MOMENT MAPS AND EQUIVARIANT COHOMOLOGY IN TORIC GEOMETRY

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Abstract. We study toric varieties and their invariants.

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1. Introduction

2. Constructing toric varieties

We describe multiple equivalent constructions of a toric variety starting from the data of a polytope P subject to certain conditions. Each of the constructions will

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yield us a space $X_i(P)$ which will all be equivariantly diffeomorphic to each other as smooth manifolds.

2.1. **Delzant polytopes.** Let V be a real vector space of dimension n and let $V_{\mathbb{Z}}$ be a lattice inside V. Given N linear functionals $a_i \in V^*$ preserving $V_{\mathbb{Z}}$ and N integers λ_i the set

$$P = \{v \in V \mid a_i(v) + \lambda_i \geq 0 \text{ for all } i\}.$$

is called a *rational polyhedron*. It is called a *rational polytope* if it is bounded. We will assume that P is a rational polytope throughout this paper. A polytope P has *facets*

$$F_i = \{v \in P \mid a_i(v) + \lambda_i = 0\}$$

and *faces* which are intersections of facets. The *vertices* of P are the 0-dimensional faces of P and the *edges* of P are the 1-dimensional faces of P. We say P is:

- *simple* if exactly n edges meet at each vertex
- *smooth* if the edges meeting at each vertex form a basis for $V_{\mathbb{Z}}$
- *Delzant* if P is simple and smooth

Example 2.1. The right triangle $P = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}$ is a Delzant polytope. The standard tetrahedron $P \subset \mathbb{R}^3$ is not Delzant because the top vertex is not simple.

We begin by stating the correspondence we are interested in.

Theorem 2.2. *[Delzant] There is a correspondence between Delzant polytopes up to* $\mathrm{GL}(n, \mathbb{Z})$ *and translation, and toric symplectic manifolds up to equivariant symplectomorphism.*

Theorem 2.3. *There is a correspondence between Delzant polytopes up to* $\mathrm{GL}(n, \mathbb{Z})$ *and translation, and smooth projective toric varieties with a particular choice of very ample line bundle, up to equivariant isomorphism.*

2.2. **Moment maps.** Let (M, ω) be a symplectic manifold. The nondegeneracy of ω allows us to pair vector fields with 1-forms. We say that a vector field X is *Hamiltonian* if the corresponding 1-form $\iota_X \omega = \omega(X, \cdot)$ is exact, in which case it is equal to dH for some smooth function H. The function H is called a *Hamiltonian* of X.

Given a Lie group G acting on M by symplectomorphisms, the Lie algebra $\mathfrak{g}$ acts on M by symplectic vector fields. This linearized action of $\mathfrak{g}$ is given by the expression

$$X_\zeta(m) = \frac{d}{dt}\Big|_{t=0} \exp(t\zeta) \cdot m$$

where we interpret the given expression via parallel transport along the flow of ζ.

Remark 2.4. In general, whenever we have a Lie group G acting on a manifold M, we get a linearized action of $\mathfrak{g}$ on $\Gamma(E)$ for any vector bundle E over M. For the trivial line bundle $E = M \times \mathbb{R}$ we have $\Gamma(E) = C^\infty(M)$ and the linearized action of $\mathfrak{g}$ on $C^\infty(M)$ is given by the Lie derivative of the function along the vector field.

Example 2.5. Consider $G = \mathrm{SL}(2, \mathbb{C})$ acting on $\mathbb{P}^1$ by linear fractional transformations. Explicitly we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [z_0 : z_1] = [az_0 + bz_1 : cz_0 + dz_1]$$

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is generated by the matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Writing down the exponential, we find that

$$\exp(tE) = I + tE \quad \text{since } E^2 = 0 \implies \exp(tE) \cdot [z_0 : z_1] = [z_0 + tz_1 : z_1]$$

On an affine chart, the action of E is given by $z \mapsto z + t$ and we compute

$$\frac{d}{dt}\Big|_{t=0} z + t = 1 \implies X_E(z) = \frac{\partial}{\partial z}$$

Similarly we compute

$$\exp(tH) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \implies \exp(tH) \cdot [z_0 : z_1] = [e^t z_0 : e^{-t} z_1]$$

which looks like $z \mapsto e^{2t} z$ on an affine chart, which gives us

$$\frac{d}{dt}\Big|_{t=0} e^{2t} z = 2z \implies X_H(z) = 2z \frac{\partial}{\partial z}$$

