

Homework 4

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Problem Let p be a prime number, and n a positive integer greater than 1. Find an example for each of the following with brief justifications.

- (1) A degree n extension of \mathbb{Q} in which p is inert (i.e. the ring of integers in the extension possesses a unique prime \mathfrak{q} above p , and the inertial degree $f_{\mathfrak{q}/p}$ is equal to n).
- (2) A degree n extension of \mathbb{Q} in which p is totally ramified.

Hint: You can apply results of Serre, I.6 after localizing at p .

Remark: There's nothing special about \mathbb{Q} . The same question can be answered similarly with any global field in place of \mathbb{Q} .

Solution:

Problem Let A be a Dedekind domain, $K = \text{Frac}(A)$. Let L/K be a finite separable extension with normal closure M of L so that M is Galois over K . Let \mathfrak{p} be a prime ideal of A . Fix a prime ideal \mathfrak{t} of M above \mathfrak{p} . (By convention, this means \mathfrak{t} is a nonzero prime in the integral closure of A in M such that \mathfrak{t} divides \mathfrak{p} .) Denote by $D_{\mathfrak{t}}(M/K)$ the decomposition group of \mathfrak{t} in M/K .

- (i) Define a map

$$\text{Gal}(M/K) \rightarrow \{\text{primes of } L \text{ above } \mathfrak{p}\}, \quad \sigma \mapsto \sigma(\mathfrak{t}) \cap L.$$

Show that this map induces a bijection

$$\text{Gal}(M/L) \backslash \text{Gal}(M/K) / D_{\mathfrak{t}}(M/K) \xrightarrow{\sim} \{\text{primes of } L \text{ above } \mathfrak{p}\}.$$

- (ii) Assume that $\text{Gal}(M/K) \simeq S_3$, the symmetric group in 3 variables, that $D_{\mathfrak{t}}(M/K)$ and $\text{Gal}(M/L)$ are order 2 subgroups of $\text{Gal}(M/K)$ which are equal (not just isomorphic). Use part (i) to verify that \mathfrak{p} does *not* split completely in L .

Remark: The point of (ii) is that when the decomposition group of \mathfrak{t} is not normal in $\text{Gal}(M/K)$, the prime \mathfrak{t} need not split completely in the decomposition field, which is L here. A concrete example for (ii) can be given when

$$K = \mathbb{Q}, \quad L = \mathbb{Q}(\sqrt[3]{2}), \quad M = \mathbb{Q}(\sqrt[3]{2}, \zeta_3).$$

By the Chebotarev density theorem, or by explicit computation, you can find \mathfrak{t} such that $(\mathfrak{t}, M/K)$ is the unique nontrivial element of $\text{Gal}(M/L)$. Then all the conditions of (ii) are satisfied.

Solution:

Neukirch Ch. I.9, Exercise 3 Continue the general setup from Problem 2. Assume the following:

- (i) L/K is solvable, meaning that $\text{Gal}(M/K)$ is a solvable group. (We are not assuming $M = L$.)
- (ii) $p = [L : K]$ is a prime number.

Now let \mathfrak{p} be a prime of K unramified in L . If there are two primes \mathfrak{q} and \mathfrak{q}' of L above \mathfrak{p} such that the inertial degrees $f_{\mathfrak{q}}$ and $f_{\mathfrak{q}'}$ are equal to 1, then show that \mathfrak{p} splits completely in L/K .

Caveat: The extension degree p has nothing to do with the prime ideal \mathfrak{p} in the problem.

Hint: Let S_p denote the symmetric group in p letters acting on $\{1, 2, \dots, p\}$. If G is a solvable subgroup of S_p acting transitively on $\{1, 2, \dots, p\}$ then every nontrivial element of G fixes at most one element in $\{1, 2, \dots, p\}$. (A reference for this fact is given in Neukirch.)

Solution: