Title

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Abstract

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1 Teleman Woodward

Let C be a smooth projective curve and $\mathfrak{M} = \mathrm{Bun}_G(C)$. Fix an "admissible class"

$$\mathcal{E} = \mathcal{L} \otimes (\text{Atiyah–Bott generators})$$

where $\mathcal{L}$ is a determinant line bundle of level $h = h_{\mathcal{L}}$, and c is the canonical level coming from $\mathcal{K} = \det(E_C^* \mathfrak{g})$. Admissible means $h + c$ is positive definite.

Goal: show the index $\mathrm{Ind}(\mathfrak{M}, \mathcal{E}) = \chi(\mathfrak{M}, \mathcal{E})$ is well-defined (finite) even though $\mathfrak{M}$ is not finite type.

1.1 Shatz stratification and finite-type exhaustions

There is the Harder–Narasimhan (Shatz) stratification

$$\mathfrak{M} = \bigsqcup_{\xi} \mathfrak{M}_{\xi}$$

indexed by dominant rational coweights ξ. The partial order on ξ gives open substacks

$$\mathfrak{M}_{\leq \xi} := \bigcup_{\mu \leq \xi} \mathfrak{M}_{\mu},$$

and a key geometric input is: each $\mathfrak{M}_{\leq \xi}$ is of finite type (and can be presented as a quotient of a quasi-projective variety by a reductive group Why is this important? Do I need a stratification with this property?). This gives an exhaustion of $\mathfrak{M}$ by finite-type opens.

1.2 Filter $R\Gamma(\mathfrak{M}, \mathcal{E})$ by local cohomology along the strata

From the increasing opens $\mathfrak{M}_{\leq \xi}$, you get a filtration of $R\Gamma(\mathfrak{M}, \mathcal{E})$ whose graded pieces are local cohomology complexes supported on the successive strata:

$$\mathrm{gr}_{\xi} \simeq R\Gamma_{\mathfrak{M}_{\xi}}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi}).$$

So finiteness of the global index reduces to:

1. each local term has finite Euler characteristic, and
2. all but finitely many ξ have zero contribution.

1.3 Purity/local duality converts local cohomology into cohomology on the stratum

Finite open unions of Shatz strata

$$\mathfrak{M}_{\leq \xi} = \bigcup_{\mu \leq \xi} \mathfrak{M}_{\mu}$$

can be presented as quotient stacks of smooth quasi–projective varieties by reductive groups. Recall that local cohomology $R\Gamma_Z(X, -)$ is defined using the closed subset $Z \subset X$:

$$R\Gamma_Z(X, \mathcal{E}) := R\mathrm{Hom}_X(R\underline{\Gamma}_Z \mathcal{O}_X, \mathcal{E}).$$

Lemma 1.1. Let $i : Z \hookrightarrow X$ be a closed immersion of schemes. By construction there is an adjunction between Ri_* and $i^!$ and so unwinding we find that local cohomology along Z may be computed as

$$R\Gamma_Z(X, \mathcal{E}) \simeq R\Gamma\big(Z, i^!(\mathcal{E})\big),$$

Lemma 1.2 (https://stacks.math.columbia.edu/tag/0AU3)**.** Let $i : Z \hookrightarrow X$ be a regular embedding of codimension d between smooth schemes, with normal bundle $N = N_{Z/X}$. One has the formula

$$i^!(\mathcal{E}) \simeq Li^*(\mathcal{E}) \otimes \omega_{Z/X}[d], \qquad \omega_{Z/X} \cong \det(N)^{-1}.$$

Applying these two lemmas together to the map of stacks $i_\xi : \mathfrak{M}_\xi \hookrightarrow \mathfrak{M}_{\leq \xi}$ (which is a regular embedding of codimension d_ξ) gives the following formula for local cohomology along the stratum $\mathfrak{M}_\xi$.

$$H^\bullet_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi}) = H^\bullet(\mathfrak{M}_\xi, \mathcal{R}_\xi \mathcal{E}),$$

where d_ξ is the codimension of $\mathfrak{M}_\xi$ and $\mathcal{R}_\xi \mathcal{E}$ denotes the sheaf of "$\mathcal{E}$–valued residues along $\mathfrak{M}_\xi$." In particular

$$\mathcal{R}_\xi \mathcal{E} \ := \ i_\xi^!(\mathcal{E}_{\leq \xi})$$

where $i_\xi : \mathfrak{M}_\xi \hookrightarrow \mathfrak{M}_{\leq \xi}$ is the inclusion and $i^!$ is the extraordinary pullback. What is the mechanism that lets us do this for stacks?

