Title

Songyu Ye

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Abstract

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Let $S = \mathrm{Spec}\,\mathbb{C}[[s]]$. Recall that Solis fixes a curve C over S whose generic fiber is smooth and whose special fiber is a nodal curve with a single node. In particular, C is classified by a morphism $f : S \to \overline{\mathcal{M}}_{g,I}$.

Solis defines an S-stack whose B-points are given by twisted local modifications C'_B of our fixed family C/S and admissible G-bundles P'_B on C'_B. He christens this stack $\mathcal{X}_G(C)$ and shows that it is algebraic over S, locally of finite type over S, and complete over S.

We spread out the stack $\mathcal{X}_G(C)$ along S to get a stack $\mathcal{X}_{G,g,I}$ over $\overline{\mathcal{M}}_{g,I}$ whose structure morphism is algebraic and locally of finite type for formal reasons.

0.1 The diagonal construction at a twisted node

Let R_n denote the rational chain of projective lines with n-components. There is an action of $\mathbb{C}^\times$ on R_n which scales each component. Let $p_0, \ldots, p_n$ denote the fixed points of this action. Let u, v

are kth roots of x, y which are our coordinates near a node.

Let p', p'' be denote the closed points of $\operatorname{Spec} \mathbb{C}[[u]]$, $\operatorname{Spec} \mathbb{C}[[v]]$ and finally let $D_n^{\frac{1}{k}}$ be the curve obtained from $\operatorname{Spec} \mathbb{C}[[u]] \sqcup R_n \sqcup \operatorname{Spec} \mathbb{C}[[v]]$ by identifying p' with p_0 and p'' with p_n.

The group μ_k acts on $D_n^{\frac{1}{k}}$ through its usual action on u, v and through the inclusion $\mu_k \subset \mathbb{C}^\times$ on the chain R_n. For an n-tuple $(\beta_0, \ldots, \beta_n) \in \operatorname{hom}(\mathbb{C}^\times, T)^n$, we can speak about the equivariant G-bundles on $D_n^{\frac{1}{k}}$ with equivariant structure at p_i determined by β_i. We refer to this equivalently as a G-bundles on $[D_n^{\frac{1}{k}}/\mu_k]$ of type $(\beta_1, \ldots, \beta_n)$. In particular, a choice of β determines by restriction to $\mu_k \subset \mathbb{C}^\times$ a μ_k–equivariant structure on the fiber of the G–bundle at p.

Further, we can also glue $[D_n^{\frac{1}{k}}/\mu_k]$ to $C_0 - p_0$ to obtain a curve $C_{n,[k]}$. Let C_n denote the coarse moduli space of $C_{n,[k]}$.

At the ith node, let the parahoric subgroup $\mathcal{P}_i$ with Levi decomposition $\mathcal{P}_i = L_i \ltimes U_i$, set

$$\mathcal{P}_i^\Delta = \Delta(L_i) \ltimes (U_i \times U_i).$$

One constructs a sheaf of groups $\mathcal{G}^\Delta$ over C_n such that

$$\mathcal{G}^\Delta(\widehat{\mathcal{O}}_{C_n, p_i}) = \mathcal{P}_i^\Delta, \qquad \mathcal{G}^\Delta|_{C_n - \{p_0, \ldots, p_n\}} = G^{\text{std}}$$

Then Solis shows there is an isomorphism

$$\mathcal{M}_{G,I}(C_{n,[k]}) \cong \mathcal{M}_{\mathcal{G}^\Delta}(C_n).$$

0.2 Constructing a vector bundle

Working on $\mathcal{M}_{G,I}(C_{n,[k]})$, let $\rho : G \to \operatorname{GL}(V)$ be a fixed representation. For a G–bundle $\mathcal{E}$ on the stacky curve $C_{n,[k]}$ we form the associated vector bundle

$$\mathcal{V} := \mathcal{E} \times^G V.$$

Away from the stacky points this is the usual associated bundle. At a stacky point p_i with stabilizer μ_k, the chosen type $\beta_i : \mathbb{C}^\times \to T \subset G$ restricts to a homomorphism $\beta_i|_{\mu_k} : \mu_k \to G$, which is precisely the local μ_k–equivariant structure on the G–torsor $\mathcal{E}$ near p_i. Applying ρ yields an induced μ_k–representation on the geometric fiber $\mathcal{V}|_{p_i}$, namely the action of μ_k on V via $\rho \circ \beta_i|_{\mu_k}$.

Consequently, fixing ρ once and for all, the universal G–torsor on the universal stacky curve $\pi : \mathcal{C} \to \mathcal{X}_{G,g,I}$ (when it exists, e.g. on the stack of G–bundles, or after the usual rigidification) determines a canonically associated vector bundle $\mathcal{V}$ on $\mathcal{C}$. We define the determinant of cohomology line bundle by $\mathcal{L}_{\det}(\rho) := \det R\pi_* \mathcal{V}$ which is a line bundle on $\mathcal{X}_{G,g,I}$.

