RECONSTRUCTION THEOREM FOR DERIVED CATEGORIES

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Abstract. In this note we give a gentle introduction to derived categories and triangulated structures. We prove the classical reconstruction theorem of Bondal-Orlov [1] for varieties with ample or anti-ample canonical bundle.

Contents

1. Derived categories and triangulated structures

Our main reference for this section is [2]. Let $\mathcal{A}$ be an abelian category. The derived category $D(\mathcal{A})$ is constructed in several steps. Consider the category $C(\mathcal{A})$ of complexes in $\mathcal{A}$, whose objects are cochain complexes and morphisms are chain maps that commute with the differentials.

Form the homotopy category $K(\mathcal{A})$ whose objects are the same as $C(\mathcal{A})$. The morphisms are chain maps modulo homotopy equivalence. Two chain maps

$$f, g : A^\bullet \to B^\bullet$$

are homotopic if there exist morphisms $h^i : A^i \to B^{i-1}$ such that

$$f^i - g^i = d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i$$

It is a routine check that two maps which are homotopic induce the same map on cohomology.

Finally form $D(\mathcal{A})$ by formally inverting all quasi-isomorphisms in $K(\mathcal{A})$. The morphisms in $D(\mathcal{A})$ are a little subtle. For example, one cannot just introduce formal inverses to quasi-isomorphisms. If X is not an injective object in $\mathcal{A}$, then

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the inclusion map $X[0] \to I^\bullet$ into an injective resolution is a quasi-isomorphism. If we formally invert by introducing $p : I^\bullet \to X[0]$ with

$$[p] \circ [i] = [\mathrm{id}_{X[0]}] \quad \text{in } K(\mathcal{A})$$
$$[i] \circ [p] = [\mathrm{id}_{I^\bullet}] \quad \text{in } K(\mathcal{A})$$

then by definition, we impose that i, p are homotopy equivalences. This is too strong, since not every quasi-isomorphism is a homotopy equivalence.

Abstractly, let S be the set of quasi-isomorphisms in $K(\mathcal{A})$. The derived category

$$D(\mathcal{A}) = K(\mathcal{A})[S^{-1}]$$

is characterized by a universal property: there is a functor

$$Q : K(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

sending every $s \in S$ to an isomorphism, and universal with that property (any other functor inverting all quasi-isomorphisms factors uniquely through Q). One can also describe morphisms in $D(\mathcal{A})$ concretely as "roofs" via Verdier localization. The bounded derived category $D^b(\mathcal{A})$ is the full subcategory of complexes with bounded cohomology.

Definition 1.1 (Mapping cone)**.** For a chain map $s : X^\bullet \to I^\bullet$ (cohomological grading), the **mapping cone** $\mathrm{Cone}(s)$ is the complex

$$\mathrm{Cone}(s)^n = I^n \oplus X^{n+1}, \qquad d(b, a) = \big(d_I b + s(a), -d_X a\big).$$

There's a short exact sequence of complexes

$$0 \to I^\bullet \xrightarrow{\iota} \mathrm{Cone}(s) \xrightarrow{\pi} X^\bullet[1] \to 0,$$

giving rise to a long exact sequence in cohomology

$$\cdots \to H^n(I^\bullet) \xrightarrow{H^n(\iota)} H^n(\mathrm{Cone}(s)) \xrightarrow{H^n(\pi)} H^{n+1}(X^\bullet) \xrightarrow{H^{n+1}(s)} H^{n+1}(I^\bullet) \to \cdots$$

Proposition 1.2. *Let $s : X^\bullet \to I^\bullet$ be a chain map in $C(\mathcal{A})$. Then:*

(1) *s is a quasi-isomorphism if and only if* $\mathrm{Cone}(s)$ *is acyclic (all cohomology groups vanish).*

(2) *s is an isomorphism in $K(\mathcal{A})$ (i.e., a homotopy equivalence) if and only if* $\mathrm{Cone}(s)$ *is contractible (chain-homotopic to 0).*

Proof.

(1) This follows from a careful examination of the segments of the long exact sequence in cohomology.

$$H^n(I^\bullet) \to H^n(\mathrm{Cone}(s)) \to H^{n+1}(X^\bullet) \xrightarrow{H^{n+1}(s)} H^{n+1}(I^\bullet),$$

(2) If s has a homotopy inverse t (so $ts \simeq \mathrm{id}_X$, $st \simeq \mathrm{id}_I$), then the triangle

$$X^\bullet \xrightarrow{s} I^\bullet \to \mathrm{Cone}(s) \to X^\bullet[1]$$

is isomorphic (in K) to

$$X^\bullet \xrightarrow{\text{id}} X^\bullet \to \text{Cone}(\text{id}_X) \to X^\bullet[1].$$

For any complex $X^\bullet$, $\text{Cone}(\text{id}_X)$ is contractible with contracting homotopy

$$H^n : X^n \oplus X^{n+1} \longrightarrow X^{n-1} \oplus X^n, \qquad H^n(x,y) = (0,x).$$

One can check that $dH + Hd = \text{id}$. Thus $\text{Cone}(s)$ is contractible. □

