Bialynicki-Birula Decomposition

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Abstract

We follow the original paper by Bialynicki-Birula and add details wherever I found them helpful.

Contents

1 Preliminaries

Let $\eta : G \times X \to X$ be an action of an algebraic group scheme G on an algebraic scheme X over an algebraically closed field k. The action η is said to be *effective* if it satisfies the following condition: if η is induced from an action

$$\eta_1 : G_1 \times X \longrightarrow X$$

(where G_1 is another algebraic group scheme) by a surjective homomorphism

$$\varphi : G \longrightarrow G_1,$$

then φ is an isomorphism. Equivalently, an action of a group G on a scheme X is effective (or faithful) if no nontrivial element of G acts as the identity on all of X.

An algebraic group scheme G is said to be linearly reductive if every G-module is semisimple. If G is an algebraic torus (i.e. $G \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$) then any G-module can be presented as a direct sum of one-dimensional G-modules; i.e., for any G-module V and a fixed isomorphism

$G \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m = \mathbb{G}_m^n$, where n is an integer, there exists a basis $\{v_i\}_{i \in I}$ of V such that for any $(\lambda_1, \ldots, \lambda_n) \in \mathbb{G}_m(k) \times \cdots \times \mathbb{G}_m(k)$ the image of v_i under $(\lambda_1, \ldots, \lambda_n)$ equals $\lambda_1^{m_{i1}} \cdots \lambda_n^{m_{in}} v_i$, where m_{ij} are integers.

We say (again for a fixed $G \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$) that the module is *positive* (resp. *negative*) if we have

(a) $m_{ij} \geq 0$ (resp. $m_{ij} \leq 0$), for all i, j.

(b) For every index $i \in I$, there exists j such that $m_{ij} \neq 0$.

The module is said to be *non-negative* (resp. *non-positive*) if (a) is satisfied. We say that the module is *fully definite* (resp. *definite*) if there exists an isomorphism $G \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ for which the module is positive (resp. non-negative). Equivalently, a module is fully definite (resp. definite) if its set of weights lies entirely in some positive (resp. non-negative) half-space of $X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $\eta : G \times X \to X$ be an action of a torus G on X and let a be a closed point from X^G. We say that the action η is *fully definite* (resp. *definite*) at a closed point $a \in X^G$ if the G-module $T_a(X)$ is fully definite (resp. definite).

It is easy to see that if X is irreducible quasi-affine and the action is definite at a closed point $a \in X^G$ then the G-module $k[X]$ is definite. Notice also that (cf. Theorem 2.1) if the action is fully definite at a then a is an isolated fixed point of η, and if X is irreducible and affine and $G = \mathbb{G}_m$ then it is fully definite.

2 Introduction

Lemma 2.1 (2.1)**.** *Let b be a non-singular closed point of an algebraic scheme. If N is a k-vector space contained in $\mathfrak{m}_b$ such that the canonical map*

$$\mathfrak{m}_b \longrightarrow \mathfrak{m}_b/\mathfrak{m}_b^2$$

maps N isomorphically onto $(N + \mathfrak{m}_b^2)/\mathfrak{m}_b^2$, then the local ring $\mathcal{O}_b/N\mathcal{O}_b$ is regular and

$$(N\mathcal{O}_b + \mathfrak{m}_b^2)/\mathfrak{m}_b^2 \;=\; (N + \mathfrak{m}_b^2)/\mathfrak{m}_b^2.$$

Moreover, if N is contained in an ideal $\mathfrak{n}$ such that $\mathcal{O}_b/\mathfrak{n}$ is regular and

$$(N\mathcal{O}_b + \mathfrak{m}_b^2)/\mathfrak{m}_b^2 \;=\; (\mathfrak{n} + \mathfrak{m}_b^2)/\mathfrak{m}_b^2,$$

then $N\mathcal{O}_b = \mathfrak{n}$.

Remark 2.2 (Reminder on Zariski tangent space)**.** *Recall for a k-point $a \in X$ with local ring $\mathcal{O}_{X,a}$ and maximal ideal $\mathfrak{m}_a$,*

$$T_a(X) \;=\; \mathrm{Der}_k(\mathcal{O}_{X,a}, k) \;\cong\; \mathrm{Hom}_k\big(\mathfrak{m}_a/\mathfrak{m}_a^2,\; k\big)$$

The G-action induces a representation on $\mathfrak{m}_a/\mathfrak{m}_a^2$ and then $T_a(X)$ carries the dual representation. If X is smooth at a, then $\dim_k T_a(X) = \dim_k(\mathfrak{m}_a/\mathfrak{m}_a^2)$ and both have dimension $\dim X$.

Theorem 2.3 (2.1)**.** *Let G be linearly reductive. Suppose that $U_0 \subset U_1$ are G-submodules of $T_a(X)$. Let X_0 be a G-invariant closed irreducible subscheme containing a, such that a is non-singular on X_0 and $T_a(X_0) = U_0$. Then one may find a closed irreducible G-invariant subscheme X_1 of X such that X_0 is a closed subscheme of X_1, a is non-singular on X_1, and $T_a(X_1) = U_1$. Moreover, if X_0 is reduced then X_1 is reduced.*

Proof. Let $\mathfrak{m}, \mathfrak{n}_0$ be the ideals of $k[X]$ corresponding to a and X_0, respectively. Then $\mathfrak{n}_0 \subset \mathfrak{m}$. Moreover, $\mathfrak{m}, \mathfrak{n}_0$ are G-submodules of $k[X]$ (recall that G acts on $k[X]$ by $g \cdot f(x) = f(g^{-1}x)$, in particular contragredient action).

Also $\mathfrak{m}\mathcal{O}_a = \mathfrak{m}_a$ (This is just saying the maximal ideal in the local ring is obtained by extending the maximal ideal from the coordinate ring under localization).

We may consider the G-submodules $U_0^\perp, U_1^\perp \subset \mathfrak{m}_a/\mathfrak{m}_a^2$. Note that $U_1^\perp \subset U_0^\perp$ since $U_0 \subset U_1$.

The canonical map

$$\varphi_a : \mathfrak{m}_a \longrightarrow \mathfrak{m}_a/\mathfrak{m}_a^2$$

induces a G-homomorphism $\mathfrak{m} \to \mathfrak{m}_a/\mathfrak{m}_a^2$, and φ_a maps $\mathfrak{m}$ onto $\mathfrak{m}_a/\mathfrak{m}_a^2$ since $\mathfrak{m}$ generates $\mathfrak{m}_a$. Moreover, φ_a sends $\mathfrak{n}_0$ onto $U_0^\perp$.

Since G is linearly reductive, we can lift the G-complement of $U_1^\perp$. Find a G-submodule N_1 of $\mathfrak{m}$ which satisfies the following property: $N_1 \subset \mathfrak{m}$, and φ_a maps N_1 isomorphically onto $U_1^\perp$.

