞利噪声)}$$

$$p(z) = \begin{cases} \frac{\alpha^b z^{b-1}}{(b-1)!} e^{-\alpha z} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases} \text{ (厄兰噪声)}$$

$$p(z) = \begin{cases} a e^{-\alpha z} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases} \text{ (指数噪声)}$$

$$p(z) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq z \leq b \\ 0 & \text{otherwise} \end{cases} \text{ (均匀噪声)}$$

$$p(z) = \begin{cases} p_a & \text{for } z = a \\ p_b & \text{for } z = b \\ 0 & \text{otherwise} \end{cases} \text{ (椒盐噪声)}$$



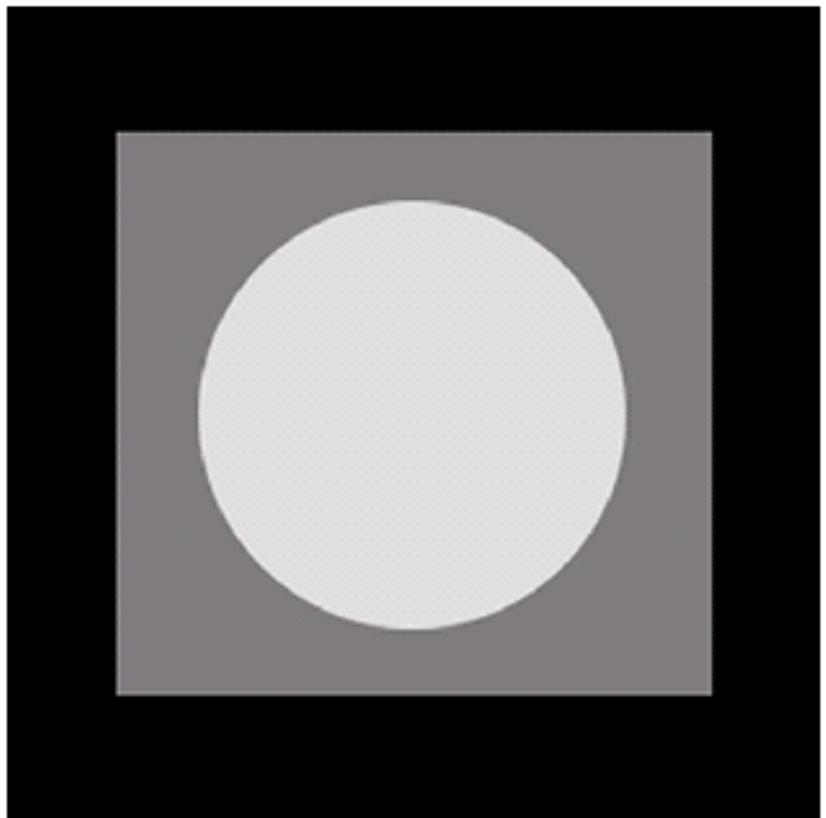
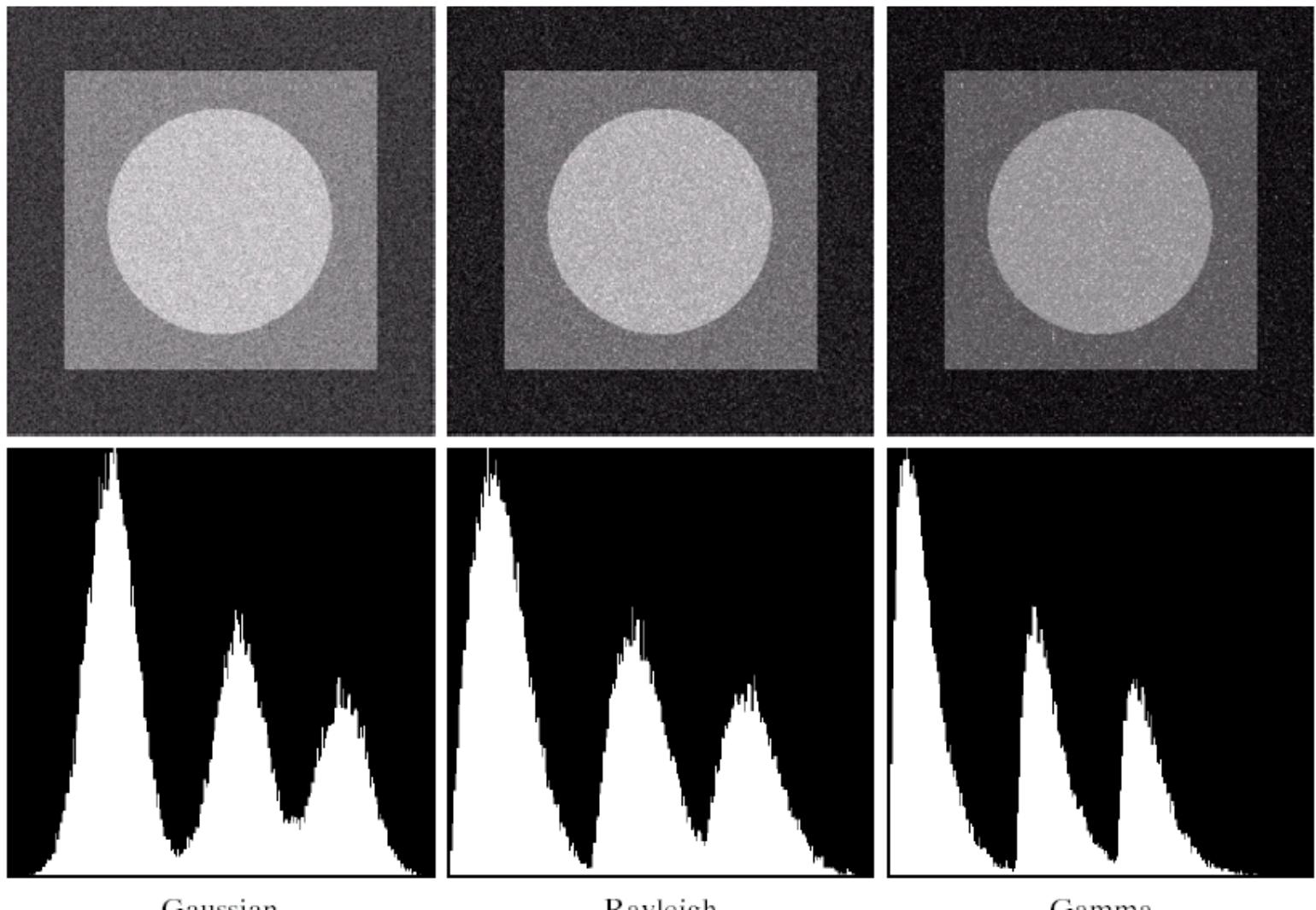


FIGURE 5.3 Test pattern used to illustrate the characteristics of the noise PDFs shown in Fig. 5.2.



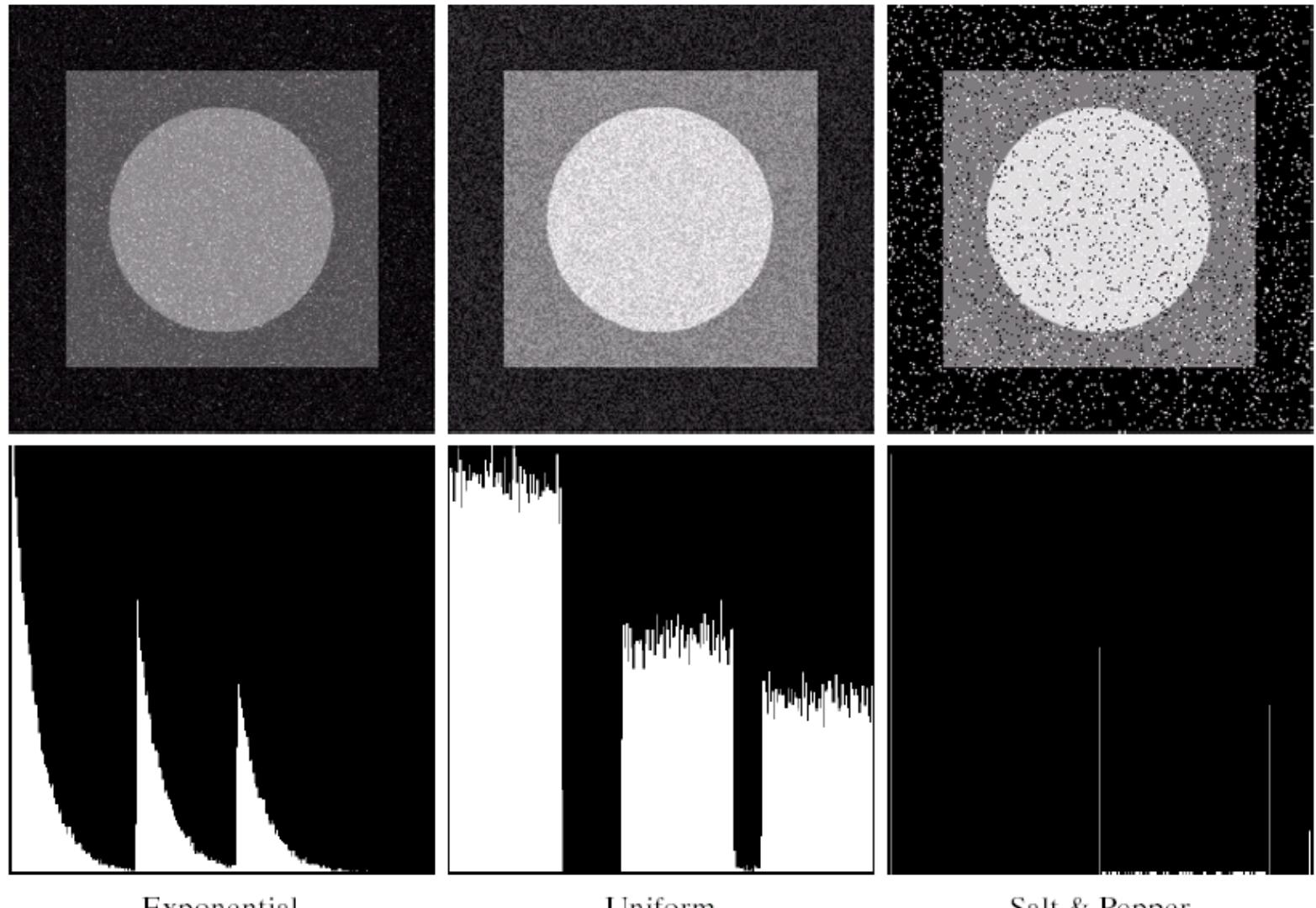
Gaussian

Rayleigh

Gamma

a b c
d e f

FIGURE 5.4 Images and histograms resulting from adding Gaussian, Rayleigh, and gamma noise to the image in Fig. 5.3.



g	h	i
j	k	l

FIGURE 5.4 (Continued) Images and histograms resulting from adding exponential, uniform, and impulse noise to the image in Fig. 5.3.

周期噪声(Periodic Noise)

a

b

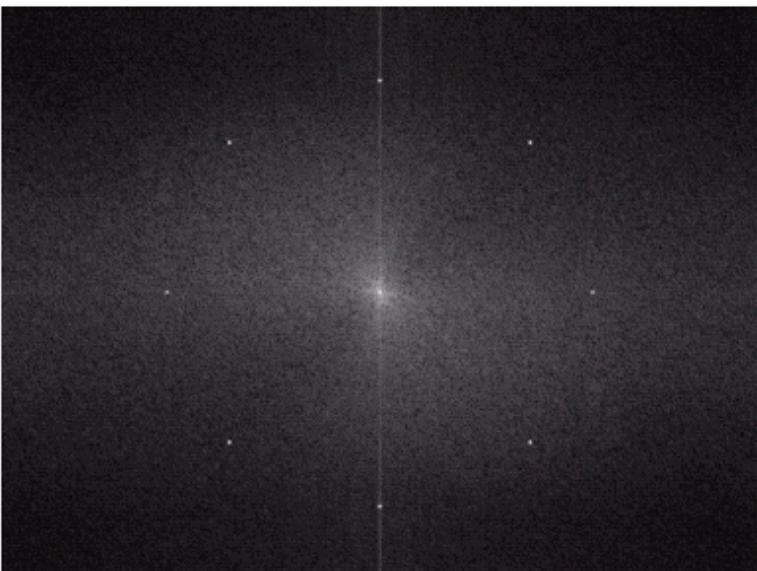
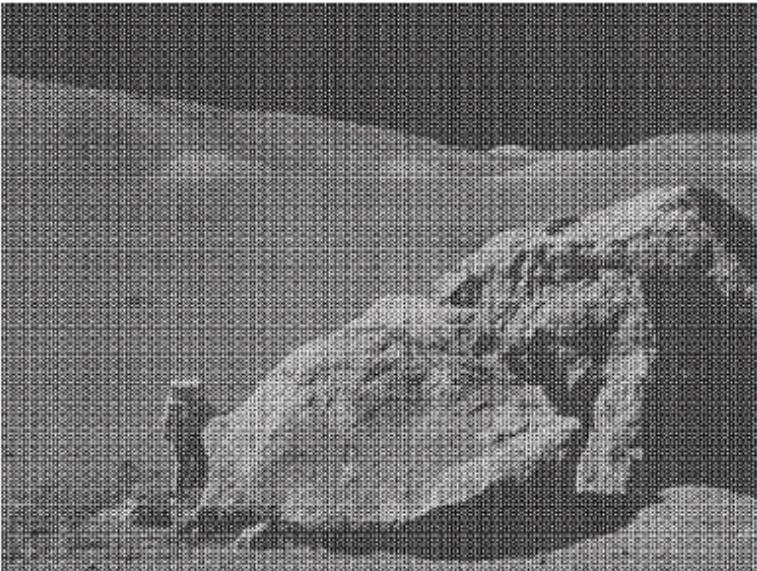
FIGURE 5.5

(a) Image corrupted by sinusoidal noise.

(b) Spectrum (each pair of conjugate impulses

corresponds to one sine wave).

(Original image courtesy of NASA.)



估计噪声参数

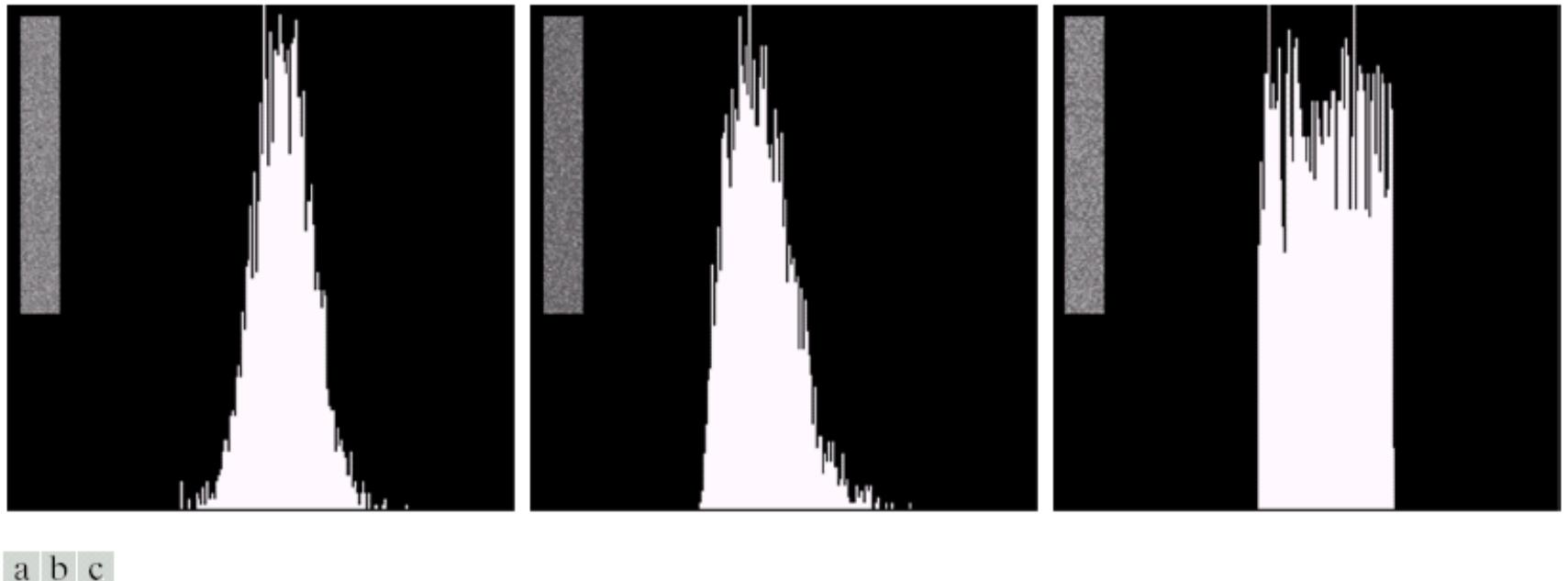


FIGURE 5.6 Histograms computed using small strips (shown as inserts) from (a) the Gaussian, (b) the Rayleigh, and (c) the uniform noisy images in Fig. 5.4.