Example 1.3. We will produce an example of an acyclic complex which is not contractible. Let $\mathcal{A} = \mathbf{Ab}$. Take the injective resolution of $\mathbb{Z}$:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{q} \mathbb{Q}/\mathbb{Z} \to 0$$

and regard $I^\bullet$ as $I^0 = \mathbb{Q}$, $I^1 = \mathbb{Q}/\mathbb{Z}$ with $d^0 = q$, and $X^\bullet = \mathbb{Z}[0]$. The resolution map $s : \mathbb{Z}[0] \to I^\bullet$ has $s^0 = i$. Computing from the definition, the mapping cone $\text{Cone}(s)$ has

$$\text{Cone}(s)^{-1} = \mathbb{Z}, \quad \text{Cone}(s)^0 = \mathbb{Q}, \quad \text{Cone}(s)^1 = \mathbb{Q}/\mathbb{Z}$$

with differentials $d^{-1} = i : \mathbb{Z} \to \mathbb{Q}$ and $d^0 = q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. So $\text{Cone}(s)$ is exactly the three-term complex sitting in degrees $-1, 0, 1$.

$$\mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{q} \mathbb{Q}/\mathbb{Z}$$

The cone is acyclic but not contractible. Indeed, the cone is acyclic since it is the cone on a short exact sequence. On the other hand, the contractibility of this 3-term exact complex is equivalent to the short exact sequence splitting (a contracting homotopy gives splittings and vice versa). But $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ does not split: if it did, $\mathbb{Z}$ would be a direct summand of the divisible group $\mathbb{Q}$, hence divisible itself, which is false.

Definition 1.4 (Triangulated category)**.** A **triangulated category** is an additive category $\mathcal{T}$ equipped with an autoequivalence $[1] : \mathcal{T} \to \mathcal{T}$ (the shift functor) and a class of distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

satisfying the following axioms:

- (TR1) For every morphism $f : X \to Y$ in $\mathcal{T}$, there exists a distinguished triangle

  $$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1].$$

  Moreover, for every object $X \in \mathcal{T}$, the triangle

  $$X \xrightarrow{\text{id}_X} X \longrightarrow 0 \longrightarrow X[1]$$

  is distinguished, and any triangle isomorphic to a distinguished triangle is distinguished.

- (TR2) A triangle
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
is distinguished if and only if the rotated triangle
$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$
is distinguished.

- (TR3) Given two distinguished triangles
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
and
$$U \xrightarrow{p} V \xrightarrow{q} W \xrightarrow{r} U[1],$$
and morphisms $a : X \to U$, $b : Y \to V$ such that $b \circ f = p \circ a$, there exists a morphism $c : Z \to W$ making the following diagram commute:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"'] & Z \arrow[r, "h"] \arrow[d, "c"'] & X[1] \arrow[d, "a[1]"] \\ U \arrow[r, "p"'] & V \arrow[r, "q"'] & W \arrow[r, "r"'] & U[1] \end{tikzcd}$$

- (TR4) (Octahedral axiom) Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{T}$, there exist distinguished triangles
$$X \xrightarrow{f} Y \xrightarrow{u} C(f) \xrightarrow{v} X[1],$$
$$Y \xrightarrow{g} Z \xrightarrow{u'} C(g) \xrightarrow{v'} Y[1],$$
and
$$X \xrightarrow{g\circ f} Z \xrightarrow{u"} C(g \circ f) \xrightarrow{v"} X[1],$$
along with morphisms $C(f) \xrightarrow{w} C(g\circ f)$ and $C(g) \xrightarrow{w'} C(g\circ f)$ such that the following diagram commutes and the rows and columns are distinguished triangles:
$$\begin{tikzcd} & Y \arrow[r, "u"] \arrow[d, "g"'] & C(f) \arrow[d, "w"] \\ X \arrow[ur, "f"] \arrow[r, "g\circ f"'] & Z \arrow[r, "u'"'] & C(g) \end{tikzcd}$$

Proposition 1.5. *This construction gives $D(\mathcal{A})$ the structure of a triangulated category, where:*

- *The shift functor* $[1]$ *moves complexes one place to the left:*
$$X^\bullet[1]^n = X^{n+1}, \quad d^n_{X[1]} = -d^{n+1}_X$$

- *Distinguished triangles come from mapping cones of chain maps, in particular, for any chain map $f : X^\bullet \to Y^\bullet$, the triangle*
$$X^\bullet \xrightarrow{f} Y^\bullet \to \mathrm{Cone}(f) \to X^\bullet[1]$$
is distinguished

- *The cohomology functors are first defined on the homotopy category as functors*
$$H^i_K : K(\mathcal{A}) \to \mathcal{A}$$
Since these functors send quasi-isomorphisms to isomorphisms, they descend through the localization map $Q : K(\mathcal{A}) \to D(\mathcal{A})$. In particular, there exists a unique functor
$$H^i_D : D(\mathcal{A}) \to \mathcal{A}$$
such that $H^i_K = H^i_D \circ Q$.

[1] work with more relaxed categories known as graded categories. In particular every triangulated category is a graded category.

Definition 1.6 (Graded categories and exact functors)**.** A **graded category** is a pair $(\mathcal{D}, T_{\mathcal{D}})$ consisting of a category $\mathcal{D}$ and a fixed autoequivalence
$$T_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathcal{D},$$
called the **translation functor**. A functor
$$F : \mathcal{D} \longrightarrow \mathcal{D}'$$
between graded categories is called **graded** if it commutes with the translation functors. More precisely, there is a fixed natural isomorphism of functors
$$t_F : F \circ T_{\mathcal{D}} \xrightarrow{\sim} T_{\mathcal{D}'} \circ F.$$
A natural transformation $\mu : F \Rightarrow G$ between graded functors is called **graded** if the following diagram commutes:
$$\begin{array}{ccc} F \circ T & \xrightarrow{t_F} & T \circ F \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle T\mu} \\ G \circ T & \xrightarrow[t_G]{} & T \circ G. \end{array}$$

A graded functor
$$F : \mathcal{D} \longrightarrow \mathcal{D}'$$
between triangulated categories is called **exact** if it sends exact triangles to exact triangles in the following sense.