Let $\mathfrak{n}_1$ be the ideal of $k[X]$ defined as the intersection of the radical of $N_1 k[X]$ and $\mathfrak{n}_0$. Let X_1 be the irreducible component containing the element a of the closed subscheme corresponding to the ideal $\mathfrak{n}_1$.

Remark 2.4 (What is happening here?)**.** *A linear subspace $U_1 \subset T_a(X)$ corresponds to its annihilator $U_1^\perp \subset \mathfrak{m}_a/\mathfrak{m}_a^2$. So specifying a subspace of tangent directions is equivalent to specifying which cotangent linear forms vanish on it. $N_1 \subset \mathfrak{m} \subset k[X]$ is chosen so that its image in $\mathfrak{m}_a/\mathfrak{m}_a^2$ is $U_1^\perp$. In other words, elements of N_1 are global functions vanishing at a whose differentials at a kill exactly the tangent vectors in U_1. N_1 gives equations that cut out, to first order at a, exactly those tangent directions outside U_1.*

The point is that we want a closed subscheme of X defined by an ideal $\mathfrak{n}_1 \subset k[X]$ such that locally at a, the generators of $\mathfrak{n}_1$ are exactly N_1. If you only used N_1, you'd get an ideal describing a subvariety that *locally has tangent space U_1. But globally, $N_1 k[X]$ might define something nonreduced or too small. So we take the radical, and intersects with $\mathfrak{n}_0$, to ensure that globally the closed set is reduced, G-stable, and still contains X_0.*

Then X_1 satisfies all desired properties.

In fact, the ideal $\mathfrak{n}_1\mathcal{O}_a$ is equal to $N_1\mathcal{O}_a$ and is contained in the ideal $\mathfrak{n}_0\mathcal{O}_a$. To see this, we start by localizing

$$\mathfrak{n}_1\mathcal{O}_a = \left(\sqrt{N_1 k[X]} \cap \mathfrak{n}_0\right)\mathcal{O}_a = \sqrt{N_1 k[X]}\mathcal{O}_a \ \cap \ \mathfrak{n}_0\mathcal{O}_a$$

But since a is in X_0, we have $\mathfrak{n}_0 \subset \mathfrak{m}$. After localizing at a, $\mathfrak{n}_0\mathcal{O}_a$ is contained in $\mathfrak{m}_a$. Meanwhile, $N_1 \subset \mathfrak{m}$, so $N_1\mathcal{O}_a \subset \mathfrak{m}_a$ as well. So we only need to understand $\sqrt{N_1 k[X]}\mathcal{O}_a$.

The general fact $\sqrt{I}\mathcal{O}_a = \sqrt{I\mathcal{O}_a}$ for any ideal $I \subset k[X]$ implies that $\sqrt{N_1 k[X]}\mathcal{O}_a = \sqrt{N_1\mathcal{O}_a}$. It remains to show that $\sqrt{N_1\mathcal{O}_a} = N_1\mathcal{O}_a$ is already radical.

Remark 2.5 (Why is $N_1\mathcal{O}_a$ radical?)**.** *We appeal to some standard commutative algebra. Let $(R, \mathfrak{m})$ be a regular local ring of dimension d. Choose elements $x_1, \dots, x_d \in \mathfrak{m}$ whose classes $\bar{x}_i$ form a k-basis of $\mathfrak{m}/\mathfrak{m}^2$; this is a regular system of parameters.*

For any $0 \le r \le d$, set $I := (x_{r+1}, \dots, x_d) \subset R$. Then:

1. *$x_{r+1}, \dots, x_d$ is a regular sequence; hence* $\operatorname{ht} I = d - r$.
2. *R/I is regular local of dimension r (indeed $R/I \cong k[[x_1, \dots, x_r]]$ after completion).*
3. *In particular, R/I is a domain, so I is prime; therefore I is radical.*

Apply this fact to N_1. We have a subspace $U_1^{\perp} \subset \mathfrak{m}_a/\mathfrak{m}_a^2$. Choose lifts $f_{r+1}, \dots, f_d \in \mathfrak{m}_a$ whose classes form a k-basis of $U_1^{\perp}$, and then extend to a basis of $\mathfrak{m}_a/\mathfrak{m}_a^2$ by adding $f_1, \dots, f_r$. Because $R = \mathcal{O}_{X,a}$ is regular, $f_1, \dots, f_d$ is a regular system of parameters. Choosing generators $f_{r+1}, \dots, f_d \in N_1$ that map to a basis of $U_1^{\perp}$, we can write that ideal as $(f_{r+1}, \dots, f_d)$, so $N_1\mathcal{O}_a = (f_{r+1}, \dots, f_d)$. By the lemma above, $R/(N_1\mathcal{O}_a)$ is regular (hence reduced, hence a domain), so $N_1\mathcal{O}_a$ is prime and radical.

Therefore:

$$\mathfrak{n}_1\mathcal{O}_a = \sqrt{N_1 k[X]}\mathcal{O}_a \cap \mathfrak{n}_0\mathcal{O}_a = \sqrt{N_1\mathcal{O}_a} \ \cap \ \mathfrak{n}_0\mathcal{O}_a = N_1\mathcal{O}_a \ \cap \ \mathfrak{n}_0\mathcal{O}_a$$

But since $N_1 \subset \mathfrak{m}$ was chosen so that $\varphi_a(N_1) = U_1^{\perp}$ extends $\varphi_a(\mathfrak{n}_0)$, the containment $N_1\mathcal{O}_a \subseteq \mathfrak{n}_0\mathcal{O}_a$ holds. So the intersection just gives $\mathfrak{n}_1\mathcal{O}_a = N_1\mathcal{O}_a$. This gives $X_0 \subset X_1$.

Since N_1 maps isomorphically onto its image in $\mathfrak{m}_a/\mathfrak{m}_a^2$, $\mathcal{O}_a/N_1\mathcal{O}_a$ is regular. Geometrically: the subscheme X_1 defined by $\mathfrak{n}_1$ is smooth at a. Therefore the point a is non-singular on X_1. Moreover (again by Lemma 2.1),

$$T_a(X_1) \;=\; \big((N_1+\mathfrak{m}_a^2)/\mathfrak{m}_a^2\big)^{\perp} \;=\; \varphi_a(N_1)^{\perp} \;=\; U_1^{\perp} \;=\; U_1.$$

Since N_1 is a G-submodule of $k[X]$, X_1 is G-invariant. Moreover, if X_0 is reduced, then the radical of $\mathfrak{n}_0$ is equal to $\mathfrak{n}_0$, hence the radical of $\mathfrak{n}_1$ is equal to $\mathfrak{n}_1$, so X_1 is reduced. □

Theorem 2.6 (2.2)**.** *Let G, U_0, U_1, X_0 be as in Theorem 2.1. Moreover assume that $U_0 = \{0\}$ (i.e. $X_0 = \{a\}$). Then there exists exactly one G-invariant reduced and irreducible closed subscheme X_1 such that a is non-singular on X_1 and $T_a(X_1) = U_1$ if and only if there exists no non-zero G-homomorphism*

$$S^r(U_1) \;\longrightarrow\; T_a(X)/U_1,$$

for any integer $r \geq 1$.