基于空域滤波的噪声恢复

- 基于空域滤波的方法仅针对加性噪声
- 与空域增强原理相同
- 几种常用滤波器
 - 均值滤波器 (Mean Filters)
 - 次序统计滤波器 (Order Statistic Filters)
 - 自适应滤波器(Adaptive Filters)



几种常用滤波器

- 算术平均滤波器(Arithmetic mean filters)

$$\hat{f}(x, y) = \frac{1}{mn} \sum_{(s,t) \in S_{xy}} g(s, t)$$

- 几何平均滤波器(Geometric mean filter)

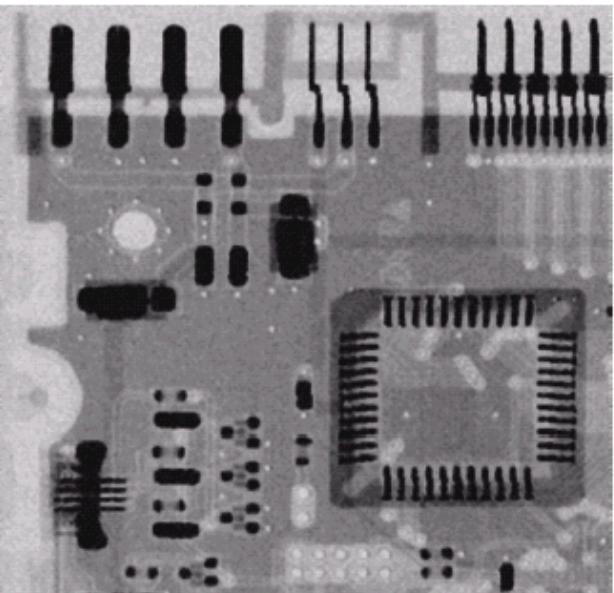
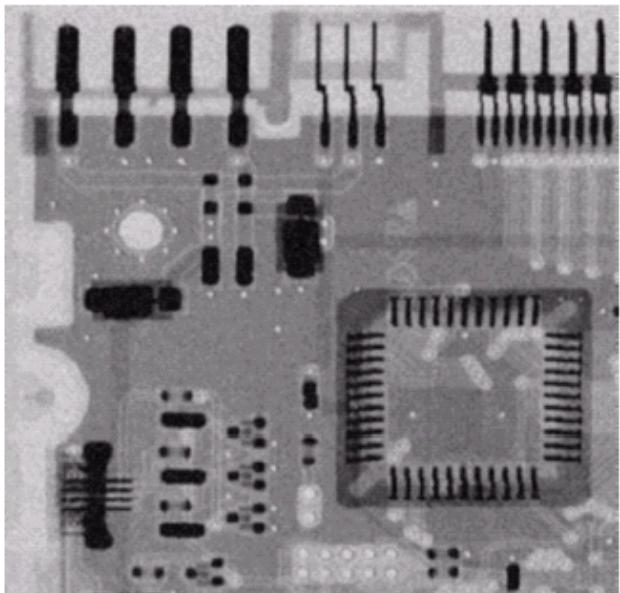
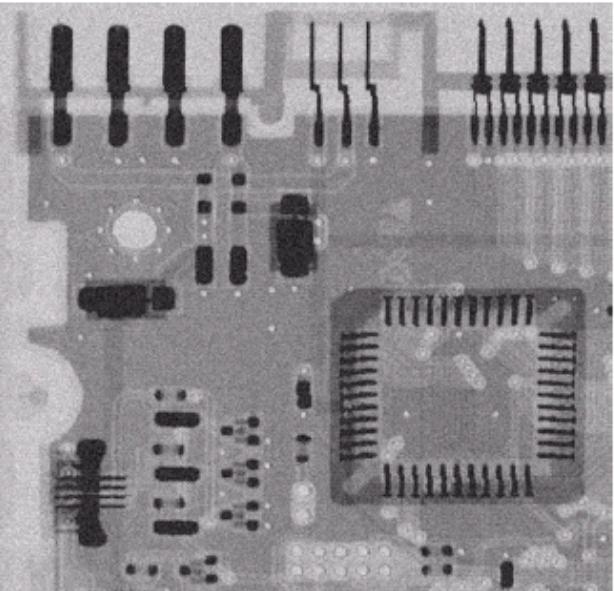
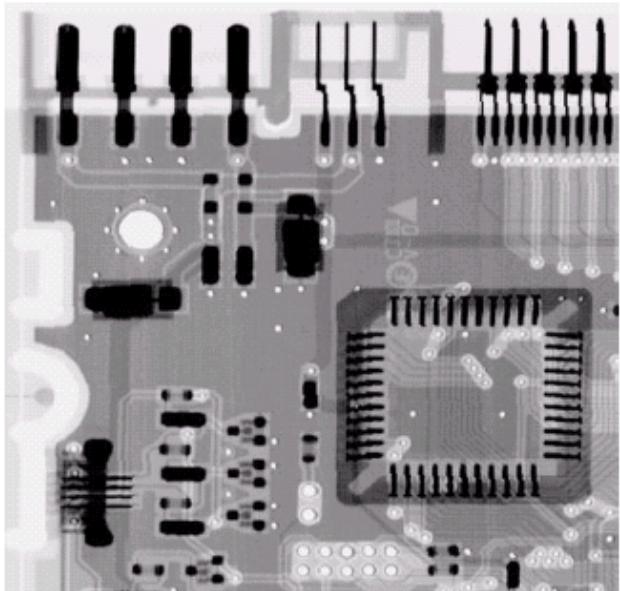
$$\hat{f}(x, y) = \left[\prod_{(s,t) \in S_{xy}} g(s, t) \right]^{\frac{1}{mn}}$$

- 调和平均滤波器(Harmonic mean filter)

$$\hat{f}(x, y) = \frac{mn}{\sum_{(s,t) \in S_{xy}} \frac{1}{g(s, t)}}$$

- 反调和平均滤波器(Contraharmonic filter)

$$\hat{f}(x, y) = \frac{\sum_{(s,t) \in S_{xy}} g(s, t)^{\frac{Q+1}{Q}}}{\sum_{(s,t) \in S_{xy}} g(s, t)^{\frac{Q}{Q}}}$$



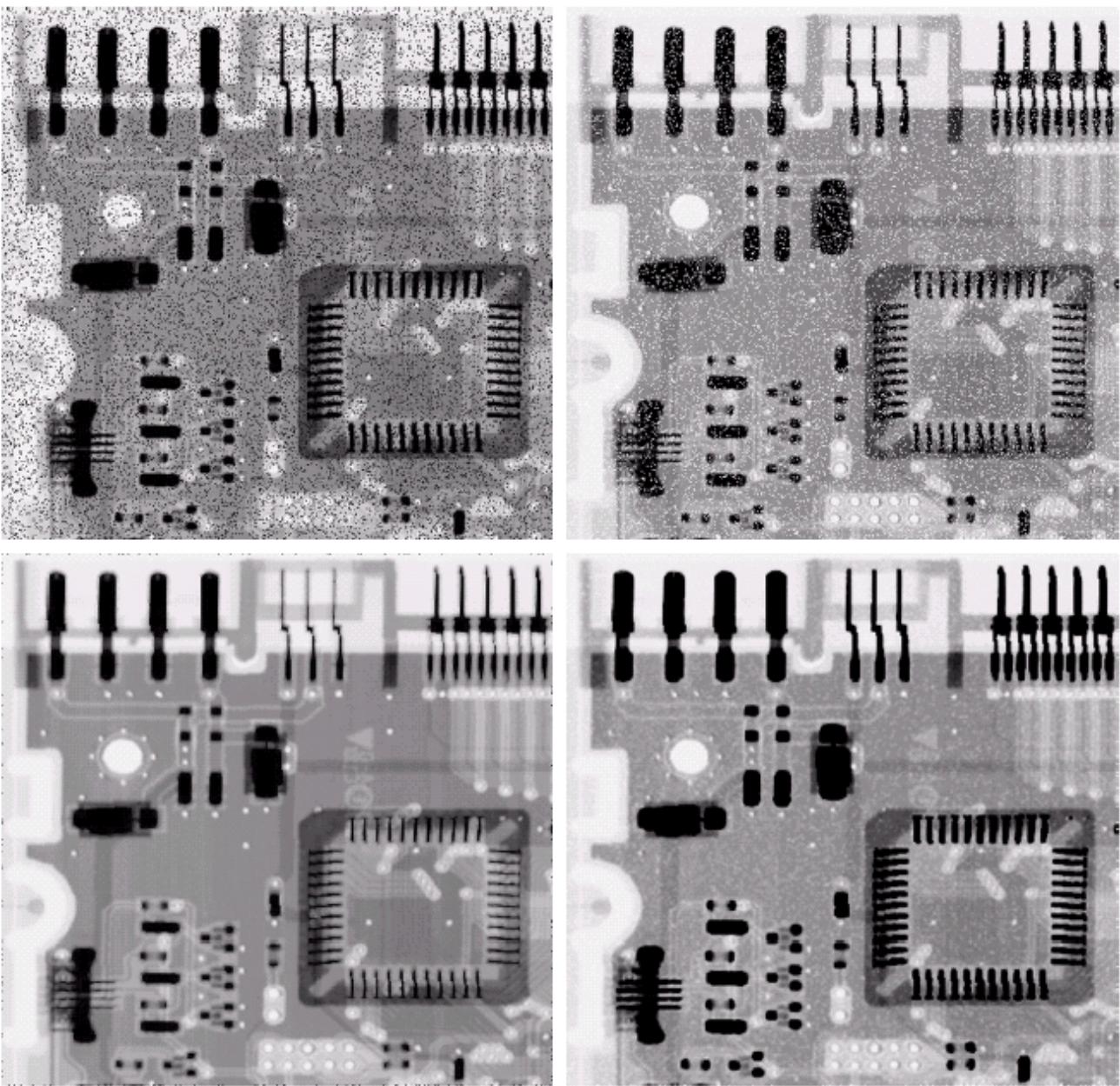
a
b
c
d

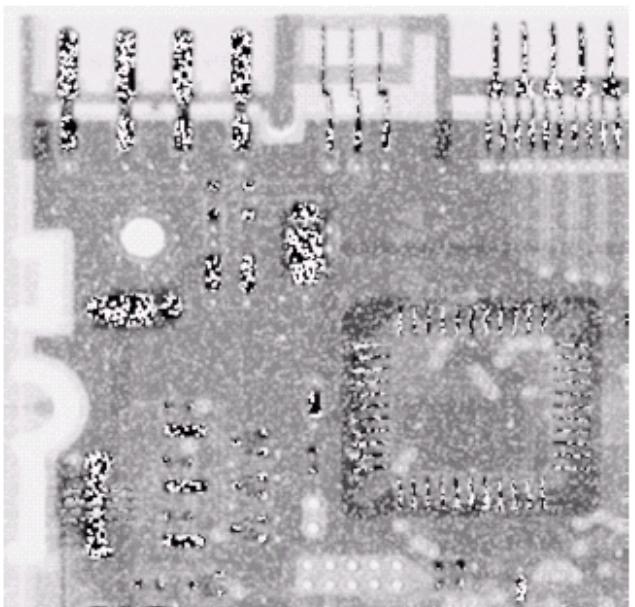
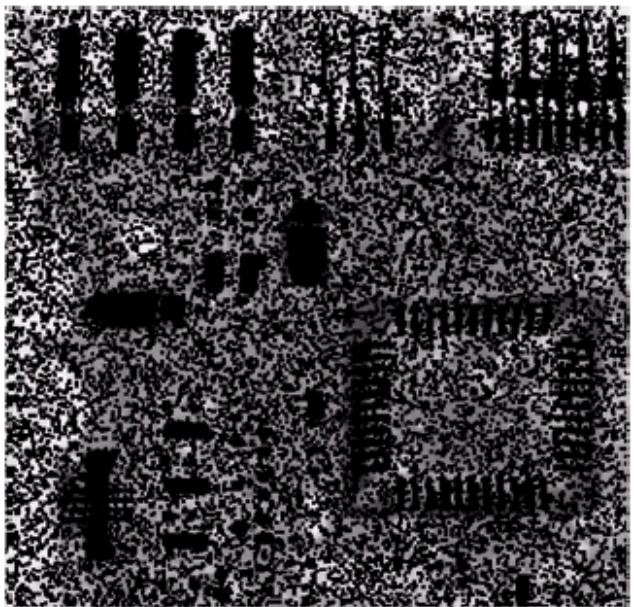
FIGURE 5.7 (a) X-ray image.
(b) Image corrupted by additive Gaussian noise. (c) Result of filtering with an arithmetic mean filter of size 3×3 . (d) Result of filtering with a geometric mean filter of the same size. (Original image courtesy of Mr. Joseph E. Pascente, Lixi, Inc.)

a
b
c
d

FIGURE 5.8

- (a) Image corrupted by pepper noise with a probability of 0.1. (b) Image corrupted by salt noise with the same probability. (c) Result of filtering (a) with a 3×3 contraharmonic filter of order 1.5. (d) Result of filtering (b) with $Q = -1.5$.





a b

FIGURE 5.9 Results of selecting the wrong sign in contraharmonic filtering. (a) Result of filtering Fig. 5.8(a) with a contraharmonic filter of size 3×3 and $Q = -1.5$. (b) Result of filtering 5.8(b) with $Q = 1.5$.



次序統計濾波器

中值濾波器(Median filter)

$$\hat{f}(x, y) = \underset{(s,t) \in S_{xy}}{\text{median}} \{g(s, t)\}$$

最大最小濾波器(Max and min filters)

$$\begin{cases} \hat{f}(x, y) = \underset{(s,t) \in S_{xy}}{\max} \{g(s, t)\} \\ \hat{f}(x, y) = \underset{(s,t) \in S_{xy}}{\min} \{g(s, t)\} \end{cases}$$

中點濾波器(Midpoint filter)

$$\hat{f}(x, y) = \frac{1}{2} \left[\underset{(s,t) \in S_{xy}}{\max} \{g(s, t)\} + \underset{(s,t) \in S_{xy}}{\min} \{g(s, t)\} \right]$$

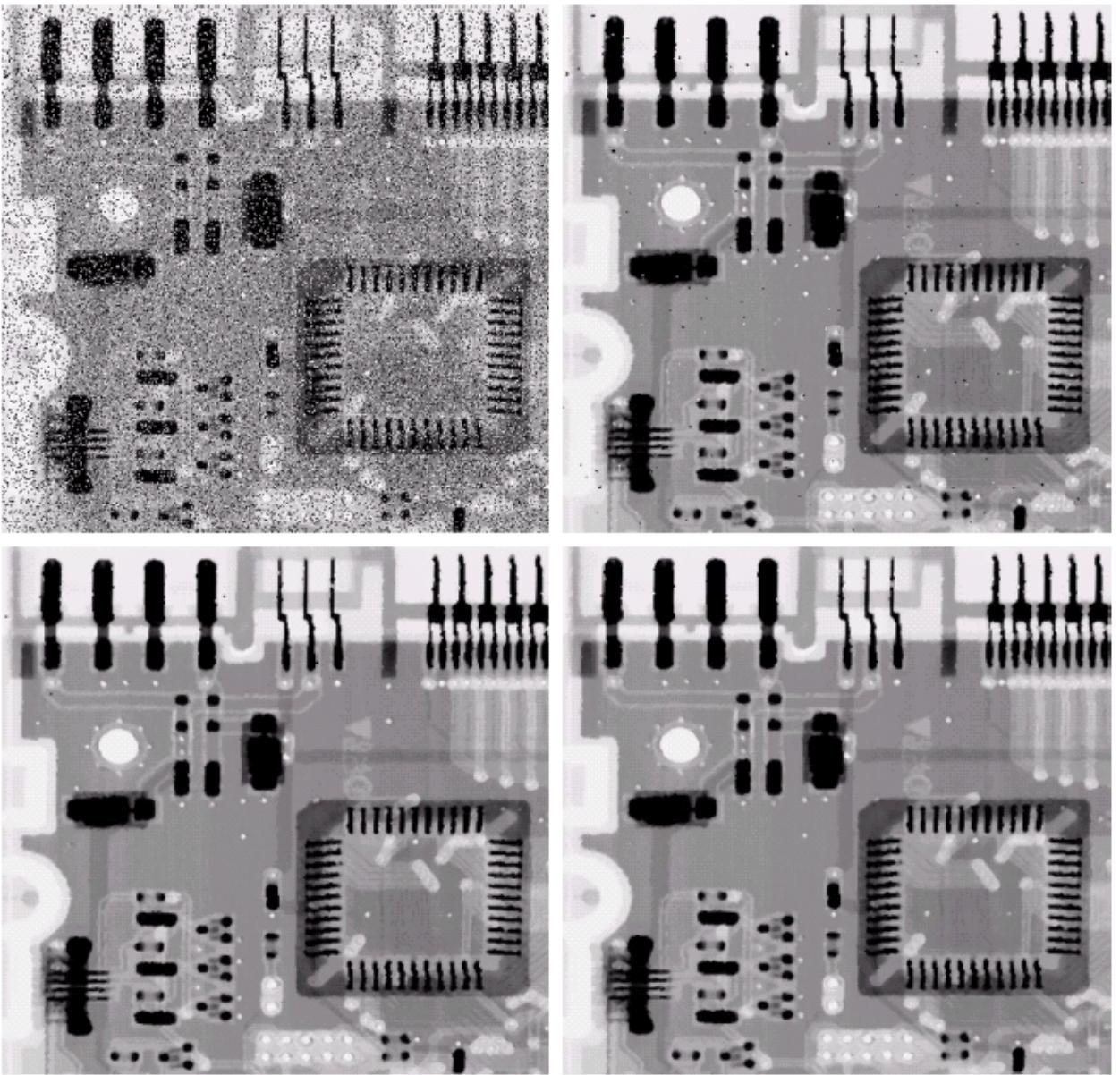
Alpha截取中值濾波器(Alpha-trimmed mean filter)

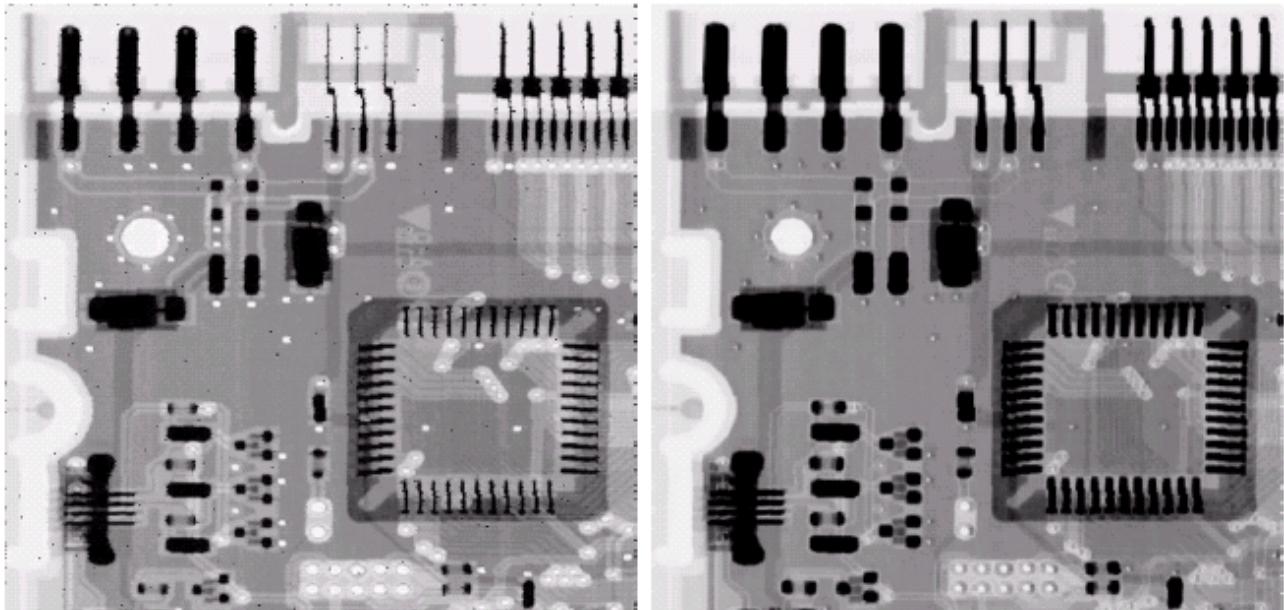
$$\hat{f}(x, y) = \frac{1}{mn-d} \sum_{(s,t) \in S_{xy}} g_r(s, t)$$

a b
c d

FIGURE 5.10

- (a) Image corrupted by salt-and-pepper noise with probabilities $P_a = P_b = 0.1$.
(b) Result of one pass with a median filter of size 3×3 .
(c) Result of processing (b) with this filter.
(d) Result of processing (c) with the same filter.





a b

FIGURE 5.11

(a) Result of filtering Fig. 5.8(a) with a max filter of size 3×3 . (b) Result of filtering 5.8(b) with a min filter of the same size.

