MOMENT MAPS AND EQUIVARIANT COHOMOLOGY IN TORIC GEOMETRY

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Abstract. We study toric varieties and their invariants.

Contents

1. Introduction

2. Constructing toric varieties

We describe multiple equivalent constructions of a toric variety starting from the data of a polytope P subject to certain conditions. Each of the constructions will yield us a space $X_i(P)$ which will all be equivariantly diffeomorphic to each other as smooth manifolds.

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2.1. **Delzant polytopes.** Let V be a real vector space of dimension n and let $V_{\mathbb{Z}}$ be a lattice inside V. Given N linear functionals $a_i \in V^*$ preserving $V_{\mathbb{Z}}$ and N integers λ_i the set

$$P = \{v \in V \mid a_i(v) + \lambda_i \geq 0 \text{ for all } i\}.$$

is called a *rational polyhedron*. It is called a *rational polytope* if it is bounded. We will assume that P is a rational polytope throughout this paper. A polytope P has *facets*

$$F_i = \{v \in P \mid a_i(v) + \lambda_i = 0\}$$

and *faces* which are intersections of facets. The *vertices* of P are the 0-dimensional faces of P and the *edges* of P are the 1-dimensional faces of P. We say P is:

- *simple* if exactly n edges meet at each vertex
- *smooth* if the edges meeting at each vertex form a basis for $V_{\mathbb{Z}}$
- *Delzant* if P is simple and smooth

Example 2.1. The right triangle $P = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}$ is a Delzant polytope. The standard tetrahedron $P \subset \mathbb{R}^3$ is not Delzant because the top vertex is not simple.

We begin by stating the correspondence we are interested in.

Theorem 2.2. *[Delzant] There is a correspondence between Delzant polytopes up to* $\mathrm{GL}(n, \mathbb{Z})$ *and translation, and toric symplectic manifolds up to equivariant symplectomorphism.*

Theorem 2.3. *There is a correspondence between Delzant polytopes up to* $\mathrm{GL}(n, \mathbb{Z})$ *and translation, and smooth projective toric varieties with a particular choice of very ample line bundle, up to equivariant isomorphism.*

2.2. **Moment maps.** Let (M, ω) be a symplectic manifold. The nondegeneracy of ω allows us to pair vector fields with 1-forms. We say that a vector field X is *Hamiltonian* if the corresponding 1-form $\iota_X \omega = \omega(X, \cdot)$ is exact, in which case it is equal to dH for some smooth function H. The function H is called a *Hamiltonian* of X.

Given a Lie group G acting on M by symplectomorphisms, the Lie algebra $\mathfrak{g}$ acts on M by symplectic vector fields. This linearized action of $\mathfrak{g}$ is given by the expression

$$X_\zeta(m) = \frac{d}{dt}\Big|_{t=0} \exp(t\zeta) \cdot m$$

where we interpret the given expression via parallel transport along the flow of ζ.

Remark 2.4. In general, whenever we have a Lie group G acting on a manifold M, we get a linearized action of $\mathfrak{g}$ on $\Gamma(E)$ for any vector bundle E over M. For the trivial line bundle $E = M \times \mathbb{R}$ we have $\Gamma(E) = C^\infty(M)$ and the linearized action of $\mathfrak{g}$ on $C^\infty(M)$ is given by the Lie derivative of the function along the vector field.

Example 2.5. Consider $G = \mathrm{SL}(2, \mathbb{C})$ acting on $\mathbb{P}^1$ by linear fractional transformations. Explicitly we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [z_0 : z_1] = [az_0 + bz_1 : cz_0 + dz_1]$$

The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is generated by the matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Writing down the exponential, we find that

$$\exp(tE) = I + tE \quad \text{since } E^2 = 0 \implies \exp(tE) \cdot [z_0 : z_1] = [z_0 + tz_1 : z_1]$$

On an affine chart, the action of E is given by $z \mapsto z + t$ and we compute

$$\frac{d}{dt}\Big|_{t=0} z + t = 1 \implies X_E(z) = \frac{\partial}{\partial z}$$