Proof. Let X_1 be a closed, reduced and irreducible subscheme of X and let $\mathfrak{n}_1$ be the ideal of X_1 in $k[X]$. The subscheme X_1 satisfies the conditions given in the theorem if and only if there exists a G-submodule $N_1 \subset \mathfrak{m}$ such that N_1 is mapped isomorphically onto $U_1^{\perp}$ under the map induced by the canonical

$$\varphi_a : \mathfrak{m}_a \;\longrightarrow\; \mathfrak{m}_a/\mathfrak{m}_a^2, \qquad \text{and } N_1\mathcal{O}_a = \mathfrak{n}_1\mathcal{O}_a.$$

In fact if such N_1 exists then X_1 satisfies the conditions of the theorem (X_1 is G-invariant since $N_1k[X]$ is a G-submodule, a is non-singular on X_1 and $T_a(X_1) = U_1$ by Lemma 2.1). On the other hand if X_1 satisfies the conditions then $\mathfrak{n}_1$ is sent by φ_a onto $U_1^{\perp}$, hence we may find a G-submodule $N_1 \subset \mathfrak{m}$ which is mapped isomorphically onto $U_1^{\perp}$. Then by Lemma 2.1,

$$N_1\mathcal{O}_a = \mathfrak{n}_1\mathcal{O}_a.$$

Therefore one may find two different subschemes satisfying the conditions of the theorem if and only if one may find two subspaces $N_1, N_2 \subset \mathfrak{m}$ satisfying the following conditions:

(0) N_1, N_2 are G-submodules of $k[X]$.

(1) N_1, N_2 are mapped by φ_a isomorphically onto $U_1^{\perp}$.

(2) $N_2\mathcal{O}_a \neq N_1\mathcal{O}_a$.

Put $\mathfrak{n}_1 = N_1\mathcal{O}_a$. Then (2) is equivalent to:

(2′) There exists an integer n such that

$$(N_2+\mathfrak{m}_a^n)/\mathfrak{m}_a^n \;\not\subset\; (\mathfrak{n}_1+\mathfrak{m}_a^n)/\mathfrak{m}_a^n.$$

On the other hand we have the following exact sequence of $\mathcal{O}_a$-modules and G-modules

$$0 \longrightarrow (\mathfrak{n}_1 + \mathfrak{m}_a^n)/\mathfrak{m}_a^n \longrightarrow \mathfrak{m}_a/\mathfrak{m}_a^n \xrightarrow{\psi} \mathfrak{m}_a/(\mathfrak{n}_1 + \mathfrak{m}_a^n) \longrightarrow 0,$$

and N_1, N_2 satisfying (0), (1) satisfy (2′) if and only if

(2″) $\psi\big((N_2 + \mathfrak{m}_a^n)/\mathfrak{m}_a^n\big) \neq 0$, for some integer $n \geq 1$.

Moreover we have the following isomorphisms of G-modules:

$$\mathfrak{m}_a/(\mathfrak{n}_1 + \mathfrak{m}_a^n) \cong \bigoplus_{r=1}^{n-1} S^r\big(\mathfrak{m}_a/(\mathfrak{n}_1 + \mathfrak{m}_a^2)\big) \cong \bigoplus_{r=1}^{n-1} S^r\big(\mathfrak{m}_a/\mathfrak{m}_a^2\big)/U_1^\perp,$$

and

$$(N_2 + \mathfrak{m}_a^n)/\mathfrak{m}_a^n \cong U_1^\perp.$$

Hence if (0), (1), (2″) are satisfied by some N_1, N_2 then there exists a non-zero G-homomorphism

$$\tau : U_1^\perp \longrightarrow \mathfrak{m}_a/(\mathfrak{n}_1 + \mathfrak{m}_a^r),$$

for some integer $r \geq 1$, or equivalently (by duality), there exists a non-zero G-homomorphism

$$S^r(U_1) \longrightarrow T_a(X)/U_1.$$

On the other hand, if such a homomorphism τ exists and N_1 satisfying (0), (1) is chosen, then we may find a non-zero G-homomorphism

$$U_1^\perp \longrightarrow \mathfrak{m}_a/(\mathfrak{n}_1 + \mathfrak{m}_a^{r+1}) \qquad \text{(where } \mathfrak{n}_1 = N_1\mathcal{O}_a\text{)}.$$

Hence we may find a G-submodule N_2' of $\mathfrak{m}_a/\mathfrak{m}_a^{r+1}$ mapped isomorphically onto $U_1^\perp$ (by the canonical map $\mathfrak{m}_a/\mathfrak{m}_a^{r+1} \to \mathfrak{m}_a/\mathfrak{m}_a^2$) but not contained in $(\mathfrak{n}_1 + \mathfrak{m}_a^{r+1})/\mathfrak{m}_a^{r+1}$. Then we may find a G-submodule N_2 of $\mathfrak{m}$ mapped isomorphically onto N_2' by the canonical map $\mathfrak{m} \to \mathfrak{m}_a/\mathfrak{m}_a^{r+1}$ (since $\mathfrak{m}$ is mapped onto $\mathfrak{m}_a/\mathfrak{m}_a^{r+1}$). Then N_1, N_2 satisfy (0), (1), (2″), and hence (0), (1), (2). Thus the proof is complete. □

Corollary 2.7. *Let $G = \mathbb{G}_m$. If U_1 is equal to one of the following subspaces of $T_a(X)$:*

$$T_a(X)^0, \quad T_a(X)^+, \quad T_a(X)^-, \quad T_a(X)^0 \oplus T_a(X)^+, \quad T_a(X)^0 \oplus T_a(X)^-,$$

then there exists exactly one closed, irreducible and reduced subscheme X_1 through a such that X_1 is G-invariant, a is non-singular on X_1 and $T_a(X_1) = U_1$.

Proof. Look at the candidate U_1's listed in the corollary. For each of these choices, $S^r(U_1)$ consists only of weight-0, weight positive, or weight negative parts. The quotient $T_a(X)/U_1$ consists of the complementary weights, and there is no nonzero homomorphism of $\mathbb{G}_m$-modules from a pure-weight representation to one of a different weight. So all the Hom spaces vanish and the uniqueness condition in Theorem 2.2 is satisfied. $\square$

Theorem 2.8 (2.3)**.** *Let X be irreducible and reduced. Let G be an algebraic torus. If the action of G on X is definite at $a \in X^G$ then X^G is irreducible.*

Proof. Since the action is definite at a, $k[X]$ as a G-module is definite. Consider the subspace $k[X]^0 \subset k[X]$. Then there exists a G-homomorphism of algebras (the map that "throws away" all positive-weight components and keeps only the weight-zero part)

$$\gamma : k[X] \longrightarrow k[X]^0$$

and γ is a homomorphism of the k-algebra $k[X]$ onto the k-algebra $k[X]^0$ (since $k[X]$ as a G-module is definite). It is an algebra homomorphism precisely because the positive-weight part is an ideal (thanks to definiteness).

The kernel of this homomorphism is the complement of $k[X]^0$ in $k[X]$ and this is exactly the ideal defining X^G (Any f in the complement transforms by a nonzero weight, so it can't take a nonzero constant value on a fixed point of G, so it must vanish on X^G).

Hence

$$k[X]^0 = k[X^G]$$

is an integral domain (because it is a subring of $k[X]$) and $X^G = \mathrm{Spec}(k[X]^0)$ is irreducible. $\square$

Corollary 2.9. *Let G and X be as in Theorem 2.3. If $\dim X = n$, $\dim G = m$, $\dim X^G = n - m$, then X^G is irreducible.*

Proof. Take $a \in X^G$. Then the tangent space at a splits as a G-module: $T_a(X) = T_a(X^G) \oplus N_a$, where $T_a(X^G)$ is the zero-weight subspace (directions fixed by G) and N_a is the "moving" part, sum of nonzero weight spaces.

Since $\dim X^G = n - m$, $\dim N_a = m = \dim G$. I claim that the weight spaces appearing in N_a span the character group $X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$. If not, then there exists a nontrivial one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ that kills all the weights appearing in N_a. Then λ acts trivially on N_a, hence on $T_a(X)$. If a whole 1-dimensional subtorus acts trivially on $T_a(X)$, then infinitesimally the fixed locus of G near a has codimension at most $(m-1)$, not m. This contradicts the assumption that $\dim X^G = n - m$. So the weights appearing in N_a span $X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$.

By changing coordinates on $G \cong (\mathbb{G}_m)^m$, we can assume those weights are the standard coordinate characters $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$. This puts us in the definite case. By Theorem 2.3, X^G is irreducible. $\square$

Now we state a gluing theorem for local models which says that if two G-schemes look the same infinitesimally at fixed points (same tangent representation), and you've already matched up some invariant subschemes through a G-isomorphism, then you can find a third scheme X_0 mapping étale-equivariantly into both, such that everything matches on the invariant subschemes. In other words, there is a "common local étale neighborhood" X_0 of a_1, a_2 that identifies the situations.

Theorem 2.10 (2.4). *Let G be linearly reductive and let G act on algebraic schemes X_1, X_2. Let $a_i \in X_i^G$ be closed and non-singular on X_i, for $i = 1, 2$. Suppose that the induced actions of $G(k)$ on $T_{a_i}(X_i)$, for $i = 1, 2$, are isomorphic and assume that there exist G-invariant closed subschemes $Y_i \subset X_i$, $i = 1, 2$, such that*

(i) $a_i \in Y_i$ and a_i is non-singular on Y_i, for $i = 1, 2$,

(ii) there exists a G-isomorphism $\alpha : Y_1 \to Y_2$ such that $\alpha(a_1) = a_2$.

Then there exist a scheme X_0, an action of G on X_0, a G-invariant subscheme Y_0 of X_0, morphisms

$$\beta_i : X_0 \to X_i, \quad i = 1, 2,$$

and a closed point $a_0 \in X_0$ such that:

(a) β_1, β_2 are étale,

(b) $\beta_i^{-1}(Y_i) = Y_0$ and β_i maps Y_0 isomorphically onto an open subscheme of Y_i, for $i = 1, 2$,

(c) β_1, β_2 are G-morphisms,

(d) a_0 is fixed under the action of G on X_0 and $\beta_i(a_0) = a_i$, for $i = 1, 2$.

Proof. We may assume that the X_i, Y_i are non-singular, for $i = 1, 2$. Consider $X_1 \times X_2$ with the action of G induced by the action on factors. Then (a_1, a_2) is fixed. The tangent space

$$T_{(a_1,a_2)}(X_1 \times X_2)$$

can be identified with $T_{a_1}(X_1) \oplus T_{a_2}(X_2)$.

Fix a G-isomorphism $\alpha : Y_1 \to Y_2$ such that $\alpha(a_1) = a_2$. Then α induces a G-isomorphism

$$\alpha^* : T_{a_1}(Y_1) \to T_{a_2}(Y_2).$$

Fix a G-isomorphism

$$\psi : T_{a_1}(X_1) \to T_{a_2}(X_2)$$

such that $\psi|_{T_{a_1}(Y_1)} = \alpha^*$ (the condition can be satisfied since G is linearly reductive). Then consider the "diagonal"

$$\Delta \subset T_{a_1}(X_1) \oplus T_{a_2}(X_2),$$

i.e. the subset composed of all vectors of the form $(v, \psi(v))$.

The subscheme Y_1 can be G-isomorphically immersed into $X_1 \times X_2$ by the map

$$i_1 \times i_2\alpha,$$

where i_1, i_2 are the closed immersions $i_1 : Y_1 \to X_1$, $i_2 : Y_2 \to X_2$. Let Y_0' be the obtained closed subscheme of $X_1 \times X_2$. Then $(a_1, a_2) \in Y_0'$, Y_0' is G-invariant, and

$$T_{(a_1,a_2)}(Y_0') \subset \Delta.$$

It follows from Theorem 2.1 that there exists a G-invariant closed irreducible subscheme X' of $X_1 \times X_2$ such that (a_1, a_2) is a non-singular point of X',

$$T_{(a_1,a_2)}(X') = \Delta, \quad X' \supset Y_0'.$$

Let $\beta_i' : X' \to X_i$ be the projection of X' into the factor X_i, for $i = 1, 2$. Then β_i' induces an isomorphism

$$T_{(a_1,a_2)}(X') = \Delta \;\to\; T_{a_i}(X_i),$$

and hence β_i' is étale at (a_1, a_2), for $i = 1, 2$. Moreover β_i' is G-invariant.

Remark 2.11 (What do we mean by etale here? What makes β_i' etale at (a_1, a_2)?)**.** *For a morphism of schemes $f : X \to Y$ and a point $x \in X$ with $y = f(x)$, we say f is étale at x if:*

1. *f is flat at x, and*
2. *the induced map on residue fields $\kappa(y) \to \kappa(x)$ is a finite separable extension,*

equivalently: if f is smooth of relative dimension 0. If $x \in X$ is a closed point, this boils down to: f is étale at x if and only if the induced map of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a local isomorphism up to completion, or equivalently $T_x(X) \cong T_y(Y)$ (isomorphism of tangent spaces at the residue field level) and $\dim X = \dim Y$ near those points. So tangent-space isomorphism is the infinitesimal characterization of being étale.

Hence the subset of all points of X' at which β_1', β_2' are étale is non-empty (hence dense) and G-invariant. Let X'' be the open subscheme determined by the subset, let $\beta_i'' = \beta_i'|_{X''}$, for $i = 1, 2$, and $a_0 = (a_1, a_2)$. Then conditions (a)–(d) are satisfied.

Let $Y_i' := (\beta_i'')^{-1}(Y_i)$ and $Y' = Y_1' \cup Y_2'$, then the morphism

$$(\beta_i''|_{Y'}) : Y_i' \to Y_i$$

is étale ($i = 1, 2$). Since Y_i is non-singular, Y_i' is non-singular (etale maps are smooth of relative dimension 0). The intersection $Y_1' \cap Y_2'$ is open and dense in Y_0' and since $\dim Y_i' = \dim Y_1 = \dim Y_0'$, we see that $Y_i' \cap Y_0'$ is full dimensional in Y_i' and closed. It follows that $Y_i' \cap Y_0'$ is the connected component of Y_i' containing a.

Moreover, since a is non-singular on Y_i, $Y_i' \cap Y_0'$ is the only irreducible component of Y_i' containing a. Take

$$X_0 := X'' - (Y' - Y_0'), \quad Y_0 := Y' \cap Y_0', \quad \beta_i := \beta_i''|_{X_0}, \quad \beta_i = \beta_i''|_{X_0},$$

then conditions (a)–(d) are satisfied. $\square$

Now we come to the main result of this section, which is an equivariant local product decomposition theorem. It roughly says near a fixed point a, a definite torus action looks locally like a product of the fixed locus X^G and a linear representation V.

Theorem 2.12 (2.5)**.** *Let G be an algebraic torus. Let the action of G on X be definite at a. Then there exists an open G-invariant neighbourhood U of a which is G-isomorphic to $(U \cap X^G) \times V$, where V is a finite-dimensional (fully definite) G-module and the action of G on $(U \cap X^G) \times V$ is induced by the trivial action of G on $U \cap X^G$ and the linear action on V (determined by the given structure of a G-module).*

Remark 2.13 (Analog with differential geometry)**.** *In differential geometry, there is a slice theorem for compact Lie group acting smoothly on manifolds. For any $x_0 \in M$, the slice theorem gives you a G_{x_0}-invariant submanifold $S \subset M$, called a slice, such that a neighborhood of the orbit $G \cdot x_0$ is diffeomorphic to $G \times_{G_{x_0}} S$, where G_{x_0} is the stabilizer of x_0.Moreover, S is transverse to the orbit $G \cdot x_0$, in the sense that*

$$T_{x_0} M = T_{x_0}(G \cdot x_0) \oplus T_{x_0} S$$

In the special case when x_0 is a fixed point, $G_{x_0} = G$ and the slice theorem says that a neighborhood of x_0 is diffeomorphic to $G \times_G S \cong S$, where $T_a S$ is the tangent space of M at x_0. Hence you can take S to be a small exponential image of $T_a M$.

Choose a G-invariant Riemannian metric (possible because G is compact), so the exponential map $\exp_a : T_a M \to M$ is G-equivariant near the origin (since G acts by isometries fixing a). Therefore the exponential map identifies a G-invariant neighborhood of $0 \in T_a M$ with a G-invariant neighborhood of $a \in M$.

In the proof of the theorem we shall use the following lemmas: The first lemma says that once U_1 contains a slice over the zero vector in V, then it contains the whole cylinder above it.

Lemma 2.14 (2.2)**.** *Let G be a torus and V a fully definite finite-dimensional G-module. For an algebraic scheme X consider the action of G on $X \times V$ induced by the trivial action of G on X and by the* *action of G on V determined by the G-module structure. Then any open G-invariant subscheme U_1 of $X \times V$ contains $(U_1 \cap (X \times \{0\})) \times V$ (where we identify X and $X \times \{0\}$).*

Proof. Assume failure. Suppose $((U_1 \cap (X \times \{0\})) \times V) - U_1 \neq \varnothing$. Pick a closed point $c = (x, v)$ in that difference. So:

- $x \in U_1 \cap (X \times \{0\})$ (so the basepoint $(x, 0) \in U_1$),
- but $(x, v) \notin U_1$ for some $v \in V$.

Use G-invariance. The set $((U_1 \cap (X \times \{0\})) \times V) - U_1$ is closed and G-invariant. Therefore the whole orbit closure $\overline{G(k) \cdot c}$ is contained in this difference. In particular, every limit point of the orbit of (x, v) stays outside U_1.

Use definiteness of V. Because V is a fully definite torus representation, every nonzero vector has an orbit whose closure meets the origin $0 \in V$. (Geometrically: all weights are positive or all are negative, so scaling by $\lambda \in \mathbb{G}_m$ drives any $v \neq 0$ toward 0.) Thus $\overline{G(k) \cdot c}$ contains some point of the form $(x, 0)$.

Contradiction. But $(x, 0) \in U_1$ by assumption ($x \in U_1 \cap (X \times \{0\})$). So we found a point $(x, 0)$ that lies both in U_1 and in the closed complement of U_1. Contradiction.

Therefore the assumption was wrong, and the whole cylinder is contained in U_1. $\square$

Lemma 2.15 (2.3). *Let $G, X, a \in X^G$ be as in Theorem 2.5. Moreover, assume that X is reduced and irreducible. Then there exists a one-dimensional connected reduced group subscheme $G_0 \subset G$ (hence $G_0 \cong \mathbb{G}_m$) such that*

(a) $X^{G_0} = X^G$,

(b) the induced action of G_0 on X is definite at a,

(c) if the action of G on X is effective then the action of G_0 on X is also effective.

Proof. Fix an isomorphism $G \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m = \mathbb{G}_m^n$ and a basis $v_1, \ldots, v_r$ of $T_a(X)$ such that for $g = (\lambda_1, \ldots, \lambda_n) \in \mathbb{G}_m^n(k)$,

$$gv_i = \lambda_1^{m_{i1}} \cdots \lambda_n^{m_{in}} v_i, \qquad m_{ij} \geq 0.$$

Let G_0 be the diagonal of $G \cong \mathbb{G}_m^n$. Then for any $g = (\lambda, \ldots, \lambda) \in G_0(k)$,

$$gv_i = \lambda^{\sum_j m_{ij}} v_i, \quad i = 1, \ldots, r.$$

Hence the induced action of G_0 on X is definite at a and by Theorem 2.3, X^{G_0} is irreducible. Since $X^{G_0} \subset X^G$ and $\dim X^G = \dim X^{G_0}$ (because $T_a(X)^{G_0} = T_a(X)^G$), it follows that $X^{G_0} = X^G$.

Remark 2.16 (Importance of definiteness). *For G: v_i is invariant iff all $m_{ij} = 0$. For G_0: v_i is invariant iff $\sum_j m_{ij} = 0$. But since all $m_{ij} \geq 0$ (definiteness assumption), $\sum_j m_{ij} = 0$ if and only if $m_{ij} = 0$ for all j. Thus the two conditions are identical and $T_a(X)^G = T_a(X)^{G_0}$.*

If the action of G is effective then by Lemma 2.4 (below) the induced action on $T_a(X)$ is effective and the sequences $(m_{i1}, \ldots, m_{in})$ span $\mathbb{Z} \times \cdots \times \mathbb{Z} = \mathbb{Z}^n$. In particular there exist integers l_i, $i = 1, \ldots, r$, such that

$$\sum_i l_i(m_{i1}, \ldots, m_{in}) = (1, 0, \ldots, 0).$$

Then

$$\sum_i l_i \Big(\sum_j m_{ij} \Big) = 1$$

and hence the $\gcd$ of $\sum_j m_{ij}$, for i running through $1, \ldots, r$, is equal to 1. Thus the action of G_0 is effective. $\square$

Let Ω be a fixed universal domain for k (i.e. Ω is a field extension of k which is algebraically closed and of infinite transcendence degree over k).

Remark 2.17 (Why do we introduce Ω?). *We want to talk about the generic point of X and its stabilizer under G. But the generic point is not an ordinary k-point of X, its local ring is the function field $\kappa(X)$ of X as opposed to a residue field of a closed point, which is a finite extension of k.*

So how do we talk about the stabilizer "at" this point in concrete terms? Instead of working directly over $\kappa(X)$, we enlarge the base field to a very large algebraically closed field Ω with infinite transcendence degree over k.

We do this because every finitely generated field over k (in particular, $\kappa(X)$) can be embedded into Ω, and that means the generic point of X (which lives over $\kappa(X)$) becomes an Ω-point of X_Ω.

So now we can think of the "geometric generic point" as an actual point $t \in X(\Omega)$. Once we have a genuine Ω-point t, we can talk about its stabilizer subgroup scheme $S_t \subseteq G_\Omega$. This stabilizer is the same as the generic stabilizer, just realized inside a big algebraically closed field.

The stabilizer at the generic point measures the global kernel of the group action. Suppose $g \in G$ fixes the generic point η. That means g fixes every rational function in $\kappa(X)$. But if g acts trivially on the function field, then g acts trivially on a dense open subset of X. Since group actions are continuous (scheme-theoretically regular), this forces g to act trivially on all of X.

Thus the stabilizer at the generic point is exactly the subgroup scheme of elements of G that act trivially on all of X. This is why in the condition in the following lemma is equivalent to "the action is effective."

Lemma 2.18 (2.4). *Let G be linearly reductive and X be irreducible and reduced. Let $a \in X^G$ be a* *closed point. Then the following conditions are equivalent:*

1. *the action of G on X is effective,*
2. *the induced action of $G(k)$ on $k[X]$ is effective,*
3. *the induced action of $G(k)$ on $T_a(X)$ is effective,*
4. *the stabilizer group $S_t \subset G$ at a generic point $t \in X(\Omega)$ is trivial.*

Proof. $(1) \Rightarrow (2)$ is obvious. $(3) \Rightarrow (4)$ follows from the fact that if an element of G fixes the generic point, it fixes all of X, hence also a. So the generic stabilizer is always contained in the stabilizer at any closed point — and in particular at a. $(4) \Rightarrow (1)$ see the above remark.

$(2) \Rightarrow (3)$ is the nontrivial part. See the remark following this proof for a sketch of why this is true. $\square$

Remark 2.19 (Elaboration on (2) implies (3))**.** *We will sketch why if the action on $T_a(X)$ is not effective, then the action on $k[X]$ is not effective.*

Assume some nontrivial subgroup $H \subset G$ acts trivially on $T_a(X)$. Then H acts trivially on $\mathfrak{m}_a/\mathfrak{m}_a^2$. Because G is linearly reductive, the functor "H-invariants" is exact. The $\mathfrak{m}_a$-adic filtration on the local ring gives graded pieces $\mathrm{gr}_{\mathfrak{m}_a}(\mathcal{O}_{X,a}) = \bigoplus_{n\geq 0} \mathfrak{m}_a^n/\mathfrak{m}_a^{n+1}$, and each $\mathfrak{m}_a^n/\mathfrak{m}_a^{n+1}$ is a quotient of the symmetric power $S^n(\mathfrak{m}_a/\mathfrak{m}_a^2)$. Since H acts trivially on $\mathfrak{m}_a/\mathfrak{m}_a^2$, it acts trivially on all $S^n(\mathfrak{m}_a/\mathfrak{m}_a^2)$, hence on every $\mathfrak{m}_a^n/\mathfrak{m}_a^{n+1}$, hence on each finite jet ring $\mathcal{O}_{X,a}/\mathfrak{m}_a^n$.

By exactness, the successive extensions split H-equivariantly, so H acts trivially on the completed local ring $\widehat{\mathcal{O}}_{X,a}$. The Krull intersection theorem guarantees that the natural map $\mathcal{O}_{X,a} \to \widehat{\mathcal{O}}_{X,a}$ is injective. Therefore the action of H is trivial on $\mathcal{O}_{X,a}$.

Now suppose X is affine near a: $X = \operatorname{Spec} A$, with a corresponding to maximal ideal $\mathfrak{m} \subset A$. Then $\mathcal{O}_{X,a} = A_{\mathfrak{m}}$. Since A is finitely generated over k, the local ring $A_{\mathfrak{m}}$ is generated as a k-algebra by finitely many elements of A. Pick generators $f_1, \ldots, f_r \in A$ whose images generate $A_{\mathfrak{m}}$. If $g^* = \mathrm{id}$ on $\mathcal{O}_{X,a}$, then in particular $g^*(f_i) = f_i$ in the localization for each generator f_i.

That means: there exists some neighborhood U_i of a (where denominators used to localize don't vanish) such that $g^*(f_i) = f_i$ as functions on U_i. Let $U = \bigcap_{i=1}^r U_i$. This is still a neighborhood of a. On $B = \mathcal{O}_X(U)$, the automorphism g^* and the identity coincide after localization. But localizing at a is injective, so they must already coincide on B. Hence g is the identity on U.

Therefore, trivial action on $\widehat{\mathcal{O}}_{X,a}$ forces H to act trivially on a Zariski neighborhood of a. If a group element acts as the identity on a nonempty open subset of an irreducible scheme, it acts as the identity everywhere. Thus every $h \in H$ acts trivially on all of X, i.e. trivially on $k[X]$.

Remark 2.20 (Using Luna's étale slice theorem)**.** *In the above remark, one can also appeal to Luna's slice theorem, which we state here without proof. However this theorem came after Bialynicki-Birula wrote his paper, and is overkill for our situation.*

> **Theorem 2.21** (Luna's étale slice theorem)**.** *Let G be a reductive algebraic group acting on an affine variety X over an algebraically closed field. Pick a point $x \in X$ with stabilizer $H \subseteq G$. Then there exists: a finite-dimensional H-representation V (the slice representation, essentially the tangent representation $T_x(X)/T_x(G\cdot x)$), and an étale, H-equivariant morphism $(G\times^H V) \longrightarrow X$ sending $[e,0]$ to x, such that étale-locally near x, the G-variety X looks like the homogeneous fiber bundle $G\times^H V$.*

Since G is reductive and a is a fixed point, Luna's étale slice theorem gives a G-equivariant étale map from a linear model to X: $\phi : (V,0) \longrightarrow (X,a)$, where V is a finite-dimensional G-representation whose linear action is the isotropy/tangent representation at a; moreover, ϕ is étale at 0 and G-equivariant: $\phi \circ g_V = g_X \circ \phi$. Since G is reductive and a is a fixed point, Luna's étale slice theorem (or, for tori, Sumihiro linearization) gives a G-equivariant étale map from a linear model to X, where V is a finite-dimensional G-representation whose linear action is the isotropy/tangent representation at a; moreover, ϕ is étale at 0 and G-equivariant.

