Let G cplx ss Lie gp, T maximal torus
B Borel.

Let Λ be the weight lattice of T.

Thm [Borel-Weil]

$$\mathrm{Pic}(G/B) \cong \Lambda$$

Let $L_\lambda := G \times_B \mathbb{C}_\lambda$ line bundle on G/B. Then

$$\Gamma(G/B, L_\lambda) = \begin{cases} V_\lambda & \text{if } \lambda \text{ dom} \\ 0 & \text{else} \end{cases}$$

and L_λ ample iff λ regular dominant.

Take $\begin{matrix} L_\lambda \\ \downarrow \\ G/B \end{matrix}$ ample and then

Properties: G ss adjoint gp, $\underbrace{\overline{G}}_{X}$ De Concini Procesi

- smooth projective variety
- $G \subset \overline{G}$ as an open dense subset
- $\overline{G} \setminus G$ is a union of smooth prime divisors w/ normal crossings.
- There are ℓ such divisors
- closures of $G \times G$ orbits on X are precisely the partial $\cap$ of the above divisors
- There is a single closed orbit, the intersection of all of them, and it is $\underbrace{G/B \times G/B^-}_{Y}$

Last time:

Prof Robles asked: what are the positive line bundles?

Thm | The restriction map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is injective w/ image

$$\{ [\underline{\mathcal{L}_Y(\lambda)}] \mid \lambda \in \Lambda \}$$

where $\mathcal{L}_Y(\lambda) = \mathcal{L}(\lambda) \boxtimes \mathcal{L}(\lambda^*)$

$\lambda^* = -w_0 \lambda$

Call $\mathcal{L}_X(\lambda)$ the line bundle on X which restricts to $\mathcal{L}_Y(\lambda)$ on Y.

Thm] $\mathcal{L}_X(\lambda)$ is generated by global sections iff λ dom.

$\mathcal{L}_X(\lambda)$ is ample iff λ regular dominant.

Finally (Serre GAGA)

Thm [Kodaira]: X cpt Kahler

L holomorphic line bundle. L is positive iff L is ample.

Part II: about the degeneration

Motivating example (where the recipe comes from)

Due to (Gonciulea & Lakshmibai)

$Gr(k,n) = G/P$, P maximal parabolic.

Take its total coordinate ring.

$$R(Gr(k,n)) = \bigoplus_{\substack{\text{all iso}\\ \text{classes of}\\ \text{line bundles}}} H^0(\mathcal{L}_\lambda)$$

$$= \bigoplus_{\lambda \in \mathbb{N}} H^0(\mathcal{L}_{\lambda\omega})$$

ω is the wt of Plücker line bundle

$R(Gr(k,n)) \circlearrowleft G$

In general, if you have λ dominant wt of $GL(n)$, V_λ corresponding irrep.

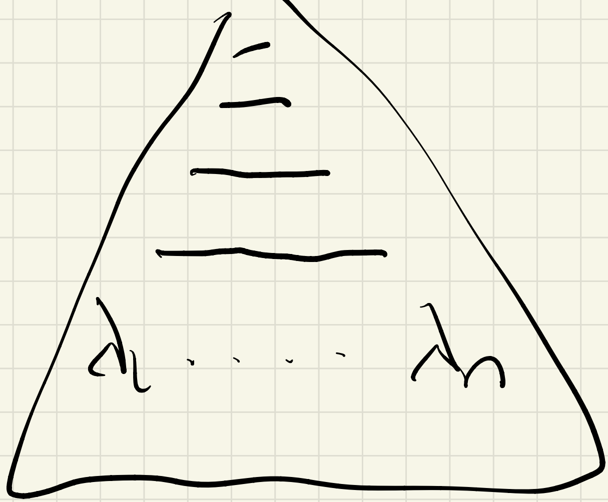
Let $GL_i \subset GL_n$ as the bottom right corner. Then V_λ decomposes as GL_{n-1} rep into $V_\lambda = \bigoplus_\mu V_\mu$ where

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$$

Do this for $GL(1) \times \cdots \times GL(1)$.

Weight spaces for V_λ

$\Longleftrightarrow$ GT patterns



Now take $R(Gr(k,n))$ and break into $GL(1) \times \cdots \times GL(1)$. There is a weight space for each lattice point of the GT polytope for ω_k and its multiples.

Algebraically: flat family over the line w/ fiber over 0 this becomes
$=$ TV whose moment polytope as above

general fiber is G/P. Geometrically, Kogan-Miller quantize this algebraic explanation in the sense of BW.

We run this machine for $\bar{G}$.

$$R(\bar{G}) = \bigoplus_{\lambda \in \Lambda} \Big(\bigoplus_{\substack{\mu \in \lambda \\ \mu \in \Lambda^+}} V_\mu \otimes V_\mu^* \Big) t^\lambda$$

λ-graded

What you get is that when you pass to this $GL(1) \times \cdots \times GL(1)$ you have 1 wt space for each data

$$(\lambda \in \Lambda, \mu \leq \lambda, \nu \in \Lambda^+)$$

two GT patterns for μ.

Then we get this degeneration $\bar{G_i}$ to
flat, equiv.

TV (of that polytope). ☐

$$(5,6,4) - (5,5,5)$$

$$(0,1,-1)$$ ✓

Should be very doable for $n=2$.