Teleman Woodward

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Abstract

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1 Overview of the finiteness theorem from Teleman Woodward

Let G be a connected reductive group and $\mathfrak{M} = \mathrm{Bun}_G(C)$ the moduli stack of G-bundles on a smooth projective curve C. The goal of Teleman-Woodward is to compute the index of certain K-theory classes on $\mathfrak{M}$, generalizing the Verlinde formula for line bundles.

However, our goal is to merely establish the finiteness of the index in the case of nodal curves. Abstracting from the paper, the finiteness theorem has the following structure:

1. Stratify the full stack $\mathfrak{M}$ by Harder–Narasimhan type $\mathfrak{M} = \bigsqcup_\xi \mathfrak{M}_\xi$. Include some explanation of HN type here.

2. Filter $R\Gamma(\mathfrak{M}, E)$ by local cohomology along the closed unions $\bigcup_{\xi' \geq \xi} \mathfrak{M}_{\xi'}$.

3. Show that admissibility forces the local terms for ξ sufficiently large to vanish.

4. Deduce that only finitely many strata contribute, hence the index is finite.

Admissible classes (Teleman–Woodward)

Let Σ be a smooth projective curve, G a connected reductive group, and $\mathfrak{M} = \mathrm{Bun}_G(\Sigma)$ the moduli stack of G–bundles. Let $\mathcal{E}$ denote the universal G–bundle on $\Sigma \times \mathfrak{M}$, and for a finite–dimensional representation V of G write $\mathcal{E}^*V$ for the associated vector bundle.

Teleman–Woodward single out the following natural K–theory classes on $\mathfrak{M}$:

(i) $E_x^*V \in K^0(\mathfrak{M})$, the restriction of $\mathcal{E}^*V$ to $\{x\} \times \mathfrak{M}$, for a point $x \in \Sigma$;

(ii) $E_C^*V := \mathcal{E}^*V/[C] \in K^{-1}(\mathfrak{M})$, the slant product of $\mathcal{E}^*V$ with a 1–cycle C on Σ;

(iii) $E_\Sigma^*V := R\pi_*(\mathcal{E}^*V \otimes \sqrt{K}) \in K^0(\mathfrak{M})$, the Dirac index bundle along Σ, where $\pi : \Sigma \times \mathfrak{M} \to \mathfrak{M}$ and $\sqrt{K}$ is a square root of the canonical bundle of Σ;

(iv) $D_\Sigma V := \det^{-1} E_\Sigma^* V$, the inverse determinant of cohomology.

The classes in (i)–(iii) are called the *Atiyah–Bott generators*. The classes in (iv) are determinant line bundles on $\mathfrak{M}$.

The first Chern class of any determinant line bundle $\mathcal{L}$ defines an invariant quadratic form

$$h_{\mathcal{L}} \in H^4(BG; \mathbb{R}) \cong \mathrm{Sym}^2(\mathfrak{g}^*)^G \cong \{\text{invariant symmetric bilinear forms on } \mathfrak{g}\},$$

called the *level* of $\mathcal{L}$. Let c denote the distinguished quadratic form corresponding to the canonical bundle $\mathcal{K} = \det E_\Sigma^* \mathfrak{g}$.

Definition 1.1 (Teleman–Woodward)**.** A line bundle $\mathcal{L}$ on $\mathfrak{M}$ is called *admissible* if the shifted quadratic form

$$h_{\mathcal{L}} + c$$

is positive definite on $\mathfrak{g}$. An *admissible class* in $K^*(\mathfrak{M})$ is any finite product of an admissible line bundle with Atiyah–Bott generators.

Shatz stratification

Recall that any G–bundle over Σ admits a canonical reduction of structure group to a standard parabolic subgroup $P \subset G$, for which the associated bundle with Levi structure group is semistable.

Remark 1.2. For a principal G-bundle P on a smooth curve Σ, there is a Harder–Narasimhan (HN) theory generalizing the usual HN filtration of vector bundles. The outcome is a canonical reduction of P to a parabolic subgroup $P \subset G$. "Canonical" means: determined functorially by P (up to unique isomorphism), not a choice. The defining property is that if you pass from P (the parabolic) to its Levi quotient $L = P/R_u(P)$, the induced L-bundle is semistable. "Standard parabolic" means: a parabolic containing a fixed Borel B (chosen once), so parabolics are indexed by subsets of simple roots.

Intuition: this parabolic reduction packages "the most destabilizing" subobject(s) of the bundle.

Topologically, this reduction is classified by a coweight of $P/[P, P]$; we identify this with a (possibly fractional) dominant coweight ξ of $\mathfrak{g}$, called the *instability type* of the original bundle. Then P is the standard parabolic subgroup defined by ξ; we denote it by P_ξ and its Levi subgroup by G_ξ. If $\mathfrak{M}_\xi$ denotes the stack of G–bundles of type ξ, we have an algebraic stratification

$$\mathfrak{M} = \bigsqcup_{\xi} \mathfrak{M}_\xi.$$

Sending a P_ξ–bundle to its associated Levi bundle defines a morphism from $\mathfrak{M}_\xi$ to the stack $\mathfrak{M}^{ss}_{G_\xi,\xi}$ of semistable principal G_ξ–bundles of type ξ; the fibres are quotient stacks of affine spaces by unipotent groups (equivalently the corresponding Lie algebra is nilpotent). The virtual normal bundle for the morphism $\mathfrak{M}^{ss}_{G_\xi,\xi} \to \mathfrak{M}$ is the complex

$$\nu_\xi = R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1].$$

Its K–theory Euler class should be the alternating sum of exterior powers

$$\lambda_{-1}(\nu_\xi^\vee) := \sum_p (-1)^p \wedge^p (\nu_\xi^\vee),$$

but for now this infinite sum is only a formal expression, whose meaning is to be spelled out.

Local cohomology

Finite open unions of Shatz strata

$$\mathfrak{M}_{\leq\xi} = \bigcup_{\mu\leq\xi} \mathfrak{M}_\mu$$

can be presented as quotient stacks of smooth quasi–projective varieties by reductive groups. The cohomology with supports over $\mathfrak{M}_\xi$ of a vector bundle $\mathcal{E}$ is

$$H^\bullet_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq\xi}, \mathcal{E}_{\leq\xi}) = H^{\bullet+d_\xi}(\mathfrak{M}_\xi, \mathcal{R}_\xi\mathcal{E}), \tag{1.9}$$

where d_ξ is the codimension of $\mathfrak{M}_\xi$ and $\mathcal{R}_\xi\mathcal{E}$ denotes the sheaf of "$\mathcal{E}$–valued residues along $\mathfrak{M}_\xi$." In particular

$$\mathcal{R}_\xi\mathcal{E} \;:=\; i_\xi^!(\mathcal{E}_{\leq\xi})[-d_\xi]$$

where $i_\xi : \mathfrak{M}_\xi \hookrightarrow \mathfrak{M}_{\leq\xi}$ is the inclusion and $i^!$ is the extraordinary pullback (local duality functor).

This is a stacky derived version of the fact that for a smooth closed subvariety $Z \subset X$, local cohomology along Z equals cohomology on Z twisted by the normal bundle and shifted by codimension.

Basically I think we need to find the right stratification and Dan HL has a machine that produces such stratifications, known as θ-stratifications.

Role of the Shatz stratification in Teleman–Woodward

The proof of the finiteness theorem in [1] is organized around the Harder–Narasimhan (Shatz) stratification of the moduli stack

$$\mathfrak{M} = \mathrm{Bun}_G(\Sigma) = \bigsqcup_\xi \mathfrak{M}_\xi,$$

indexed by dominant rational coweights ξ. This stratification plays the role of a Morse stratification for the Yang–Mills functional, and replaces compactness in the non–finite–type stack $\mathfrak{M}$.

(1) Filtration by supports. The partial order on instability types defines an increasing filtration by open substacks

$$\mathfrak{M}_{\leq\xi} := \bigcup_{\mu\leq\xi} \mathfrak{M}_\mu.$$

For any sheaf or complex $\mathcal{E}$ on $\mathfrak{M}$, this filtration produces a filtration on derived global sections $R\Gamma(\mathfrak{M},\mathcal{E})$ by the subcomplexes $R\Gamma_{\mathfrak{M}_{\leq\xi_k}}(\mathfrak{M},\mathcal{E})$.