Because ϕ is étale at 0, it induces an isomorphism of completed local rings

$$\widehat{\mathcal{O}}_{X,a} \xleftarrow[\cong]{\phi^\#} \widehat{\mathcal{O}}_{V,0}$$

By the same argument as above we again see that g acts trivially on a Zariski neighborhood of 0 in V, which we identify with a neighborhood of a in X via ϕ.

Definition 2.22. *Let Y be a scheme over k. We say Y is **geometrically unibranched** if for every point $y \in Y$: (1) the local ring $\mathcal{O}_{Y,y}$ is reduced and has a unique minimal prime (so the germ of Y at y is irreducible), and (2) the normalization of $\mathcal{O}_{Y,y}$ is still local (i.e. has only one maximal ideal).*

> Y may have singularities, but at each point there is only one "branch" of the variety passing through that point. One can't "split" the normalization into multiple components over that point. Contrast with something like a node (e.g. $xy = 0$ in $\mathbb{A}^2$), where two branches meet: that is not unibranched.

Lemma 2.23 (2.5)**.** *Let $G = \mathbb{G}_m$ and $t \in X(\Omega)$. Then the algebraic scheme $\overline{G(\Omega)\cdot t}$ (over Ω) defined as the closure of $G(\Omega)\cdot t$ in X_Ω is geometrically unibranched.*

> *Proof.* If t is fixed for the action of G on X then the lemma is obvious. Suppose that t is not fixed. Since X is quasi-affine, $\overline{G(\Omega)\cdot t}$ is also quasi-affine. This is because if X is quasi-affine and G acts on X, then every G-orbit is quasi-affine (classical fact: a quasi-affine open subset of an affine variety remains quasi-affine, and orbits are locally closed). Thus the normalization of $\overline{G(\Omega)\cdot t}$ is quasi-affine (General fact: the normalization of a quasi-affine scheme is again quasi-affine.)

Since the orbit $G_\circ\Omega t$ is open in $\overline{G(\Omega)\cdot t}$ and is isomorphic to $\operatorname{Spec}\Omega[x,1/x]$, the normalization of $\overline{G(\Omega)\cdot t}$ is either equal to $\overline{G(\Omega)\cdot t}$ (in case $G_\Omega t = G(\Omega)\cdot t$) or is isomorphic to $\operatorname{Spec}\Omega[x]$ (in case $G_\Omega t \neq G(\Omega)\cdot t$). In both cases the map of the normalization onto $\overline{G(\Omega)\cdot t}$ is one-to-one.

In other words, we are starting with orbit $G_\Omega \cdot t$, which is isomorphic to $\operatorname{Spec}\Omega[x,1/x]$. Its closure in a quasi-affine space must be quasi-affine, 1-dimensional, and normalizable to something quasi-affine. There are only two possibilities: stay $\mathbb{G}_m$ (if the orbit is closed), or compactify to $\mathbb{A}^1$ (if closure adds one point). Hence the lemma is proved. $\square$

Proof of Theorem 2.5. Replacing X by a G-invariant reduced and irreducible neighbourhood of a, we may restrict considerations to the case where X is reduced and irreducible. Moreover, we may assume that the action is effective. Let V be the G-submodule complement of $T_a(X^G)$ in $T_a(X)$, i.e. let $T_a(X^G)\oplus V = T_a(X)$. Then the G-module V is fully definite. Apply Theorem 2.4 to the case where

$$X_1 = X, \qquad X_2 = X^G \times V$$

(with the action of G induced by the trivial action on X^G and the action on V determined by the G-module structure of V),

$$Y_1 = X^G \subset X = X_1, \qquad Y_2 = X^G \times \{0\} \subset X_2 = X^G \times V, \qquad a_1 = a,\ a_2 = (a,0).$$

The assumptions of the theorem are satisfied and hence we may find and fix $X_0, Y_0, \beta_1, \beta_2, a_0 \in X_0$ satisfying conditions (a)–(d). We shall show that β_1, β_2 are open immersions.

First let us check that this will prove the theorem. If $\beta_2 : X_0 \hookrightarrow X_2 = X^G \times V$ is an open immersion, so $\beta_2(X_0)$ is an open subset of $X^G \times V$. Because β_2 is étale and G-equivariant, the set $U_0 := \beta_2(X_0)\cap X_2^G$ is just an open neighbourhood of a_2 in X_2^G since $X_2^G = X^G \times \{0\}$. So $U_0 \subset X^G$ is an open neighbourhood of a. Here we are invoking the fact that étale maps are open.

Since $\beta_2(X_0)$ is G-invariant (because β_2 is a G-map) and intersects the fixed locus in U_0, the product structure of $X_2 = X^G \times V$ forces: $\beta_2(X_0) \supset U_0 \times V$. Here we use Lemma 2.2.

Because β_2 is an isomorphism onto its image, we can identify $X_0' := \beta_2^{-1}(U_0 \times V) \subset X_0$. Then X_0' is G-isomorphic to $U_0 \times V$. Composing with the other open immersion β_1, define $U := \beta_1(X_0') \subset X$. Then U is an open G-invariant neighbourhood of a, and $U \simeq_G (U_0 \times V)$

Since β_1, β_2 are étale it suffices to show that they are birational (since an étale and birational morphism of algebraic schemes is an open immersion). To prove this we may (and will) assume in the sequel that $G = \mathbb{G}_m$. In fact, let $G_0 \subset G$ be as in Lemma 2.3, then $X^{G_0} = X^G$ and hence $X_0, \beta_1, \beta_2, a_0$ fixed above satisfy conditions (a)–(d) for G replaced by G_0.

The set $\beta_2(X_0)$ is open and, because it is also G-invariant, by Lemma 2.2 it contains the non-empty and open subscheme

$$(\beta_2(X_0)\cap X_2^G)\times V.$$

Let $t \in [\beta_2(X_0)](\Omega)$ be generic over k (then $t \in (\beta_2(X_0) \cap X_2^G) \times V(\Omega)$). In particular, this means that t corresponds to the generic point of the open subset $\beta_2(X_0)$ of X_2.

The set $\beta_2^{-1}(t)$ is non-empty and finite. It is nonempty because t was chosen inside the image $\beta_2(X_0)$. It is finite because β_2 is étale. Hence it is relative dimension 0 and its fibers are finite type, zero-dimensional, and reduced.