Finally we compute

$$\exp(tF) = 1 + tF \implies \exp(tF) \cdot [z_0 : z_1] = [z_0 : z_1 + tz_0]$$

On an affine chart this transformation looks like $z \mapsto z/(1 + tz)$ and we compute

$$\frac{d}{dt}\Big|_{t=0} \frac{z}{1 + tz} = -z^2 \implies X_F(z) = -z^2 \frac{\partial}{\partial z}$$

In particular we get a map $\mathfrak{sl}(2, \mathbb{C}) \to \Gamma(T\mathbb{P}^1)$ given by

$$\begin{aligned} E &\mapsto \frac{\partial}{\partial z} \\ H &\mapsto 2z \frac{\partial}{\partial z} \\ F &\mapsto -z^2 \frac{\partial}{\partial z} \end{aligned}$$

which is the standard action of $\mathfrak{sl}(2, \mathbb{C})$ on $T\mathbb{P}^1$. This map in fact extends to an isomorphism between the enveloping algebra $U(\mathfrak{sl}(2, \mathbb{C}))$ and the algebra of differential operators on $\mathcal{D}_{\mathrm{hol}}(\mathbb{P}^1)$. This is a shadow of the Beilinson-Bernstein localization theorem (cf. [7]).

We say that the action of G is *weakly Hamiltonian* if for every $\zeta \in \mathfrak{g}$ the corresponding vector field X_ζ is Hamiltonian, i.e. $\iota_{X_\zeta} \omega = dH_\zeta$ for some smooth function H_ζ. The H_ζ is determined only up to a constant, so choose the map $\mathfrak{g} \to C^\infty(M)$ given by $\zeta \mapsto H_\zeta$ to be linear. If the map can be chosen to be equivariant with respect to the adjoint action of G on $\mathfrak{g}$, then the action of G is called *Hamiltonian.* In this case, there is a map $\mu : M \to \mathfrak{g}^*$ called the *moment map* defined by

$$H_\zeta(m) := \langle \mu(m), \zeta \rangle$$

If the action of G is Hamiltonian, then the moment map is G-equivariant and unique up to the addition of a constant. When $G = T$ is a torus, the adjoint action is trivial and the moment map $\mu : M \to \mathfrak{t}^*$ is T-invariant.

Theorem 2.6. *[Atiyah-Guillemin-Sternberg Convexity Theorem] Let M be a compact connected symplectic manifold with a Hamiltonian T-action. Then the image of the moment map is a convex polytope in $\mathfrak{t}^*$ whose vertices are the image of the fixed points of the T-action.*

Proof. See [9]. $\square$

We say M is a *toric symplectic manifold* in the sense of Theorem 2.2 if (M, ω) is a compact connected symplectic manifold with a effective (meaning no element of T acts trivially) Hamiltonian T-action.

Proposition 2.7. *Let $M \subset \mathbb{CP}^n$ be a smooth projective toric variety embedded by a line bundle. Then M is equivariantly symplectomorphic to a toric symplectic manifold.*

Proof. $\mathbb{CP}^n$ carries a natural symplectic form ω called the Fubini-Study form. Any smooth projective toric variety M embedded in projective space carries a symplectic form ω induced by pulling back the Fubini-Study form. Moreover, the action of T on M is Hamiltonian with respect to ω. $\square$

Conversely, given a toric symplectic manifold (M, ω), we can associate a smooth projective toric variety to the moment polytope $\mu(M)$ which will be equivariantly symplectomorphic to M.

2.3. **Symplectic reduction.** We describe how to construct M as the symplectic reduction of affine space $\mathbb{C}^N$ for a particular moment map. In particular, M carries a natural symplectic form ω and a Hamiltonian T-action. This section follows [1].

Let P be a Delzant polytope. There are maps

$$\pi : \mathbb{R}^N \to \mathbb{R}^n$$
$$e_i \mapsto a_i$$

and the induced map

$$\pi : \mathbb{R}^N/\mathbb{Z}^N \to \mathbb{R}^n/\mathbb{Z}^n$$

of tori, which give rise to the following short exact sequences.

$$1 \to K \to \mathbb{T}^N \to \mathbb{T}^n \to 1 \tag{2.8}$$

$$0 \to k \to \mathbb{R}^N \to \mathbb{R}^n \to 0$$

The dual of the second sequence gives

$$0 \to (\mathbb{R}^n)^* \to (\mathbb{R}^N)^* \to k^* \to 0$$

and denote the map $i^* : (\mathbb{R}^N)^* \to k^*$. Now consider $\mathbb{C}^N$ with the standard symplectic form $\omega = \sum dz_i \wedge d\bar{z}_i$ and the standard Hamiltonian torus action

$$(e^{i\theta_1}, \dots, e^{i\theta_N}) \cdot (z_1, \dots, z_N) = (e^{i\theta_1} z_1, \dots, e^{i\theta_N} z_N)$$

and corresponding moment map

$$\phi : \mathbb{C}^N \to (\mathbb{R}^N)^*$$
$$\phi(z_1, \ldots, z_N) = -\pi(|z_1|^2, \ldots, |z_N|^2) + (\lambda_1, \ldots, \lambda_N)$$

The subtorus K acts on $\mathbb{C}^N$ via restriction and the restricted action is Hamiltonian. Moreover, the moment map for the action of K is given by $i^* \circ \phi : M \to k^*$.

