Equivariant Derived Categories of Coherent Sheaves

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September 27, 2025

Abstract

Notes for a talk I'm giving on equivariant derived categories of coherent sheaves.

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1 Preliminaries on GIT

1.1 Generalities on GIT quotients

Let $X \subset \mathbb{P}^n$ be a projective variety, and let $\widetilde{X} \subset \mathbb{C}^{n+1}$ be the corresponding affine cone. Since X is the space of lines in $\widetilde{X}$, it has a tautological line bundle

$$\mathcal{O}_X(-1) = \mathcal{O}_{\mathbb{P}^n}(-1)\big|_X$$

over it whose fibre over a point in X is the corresponding line in $\widetilde{X} \subset \mathbb{C}^{n+1}$. The total space of $\mathcal{O}_X(-1)$ therefore has a tautological map to $\widetilde{X}$ which is an isomorphism away from the zero section $X \subset \mathcal{O}_X(-1)$, which is all contracted down to the origin in $\widetilde{X}$. In fact the total space of $\mathcal{O}_X(-1)$ is the **blow up** of $\widetilde{X}$ in the origin.

Linear functions on $\mathbb{C}^{n+1}$ like x_i, restricted to $\widetilde{X}$ and pulled back to the total space of $\mathcal{O}_X(-1)$, give functions which are linear on the fibres, so correspond to sections of the **dual** line bundle

$\mathcal{O}_X(1)$. Similarly degree k homogeneous polynomials on $\widetilde{X}$ define functions on the total space of $\mathcal{O}_X(-1)$ which are of degree k on the fibres, and so give sections of the kth tensor power $\mathcal{O}_X(k)$ of the dual of the line bundle $\mathcal{O}_X(-1)$.

So the grading that splits the functions on $\widetilde{X}$ into homogeneous degree (or $\mathbb{C}^*$-weight spaces) corresponds to sections of different line bundles $\mathcal{O}_X(k)$ on X. So

$$\bigoplus_{k\geq 0} H^0(\mathcal{O}_X(k))$$

considered a graded ring by tensoring sections $\mathcal{O}(k)\otimes\mathcal{O}(l)\cong\mathcal{O}(k+l)$. For the line bundle $\mathcal{O}_X(1)$ sufficiently positive, this ring will be generated in degree one. It is often called the (homogeneous) coordinate ring of the **polarised** (i.e. endowed with an ample line bundle) variety $(X, \mathcal{O}_X(1))$.

The degree one restriction is for convenience and can be dropped (by working with varieties in weighted projective spaces), or bypassed by replacing $\mathcal{O}_X(1)$ by $\mathcal{O}_X(p)$, i.e. using the ring

$$R^{(p)} = \bigoplus_{k\geq 0} R_{kp}; \qquad \text{for } p \gg 0 \text{ this will be generated by its degree one piece } R_p.$$

The choice of generators of the ring is what gives the embedding in projective space. In fact the sections of any line bundle L over X define a (rational) map

$$X \dashrightarrow \mathbb{P}\big(H^0(X,L)^*\big), \qquad x \mapsto ev_x, \quad ev_x(s) := s(x), \tag{1}$$

which in coordinates maps x to $(s_0(x) : \cdots : s_n(x)) \in \mathbb{P}^n$, where s_i form a basis for $H^0(L)$. This map is only defined for those x with $ev_x \neq 0$, i.e. for which $s(x)$ is not zero for every s.

Now suppose we are in the following situation, of G acting on a projective variety X through SL transformations of the projective space.

$$\begin{array}{ccc} G & \curvearrowright & X \\ \downarrow & & \downarrow \\ SL(n+1,\mathbb{C}) & \curvearrowright & \mathbb{P}^n \end{array}$$

Since we have assumed that G acts through $SL(n+1,\mathbb{C})$, the action lifts from X to one covering it on $\mathcal{O}_X(-1)$. In other words we don't just act on the projective space (and X therein) but on the vector space overlying it (and the cone $\widetilde{X}$ on X therein). This is called a **linearisation** of the action. Thus G acts on each $H^0(\mathcal{O}_X(r))$.

