# Homework 8

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Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and choose a "compatible" system of $p$-power roots $p^{1/p}, p^{1/p^2}, \ldots$ in $\overline{\mathbb{Q}}_p$ such that $(p^{1/p^{i+1}})^p = p^{1/p^i}$ for all $i$. Similarly, choose a compatible system of $p$-power roots of unity $\zeta_p, \zeta_{p^2}, \ldots$ such that $(\zeta_{p^{i+1}})^p = \zeta_{p^i}$ for all $i$. Define

$$\mathbb{Q}_p(p^{1/p^\infty}) = \bigcup_{i \geq 1} \mathbb{Q}_p(p^{1/p^i}), \qquad \mathbb{Q}_p(\zeta_{p^\infty}) := \bigcup_{i \geq 1} \mathbb{Q}_p(\zeta_{p^i}).$$

**[Fact]** (you're encouraged to verify, but you're welcome to take for granted): These are infinite extensions of $\mathbb{Q}_p$ which are totally ramified in that every finite subextension is totally ramified, or equivalently in that the residue fields of the two extensions are still $\mathbb{F}_p$. Their completions, denoted by $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$, are perfectoid fields.

On the other hand, the valuation on the CDVF $\mathbb{F}_p((t))$ extends uniquely to

$$\mathbb{F}_p\big((t^{1/p^\infty})\big) = \bigcup_{i \geq 1} \mathbb{F}_p\big((t^{1/p^i})\big),$$

where $t^{1/p}, t^{1/p^2}, \ldots$ form a compatible system of $p$-power roots of $t$. (Another way to think of $\mathbb{F}_p((t^{1/p^\infty}))$ is that it is the perfection of $\mathbb{F}_p((t))$ in the same way $\mathbb{F}_p[X^{p^{-\infty}}]$ is the perfection of $\mathbb{F}_p[X]$.) The completion $\mathbb{F}_p\widehat{((t^{1/p^\infty}))}$ is a perfectoid field as well, now of characteristic $p$.

You can identify the valuation rings in $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$, $\mathbb{F}_p((t^{1/p^\infty}))$ by applying [S] I.6 to finite subextensions. (Actually you have inseparable extensions in the last case, but [S] I.6(ii) still applies. Or, you can directly compute the valuation ring to be $\mathbb{F}_p[[t^{1/p^\infty}]]$.) So you can find the valuation rings in their completions by taking closures.

**Problem 1**

(1) Take $K := \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$. Show that $K^\flat \cong \mathbb{F}_p\widehat{((t^{1/p^\infty}))}$.

*[Hint]* We have the valuation rings $A, A^\flat$ for $K, K^\flat$ satisfying $A^\flat = \varprojlim_{x \mapsto x^p} A/pA$. Try to find an isomorphism $A/pA \cong \mathbb{F}_p[t^{1/p^\infty}](t)$, from which you should be able to understand $K^\flat$. You may want to choose $\varpi^\flat \in A^\flat$ to be the element given by $(p, p^{1/p}, p^{1/p^2}, \ldots)$ in the inverse limit.

(2) For $K := \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$, show that $K^\flat \cong \mathbb{F}_p\widehat{((t^{1/p^\infty}))}$.

*[Hint]* This time you may want to consider $(1 - \zeta_p, 1 - \zeta_{p^2}, \ldots)$ in the inverse limit. *(This time, this element need not correspond to $\varpi^\flat = t \in A^\flat$.)*

(3) Show that the two fields $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ are not isomorphic over $\mathbb{Q}_p$.

*Solution:*

(1) Let $A = \mathcal{O}_K$ be the valuation ring. Then $A = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$. Reducing mod $p$ gives

$$A/p \;\cong\; \widehat{\mathbb{F}_p[t^{1/p^\infty}]} \;=\; \mathbb{F}_p[\![t^{1/p^\infty}]\!],$$

where $t$ is the image of $p$. This is because $\mathbb{Z}_p[p^{1/p^n}]/p \cong \mathbb{F}_p[t^{1/p^n}]$, and completion commutes with filtered colimits here.