If
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$$
is an exact triangle in $\mathcal{D}$, then one replaces the segment
$$FT(X)$$

by

$$TF(X)$$

via the natural isomorphism $t_F : FT \xrightarrow{\sim} TF$, and requires that the resulting sequence

$$FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{t_F(Fh)} TFX$$

be an exact triangle in $\mathcal{D}'$. We call a morphism between graded exact functors a **graded natural transformation**.

Proposition 1.7. *Let $F : \mathcal{D} \longrightarrow \mathcal{D}'$ be a graded functor between graded categories, and let $G : \mathcal{D}' \longrightarrow \mathcal{D}$ be its left adjoint, so that the unit and counit of the adjunction are the natural transformations*

$$\mathrm{id}_{\mathcal{D}'} \xrightarrow{\alpha} F \circ G, \qquad G \circ F \xrightarrow{\beta} \mathrm{id}_{\mathcal{D}}\,.$$

Then G can be canonically endowed with the structure of a graded functor, so that the unit and counit of the adjunction become morphisms of graded functors. If, in addition, F is an exact functor between triangulated categories, then G also becomes an exact functor.

Definition 1.8. Let $\mathcal{D}$ be a k-linear category with finite-dimensional Hom's. A covariant additive functor

$$S : \mathcal{D} \longrightarrow \mathcal{D}$$

is called a **Serre functor** if it is a category equivalence and there are given bifunctorial isomorphisms

$$\varphi_{A,B} : \mathrm{Hom}_{\mathcal{D}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(B, SA)^*$$

for all $A, B \in \mathcal{D}$, such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(A, B) & \xrightarrow{\varphi^{A,B}} & \mathrm{Hom}_{\mathcal{D}}(B, SA)^* \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}}(SA, SB) & \xrightarrow[\varphi^{SA,SB}]{} & \mathrm{Hom}_{\mathcal{D}}(SB, S^2A)^* \end{array}$$

The vertical isomorphisms in this diagram are those induced by S.

A Serre functor in a category $\mathcal{D}$, if it exists, is unique up to a graded natural isomorphism. Serre functors are also natural in the following sense.

Proposition 1.9. *Any autoequivalence*

$$\Phi : \mathcal{D} \longrightarrow \mathcal{D}$$

commutes with a Serre functor, i.e. there exists a natural graded isomorphism of functors

$$\Phi \circ S \xrightarrow{\sim} S \circ \Phi.$$

Proof. For any pair of objects $A, B \in \mathcal{D}$, we have a system of natural isomorphisms:

$$\begin{aligned} \operatorname{Hom}(\Phi A, \Phi SB) &\cong \operatorname{Hom}(A, SB) \\ &\cong \operatorname{Hom}(B, A)^* \\ &\cong \operatorname{Hom}(\Phi B, \Phi A)^* \\ &\cong \operatorname{Hom}(\Phi A, S\Phi B). \end{aligned}$$

Since Φ is an equivalence, the essential image of Φ covers all of $\mathcal{D}$; that is, every object is isomorphic to some ΦA. Hence we have isomorphisms of contravariant functors represented by the objects ΦSB and $S\Phi B$. Morphisms between representable functors correspond bijectively to morphisms between their representing objects. This yields a canonical isomorphism

$$\Phi SB \xrightarrow{\sim} S\Phi B,$$

which is in fact natural in B. □

Finally we recall an important computation tool in derived categories: spectral sequences arising from filtered complexes. General spectral sequence theory for filtered complexes says if $(K^\bullet, F^\bullet)$ is a filtered complex in an abelian category (or more generally in a suitable derived context), there is a spectral sequence

$$E_1^{p,q} = H^{p+q}\big(\operatorname{Gr}_F^p K^\bullet\big) \Longrightarrow H^{p+q}(K^\bullet)$$

Here the associated graded pieces are the complexes $\operatorname{Gr}_F^p K^\bullet = F^p K^\bullet / F^{p-1} K^\bullet$ obtained by taking successive quotients of the filtration. The differentials in the spectral sequence come from the differentials in the original complex $K^\bullet$ and from the filtration structure. We refer to [3] for more details and a precise discussion of the following proposition.

Proposition 1.10. *Let $\mathcal{A}$ be an abelian category, and let $P^\bullet \in D^b(\mathcal{A})$ be a bounded complex. There is a convergent spectral sequence with E_1–page*

$$E_1^{p,q} \cong \operatorname{Ext}^q_{\mathcal{A}}\big(\mathcal{H}^p(P^\bullet), P^\bullet\big)$$

and E_2–page

$$E_2^{p,q} \cong \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^p_{\mathcal{A}}\big(\mathcal{H}^i(P^\bullet), \mathcal{H}^{i+q}(P^\bullet)\big)$$

converging to $\operatorname{Hom}^{p+q}(P^\bullet, P^\bullet)$.

2. Point objects and invertible objects

Let X be a smooth projective variety over a field k with either ample or antiample canonical sheaf ω_X. Let $n = \dim X$ and $\mathcal{D} = D^b_{\mathrm{coh}}(X)$ be the bounded derived category of coherent sheaves on X.