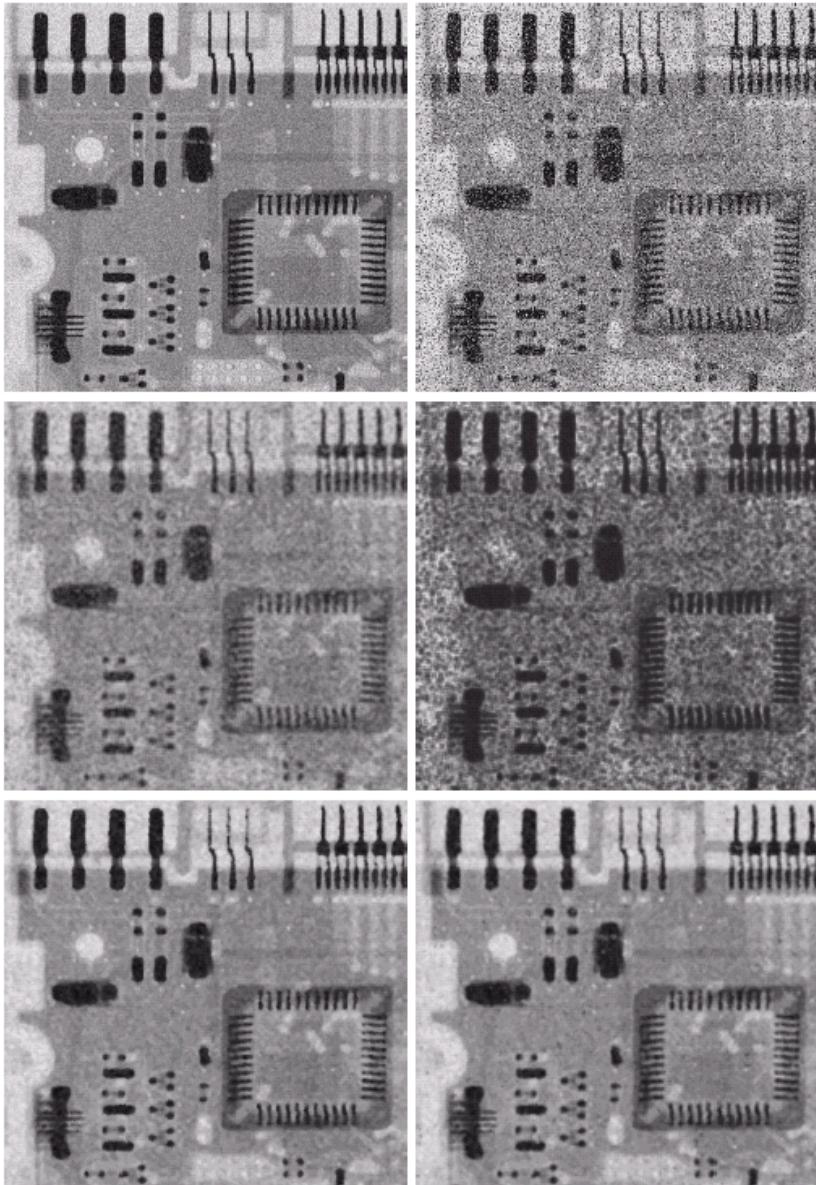


FIGURE 5.12 (a) Image corrupted by additive uniform noise. (b) Image additionally corrupted by additive salt-and-pepper noise. Image in (b) filtered with a 5×5 : (c) arithmetic mean filter; (d) geometric mean filter; (e) median filter; and (f) alpha-trimmed mean filter with $d = 5$.



循环噪声频域滤波

Periodic Noise Reduction by Frequency Domain Filtering

- 带阻滤波器(Bandreject Filters)
- 带通滤波器(Bandpass Filters)
- 槽口滤波器(Notch Filters)



帶阻濾波器 (Bandreject Filters)

理想帶阻濾波器(Ideal Bandreject Filters)

$$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) < D_0 - \frac{W}{2} \\ 0 & \text{if } D_0 - \frac{W}{2} \leq D(u, v) \leq D_0 + \frac{W}{2} \\ 1 & \text{if } D(u, v) > D_0 + \frac{W}{2} \end{cases}$$

Butterworth帶阻濾波器(Butterworth Bandreject Filters)

$$H(u, v) = \frac{1}{1 + \left[\frac{D(u, v)W}{D^2(u, v) + D_0^2} \right]^{2n}}$$

高斯帶阻濾波器(Gaussian Bandreject Filters)

$$H(u, v) = 1 - e^{-\frac{1}{2} \left[\frac{D^2(u, v) + D_0^2}{D(u, v)W} \right]^2}$$

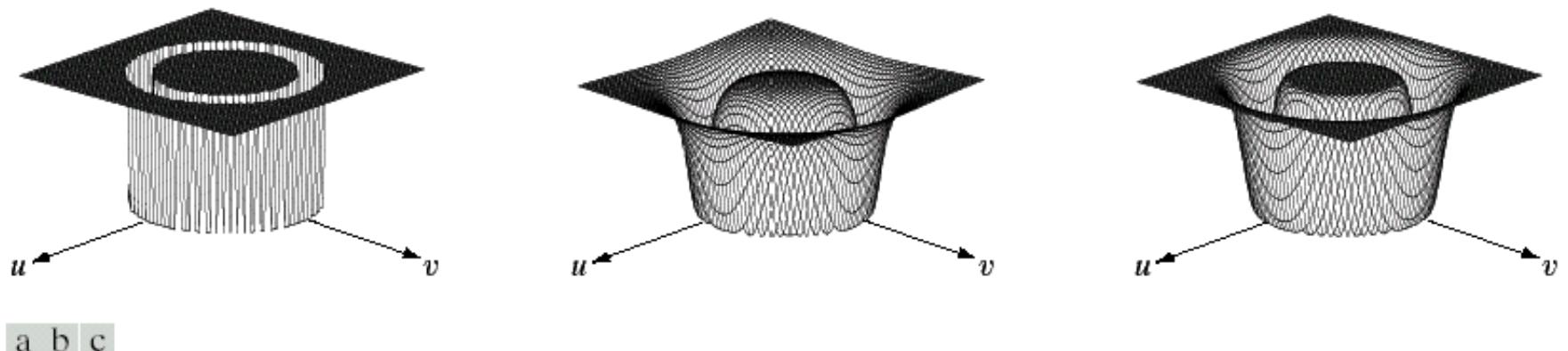
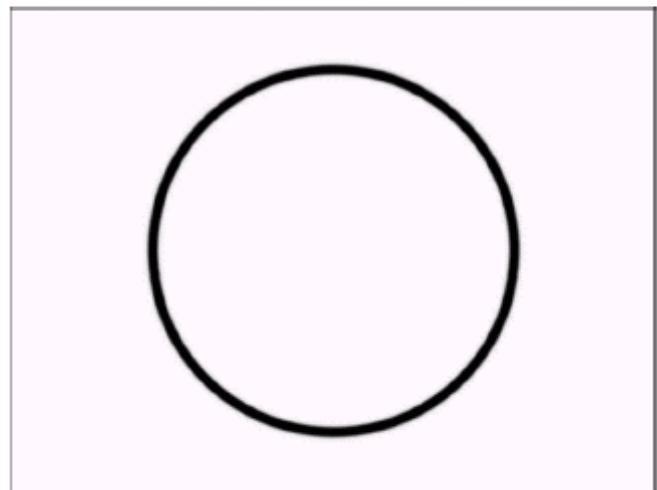
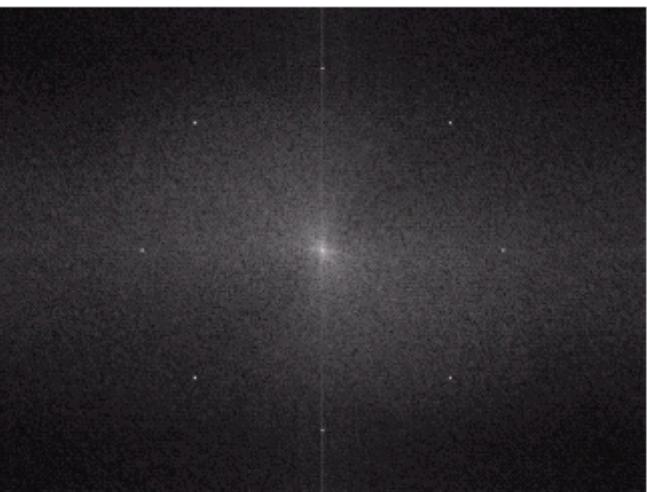
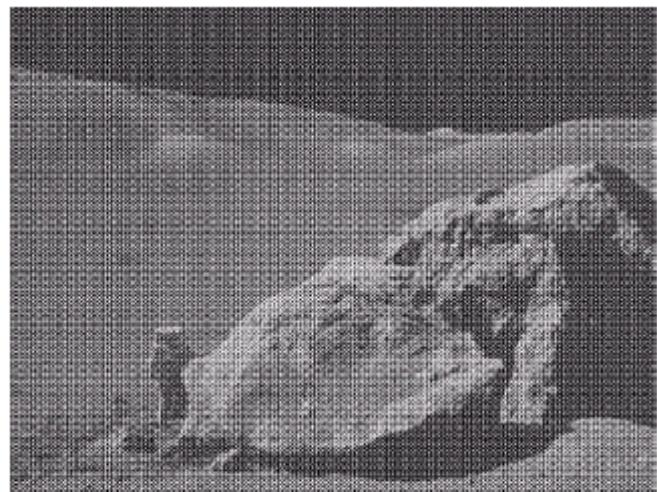


FIGURE 5.15 From left to right, perspective plots of ideal, Butterworth (of order 1), and Gaussian bandreject filters.



a
b
c
d

FIGURE 5.16

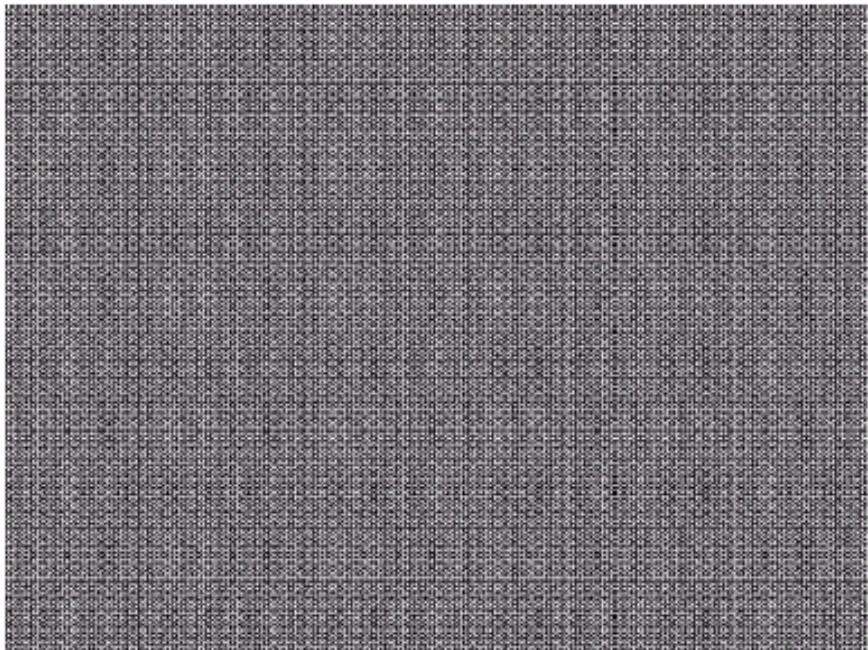
- (a) Image corrupted by sinusoidal noise.
(b) Spectrum of (a).
(c) Butterworth bandreject filter (white represents 1). (d) Result of filtering. (Original image courtesy of NASA.)

带通滤波器(Bandpass Filters)

带通滤波器与带阻滤波器功能相反

$$H_{bp}(u, v) = 1 - H_{br}(u, v)$$

FIGURE 5.17
Noise pattern of
the image in
Fig. 5.16(a)
obtained by
bandpass filtering.



槽口滤波器

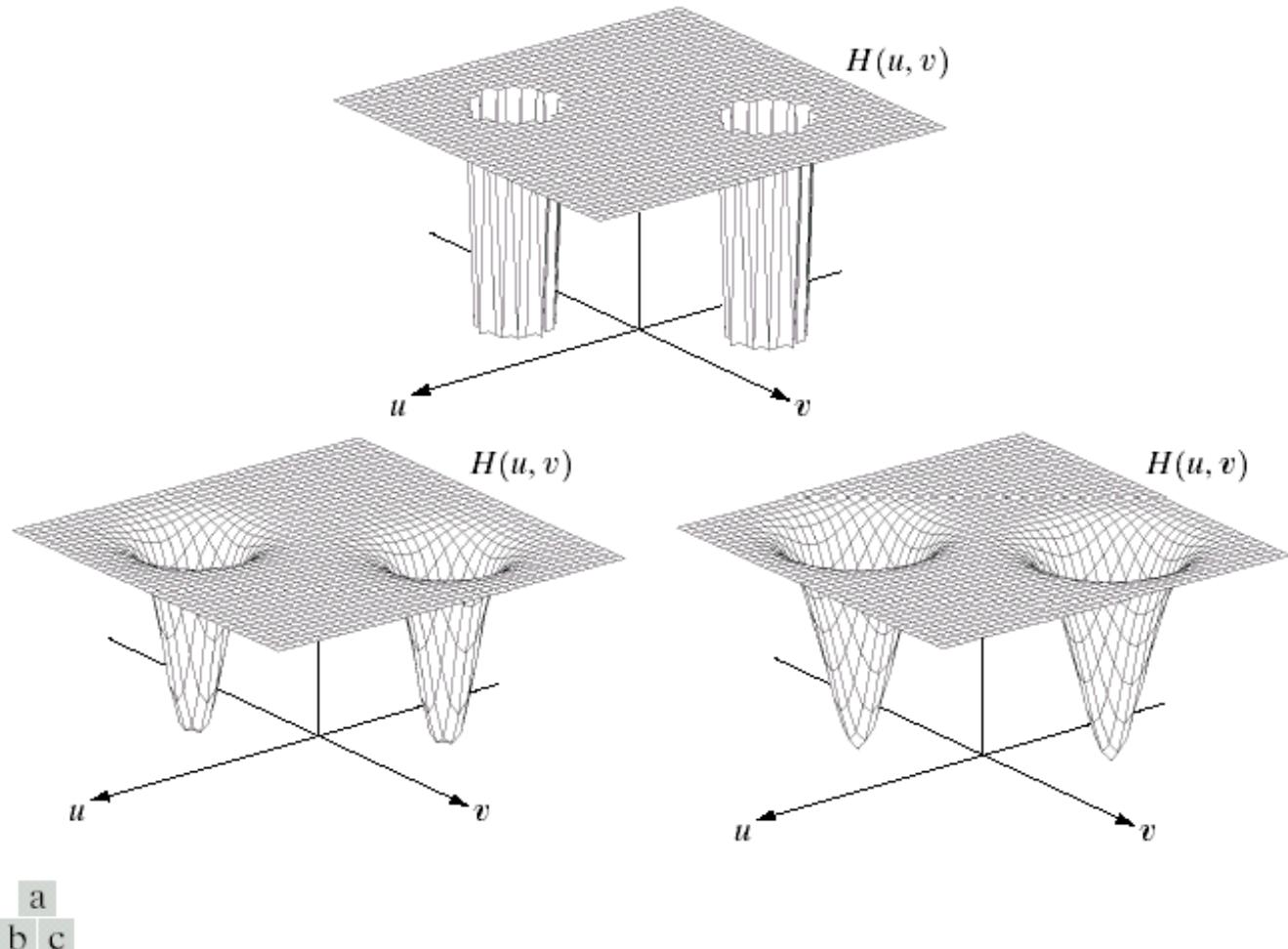
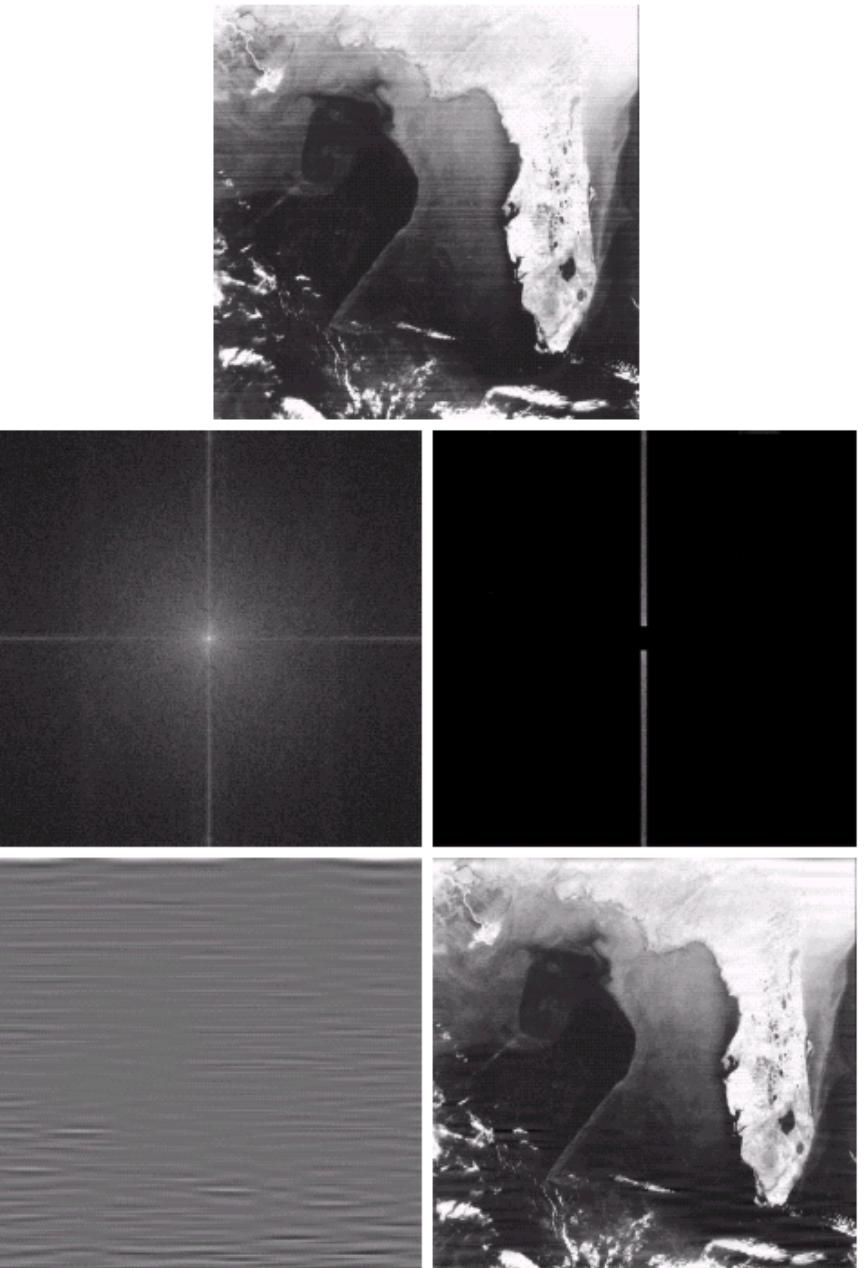


