Homework 3

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Problem 1 Which of the following is a Galois cover of the complex z-plane?

(a) $w^2 = 4z^3 - g_2 z - g_3$;

(b) $w^n - z^n = 1$;

(c) $w^3 + z + z^2 = w^2 + wz$; *Hint: look at the fiber over* 0.

(d) $w^2 - 2zw + z^3 = 1$.

Solution:

1. The Riemann surface is degree two over the z-plane, and one sees by inspection that $w \mapsto -w$ is a deck transformation. So the cover has Deck group at least the size of the degree, hence is Galois.

2. The Riemann surface is degree n over the z-plane, and one sees by inspection that $w \mapsto \zeta w$ ($\zeta^n = 1$) is a deck transformation. So the cover has Deck group at least the size of the degree, hence is Galois.

3. Look at the fiber over $z = 0$. The point $(0, 0)$ has valency 2 with respect to the projection and hence the cover must fix it. In particular any deck transformation must also fix $(1, 0)$ which is a point of valency 1. So it must be the identity and the cover is not Galois.

4. Set $y = w - z$. Then the equation becomes $y^2 = 1 + z^2 - z^3$ is a degree two cover of the z-plane, and one sees by inspection that $y \mapsto -y$ is a deck transformation. So the cover has Deck group at least the size of the degree, hence is Galois.

Problem 2 Let V be a rank 2 (for simplicity) vector bundle over a Riemann surface R. Assume that V has two meromorphic sections s_1, s_2 which, at some point, are holomorphic and span the fiber.

(a) Show that this will be the case everywhere except at a set of isolated points.

(b) At an exceptional point, show that we can modify V by a finite sequence of elementary transformations so that s_1 and s_2 form a holomorphic frame of the new bundle.

Suggestion: First make the sections holomorphic, then find some numerical measure for their failure to give a basis. Then find a way to reduce that number.

Remark: The argument generalizes to any dimension. If R is compact, it follows that we can trivialize V by a finite number of elementary transformations. If R is non-compact, one can show that every vector bundle is in fact trivial.

Solution: Let s_1, s_2 be two meromorphic sections of a rank 2 vector bundle V over a Riemann surface R. Since V is a holomorphic vector bundle, there exists a local trivialization of V around p.

$$V|_U \cong \mathcal{O}_U e_1 \oplus \mathcal{O}_U e_2$$

and we can write

$$s_1 = f_1 e_1 + f_2 e_2, \quad s_2 = g_1 e_1 + g_2 e_2$$

where f_i, g_i are meromorphic functions on U. The failure of s_1, s_2 to span the fiber at a point $q \in U$ is given by the vanishing of the determinant

$$D(q) = f_1(q) g_2(q) - f_2(q) g_1(q).$$

which is a meromorphic function on U. The zeroes of a meromorphic function are isolated unless the function is identically zero. Since s_1, s_2 span the fiber at p, D is not identically zero. Therefore, the set of points where s_1, s_2 fail to be holomorphic or fail to span the fiber is a discrete set of isolated points in R, because meromorphic functions can only have isolated singularities and the determinant D is meromorphic.

Let D be the effective divisor of the poles of s_1, s_2. We can make s_1, s_2 holomorphic by twisting V with the line bundle $\mathcal{O}(D)$, i.e. consider the new vector bundle

$$V(D) = V \otimes \mathcal{O}(D)$$

Then s_1, s_2 are holomorphic sections of $V(D)$. Now consider a point p where s_1, s_2 fail to span the fiber of $V(D)$. If $s_1(p)$ and $s_2(p)$ both vanish, then twist by an appropriate power of $\mathcal{O}(-p)$ to make at least one of them non-vanishing at p, say $s_1(p) \neq 0$. In a chart near $V(D)$ we have a local trivialization $V(D)|_U \cong \mathcal{O}_U e_1 \oplus \mathcal{O}_U e_2$ so that $s_1 = e_1$ and $s_2 = f(z) e_1 + g(z) e_2$ for some holomorphic functions $f(z), g(z)$. Let $L = \mathbb{C} e_1 \subset V_p$. We can perform an elementary transformation of $V(D)$ at p with respect to L to obtain a new vector bundle V' which fits into the short exact sequence of coherent sheaves

$$0 \to V' \to V(D) \to (V(D)_p / L) \otimes \mathcal{O}_p \to 0. \tag{1}$$

The wedge product of the sections is given by

$$s_1 \wedge s_2 = g(z) e_1 \wedge e_2.$$

Since s_1, s_2 fail to span the fiber at p, we have $g(0) = 0$, so we can write $g(z) = z^n h(z)$ for some $n \geq 1$ and unit $h(0) \neq 0$. After absorbing the unit $h(z)$ into e_2, we can assume $g(z) = z^n$. Then we have in local coordinates sections $s_1 = e_1$ and $s_2 = f(z) e_1 + z^n e_2$.