Let $Z = (i*^\circ \phi)^{-1}(0)$ be the zero level set of the moment map. The following claims are all justified in [1].

Lemma 2.9. *Z is compact and K freely acts on Z.*

The following theorem tells us that the orbit space Z/K is a symplectic manifold.

Theorem 2.10. *[Marsden-Weinstein-Meyer] Let G be a compact group and let (M, ω) be a symplectic manifold with a Hamiltonian G-action. Let $i : \mu^{-1}(0) \to M$ be the inclusion of the zero level set of the moment map. Assume G acts freely on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a smooth manifold*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle*
- *there is a unique symplectic form ω_{red} on M_{red} such that $\pi^*\omega_{red} = i^*\omega$*

Symplectic reduction realizes one direction of Delzant's correspondence.

Proposition 2.11. *The reduced space $X_1(P) := Z/K$ is a toric symplectic manifold with moment map image P.*

2.4. **Monoid algebra.** Let P be a Delzant polytope. Consider the cone

$$\sigma_P = \{(v, t) \in V \times \mathbb{R} \mid v \in tP, t \geq 0\}$$

The lattice points of this cone form a semigroup S_P. Define the space

$$X_2(P) = \operatorname{Proj} \mathbb{C}[S_P]$$

Proposition 2.12. *The space $X_2(P)$ is a smooth projective toric variety.*

This follows from general theory of projective toric varieties. We refer the reader to chapter 2 of [2] for a detailed exposition.

2.5. **Projective GIT.** Let P be a Delzant polytope. Complexifying (2.8), we get

$$1 \to K_{\mathbb{C}} \to T^N_{\mathbb{C}} \to T^n_{\mathbb{C}} \to 1$$

Let F_i denote the facets of Δ and for any $z = (z_1, \ldots, z_N) \in \mathbb{C}^n$ let $F_z := \cap_{z_i=0} F_i$. Consider the set

$$U = \{z \in \mathbb{C}^n : F_z \neq \emptyset\}$$

Then the quotient $X_3(P) = U/K_{\mathbb{C}}$ is a manifold with an action of $T^N_{\mathbb{C}}/K_{\mathbb{C}} = T^n_{\mathbb{C}}$. It is a smooth projective toric variety because it is a projective GIT quotient.

Remark 2.13. There is a surjective map from X^{ss} to $X//G$. Two points in X^{ss} lie in the same fiber of this map if and only if the closures of their G-orbits intersect. In this case, the $K_{\mathbb{C}}$ orbits are closed. See [10] for more details.

2.6. **Kempf-Ness theorem.** We want to compare the symplectic quotient to the projective GIT quotient. First recall that if K is a real compact group, then its complexification $G := K_{\mathbb{C}}$ is a complex Lie group which contains K and $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ is the complexification of $\mathfrak{k}$. See [6] for more details about the following theorems.

Theorem 2.14. *Complexification defines a bijection between the isomorphism classes of compact real Lie groups and complex reductive groups.*

The following theorem of Kempf-Ness states a relationship between the symplectic quotient and the GIT quotient.

Theorem 2.15. *[Kempf-Ness] Let G be a complex reductive group acting on a smooth complex projective variety $X \subset \mathbb{P}^n$. Let K be a maximal compact subgroup of G and suppose K is connected and acts on X Hamiltonianly. Let $\mu : X \to \mathfrak{k}^*$ be the moment map. Then the inclusion $\mu^{-1}(0) \to X$ induces a homeomorphism*

$$\mu^{-1}(0)/K \to X//G$$

Remark 2.16. We should say more about this theorem

2.7. **Fans and abstract toric varieties.** Let T be an n-dimensional torus with character group M, and let $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be the dual lattice, with pairing denoted $\langle , \rangle$. Recall that Theorem 2.2 gives a correspondence between Delzant polytopes and smooth projective toric varieties equipped with a very ample line bundle. Forgetting the embedding, we pass to the abstract toric variety, whose combinatorics ends up being encoded in the data of a fan.

