Quintic Threefolds

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Abstract

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Take a smooth degree-5 hypersurface $X \subset \mathbb{P}^4$. For a generic one hodge numbers?

$$h^{1,1}(X) = 1, h^{2,1}(X) = 101, h^{3,0}(X) = h^{0,3}(X) = 1$$

and all other $h^{p,q}$ vanish. (The lone $h^{1,1}$ is the Kähler class; $h^{2,1}$ is complex-structure deformations.)

There is a standard 1-parameter family (the Dwork pencil)

$$X_\psi : \ x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \ = \ 5\psi \, x_1 x_2 x_3 x_4 x_5 \subset \mathbb{P}^4$$

1. Picard–Fuchs for periods. Let $z = (5\psi)^{-5}$, $\theta = z\frac{d}{dz}$. Periods $\Pi(z)$ of the holomorphic 3-form satisfy $\mathcal{L}\Pi = 0$, where $\mathcal{L} = \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)$. Solve near the large complex structure limit $z = 0$: $\Pi_0(z) = \sum_{n\geq 0} \frac{(5n)!}{(n!)^5} z^n$, $\Pi_1 = \Pi_0 \log z + \cdots$ Extract the mirror map $q = \exp(\Pi_1/\Pi_0)$.

2. Yukawa coupling & GW invariants. Compute the Yukawa coupling C_{zzz} from the PF system, convert to C_{ttt} in the flat coordinate $t = \frac{1}{2\pi i}\log q + \cdots$, expand $C_{ttt} = 5 + \sum_{d\geq 1} \frac{n_d \, d^3 q^d}{1-q^d}$, and read off the genus-0 instanton numbers n_d (curve counts on X): $n_1 = 2875$ lines, $n_2 = 609,250$, $n_3 = 317,206,375$, etc.

3. Monodromy. Compute monodromies around $z = 0$ (maximally unipotent), the conifold point $z = 5^{-5}$, and the Gepner point $\psi = 0$. Check that one monodromy is maximally unipotent (mirror criterion).

4. Kähler vs complex moduli. Identify the complex moduli of Y with the 1-parameter Dwork modulus, and the Kähler moduli of X with the q-coordinate you built. This is the mirror map statement in practice.

5. (Optional) Toric re-derivation. Rebuild the whole story via Batyrev's reflexive polytopes for the quintic and its polar dual; compute Hodge numbers from lattice point counts to see the $(1, 101) \leftrightarrow (101, 1)$ swap without period theory.

1 Basic setup

Let $Y = \mathbb{C}^n$ and $f : Y \to \mathbb{C}$ a holomorphic function with an isolated critical point at 0.

Define the local algebra $H_f = \mathbb{C}[y_1, \ldots, y_n]/(\partial f/\partial y_i)$, called the Milnor ring or Jacobian algebra. It's a finite-dimensional vector space of dimension μ (the Milnor number).

Choose a monomial basis $a_1, \ldots, a_\mu$ representing classes in H_f. Then consider a versal deformation (a general perturbation)

$$f_\lambda(y) = f(y) + \lambda_1 a_1(y) + \cdots + \lambda_\mu a_\mu(y)$$

This gives a μ-dimensional parameter space with coordinates $\lambda = (\lambda_1, \ldots, \lambda_\mu)$.

Define $I_i(\lambda) = \int e^{f_\lambda(y)/h}\, a_i(y)\, \omega$ where $\omega = dy_1 \wedge \cdots \wedge dy_n$. As $h \to 0$, the integral is dominated by critical points of f_λ; so by stationary phase

$$I_i(\lambda) \sim \sum_{y_*(\lambda)} \frac{a_i(y_*(\lambda))}{\sqrt{J_\lambda(y_*(\lambda))}} e^{f_\lambda(y_*(\lambda))/h},$$

where $J_\lambda = \det(\partial^2 f_\lambda/\partial y_i \partial y_j)$. Each critical point contributes an exponential term with phase $f_\lambda(y_*)/h$.

The key observation is that the functions $I_i(\lambda)$ satisfy a system of differential equations in the parameters

$$\lambda_j : h\frac{\partial I_i}{\partial \lambda_j} = \sum_k c_{ij}^k(\lambda) I_k$$

where the $c_{ij}^k(\lambda)$ are the structure constants of the algebra $a_i a_j = \sum_k c_{ij}^k(\lambda) a_k$ in H_{f_λ}.

This has something to do with the Gauss Manin connection. Formally, you can think of the family of vector spaces $\mathcal{H}_\lambda = H_n(\mathbb{C}^n, \Re f_\lambda = -\infty)$ as forming a flat vector bundle over the parameter space of λ.

The integrals $I_i(\lambda)$ can be viewed as flat sections of the dual bundle $\mathcal{H}_\lambda^*$. The differential equations satisfied by the $I_i(\lambda)$ reflect the flatness of this connection.

The family of quintic-mirrors Y_λ is one of the examples for which one can construct flat coordinates on moduli spaces of complex structures.

2 Givental Hori Vafa

Let X be a compact toric Fano variety. Let $\mathcal{F}(X)$ be its **Fukaya category** which is $D(\mathbb{Z}/2c_1(X)\mathbb{Z})$-graded. Then conjecturally

$$HH^*(\mathcal{F}(X)) \cong QH^*(X)$$

Write $X = \mathbb{C}^n//H$ where

$$0 \to H \to (\mathbb{C}^*)^n \to T_\mathbb{C} \to 0$$
$$0 \to T_\mathbb{C}^\vee \to (\mathbb{C}^*)^{n\vee} \to H^\vee \to 0$$

Consider the fiber $T_h^\vee$ of $h \in H^\vee$. Take the **superpotential** function

$$W = x_1 + \cdots + x_n : T_h^\vee \to \mathbb{C}$$

Remark 2.1. *Joe made the remark that if you try to make the naive statement that there are two derived catgories on the A and B side of mirror symmetry which are equivalent, then this cannot possibly work and one needs to introduce extra structures, such as the superpotential W here.*

Then by homological mirror symmetry this defines a matrix factorization category $MF(T_h^\vee, W)$ with a "map"

$$MF(T_h^\vee, W) \to T_h^\vee$$

Then we have the following theorem:

Theorem 2.2.

1. *There is an equivalence of categories*

$$MF(T_h^\vee, W) \cong \mathcal{F}(X, \mathfrak{h})$$

2. *$MF(T_h^\vee, W)$ is a module category over $\mathbb{C}[T_h^\vee]$ and the Fourier modes are the Seidel shift operators.*

3. *There is an isomorphism of algebras*

$$HH^*(MF(T_h^\vee, W)) \cong Jac(W) \cong QH^*(X, \mathfrak{h})$$