Songyu Ye
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1 Canonical generators for equivariant cohomology

I'm trying to understand the following idea. If I have a T-invariant subvariety $Y \subset X$ then this defines a class $[Y]$ in the T-equivariant cohomology $H_T^*(X)$. The idea is that we can then view them inside the ring $H_T^*(X^T)$ as lists of polynomials indexed by the T-fixed points.

The ring $H_T^*(X^T)$ has canonical generators also indexed by the T-fixed points given by the Bialynicki-Birula decomposition. The idea is that every fixed point p gives rise to an attracting set X_p which is a T-invariant subvariety, and then we can pullback the class $[X_p]$ to X^T and this gives us a canonical generator.

This decomposition also gives us a partial order on the fixed points, where we say that $v \leq w$ if $X_v \subset \overline{X_w}$. For the flag variety in $\mathrm{GL}(n)$ with the torus action of conjugation, the fixed points are indexed by the Weyl group S^n and the partial order is given by the Bruhat order.

There is this related idea of looking at the B-orbits on X. It seems that there is a relationship between the B-orbits and the attracting sets of the T-fixed points. They are both defining elements in the T equivariant cohomology rings of X and there are results saying that they both form a basis. Should they be the same? In fact why is one even studying the B-orbits? We are interested in the T-equivariant cohomology ring, so why are we looking at the B-orbits. Because B deformation retracts onto T?

Tymoczko describes an algorithm for realizing canonical generators for $H_T^*(X)$ in the ring $H_T^*(X^T)$. The algorithm is as follows. Recall the partial order observed above.

Fix $v \in X^T$. I will describe the canonical generator S_v associated to v. For all $u < v$ the generator restricts to 0. Now consider $w \in X^T$ so that every $u < w$ is known. Then $S_v|_w$ is the minimal degree guy that satisfies all the relations, in particular it should be zero whenever possible (except for when $w = v$ in which case we insist that it is not zero).

The claim is that the output of this algorithm is the pullback of the class $[X_v]$ to X^T. This output is unique up to a scalar whenever the moment graph can be drawn in the plane so that hte vertex v lies above its neighbor u if and only if v has more downward edges than u.

One question is what does it even mean to restrict a class of the cell decomposition to a fixed point? If X_v is a cell (one of the basis elements for the T-equivariant cohomology ring) then when I restrict it to w I should get a polynomial in the weights of T. What data does this even mean?

2 Example

Consider the $T \times T$ action on $\overline{\mathrm{PGL}(2)} \cong \mathbb{P}^3$. There are two $G \times G$ orbits, the open one and the closed one corresponding to taking no simple roots or taking all of the simple roots.

Now we can ask about the $B \times B^-$ orbits in each. In the open orbit $\cong \mathrm{PGL}(2)$ this is precisely the Bruhat decomposition.

$$\mathrm{PGL}(2) = B \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} B^- \cup B \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} B^-$$

In the closed orbit $\cong \mathbb{P}^1 \times \mathbb{P}^1$, recall that every projective 2×2 matrix of rank 1 is just a row vector times a column vector (projectively)

$$M = r \cdot c$$

and so thinking about the $B \times B^-$ orbits

$$BMB^- = (Br)(cB^-)$$

and so there are four $B \times B^-$-orbits, the product of the situations when

$$r = \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ * \end{bmatrix}$$

So there are four orbits which I will denote

$$\infty \times \infty, \infty \times \mathbb{A}^1, \mathbb{A}^1 \times \infty, \mathbb{A}^1 \times \mathbb{A}^1$$