Moreover, the residue object $\mathcal{R}_\xi \mathcal{E}$ may be expressed formally as

$$\mathcal{R}_\xi \mathcal{E} = \mathcal{E}|_{\mathfrak{M}_\xi} \otimes \mathrm{Eul}(\nu_\xi)^{-1}[d_\xi]$$

where ν_ξ is the virtual normal complex

$$\nu_\xi = R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1]$$

1.4 Identify the virtual normal complex ν_ξ

The virtual normal complex is identified via deformation theory. Each stratum $\mathfrak{M}_\xi$ maps to a finite-type "semistable Levi core"

$$q_\xi : \mathfrak{M}_\xi \longrightarrow \mathfrak{M}^{\mathrm{ss}}_{G_\xi, \xi},$$

and the transverse deformation theory is governed by the perfect complex

$$\nu_\xi = R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1] \quad \text{on } \mathfrak{M}^{\mathrm{ss}}_{G_\xi, \xi}.$$

We might just have to think really hard about deformation theory to work this out, once I identify a line bundle on the moduli stack that gives me a notion of stability.

2 Solis compactification

Definition 2.1 ($\mathcal{P}$–parahoric G–bundles at a point)**.** Fix a smooth curve C over $\mathbb{C}$ and a point $p \in C$ with a choice of formal parameter z at p, so that the completed local ring is $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[z]]$ and the punctured disc is $D^\times = \operatorname{Spec} \mathbb{C}((z))$. Let

$$\mathcal{P} \subset G((z))$$

be a *parahoric* subgroup (for instance a *maximal* parahoric, i.e. one corresponding to a vertex of the fundamental alcove).

Define a sheaf of groups $\mathcal{G}^{\mathcal{P}}$ on C by gluing the standard sheaf $\mathcal{G}^{\mathrm{std}} = \mathrm{Hom}_{\mathrm{Sch}}(-, G)$ away from p with the local sheaf determined by $\mathcal{P}$ at p as follows:

- if $U \subset C$ is an open subset with $p \notin U$, set $\mathcal{G}^{\mathcal{P}}(U) := \mathcal{G}^{\mathrm{std}}(U) = \mathrm{Hom}_{\mathrm{Sch}}(U, G)$;
- for the formal disc $D = \operatorname{Spec} \mathbb{C}[[z]]$ and punctured disc $D^\times = \operatorname{Spec} \mathbb{C}((z))$, set
$$\mathcal{G}^{\mathcal{P}}(D) := \mathcal{P}, \qquad \mathcal{G}^{\mathcal{P}}(D^\times) := G((z)),$$
with restriction map given by the inclusion $\mathcal{P} \hookrightarrow G((z))$;
- on the overlap $(C \setminus \{p\}) \cap D^\times \simeq D^\times$, we identify both restrictions with $G((z))$ and glue.

A *$\mathcal{P}$–parahoric G–bundle on C (with parahoric structure of type $\mathcal{P}$ at p)* is a $\mathcal{G}^{\mathcal{P}}$–torsor on C. Equivalently, it is the data of:

(i) a principal G–bundle E on $C \setminus \{p\}$;

(ii) a $\mathcal{P}$–torsor E_D on the formal disc D (i.e. a principal homogeneous space under the group $\mathcal{P} = \mathcal{G}^{\mathcal{P}}(D)$);

(iii) an identification of the induced $G((z))$–torsors over $D^\times$.

When $\mathcal{P} = L_P^+ G = \{\gamma \in G[[z]] \mid \gamma(0) \in P\}$ for a parabolic $P \subset G$, this recovers the usual notion of a quasi–parabolic G–bundle with a P–reduction at p. When $\mathcal{P}$ is a maximal parahoric corresponding to a vertex η_i of the fundamental alcove, we say the parahoric structure at p is *of type* η_i.

2.1 Refined stratification of $\mathcal{X}_{G,g,I}$

Let C/B be a prestable curve with dual graph Γ, and let (C'_B, P_B) be an object of $\mathcal{X}_{G,g,I}$; that is, C'_B is a twisted modification of C_B and P_B is an admissible G–bundle on C'_B.