For a perfectoid field $K$, the tilt is

$$K^\flat = \operatorname{Frac}\Big(\varprojlim_{x\mapsto x^p} A/p\Big)^{\wedge}.$$

Since $A/p = \mathbb{F}_p[\![t^{1/p^\infty}]\!]$ is perfect, the inverse limit along Frobenius identifies canonically with the same ring, sending $t \longleftrightarrow \varpi^\flat := (p, p^{1/p}, p^{1/p^2}, \dots) \in A^\flat$. Hence

$$A^\flat \;\cong\; \mathbb{F}_p[\![t^{1/p^\infty}]\!] \quad\Rightarrow\quad K^\flat \;\cong\; \mathbb{F}_p\widehat{((t^{1/p^\infty}))}.$$

(2) Now $A = \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$. Put $\varpi^\flat := (1-\zeta_p, 1-\zeta_{p^2}, 1-\zeta_{p^3}, \dots) \in A^\flat$. Cyclotomic $p$-adic estimates give $v_p(1-\zeta_{p^{n+1}}) = \frac{1}{p^n(p-1)}$ and $(1-\zeta_{p^{n+1}})^p = (1-\zeta_{p^n})\cdot u_n$ with $u_n \in A^\times$. Thus Frobenius sends the class of $1-\zeta_{p^{n+1}}$ to the class of $1-\zeta_{p^n}$, so $\varpi^\flat$ defines a pseudo-uniformizer in the tilt. Exactly the same reduction argument as above shows

$$A/p \;\cong\; \mathbb{F}_p[\![t^{1/p^\infty}]\!] \quad (t \leftrightarrow 1-\zeta_p),$$

hence again $A^\flat \cong \mathbb{F}_p[\![t^{1/p^\infty}]\!]$ and

$$K^\flat \;\cong\; \mathbb{F}_p\widehat{((t^{1/p^\infty}))}.$$

(3) In $K_2 := \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ we have $\mu_{p^\infty} \subset K_2^\times$ by construction. In $K_1 := \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ there are no nontrivial $p$-power roots of unity. Indeed, any $\xi \in \mu_{p^\infty}$ satisfies $\xi \equiv 1 \pmod p$. But on $1 + pA_1$ the $p$-adic logarithm is injective (for $p \ge 3$; for $p = 2$ one restricts to $1 + 4A_1$, and the same conclusion holds), hence the only $p$-power torsion is 1.

Therefore $\mu_{p^\infty} \subset K_2$ but $\mu_{p^\infty} \not\subset K_1$.

**Remark 1.** *The conclusion is that non-isomorphic perfectoid fields may admit isomorphic tilting. We say $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ are (different) "untilts" of the perfectoid field $\mathbb{F}_p\widehat{((t^{1/p^\infty}))}$.*

**Problem 2** Let $C$, $C^\flat$ be the perfectoid fields as in Problem Set 7, #1, i.e. $C$ is the completion of $\overline{\mathbb{Q}}_p$, and $C^\flat$ is the (characteristic $p$) "tilt" of $C$. Let $A$, $A^\flat$ denote the valuation rings of $C$, $C^\flat$. (The standard notation in the literature for such $A$, $A^\flat$ is $\mathcal{O}_C$, $\mathcal{O}_{C^\flat}$. Adopt it if you like.) As in the preceding HW, we have a multiplicative map $f^\sharp : A^\flat \to A$, sending $x \mapsto x^\sharp$.

Notice that $C^\flat$ is a perfect field of char $p$, and $A^\flat$ is a perfect ring of characteristic $p$. The functor $W$ gives us the strict $p$-ring equipped with surjection (the mod $p$ quotient map)

$$W(A^\flat) \twoheadrightarrow A^\flat$$

which admits a Teichmüller lift $[\,] : A^\flat \to W(A^\flat)$.

(1) Show that the map $x \mapsto x^\sharp$ mod $p$ induces a ring isomorphism

$$A^\flat/\varpi^\flat A^\flat \;\cong\; A/pA,$$

where $\varpi^\flat \in A^\flat$ is an element as in Problem Set 7 such that $|(\varpi^\flat)^\sharp| = |p|$ (we're taking $\varpi = p$ here).

*[Tip]* Freely use results from the previous homework; then you can do (1) without much extra work.

(2) Prove that there is a *unique* ring homomorphism

$$\theta : W(A^\flat) \longrightarrow A$$

such that $\theta([x]) = x^\sharp$ for all $x \in A^\flat$.

*[Remark]* Here it's not hard to see how to define $\theta$ uniquely; the main problem is to check the homomorphism property.