Remark 1.3 (General mechanism of local cohomology filtration)**.** Suppose you have a space/stack X and an increasing sequence of open substacks $\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset X$ with closed complements $Z_k := X \setminus U_k$. In our situation $X = \mathfrak{M}$, $U_k = \mathfrak{M}_{\leq\xi_k}$, $Z_k = \bigcup_{\mu>\xi_k} \mathfrak{M}_\mu$. For any sheaf or complex $\mathcal{E}$ on X, there is a canonical exact triangle

$$R\Gamma_{Z_k}(X,\mathcal{E}) \;\to\; R\Gamma(X,\mathcal{E}) \;\to\; R\Gamma(U_k,\mathcal{E}|_{U_k}) \;\to$$

This triangle is the definition of local cohomology with supports in Z_k.

The decreasing family of closed substacks $Z_k = X\backslash U_k$ induces a decreasing filtration $F^k := R\Gamma_{Z_k}(X,\mathcal{E})$ of $R\Gamma(X,\mathcal{E})$. Its successive graded pieces are

$$\mathrm{gr}^k F \simeq R\Gamma_{Z_k\backslash Z_{k+1}}(U_{k+1},\mathcal{E}|_{U_{k+1}}).$$

In the Shatz situation (refining the indexing so that $U_k = U_{k-1} \sqcup \mathfrak{M}_{\xi_k}$), this becomes

$$\mathrm{gr}^{k-1} F \simeq R\Gamma_{\mathfrak{M}_{\xi_k}}(\mathfrak{M}_{\leq \xi_k}, \mathcal{E}_{\leq \xi_k}).$$

Equivalently there is a spectral sequence with

$$E_1^{\xi,*} = R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi}) \quad \Longrightarrow \quad R\Gamma(\mathfrak{M}, \mathcal{E}).$$

(2) Reduction to semistable Levi moduli. Each stratum $\mathfrak{M}_\xi$ carries a canonical morphism

$$\mathfrak{M}_\xi \longrightarrow \mathfrak{M}^{ss}_{G_\xi,\xi}$$

to the moduli stack of semistable principal G_ξ–bundles of fixed topological type, whose fibres are quotient stacks of affine spaces by unipotent groups. This identifies $\mathfrak{M}_\xi$ as a bundle of unstable directions over a semistable core.

(3) Virtual normal complex. The stratification provides a uniform description of the virtual normal complex of $\mathfrak{M}^{ss}_{G_\xi,\xi}$ in $\mathfrak{M}$:

$$\nu_\xi = R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1].$$

Consequently, local cohomology along $\mathfrak{M}_\xi$ may be expressed formally as

$$R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}) \simeq R\Gamma\big(\mathfrak{M}_\xi, \mathcal{E} \otimes \lambda_{-1}(\nu_\xi^\vee)^{-1}\big),$$

where $\lambda_{-1}(\nu_\xi^\vee)$ is the K–theoretic Euler class of the normal complex.

(4) Weight decomposition and admissibility. The representation $\mathfrak{g}/\mathfrak{g}_\xi$ decomposes into positive ξ–weight spaces. This induces a natural grading on ν_ξ and hence on $\lambda_{-1}(\nu_\xi^\vee)^{-1}$. Admissibility of the twisting line bundle forces all sufficiently unstable types ξ to contribute only strictly negative weights, so that the formal inverse Euler class becomes summable and the local contributions vanish for ξ sufficiently large.

(5) Finiteness. Since only finitely many instability types can contribute nontrivially, the local–cohomology filtration terminates after finitely many steps. This yields the finiteness of the index.

In summary, the Shatz stratification supplies a canonical filtration, a reduction to semistable Levi moduli, and a uniform normal complex whose weight decomposition is controlled by admissibility. All finiteness statements in [1] are ultimately consequences of this structure.

2 Toy model: $G = \mathrm{GL}_2$ on a smooth curve

Let Σ be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$, and let

$$\mathfrak{M} = \mathrm{Bun}_{\mathrm{GL}_2}(\Sigma)$$

be the moduli stack of rank 2 vector bundles on Σ. We explain explicitly the Shatz stratification, the Levi description of strata, the virtual normal complex, and the weight bookkeeping behind the Teleman–Woodward finiteness mechanism in this case.

2.1 Harder–Narasimhan type and Shatz strata

Every $E \in \mathfrak{M}$ admits a unique Harder–Narasimhan filtration

$$0 \subset L \subset E, \qquad M := E/L,$$

where L is a line subbundle of maximal slope. Write

$$\deg(L) = d_1, \qquad \deg(M) = d_2, \qquad m := d_1 - d_2 \geq 0.$$

Then E is semistable iff $m = 0$ (equivalently $d_1 = d_2$).

The Shatz (HN) stratum of type (d_1, d_2) is the locally closed substack

$$\mathfrak{M}_{d_1,d_2} \subset \mathfrak{M}$$

parametrizing bundles whose HN filtration has graded pieces (L, M) of degrees (d_1, d_2) (so $m > 0$ on unstable strata). One has the stratification

$$\mathfrak{M} = \bigsqcup_{d_1 \geq d_2} \mathfrak{M}_{d_1,d_2}.$$

2.2 Parabolic and Levi

Fix the standard Borel $B \subset \mathrm{GL}_2$ of upper triangular matrices. For $m > 0$, the destabilizing reduction is to the standard parabolic

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\},$$

with Levi subgroup

$$G_\xi \cong \mathrm{GL}_1 \times \mathrm{GL}_1 \quad \text{(diagonal matrices)}.$$

Equivalently, the associated dominant coweight (instability type) may be taken as

$$\xi = \left(\frac{m}{2}, -\frac{m}{2} \right) \in \mathfrak{t}_{\mathbb{Q}},$$

so that the positive root α satisfies $\alpha(\xi) = m$.

2.3 Semistable Levi core and structure of the stratum

A principal G_ξ–bundle is the same as a pair of line bundles (L, M), hence the moduli stack of semistable G_ξ–bundles of type (d_1, d_2) is

$$\mathfrak{M}^{ss}_{G_\xi,\xi} \;\cong\; \mathrm{Pic}^{d_1}(\Sigma) \times \mathrm{Pic}^{d_2}(\Sigma).$$

There is a canonical morphism

$$q : \mathfrak{M}_{d_1,d_2} \longrightarrow \mathrm{Pic}^{d_1}(\Sigma) \times \mathrm{Pic}^{d_2}(\Sigma), \qquad E \mapsto (L, E/L).$$

Fixing (L, M), the fibre of q over (L, M) classifies extensions

$$0 \to L \to E \to M \to 0,$$

hence is governed by

$$\mathrm{Ext}^1(M, L) \cong H^1(\Sigma, L \otimes M^{-1}).$$

Automorphisms of a given extension (fixing (L, M)) come from

$$\mathrm{Hom}(M, L) \cong H^0(\Sigma, L \otimes M^{-1}),$$

which is a unipotent group (additively) acting on the affine space $H^1(\Sigma, L \otimes M^{-1})$ by the usual change-of-splitting. Thus the fibre is a quotient stack $\left[H^1(\Sigma, L \otimes M^{-1})\middle/\; H^0(\Sigma, L \otimes M^{-1})\right]$ making $\mathfrak{M}_{d_1,d_2}$ a (stacky) affine fibration over the semistable Levi core.

2.4 Virtual normal complex

The tangent complex of Bun_G at a G–bundle E is

$$T_{\mathfrak{M},E} \simeq R\Gamma(\Sigma, \mathrm{End}(E))[1].$$

Over the Levi core (L, M) the adjoint representation decomposes as

$$\mathrm{End}(L \oplus M) \;=\; \underbrace{\mathrm{End}(L) \oplus \mathrm{End}(M)}_{\mathfrak{g}_\xi} \;\oplus\; \underbrace{\mathrm{Hom}(M, L) \oplus \mathrm{Hom}(L, M)}_{\mathfrak{g}/\mathfrak{g}_\xi}.$$

Along the stratum $\mathfrak{M}_{d_1,d_2}$, the relevant (unstable) normal directions are governed by the positive–weight root space $\mathrm{Hom}(M, L)$, and the virtual normal complex for the inclusion of the Levi moduli into $\mathfrak{M}$ is

$$\nu_\xi \;\simeq\; R\Gamma(\Sigma, L \otimes M^{-1})[1].$$

Equivalently, ν_ξ has cohomology

$$H^{-1}(\nu_\xi) \cong H^0(\Sigma, L \otimes M^{-1}), \qquad H^0(\nu_\xi) \cong H^1(\Sigma, L \otimes M^{-1}).$$

2.5 K–theoretic Euler class and its formal inverse

Formally, the K–theory Euler class of the dual normal complex is

$$\lambda_{-1}(\nu_\xi^\vee) = \sum_{p\geq 0}(-1)^p \wedge^p (\nu_\xi^\vee).$$

Because ν_ξ is a shifted cohomology complex, its inverse Euler class expands into exterior powers of the H^0–piece and symmetric powers of the H^1–piece. Schematically one may think of

$$\lambda_{-1}(\nu_\xi^\vee)^{-1} \sim \frac{\mathrm{Sym}^\bullet\big(H^1(\Sigma, L\otimes M^{-1})^\vee\big)}{\Lambda^\bullet\big(H^0(\Sigma, L\otimes M^{-1})^\vee\big)}$$

an infinite sum in ordinary K–theory which is made meaningful in Teleman–Woodward by working in a suitable completion determined by ξ–weights.