Let us take $\rho : G \to \operatorname{Lie}(G)$ to be the adjoint representation. This is the line bundle we will use to construct the Θ-stratification on $\mathcal{X}_{G,g,I}$.

0.3 Very close degenerations

One can reformulate the numerical Hilbert–Mumford criterion in terms of stacks. The quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ has two geometric points 1 and 0 which are the images of the points of the same name in $\mathbb{A}^1$. For any algebraic stack $\mathcal{M}$ and $f : [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{M}$ we will write $f(0), f(1) \in \mathcal{M}(k)$ for the points given by the images of $0, 1 \in \mathbb{A}^1(k)$.

Definition 0.1 (Very close degenerations)**.** Let $\mathcal{M}$ be an algebraic stack over k and $x \in \mathcal{M}(K)$ a geometric point for some algebraically closed field K/k. A *very close degeneration* of x is a morphism $f : [\mathbb{A}^1_K/\mathbb{G}_{m,K}] \to \mathcal{M}$ with $f(1) \simeq x$ and $f(0) \not\simeq x$.

We emphasize that $f(0)$ is an object that lies in the closure of a K point of $\mathcal{M}_K$, which only happens for stacks and orbit spaces, but if $X = \mathcal{M}$ is a scheme, then there are no very close degenerations.

Definition 0.2 ($\mathcal{L}$-stability)**.** Let $\mathcal{M}$ be an algebraic stack over k, locally of finite type with affine diagonal and $\mathcal{L}$ a line bundle on $\mathcal{M}$. A geometric point $x \in \mathcal{M}(K)$ is called $\mathcal{L}$-*stable* if

1. for all very close degenerations $f : [\mathbb{A}^1_K/\mathfrak{G}_{m,K}] \to \mathcal{M}$ of x we have

$$\mathrm{wt}(f^*\mathcal{L}) < 0$$

   and

2. $\dim_K(\mathrm{Aut}_{\mathcal{M}}(x)) = 0$.

We can also introduce the notion of $\mathcal{L}$-semistable points, by requiring only $\leq$ in (1) and dropping condition (2).

0.4 Very close degenerations of G-bundles

For a cocharacter

$$\lambda : \mathbb{G}_m \to G$$

we denote by P_λ, U_λ, L_λ the corresponding parabolic subgroup, its unipotent radical and the Levi subgroup.

The source of degenerations is the following analog of the Rees construction. Given $\lambda : \mathbb{G}_m \to G$ we obtain a homomorphism of group schemes over $\mathbb{G}_m$:

$$\mathrm{conj}_\lambda : P_\lambda \times \mathbb{G}_m \longrightarrow P_\lambda \times \mathbb{G}_m, \qquad (p, t) \longmapsto (\lambda(t)p\lambda(t)^{-1}, t).$$

This homomorphism extends to a morphism of group schemes over $\mathbb{A}^1$:

$$\mathrm{gr}_\lambda : P_\lambda \times \mathbb{A}^1 \longrightarrow P_\lambda \times \mathbb{A}^1$$

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in such a way that

$$\mathrm{gr}_\lambda(p,0) = \lim_{t\to 0} \lambda(t)p\lambda(t)^{-1} \in L_\lambda \times 0.$$

These morphisms are $\mathbb{G}_m$-equivariant with respect to the action $(\mathrm{conj}_\lambda, \mathrm{act})$ on $P_\lambda \times \mathbb{A}^1$.

Given a P_λ-bundle $\mathcal{E}_\lambda$ on a scheme X, this morphism defines a P_λ-bundle on $X \times [\mathbb{A}^1/\mathbb{G}_m]$ by

$$\mathrm{Rees}(\mathcal{E}_\lambda, \lambda) := \big[((\mathcal{E}_\lambda \times \mathbb{A}^1) \times_{\mathbb{A}^1}^{\mathrm{gr}_\lambda} (P_\lambda \times \mathbb{A}^1))/\mathbb{G}_m\big],$$

where $\times_{\mathbb{A}^1}^{\mathrm{gr}_\lambda}$ denotes the bundle induced via the morphism gr_λ, i.e. we take the product over $\mathbb{A}^1$ and divide by the diagonal action of the group scheme $P_\lambda \times \mathbb{A}^1/\mathbb{A}^1$, which acts on the right factor via gr_λ.

By construction this bundle satisfies

$$\mathrm{Rees}(\mathcal{E}_\lambda, \lambda)|_{X\times 1} \cong \mathcal{E}_\lambda$$

and

$$\mathrm{Rees}(\mathcal{E}_\lambda, \lambda)|_{X\times 0} \cong \mathcal{E}_\lambda/U_\lambda \times_{L_\lambda} P_\lambda,$$

which is the analog of the associated graded bundle.