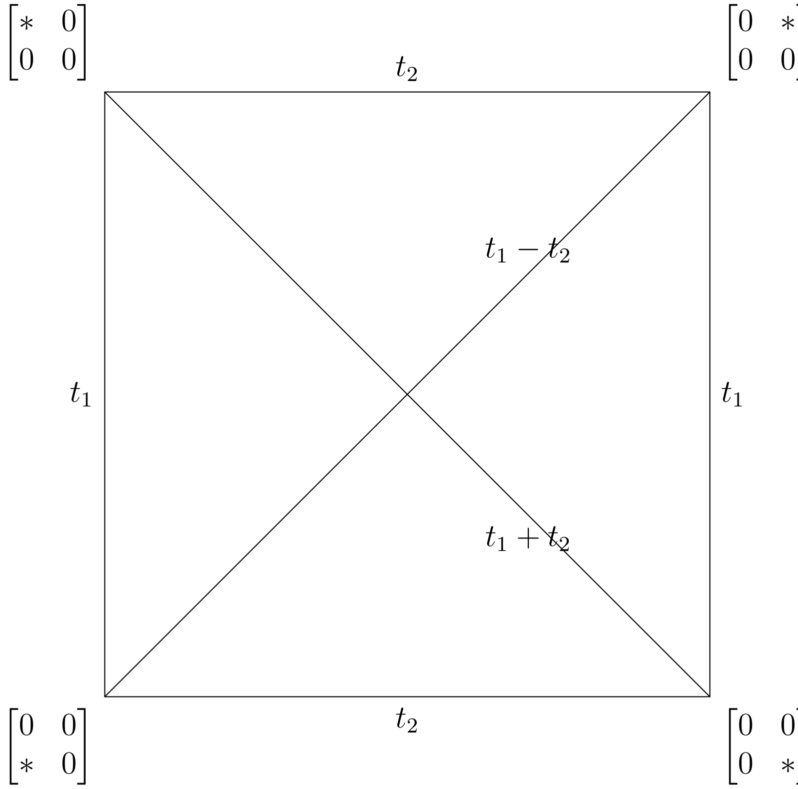
Previously I said that there were maps

$$H_T^*(X) \to H_T^*(\mathbb{P}^1 \times \mathbb{P}^1) \to H_T^*(X^{T \times T})$$

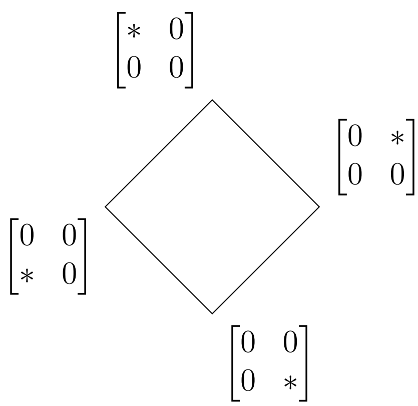
and we know that the second map is an injection. The first map is supposed to correspond to take the $B \times B^-$ orbits in the open orbit, take their closures, and see which $B \times B^-$ orbits in the closed orbit appear. If you are a $B \times B^-$ orbit in the closed orbit, then the first map is the identity on the class of your closure. But then the first map is not injective as it is supposed to be. I've gone wrong somewhere. There are more $B \times B^-$ orbits in X than there are $T \times T$ fixed points. Is it just the case that the $B \times B^-$ orbits do not form a basis, but perhaps they are only generators.

Let's continue with the calculation by writing down generators in the GKM ring. What are the attracting sets?

We can just use the algorithm: The GKM graph with labels looks like

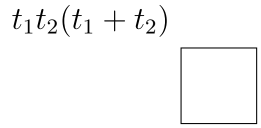


The partial ordering on these guys looks like

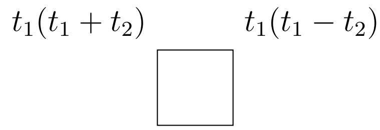


The algorithm says that the canonical generators are

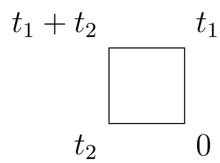
$$X_{\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}} =$$



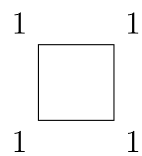
$$X_{\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}} =$$



$$X_{\begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}} =$$



$$X_{\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}} =$$



It felt like I had to make ad hoc choices along the way. For example it feels like rows 2 and 3 are symmetric in Bruhat order so they should either both be degree 2 or both be degree 1. However we know the Betti numbers of $\mathbb{CP}^3$. waht should the betti numbers of other wonderful compactifications be? Schubert calculus seems to be really stupid in the case that all of the Betti numbers are 1

To continue this calculation we can think about what is happening with the $B \times B^-$ orbits.

3 B orbits and T attracting sets

Indeed there is a corresponense that I was concerned about. First an example.

Example 3.1 (Allen).

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While I basically agree with Kevin Buzzard that this is something to find in a textbook rather than on mathoverflow, I'll take the opportunity to give a totally nonstandard description, inspired by Shizuo Zhang's comment.

Given an action of the circle group $S = \mathbb{G}m$ on a smooth variety X, with isolated fixed points X^S, we can define a Bialynicki-Birula decomposition

$$X = \coprod_{f \in X^S} X_f, \qquad X_f := \{x \in X : \lim_{z \to 0} S(z) \cdot x = f\}.$$

Part of B-B's theorem is that each X_f is a copy of affine space.