Proposition 2.1. *$\mathcal{D}$ has a Serre functor S given by*

$$S(-) = - \otimes \omega_X[n]$$

Proof. Grothendieck-Serre duality gives bifunctorial isomorphisms

$$\mathrm{Ext}^i_X(F,G) \cong \mathrm{Ext}^{n-i}_X(G, F \otimes \omega_X)^*$$

for all coherent sheaves F, G on X. This extends to complexes in $\mathcal{D}$ by taking injective resolutions. Thus S is a Serre functor. □

Definition 2.2 (Point object)**.** An object $P \in \mathcal{D}$ is called a **point object of codimension** $n(P)$ if

(1) $S_{\mathcal{D}}(P) \simeq P[n(P)]$,

(2) $\mathrm{Hom}^{<0}(P,P) = 0$,

(3) $\mathrm{Hom}^0(P,P) = k(P)$,

where $k(P)$ is a field, necessarily a finite extension of the base field k.

Lemma 2.3. *If $\mathcal{F}$ is a coherent sheaf on a projective variety X such that $\mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}$ for an ample line bundle $\mathcal{L}$, then $\mathcal{F}$ is supported at finitely many points.*

Proof. Examining the Hilbert polynomial of $\mathcal{H}^i \otimes \omega_X^{\otimes m}$ for $m \gg 0$ shows that the dimension of the support of $\mathcal{F}$ must be zero. □

Proposition 2.4. *Let X be a smooth algebraic variety of dimension n with ample canonical or anticanonical sheaf. Then an object $P \in D^b_{\mathrm{coh}}(X)$ is a point object if and only if*

$$P \cong \mathcal{O}_x[r], \qquad r \in \mathbb{Z},$$

where $\mathcal{O}_x$ is the skyscraper sheaf of a closed point $x \in X$ (up to translation).

Proof. Since X has an ample invertible sheaf, it is projective. Any skyscraper sheaf of a closed point obviously satisfies the conditions of a point object with codimension equal to the dimension of the variety.

Suppose now that for some $P \in D^b_{\mathrm{coh}}(X)$ we have that P is a point object of codimension s. Let $\mathcal{H}^i$ be the cohomology sheaves of P.

From (i) we obtain $s = n$. From the Serre functor formula, we have

$$P \otimes \omega_X[n] \simeq P[s]$$

Because tensoring with an invertible sheaf is an exact functor on the abelian category of coherent sheaves, we can take cohomlogy sheaves

$$\mathcal{H}^i(P \otimes \omega_X) \cong \mathcal{H}^i(P) \otimes \omega_X \cong \mathcal{H}^{i+t}(P)$$

If $t = s - n \neq 0$, then for any i we can iterate this isomorphism to get that infinitely many $\mathcal{H}^j(P)$ are nonzero, contradicting the boundedness of P. Thus $t = 0$.

We also get that $\mathcal{H}^i \otimes \omega_X \cong \mathcal{H}^i$. Since ω_X is either ample or antiample, it follows from Lemma 2.3 that each $\mathcal{H}^i$ is a finite-length sheaf, i.e. its support consists of isolated points.

Sheaves supported at different points are homologically orthogonal in the sense that if $\mathcal{F}, \mathcal{G}$ are coherent sheaves with disjoint supports, then

$$\mathrm{Ext}^p_X(\mathcal{F}, \mathcal{G}) = 0$$

for all p. This is because Ext groups are computed locally, i.e. for every open $U \subset X$,

$$\mathcal{E}xt^p_X(\mathcal{F}, \mathcal{G})|_U \cong \mathcal{E}xt^p_U(\mathcal{F}|_U, \mathcal{G}|_U),$$

and the support of $\mathcal{E}xt^p_X(\mathcal{F}, \mathcal{G})$ is contained in $\mathrm{Supp}(\mathcal{F}) \cap \mathrm{Supp}(\mathcal{G})$. Thus P decomposes into a direct sum of components supported at single points.

By (iii), P is indecomposable. In particular, if $P = P_1 \oplus P_2$ with P_1, P_2 supported at different points, then $\mathrm{End}(P)$ would contain nontrivial idempotents, contradicting (iii).

Applying Proposition 1.10 gives us a spectral sequence coming from the stupid filtration on P computing self-Exts of P from Exts between its cohomology sheaves:

$$E_2^{p,q} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^p\big(\mathcal{H}^i,\ \mathcal{H}^{i+q}\big) \Longrightarrow \mathrm{Hom}^{p+q}(P, P).$$

If two cohomology sheaves are nonzero, a negative-degree class appears. Assume for contradiction that $\mathcal{H}^i$ and $\mathcal{H}^j$ are both nonzero for some $i < j$. Since all $\mathcal{H}^k$ are supported at the same closed point, the sheaves $\mathcal{H}^i$ and $\mathcal{H}^j$ are finite–length $\mathcal{O}_{X,x}$–modules. For such modules it is standard that

$$\mathrm{Hom}(\mathcal{H}^j, \mathcal{H}^i) \neq 0,$$

because any nonzero finite–length module possesses a simple quotient, and any nonzero finite–length module contains a copy of that simple module.

Such a map $\phi : \mathcal{H}^j \to \mathcal{H}^i$ determines a nonzero class

$$0 \neq [\phi] \in E_2^{0,i-j}$$

where $i - j < 0$. Among all nonzero classes in $E_2^{0,q}$ with $q < 0$, choose one with q_0 *minimal*. We will show that this class cannot be killed by any differential. The possible outgoing differentials from E_r^{0,q_0} have targets

$$E_r^{r,q_0-r+1}, \qquad r \geq 2$$

But $q_0 - r + 1 < q_0$, and by minimality of q_0 there are *no* nonzero entries with $q < q_0$ at the E_2–page, hence none at any later page. Therefore all outgoing differentials vanish.

The possible incoming differentials come from

$$E_r^{-r,q_0+r-1}$$

but $p = -r < 0$ forces $\mathrm{Ext}^p(-,-) = 0$, so these groups are always zero. Thus there are no incoming differentials either. Hence the class $[\phi]$ survives to the limit:

$$0 \neq [\phi] \in E_\infty^{0,q_0}$$

Since the spectral sequence abuts to $\mathrm{Hom}^m(P, P)$ with $m = p + q$, our surviving class contributes

$$0 \neq [\phi] \in \mathrm{Hom}^{\,q_0}(P, P)$$

But $q_0 < 0$, contradicting the assumption that $\mathrm{Hom}^m(P,P) = 0$ for all negative m. Thus it is impossible for two distinct cohomology sheaves $\mathcal{H}^i$ and $\mathcal{H}^j$ to be nonzero and so P has a single nonzero cohomology sheaf:

$$P \simeq \mathcal{H}^r(P)[-r]$$

Since $\mathrm{End}(P) = \mathrm{End}(\mathcal{H}^r)$ is a field, the sheaf $\mathcal{H}^r$ must be an indecomposable finite–length $\mathcal{O}_{X,x}$–module whose endomorphism ring has no nontrivial idempotents. The only such modules are the simple ones. Thus $\mathcal{H}^r \cong k(x)$ is a skyscraper sheaf at a closed point. $\square$

Definition 2.5 (Invertible object)**.** An object $L \in \mathcal{D}$ is called *invertible* if for any point object $P \in \mathcal{D}$ there exists an $s \in \mathbb{Z}$ such that

(i) $\mathrm{Hom}^s(L,P) = k(P)$,

(ii) $\mathrm{Hom}^i(L,P) = 0$ for $i \neq s$.

Proposition 2.6. *Let X be a smooth irreducible algebraic variety. Assume that all point objects have the form $\mathcal{O}_x[s]$ for some $x \in X$, $s \in \mathbb{Z}$. Then an object $L \in \mathcal{D}$ is invertible if and only if $L \simeq \mathcal{L}[t]$ for some invertible sheaf $\mathcal{L}$ on X and some $t \in \mathbb{Z}$.*

Proof. For an invertible sheaf $\mathcal{L}$ we have

$$\mathrm{Hom}(\mathcal{L}, \mathcal{O}_x) = k(x), \qquad \mathrm{Ext}^i(\mathcal{L}, \mathcal{O}_x) = 0, \quad \text{if } i \neq 0.$$

Therefore, if $L = \mathcal{L}[s]$, then it is an invertible object. Now suppose L is an invertible object in $D^b(X)$ and let m be maximal such that $\mathcal{H}^m := \mathcal{H}^m(L) \neq 0$.

From the truncation triangle

$$\tau_{\le m-1}L \longrightarrow L \longrightarrow \mathcal{H}^m[-m]$$

and the assumption that m is maximal with $\mathcal{H}^m(L) \neq 0$, one knows that $\tau_{\le m-1}L$ has cohomology only in degrees $< m$. Thus applying $\mathrm{Hom}(-, \mathcal{O}_{x_0})$ shows that $\mathrm{Hom}(\tau_{\le m-1}L, k(x_0)[t]) = 0$ for $t \ge -m$ and in particular the map $L \longrightarrow \mathcal{H}^m[-m]$ induces isomorphisms on all $\mathrm{Hom}(-, k(x_0)[t])$ for $t \ge -m$.

Pick a point $x_0 \in \mathrm{supp}(\mathcal{H}^m)$. Then there exists a nontrivial homomorphism

$$\mathcal{H}^m \longrightarrow k(x_0).$$

This is because the stalk $M := \mathcal{H}^m_{x_0}$ is a nonzero finitely generated $\mathcal{O}_{X,x_0}$–module. Let $R := \mathcal{O}_{X,x_0}$, with maximal ideal $\mathfrak{m}_{x_0}$ and residue field $k(x_0) = R/\mathfrak{m}_{x_0}$. By Nakayama, $M/\mathfrak{m}_{x_0}M \neq 0$, so $M/\mathfrak{m}_{x_0}M$ is a nonzero finite–dimensional $k(x_0)$–vector space. Choose a nonzero $k(x_0)$–linear functional

$$\ell : M/\mathfrak{m}_{x_0}M \to k(x_0),$$

and compose with the natural surjection $M \twoheadrightarrow M/\mathfrak{m}_{x_0}M$ to obtain a nonzero R–linear map $M \to k(x_0)$. Using the identification

$$\mathrm{Hom}_X(\mathcal{H}^m, k(x_0)) \cong \mathrm{Hom}_R(M, k(x_0)),$$

this gives a nontrivial homomorphism of sheaves $\mathcal{H}^m \to k(x_0)$.

Hence

$$0 \neq \mathrm{Hom}\big(\mathcal{H}^m, k(x_0)\big) = \mathrm{Hom}\big(L, k(x_0)[-m]\big),$$

and the nonvanishing of this group forces the codimension of this point object $n_{k(x_0)} = -m$. Apply the same spectral sequence (Proposition 1.10) to deduce

$$E_2^{1,-m} = \mathrm{Hom}(\mathcal{H}^m, k(x_0)[1]) = \mathrm{Hom}\big(L, k(x_0)[1 + n_{k(x_0)}]\big) = 0.$$

Thus, as soon as $x_0 \in X$ is in the support of $\mathcal{H}^m$, we obtain

$$\mathrm{Ext}^1(\mathcal{H}^m, k(x_0)) = 0.$$

Next, we shall apply the following standard result in commutative algebra: Any finite module M over an arbitrary noetherian local ring $(A, \mathfrak{m})$ with $\mathrm{Ext}^1_A(M, A/\mathfrak{m}) = 0$ is free.