Similarly we compute

$$\exp(tH) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \implies \exp(tH) \cdot [z_0 : z_1] = [e^t z_0 : e^{-t} z_1]$$

which looks like $z \mapsto e^{2t} z$ on an affine chart, which gives us

$$\frac{d}{dt}\Big|_{t=0} e^{2t} z = 2z \implies X_H(z) = 2z\frac{\partial}{\partial z}$$

Finally we compute

$$\exp(tF) = 1 + tF \implies \exp(tF) \cdot [z_0 : z_1] = [z_0 : z_1 + tz_0]$$

On an affine chart this transformation looks like $z \mapsto z/(1 + tz)$ and we compute

$$\frac{d}{dt}\Big|_{t=0} \frac{z}{1+tz} = -z^2 \implies X_F(z) = -z^2\frac{\partial}{\partial z}$$

In particular we get a map $\mathfrak{sl}(2,\mathbb{C}) \to \Gamma(T\mathbb{P}^1)$ given by

$$\begin{aligned} E &\mapsto \frac{\partial}{\partial z} \\ H &\mapsto 2z\frac{\partial}{\partial z} \\ F &\mapsto -z^2\frac{\partial}{\partial z} \end{aligned}$$

which is the standard action of $\mathfrak{sl}(2,\mathbb{C})$ on $T\mathbb{P}^1$. This map in fact extends to an isomorphism between the enveloping algebra $U(\mathfrak{sl}(2,\mathbb{C}))$ and the algebra of differential operators on $\mathcal{D}_{\text{hol}}(\mathbb{P}^1)$. This is a shadow of the Beilinson-Bernstein localization theorem (cf. [5]).

We say that the action of G is *weakly Hamiltonian* if for every $\zeta \in \mathfrak{g}$ the corresponding vector field X_ζ is Hamiltonian, i.e. $\iota_{X_\zeta}\omega = dH_\zeta$ for some smooth function H_ζ. The H_ζ is determined only up to a constant, so choose the map $\mathfrak{g} \to C^\infty(M)$ given by $\zeta \mapsto H_\zeta$ to be linear. If the map can be chosen to be equivariant with respect to the adjoint action of G on $\mathfrak{g}$, then the action of G is called *Hamiltonian*. In this case, there is a map $\mu : M \to \mathfrak{g}^*$ called the *moment map* defined by

$$H_\zeta(m) := \langle \mu(m), \zeta \rangle$$

If the action of G is Hamiltonian, then the moment map is G-equivariant and unique up to the addition of a constant. When $G = T$ is a torus, the adjoint action is trivial and the moment map $\mu : M \to \mathfrak{t}^*$ is T-invariant.

Theorem 2.6. *[Atiyah-Guillemin-Sternberg Convexity Theorem] Let M be a compact connected symplectic manifold with a Hamiltonian T-action. Then the image of the moment map is a convex polytope in $\mathfrak{t}^*$ whose vertices are the image of the fixed points of the T-action.*

Proof. See [7]. $\square$

We say M is a *toric symplectic manifold* in the sense of Theorem 2.2 if (M, ω) is a compact connected symplectic manifold with a effective (meaning no element of T acts trivially) Hamiltonian T-action.

Proposition 2.7. *Let $M \subset \mathbb{CP}^n$ be a smooth projective toric variety embedded by a line bundle. Then M is equivariantly symplectomorphic to a toric symplectic manifold.*

Proof. $\mathbb{CP}^n$ carries a natural symplectic form ω called the Fubini-Study form. Any smooth projective toric variety M embedded in projective space carries a symplectic form ω induced by pulling back the Fubini-Study form. Moreover, the action of T on M is Hamiltonian with respect to ω. $\square$

Conversely, given a toric symplectic manifold (M, ω), we can associate a smooth projective toric variety to the moment polytope $\mu(M)$ which will be equivariantly symplectomorphic to M.

2.3. **Symplectic reduction.** We describe how to construct M as the symplectic reduction of affine space $\mathbb{C}^N$ for a particular moment map. In particular, M carries a natural symplectic form ω and a Hamiltonian T-action. This section follows [1].