Lemma 0.3. Let G be a split reductive group over k. Given a very close degeneration

$$f : [\mathbb{A}^1/\mathbb{G}_m] \to \mathrm{Bun}_G$$

corresponding to a family $\mathcal{E}$ of G-bundles on $X \times [\mathbb{A}^1/\mathbb{G}_m]$, there exist:

1. a cocharacter $\lambda : \mathbb{G}_m \to G$, canonical up to conjugation,

2. a reduction $\mathcal{E}_\lambda$ of the bundle $\mathcal{E}$ to P_λ,

3. an isomorphism
$$\mathcal{E} \cong \mathrm{Rees}(\mathcal{E}_\lambda|_{X\times 1}, \lambda).$$

Theorem 0.4. A G-bundle E is $\mathcal{L}_{\mathrm{det}}$-stable if and only if for all reductions E_P to maximal parabolic subgroups $P \subset G$ we have $\deg(E_P \times_P \mathrm{Lie}(P)) < 0$.

This theorem of Heinloth [**?**] is precisely the way we need to generalize the notion of stability to our compactified moduli stack $\mathcal{X}_{G,g,I}$. We need to understand the very close degenerations of points of $\mathcal{X}_{G,g,I}$, and then apply the numerical criterion to determine which points are semistable with respect to the determinant line bundles we have constructed.

Note that we need to know that the stack $\mathcal{X}_{G,g,I} \to \overline{\mathcal{M}}_{g,I}$ has affine diagonal in order to apply Heinloth's theory of $\mathcal{L}$-stability.

1 Monday, Feb 16, 2026

Fix a geometric point

$$x_0 \in \mathcal{X}_{G,g,I}(k)$$

packaged as a stacky curve C_0' together with a G-bundle $\mathcal{P}$ whose local isotropy data at the stacky nodes are prescribed by η_i.

A very close degeneration of x_0 is a morphism

$$f : \Theta := [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{X}_{G,g,I}$$

with $f(1) \simeq x_0$ and $f(0) \not\simeq x_0$.

By construction we get $\mathbb{G}_m \to \mathrm{Aut}(x_0)$, where $\mathrm{Aut}(x_0)$ is those bundle automorphisms of $\mathcal{P}$ covering the stacky curve C_0'. By restricting to the stacky node p_i, we get a homomorphism

$$\lambda_{f,i} := \mathrm{ev}_{p_i} \circ \lambda_f : \mathbb{G}_m \to Z_G(\eta_i).$$

well defined up to conjugation by $Z_G(\eta_i)$. Also by restricting to the generic point of C_0', we get a homomorphism

$$\lambda_{f,\mathrm{gen}} : \mathbb{G}_m \to G$$

well defined up to conjugation by G.

Choose an invariant norm $\| \cdot \|$ on $X_*(T)_{\mathbb{R}}$. We should take the sum of the squares of the norms of $\lambda_{f,i}$ and $\lambda_{f,\mathrm{gen}}$ to get a norm $\|\lambda_f\|_{sum}$ on the space of very close degenerations of x_0.

Thus we can define the numerical invariant

$$M_{\mathcal{L}}(f) := \frac{-\mu_{\mathcal{L}}(f)}{\|\lambda_f\|_{sum}}$$

A maximally destabilizing Θ-filtration of x (with respect to $\mathcal{L}$ and $\| \cdot \|$) is a map $f_{\mathrm{HN}} : \Theta \to \mathcal{X}$ with $f_{\mathrm{HN}}(1) = x$ such that

$$M_{\mathcal{L}}(f_{\mathrm{HN}}) = \sup \Big\{ M_{\mathcal{L}}(f) \ : \ f : \Theta \to \mathcal{X}, \ f(1) = x, \ f \text{ nontrivial} \Big\}.$$

The theorem that will give us a stratification is the following:

Theorem 1.1 (Need to prove)**.** For any $x \in \mathcal{X}_{G,g,I}(k)$, there exists a unique maximally destabilizing Θ-filtration $f_{\mathrm{HN}} : \Theta \to \mathcal{X}_{G,g,I}$ of x with respect to $\mathcal{L}_{\mathrm{det}}$ and any invariant norm on $X_*(T)_{\mathbb{R}}$.

1.1 Worked example for $\mathrm{SL}(3)$

We consider the case where $G = \mathrm{SL}(3)$ and we expand the nodal curve C_0 with a single node p by a single copy of $\mathbb{P}^1$. We take the local isotropy data at p to be given by η_1 and η_2 which are the two nontrivial vertices of the fundamental alcove of $\mathrm{SL}(3)$. Recall that our line bundle is the determinant of cohomology line bundle associated to the adjoint representation.

2 Meeting remarks

- Constantin emphasized the ordering of the labels and simple roots, recommended looking in Kausz to see if the data is ordered or not.

- Constantin said that the uniqueness part of the theorem is not necessary and that the key part of the setup is to have something as depicted in Kirwan. Dan also has remarks about this if I want to make it precise. He mentioned that in Quantization Conjecture Revisited, he describes a general machine that is enough to establish the finiteness of the pushforward.

- Constantin also said that the Frenkel Teleman Tolland paper admits generalization to larger tori, and that there is still work to be done there and to think about how the bubbling data should be organized.

- Constantin said to work on the $\mathrm{SL}(2)$ case for the Solis moduli problem, where things are concrete and labels are just integers.