Definition 2.17. A *fan* Σ in a real vector space N is a collection of cones σ such that

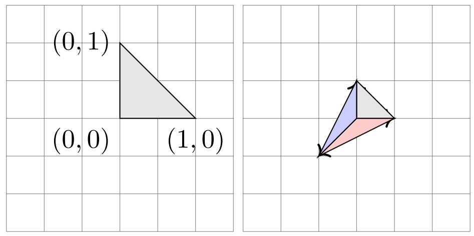
- σ is a strongly convex polyhedral cone
- if $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$
- the intersection of any two cones in Σ is a face of each

Given a Delzant polytope P, there is a fan Σ_P in $N_{\mathbb{R}}$ obtained by taking normal directions to the facets of P. The fan Σ_P is called the *normal fan* of P and it is a combinatorial object which encodes an equivariant atlas of charts for the toric variety $X(P)$.

Definition 2.18. A fan Σ is *complete* if the union of the cones in Σ is all of $N_{\mathbb{R}}$. A fan Σ is *nonsingular* if for each k-dimensional cone $\sigma \in \Sigma$, there exist k lattice vectors $v_1, \dots, v_k$ such that $\{v_1, \dots, v_k\}$ generate σ and $v_1, \dots, v_k$ can be extended to a basis of N. A fan Σ is *projective* if there is a rational polytope P such that Σ is the normal fan of P.

As the geometric language suggests, the toric variety $X(\Sigma)$ corresponding to a fan Σ is complete if and only if the fan is complete, and $X(\Sigma)$ is smooth if and only if the fan is nonsingular.
See [2] for more details.

Example 2.19. Consider the unit right triangle with corresponding normal fan



Note that the fan has three 2-dimensional cones which are filled in. These cones represent the three standard coordinate charts of $\mathbb{P}^2$ given by $x_i \neq 0$ for $i = 0, 1, 2$. The isosceles right triangle with side lenghth a corresponds to the a-th Veronese embedding of $\mathbb{P}^2$.

Fans are more friendly objects for algebraic geometers and one can read further about them in [2]. The data of a fan, and in particular the primitive edge vectors (defined as the generators of the rays of the fan), will prove important in our discussion on equivariant cohomology.

2.8. **Prequantization.** Manifolds equipped with integral symplectic forms admit prequantization line bundles. In particular we have the following theorem.

Theorem 2.20. *Let (M, ω) be a symplectic manifold. Suppose that $[\omega]$ is integral. Then there exists a "prequantization" line bundle $\mathcal{L} \to M$ with $c_1(\mathcal{L}) = [\omega]$ and a Hermitian connection α whose corresponding curvature form is ω. Moreover $\mathcal{L}$ is unique up to isomorphism (but still requires a particular choice).*

Remark 2.21. I don't quite understand the so-what of this theorem. What comes after "prequantization"?

Proposition 2.22. *The line bundle $\mathcal{L} = U \times_{K_C} \mathbb{C}$ where $K_{\mathbb{C}}$ acts on $\mathbb{C}$ with weight $\nu = L(-\lambda)$ is a prequantization line bundle for $M = U/K_{\mathbb{C}}$. Note that $\nu \in k^*$ is the dual Lie algebra of $K_{\mathbb{C}}$.*

Proof. See [4]. $\square$

Symplectic reduction realizes a Kahler form on the reduced space, in particular M and $\mathcal{L}$ actually carry complex structures. The following theorem is about the space of holomorphic sections of $\mathcal{L}$.

Theorem 2.23. *With the setup above, we have*

$$\dim H^0(M, \mathcal{L}) = \#(\textit{integer points in } P)$$

Proof. A holomorphic section of $\mathcal{L}$ over M corresponds to a $K_{\mathbb{C}}$-equivariant holomorphic function $f : U \to \mathbb{C}$. Such f extends to all of $\mathbb{C}^N$ because of Hartog's theorem (A holomorphic function on $\mathbb{C}^N$ for $N > 1$ canont have an isolated singularity and therefore cannot have a singularity on a submanifold of codimension ≥ 2).

Write such a function as its Taylor series so that

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

Consider the equivariance one term at a time. Thinking about the monomial $f(z) = z^I$ we see that

$$f(k \cdot z) = f(i(k) \cdot z) = (i(k) \cdot z)^I = i(k)^I z^I = k^{i^*(I)} z^I$$
$$k \cdot f(z) = k^\nu z^I$$

and therefore a basis for the space of equivariant functions $f : U \to \mathbb{C}$ is given by the monomials corresponding to lattice points in P. $\square$

3. Equivariant cohomology

We introduce equivariant cohomology and some classical results about the T-equivariant cohomology of smooth projective toric varieties.