Then, just as $(X, \mathcal{O}_X(1))$ is determined by its graded ring of sections of $\mathcal{O}(r)$ (i.e. the ring of functions on $\widetilde{X}$),

$$(X, \mathcal{O}(1)) \longleftrightarrow \bigoplus_r H^0(X, \mathcal{O}(r))$$

we simply **construct** X/G (with a line bundle on it) from the ring of **invariant** sections:

$$X/G \longleftrightarrow \bigoplus_r H^0(X, \mathcal{O}(r))^G$$

This is sensible, since if there is a good quotient then functions on it pullback to give G-invariant functions on X, i.e. functions constant on the orbits, the fibres of $X \to X/G$. For it to work we need:

Lemma 1.1. *$\bigoplus_r H^0(X, \mathcal{O}(r))^G$ is finitely generated.*

Proof. Since $R := \bigoplus_r H^0(X, \mathcal{O}(r))$ is Noetherian, Hilbert's basis theorem tells us that the ideal $R \cdot \big(\bigoplus_{r>0} H^0(X, \mathcal{O}(r))^G\big)$ generated by $R_+^G := \bigoplus_{r>0} H^0(X, \mathcal{O}(r))^G$ is generated by a finite number of elements $s_0, \ldots, s_k \in R_+^G$.

Thus any element $s \in H^0(X, \mathcal{O}(r))^G$, $r > 0$, may be written $s = \sum_{i=0}^k f_i s_i$ for some $f_i \in R$ of degree $< r$. To show that the s_i generate R_+^G as an algebra we must show that the f_i can be taken to lie in R^G.

We now use the fact that G is the complexification of the compact group K. Since K has an invariant metric, we can average over it and use the facts that s and s_i are invariant to give

$$s = \sum_{i=0}^k \mathrm{Av}(f_i)\, s_i,$$

where $\mathrm{Av}(f_i)$ is the (K-invariant) K-average of f_i. By complex linearity $\mathrm{Av}(f_i)$ is also G-invariant (for instance, since G has a polar decomposition $G = K \exp(i\mathfrak{t})$). The $\mathrm{Av}(f_i)$ are also of degree $< r$, and so we may assume, by an induction on r, that we have already shown that they are generated by the s_i in R_+^G. Thus s is also. $\square$

Definition 1.2 (Projective GIT quotient)**.** *Let X be a projective variety with an action of a reductive group G linearised by a line bundle $\mathcal{O}_X(1)$. We define X/G to be*

$$\mathrm{Proj} \bigoplus_r H^0(X, \mathcal{O}(r))^G.$$

If X is a variety (rather than a scheme) then so is X/G, as its graded ring sits inside that of X and so has no zero divisors.

Definition 1.3 (Affine GIT quotient)**.** *Let $X = \mathrm{Spec}\, R$ be an affine variety with an action of a reductive group G. We define the affine GIT quotient X/G to be $\mathrm{Spec}(R^G)$, where R^G is the ring of G-invariant regular functions on X.*

In some cases, this does not work so well. For instance, under the scalar action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ the only invariant polynomials in $\mathbb{C}[x_0, \dots, x_n]$ are the constants and this recipe for the quotient gives a single point. In the language of the next section, this is because there are no stable points in this example, and all semistable orbits' closures intersect (or equivalently, there is a unique polystable point, the origin). More generally in any affine case all points are always at least semistable (as the constants are always G-invariant functions) and so no orbits gets thrown away in making the quotient (though many may get identified with each other — those whose closures intersect which therefore cannot be separated by invariant functions). But for the scalar action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ we clearly need to remove at least the origin to get a sensible quotient.

So we should change the linearisation, from the trivial linearisation to a nontrivial one, to get a bigger quotient. This is demonstrarted in the following example.

Example 1.4 (Projective space as a GIT quotient)**.** *Consider the trivial line bundle on $\mathbb{C}^{n+1}$ but with a nontrivial linearisation, by composing the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ by a character $\lambda \mapsto \lambda^p$ of $\mathbb{C}^*$ acting on the fibres of the trivial line bundle over $\mathbb{C}^{n+1}$. The invariant sections of this no longer form a ring; we have to take the direct sum of spaces of sections of **all powers** of this linearisation, just as in the projective case, and take Proj of the invariants of the resulting graded ring.*

We calculate the invariant sections for general p. Look at the k-th tensor power of the linearised line bundle. Sections are homogeneous polynomials $f(x_0, \dots, x_n)$ of some degree. Under λ, such an f transforms as

$$f(x_0, \dots, x_n) \mapsto f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n),$$

where $d = \deg f$.