The local-to-global spectral sequence

$$E_2^{p,q} = H^p\left(X, \mathcal{E}xt^q(\mathcal{H}^m, k(x_0))\right) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{H}^m, k(x_0))$$

allows us to pass from the global vanishing $\mathrm{Ext}^1(\mathcal{H}^m, k(x_0)) = 0$ to the local one $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0)) = 0$. More precisely, as $\mathcal{E}xt^0(\mathcal{H}^m, k(x_0))$ is concentrated at $x_0 \in X$, one has

$$E_2^{2,0} = H^2\left(X, \mathcal{E}xt^0(\mathcal{H}^m, k(x_0))\right) = 0.$$

since sheaves with zero-dimensional support have vanishing higher cohomology. Hence, there are no nontrivial differentials and so

$$E_2^{0,1} = E_\infty^{0,1}$$

Moreover, since $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0))$ is also concentrated at $x_0 \in X$, it is a globally generated sheaf because it is precisely the data of its stalk at x_0. Hence,

$$H^0\left(X, \mathcal{E}xt^1(\mathcal{H}^m, k(x_0))\right) = E_2^{0,1} = 0$$

implies $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0)) = 0$. But then the aforementioned result from commutative algebra shows that $\mathcal{H}^m$ is free in a neighbourhood of $x_0 \in X$.

Since X is irreducible, we have in particular $\mathrm{supp}(\mathcal{H}^m) = X$. Thereby, there exists for any $x \in X$ a surjection $\mathcal{H}^m \twoheadrightarrow k(x)$. Hence,

$$\mathrm{Hom}(L, k(x)[-m]) = \mathrm{Hom}(\mathcal{H}^m, k(x)) \neq 0.$$

In particular, $n_{k(x)}$ does not depend on x. As by assumption,

$$k(x) = \mathrm{Hom}\big(L, k(x)[-m]\big) = \mathrm{Hom}(\mathcal{H}^m, k(x)),$$

the sheaf $\mathcal{H}^m$ has constant fibre dimension one. Hence $\mathcal{H}^m$ is a line bundle. $\square$

Definition 2.7. We say a set Ω is a **spanning class** if for any $E \in D^b(Y)$,

(1) if $\mathrm{Hom}(A, E[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $E = 0$;

(2) if $\mathrm{Hom}(E[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $E = 0$.

Lemma 2.8. *Let Y be a smooth projective variety over a field k. Then the set*

$$\Omega = \{k(y)[m] \mid y \in Y \ \text{closed point}, m \in \mathbb{Z}\}$$

is a spanning class for $D^b_{\mathrm{coh}}(Y)$.

Proof. Assume $\operatorname{Hom}\big(k(y)[m], E\big) = 0$ for all y, m. Let i be minimal such that $\mathcal{H}^i(E) \neq 0$ (if no such i exists, then the natural map $E \to 0$ is an isomorphism). Choose a closed point $y \in \operatorname{Supp} \mathcal{H}^i(E)$.

For coherent sheaves there is a standard identification

$$\operatorname{Hom}_Y(k(y), \mathcal{H}^i(E)) \cong \operatorname{Hom}_{\mathcal{O}_{Y,y}}\big(k(y), \mathcal{H}^i(E)_y\big).$$

Since $\mathcal{H}^i(E)_y \neq 0$ over the local ring $\mathcal{O}_{Y,y}$, the simple module $k(y)$ occurs as a quotient of some submodule, so $\operatorname{Hom}_Y(k(y), \mathcal{H}^i(E)) \neq 0$. Now use the natural map $\mathcal{H}^i(E)[-i] \to E$: composing $k(y)[-i] \longrightarrow \mathcal{H}^i(E)[-i] \longrightarrow E$ gives a nonzero element of $\operatorname{Hom}(k(y)[-i], E)$, contradicting the assumption. Hence no such i exists and $E = 0$.

Assume $\operatorname{Hom}\big(E, k(y)[m]\big) = 0$ for all y, m. Let i be maximal such that $\mathcal{H}^i(E) \neq 0$. Consider the truncation triangle

$$\tau_{<i} E \longrightarrow E \longrightarrow \mathcal{H}^i(E)[-i] \xrightarrow{+1} .$$

Apply $\operatorname{Hom}(-, k(y)[m])$. Using the long exact sequence of Hom's and the hypothesis, we get

$$\operatorname{Hom}\big(\mathcal{H}^i(E)[-i], k(y)[m]\big) = 0$$

for all y, m. Taking $m = i$, we have

$$\operatorname{Hom}\big(\mathcal{H}^i(E), k(y)\big) = 0 \quad \text{for all } y.$$

Recall that if F is a coherent sheaf with $\operatorname{Hom}(F, k(y)) = 0$ for all closed y, then $F = 0$.

To see this, suppose $F \neq 0$ and choose y in $\operatorname{Supp} F$. Then $F_y \neq 0$ as an $\mathcal{O}_{Y,y}$-module. Since $\mathcal{O}_{Y,y}$ is local Noetherian, there is a surjection $F_y \twoheadrightarrow k(y)$, which corresponds exactly to a nonzero morphism $F \to k(y)$, a contradiction. Applying this to $F = \mathcal{H}^i(E)$, we conclude $\mathcal{H}^i(E) = 0$, contradicting the choice of i. Hence all cohomology sheaves vanish and $E = 0$. □

3. The reconstruction theorem

We are now ready to state and prove the reconstruction theorem.