The elementary transformation V' is locally generated by the sections $s_1' = e_1$ and $s_2' = ze_2$. This is because $V'(U)$ consists of sections of $V(D)(U)$ whose value at p lies in $L = \mathbb{C}e_1$. Any section of $V(D)(U)$ can be written as $a(z)e_1 + b(z)e_2$ for some holomorphic functions $a(z), b(z)$. The condition that the value at p lies in L means that $b(0) = 0$, so we can write $b(z) = zc(z)$ for some holomorphic function $c(z)$. Therefore, sections of $V'(U)$ are of the form

$$a(z)e_1 + zc(z)e_2, \quad a(z), c(z) \in \mathcal{O}_U$$

which means $V'(U)$ is a $\mathcal{O}_U$-module freely generated by e_1 and ze_2. In particular, the bundle V' is locally trivialized by the sections e_1 and $e_2' = ze_2$. In the new bundle V', the sections s_1 and s_2 have wedge product

$$s_1' \wedge s_2' = z^{n-1} e_1 \wedge e_2'.$$

Thus, the order of vanishing of the wedge product at p has decreased by 1. By repeating this process a finite number of times, we can obtain a vector bundle where s_1, s_2 span the fiber at p. By performing this procedure at each point where s_1, s_2 fail to span the fiber, we can obtain a vector bundle where s_1, s_2 form a holomorphic frame everywhere.

Problem 3

(a) Consider the vector bundle V with sheaf of sections $\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_k)$ over $\mathbb{P}^1$, with $n_1 \leq \cdots \leq n_k$. Show that the sequence of integers n_i is uniquely determined by V.

(b) In contrast with (a), show that $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O}$ are isomorphic as topological vector bundles.

(c) Show that there is a holomorphic automorphism of V which takes the vector $[1, 0, \ldots, 0]$ in the fiber over 0 to $[1, 1, \ldots, 1]$.

(d) Assuming the fact that every rank k holomorphic vector bundle on $\mathbb{P}^1$ can be constructed from $\mathcal{O}^{\oplus k}$ by elementary transformations, show that it must be isomorphic to one of the form in (a).

Solution:

(a) Suppose also $V \simeq \bigoplus_{j=1}^k \mathcal{O}(m_j)$ with $m_1 \leq \cdots \leq m_k$. Recall

$$\mathrm{Hom}(\mathcal{O}(a), \mathcal{O}(b)) \cong H^0(\mathcal{O}(b-a))$$

which is 0 if $b < a$ and nonzero if $b \geq a$.

Consider the r maps ϕ_r given by the composition

$$\mathcal{O}(n_k) \hookrightarrow V \cong \bigoplus_{j=1}^{k} \mathcal{O}(m_j) \twoheadrightarrow \mathcal{O}(m_r)$$

At least one composite $\mathcal{O}(n_k) \to \mathcal{O}(m_r)$ is nonzero (if not then the inclusion would be zero), forcing $m_r \geq n_k$. With the orderings this gives $m_k \geq n_k$. By symmetry (swap the roles of n, m) we also get $n_k \geq m_k$. Hence $n_k = m_k$. Cancel that summand and argue by induction on k.

(b) Complex vector bundles of rank 2 on $\mathbb{P}^1 \cong S^2$ are topologically classified by homotopy classes of maps from S^2 to the classifying space $BU(2)$. Since

$$\pi_2(BU(2)) \cong \pi_1(U(2)) \cong \mathbb{Z}$$

the isomorphism classes of rank 2 complex vector bundles on S^2 are classified by an integer, which is the first Chern class of the bundle. The first Chern class is additive under direct sum, and $c_1(\mathcal{O}(n)) = n$. Thus, both $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O}$ have first Chern class 0, so they are isomorphic as topological vector bundles.