Let P be a Delzant polytope. There are maps

$$\pi : \mathbb{R}^N \to \mathbb{R}^n$$
$$e_i \mapsto a_i$$

and the induced map

$$\pi : \mathbb{R}^N/\mathbb{Z}^N \to \mathbb{R}^n/\mathbb{Z}^n$$

of tori, which give rise to the following short exact sequences.

$$1 \to K \to \mathbb{T}^N \to \mathbb{T}^n \to 1 \tag{2.8}$$

$$0 \to k \to \mathbb{R}^N \to \mathbb{R}^n \to 0$$

The dual of the second sequence gives

$$0 \to (\mathbb{R}^n)^* \to (\mathbb{R}^N)^* \to k^* \to 0$$

and denote the map $i^* : (\mathbb{R}^N)^* \to k^*$. Now consider $\mathbb{C}^N$ with the standard symplectic form $\omega = \sum dz_i \wedge d\bar{z}_i$ and the standard Hamiltonian torus action

$$(e^{i\theta_1}, \dots, e^{i\theta_N}) \cdot (z_1, \dots, z_N) = (e^{i\theta_1} z_1, \dots, e^{i\theta_N} z_N)$$

and corresponding moment map

$$\phi : \mathbb{C}^N \to (\mathbb{R}^N)^*$$
$$\phi(z_1, \dots, z_N) = -\pi(|z_1|^2, \dots, |z_N|^2) + (\lambda_1, \dots, \lambda_N)$$

The subtorus K acts on $\mathbb{C}^N$ via restriction and the restricted action is Hamiltonian. Moreover, the moment map for the action of K is given by $i^* \circ \phi : M \to k^*$.

Let $Z = (i*^\circ \phi)^{-1}(0)$ be the zero level set of the moment map. The following claims are all justified in [1].

Lemma 2.9. *Z is compact and K freely acts on Z.*

The following theorem tells us that the orbit space Z/K is a symplectic manifold.

Theorem 2.10. *[Marsden-Weinstein-Meyer] Let G be a compact group and let (M, ω) be a symplectic manifold with a Hamiltonian G-action. Let $i : \mu^{-1}(0) \to M$ be the inclusion of the zero level set of the moment map. Assume G acts freely on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a smooth manifold*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle*
- *there is a unique symplectic form ω_{red} on M_{red} such that $\pi^*\omega_{red} = i^*\omega$*

Symplectic reduction realizes one direction of Delzant's correspondence.

Proposition 2.11. *The reduced space $X_1(P) := Z/K$ is a toric symplectic manifold with moment map image P.*

2.4. **Monoid algebra.** Let P be a Delzant polytope. Consider the cone

$$\sigma_P = \{(v, t) \in V \times \mathbb{R} \mid v \in tP, t \geq 0\}$$

The lattice points of this cone form a semigroup S_P. Define the space

$$X_2(P) = \operatorname{Proj} \mathbb{C}[S_P]$$

Proposition 2.12. *The space $X_2(P)$ is a smooth projective toric variety.*

This follows from general theory of projective toric varieties. We refer the reader to chapter 2 of [2] for a detailed exposition.

2.5. **Projective GIT.** Let P be a Delzant polytope. Complexifying (2.8), we get

$$1 \to K_{\mathbb{C}} \to T^N_{\mathbb{C}} \to T^n_{\mathbb{C}} \to 1$$

Let F_i denote the facets of Δ and for any $z = (z_1, \ldots, z_N) \in \mathbb{C}^n$ let $F_z := \cap_{z_i=0} F_i$. Consider the set

$$U = \{z \in \mathbb{C}^n : F_z \neq \emptyset\}$$

Then the quotient $X_3(P) = U/K_{\mathbb{C}}$ is a manifold with an action of $T^N_{\mathbb{C}}/K_{\mathbb{C}} = T^n_{\mathbb{C}}$. It is a smooth projective toric variety because it is a projective GIT quotient.

Remark 2.13. There is a surjective map from X^{ss} to $X//G$. Two points in X^{ss} lie in the same fiber of this map if and only if the closures of their G-orbits intersect. In this case, the $K_{\mathbb{C}}$ orbits are closed. See [8] for more details.

2.6. **Kempf-Ness theorem.** We want to compare the symplectic quotient to the projective GIT quotient. First recall that if K is a real compact group, then its complexification $G := K_{\mathbb{C}}$ is a complex Lie group which contains K and $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ is the complexification of $\mathfrak{k}$. See [4] for more details about the following theorems.