FIGURE 5.18 Perspective plots of (a) ideal, (b) Butterworth (of order 2), and (c) Gaussian notch (reject) filters.

a
b
c
d
e

FIGURE 5.19 (a) Satellite image of Florida and the Gulf of Mexico (note horizontal sensor scan lines). (b) Spectrum of (a). (c) Notch pass filter shown superimposed on (b). (d) Inverse Fourier transform of filtered image, showing noise pattern in the spatial domain. (e) Result of notch reject filtering. (Original image courtesy of NOAA.)





线性位置不变降质模型

降质过程:

$$g(x, y) = H[f(x, y)] + \eta(x, y)$$

线性系统:

$$H[af_1(x, y) + bf_2(x, y)] = aH[f_1(x, y)] + bH[f_2(x, y)]$$

位置不变系统:

$$H[f(x - \alpha, y - \beta)] = g(x - \alpha, y - \beta)$$

线性位置不变降质模型

$$g(x, y) = h(x, y) * f(x, y) + \eta(x, y)$$

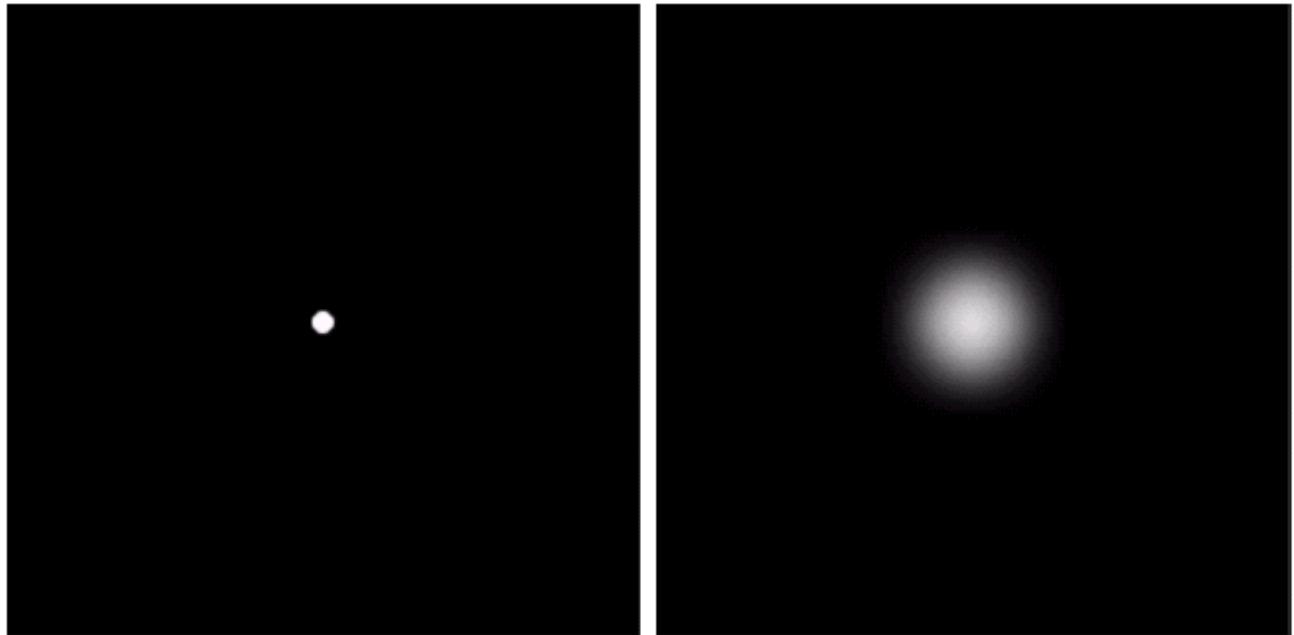
$$G(u, v) = H(u, v)F(u, v) + N(u, v)$$



估计降质函数

Estimating the Degradation Function

- 通过观测图像估计(Estimation by Image Observation)
- 通过实验估计(Estimation by Experimentation)
- 通过数学建模估计(Estimation by Modeling)

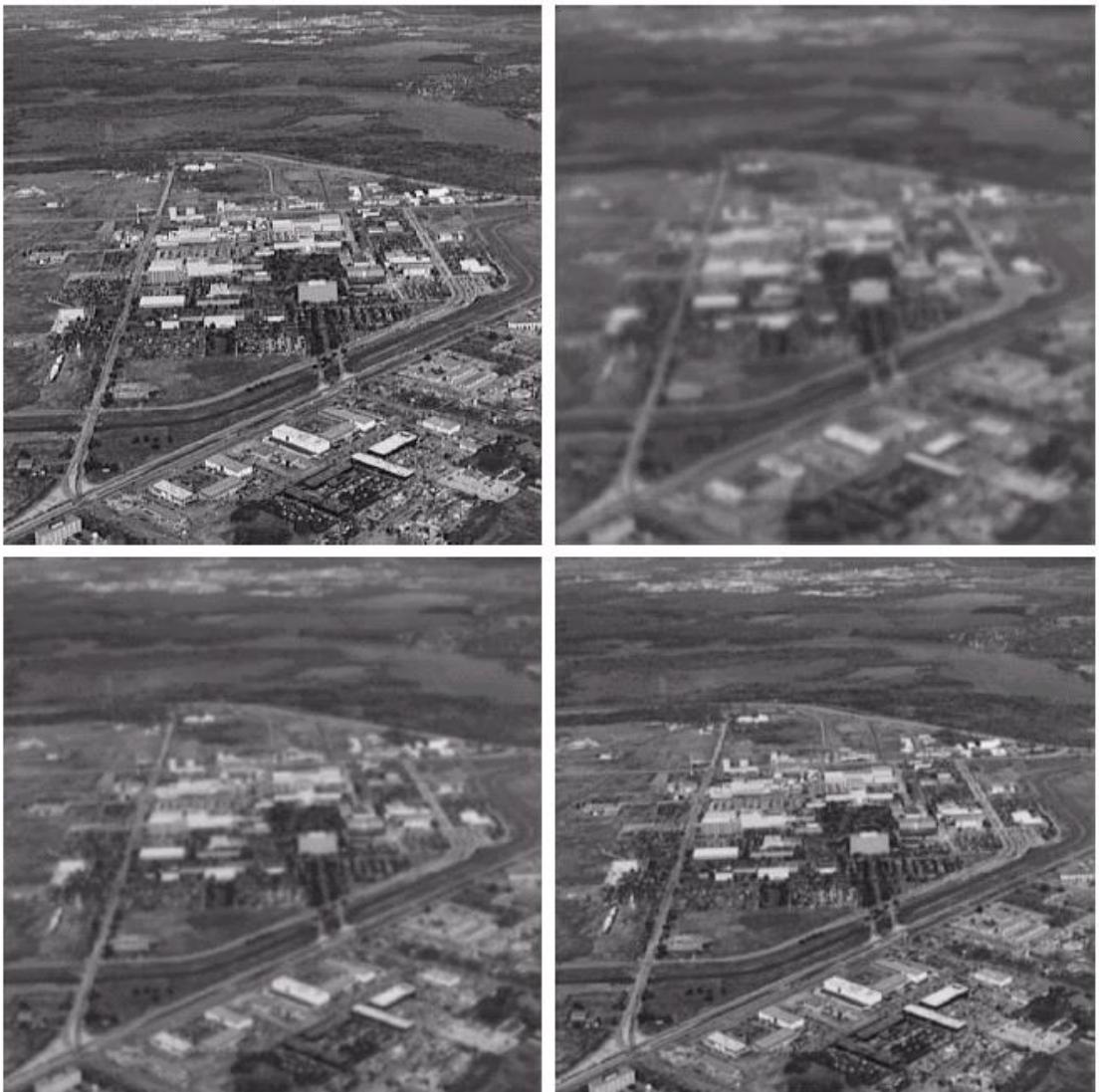


a b

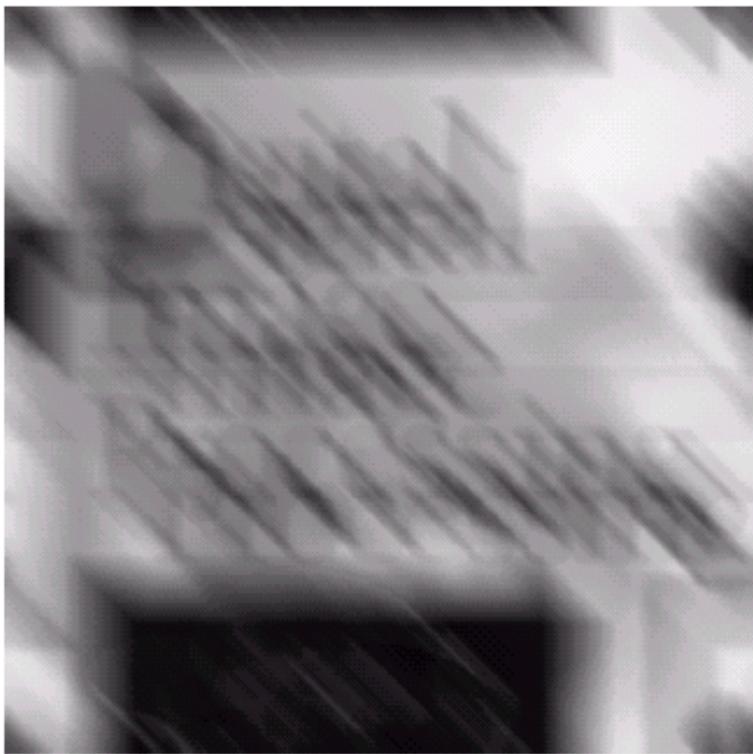
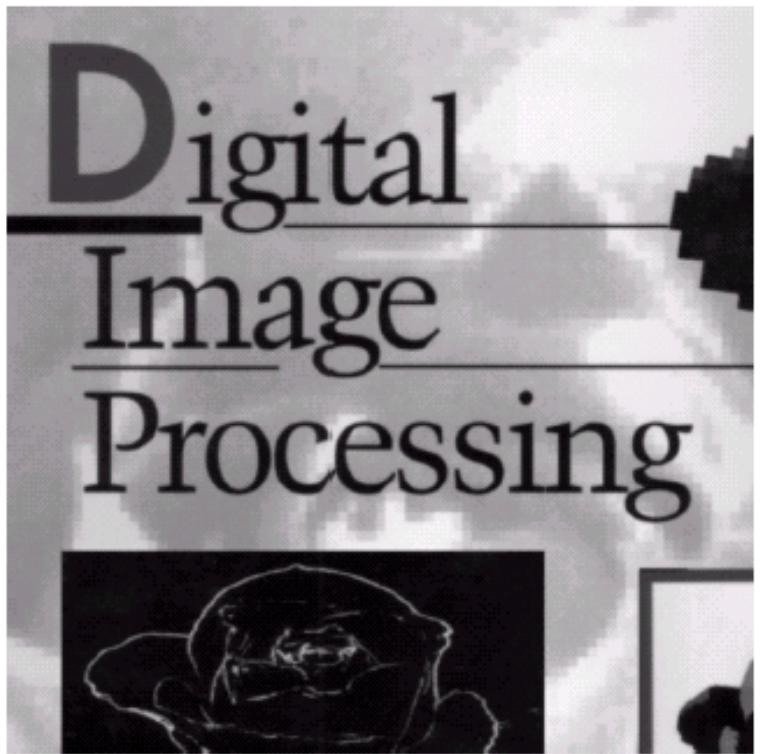
FIGURE 5.24
Degradation estimation by impulse characterization.
(a) An impulse of light (shown magnified).
(b) Imaged (degraded) impulse.

a b
c d

FIGURE 5.25
 Illustration of the atmospheric turbulence model.
 (a) Negligible turbulence.
 (b) Severe turbulence,
 $k = 0.0025$.
 (c) Mild turbulence,
 $k = 0.001$.
 (d) Low turbulence,
 $k = 0.00025$.
 (Original image courtesy of NASA.)



$$H(u, v) = e^{-k(u^2 + v^2)^{5/6}}$$



a b

FIGURE 5.26 (a) Original image. (b) Result of blurring using the function in Eq. (5.6-11) with $a = b = 0.1$ and $T = 1$.

$$H(u, v) = \frac{T}{\pi(ua + vb)} \sin[\pi(ua + vb)] e^{-j\pi(ua + vb)}$$

$$x(t) = at / T \quad y(t) = bt / T$$



逆滤波(Inverse Filtering)

- 最简单且粗糙的恢复方法
- 降质图像频谱直接除以降质函数：

$$\hat{F}(u, v) = \frac{G(u, v)}{H(u, v)} \quad \hat{F}(u, v) = F(u, v) + \frac{N(u, v)}{H(u, v)}$$

- 即使得到降质函数，难以精确重建
- 解决方法：限制频谱范围

a b
c d

FIGURE 5.27

Restoring Fig. 5.25(b) with Eq. (5.7-1).
(a) Result of using the full filter. (b) Result with H cut off outside a radius of 40; (c) outside a radius of 70; and (d) outside a radius of 85.





维纳滤波

最小均方差准则(Minimum Mean Square Error)

$$e^2 = E\{(f - \hat{f})^2\}$$

$$F(u, v) = \left[\frac{1}{H(u, v)} \frac{|H(u, v)|^2}{|H(u, v)|^2 + S_\eta(u, v) / S_f(u, v)} \right] G(u, v)$$

$$F(u, v) = \left[\frac{1}{H(u, v)} \frac{|H(u, v)|^2}{|H(u, v)|^2 + K} \right] G(u, v)$$



a b c

FIGURE 5.28 Comparison of inverse- and Wiener filtering. (a) Result of full inverse filtering of Fig. 5.25(b). (b) Radially limited inverse filter result. (c) Wiener filter result.