(c) Order $n_1 \leq \cdots \leq n_k$. For $i > 1$ pick the constant section $s_{i1} \in H^0\big(\mathcal{O}(n_i - n_1)\big)$ equal to 1 (exists since $n_i - n_1 \geq 0$). View s_{i1} as a morphism $\mathcal{O}(n_1) \to \mathcal{O}(n_i)$.

Define an endomorphism of V by the lower-triangular matrix with ones on the diagonal and s_{i1} in the $(i, 1)$-entry, zeros elsewhere. It is clearly an automorphism. On the fiber over 0 it sends $[1, 0, \ldots, 0]$ to $(1, s_{21}(0), \ldots, s_{k1}(0)) = (1, 1, \ldots, 1)$.

(d) Let L be the span of the first basis vector in the fiber of $\mathcal{O}(n_1)$ at p. On sections near p, $0 \to \ker\big(\mathcal{O}(n_1) \xrightarrow{\mathrm{ev}_p} \mathbb{C}_p\big) \cong \mathcal{O}(n_1 - 1) \to \mathcal{O}(n_1) \to \mathbb{C}_p \to 0$, while the other summands are untouched. Therefore $\mathrm{Elm}_{p,L}(\bigoplus_i \mathcal{O}(n_i)) \cong \mathcal{O}(n_1 - 1) \oplus \mathcal{O}(n_2) \oplus \cdots \oplus \mathcal{O}(n_k)$. Dually, the inverse elementary transform raises one chosen summand by $+1$. Hence any sequence of elementary transforms starting from $\mathcal{O}^{\oplus k}$ stays split and merely adjusts integers on the summands by ± 1. After reordering to $n_1 \leq \cdots \leq n_k$ you land in a bundle of the form in (a).

Problem 4 Show that on a compact Riemann surface R of genus g and a line bundle L of degree $> 2g - 2$, we have $H^1(R; \mathcal{O}(L)) = 0$. Find a counterexample to this if L is a vector bundle instead.
Remark: For noncompact Riemann surfaces, H^1 vanishes for any vector bundle.

Solution: Serre duality gives $H^1(R; \mathcal{O}(L)) \cong H^0(R; \mathcal{O}(K \otimes L^{-1}))^*$ where K is the canonical bundle of R. Since $\deg(K) = 2g - 2$, we have $\deg(K \otimes L^{-1}) = 2g - 2 - \deg(L) < 0$. A line bundle of negative degree has no nontrivial global sections, so $H^0(R; \mathcal{O}(K \otimes L^{-1})) = 0$. Therefore, $H^1(R; \mathcal{O}(L)) = 0$.

Take any R with $g \geq 1$. Pick a line bundle A with $\deg A > 2g - 2$ (there are plenty). Set $E = \mathcal{O} \oplus A$. Then $\deg E = \deg A > 2g - 2$ and

$$H^1(R, E) = H^1(R, \mathcal{O}) \oplus H^1(R, A)$$

By the claim above $H^1(R, A) = 0$, while $H^1(R, \mathcal{O}) \cong \mathbb{C}^g \neq 0$ for $g \geq 1$. Hence $H^1(R, E) \neq 0$ even though $\deg E > 2g - 2$.

Problem 5 Prove that every compact Riemann surface of genus 2 is *hyperelliptic*, meaning that it can be realized as a double (branched) cover of $\mathbb{P}^1$.
Hint: Use differentials.

Solution: Being genus 2 means that the space of holomorphic differentials $H^0(R, K)$ is 2-dimensional. Pick a basis ω_1, ω_2. Since K has degree $2g - 2 = 2$, the differentials ω_i each have two zeroes (counted with multiplicity). If they had a common zero, then they would be linearly dependent (since they are sections of a line bundle with a common zero), contradicting the choice of basis. Therefore, ω_1 and ω_2 have no common zeroes.

Define a meromorphic function $f = \omega_1/\omega_2$. Since ω_1 and ω_2 have no common zeroes, f is well-defined and meromorphic on R. The poles of f are the zeroes of ω_2, and the zeroes of f are the zeroes of ω_1. Each differential has two zeroes, so f has two poles and two zeroes, counting multiplicities. Therefore, f is a degree 2 meromorphic function on R, which defines a double cover of $\mathbb{P}^1$ (the Riemann sphere).