But the linearisation introduces an extra factor λ^{-pk} when we act on the fibre of the k-th tensor power. By definition, the G-action on a section s is

$$(g \cdot s)(x) \ = \ g \cdot \big(s(g^{-1} \cdot x)\big).$$

Take a polynomial f homogeneous of degree d. View the section as

$$s(x) = f(x) \cdot e$$

where e is a trivialising section of the fibre. When we apply the group action:

$$(g \cdot s)(x) = g \cdot (f(g^{-1} \cdot x) \cdot e) = \big(\lambda^{-d} f(x)\big) \cdot \lambda^{pk} e = \lambda^{-d+pk)} f(x) \cdot e.$$

For invariance, we need the weight to vanish, i.e.

$$d = pk.$$

So only polynomials of degree exactly pk survive as invariants in the degree k graded piece.

If $p < 0$ then there are no invariant sections and the quotient is empty. We have seen that for $p = 0$ the quotient is a single point. For $p > 0$ the invariant sections of the kth power of the linearisation are the homogeneous polynomials on $\mathbb{C}^n$ of degree kp. So for $p = 1$ we get the quotient

$$\mathbb{C}^{n+1}/\mathbb{C}^* = \operatorname{Proj} \bigoplus_{k \geq 0} \big(\mathbb{C}[x_0, \ldots, x_n]_k\big) = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n] = \mathbb{P}^n. \tag{2}$$

For $p \geq 1$ we get the same geometric quotient but with the line bundle $\mathcal{O}(p)$ on it instead of $\mathcal{O}(1)$.

Another way to derive this is to embed $\mathbb{C}^{n+1}$ in $\mathbb{P}^{n+1}$ as $x_{n+1} = 1$, act by $\mathbb{C}^*$ on the latter by $\operatorname{diag}(\lambda, \ldots, \lambda, \lambda^{-(n+1)}) \in SL(n+2, \mathbb{C})$, and do projective GIT. This gives, on restriction to $\mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$, the $p = n+1$ linearisation above. The invariant sections of $\mathcal{O}((n+2)k)$ are of the form $x_{n+1}^k f$, where f is a homogeneous polynomial of degree $(n+1)k$ in $x_1, \ldots, x_n$. Therefore the quotient is

$$\operatorname{Proj} \bigoplus_{k \geq 0} \big(\mathbb{C}[x_1, \ldots, x_n]_{(n+1)k}\big) = \operatorname{Proj} \big(\mathbb{C}[x_1, \ldots, x_n], \mathcal{O}(n+1)\big).$$

Definition 1.5 (Semistable points)**.** *A point $x \in X$ is* ***semistable*** *iff there exists $s \in H^0(X, \mathcal{O}(r))^G$ with $r > 0$ such that $s(x) \neq 0$. Points which are not semistable are* ***unstable****.*

So semistable points are those that the G-invariant functions "see." The map

$$\begin{aligned} X^{ss} &\to \mathbb{P}(H^0(X, \mathcal{O}(r))^G)^* \\ x &\mapsto ev_x \end{aligned}$$

is well defined on the (Zariski open, though possibly empty) locus $X^{ss} \subseteq X$ of semistable points, and it is clearly constant on G-orbits, i.e. it factors through the set-theoretic quotient X^{ss}/G. But it may contract more than just G-orbits, so we need another definition.

Definition 1.6 (Stable points)**.** *A semistable point x is* ***stable*** *if and only if $\bigoplus_r H^0(X, \mathcal{O}(r))^G$ separates orbits near x and the stabiliser of x is finite.*

1.2 Relevant example

We recall the basic set-up of Geometric Invariant Theory (GIT) quotients relevant to our situation.

Let $V = \mathbb{C}^4$ with coordinates x_1, x_2, y_1, y_2 and consider the $\mathbb{C}^*$-action given by

$$t \cdot (x_1, x_2, y_1, y_2) = (tx_1, tx_2, t^{-1}y_1, t^{-1}y_2)$$

We linearize this action by a character $\chi_m : t \mapsto t^m$ with $m \in \mathbb{Z} \setminus \{0\}$. Since V is affine, the GIT quotient for χ_m is $\operatorname{Proj} R^{(m)}$, where

$$R^{(m)} = \bigoplus_{d \geq 0} \Gamma(V, \mathcal{O}_V)^{\mathbb{C}^*, \chi_m^{\otimes d}} = \bigoplus_{d \geq 0} \{\, f \in \mathbb{C}[x_1, x_2, y_1, y_2] \mid t \cdot f = t^{md} f \,\}$$

In other words, $R_d^{(m)}$ is spanned by monomials whose total $\mathbb{C}^*$-weight is md, where the weight of a monomial $x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}$ is $w = a_1 + a_2 - (b_1 + b_2)$.