2.6 Weight bookkeeping: linear vs. quadratic growth

The one–parameter subgroup ξ acts on $\mathrm{Hom}(M, L)$ with weight $\alpha(\xi) = m$. Consequently, ξ acts on $H^i(\Sigma, L\otimes M^{-1})$ with weight m, and hence on the graded summand

$$\mathrm{Sym}^p\big(H^1(\Sigma, L\otimes M^{-1})^\vee\big)$$

with weight $p\,m$. This is the *linear* growth in the instability parameter m.

On the other hand, a determinant line bundle $\mathcal{L}$ on $\mathfrak{M}$ has a level $h_{\mathcal{L}} \in \mathrm{Sym}^2(\mathfrak{g}^*)^G$, and Teleman–Woodward introduce the canonical correction c coming from $\mathcal{K} = \det E_\Sigma^*\mathfrak{g}$. For an *admissible* $\mathcal{L}$, the form $h_{\mathcal{L}} + c$ is positive definite, so

$$(h_{\mathcal{L}} + c)(\xi, \xi) \to +\infty \quad \text{as } \|\xi\| \to \infty.$$

In the GL_2 normalization $\xi = (m/2, -m/2)$ and the standard invariant form $(X, Y) = \mathrm{tr}(XY)$ on diagonal matrices gives

$$(\xi, \xi) = \frac{m^2}{2},$$

so $(h_{\mathcal{L}} + c)(\xi, \xi)$ grows like a positive constant times m^2. In Teleman–Woodward's local cohomology calculation, twisting by $\mathcal{L}$ shifts the ξ–weight spectrum by a *negative* amount with leading term

$$-(h_{\mathcal{L}} + c)(\xi, \xi) \sim -\kappa m^2 \qquad (\kappa > 0).$$

Thus, on the ξ–stratum, the inverse Euler class contributes graded pieces with weights growing at most *linearly* in m (e.g. $p\,m$), while an admissible twist shifts weights by a *quadratic* negative term $\sim -\kappa m^2$. This is the mechanism behind the eventual vanishing of sufficiently unstable strata in the Teleman–Woodward finiteness theorem.

Why ξ–invariants control finiteness of the index

Let Σ be a smooth projective curve and $\mathfrak{M} = \mathrm{Bun}_G(\Sigma)$. For a class $\mathcal{E} \in K^*(\mathfrak{M})$ one defines its index by the Euler characteristic

$$\mathrm{Ind}(\mathfrak{M}, \mathcal{E}) \;:=\; \chi(\mathfrak{M}, \mathcal{E}) \;=\; \sum_i (-1)^i \dim H^i(\mathfrak{M}, \mathcal{E}),$$

whenever the right-hand side is finite.

Since $\mathfrak{M}$ is not of finite type, finiteness is proved by filtering $\mathfrak{M}$ by finite-type open substacks using the Shatz stratification

$$\mathfrak{M} = \bigsqcup_{\xi} \mathfrak{M}_\xi, \qquad \mathfrak{M}_{\leq \xi} := \bigcup_{\mu \leq \xi} \mathfrak{M}_\mu.$$

The open substacks $\mathfrak{M}_{\leq \xi}$ form an increasing filtration of $\mathfrak{M}$, and for any complex $\mathcal{E}$ on $\mathfrak{M}$ this induces a filtration of $R\Gamma(\mathfrak{M}, \mathcal{E})$ by local cohomology with supports in the complements. Equivalently, there is a spectral sequence whose E_1–page is built from the local cohomology complexes

$$E_1^{\xi,*} \;=\; R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi}) \quad\Longrightarrow\quad R\Gamma(\mathfrak{M}, \mathcal{E}).$$

In particular, finiteness of $\mathrm{Ind}(\mathfrak{M}, \mathcal{E})$ follows once one knows:

(a) for each ξ, the contribution of $R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi})$ to Euler characteristic is finite-dimensional; and

(b) all but finitely many ξ contribute trivially.

Teleman–Woodward identify each local term by a purity/local-duality statement:

$$R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi}) \simeq R\Gamma\big(\mathfrak{M}_\xi, \mathcal{R}_\xi \mathcal{E}\big)[d_\xi], \qquad \mathcal{R}_\xi \mathcal{E} := i_\xi^!(\mathcal{E}_{\leq \xi})[-d_\xi],$$

where $d_\xi = \mathrm{codim}(\mathfrak{M}_\xi, \mathfrak{M}_{\leq \xi})$ and $i_\xi : \mathfrak{M}_\xi \hookrightarrow \mathfrak{M}_{\leq \xi}$ is the inclusion.

Moreover, the residue object $\mathcal{R}_\xi \mathcal{E}$ may be expressed formally in terms of the virtual normal complex

$$\nu_\xi = R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1]$$

as

$$\mathcal{R}_\xi \mathcal{E} \;\sim\; \mathcal{E}|_{\mathfrak{M}_\xi} \otimes \mathrm{Eul}(\nu_\xi)^{-1}.$$

Remark 2.1. Let X be an algebraic stack, let $i : Z \hookrightarrow X$ be a closed immersion, and let $\mathcal{E}$ be a (bounded) complex of coherent sheaves on X. Recall that the *local cohomology* of $\mathcal{E}$ with supports in Z is defined by the exact triangle

$$R\Gamma_Z(X, \mathcal{E}) \;\longrightarrow\; R\Gamma(X, \mathcal{E}) \;\longrightarrow\; R\Gamma(X \setminus Z, \mathcal{E}|_{X \setminus Z}) \;\longrightarrow,$$

or, equivalently, by the functor of sections with supports $R\Gamma_Z(X,-) = R\Gamma(X, R\Gamma_Z(-))$. Local duality packages these groups as ordinary cohomology on Z: one defines the *residue object* along Z by

$$\mathcal{R}_Z(\mathcal{E}) \;:=\; i^!(\mathcal{E})[-\operatorname{codim}(Z)],$$

so that (under standard hypotheses, e.g. i a local complete intersection)

$$R\Gamma_Z(X,\mathcal{E}) \;\simeq\; R\Gamma\big(Z, \mathcal{R}_Z(\mathcal{E})\big)[\operatorname{codim}(Z)].$$

In Teleman–Woodward's setting one takes $X = \mathfrak{M}_{\leq\xi}$ and $Z = \mathfrak{M}_\xi$, so

$$\mathcal{R}_\xi\mathcal{E} := i^!_\xi(\mathcal{E}_{\leq\xi})[-d_\xi], \qquad d_\xi = \operatorname{codim}(\mathfrak{M}_\xi, \mathfrak{M}_{\leq\xi}).$$

Why an Euler factor appears. If $i : Z \hookrightarrow X$ is a *regular embedding* between smooth schemes with normal bundle $N_{Z/X}$, then $i^!$ is controlled by the normal directions. At the level of K–theory one has the standard identity

$$\big[i^!(\mathcal{E})\big] = \big[i^*(\mathcal{E})\big] \cdot \lambda_{-1}(N^\vee_{Z/X})^{-1} \qquad \text{in } K^0(Z), \tag{1}$$

where

$$\lambda_{-1}(W) := \sum_{p\geq 0}(-1)^p[\wedge^p W]$$

is the K–theoretic Euler class. Heuristically, local cohomology measures "principal parts along Z", and principal parts are obtained by expanding in the normal directions; the inverse Euler class $\lambda_{-1}(N^\vee_{Z/X})^{-1}$ is the K–theoretic avatar of this expansion.