Write $t = (x, v)$ with $x \in X^G(\Omega)$ and $v \in V(\Omega)$. Because G acts trivially on X^G, the G-orbit of t is

$$G(\Omega) \cdot t = \{(x, g \cdot v) : g \in G(\Omega)\} \subset X_2(\Omega)$$

Since $G = \mathbb{G}_m$ acting linearly on V with weights of definite sign, we know that $\lim_{\lambda \to 0} \lambda \cdot v = 0$. That is, all points in V flow to 0 under the torus action. Therefore, the orbit closure in V is $\overline{G(\Omega) \cdot v} = (G(\Omega) \cdot v) \cup \{0\}$. Lifting this to $X^G \times V$, we get $\overline{G(\Omega) \cdot t} = \big(G(\Omega) \cdot t\big) \cup \big(X^G(\Omega) \times \{0\}\big)$.

Because V is fully definite, all nontrivial one-parameter subgroups of G all contract V toward 0. Hence $X_2(\Omega) \cap \overline{G(\Omega) \cdot t} = (G(\Omega) \cdot t) \cup \{b\}$.

Consider the map

$$\beta_2^t := \beta_2^{-1}(G(\Omega) \cdot t) \longrightarrow G(\Omega) \cdot t.$$

Notice that it follows from (c) that $\beta_2^t(b)$ is composed of exactly one point; denote it by b'. Moreover

$$\beta_2^{-1}(X_2(\Omega) \cap \overline{G(\Omega) \cdot t}) = \beta_2^{-1}(G(\Omega) \cdot t) \cup \beta_2^{-1}(b) = \bigcup_{t_i \in \beta_2^{-1}(t)} (X_0(\Omega) \cap \overline{G(\Omega) \cdot t_i}) \cup \{b'\}.$$

(The last equality follows from the fact that $\beta_2'^{-1}(G(\Omega) \cdot t)$ is closed.) Hence $b' \in \bigcup_i \overline{G(\Omega) \cdot t_i}$. Otherwise $\{b'\}$ would be open, so $\{b'\}$ would be open in $\overline{G(\Omega) \cdot t}$, a contradiction. Notice that $G(\Omega) \cdot t_i \cap G(\Omega) \cdot t_j = \varnothing$ for $t_i \neq t_j$ since the isotropy group S_t of the point t is trivial. In fact, if $G(\Omega) \cdot t_i \cap G(\Omega) \cdot t_j \neq \varnothing$ then $G(\Omega) \cdot t_i = G(\Omega) \cdot t_j$. Hence there exists $g \in G(\Omega)$ such that $g(t_i) = t_j$ and thus

$$t = \beta_2(g(t_i)) = g(\beta_2(t_i)) = g(t),$$

and $g \in S_t(\Omega)$. If $t_i \neq t_j$ then $g \neq e \in G(\Omega)$ and hence S_t is non-trivial, which by Lemma 2.4 contradicts the assumption that the action is effective. Hence $b' \in \overline{G(\Omega) \cdot t_i}$ for exactly one t_i (by Lemma 2.5 and Proposition 17.5.7, Ch. IV of [**?**]). Since $t \in X_2(\Omega)$ is generic (over k) and the property $\overline{G(\Omega) \cdot t_0} \cap X_2^G(\Omega) \neq \varnothing$ holds for any generic (over k) $t_0 \in X_2(\Omega)$, in particular for any $t_j \in \beta_2'^{-1}(t)$, $\overline{G(\Omega) \cdot t_j} \cap X_2^G(\Omega) \neq \varnothing$. But

$$\overline{G(\Omega) \cdot t_j} \cap X_2^G(\Omega) = (G(\Omega) \cdot t_j \cup \{b'\}) \cap X_2^G(\Omega) = \{b'\},$$

hence $t_j = t_i$ for all $t_j \in \beta_2'^{-1}(t)$ (by the uniqueness of t_i proved above). Thus $\beta_2'^{-1}(t)$ is a one-element set and therefore β_2 is birational (since β_2 is also étale).

Now consider β_1. Take $u \in X_1(\Omega)$, generic for X_1 over k. Then $\beta_1^{-1}(u)$ is non-empty and finite. Let $u_1, u_2 \in \beta_1^{-1}(u)$. Then

$$X_1(\Omega) \cap \overline{G(\Omega) \cdot u_1} = G(\Omega) \cdot u_1 \cup \{s_1\}, \qquad X_1(\Omega) \cap \overline{G(\Omega) \cdot u_2} = G(\Omega) \cdot u_2 \cup \{s_2\},$$

where $s_1, s_2 \in X_1^{G^\circ}(\Omega)$ (because we have shown already that X_0 contains an open G-invariant subscheme which is G-isomorphic to $U_0 \times V$, where U_0 is an open neighbourhood of a in X^G and the G-module V is fully definite).

$$G(\Omega) \cdot u_i \subset \beta_1^{-1}(G(\Omega) \cdot u), \quad i = 1, 2,$$

hence $\beta_1(s_i) = \beta_2(s_i)$ and

$$X_1(\Omega) \cap \overline{G(\Omega) \cdot u} = G(\Omega) \cdot u \cup \{s_i\}.$$

Moreover, since $\beta_1|_{X_1^{G^\circ}}$ is one-one (condition (b) of Theorem 2.4), $s_1 = s_2$. Therefore $u_1 = u_2$ (by an argument as in the first part of the proof, where we have shown $t_j = t_i$ for all $t_j \in \beta_2'^{-1}(t)$). Since β_1 is étale this shows that β_1 is birational. $\square$

Corollary 2.24. *Let G be an n-dimensional torus and let $\dim X = n$. If the action of G on X is effective and there exists a point $a \in X^G$, then X contains an open G-invariant neighbourhood of a which is G-isomorphic to a k-vector space V with a linear action of G.*

Proof. Since the action is effective and $\dim G = \dim X$, for any closed point $a \in X$ the action is fully definite at a. Hence X contains an open G-invariant neighbourhood of a which is G-isomorphic to a k-vector space V with the action of G induced by a G-module structure. $\square$

Corollary 2.25. *Any effective action of an n-torus G on an n-dimensional vector space V is equivalent to a linear action; i.e. V with the action of G is G-isomorphic to V with an action of G determined by a G-module structure on V.*

Proof. Every algebraic torus acting effectively on affine space $\mathbb{A}^n$ has a fixed point. Then from the above corollary we obtain that there exists an open G-invariant subscheme U which is isomorphic to V with an action of G determined by a G-module structure on V. No proper open subscheme of V can be isomorphic to V, hence $U = V$. $\square$

Remark 2.26. *Bialynicki-Birula gets the existence of a fixed point using an argument from one of his earlier papers, which I didn't hunt down. But the sort of result he needs reminded me of the Borel fixed point theorem.*

Theorem 2.27 (Borel fixed point theorem)**.** *If G is a connected solvable linear algebraic group acting regularly on a non-empty, complete algebraic variety V over an algebraically closed field k, then G has a fixed point on V.*

However V is not complete in our situation, so we cannot use this theorem to guarantee the existence of a fixed point.

3 References

1. Bialynicki-Birula, A. Some theorems on actions of algebraic groups. Ann. of Math. (2) 98 (1973), 480–497.

2. On the Existence of Slices for Actions of Non-Compact Lie Groups Richard S. Palais