Theorem 3.1 (Reconstruction theorem [1])**.** *Let X and Y be smooth projective varieties over a field k with either ample or antiample canonical sheaf. If there is an exact equivalence of triangulated categories*

$$D^b_{\mathrm{coh}}(X) \xrightarrow{\sim} D^b_{\mathrm{coh}}(Y),$$

then X is isomorphic to Y.

Proof. Assume that under an equivalence

$$F : D^b(X) \xrightarrow{\sim} D^b(Y)$$

the structure sheaf $\mathcal{O}_X$ is mapped to $\mathcal{O}_Y$. Since any equivalence is compatible with Serre functors and $\dim(X) = \dim(Y) =: n$, this proves

$$F(\omega_X^k) = F\big(S_X^k(\mathcal{O}_X)[-kn]\big) \simeq S_Y^k\big(F(\mathcal{O}_X)\big)[-kn] \simeq S_Y^k(\mathcal{O}_Y)[-kn] = \omega_Y^k.$$

Using that F is fully faithful, we conclude from this that

$$H^0(X,\omega_X^k) = \mathrm{Hom}(\mathcal{O}_X,\omega_X^k) \simeq \mathrm{Hom}(F(\mathcal{O}_X),F(\omega_X^k)) = \mathrm{Hom}(\mathcal{O}_Y,\omega_Y^k) = H^0(Y,\omega_Y^k)$$

for all k.

Write the product in $\bigoplus H^0(X,\omega_X^k)$ as follows: for $s_i \in \mathrm{Hom}(\mathcal{O}_X,\omega_X^{k_i})$ one has

$$s_1 \cdot s_2 = S_X^{k_1}\big(s_2\big)[-k_1 n] \circ s_1$$

and similarly for sections on Y. Hence, the induced bijection

$$\bigoplus_k H^0(X,\omega_X^k) \simeq \bigoplus_k H^0(Y,\omega_Y^k)$$

is a ring isomorphism. If the (anti-)canonical bundle of Y is also ample, then this shows

$$X \simeq \mathrm{Proj}\left(\bigoplus\nolimits_k H^0(X,\omega_X^k)\right) \simeq \mathrm{Proj}\left(\bigoplus\nolimits_k H^0(Y,\omega_Y^k)\right) \simeq Y.$$

Thus, under the two assumptions that $F(\mathcal{O}_X) \simeq \mathcal{O}_Y$ and that ω_Y (or ω_Y^*) is ample, we have proved the assertion.

We now explain how to reduce to this situation. As the notions of pointlike and invertible objects in D^b are intrinsic, an exact equivalence

$$F : D^b(X) \longrightarrow D^b(Y)$$

induces bijections

$$\begin{array}{ccc} \{\text{pointlike objects in } D^b(X)\} & \overset{(*)}{\longleftrightarrow} & \{\text{pointlike objects in } D^b(Y)\} \\ \| & & \uparrow \\ \{\, k(x)[m] \mid x \in X,\, m \in \mathbb{Z}\,\} & & \{\, k(y)[m] \mid y \in Y,\, m \in \mathbb{Z}\,\} \end{array}$$

and

$$\begin{array}{ccc} \{\text{invertible objects in } D^b(X)\} & \overset{(**)}{\longleftrightarrow} & \{\text{invertible objects in } D^b(Y)\} \\ \| & & \downarrow \\ \{\, L[m] \mid L \in \mathrm{Pic}(X)\,\} & & \{\, M[m] \mid M \in \mathrm{Pic}(Y)\,\}. \end{array}$$

The pointlike objects in $D^b(X)$ are all of the form $k(x)[m]$ for $x \in X$ a closed point and $m \in \mathbb{Z}$. Any line bundle L, in particular $L = \mathcal{O}_X$, defines an invertible object in $D^b(X)$. Thus, by $(**)$ also $F(\mathcal{O}_X)$ is an invertible object in $D^b(Y)$ and hence of the form $M[m]$ for some line bundle M on Y.

Compose F with the two equivalences given by $M^* \otimes (\,)$ and then $[-m]$ to obtain a new equivalence, which we also call F. It satisfies

$$F(\mathcal{O}_X) \simeq \mathcal{O}_Y.$$

In order to prove the ampleness of the (anti-)canonical bundle ω_Y, we shall first prove that point like objects in $D^b(Y)$ are of the form $k(y)[m]$. We will conclude this, without assuming any positivity of ω_Y, simply from the existence of the equivalence F.

Due to $(*)$, one finds for any closed point $y \in Y$ a closed point $x_y \in X$ and an integer m_y such that
$$k(y) \simeq F\big(k(x_y)[m_y]\big).$$

Suppose there exists a point like object $P \in D^b(Y)$ which is not of the form $k(y)[m]$ and denote by $x_P \in X$ the closed point with
$$F\big(k(x_P)[m_P]\big) \simeq P$$
for a certain $m_P \in \mathbb{Z}$. Note that $x_P \neq x_y$ for all $y \in Y$. Hence we have for all $y \in Y$ and all $m \in \mathbb{Z}$
$$\begin{aligned} \operatorname{Hom}\big(P, k(y)[m]\big) &= \operatorname{Hom}\big(F(k(x_P))[m_P],\ F(k(x_y))[m_y + m]\big) \\ &= \operatorname{Hom}\big(k(x_P), k(x_y)[m_y + m - m_P]\big) \\ &= 0. \end{aligned}$$
This implies that $P \simeq 0$ because the objects $k(y)[m]$ form a spanning class in $D^b(Y)$ by 2.8. This is a contradiction so point like objects in $D^b(Y)$ are exactly the objects of the form $k(y)[m]$.