3.1. **Basic properties.** Let G be a Lie group. The equivariant cohomology ring $H^*_G(X)$ of a G-space X is defined as the singular cohomology of the Borel construction $X \times_G EG$, where EG is a contractible space on which G acts freely. Such a space always exists and is unique up to homotopy equivalence [5].

Here are some important properties to know about equivariant cohomology:

(1) functoriality;
(2) a ring structure;
(3) excision;
(4) the Mayer-Vietoris sequence;
(5) the Künneth formula;
(6) the Leray spectral sequence;
(7) for smooth orientable X, Poincaré duality; and
(8) existence of Chern classes,

where all subsets and maps are assumed to be equivariant. The ring $H^*_T(X)$ is a module over $H^*_T(\mathrm{pt})$ via the map $X \to \mathrm{pt}$.

We now introduce a notion of equivariant formality which is a particularly nice algebraic notion. It implies that the equivariant cohomology of X is a free module over the equivariant cohomology of a point.

Definition 3.1. A G-space X is called **equivariantly formal** if the Leray spectral sequence associated to the fibration $X \to X \times_T ET \to BT$ collapses at the E_2-page.

By definition, we have that

Proposition 3.2. *If X is equivariantly formal, then*

$$H^*_G(X) \cong H^*_G(pt) \otimes H^*(X).$$

When X is equivariantly formal, the ordinary cohomology can be recovered from equivariant cohomology as the quotient

$$H^*(X) = \frac{H^*_T(X)}{M \cdot H^*_T(X)}$$

which in effect simply sets each $t_i = 0$. In particular, $H^*_T(X)$ is a free module over $H^*_T(\mathrm{pt})$.

Many varieties of interest are equivariantly formal, such as all of the following:

(1) a smooth complex projective variety (with respect to any linear algebraic T-action);
(2) a variety whose ordinary cohomology vanishes in odd degree (with respect to any T-action);
(3) a compact symplectic manifold with a Hamiltonian T-action, where T is a compact torus.

See [3] for a good discussion of these facts.

3.2. **Examples.** We introduce some examples.

Example 3.3. We can identify $U(n)$ as those complex matrices preserving the standard Hermitian form on $\mathbb{C}^n$. The group $U(n)$ acts on S^{2n-1} transitively and the stabilizer of the point $(1, 0, \dots, 0)$ is $U(n-1)$.
Therefore there is a canonical action of $U(1)$ acting as scalar matrices on S^{2n-1} inherited from the action on $U(n)$. None of these odd dimensional spheres are contractible, but S^∞ is contractible. In particular $EU(1) = S^\infty$ and

$$BU(1) = S^\infty/U(1) = \mathbb{CP}^\infty$$

Therefore

$$H^*_{U(1)}(pt) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[t]$$

where t is the first Chern class of the tautological line bundle and $\deg t = 2$.

Example 3.4. In general, $H^*_T(pt)$ can be identified with the representation ring of T.

$$BT \cong \prod_{\operatorname{rank} T} \mathbb{CP}^\infty$$

Given a representation V of T, we can form a vector bundle on the classifying space whose total Chern class is equal to the class of V in the representation ring, c.f. [8].

3.3. **Danilov's theorem.** The geometry of a smooth projective toric variety X is particularly special because it is encoded in the combinatorics of the fan of X. Danilov's theorem gives a presentation of the ordinary cohomology ring of X in terms of the combinatorics of the fan of X.

Theorem 3.5. *[Danilov] Let X be a smooth projective toric variety. Then the ordinary cohomology ring of X is*

$$H^*(X) \cong \mathbb{Q}[\alpha_1, \dots, \alpha_n]/(I + J)$$

where I is the ideal generated by the linear relations among the α_i

$$\sum_{i=1}^n \langle v_i, u\rangle \alpha_i \in I$$

for $u \in M$ and $v_1, \dots, v_n$ the fundamental vectors of the rays of the fan of X, and J is the ideal generated by the Stanley-Reisner relations.

$$\alpha_{i_1} \cdots \alpha_{i_k} \in J \text{ if } \{i_1, \dots, i_k\} \text{ is not a cone of the fan of } X.$$

Danilov's original proof of the theorem is quite involved and invokes a lot of intersection theory, but using facts from equivariant cohomology and toric geometry, we can give a much simpler proof. In particular smooth projective toric varieties are equivariantly formal. The proof of theorem 3.5 will be deferrred to the next section where we introduce the localization theorem. It is a good example of how the theory of equivariant cohomology enriches ordinary cohomology.