For each vertex $v \in V(\Gamma)$ let ξ_v denote the Harder–Narasimhan type of the restriction of P_B to the normalization component indexed by v. For each node $e \in E(\Gamma)$ we have two additional pieces of boundary data:

1. a *parahoric type*
$$I_e \subset \{0, \dots, r\},$$
specifying the parahoric subgroup $\mathcal{P}_{I_e} \subset G((z))$ which governs the local structure of the bundle at e;

2. a *relative position label*
$$\mathbf{w}_e \in W_{I_e} \backslash \widetilde{W} / W_{I_e},$$
equivalently an orbit $O_e \subset \mathcal{P}_{I_e} \backslash G((z)) / \mathcal{P}_{I_e}$ describing the gluing of the two branches at e. We include this data to mod out by changes of trivialization on each side.

Let τ_e denote the length of the modification chain over the node e. Collect the data into
$$\alpha = (\Gamma, \tau, \mathbf{I}, \mathbf{w}, \boldsymbol{\xi}), \qquad \mathbf{I} = (I_e)_{e \in E(\Gamma)}, \ \mathbf{w} = (\mathbf{w}_e)_{e \in E(\Gamma)}, \ \boldsymbol{\xi} = (\xi_v)_{v \in V(\Gamma)}.$$

Definition 2.2. The *refined stratum* of type α is the locally closed substack
$$\mathcal{X}_\alpha \subset \mathcal{X}_{G,g,I}$$
consisting of objects (C'_B, P_B) such that:

(i) the coarse curve C_B has dual graph Γ and the modification lengths at the nodes are τ_e;

(ii) for every vertex v, the restriction of P_B to the corresponding normalization component has Harder–Narasimhan type ξ_v;

(iii) at each node e, the parahoric structure of P_B is of type I_e, and the gluing of the two branches lies in the orbit O_e corresponding to $\mathbf{w}_e$.

The collection $\{\mathcal{X}_\alpha\}_\alpha$ forms a stratification of $\mathcal{X}_{G,g,I}$, and its closure relations are governed by:

- the usual specialization of dual graphs and modification lengths;
- the dominance order on the HN types ξ_v;
- the Bruhat order on the double cosets $W_{I_e} \backslash \widetilde{W} / W_{I_e}$.

Remark 2.3. For a smooth curve the Shatz stratification indexed by HN types ξ is adequate, because a G–bundle has no additional local structure. On a nodal curve, however, two bundles with identical Shatz types on the normalization can differ essentially at the node.

Consider two objects: (1) a genuine G-bundle on the nodal curve with gluing element $g \in G \subset G((z))$, and (2) a limit object where the gluing is $z^\lambda \in G((z))$ with $\lambda > 0$.

Both have the same normalization bundles, the same HN type $\xi_v = 0$, and the same parahoric type $I_e = \{0\}$. However, (1) lies in the open stratum corresponding to actual G-bundles, while (2) lies in a boundary stratum of the wonderful compactification.

Lemma 2.4. For any fixed combinatorial type (graph Γ + expansion length bound + allowed parahoric types I_e in a finite set) and any bound on $(\boldsymbol{\xi}_v, \mathbf{w}_e)$, the open substack $\mathcal{X}_{\leq \alpha}$ is algebraic and of finite type over an étale chart of $\overline{\mathfrak{M}}_{g,I}$.

1. Do I need to find a presentation of this stratum as a quotient of a quasi-projective variety by a reductive group? The answer is yes this is important because once you show that only finitely many strata contribute, this is how each contribution has finitely many weight spaces.

2. Do you expect the formula

$$i^!(\mathcal{E}) \simeq Li^*(\mathcal{E}) \otimes \omega_{Z/X}[d], \qquad \omega_{Z/X} \cong \det(N)^{-1}.$$

   to hold for me in this situation? What do I need to know about the embedding $i : \mathcal{X}_\alpha \hookrightarrow \mathcal{X}_{\leq\alpha}$ to be able to apply this? Shouldn't be a problem because deformation theory is controlled by H^2 of the stack which vanishes because we are on a curve.

3. How do I identify the virtual normal complex ν_α? Atiyah Bott.

Apparently there should be a completely canonical way of writing down a stratification, once you choose a line bundle on the moduli stack $\mathcal{X}_{G,g,I}$. This line bundle should give you a notion of stability, and the strata should be indexed by the instability types. How do I write down this line bundle? How is it done in the case of G-bundles on a smooth curve?