Virtual normal complex. In the Shatz situation, $\mathfrak{M}_\xi \hookrightarrow \mathfrak{M}_{\leq\xi}$ is not presented globally as a single regular embedding into a smooth ambient space. Instead, Teleman–Woodward use the fact that $\mathfrak{M}_\xi$ maps to a finite–type semistable Levi stack $\mathfrak{M}^{ss}_{G_\xi,\xi}$ with fibres quotient stacks of affine spaces by unipotent groups, and there is a canonical *virtual normal complex* (perfect complex playing the role of $N_{Z/X}$)

$$\nu_\xi \;=\; R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1] \qquad \text{on } \mathfrak{M}^{ss}_{G_\xi,\xi},$$

whose pullback to $\mathfrak{M}_\xi$ controls the unstable directions transverse to the Levi moduli. Consequently, the K–theory Euler class is defined by

$$\operatorname{Eul}(\nu^\vee_\xi) \;:=\; \lambda_{-1}(\nu^\vee_\xi),$$

and the same formal identity as (1) holds with $N_{Z/X}$ replaced by ν_ξ:

$$\big[\mathcal{R}_\xi\mathcal{E}\big] = \big[\mathcal{E}|_{\mathfrak{M}_\xi}\big] \cdot \lambda_{-1}(\nu^\vee_\xi)^{-1}. \tag{2}$$

2.6.1 The polarized inverse Euler class and the formula for $\operatorname{Eul}(\nu_\xi)^{-1}_+$

We explain the origin of Teleman–Woodward's formula

$$\operatorname{Eul}(\nu_\xi)^{-1}_+ := \operatorname{Sym}\Big(R\pi_*\mathcal{E}^*(\mathfrak{p}_\xi/\mathfrak{g}_\xi)[1]^\vee \;\oplus\; R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big) \;\otimes\; \det\Big(R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big)[d_\xi], \quad (*)$$

It is a formal inverse to the Euler class with a weight decomposition satisfing the key properties that only weights ≤ 0 occur and each weight space is finite.

(1) ξ–weights and the parabolic splitting. Fix a maximal torus $T \subset G$ and a Borel $B \supset T$. For a dominant rational coweight $\xi \in X_*(T) \otimes \mathbb{Q}$, let P_ξ be the associated *standard* parabolic subgroup (so $B \subset P_\xi$), and let G_ξ be its Levi subgroup. At the Lie algebra level one has a canonical ξ–weight decomposition

$$\mathfrak{g} = \mathfrak{g}_\xi \oplus \mathfrak{n}_\xi \oplus \mathfrak{n}_\xi^-, \qquad \mathfrak{n}_\xi = \bigoplus_{\langle \alpha, \xi \rangle > 0} \mathfrak{g}_\alpha, \quad \mathfrak{n}_\xi^- = \bigoplus_{\langle \alpha, \xi \rangle < 0} \mathfrak{g}_\alpha.$$

Equivalently,

$$\mathfrak{p}_\xi = \mathfrak{g}_\xi \oplus \mathfrak{n}_\xi, \qquad \mathfrak{g}/\mathfrak{g}_\xi \cong (\mathfrak{p}_\xi/\mathfrak{g}_\xi) \oplus (\mathfrak{g}/\mathfrak{p}_\xi),$$

where $\mathfrak{p}_\xi/\mathfrak{g}_\xi \cong \mathfrak{n}_\xi$ carries strictly *positive* ξ–weights and $\mathfrak{g}/\mathfrak{p}_\xi \cong \mathfrak{n}_\xi^-$ carries strictly *negative* ξ–weights.

(2) The virtual normal complex and its ξ–grading. Let $\mathfrak{M}_\xi$ be the Shatz stratum of instability type ξ and let $\mathfrak{M}^{ss}_{G_\xi,\xi}$ be the semistable Levi moduli.

Sending a P_ξ–bundle to its associated Levi bundle defines a morphism $q_\xi : \mathfrak{M}_\xi \to \mathfrak{M}^{ss}_{G_\xi,\xi}$. The fibres are quotient stacks of affine spaces by unipotent groups. Whenever we define our stratification, we need to make sure this is true. The deformation theory transverse to the Levi directions is governed by the perfect complex on $\mathfrak{M}^{ss}_{G_\xi,\xi}$

$$\nu_\xi := R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1],$$

How did Teleman-Woodward identify this? Is there some general theory? whose pullback along q_ξ controls the virtual normal directions along the stratum and where

$$\pi : \Sigma \times \mathfrak{M}^{ss}_{G_\xi,\xi} \to \mathfrak{M}^{ss}_{G_\xi,\xi}$$

and $\mathcal{E}$ is the universal bundle. Using the splitting above,

$$\nu_\xi \simeq \nu_\xi^+ \oplus \nu_\xi^-, \qquad \nu_\xi^+ := R\pi_* \mathcal{E}^*(\mathfrak{p}_\xi/\mathfrak{g}_\xi)[1], \quad \nu_\xi^- := R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1].$$

Thus ν_ξ carries a canonical $\mathbb{Z}$–grading by ξ–weights: ν_ξ^+ has strictly positive weights and ν_ξ^- has strictly negative weights.

(3) Why an "inverse Euler class" is not a genuine K–class. For a vector bundle W, the K–theoretic Euler class is

$$\lambda_{-1}(W^\vee) = \sum_{p \ge 0} (-1)^p [\wedge^p W^\vee].$$

Even for a line bundle L, the inverse of $1 - L^\vee$ is *not* a finite K–class:

$$(1 - L^\vee)^{-1} = \sum_{n \ge 0} (L^\vee)^n \qquad \text{(a formal geometric series).}$$

Teleman–Woodward therefore work in a *completion* of equivariant K–theory determined by the ξ–weights. In such a completion one is allowed to expand $1 - L^\vee$ as a geometric series *in whichever direction is convergent in the chosen completion*. This is the meaning of the phrase "prefers ξ–negative eigenvalues."

(4) The basic one–dimensional identity and the determinant correction. Let $\mathfrak{G}_m$ act on a one–dimensional representation of weight $w \neq 0$, so the character is t^w. Then

$$(1 - t^w)^{-1} = \sum_{n \geq 0} t^{nw} \qquad \text{as a formal series in the direction of weights } w, w, 2w, \ldots.$$

If we instead want an expansion which involves only *nonpositive* weights (i.e. which "prefers negative eigenvalues"), we rewrite

$$(1 - t^w)^{-1} = -t^{-w}\,(1 - t^{-w})^{-1} = -t^{-w} \sum_{n \geq 0} t^{-nw}.$$

Compared to the naive geometric series, this introduces a prefactor $-t^{-w}$. In higher rank, multiplying these prefactors over all negative–weight lines produces a *determinant factor*.

(5) From weights to a polarized inverse for a split complex. Suppose a perfect complex K carries a ξ–grading and splits as

$$K \simeq K^+ \oplus K^-,$$

where all ξ–weights in K^+ are > 0 and all ξ–weights in K^- are < 0. Then

$$\lambda_{-1}(K^\vee) = \lambda_{-1}((K^+)^\vee) \cdot \lambda_{-1}((K^-)^\vee)$$

and one defines a *polarized inverse* $\lambda_{-1}(K^\vee)_+^{-1}$ by inverting each factor in the completion which expands in the direction of *ξ–negative weights*. The outcome is the standard schematic identity

$$\lambda_{-1}(K^\vee)_+^{-1} \;=\; \mathrm{Sym}\big((K^+)^\vee \oplus K^-\big) \;\otimes\; \det(K^-)\,[\text{shift}], \tag{3}$$

where:

- $\mathrm{Sym}(-)$ denotes the total symmetric algebra $\mathrm{Sym}^\bullet(-) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(-)$, interpreted in K–theory (or in the corresponding completed K–group) as a formal power series;
- $\det(K^-)$ is the determinant line of the perfect complex K^-, which precisely packages the product of the one–dimensional prefactors in **(4)**;
- $[\text{shift}]$ is the cohomological degree shift dictated by local duality/purity (and in the Shatz situation becomes $[d_\xi]$).