If $Y \subseteq X$ is S-invariant, then Y acquires a similar decomposition, and $Y_f = X_f \cap Y$ for each $f \in Y^S \subseteq X^S$ (very easy to prove).

Consider the embedding $Y := GL_n/B = Flags(n) \to \prod_{k=1}^n Gr(k,n) \to \prod_{k=1}^n \mathbb{P}(Alt^k\ \mathbb{C}^n) =: X$, where the second map is made of Plucker embeddings, and take S acting on $\mathbb{C}^n$ by $z \mapsto diag(z, z^2, z^3, \ldots, z^n)$, AKA the $\check{\rho}$ coweight. Then its fixed points on each $\mathbb{P}(Alt^k\ \mathbb{C}^n)$ are indexed by k-element subsets of $1 \ldots n$. So X^S is lists of subsets, and Y^S is increasing lists of subsets, or equivalently permutations.

Ergo, there exists a decomposition of GL_n/B into affine spaces, indexed by permutations. (It's not obvious from this description that they are the B-orbits, but maybe that's okay, since more spaces have these S-actions than have finitely many B-orbits.)

The point is that in this case the B orbits on $\mathrm{GL}(n)/B$ are precisely in correspondense with the fixed points of the torus action on the flag variety, and moreover the B orbits are the attracting sets of the fixed points.

In general, we appeal to more facts about the Bialynicki-Birula decomposition.

Recall the setup. We have a connected reductive algebraic group G with a maximal torus T and Borel B, fixed. Let X be smooth projective variety with an action of G so X has finitely many G-orbits.

For a one parameter subgroup $\lambda : \mathbb{C}^* \to T$ we can consider the attracting set for $y \in X^T$

$$X^\lambda(y) = \{x \in X : \lim_{t \to 0} \lambda(t) \cdot x = y\}$$

If we choose λ in the interior of the Weyl chamber, then $X_\lambda(y)$ is B -stable. For sufficiently general λ, then $X^{\mathbb{C}^*} = X^T$. This is because we can think of X as sitting in some projective space $\mathbb{P}(V)$ and then T acts on V via characters

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi$$

. Now let λ sufficiently general so that $\langle \lambda, \chi \rangle \neq 0$ for all χ which appear. Therefore λ and T induce the same eigendecomposition of V and so a point $x \in \mathbb{P}(V)$ is fixed by one if and only if it is fixed by the other.
We say that a G-variety is spherical if it is normal and some Borel B has an open dense orbit. We say that a spherical G-variety with open G-orbit X_G^0 is toroidal if the closure of every B-stable divisor in X_G^0 contains no G-orbit.

Complete symmetric spaces and toric varieties are toroidal.

Theorem 3.2. *Let X be a toroidal complete G-variety. Then the intersection of any cell $X^\lambda(y)$ with any G-orbit is empty or a single B-orbit.*

4 Our story

We have the torus $T \times T$ acting on the wonderful compactification $\overline{\mathrm{PGL}}(2)$. The fixed points are indexed by elements of $S_2 \times S_2$ and they tell us how to decompose the wonderful compactification into cells. The attracting sets are the $B \times B^-$ orbits. However there are attacting sets which do not contain $T \times T$ fixed points. So it is not true that the map

$$H_T^*(\overline{\mathrm{PGL}}(2)) \to H_T^*(\mathbb{P}^1 \times \mathbb{P}^1)$$

is injective.

5 Loop group of $\mathrm{SU}(2)$

Is there a hope of doing Schubert calculus in the loop group of $\mathrm{SU}(2)$? What indexes the fixed points? What are the attracting sets? Is there a corresponding Bialynicki-Birula decomposition?

There is a corresponding notion of Bruhat decomposition and Schubert varieties in Kac Moody groups. People have also studied the closure patterns of these decompositions. Have people thought about the equivariant cohomology of these things? One can start with the paper of Billig and Dyer.

6 Affine flag varieties and their Schubert calculus

A good place to start thinking about these questions is the paper of Braden and MacPherson. We should read this and be prepared to talk with Tara on Thursday about it.

7 References

- https://mathoverflow.net/questions/358450/bialynicki-birula-decompositions-and-fixed-points

- https://mathoverflow.net/questions/284894/bialynicki-birula-decomposition-and-moment-polytopes-graphs