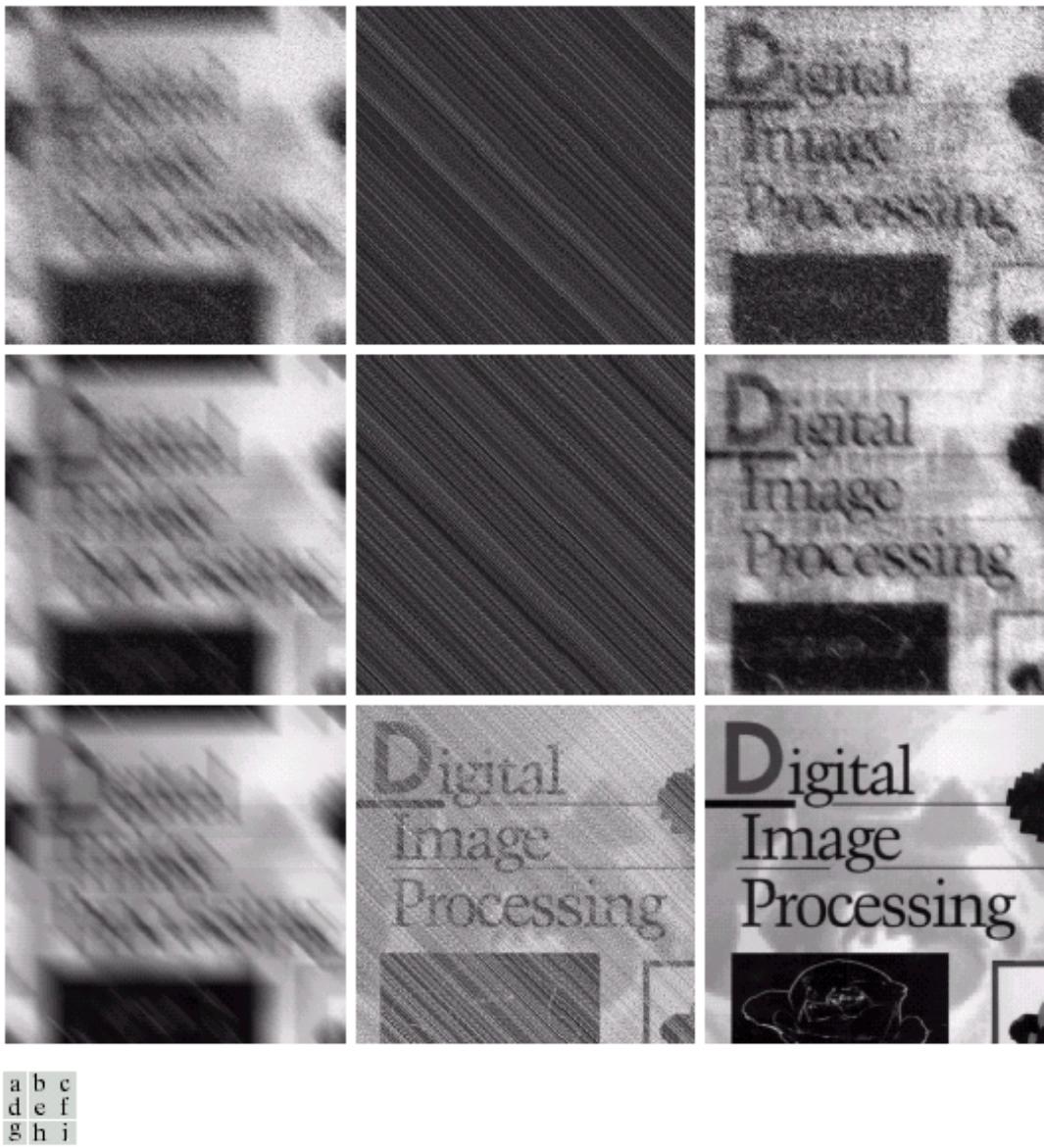
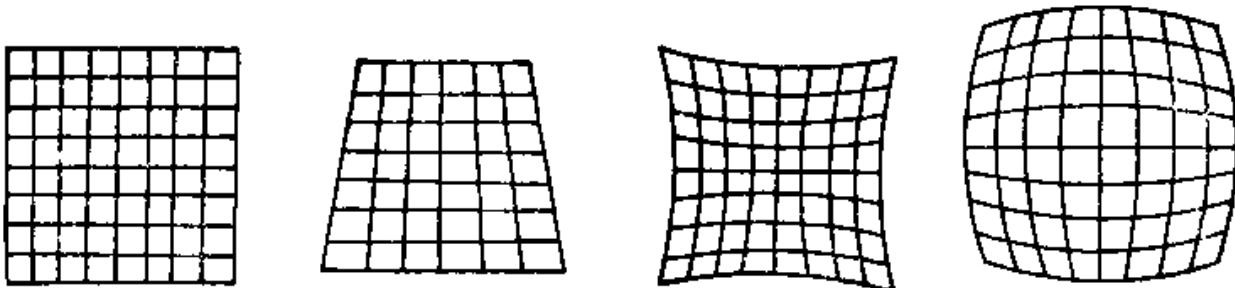


FIGURE 5.29 (a) Image corrupted by motion blur and additive noise. (b) Result of inverse filtering. (c) Result of Wiener filtering. (d)–(f) Same sequence, but with noise variance one order of magnitude less. (g)–(i) Same sequence, but noise variance reduced by five orders of magnitude from (a). Note in (h) how the deblurred image is quite visible through a “curtain” of noise.

几何校正

- 恢复或校正几何畸变，不改变灰度
- 通常分两步进行：
 1. 坐标的几何校正(空间变换)
 2. 象素灰度值的估计(灰度插值)





几何校正

□ 空间变换一般形式:

$$g(x,y) = f(x',y') = f [a(x,y), b(x,y)]$$

输入图

$$f(x',y')$$

输出图

$$g(x,y)$$



几何校正

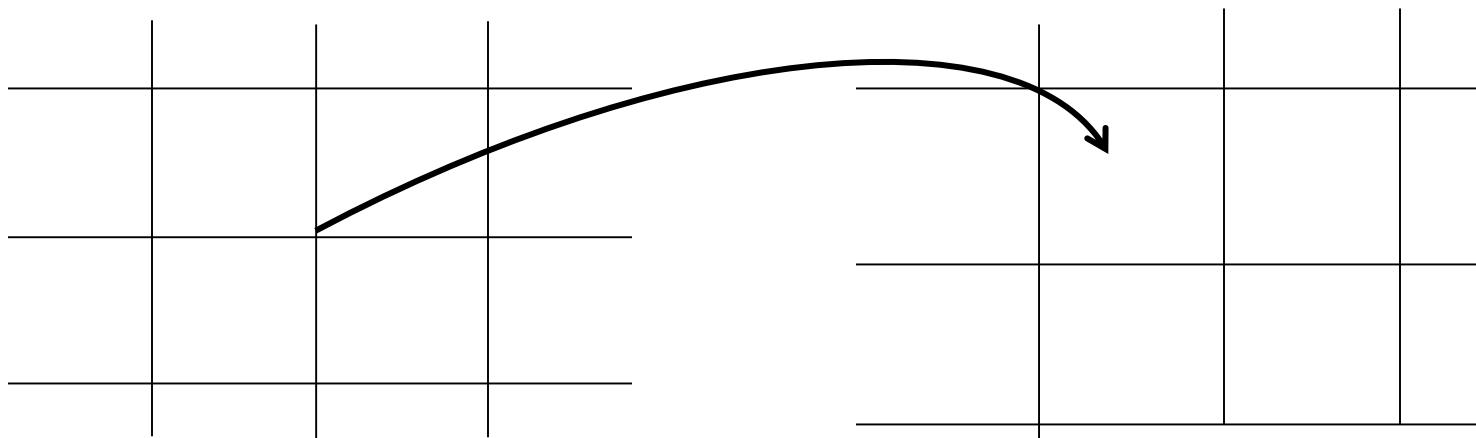
□ 灰度插值的原因是：

- 输入输出图像坐标间的映射通常不会是整数转换到整数坐标，即输入图像的像素往往映射到输出图像几个像素之间。

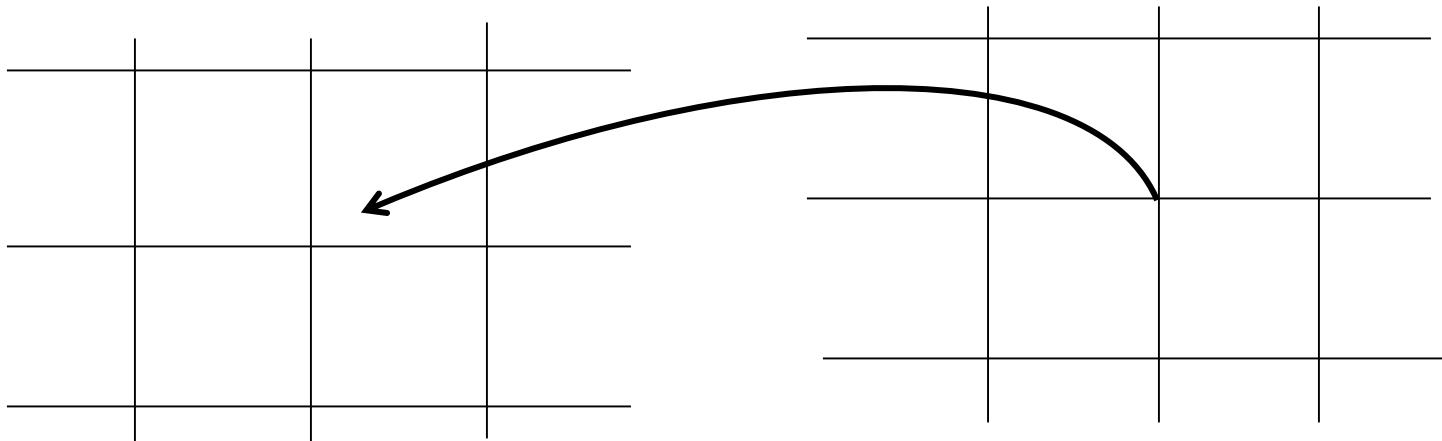
□ 灰度插值的实现有两种方法：

- 前向映射法
- 后向映射法（常用）

灰度插值的实现



前向映射法



后向映射法



1. 坐标的几何校正

- 建立正确图象网格坐标与失真图象网格坐标之间的对应关系
- 这一对应关系可以用多项式来表示。一般用二次多项式
- 为了求得变换多项式的系数，必须要有足够多、合适的控制点。
- 变换式和控制点的选取关系到校正的精度和计算量



确定多项式变换式系数

$$p = a_0 + a_1 i + a_2 j + a_3 i^2 + a_4 j^2 + a_5 ij$$

$$q = b_0 + b_1 i + b_2 j + b_3 i^2 + b_4 j^2 + b_5 ij$$

控制点：

$$\vec{p} = [p_1, p_2, \dots, p_M]^T$$

$$\vec{q} = [q_1, q_2, \dots, q_M]^T$$

由一组控制点， 得到变换多项式系数。



2. 象素灰度值的估计

1) 最邻近赋值

- 也称零阶插值，把靠近点的像素值赋予该点
- 计算简单，结果一般可以接受，但细结构处质量差

2) 双向线性插值

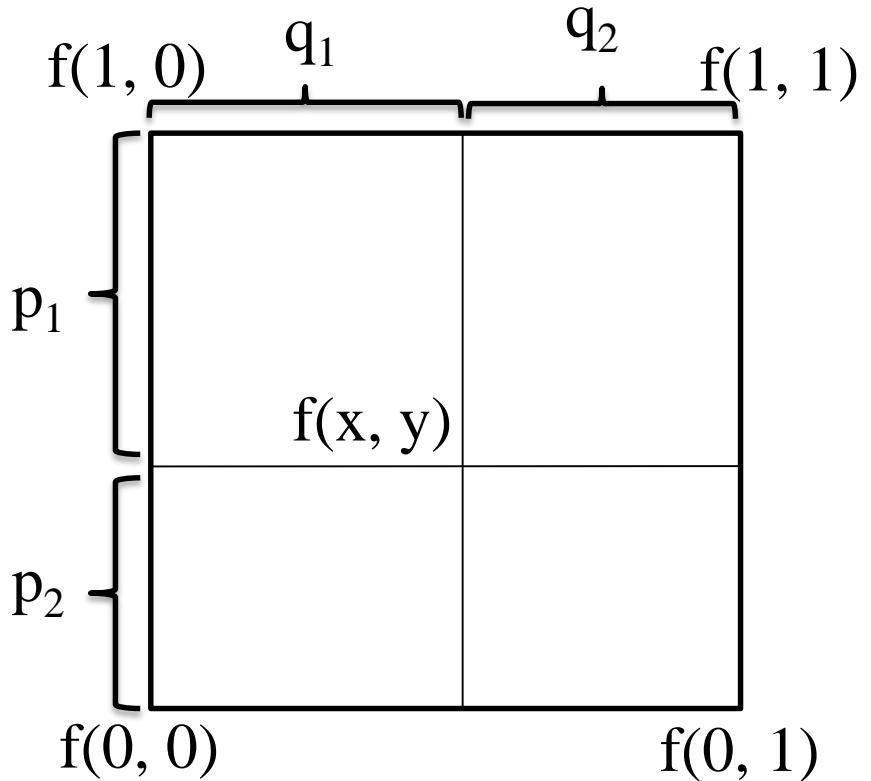
- 也称一阶插值
- 计算稍复杂，结果比零阶插值好，虽然在边界处是连续的，但是其导数不连续

3) 高阶插值

- 计算复杂，可以克服边界处导数的不连续性

双向线性插值计算式

$$\begin{aligned} p_1 + p_2 &= 1 \\ q_1 + q_2 &= 1 \end{aligned}$$



$$f(x,y) =$$

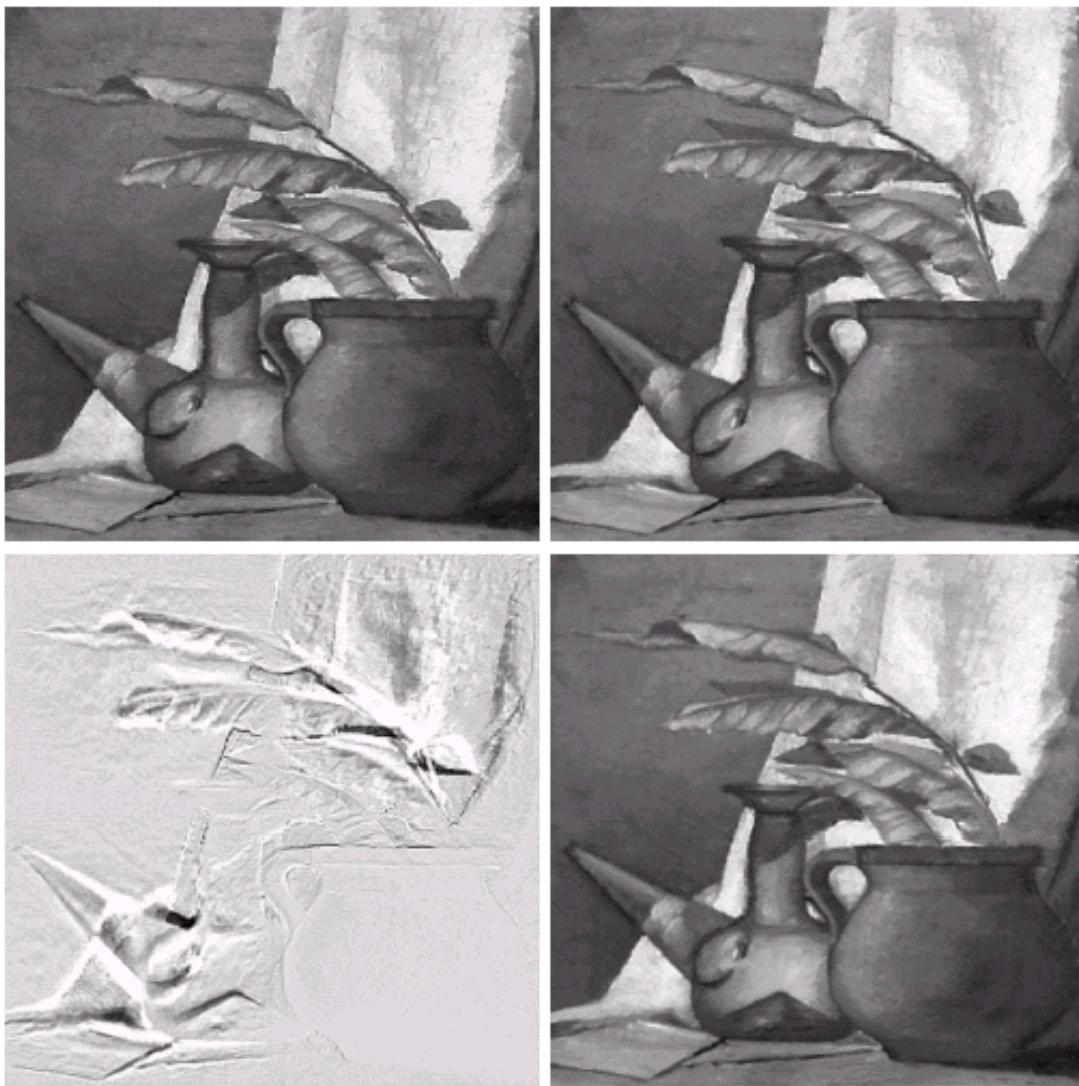
$$[f(0,0)p_1q_2+f(1,0)p_2q_2+f(0,1)p_1q_1+f(1,1)p_2q_1]$$

$$[(p_1+p_2)(q_1+q_2)]$$



几何运算的应用

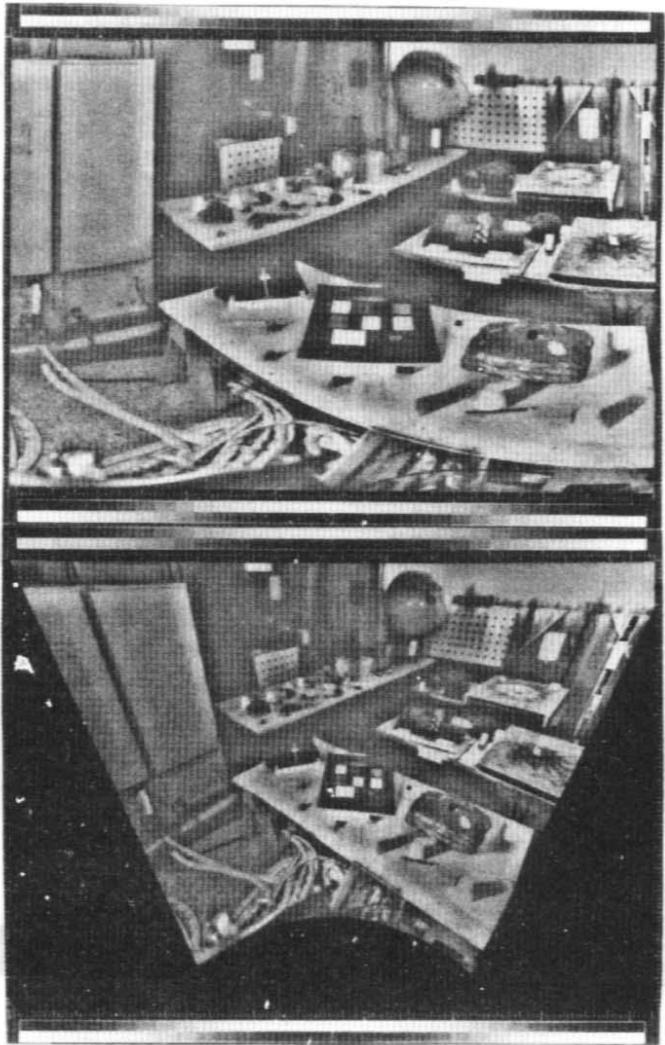
- 几何校正是几何运算的一种应用
- 几何运算还用于：图像配准，图像样式转换，地图投影和变形等



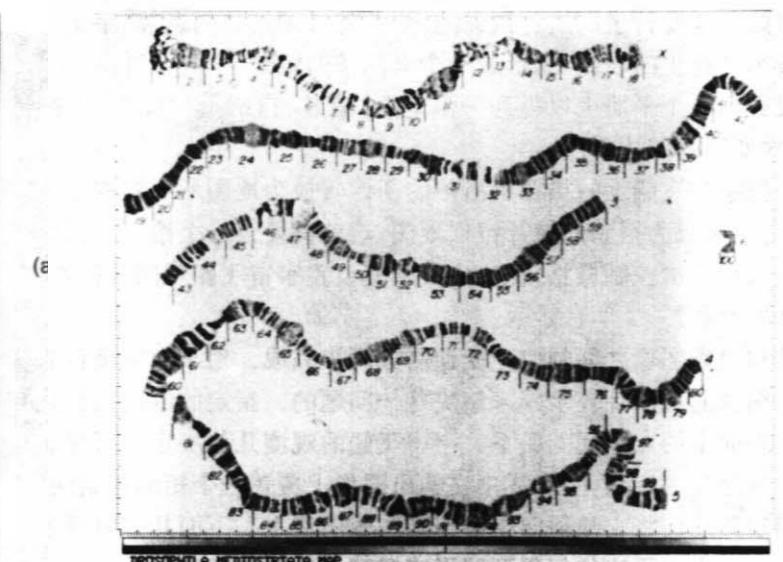
a b
c d

FIGURE 5.35 (a) An image before geometric distortion. (b) Image geometrically distorted using the same parameters as in Fig. 5.34(e). (c) Difference between (a) and (b). (d) Geometrically restored image.