Remark 2.2 (Origin of the determinant factor in the polarized inverse)**.** Let $\mathfrak{G}_m$ act with a $\mathbb{Z}$–grading, and let W be a finite–rank $\mathfrak{G}_m$–equivariant vector bundle with *strictly negative* weights. Write the K–theoretic Euler class as

$$\lambda_{-1}(W^\vee) = \prod_i (1 - L_i^\vee),$$

after splitting $W = \bigoplus_i L_i$ into $\mathfrak{G}_m$–eigenlines (locally on the base). Formally,

$$(1 - L_i^\vee)^{-1} = \sum_{n \geq 0} (L_i^\vee)^n$$

is the geometric expansion in nonnegative powers of $L_i^\vee$. However, if L_i has *negative* ξ–weight, then $L_i^\vee$ has *positive* weight, so this expansion lives in the completion which prefers *positive* weights. To invert in the opposite completion (the one preferring negative weights), we rewrite

$$(1 - L_i^\vee)^{-1} = -\, L_i \cdot (1 - L_i)^{-1} = -\, L_i \sum_{n \geq 0} L_i^n,$$

which is now a series in nonnegative powers of L_i (hence in nonpositive weights). The price paid for using this expansion is the prefactor $(-L_i)$.

Multiplying over i gives

$$\lambda_{-1}(W^\vee)^{-1}\Big|_{\text{prefer negative}} = \Big(\prod_i (-L_i)\Big) \cdot \prod_i (1 - L_i)^{-1} = (-1)^{\operatorname{rank} W} \det(W) \cdot \operatorname{Sym}(W),$$

where $\operatorname{Sym}(W) := \bigoplus_{n \geq 0} \operatorname{Sym}^n(W)$. Up to the harmless sign $(-1)^{\operatorname{rank} W}$ (often suppressed in K–theory conventions), this explains the appearance of the factor $\det(W)$ in the polarized inverse.

For a perfect complex K^- of strictly negative weights, the same argument applied to any local splitting into graded line bundles (together with the multiplicativity of λ_{-1} in K–theory) produces the factor $\det(K^-)$ in the polarized inverse $\lambda_{-1}(K^\vee)_+^{-1}$.

(6) Specialization to ν_ξ. Apply (3) to $K = \nu_\xi$ and the splitting $\nu_\xi \simeq \nu_\xi^+ \oplus \nu_\xi^-$ from **(2)**. Then $(K^+)^\vee = (\nu_\xi^+)^\vee$ and $K^- = \nu_\xi^-$, and we obtain

$$\operatorname{Eul}(\nu_\xi)_+^{-1} := \lambda_{-1}(\nu_\xi^\vee)_+^{-1} = \operatorname{Sym}\big((\nu_\xi^+)^\vee \oplus \nu_\xi^-\big) \otimes \det(\nu_\xi^-)\,[d_\xi].$$

Unwinding the definitions of $\nu_\xi^\pm$ gives exactly the formula $(*)$ above.

Weight bookkeeping and the finiteness mechanism

Fix an instability type ξ and consider the local contribution supported on the Shatz stratum $\mathfrak{M}_\xi$. Teleman–Woodward control this contribution by analysing the ξ–weight decomposition of the *polarized inverse Euler factor* $\operatorname{Eul}(\nu_\xi)_+^{-1}$ and its tensor product with an admissible class $\mathcal{E}$.

(A) The determinant weight and the quadratic form $c(\xi,\xi)$. Recall the polarized inverse Euler factor (cf. [1, §1.10–1.11])

$$\mathrm{Eul}(\nu_\xi)_+^{-1} := \mathrm{Sym}\Big(R\pi_*\mathcal{E}^*(\mathfrak{p}_\xi/\mathfrak{g}_\xi)[1]^\vee \;\oplus\; R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big) \;\otimes\; \det\Big(R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big)[d_\xi]. \quad (*)$$

The second tensor factor is a determinant line bundle

$$\det\Big(R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big).$$

Because $\mathfrak{g}/\mathfrak{p}_\xi$ is a direct sum of root spaces $\mathfrak{g}_\alpha$ with $\langle\alpha,\xi\rangle < 0$, the one–parameter subgroup determined by ξ acts on $\mathfrak{g}/\mathfrak{p}_\xi$ with strictly *negative* weights. Consequently, the induced $\mathfrak{G}_m$–action on the determinant line above has a well-defined ξ–weight which may be computed as a signed sum of these negative integers, counted with the appropriate cohomological multiplicities coming from $R\pi_*$.

Teleman–Woodward package this total determinant weight by a distinguished invariant quadratic form

$$c \in \mathrm{Sym}^2(\mathfrak{g}^*)^G,$$

namely the quadratic form attached (via Grothendieck–Riemann–Roch) to the canonical bundle

$$\mathcal{K} := \det(E_\Sigma^*\mathfrak{g}) \qquad \text{on } \mathfrak{M}.$$

With this notation, the determinant factor in $(*)$ has ξ–weight

$$\mathrm{wt}_\xi\Big(\det(R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1])\Big) = c(\xi,\xi). \tag{4}$$

In Teleman–Woodward's conventions, $c(\xi,\xi)$ is *negative* when ξ is viewed in the compact real form $i\mathfrak{t}_k$; equivalently, c is negative definite on $i\mathfrak{t}_k$.

Justification of (4). Fix ξ and let $\lambda_\xi : \mathfrak{G}_m \to G$ be the canonical one–parameter subgroup. Consider the determinant line

$$\mathcal{D}_\xi = \det\Big(R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big)$$

which appears in the polarized inverse Euler class $\mathrm{Eul}(\nu_\xi)_+^{-1}$.

As a T–module (and hence as a λ_ξ–module),

$$\mathfrak{g}/\mathfrak{p}_\xi \;\cong\; \bigoplus_{\langle\alpha,\xi\rangle<0} \mathfrak{g}_\alpha,$$

a direct sum of root spaces on which λ_ξ acts with weights $\langle\alpha,\xi\rangle < 0$. The corresponding trace form is

$$\mathrm{Tr}_{\mathfrak{g}/\mathfrak{p}_\xi}(\eta,\eta) \;=\; \sum_{\langle\alpha,\xi\rangle<0} \langle\alpha,\eta\rangle^2 \qquad (\eta\in\mathfrak{t}),$$

because η acts on $\mathfrak{g}_\alpha$ by the scalar $\langle \alpha, \eta \rangle$.

Now compare with the adjoint trace form:

$$\mathrm{Tr}_{\mathfrak{g}}(\eta, \eta) \; = \; \sum_{\alpha \in \Phi} \langle \alpha, \eta \rangle^2,$$

since η acts trivially on $\mathfrak{t}$ and by $\langle \alpha, \eta \rangle$ on $\mathfrak{g}_\alpha$. The root system is symmetric $\alpha \leftrightarrow -\alpha$, so the sum over $\{\alpha : \langle \alpha, \xi \rangle < 0\}$ is exactly half the sum over all roots:

$$\mathrm{Tr}_{\mathfrak{g}/\mathfrak{p}_\xi}(\eta, \eta) \; = \; \frac{1}{2} \, \mathrm{Tr}_{\mathfrak{g}}(\eta, \eta).$$

Therefore the level of $\mathcal{D}_\xi$ is

$$\mathrm{lev}(\mathcal{D}_\xi) \; = \; \mathrm{Tr}_{\mathfrak{g}/\mathfrak{p}_\xi} \; = \; \frac{1}{2} \, \mathrm{Tr}_{\mathfrak{g}} \, .$$

There is the classical indentification of the level of the Pfaffian square root $\mathcal{K}^{-1/2}$

$$c \; := \; -\frac{1}{2} \, \mathrm{Tr}_{\mathfrak{g}},$$

where $\mathcal{K} = \det(E_\Sigma^* \mathfrak{g})$ is the canonical bundle on $\mathfrak{M}$. Combining with the computation above gives

$$\mathrm{lev}(\mathcal{D}_\xi) = -c.$$

However the shift $[1]$ in the determinant line $\mathcal{D}_\xi$:

$$\det\Big(R\pi_* \mathcal{E}^* (\mathfrak{g}/\mathfrak{p}_\xi)[1] \Big) = \det\Big(R\pi_* \mathcal{E}^* (\mathfrak{g}/\mathfrak{p}_\xi) \Big)^{-1}$$

leaves us with c as desired. $\square$

(B) Tensoring by an admissible class $\mathcal{E}$. Lemma [1, §1.11] concerns the ξ–invariant part of

$$\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}, \qquad \big(\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1} \big)^{\xi\text{-inv}},$$

i.e. the weight–0 subobject for the $\mathfrak{G}_m$–action defined by ξ.