Theorem 2.14. *Complexification defines a bijection between the isomorphism classes of compact real Lie groups and complex reductive groups.*

The following theorem of Kempf-Ness states a relationship between the symplectic quotient and the GIT quotient.

Theorem 2.15. *[Kempf-Ness] Let G be a complex reductive group acting on a smooth complex projective variety $X \subset \mathbb{P}^n$. Let K be a maximal compact subgroup of G and suppose K is connected and acts on X Hamiltonianly. Let $\mu : X \to \mathfrak{k}^*$ be the moment map. Then the inclusion $\mu^{-1}(0) \to X$ induces a homeomorphism*

$$\mu^{-1}(0)/K \to X//G$$

Remark 2.16. We should say more about this theorem

2.7. **Prequantization.** Manifolds equipped with integral symplectic forms admit prequantization line bundles. In particular we have the following theorem.

Theorem 2.17. *Let (M, ω) be a symplectic manifold. Suppose that $[\omega]$ is integral. Then there exists a "prequantization" line bundle $\mathcal{L} \to M$ with $c_1(\mathcal{L}) = [\omega]$ and a Hermitian connection α whose corresponding curvature form is ω. Moreover $\mathcal{L}$ is unique up to isomorphism (but still requires a particular choice).*

Remark 2.18. I don't quite understand the so-what of this theorem. What comes after "prequantization"?

Proposition 2.19. *The line bundle $\mathcal{L} = U \times_{K_C} \mathbb{C}$ where $K_{\mathbb{C}}$ acts on $\mathbb{C}$ with weight $\nu = L(-\lambda)$ is a prequantization line bundle for $M = U/K_{\mathbb{C}}$. Note that $\nu \in k^*$ is the dual Lie algebra of $K_{\mathbb{C}}$.*

Proof. See [3]. □

Symplectic reduction realizes a Kahler form on the reduced space, in particular M and $\mathcal{L}$ actually carry complex structures. The following theorem is about the space of holomorphic sections of $\mathcal{L}$.

Theorem 2.20. *With the setup above, we have*

$$\dim H^0(M, \mathcal{L}) = \#(\textit{integer points in } P)$$

Proof. A holomorphic section of $\mathcal{L}$ over M corresponds to a $K_{\mathbb{C}}$-equivariant holomorphic function $f : U \to \mathbb{C}$. Such f extends to all of $\mathbb{C}^N$ because of Hartog's theorem (A holomorphic function on $\mathbb{C}^N$ for $N > 1$ canont have an isolated singularity and therefore cannot have a singularity on a submanifold of codimension ≥ 2).

Write such a function as its Taylor series so that

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

Consider the equivariance one term at a time. Thinking about the monomial $f(z) = z^I$ we see that

$$f(k \cdot z) = f(i(k) \cdot z) = (i(k) \cdot z)^I = i(k)^I z^I = k^{i^*(I)} z^I$$
$$k \cdot f(z) = k^\nu z^I$$

and therefore a basis for the space of equivariant functions $f : U \to \mathbb{C}$ is given by the monomials corresponding to lattice points in P. $\square$

3. Equivariant cohomology

We introduce equivariant cohomology and some classical results about the T-equivariant cohomology of smooth projective toric varieties.

3.1. **Basic properties.** Let G be a Lie group. in terms of exposition, want to say something about principle G-bundles over X The equivariant cohomology $H^*_G(X)$ of a G-space X is defined as the singular cohomology of the Borel construction $X \times_G EG$, where EG is a contractible space on which G acts freely. Such a space always exists and is unique up to homotopy equivalence [?]. The equivariant cohomology ring $H^*_G(X)$ is a module over the ordinary cohomology ring $H^*(BG)$ of the classifying space BG of G.

Here are some important things to know about equivariant cohomology:

- A G-equivariant map $f : X \to Y$ induces a pullback map of $H^*(BG)$-modules $f_* : H^*_G(Y) \to H^*_G(X)$.
- (Mayer-Vietoris)
- Setting all the variables $= 0$

3.2. **Examples.** We introduce some examples.

Example 3.1. We compute the G-equivariant cohomology of a point when $G = T$ is a torus and when $G = \mathrm{GL}(n, \mathbb{C})$. It is equal to the representation ring of T and the representation ring of $\mathrm{GL}(n, \mathbb{C})$, respectively.