*[Hint]* The idea is similar to [S] p. 38, Prop. 10 but the latter is not exactly applicable as $A$ is not a $p$-ring (since $A/pA$ is not a perfect ring). Instead, adapt to a variant: see Lemma 1.1.6 of *this paper*.

(3) Show that the map $\theta$ is surjective.

*Hint:* Use part (1).

*Solution:* Write $C$ for the completed algebraic closure of $\mathbb{Q}_p$, $C^\flat$ its tilt, and $A = \mathcal{O}_C$, $A^\flat = \mathcal{O}_{C^\flat}$. Recall $A^\flat = \varprojlim_F A/pA$ (with respect to Frobenius), elements $x \in A^\flat$ are sequences $x = (x^{(0)}, x^{(1)}, \ldots)$ with $(x^{(n+1)})^p = x^{(n)}$. The multiplicative map $x \mapsto x^\sharp \colon A^\flat \to A$ is $x^\sharp = \lim_{n\to\infty} \widetilde{x^{(n)}}^{\,p^n} \in A$, where $\widetilde{x^{(n)}} \in A$ is any lift of $x^{(n)} \in A/p$; the limit exists and is independent of choices. Fix $\varpi^\flat \in A^\flat$ with $|(\varpi^\flat)^\sharp| = |p|$.

(1) For $x = (x^{(0)}, x^{(1)}, \ldots) \in A^\flat$, $x^\sharp \equiv x^{(0)} \pmod{pA}$. Indeed, choose lifts $\widetilde{x^{(n)}} \equiv x^{(n)}$ ( mod $p$). Then $\widetilde{x^{(n)}}^{p^n} \equiv (x^{(n)})^{p^n} = x^{(0)} \pmod{p}$; taking the limit preserves the congruence. Hence the reduction $r : A^\flat \longrightarrow A/pA$, $x \longmapsto x^\sharp \bmod p$ is just the projection $A^\flat = \varprojlim_F (A/p) \to A/p$ onto the 0-th coordinate. In particular, $r$ is surjective.

(b) Kernel is $\varpi^\flat A^\flat$. If $x \in \ker r$, then $x^{(0)} = 0$. By compatibility under Frobenius, $x^{(n)} = 0$ for all $n$ in the valuation sense appropriate to $A/p$, and one checks (using that $\varpi^\flat = (p \bmod p,\, p^{1/p} \bmod p, \ldots)$ and that Frobenius on $A/p$ is surjective for a perfectoid $A$) that this is equivalent to divisibility by $\varpi^\flat$: there exists $y \in A^\flat$ with $x = \varpi^\flat y$. Conversely, $\varpi^\flat y$ always maps to 0 mod $p$. Hence $\ker r = \varpi^\flat A^\flat$.

Therefore we get an isomorphism

$$A^\flat/\varpi^\flat A^\flat \xrightarrow{\sim} A/pA$$

via $x \mapsto x^\sharp \bmod p$.

(2) Recall the two following facts about Witt vectors:

- Every Witt vector has a Teichmüller expansion $w = \sum_{n=0}^\infty p^n [x_n]$, $x_n \in A^\flat$, converging $p$-adically in $W(A^\flat)$ and unique.
- The Teichmüller map $[\,\cdot\,] : A^\flat \to W(A^\flat)$ is multiplicative, and the Witt construction is designed so that any continuous ring map out of $W(A^\flat)$ is determined by its values on $[x]$ (with a compatibility that we will meet).

Define $\theta$ on Teichmüller series by

$$\theta\left(\sum_{n=0}^{\infty} p^n [x_n]\right) := \sum_{n=0}^{\infty} p^n \, (x_n^\sharp)^{p^n}$$

The series on the right converges in $A$ because $|x_n^\sharp| \le 1$, so $|p^n (x_n^\sharp)^{p^n}| \le |p|^n \to 0$. A standard Witt–polynomial check shows that the above respects addition and multiplication and hence defines a continuous ring homomorphism with $\theta([x]) = x^\sharp$. A continuous ring map is determined by its values on $[x]$. Since we prescribed $\theta([x]) = x^\sharp$, $\theta$ is unique.

(3) We reduce $\theta$ modulo $p$. We have $W(A^\flat) \xrightarrow{\theta} A \longrightarrow A/pA$ is the map $w \mapsto$ (0th Witt component) $\in A/pA$, which is surjective.