Write $\mathcal{E}$ as a product

$$\mathcal{E} \; = \; \mathcal{L} \; \otimes \; (\text{Atiyah–Bott generators}),$$

where $\mathcal{L}$ is a determinant line bundle and the remaining factors are built from the Atiyah–Bott generators $E_x^* V$, $E_C^* V$, $E_\Sigma^* V$.

(B1) Quadratic shift from $\mathcal{L}$. By Grothendieck–Riemann–Roch, the first Chern class of $\mathcal{L}$ determines an invariant quadratic form

$$h = h_{\mathcal{L}} \in \mathrm{Sym}^2(\mathfrak{g}^*)^G,$$

called the *level* of $\mathcal{L}$. Teleman–Woodward's GRR calculation shows that, on the ξ–stratum, the ξ–weight contributed by $\mathcal{L}$ has leading behaviour controlled by this level:

$$\mathrm{wt}_\xi(\mathcal{L}) \sim h(\xi, \xi), \qquad \text{quadratic in } \xi. \tag{5}$$

Combining (5) with the determinant contribution (4) coming from $\mathrm{Eul}(\nu_\xi)_+^{-1}$, the *net* quadratic behaviour is governed by

$$(h + c)(\xi, \xi). \tag{6}$$

Recall that $\mathcal{L}$ is *admissible* precisely when $h + c$ is positive definite on $\mathfrak{g}$, equivalently when

$$(h + c)(\xi, \xi) \to +\infty \qquad \text{as } \|\xi\| \to \infty.$$

(B2) Linear perturbation from Atiyah–Bott factors. The remaining factors in $\mathcal{E}$ are Atiyah–Bott generators attached to representations V of G. Their ξ–weights are governed by ordinary representation theory: if λ is a weight of V, then ξ acts with weight $\langle \lambda, \xi \rangle$. In particular, these contributions are at most *linear* in ξ:

$$\mathrm{wt}_\xi(\text{Atiyah–Bott factors}) = O(\|\xi\|). \tag{7}$$

(C) Finite-dimensionality of the ξ–invariant part. The polarized inverse Euler factor $\mathrm{Eul}(\nu_\xi)_+^{-1}$ has a ξ–weight decomposition with two crucial properties:

(i) only weights ≤ 0 occur; and

(ii) each weight space has finite multiplicity.

These follow from the fact that in $(*)$ the symmetric algebra is generated by strictly negative ξ–weight summands.

Fix ξ. The weight–0 piece of $\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}$ is obtained by summing those weight spaces of $\mathrm{Eul}(\nu_\xi)_+^{-1}$ whose weights cancel the (finite) set of weights appearing in $\mathcal{E}$. Since each weight space of $\mathrm{Eul}(\nu_\xi)_+^{-1}$ has finite multiplicity, it follows that

$$\left(\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}\right)^{\xi\text{-inv}} \text{ is finite-dimensional.} \tag{8}$$

This is the first conclusion of [1, §1.11].

(D) Eventual vanishing for $\|\xi\| \gg 0$. Now let ξ vary in the dominant cone. The symmetric algebra part of $\mathrm{Eul}(\nu_\xi)_+^{-1}$ produces weights by taking symmetric powers of negative–weight generators. The possible weights contributed in this way move away from 0 in steps controlled by the individual ξ–weights of the generators; these steps scale *linearly* in ξ (because root weights $\langle \alpha, \xi \rangle$ are linear in ξ).

On the other hand, twisting by an admissible line bundle $\mathcal{L}$ produces the quadratic shift (6). Combining with the linear perturbation (7) from Atiyah–Bott generators, the net effect is that the set of ξ–weights appearing in $\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}$ is translated by a term which grows like $(h+c)(\xi, \xi)$, up to linear error. Since $(h+c)(\xi, \xi)$ grows quadratically while all available "corrections" coming from symmetric powers grow at most linearly, it follows that for $\|\xi\|$ sufficiently large the total ξ–weight 0 cannot occur. Equivalently, there exists $B > 0$ such that

$$\|\xi\| > B \quad \Longrightarrow \quad \left(\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}\right)^{\xi\text{-inv}} = 0. \tag{9}$$

This is the second conclusion of [1, §1.11].

(E) Consequence for finiteness of the index. The local cohomology filtration of $R\Gamma(\mathfrak{M}, \mathcal{E})$ by Shatz supports has graded pieces controlled by the strata $\mathfrak{M}_\xi$. Identifying the residue contribution along $\mathfrak{M}_\xi$ with the ξ–invariant part of $\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}$, the finiteness statement (8) gives finite-dimensionality of each stratum contribution, while the vanishing (9) shows that only finitely many ξ contribute. Therefore the local-to-global spectral sequence has only finitely many nonzero columns, and the index $\mathrm{Ind}(\mathfrak{M}, \mathcal{E})$ is finite.

3 General idea

Let $S = \mathbb{C}[[s]], S^* = \mathbb{C}((s))$ and B be an S-scheme. Let $C_S \to S$ be a projective flat family of curves with generic fiber $\mathbb{C}_{S^*}$ smooth and special fiber C_0 nodal with unique node p. Let $C_B = C_S \times_S B$.

Solis [**?**] defines the S-stack $\mathcal{X}_G(C_S)$ whose points evaluated at a test scheme B/S are given by elements (C'_B, P_B) where C'_B is a twisted modification of C_B and P_B is an admissible G-bundle on C'_B. This stack is over a fixed curve C_S and Solis shows that it is algebraic, locally of finite type, and complete over S. It contains $M_G(C_S)$ and $M_G(C_{S^*})$ as dense open substacks, and the complement of $M_G(C_{S^*})$ is a divisor with normal crossings.

In this section, we discuss how to generalize Solis' construction to families of curves by working over the universal curve over the moduli stack of stable curves $\overline{\mathfrak{M}}_{g,I}$. Let $\pi : \overline{\mathcal{C}_{g,I}} \to \overline{\mathfrak{M}}_{g,I}$ be the universal curve over the moduli stack of stable curves of genus g with I marked points.

Let $\pi : C \to B$ be a prestable family of nodal curves. Let

$$\Sigma := \mathrm{Sing}(C/B) \subset C$$

be the relative singular locus. It is finite étale over B after restricting to the locus where the number of nodes is constant; globally it is at least finite unramified in good situations.

Definition 3.1. A **modification of** C/B is a proper morphism $m : C' \to C$ over B such that:

1. $C' \to B$ is flat prestable curve, and m is finitely presented and projective.

2. m is an isomorphism away from the nodes:

$$m: \ C' \setminus m^{-1}(\Sigma) \ \xrightarrow{\sim} \ C \setminus \Sigma.$$

3. For every geometric point $b \to B$ and every node $p \in \Sigma_b \subset C_b$, the fiber of $m_b : C'_b \to C_b$ over p is either a point (no modification at that node) or a chain of $\mathbb{P}^1$'s meeting the two branches in the standard way, and m_b contracts that chain to p and is an isomorphism elsewhere.

A **length** $\leq n$ **condition** can be stated as:

- for every b and every node $p \in \Sigma_b$, the chain over p has at most n components.

Definition 3.2 (Twisted nodal curves over a base). Let B be a scheme over $\mathbb{C}$. A **twisted nodal curve over** B is a proper Deligne–Mumford stack

$$\pi : \mathcal{C} \longrightarrow B$$

such that:

1. The geometric fibers of π are connected, one–dimensional, and the coarse moduli space $\overline{\mathcal{C}}$ is a nodal curve.

2. Let $\mathcal{U} \subset \mathcal{C}$ be the complement of the relative singular locus $\mathrm{Sing}(\mathcal{C}/B)$. Then the restriction

$$\mathcal{U} \hookrightarrow \mathcal{C}$$

is an open immersion.

3. For any geometric point $p : \operatorname{Spec} k \to \mathcal{C}$ mapping to a node of the fiber over $b \in B$, there exists an integer $k \geq 1$ and an element $t \in \mathfrak{m}_{B,b}$ such that, étale-locally on B at b and strictly henselian locally on $\mathcal{C}$ at p, there is an isomorphism

$$\operatorname{Spec} \mathcal{O}^{sh}_{\mathcal{C},p} \ \cong \ \Big[\operatorname{Spec}\big(\mathcal{O}^{sh}_{B,b}[u,v]/(uv - t)\big) \ / \ \mu_k \Big],$$

where $\zeta \in \mu_k$ acts by

$$(u, v) \longmapsto (\zeta u, \zeta^{-1} v).$$

Definition 3.3. A **twisted modification of** C/B is a twisted nodal curve $\mathcal{C} \to B$ whose coarse moduli space $\overline{\mathcal{C}}$ is a modification of C/B.