A point $v \in V$ is χ_m-semistable iff there exists $d > 0$ and $f \in R_d^{(m)}$ with $f(v) \neq 0$.

Here $R_d^{(m)}$ consists of polynomials whose monomials have positive weight $w = md > 0$. Such a monomial must contain at least one x (indeed, more x's than y's), so it vanishes at any point with $x_1 = x_2 = 0$. Therefore no section in $R_d^{(m)}$ can be nonzero at a point with $x_1 = x_2 = 0 \Rightarrow$ those points are unstable.

Conversely, if $(x_1, x_2) \neq (0, 0)$, then pick d and the monomial $f = x_i^{md}$ with $x_i \neq 0$. It has weight md and $f(v) \neq 0$, so v is semistable.

Therefore, for $m > 0$,

$$V^{ss}(\chi_m) = V \setminus \{x_1 = x_2 = 0\}.$$

The quotient is $(V \setminus \{x_1 = x_2 = 0\})/\mathbb{C}^*$, i.e. the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1_{[x_1:x_2]}$. Similarly, for $m < 0$, we have

$$V^{ss}(\chi_m) = V \setminus \{y_1 = y_2 = 0\}.$$

The quotient is $(V \setminus \{y_1 = y_2 = 0\})/\mathbb{C}^*$, i.e. the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1_{[y_1:y_2]}$.

1.3 Quotient stack

Let $\mathcal{S}$ be a category and $p : \mathcal{X} \to \mathcal{S}$ be a functor of categories. We visualize this data as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \\ \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & b \arrow[d] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

where the lower case letters a, b are objects of $\mathcal{X}$ and the upper case letters S, T are objects of $\mathcal{S}$. We say that a is over S and that a morphism $\alpha : a \to b$ is over $f : S \to T$.

Definition 1.7 (Prestacks). *A functor $p : \mathcal{X} \to \mathcal{S}$ is a **prestack over a category** $\mathcal{S}$ if*

*(1) (**pullbacks exist**) for every diagram*

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d] & b \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a morphism $a \to b$ over $S \to T$; and

*(2) (**universal property for pullbacks**) for every diagram*

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & b \arrow[r] \arrow[d] & c \arrow[d] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

*Prestacks are also referred to as **categories fibered in groupoids**.*

Definition 1.8 (Fiber categories). *If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the **fiber category** $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over S with morphisms over id_S.*

Given an action of an algebraic group G on a scheme X, the **quotient prestack** $[X/G]^{\mathrm{pre}}$ is the prestack whose fiber category $[X/G]^{\mathrm{pre}}(S)$ over a scheme S is the quotient groupoid (or the moduli groupoid of orbits) $[X(S)/G(S)]$ This will not satisfy the gluing axioms of a stack; even when the action is free, the quotient functor $\mathrm{Sch} \to \mathrm{Sets}$ defined by $S \mapsto X(S)/G(S)$ is not a sheaf in general. Put another way, we define:

Definition 1.9 (Quotient prestacks). *Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. The **quotient prestack** $[U/G]^{\mathrm{pre}}$ of an action of a smooth affine group scheme $G \to S$ on an S-scheme U is the category over Sch/S consisting of pairs (T, u) where T is an S-scheme and $u \in U(T)$. An element $g \in G(T')$ acts by $(T', u') \to (T, u)$ via the data of a map $f : T' \to T$ of S-schemes and an element $g \in G(T')$ such that $f^*u = g \cdot u'$. Note that the fiber category $[U(T)/G(T)]$ is identified with the quotient groupoid.*

It turns out that the stackification of $[U/G]^{\mathrm{pre}}$ is the quotient stack $[U/G]$, hence the name is justified.

Definition 1.10 (Quotient stacks). *The **quotient stack** $[U/G]$ is the prestack over Sch/S consisting of diagrams*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of S-schemes.

A morphism

$$(T' \longleftarrow P' \longrightarrow U) \longrightarrow (T \longleftarrow P \longrightarrow U)$$

consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and the left square is cartesian.