Note that together with $F(\mathcal{O}_X) \simeq \mathcal{O}_Y$ this also implies that for any closed point $x \in X$ there exists a closed point $y \in Y$ such that $F(k(x)) \simeq k(y)$. This is because in $D^b(Y)$, for any complex E, we have
$$\operatorname{Hom}_{D^b(Y)}(\mathcal{O}_Y, E[m]) \cong H^m(Y, E),$$
where the right-hand side denotes the m-th sheaf cohomology group of E. This follows from the fact that $\operatorname{Hom}(\mathcal{O}_Y, E) = \Gamma(E)$.

Now $k(y)$ is a skyscraper sheaf at a single closed point. Thus its sheaf cohomology is
$$\operatorname{Hom}(\mathcal{O}_Y, k(y)[m]) \cong H^m(Y, k(y)) = \begin{cases} k & m = 0, \\ 0 & m \neq 0. \end{cases}$$
This gives us
$$\operatorname{Hom}(\mathcal{O}_Y, k(y)[m]) \neq 0 \quad \Longleftrightarrow \quad m = 0.$$

From the point-object discussion above, we already know that for each closed point $x \in X$ there exist a closed point $y \in Y$ and an integer m such that $F(k(x)) \simeq k(y)[m]$.

Now assume additionally that $F(\mathcal{O}_X) \simeq \mathcal{O}_Y$. Because F is an equivalence, it preserves Hom-spaces. In particular, for each x, we have
$$\operatorname{Hom}(\mathcal{O}_X, k(x)) \cong \operatorname{Hom}(F(\mathcal{O}_X), F(k(x))) \cong \operatorname{Hom}(\mathcal{O}_Y, k(y)[m]).$$

The left-hand side is clearly nonzero: there is a nonzero surjective map $\mathcal{O}_X \twoheadrightarrow k(x)$ obtained by taking the quotient by the maximal ideal at x. Therefore the right-hand side is also nonzero:
$$\operatorname{Hom}(\mathcal{O}_Y, k(y)[m]) \neq 0.$$
By the computation above, this can only happen if $m = 0$.

Now we will show that some power ω_Y^k separates points and tangents and thus ω_Y is ample. We continue to use that for any $k(y)$, with $y \in Y$ a closed point, there exists a closed point $x_y \in X$ with $F(k(x_y)) = k(y)$ and that $F(\omega_X^k) = \omega_Y^k$ for all $k \in \mathbb{Z}$. The line bundle ω_Y^k separates points if for any two points $y_1 \neq y_2 \in Y$ the restriction map

$$r_{y_1,y_2} : \omega_Y^k \longrightarrow \omega_Y^k(y_1) \oplus \omega_Y^k(y_2) \simeq k(y_1) \oplus k(y_2)$$

induces a surjection

$$H^0(r_{y_1,y_2}) : H^0(Y, \omega_Y^k) \longrightarrow H^0\big(k(y_1) \oplus k(y_2)\big).$$

Let us denote $x_i := x_{y_i}$, $i = 1, 2$. Then

$$\begin{aligned} r_{y_1,y_2} &\in \mathrm{Hom}\big(\omega_Y^k,\, k(y_1) \oplus k(y_2)\big) \\ &\simeq \mathrm{Hom}\big(F(\omega_X^k),\, F(k(x_1) \oplus k(x_2))\big) \\ &\simeq \mathrm{Hom}\big(\omega_X^k,\, k(x_1) \oplus k(x_2)\big). \end{aligned}$$

It indeed corresponds to the restriction map

$$r_{x_1,x_2} : \omega_X^k \longrightarrow k(x_1) \oplus k(x_2)$$

as there is only one non-trivial homomorphism $\omega_X^k \to k(x_i)$ up to scaling. Altogether this yields the commutative diagram:

$$\begin{tikzcd} H^0(Y, \omega_Y^k) \arrow[r, "H^0(r_{y_1,y_2})"] \arrow[d, equal] & H^0\big(Y,\, k(y_1) \oplus k(y_2)\big) \arrow[d, equal] \\ \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^k) \arrow[r, "r^0_{y_1,y_2}"] \arrow[d, equal] & \mathrm{Hom}\big(\mathcal{O}_Y,\, k(y_1) \oplus k(y_2)\big) \arrow[d, equal] \\ \mathrm{Hom}(\mathcal{O}_X, \omega_X^k) \arrow[r, "r^0_{x_1,x_2}"] \arrow[d, equal] & \mathrm{Hom}\big(\mathcal{O}_X,\, k(x_1) \oplus k(x_2)\big) \arrow[d, equal] \\ H^0(X, \omega_X^k) \arrow[r, "H^0(r_{x_1,x_2})"] & H^0\big(X,\, k(x_1) \oplus k(x_2)\big). \end{tikzcd}$$

As, by assumption, the line bundle ω_X^k is very ample for $k \gg 0$ (or $k \ll 0$) and, in particular, separates points, the map

$$H^0(r_{x_1,x_2})$$

is surjective. The commutativity of the diagram allows us to conclude that also $H^0(r_{y_1,y_2})$ is surjective.

One proceeds in a similar fashion to prove that ω_Y^k separates tangent directions if ω_X^k does. Thus, we have proved that ω_Y (or ω_Y^*) is ample and this completes the proof of the theorem. $\square$

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