Let $r = \mathrm{rk}(G)$. The ordered simple roots $\{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ determine ordered vertices $\{\eta_0, \ldots, \eta_r\}$ determined by the conditions

$$\langle \eta_i, \alpha_j \rangle = 0 \text{ for } i \neq j \quad \text{and} \quad \langle \eta_0, \alpha_0 \rangle = 1.$$

If we write $\theta = \sum_{i=1}^{r} n_i \alpha_i$ and set $n_0 = 1$ then one can check these conditions can be expressed as

$$\langle \alpha_i, \eta_j \rangle = \frac{1}{n_i} \delta_{i,j}. \tag{10}$$

Following [?], if C'_B is a twisted modification of length $\leq r$, then a G–bundle on C'_B is called **admissible** if the co–characters determining the equivariant structure at all nodes are linearly independent over $\mathbb{Q}$ and are given by a subset of $\{\eta_0, \ldots, \eta_r\}$.

Definition 3.4. We define a stack $\mathcal{X}_{G,g,I}$ over $\overline{\mathfrak{M}}_{g,I}$ whose points over a test scheme $B \to \overline{\mathfrak{M}}_{g,I}$ are given by pairs (C'_B, P_B) where C'_B is a twisted modification of the pullback C_B of the universal curve $\overline{\mathfrak{C}_{g,I}}$ to B, and P_B is an admissible G-bundle on C'_B.

Proposition 3.5. The projection

$$F : \mathcal{X}_{G,g,I} \to \overline{\mathcal{M}}_{g,I}$$

is algebraic and locally of finite type.

3.1 The PGL_2 toy model: gluing at a node and the wonderful compactification

Let $\widetilde{C}$ be a smooth connected curve and fix two distinct points $p, q \in \widetilde{C}$. Let C be the nodal curve obtained by identifying $p \sim q$, and write $\nu : \widetilde{C} \to C$ for the normalization; the node $x \in C$ satisfies $\nu^{-1}(x) = \{p, q\}$.

1. A point of G produces a G–bundle on the nodal curve

Let $G = \mathrm{PGL}_2(\mathbb{C})$ (or any algebraic group). Fix a principal G–bundle E on $\widetilde{C}$ together with framings (trivializations of the fibres as G–torsors)

$$f_p : E|_p \xrightarrow{\sim} G, \qquad f_q : E|_q \xrightarrow{\sim} G.$$

Given $g \in G$, define a G–equivariant isomorphism of G–torsors

$$\phi_g : E|_p \longrightarrow E|_q$$

by the composite

$$E|_p \xrightarrow{f_p} G \xrightarrow{\cdot g} G \xrightarrow{f_q^{-1}} E|_q,$$

where $x \mapsto xg$ denotes right multiplication (any consistent left/right convention works).

Using ϕ_g, one descends E from $\widetilde{C}$ to a principal G–bundle on the nodal curve C: informally, one glues the two fibres $E|_p$ and $E|_q$ over the branches of the node using the identification ϕ_g. Denote the resulting descended bundle by $E(\phi_g)$.

Equivalently, a principal G–bundle on C is the same as a principal G–bundle on $\widetilde{C}$ together with an identification of the fibres over p and q; the isomorphism ϕ_g is exactly such an identification. Thus, once the data (E, f_p, f_q) is fixed, the element $g \in G$ determines a principal G–bundle on C.

2. Rephrasing the gluing as a $\Delta(G)$–reduction

Consider the $G \times G$–torsor $E|_p \times E|_q$ over $\operatorname{Spec} \mathbb{C}$. An isomorphism $\phi : E|_p \to E|_q$ is equivalent to the choice of a point in the contracted product

$$(E|_p \times E|_q)/\Delta(G),$$

since $\Delta(G)$ acts by simultaneous change of trivializations, and the graph of ϕ is a $\Delta(G)$–orbit.

Equivalently, giving ϕ is the same as giving a reduction of structure group

$$E|_p \times E|_q \quad \text{from } G \times G \text{ to } \Delta(G) \subset G \times G.$$

This is the sense in which one identifies $G \simeq (G \times G)/\Delta(G)$ and interprets the gluing map ϕ_g as a $\Delta(G)$–reduction at the pair (p, q).

3. One–parameter families and the need for compactification

A morphism

$$\gamma : \operatorname{Spec} \mathbb{C}[t^{\pm 1}] \longrightarrow G$$

gives a family of gluing maps $\phi_{\gamma(t)}$, hence a family of principal G–bundles on the fixed nodal curve C parametrized by $t \in \mathbb{C}^\times$. However, γ may fail to extend to $t = 0$ as a morphism $\operatorname{Spec} \mathbb{C}[t] \to G$. A basic example is

$$t \longmapsto \operatorname{diag}(t, t^{-1}) \in \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{C}),$$

which “goes to infinity” in G. In that case the family of glued bundles on C has no *a priori* limit inside the original moduli problem. One remedy is to enlarge the parameter space G to a compactification $\overline{G}$, so that γ extends and the boundary value can be given a modular interpretation.

4. For $G = \mathrm{PGL}_2$, the wonderful compactification is $\mathbb{P}^3$

An element of $G = \mathrm{PGL}_2(\mathbb{C})$ may be represented by a 2×2 matrix up to overall scaling,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad (a, b, c, d) \neq 0,$$

giving an open embedding

$$\mathrm{PGL}_2 \hookrightarrow \mathbb{P}^3, \qquad [a:b:c:d],$$

whose image is the open subset $\{ad - bc \neq 0\}$. The boundary is the quadric

$$\{ad - bc = 0\} \subset \mathbb{P}^3,$$

which is the locus of rank ≤ 1 matrices. Concretely,

$$\{ad - bc = 0\} = \{\text{rank} \leq 1\} = \big\{[uv^T] : u \in \mathbb{C}^2 \setminus \{0\},\, v \in (\mathbb{C}^2)^\vee \setminus \{0\}\big\}/\mathbb{C}^\times \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

sending a rank–one matrix uv^T to the pair $([u],[v])$.

5. Modular meaning of boundary points: Borel reductions at p and q

Let $B \subset \mathrm{PGL}_2$ be a Borel subgroup (e.g. the image of upper triangular matrices). Then

$$G/B \cong \mathbb{P}^1$$

is the flag variety. In this case, the boundary of the wonderful compactification admits an identification

$$\partial\overline{G} := \overline{G} \setminus G \;\cong\; (G/B) \times (G/B) \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Given a point $(s_p, s_q) \in (G/B) \times (G/B)$, the framings f_p, f_q identify the fibres $E|_p \simeq G$ and $E|_q \simeq G$, and hence:

- the point $s_p \in G/B$ is the same as a B–reduction of the framed fibre $E|_p$;
- the point $s_q \in G/B$ is the same as a B–reduction of the framed fibre $E|_q$.

Thus allowing the gluing parameter to land in the boundary replaces an honest identification $E|_p \simeq E|_q$ (equivalently a $\Delta(G)$–reduction of $E|_p \times E|_q$) by weaker boundary data: a $B \times B$–reduction of the $G \times G$–torsor $E|_p \times E|_q$.

In other words, the "completed" moduli problem includes:

- a principal G–bundle E on $\widetilde{C}$, and
- either a $\Delta(G)$–reduction at (p,q) (giving a genuine G–bundle on C),
- or a boundary datum consisting of B–reductions at p and q (i.e. a $B \times B$–reduction).

This is the simplest instance of completing a moduli problem by allowing controlled degenerations at the node.

6. Relation to the affine/loop–group story

In the affine story one replaces the finite–dimensional parameter space G by loop–group data (often together with loop rotation $\mathbb{C}^\times$). The boundary of an affine compactification encodes parahoric reductions (equivalently, coweight data) at the node; this is where affine Weyl combinatorics enters. For $G = \mathrm{PGL}_2$ in the finite–dimensional toy model, the boundary is merely $(G/B)^2$, whereas in the affine case the boundary stratifies by affine Weyl data and leads to admissibility constraints.