A stack over a site $\mathcal{S}$ is a prestack $\mathcal{X}$ where the objects and morphisms glue uniquely in the Grothendieck topology of $\mathcal{S}$.

Definition 1.11 (Stack)**.** *A **stack** $\mathcal{X}$ over a site $\mathcal{C}$ is a prestack over $\mathcal{C}$ satisfying the following descent conditions:*

- *(Descent for morphisms) For any $U \in \mathcal{C}$, any covering $\{f_i : U_i \to U\}$, and any $x, y \in \mathcal{X}(U)$, the presheaf*

  $$\underline{\mathrm{Hom}}(x, y) : (V \to U) \mapsto \mathrm{Hom}_{\mathcal{X}(V)}(f^*x, f^*y)$$

  is a sheaf on $\mathcal{C}/U$.

- *(Descent for objects) For any $U \in \mathcal{C}$, any covering $\{f_i : U_i \to U\}$, and any descent datum (x_i, ϕ_{ij}) relative to $\{f_i : U_i \to U\}$, there exists an object $x \in \mathcal{X}(U)$ and isomorphisms $\psi_i : f_i^*x \xrightarrow{\sim} x_i$ such that $\phi_{ij} \circ f_j^*\psi_j = f_i^*\psi_i$.*

Definition 1.12 (Substack)**.** *A **substack** $\mathcal{Y} \subseteq \mathcal{X}$ is given by:*

- *For each $U \in \mathcal{C}$, a full subcategory $\mathcal{Y}(U) \subseteq \mathcal{X}(U)$.*

- *Stability under restriction: If $y \in \mathcal{Y}(U)$ and $f : V \to U$ is a morphism in the site, then the pullback $f^*y \in \mathcal{X}(V)$ must lie in $\mathcal{Y}(V)$.*

- *Stack condition: The collection $\mathcal{Y}$ is itself a stack (i.e. satisfies descent for objects and morphisms).*

Definition 1.13 (Open and closed substacks)**.** *A substack $\mathcal{T} \subseteq \mathcal{X}$ of a stack over $\mathrm{Sch}_{\text{ét}}$ is called an **open substack** (resp. **closed substack**) if the inclusion $\mathcal{T} \to \mathcal{X}$ is representable by schemes and an open immersion (resp. closed immersion).*

2 Introduction

Let $V = \mathbb{C}^4$ with co-ordinates x_1, x_2, y_1, y_2, and let $\mathbb{C}^*$ act on V with weight 1 on each x_i and weight -1 on each y_i. There are two possible GIT quotients X_+ and X_-, depending on whether we choose a positive or negative character of $\mathbb{C}^*$. Both are isomorphic to the total space of the bundle $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{P}^1$.

Both are open substacks of the Artin quotient stack

$$\mathcal{X} = [V/\mathbb{C}^*]$$

given by the semi-stable locus for either character. Let

$$\iota_\pm : X_\pm \hookrightarrow \mathcal{X}$$

denote the inclusions.

Remark 2.1 (The quotient stack and its open substacks)**.** *Recall that via the functor of points perspective, its objects are pairs (P, ϕ), where P is a principal $\mathbb{C}^*$-bundle and $\phi : P \to V$ is $\mathbb{C}^*$-equivariant.*

For a given choice of character χ_m, the semistable locus $V^{ss}(\chi_m)$ is an open subset of V. It is open because it is defined by the nonvanishing of some semi-invariant sections. The corresponding GIT quotient is $[V^{ss}(\chi_m)/\mathbb{C}^*]$ as a substack. Thus:

$$X_\pm = [V^{ss}(\pm 1)/\mathbb{C}^*] \ \subset \ [V/\mathbb{C}^*] = \mathcal{X}$$

It turns out that open substacks of quotient stacks $[V/G]$ are exactly those substacks which are of the form $[U/G]$ where $U \subseteq V$ is a G-invariant open subscheme. Here $V^{ss}(\chi_m) \subset V$ is G-invariant and open, so $[V^{ss}(\chi_m)/\mathbb{C}^*] \hookrightarrow [V/\mathbb{C}^*]$ is exactly an open immersion of stacks.

This stacky point of view makes it clear that there are (exact) restriction functors

$$\iota_\pm^* : D^b(\mathcal{X}) \ \to \ D^b(X_\pm).$$

By $D^b(\mathcal{X})$ we mean the derived category of the category of $\mathbb{C}^*$-equivariant sheaves on V. This contains the obvious equivariant line bundles $\mathcal{O}(i)$ associated to the characters of $\mathbb{C}^*$.