3.3. **Danilov's theorem.**

3.4. **ABBV localization.**

3.5. **Borel presentation.**

3.6. **GKM theorem.**

4. Sheaf cohomology

5. Toric degenerations

6. Appendix A: Vector bundles and connections

We provide a brief introduction to connections on vector bundles.

6.1. **Parallel transport.** One way to think about a connection is to consider parallel transport. You want to be able to differentiate sections of a vector bundle along paths. When we are dealing with functions, we can form the directional derivative

$$ds(x)X = \lim_{t\to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}$$

for any smooth path γ representing the tangent vector $X \in T_xM$ and this expression gives us a linear map $ds(x) : T_xM \to E_x$.

However if E is a nontrivial bundle, then this expression does not make sense because the summands live in different fibers. There is unfortunately no natural way to compare vectors in different fibers. Therefore we need to introduce additional structure to be able to compare these fibers.

For a general vector bundle $E \to M$, we want to associate, to a path γ in M, a smooth family of parallel transport isomorphisms $P^t_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$ such that

- $P^0_\gamma = \mathrm{id}$
- $P^t_{\gamma_1\cdot\gamma_2} = P^t_{\gamma_2} \circ P^t_{\gamma_1}$

for any paths γ_1, γ_2 and $t \in \mathbb{R}$.

Such a choice would allow us to define the directional (”covariant”) derivative of a section s along a path γ as before. We shuold require that

- The directional derivative depends only on s and $X \in T_xM$, not the particular choice of γ.
- The map $\nabla s(x) : T_xM \to E_x$ is $\mathbb{C}$-linear.

This will give us the richest notion of a connection on a vector bundle.

6.2. **Connections.** In this section, we consider M a real manifold and $\pi : E \to M$ complex vector bundle. Let $\mathcal{A}^i(E) = \Omega^i(M) \otimes E$ denote the sheaf of smooth i-forms with values in E.

Definition 6.1. A **connection** on E is a $\mathbb{C}$-linear map of sheaves $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

We can interpret this definition in the sense of parallel transport. Given a section $s \in \mathcal{A}^0(E)$, we can differentiate it along a path γ to get another section of E, i.e. $\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}(TM, E))$.

Theorem 6.2. *The space of all connections $\mathcal{A}(E)$ is an affine space modelled on $\mathcal{A}^1(\mathrm{End}\, E)$. In particular*

- *$\mathcal{A}(E)$ is nonempty*
- *For any two connections ∇_1, ∇_2 the difference $\nabla_1 - \nabla_2$ is a global section of $\mathcal{A}^1(\mathrm{End}\, E)$.*
- *$(\nabla + a)s := \nabla s + as$ is a connection whenever ∇ is a connection and $a \in \mathcal{A}^1(\mathrm{End}\, E)$.*

Proof. See [6]. $\square$

The idea of a connection generalizes the exterior differential to sections of general vector bundles. However, a connection need not satisfy $\nabla^2 = 0$ in general. The obstruction for a connection define a differential is measured by its curvature. We explain this now.

6.3. **Curvature.** A connection $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ induces "differentials"

$$\nabla : \mathcal{A}^i(E) \to \mathcal{A}^{i+1}(E)$$

given by the formula

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^i \alpha \wedge \nabla s$$

Definition 6.3. The **curvature** F_∇ of a connection ∇ is the composition

$$F_\nabla := \nabla^2 : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$$

In particular F_∇ is a global section of $\mathcal{A}^2(\operatorname{End} E)$. This is because the curvature homomorphism is $\mathcal{A}^0$-linear.

Example 6.4. Consider the connections on the trivial bundle $M \times \mathbb{C}^r$. If $\nabla = d$ is the trivial connection then $F_\nabla = 0$.

Any other connection is of the form $\nabla = d + A$ where A is a matrix of 1-forms. For a section s we compute

$$\begin{aligned} F_\nabla(s) &= (d + A)(d + A)(s) \\ &= d^2 s + dAs + Ads + AAs \\ &= d(A)s + A \wedge As \end{aligned}$$

and therefore

$$F_\nabla = dA + A \wedge A$$

For line bundles we get that $F_\nabla = dA$ is an ordinary 2-form.

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