3.4. **ABBV localization.** At its core, the ABBV localization formula arises from the principle that, under certain conditions, integrals over a compact space with a torus action can be "localized" to the fixed points of the action. This idea can be traced back to the stationary phase approximation in physics, where integrals are approximated by contributions from critical points. In the equivariant cohomology setting, the fixed points of the torus action play a similar role.

Theorem 3.6 (Atiyah-Bott, Berline-Vergne)**.** *Suppose a torus T of dimensionl l acts on a compact oriented manifold M with fixed point set F. If $\phi \in H_T^*(M)$, then*

$$\int_M \phi = \sum_{x \in F} \frac{i_x^* \phi}{e_x}$$

where e_x is the equivariant Euler class of the normal bundle of x in M.

In particular, we will apply this theorem when the fixed point set is a finite set of points, in which case the localization formula becomes

$$\int_M \phi = \sum_{x \in F} \frac{\phi(x)}{e_x}$$

where e_x is the equivariant Euler class of $T_x M$. In particular, e_x is the product of the weights of the action of T on $T_x M$. Moreover, the weights are distinct because the fixed point set is isolated.

We will now consider the localization theorem in an example. In general, [11] contains a proof of the ABBV localization formula for S^1-actions.

Example 3.7. Consider the two sphere S^2 with real coordinates (x, y, z) and the S^1-action

$$e^{i\theta} \cdot (x, y, z) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z).$$

which rotates the sphere about the z-axis. The fixed points of this action are the north and south poles.

The Cartan model of equivariant cohomology suggests that we can consider integrating an equivariant cohomology class over S^2 by integrating corresponding *equivariant differential forms* over S^2. In particular if we consider the equivariant symplectic form $\alpha = \omega + 2\pi\mu$ where $\mu(x, y, z) = z$ is the moment map, viewed as a degree 0 form. α is equivariantly closed because in the Cartan model, the equivariant differential is given by

$$d_X = d - u i_X$$

where i_X is the contraction with the vector field X generating the S^1-action and u is the standard representation of S^1, the "equivariant parameter".
Therefore α represents a class in $H^2_{S^1}(S^2)$. We can compute the integral of α over S^2 using the localization formula.

$$\begin{aligned}\int_{S^2} \alpha &= \frac{\mu(N)}{\text{weight of } U(1) \text{ action on } T_N S^2} + \frac{\mu(S)}{\text{weight of } U(1) \text{ action on } T_S S^2} \\ &= 2\pi\mu(N) - 2\pi\mu(S) = 4\pi\end{aligned}$$

where N and S are the north and south poles respectively. Indeed the surface area of S^2 is 4π.

3.5. **Proof of Danilov's theorem.**

3.5.1. *Cone-orbit correspondence.* Let T be an n-dimensional torus with character group M. Let $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice, their pairing is denoted by $\langle \cdot, \cdot \rangle$. Let $X = X(\Sigma)$ be a smooth complete toric variety, which are in bijection with complete nonsingular fans Σ in $N_{\mathbb{R}}$.

For any convex cone $\sigma \subset N_{\mathbb{R}}$, the *dual cone* in $M_{\mathbb{R}}$ is

$$\sigma^\vee = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

By intersecting with the lattice, we obtain a semigroup $\sigma^\vee \cap M$ with corresponding semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$. The toric variety X is covered by T-invariant open affine sets

$$U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$$

The affine charts corresponding to the top-dimensional cones of Σ are enough to cover X, and the intersection of cones corresponds to the intersection of affine charts.

Each cone τ of the fan also defines a torus-invariant subvariety $V(\tau)$ of X of codimension $\dim \tau$. On open affines, the subvariety looks like

$$V(\tau) \cap U_\sigma = \operatorname{Spec} \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M] \hookrightarrow \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$$

and so elements of the dual lattice N can be thought of as rational functions on X.

3.5.2. *Bialynicki-Birula decomposition.* The Bialynicki-Birula decomposition is a generalization of the Morse theory for torus actions. Suppose that $\mathbb{C}^*$ acts on a smooth projective variety X with finitely many fixed points $p_1, \ldots, p_k$.

Then each $T_{p_i}X$ is a representation of $\mathbb{C}^*$, and so we can decompose into weight spaces

$$T_{p_i}X = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$$

where $V_\lambda = \{v \in T_{p_i}X \mid t \cdot v = t^\lambda v\}$. Note that $\lambda \neq 0$ because the fixed point set is isolated.
Define the attracting set

$$C_i = \{x \in X \mid \lim_{t \to 0} t \cdot x = p_i\}$$

Theorem 3.8 (Bialynicki-Birula)**.** *There exists a filtration of X by closed subschemes*

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0$$

such that each $X_i \backslash X_{i-1}$ is a disjoint union of affine spaces called cells, in particular the attracting sets.