例

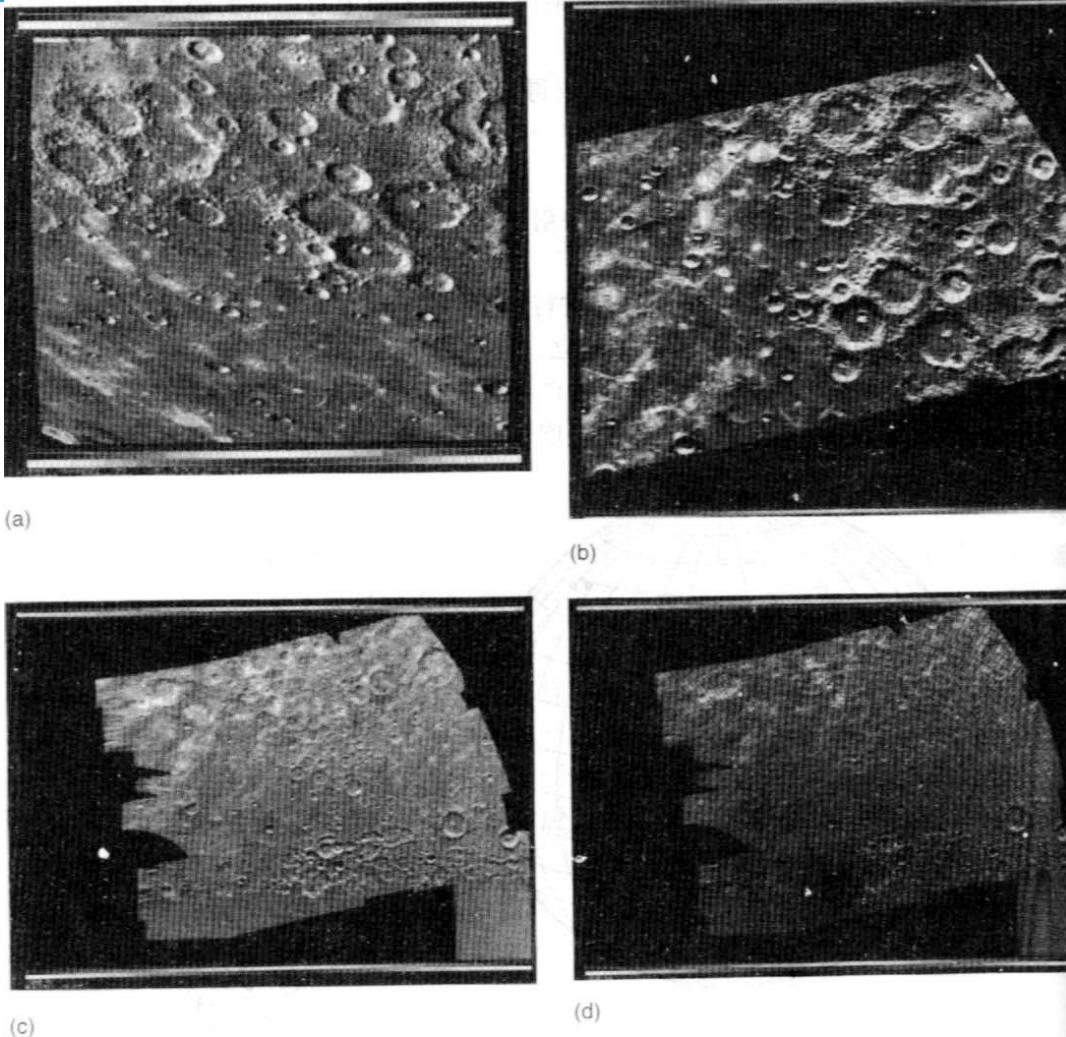


Viking Lander 摄像机的校正
(a)校正之前;(b)校正之后(NASA-JPL 提供)



果蝇染色体图
(a)原始图像;(b)拉直的图像(NASA-JPL 提供)

例

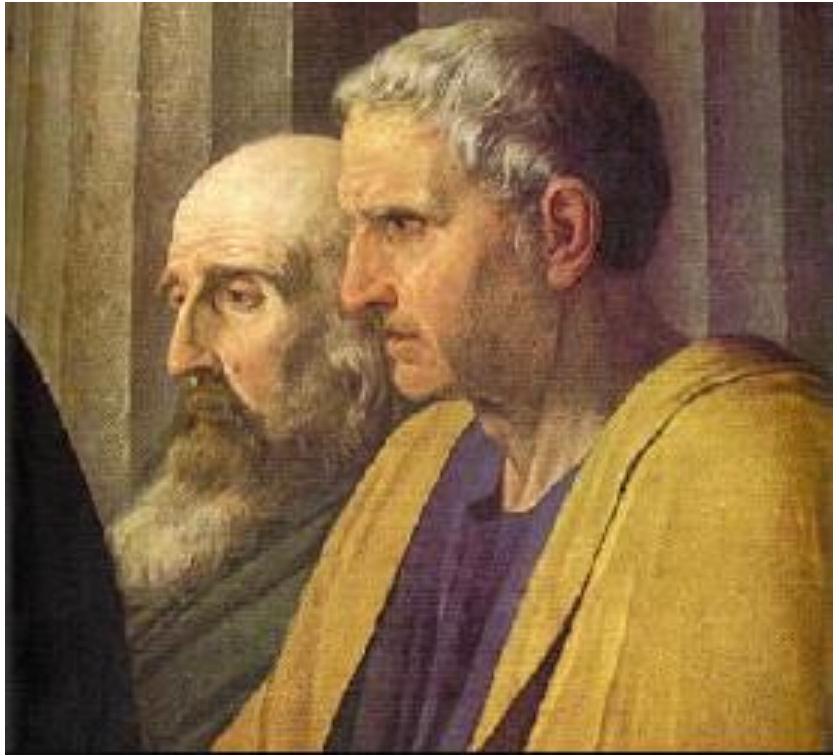
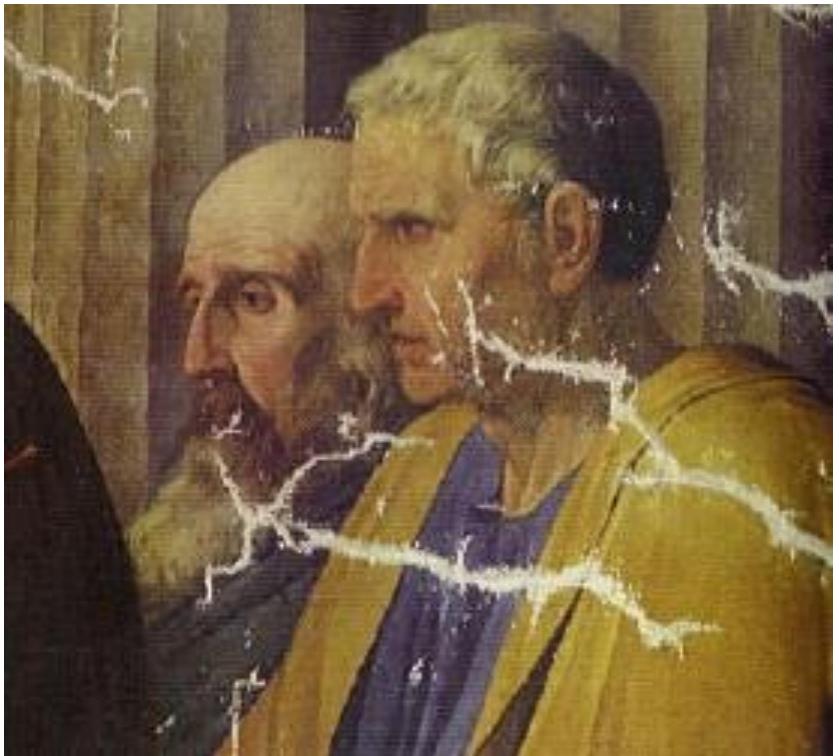


Mariner 10 地图投影示例

(a)原始图像;(b)正交投影;(c)几个投影图的镶嵌图;(d)网络覆盖(NASA-JPL 提供)

图像修复(Image Inpainting)

- 按照非检测的形式修改图像





应用

- 古书画的数字化修复
- 修复损坏的图片或者电影
- 去除图像或者视频中不想要的物体
- 视频传输过程中的差错消除
- 计算机视觉研究中的去除遮蔽
- 数码变焦
- 图像和视频的压缩

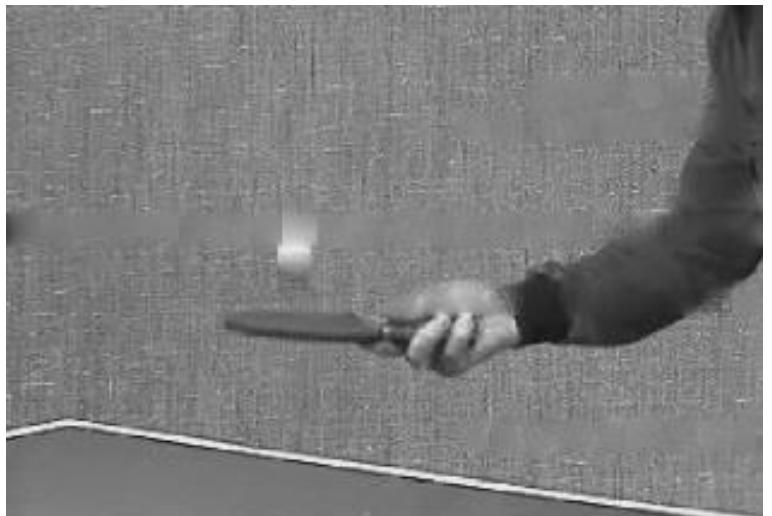
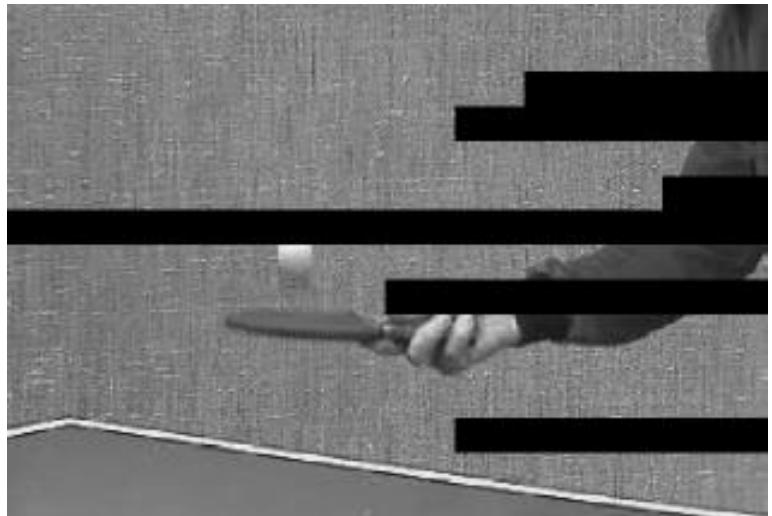
例:古书画的数字化修复



例:去除图像或者视频中不想要的物体



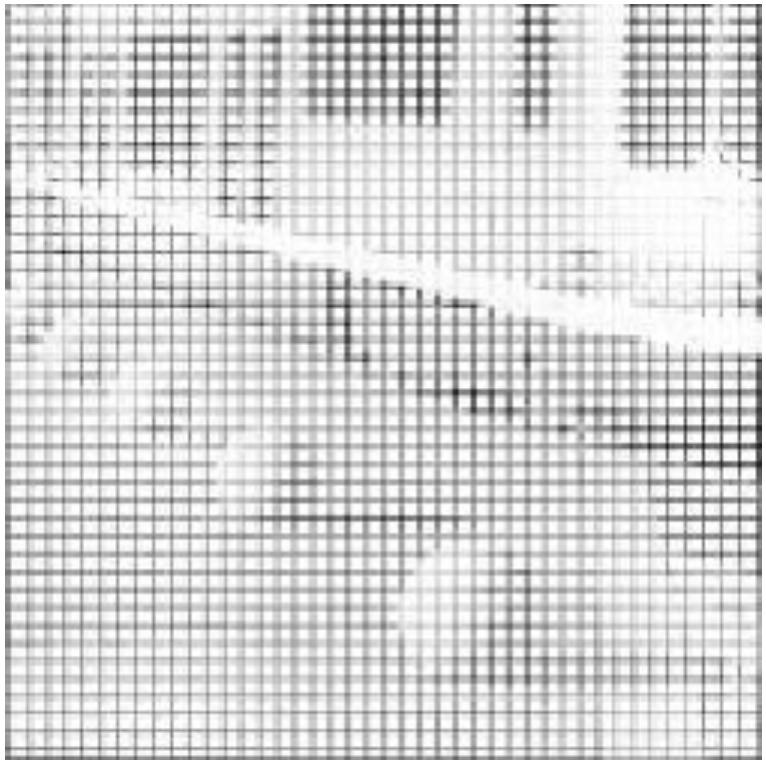
例:视频传输过程中的差错消除



例: 数码变焦



例:图像和视频的压缩



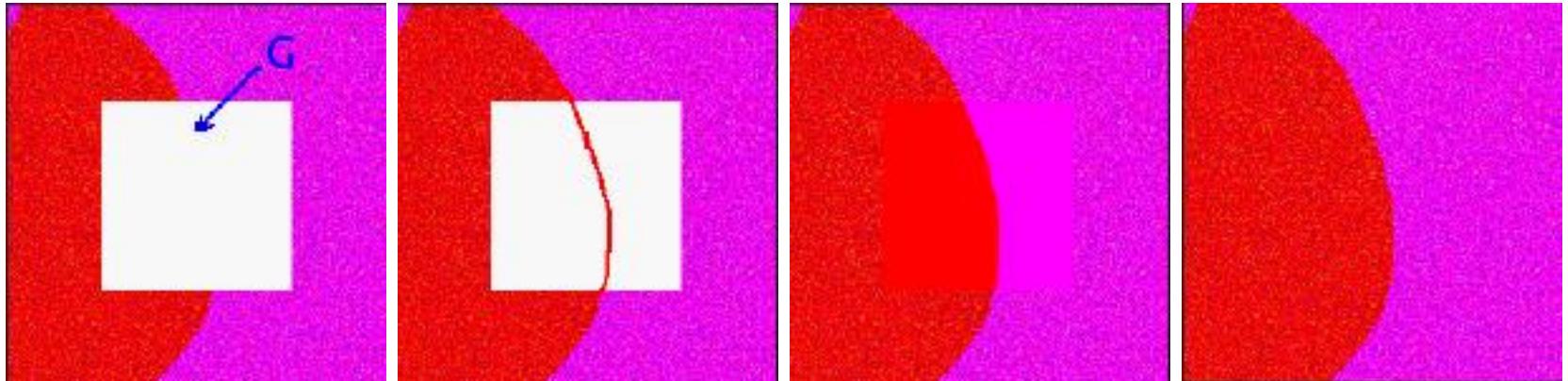
左边: 取图像行和列的 $1/6$, 对图像进行降采样得到。
右边: 修复得到的图像。



图像修复的相关工作

- 基于PDE的修复方法
- 基于水平线的修复方法
- 基于结构传播的修复方法
- 纹理融合

怎么人工的进行修复



根据明尼阿波利斯(Minneapolis) 艺术学院管理者的经验:

1. 确定怎么去修复白色区域同时保持图像的一致性。
2. 该区域周围的结构一直持续到间隙。
3. 填充不同的区域。
4. 添加纹理信息。



基于PDE的修复理论(1) —— 水平集方法

假设: $\phi(x, y, t) = C$

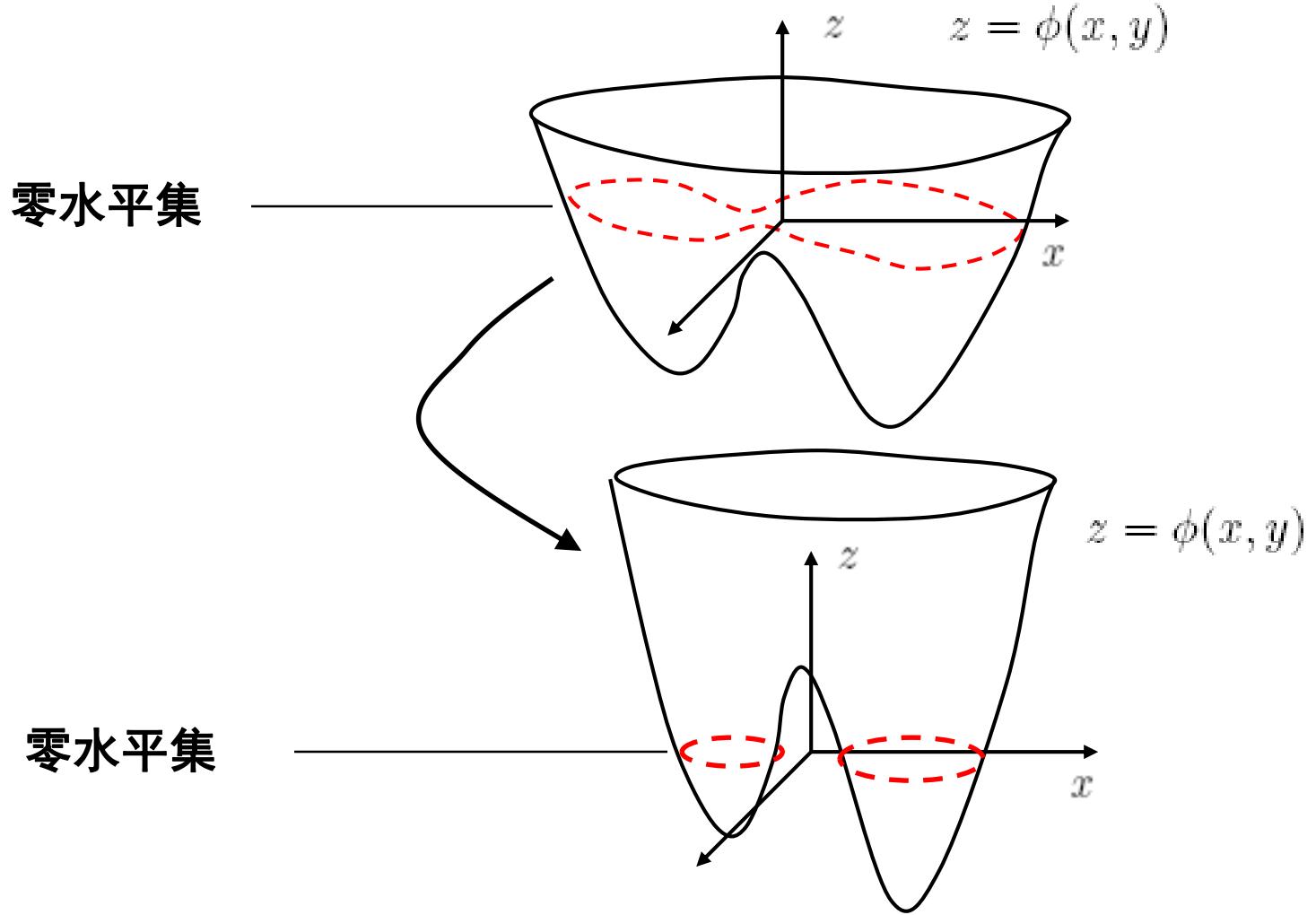
$$\frac{d\phi}{dt} = 0 \Rightarrow \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial\phi}{\partial y} \frac{\partial y}{\partial t} = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} v_x + \frac{\partial\phi}{\partial y} v_y = 0$$

初级水平集方程:

$$\frac{\partial\phi}{\partial t} + \nabla\phi \cdot \mathbf{v} = 0$$

曲线的水平集描述



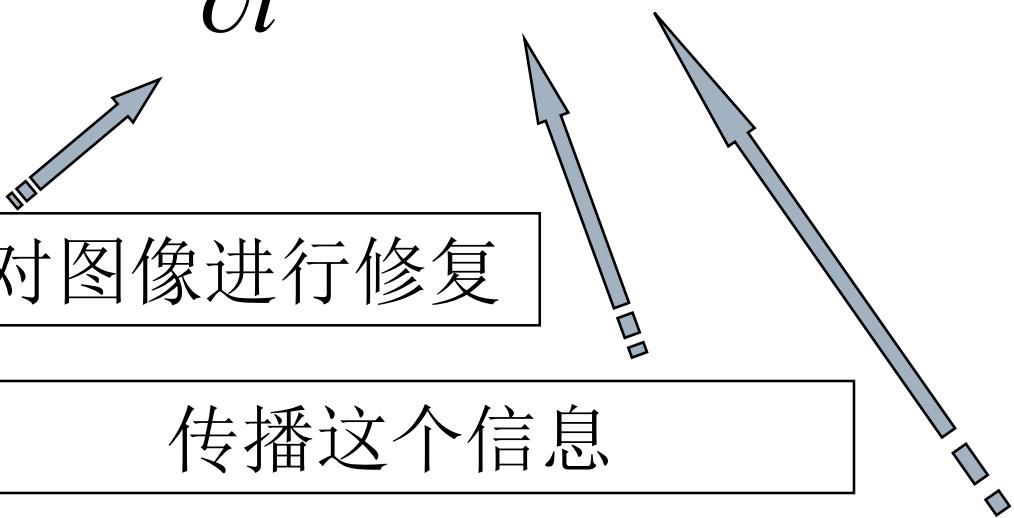
水平集方法

$$\frac{\partial \phi}{\partial t} + \nabla \phi \cdot \mathbf{v} = 0$$

通过时间对图像进行修复

传播这个信息

沿着正确的方向



水平集方法的一个例子

- Bertalmio等人于2000年提出
- 演化方程: $I_t^n(i, j) = \overrightarrow{\delta L}(i, j) \cdot \overrightarrow{N}^n(i, j)$

$$\overrightarrow{N}(i, j, n) = \nabla^\perp I^n(i, j)$$





基于PDE的修复理论(2)——变分图像的修复



图像修复: 一个病态的问题

我们没有足够的信息。

怎么解决这个问题?

增加约束条件:

- 光滑

- 边缘长度最小

.....

然后建立一个模型，通常是用待修复区域的积分函数，例如：



变分图像的修复

我们只需要找到 I_{opt} 让 J 最小

$$J(I) = \int_{\Omega} f(I, I^0) d\Omega, \quad \frac{\partial J(I)}{\partial I} \Big|_{I=I_{opt}} = 0$$

□ TV (Total Variation) 模型:

$$J(I) = \int_{\Omega} |\nabla I| + \lambda |I - I^0|^2 d\Omega,$$

□ Mumford-Shah 模型:

$$J(I) = \int_{\Omega} \left(\gamma |\nabla I|^2 + \lambda |I - I^0|^2 \right) d\Omega + \alpha \cdot \text{length}(\partial\Omega),$$

TV 方法的一个例子

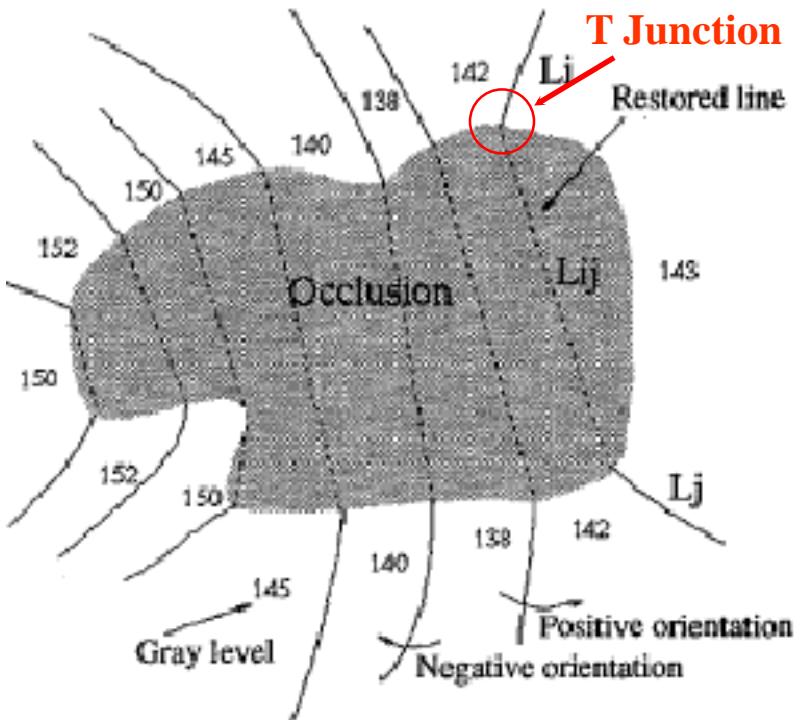




基于水平线(Level Line)的方法

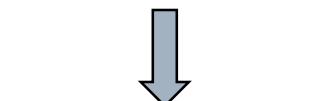
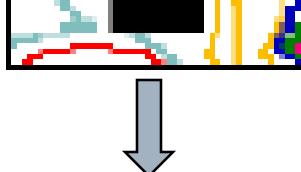
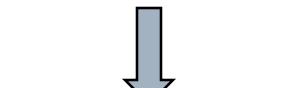
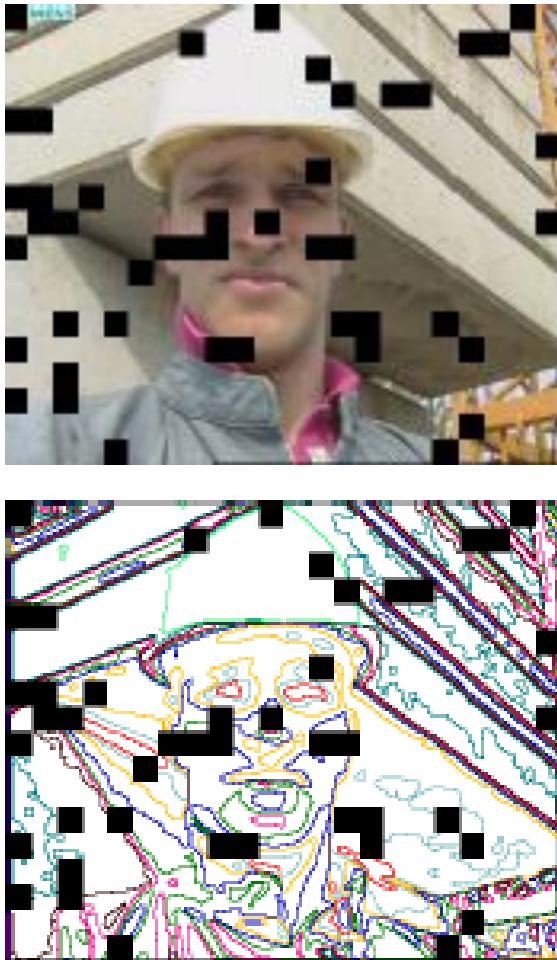
- Masnou and Morel, “Level Lines Based Disocclusion”, 1998
- Masnou, “Disocclusion: A Variational Approach Using Level Lines”, 2002
- Kim et al, “Efficient error concealment using best neighborhood matching and genetic algorithms (BNM-GA)”, 2003

基于水平线的方法



- 根据水平线将图像分割
- 产生T型连接点表
- 连接T型结构对
- 修补该区域

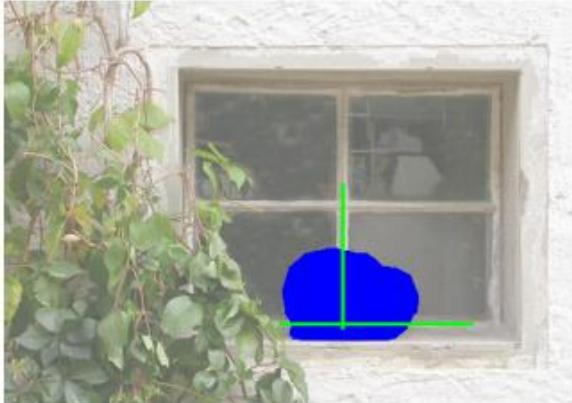
基于水平线方法的例子(1)



基于水平线方法的例子(2)



基于结构传播的图像修复

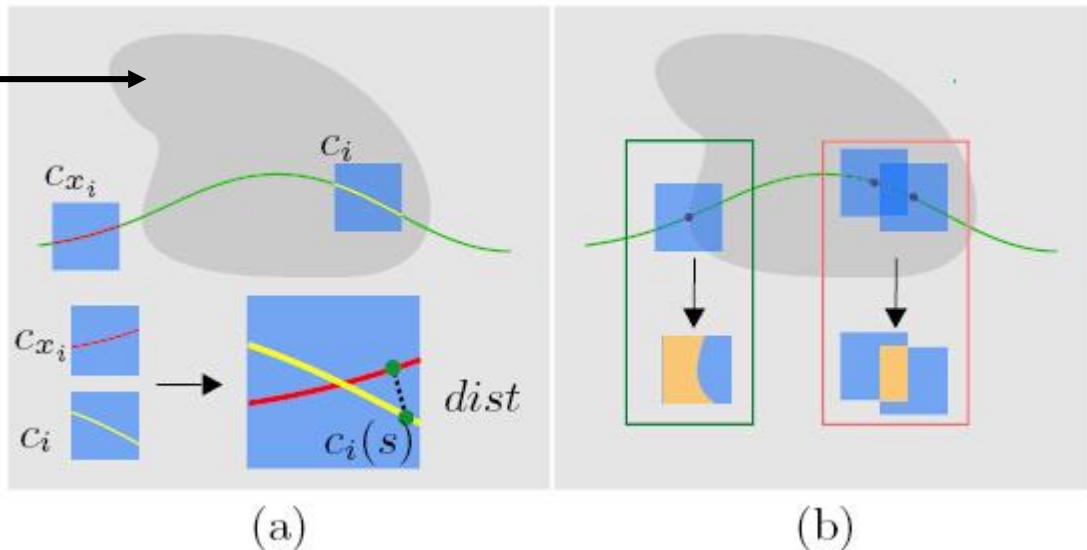


基于结构传播的图像修复

□ 结构传播代价函数 (Sun et al, SIGGRAPH 2005)

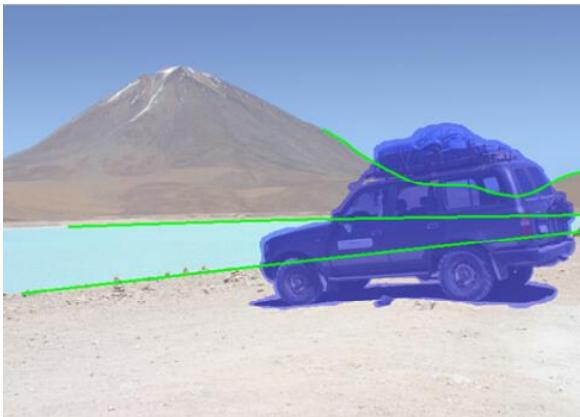
$$E(X) = \sum_{i \in v} E_1(x_i) + \sum_{(i,j) \in e} E_2(x_i, x_j), \quad E_1(x_i) = k_s E_s(x_i) + k_I E_I(x_i)$$

灰色区域为
待修复区域

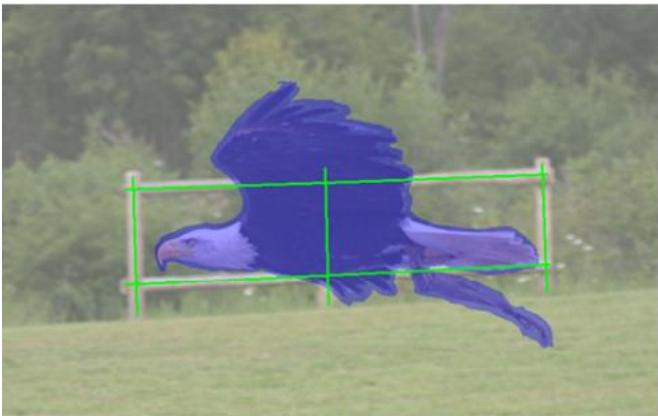


(a) Curve segments c_{x_i} (red) in the source patch, and curve segments c_i (yellow) in the target rectangle. $E_s(x_i)$ measures the structure similarity between c_{x_i} and c_i . $dist$ is the shortest distance (black dotted line) from point $c_i(s)$ on segment c_i to segment c_{x_i} . (b) The green box shows the cost $E_I(x_i)$ on the boundary of the unknown region. The red box shows the cost $E_2(x_i, x_j)$ for neighboring patches.