Remark 2.2 (General fact about open immersions)**.** *If $j : U \hookrightarrow X$ is an open immersion of schemes, then there is an exact restriction functor $j^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$. This is because $j^*\mathcal{F}$ has the same stalk as $\mathcal{F}$ at points of U.*

Althernatively, exactness comes from the fact that restricting a quasi-coherent sheaf to an open set is just tensoring with $\mathcal{O}_U$, which is flat (in general localisation is flat).

Passing to derived categories, you still have $j^*: D^b(\mathrm{QCoh}(X)) \to D^b(\mathrm{QCoh}(U))$ which has no higher derived functors since j^* is exact.The exact same holds in the stack setting: if $\iota : \mathcal{U} \hookrightarrow \mathcal{X}$ is an open immersion of stacks, you get $\iota^* : D^b(\mathcal{X}) \to D^b(\mathcal{U})$.

Remark 2.3 (General dictionary for quotient stacks and equivariant geometry)**.** *There is a general dictionary relating the stack-theoretic concepts and the equivariant geometry of X. Here G is a reductive algebraic group acting on a scheme X and $[X/G]$ is the quotient stack.*

Geometry of $[X/G]$	G***-equivariant geometry of*** X
$\mathbb{C}$*-point* $\bar{x} \in [X/G]$	*orbit* Gx *of* $\mathbb{C}$*-point* $x \in X$ *(with* $\bar{x}$ *the image of* x *under* $X \to [X/G]$*)*
automorphism group $\mathrm{Aut}(\bar{x})$	*stabilizer* G_x
function $f \in \Gamma([X/G], \mathcal{O}_{[X/G]})$	G*-equivariant function* $f \in \Gamma(X, \mathcal{O}_X)^G$
map $[X/G] \to Y$ *to a scheme* Y	G*-equivariant map* $X \to Y$
line bundle	G*-equivariant line bundle (or* G*-linearization)*
quasi-coherent sheaf	G*-equivariant quasi-coherent sheaf*
tangent space $T_{[X/G],\bar{x}}$	*normal space* $T_{X,x}/T_{Gx,x}$ *to the orbit*
coarse moduli space $[X/G] \to Y$	*geometric quotient* $X \to Y$
good moduli space $[X/G] \to Y$	*good GIT quotient* $X \to Y$

The unstable locus for the negative character is the set $\{y_1 = y_2 = 0\} \subset V$. Consider the Koszul resolution of the associated sky-scraper sheaf:

$$K_- = \mathcal{O}(2) \xrightarrow{(y_2,-y_1)} \mathcal{O}(1)^{\oplus 2} \xrightarrow{(y_1,y_2)} \mathcal{O}.$$

Then $\iota_- K_-$ is exact, it is the pull-up of the Euler sequence from $\mathbb{P}^1_{y_1:y_2}$. On the other hand $\iota_+ K_-$ is a resolution of the sky-scraper sheaf $\mathcal{O}_{\mathbb{P}^1_{x_1:x_2}}$ along the zero section. Similar comments apply for the Koszul resolution K_+ of the set $\{x_1 = x_2 = 0\}$.

Let
$$\mathcal{G}_t \subset D^b(\mathcal{X})$$
be the triangulated subcategory generated by the line bundles $\mathcal{O}(t)$ and $\mathcal{O}(t+1)$. This is the **grade restriction rule** of [**?**], we are restricting to characters lying in the "window" $[t, t+1]$.

Claim 2.4. *For any $t \in \mathbb{Z}$, both ι_+^* and ι_-^* restrict to give equivalences*

$$D^b(X_+) \xleftarrow{\sim} \mathcal{G}_t \xrightarrow{\sim} D^b(X_-).$$

To see that these functors are fully-faithful it suffices to check what they do to the maps between the generating line-bundles, so we just need to check that

$$\operatorname{Ext}^\bullet_{\mathcal{X}}(\mathcal{O}(t+k), \mathcal{O}(t+l)) = \operatorname{Ext}^\bullet_{X_\pm}(\mathcal{O}(t+k), \mathcal{O}(t+l))$$

for $k, l \in [0, 1]$, i.e.