Corollary 3.9. *Let X as above. Then*

(1) $H_{2i+1}(X) = 0$ *for all i;*

(2) *$H_{2i}(X)$ is a $\mathbb{Z}$-module freely generateed by the classes of the closures of the i-dimensional cells.*

3.5.3. *Shellings.* In this section, we apply the setup of the Bialynicki-Birula decomposition to the context of toric varieties.

Let $X = X(\Sigma)$ be projective, with P a polytope whose normal fan is Σ. Choosing a general vector $v \in N_{\mathbb{R}}$ we obtain an ordering of the vertices $u_1, \ldots, u_s$ of P by the order of the inner products $\langle v, u_i \rangle$. Geometrically, we are choosing a 1-parameter subgroup of the torus T which acts on X. The correpsonding sub-moment map turns out to be a perfect Morse-Bott function on X.

By the polytope-fan correspondence, we get an ordering of the maximal cones $\sigma_1, \ldots, \sigma_s$ of Σ. For $1 \leq i \leq s$ let

$$\tau_i = \bigcap_{j>i, \dim(\sigma_j \cap \sigma_i)=n-1} \sigma_j \cap \sigma_i$$

so that $\tau_1 = \{0\}, \tau_s = \sigma_s$ and $\tau_p \subset \tau_q$ implies $p \leq q$. Such an ordering of cones is called a *shelling* of Σ.

Proposition 3.10. *A shelling gives a cellular decomposition of X, with the closures of the cells being $V(\tau_i)$. In particular, the classes*

$$\alpha_i = [V(\tau_i)] \in H^{2(n-\dim \tau_i)}(X)$$

form an additive $\mathbb{Z}$-basis of $H^*(X)$. Moreover, the corresponding equivariant cohomology classes

$$\alpha_i^T = [V(\tau_i)]^T \in H_T^{2(n-\dim \tau_i)}(X)$$

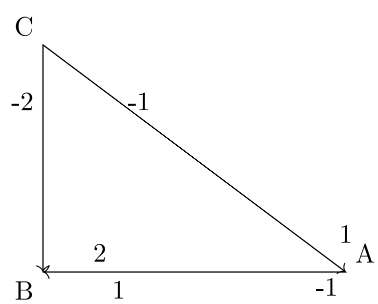
form an additive $\mathbb{Z}$-basis of $H_T^*(X)$.

The $V(\tau_i)$ are precisely the closures of the attracting sets of the chosen $\mathbb{C}^*$-action on X. We will demonstrate this in a particularly pleasant example.

Example 3.11 (Morse theory on $\mathbb{CP}^2$). Classically, recall that the Chow ring (or cohomology ring) of $\mathbb{CP}^2$ is

$$H^*(\mathbb{CP}^2) = \mathbb{Z}[0] \oplus \mathbb{Z}[\mathbb{P}^1] \oplus \mathbb{Z}[\mathbb{P}^2]$$

Consider the standard action of T^2 on $\mathbb{CP}^2$ and consider the 1-parameter subgroup acting by $t \cdot [x : y : z] = [tx : t^2 y : z]$. Consider the following moment image, whose edges are labeled by the weights of the action on the tangent space at the fixed points:



Based on our choice of 1-parameter subgroup, we have the following decomposition:

$$\mathbb{CP}^2 = C \coprod (\mathbb{P}^1 \backslash C) \coprod \mathbb{P}^2 \backslash \mathbb{P}^1$$

One sees this decomposition by considering the attracting sets of the action for each fixed point. For C the attracting set is C itself, for A the attracting set is the line $\mathbb{P}^1$, all the points of $\mathbb{P}^1$ except C are attracted to A, and for B the attracting set is $\mathbb{P}^2 \backslash \mathbb{P}^1$, where the $\mathbb{P}^1$ is the T-invariant curve which joins the fixed points A and C. These cells are all affine spaces, and their closures are precisely $V(\tau_i) = \mathbb{P}^i$.

The equivariant statement follows from applying the Bialynicki-Birula decomposition to the mixing space, which admits a finite dimensional approximation by smooth projective varieties. See [3] for more details.

If X is not projective, then one can subdivide cones and produce a refinement Σ' of Σ so that the corresponding map

$$\pi : X(\Sigma') \to X(\Sigma)$$

is a surjective birational T-equivariant morphism and $X(\Sigma')$ is smooth and projective. The composition

$$\pi_* \circ \pi^* : H^*(X(\Sigma)) \to H^*(X(\Sigma))$$

is the identity on $H^*(X(\Sigma))$ and on $H^*_T(X(\Sigma))$ and therefore π^* is injective and π_* is surjective. Assembling the results of the previous sections, we obtain the following proposition.