例子(1)



例子(2)





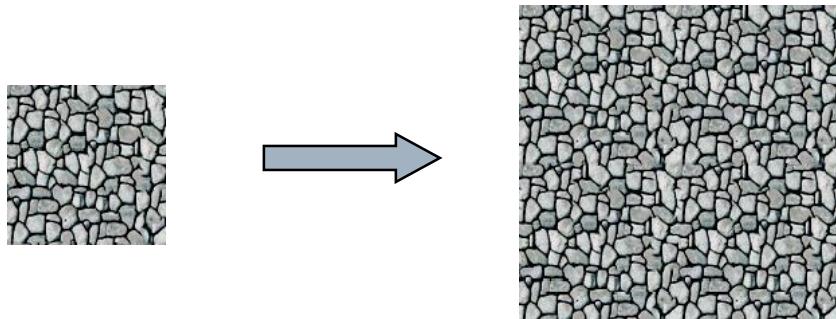
纹理合成

- Hirani and Totsuka, “Combining Frequency and Spatial Domain Information for Fast Interactive Image Noise Removal”, 1996
- Criminisi et al, “Object Removal by Exemplar-Based Inpainting”, 2003
- Wei and Levoy, “Fast texture synthesis using tree-structured vector quantization”, 2000
- Harrison, “A non-hierarchical procedure for re-synthesis of complex texture”, 2001

纹理合成的主要思想

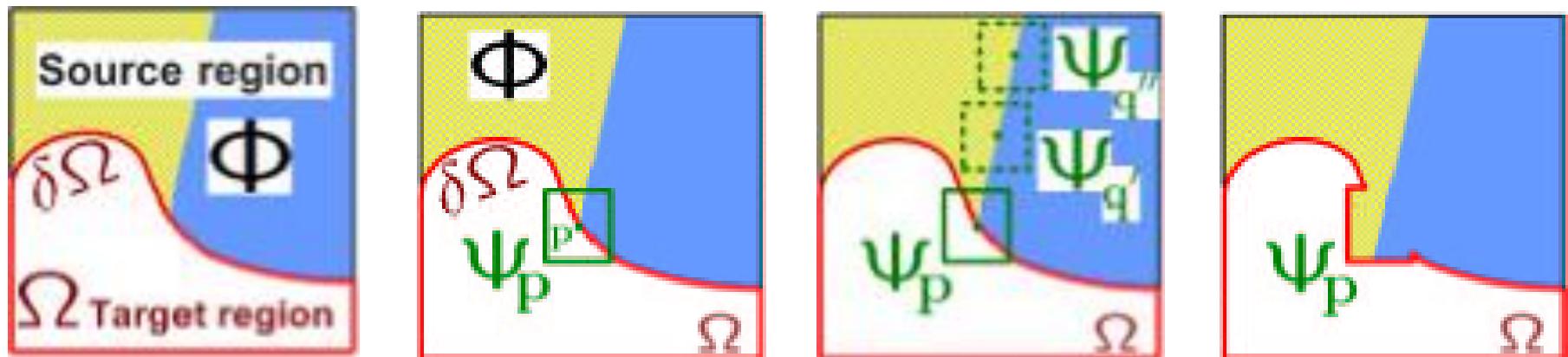
- 根据提供的样本纹理 I_{in} , 融合出纹理 I_{out} , 其纹理和样本相似, 但不相同。
- 流行的模型:

马尔科夫随机场(*Markov Random Field, MRF*)



怎么将纹理合成应用到图像修复中？

1. 选择待修复区域。
2. 根据一定的标准，选择样本纹理。
3. 纹理合成



例子(1)



例子(2)

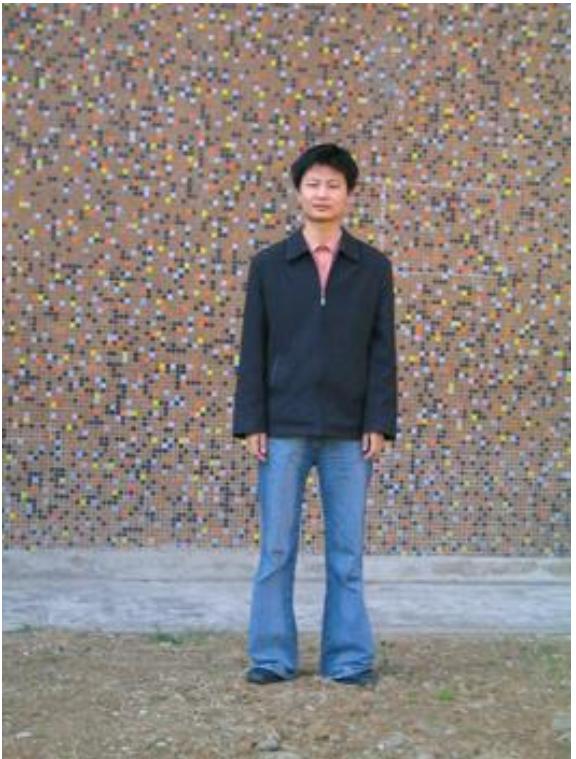


纹理合成
的实验结果



基于PDE方法
的实验结果

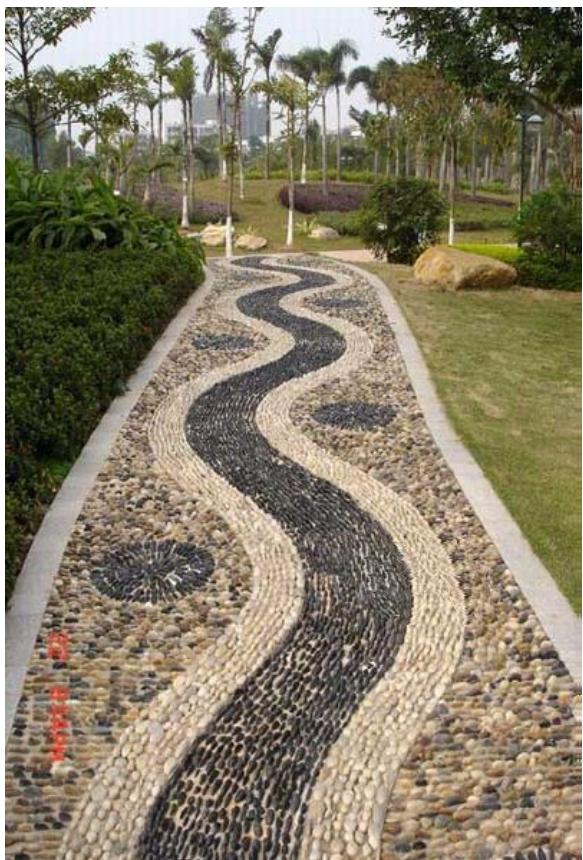
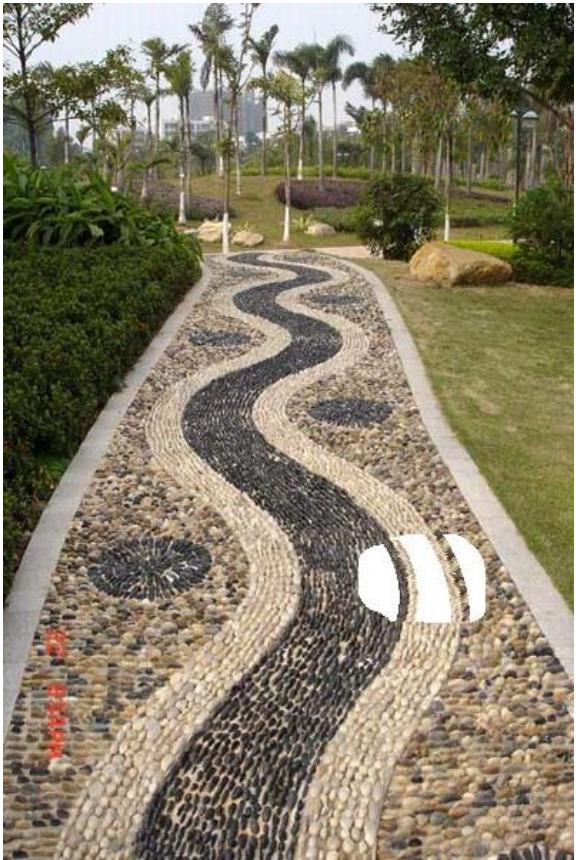
我们的工作



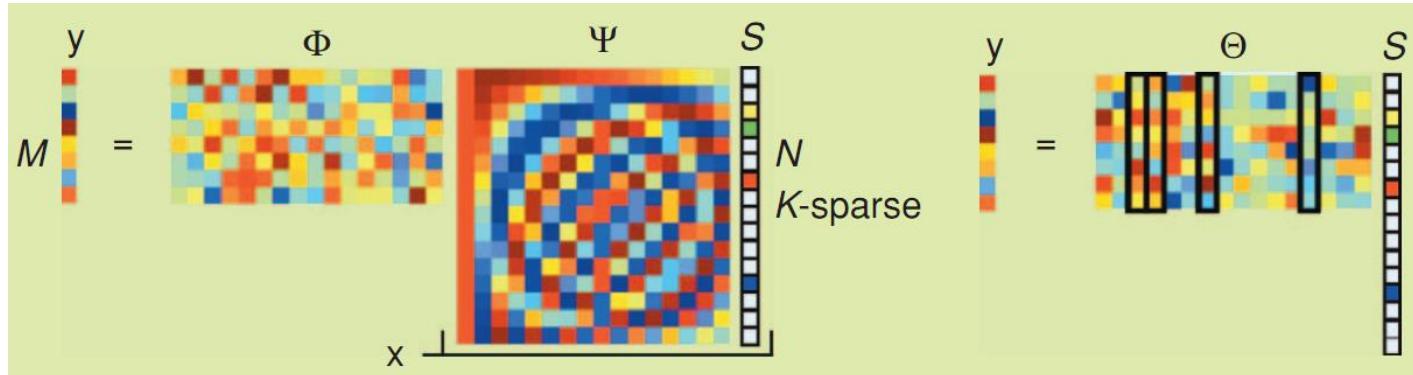
我们的工作



实验结果



基于压缩感知的图像恢复



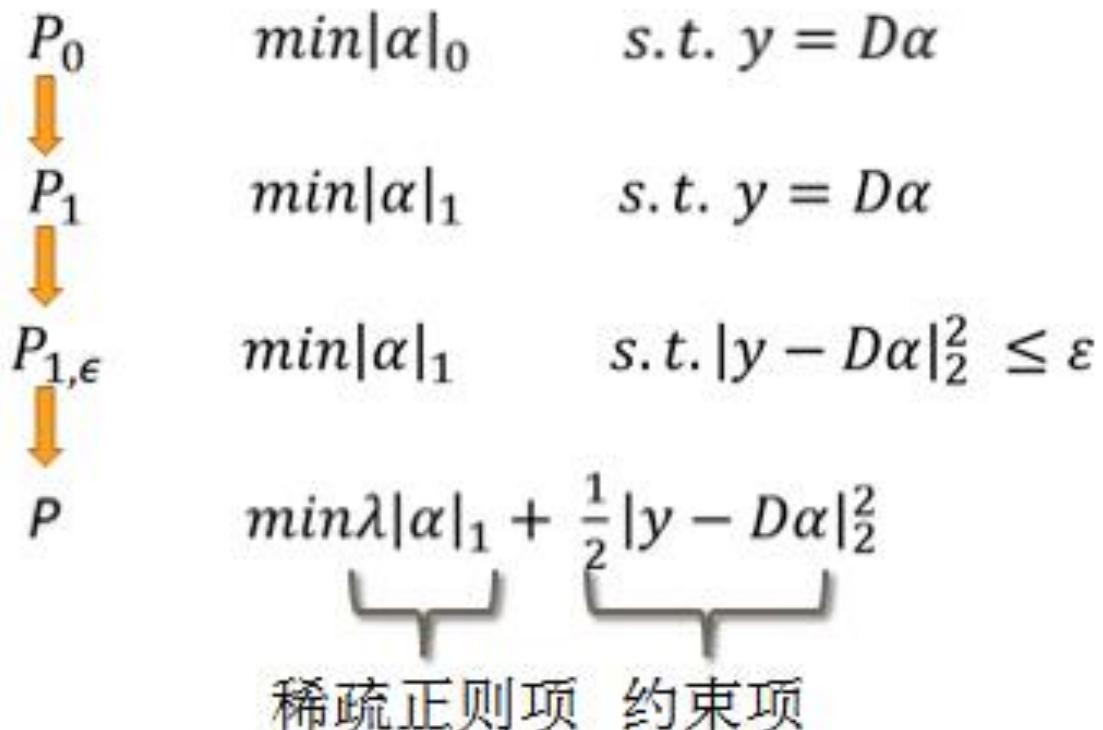
- 假设 $x \in R^N$ 在字典 Ψ 上是稀疏的，即

$$x = \Psi S, \quad \|S\|_0 = k \ll N$$
- 表示 x 在 Ψ 上是可压缩的，它可被 Ψ 中的 k 序列很好地表达
- 假设有一个 $M \times N$ 大小的观测矩阵 Φ ，此处 $M < N$ ，我们得到的观测数据为 y ，即

$$y = \Phi x = \Phi \Psi S = \Theta S$$

稀疏表达

- 稀疏表达也是在稀疏性的假设下，通过求解优化问题进行信号恢复





稀疏表达和MAP的关系

- 假设稀疏系数符合Laplace分布，噪声符合高斯分布

$$\begin{aligned}y - D\alpha &\simeq N(0, \sigma^2) \quad \alpha \simeq L(0, b) \\&\quad \text{似然项} \quad \text{先验项} \\ \hat{\alpha}_{MAP} &= \operatorname{argmax} p(\alpha|y) \propto \operatorname{argmax} p(y|\alpha) \times p(\alpha) \\&= \operatorname{argmax} \{-|y - D\alpha|^2 - \lambda|\alpha|_1\} \quad \text{取对数} \\&= \operatorname{argmin} \{|y - D\alpha|^2 + \lambda|\alpha|_1\} \\&\quad \text{稀疏表达的优化方程}\end{aligned}$$

- 基于稀疏表达的信号恢复就是基于稀疏系数是Laplace分布假设的MAP



稀疏表达的求解

- 求解稀疏表达的问题的核心是如何求解

$$\min \lambda |\alpha|_1 + \frac{1}{2} \|y - D\alpha\|_2^2$$

- 1) Gradient Projection Methods
- 2) Homotopy Methods
- 3) Iterative Shrinkage-Thresholding Methods
- 4) Proximal Gradient Methods
- 5) Alternating Direction Methods

Allen Y. Yang, A. Ganesh, S. Shankar Sastry, Yi Ma “Fast L1 Minimization Algorithms and An Application in Robust Face Recognition: A Review”, ICIP 2010



稀疏字典D

设计的字典

- Wavelets, curvelets, wedgelets, bandlets, Lets

训练的字典

- k-means + PCA 得到字典

- 学习得到字典

- 求解

$$\min_{\alpha \in \mathbb{R}^{p \times n}, D \in \mathbb{C}} \sum_{i=1}^n \frac{1}{2} \|y_i - D\alpha_i\|_2^2 + \lambda \|\alpha_i\|_1$$

$$\mathbb{C} \triangleq \{D \in \mathbb{R}^{m \times p} \text{ s.t. } \forall i = 1, \dots, p, \|d_i\|_2 \leq 1\}$$



稀疏字典D

- $$\min_{\alpha \in \mathbb{R}^{p \times n}, D \in \mathbb{C}} \sum_{i=1}^n \frac{1}{2} \|y_i - D\alpha_i\|_2^2 + \lambda \|\alpha_i\|_1$$
$$\mathbb{C} \triangleq \{D \in \mathbb{R}^{m \times p} \text{ s.t. } \forall i = 1, \dots, p, \|d_i\|_2 \leq 1\}$$

► 通常算法，迭代优化 D, α

- 1) 固定 D , 优化 α
 - l_1 优化问题
- 2) 固定 α , 优化 D
 - l_2 优化问题

Aharon M, Elad M, Bruckstein A. “K-SVD: An algorithm for designing overcomplete dictionaries for sparse representation”, IEEE TSP, 54 (11), 2006

Lee, H., Battle, A., Raina, R., Ng. A. Y. “Efficient sparse coding algorithms”, NIPS 2006

W. Dong, L. Zhang, G. Shi, and X. Wu, “Image deblurring and super-resolution by adaptive sparse domain selection and adaptive regularization,” IEEE TIP, 20(7), 2011.



稀疏表达 —— 图像恢复

□ 基本框架

$$\begin{aligned} & \min_{\alpha \in \mathbb{R}^{p \times n}} \frac{1}{2} \|Y - AX\|_2^2 + \lambda \sum_{i=1}^n |\alpha_i|_1 \\ & s.t. \quad D\alpha_i = R_i X \end{aligned} \tag{1}$$

A 表示降质模型， R_i 表示从图像中抽取一个小块，假设观测数据受高斯干扰。
上式的物理意义是：希望恢复的图像既能在字典上稀疏表达，也符合观测模型

- 去高斯噪声
- 去模糊
- Inpainting
- 超分辨

稀疏表达 —— 图像恢复 —— 去噪

- 对于去除高斯噪声，(1)式中 $A=I$ （单位矩阵）



稀疏表达 —— 图像恢复 —— 去噪

- 对于去除高斯噪声，(1)式中 $A=I$ （单位矩阵）



$$\sigma = 25$$

稀疏表达 —— 图像恢复 —— 去模糊

- 此时式(1)中，A是由于卷积作用而产生的矩阵，如果是空间不变的卷积核，A为一个循环矩阵



高斯卷积核 $\sigma = 2$

高斯卷积核 $\sigma = \sqrt{2}$

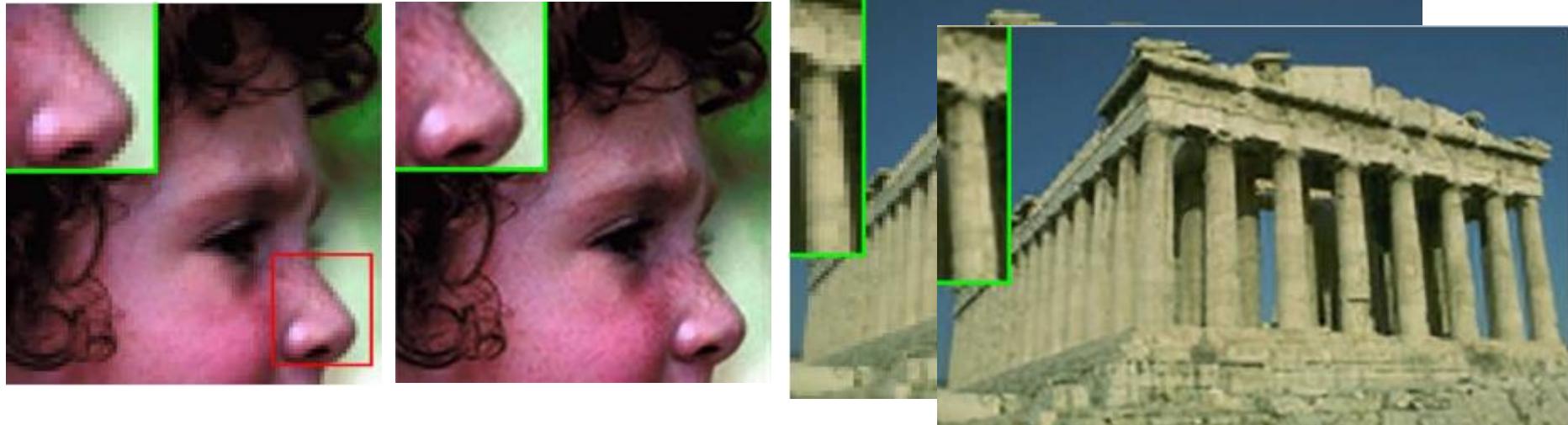
稀疏表达 —— 图像恢复 —— inpainting

- 此时式(1)中， A是由于模板而产生的矩阵



稀疏表达 —— 图像恢复 —— 超分辨

- 此时式(1)中，A是由于卷积然后下采样作用而产生的矩阵



超分辨率系数3