$$H^\bullet_{\mathcal{X}}(\mathcal{O}(i)) = H^\bullet_{X_\pm}(\mathcal{O}(i))$$

for $i \in [-1, 1]$, and this is easily verified. To see that they are essentially surjective we need to know that the two given line bundles generate $D^b(X_\pm)$. This is essentially a corollary of Beilinson's theorem [?]. One way to see it is to first observe that the set $\{\mathcal{O}(i),\ i \in \mathbb{Z}\}$ generates $D^b(X_\pm)$ because $X_\pm$ is quasi-projective, then use twists of the exact sequence $\iota_\pm K_\mp$ repeatedly to resolve any $\mathcal{O}(i)$ by a complex involving only $\mathcal{O}(t)$ and $\mathcal{O}(t+1)$.

So for any $t \in \mathbb{Z}$ we have a derived equivalence

$$\Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_-)$$

passing through $\mathcal{G}_t$. Composing these, we get auto-equivalences

$$\Phi_{t+1}^{-1}\Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_+).$$

To see what these do, we need to check them on the generating set of line-bundles $\{\mathcal{O}(t), \mathcal{O}(t+1)\}$. Applying Φ_t to this set is easy, it just sends them to the same line-bundles on X_-. To apply Φ_{t+1}^{-1} however, we first have to resolve $\mathcal{O}(t)$ in terms of $\mathcal{O}(t+1)$ and $\mathcal{O}(t+2)$. We do this using the exact sequence $\iota_- K_-(t)$. The result is that $\Phi_{t+1}^{-1}\Phi_t$ sends

$$\mathcal{O}(t) \mapsto \big[\mathcal{O}(t+2) \xrightarrow{(-y_2, y_1)} \mathcal{O}(t+1)^{\oplus 2}\big], \qquad \mathcal{O}(t+1) \mapsto \mathcal{O}(t+1).$$

Claim 2.5. *$\Phi_{t+1}^{-1}\Phi_t$ is an inverse spherical twist around $\mathcal{O}_{\mathbb{P}^1_{x_1:x_2}}(t)$.*

A spherical twist is an autoequivalence discovered by [?] associated to any spherical object in the derived category, i.e. an object S such that

$$\operatorname{Ext}(S, S) = \mathbb{C} \oplus \mathbb{C}[-n]$$

for some n (i.e. the homology of the n-sphere). It sends any object $\mathcal{E}$ to the cone on the evaluation map

$$\big[\operatorname{RHom}(S, \mathcal{E}) \otimes S \longrightarrow \mathcal{E}\big].$$

The inverse twist sends $\mathcal{E}$ to the cone on the dual evaluation map

$$\big[\mathcal{E} \longrightarrow \operatorname{RHom}(\mathcal{E}, S)^\vee \otimes S\big].$$

The object $\mathcal{O}_{\mathbb{P}^1_{x_1:x_2}}(t) \simeq \iota_+ K_-(t)$ is spherical, and the inverse twist around it sends $\mathcal{O}(t+1)$ to itself and $\mathcal{O}(t)$ to the cone

$$\big[\mathcal{O}(t) \longrightarrow \iota_+ K_-(t)\big] \simeq \big[\mathcal{O}(t+2) \xrightarrow{(-y_2,y_1)} \mathcal{O}(t+1)^{\oplus 2}\big],$$

which agrees with $\Phi_{t+1}^{-1}\Phi_t$. To complete the proof of the claim we would just need to check that the two functors also agree on the Hom-sets between $\mathcal{O}(t)$ and $\mathcal{O}(t+1)$.

Now instead let $V = \mathbb{C}^{p+q}$ with co-ordinates $x_1, \ldots, x_p, y_1, \ldots, y_q$. Let $\mathbb{C}^*$ act linearly on V with positive weights on each x_i and negative weights on each y_i. The two GIT quotients X_+ and X_- are both the total spaces of orbi-vector bundles over weighted projective spaces.

We must assume the Calabi–Yau condition that $\mathbb{C}^*$ acts through $SL(V)$. Let d be the sum of the positive weights, so the sum of the negative weights is $-d$. The above argument goes through word-for-word, where now

$$\mathcal{G}_t = \langle \mathcal{O}(t), \ldots, \mathcal{O}(t+d-1) \rangle.$$

3 References

NOTES ON GIT AND SYMPLECTIC REDUCTION FOR BUNDLES AND VARIETIES by R. P. THOMAS

Alper Moduli

Segal paper