Proposition 3.12. *For any complete smooth toric variety X, the cohomology ring $H^*(X)$ is generated by the classes $[V(\tau_i)]$ of the closures of the attracting sets as $\mathbb{Z}$-module, and the equivariant cohomology ring $H^*_T(X)$ is generated by the classes $[V(\tau_i)]^T$ as a module over $H^*_T(pt)$.*

3.5.4. *The Stanley-Reisner ring.*

3.6. **GKM Theory.**

4. Sheaf cohomology

5. Toric degenerations

6. Appendix A: Vector bundles and connections

We provide a brief introduction to connections on vector bundles.

6.1. **Parallel transport.** One way to think about a connection is to consider parallel transport. You want to be able to differentiate sections of a vector bundle along paths. When we are dealing with functions, we can form the directional derivative

$$ds(x)X = \lim_{t\to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}$$

for any smooth path γ representing the tangent vector $X \in T_xM$ and this expression gives us a linear map $ds(x) : T_xM \to E_x$.

However if E is a nontrivial bundle, then this expression does not make sense because the summands live in different fibers. There is unfortunately no natural way to compare vectors in different fibers. Therefore we need to introduce additional structure to be able to compare these fibers.

For a general vector bundle $E \to M$, we want to associate, to a path γ in M, a smooth family of parallel transport isomorphisms $P_\gamma^t : E_{\gamma(0)} \to E_{\gamma(t)}$ such that

- $P_\gamma^0 = \mathrm{id}$
- $P_{\gamma_1 \cdot \gamma_2}^t = P_{\gamma_2}^t \circ P_{\gamma_1}^t$

for any paths γ_1, γ_2 and $t \in \mathbb{R}$.

Such a choice would allow us to define the directional ("covariant") derivative of a section s along a path γ as before. We shuold require that

- The directional derivative depends only on s and $X \in T_xM$, not the particular choice of γ.
- The map $\nabla s(x) : T_xM \to E_x$ is $\mathbb{C}$-linear.

This will give us the richest notion of a connection on a vector bundle.

6.2. **Connections.** In this section, we consider M a real manifold and $\pi : E \to M$ complex vector bundle. Let $\mathcal{A}^i(E) = \Omega^i(M) \otimes E$ denote the sheaf of smooth i-forms with values in E.

Definition 6.1. A **connection** on E is a $\mathbb{C}$-linear map of sheaves $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

We can interpret this definition in the sense of parallel transport. Given a section $s \in \mathcal{A}^0(E)$, we can differentiate it along a path γ to get another section of E, i.e. $\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}(TM, E))$.

Theorem 6.2. *The space of all connections $\mathcal{A}(E)$ is an affine space modelled on $\mathcal{A}^1(\mathrm{End}\, E)$. In particular*

- *$\mathcal{A}(E)$ is nonempty*
- *For any two connections ∇_1, ∇_2 the difference $\nabla_1 - \nabla_2$ is a global section of $\mathcal{A}^1(\mathrm{End}\, E)$.*
- *$(\nabla + a)s := \nabla s + as$ is a connection whenever ∇ is a connection and $a \in \mathcal{A}^1(\mathrm{End}\, E)$.*

Proof. See [8]. □

The idea of a connection generalizes the exterior differential to sections of general vector bundles. However, a connection need not satisfy $\nabla^2 = 0$ in general. The obstruction for a connection define a differential is measured by its curvature. We explain this now.

6.3. **Curvature.** A connection $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ induces "differentials"

$$\nabla : \mathcal{A}^i(E) \to \mathcal{A}^{i+1}(E)$$

given by the formula

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^i \alpha \wedge \nabla s$$

Definition 6.3. The **curvature** F_∇ of a connection ∇ is the composition

$$F_\nabla := \nabla^2 : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$$

In particular F_∇ is a global section of $\mathcal{A}^2(\operatorname{End} E)$. This is because the curvature homomorphism is $\mathcal{A}^0$-linear.

Example 6.4. Consider the connections on the trivial bundle $M \times \mathbb{C}^r$. If $\nabla = d$ is the trivial connection then $F_\nabla = 0$.

Any other connection is of the form $\nabla = d + A$ where A is a matrix of 1-forms. For a section s we compute

$$\begin{aligned} F_\nabla(s) &= (d + A)(d + A)(s) \\ &= d^2 s + dAs + Ads + AAs \\ &= d(A)s + A \wedge As \end{aligned}$$

and therefore

$$F_\nabla = dA + A \wedge A$$

For line bundles we get that $F_\nabla = dA$ is an ordinary 2-form.

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