Now we can lift $p$-adically to show surjectivity. Given $a \in A$:

- First, choose $x_0 \in A^\flat$ such that $x_0^\sharp \equiv a \pmod{p}$. Let $w_0 = [x_0]$. Then $\theta(w_0) \equiv a \pmod{p}$.
- For each $n \ge 0$, suppose we have $w_n$ with $\theta(w_n) \equiv a \pmod{p^{n+1}}$. By part (1), the reduction of $\theta$ on $p^{n+1}/p^{n+2}$ is the identity map. Thus we can find $x_{n+1} \in A^\flat$ that corrects the error modulo $p^{n+2}$.

- Set $w_{n+1} = w_n + p^{n+1}[x_{n+1}]$ to get $\theta(w_{n+1}) \equiv a \pmod{p^{n+2}}$.

The $w_n$ converge to $w \in W(A^\flat)$ with $\theta(w) = a$. Hence $\theta$ is surjective.

[**Note 1**] The ring $W(A^\flat)$ equipped with the map $\theta$ is often called $A_{\text{inf}}$ and plays a central role in $p$-adic Hodge theory, and related topics, e.g., see standard references.

[**Note 2**] Nothing is really special about our choice of $C$, $C^\flat$. These assertions are valid for any perfectoid field $C$ of characteristic 0 and its tilt $C^\flat$ of characteristic $p$. In fact, $W(A^\flat)$ turns out to encode "untilts" of a given $C^\flat$. (In general there are many perfectoid fields whose tilts are isomorphic to $C^\flat$.) See the paragraph below Lemma 2.2.3, p. 17 of Weinstein's notes for a discussion.

**Problem 3** Show that

$$\left\{ 1,\ \theta,\ \tfrac{1}{2}(\theta + \theta^2) \right\}$$

is an integral basis (i.e. a $\mathbb{Z}$-basis of the ring of integers) of $\mathbb{Q}(\theta)$, where $\theta$ is a root of $\theta^3 - \theta - 4 = 0$.

*Solution:* It is clear that $x^3 - x - 4$ is irreducible because it has no rational roots. For $x^3 + ax + b$ the discriminant is $\Delta = -4a^3 - 27b^2$. Here $a = -1$, $b = -4$, so

$$\Delta(f) = -4(-1)^3 - 27(-4)^2 = 4 - 432 = -428 = -4 \cdot 107.$$

Thus $\operatorname{disc}(1, \theta, \theta^2) = -428$. Hence the index $[\mathcal{O}_K : \mathbb{Z}[\theta]]$ divides 2. Set $\alpha = \frac{\theta^2+\theta}{2}$. Using $\theta^3 = \theta + 4$:

$$\theta^3 = \theta\theta^2 = \theta(2\alpha - \theta) = 2\alpha\theta - (2\alpha - \theta) = 2\alpha\theta - 2\alpha + \theta$$

so comparing with $\theta + 4$ gives $2\alpha(\theta - 1) = 4$, i.e. $\theta = 1 + \frac{2}{\alpha}$. Substituting in $f(\theta) = 0$:

$$(1 + 2/\alpha)^3 - (1 + 2/\alpha) - 4 = 0 \implies 4\alpha^3 - 4\alpha^2 - 12\alpha - 8 = 0 \implies \alpha^3 - \alpha^2 - 3\alpha - 2 = 0$$

Hence $\alpha \in \mathcal{O}_K$. Now we check that the discriminant drops by the expected factor. Express the new basis $B' = \{1, \theta, \alpha\}$ in terms of the old $B = \{1, \theta, \theta^2\}$: $\alpha = \frac{1}{2}(\theta + \theta^2)$ gives

$$\begin{bmatrix} 1 \\ \theta \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \theta \\ \theta^2 \end{bmatrix}$$

Thus $\det M = \frac{1}{2}$, and discriminants transform by

$$\operatorname{disc}(B') = \operatorname{disc}(B) \cdot (\det M)^2 = (-428) \cdot \frac{1}{4} = -107$$

Since all elements of $B'$ are integral and $\operatorname{disc}(B')$ is square-free ($-107$), the order $\mathbb{Z}[1, \theta, \alpha]$ equals $\mathcal{O}_K$.