7. A concrete limit computation

Let $\gamma(t) = [\mathrm{diag}(t, t^{-1})] \in \mathrm{PGL}_2$. In homogeneous coordinates $[a : b : c : d]$ on $\mathbb{P}^3$, this is

$$[t : 0 : 0 : t^{-1}] = [t^2 : 0 : 0 : 1].$$

As $t \to 0$ this tends to $[0 : 0 : 0 : 1]$, a rank–one matrix, hence a boundary point. Under $\partial\overline{G} \cong \mathbb{P}^1 \times \mathbb{P}^1$, this point corresponds to a pair of flags, i.e. to a choice of Borels at p and q, which is exactly the $B \times B$–reduction boundary datum described above.

Let $S = \mathbb{C}[[t]]$ and take a family $C \to S$ smoothing a node, locally $xy = t$. Near the node you have two formal branches and the "middle" is a punctured disc in the generic fiber. To give a G-bundle on a curve, it is enough to give it on the complement of a point and on the formal disc, plus an identification on the punctured disc.

That identification is a transition function in $G(\!(z)\!)$, i.e. an element of LG. In particular, if we allow limits of such transition functions, we are forced into allowing other "integral models" on the disc, i.e. parahorics.

The classification of parahorics is affine-Weyl/alcove combinatorics. In partiuclar, parahorics are classified by facets of the fundamental alcove, and maximal parahorics correspond to vertices of the alcove, which are labeled by the η_i.

Remark 3.6 (Why the η_i should be viewed as *parahoric types*)**.** The slogan is that in Solis' compactification one is not parametrizing ordinary G–bundles alone, but rather torsors under a *sheaf of groups* $\mathcal{G}$ which agrees with G away from the nodes and is replaced by a *parahoric* subgroup of the loop group near each node. The labels η_i encode precisely which parahoric is allowed.

(1) G–bundles as torsors for a sheaf of groups. Let C be a smooth curve. Write

$$\mathcal{G}^{\mathrm{std}}(U) := \mathrm{Hom}_{\mathrm{Sch}}(U, G)$$

for the usual sheaf of groups on C. A principal G–bundle E determines a $\mathcal{G}^{\mathrm{std}}$–torsor by

$$\mathcal{F}_E(U) := \Gamma(U, E|_U),$$

and conversely $\mathcal{G}^{\mathrm{std}}$–torsors are the same thing as principal G–bundles (this is the standard equivalence between G–bundles and G–torsors).

(2) Modifying the local structure group: parabolics and parahorics. Fix a point $p \in C$ with a formal parameter z. Let $D = \operatorname{Spec}\mathbb{C}[[z]]$ and $D^\times = \operatorname{Spec}\mathbb{C}((z))$. A principal G–bundle may be described by giving it on $C \setminus \{p\}$ and on D, together with a gluing isomorphism on the overlap $D^\times$; after choosing trivialisations this gluing is an element of the loop group

$$G((z)) = LG.$$

Now let $P \subset G$ be a parabolic subgroup. Set

$$L_P^+ G := \{\gamma \in G[[z]] \mid \gamma(0) \in P\} \subset G[[z]].$$

One may package the condition "a G–bundle with reduction to P at p" by replacing the local gauge group $G[[z]]$ on the disc by L_P^+G. Equivalently, define a new sheaf of groups $\mathcal{G}^P$ on a neighborhood of p by

$$\mathcal{G}^P(D) = L_P^+G, \qquad \mathcal{G}^P(D^\times) = G((z)),$$

and gluing this with $\mathcal{G}^{\mathrm{std}}|_{C\setminus\{p\}}$ along the overlap $D^\times$. Then $\mathcal{G}^P$–torsors are exactly quasi–parabolic G–bundles (i.e. G–bundles on C with a P–reduction at p).

More generally one may replace L_P^+G by any *parahoric* subgroup $K \subset G((z))$ (in the sense of Bruhat–Tits), obtaining a sheaf of groups $\mathcal{G}^K$ whose torsors are "G–bundles with parahoric structure at p". For *exotic* parahorics these torsors need not be identifiable with ordinary G–bundles plus extra structure; they are genuinely new objects.

(3) Where the η_i enter. Parahoric subgroups of $G((z))$ are classified by facets of the affine alcove, and the *vertices* of the fundamental alcove correspond to *maximal* parahorics. The affine simple roots $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$ have dual "vertex" coweights $\{\eta_0, \dots, \eta_r\}$, and one may view η_i as labels for these maximal parahorics:

$$\eta_i \quad \longleftrightarrow \quad \text{a maximal parahoric } \mathcal{P}_{\eta_i} \subset G((z)).$$

Thus, specifying "η_i at a node" should be interpreted as specifying that the local structure group near that node is the parahoric $\mathcal{P}_{\eta_i}$, i.e. that we are working with torsors for the corresponding modified sheaf of groups.

(4) Conceptual rephrasing of admissibility. In this language, Solis' admissibility condition is a finiteness restriction on the allowed local group schemes at the nodes: on a twisted modification one only allows those parahoric types indexed by the finite set $\{\eta_0, \dots, \eta_r\}$ (and imposes a further independence condition when several nodes occur). This is why it is natural to think of the η_i as *parahoric types*.

Definition 3.7 ($\mathcal{P}$–parahoric G–bundles at a point)**.** Fix a smooth curve C over $\mathbb{C}$ and a point $p \in C$ with a choice of formal parameter z at p, so that the completed local ring is $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[z]]$ and the punctured disc is $D^\times = \operatorname{Spec}\mathbb{C}((z))$. Let

$$\mathcal{P} \subset G((z))$$

be a *parahoric* subgroup (for instance a *maximal* parahoric, i.e. one corresponding to a vertex of the fundamental alcove).

Define a sheaf of groups $\mathcal{G}^{\mathcal{P}}$ on C by gluing the standard sheaf $\mathcal{G}^{\mathrm{std}}$ away from p with the local sheaf determined by $\mathcal{P}$ at p as follows:

- if $U \subset C$ is an open subset with $p \notin U$, set $\mathcal{G}^{\mathcal{P}}(U) := \mathcal{G}^{\mathrm{std}}(U) = \mathrm{Hom}_{\mathrm{Sch}}(U, G)$;
- for the formal disc $D = \operatorname{Spec} \mathbb{C}[[z]]$ and punctured disc $D^{\times} = \operatorname{Spec} \mathbb{C}((z))$, set

  $$\mathcal{G}^{\mathcal{P}}(D) := \mathcal{P}, \qquad \mathcal{G}^{\mathcal{P}}(D^{\times}) := G((z)),$$

  with restriction map given by the inclusion $\mathcal{P} \hookrightarrow G((z))$;
- on the overlap $(C \setminus \{p\}) \cap D^{\times} \simeq D^{\times}$, we identify both restrictions with $G((z))$ and glue.

A *$\mathcal{P}$–parahoric G–bundle on C (with parahoric structure of type $\mathcal{P}$ at p)* is a $\mathcal{G}^{\mathcal{P}}$–torsor on C. Equivalently, it is the data of:

(i) a principal G–bundle E on $C \setminus \{p\}$;

(ii) a $\mathcal{P}$–torsor E_D on the formal disc D (i.e. a principal homogeneous space under the group $\mathcal{P} = \mathcal{G}^{\mathcal{P}}(D)$);

(iii) an identification of the induced $G((z))$–torsors over $D^{\times}$.

When $\mathcal{P} = L_P^+G = \{\gamma \in G[[z]] \mid \gamma(0) \in P\}$ for a parabolic $P \subset G$, this recovers the usual notion of a quasi–parabolic G–bundle with a P–reduction at p. When $\mathcal{P}$ is a maximal parahoric corresponding to a vertex η_i of the fundamental alcove, we say the parahoric structure at p is *of type* η_i.

Proposition 3.8 (Normalization description of parahoric bundles)**.** Let C be a nodal curve with normalization $\nu : \widetilde{C} \to C$ and nodes x_i with preimages $\nu^{-1}(x_i) = \{p_i, q_i\}$. Fix parahoric subgroups $\mathcal{P}_i \subset G((z_i))$ at each node.

A $\mathcal{G}^{\mathcal{P}}$–torsor on C is equivalent to:

1. a principal G–bundle E on $\widetilde{C}$;
2. for each i, $\mathcal{P}_i$–torsors E_{p_i} and E_{q_i} on the formal discs at p_i, q_i whose restrictions to the punctured discs agree with $E|_{\widetilde{C}\setminus\{p_i,q_i\}}$;
3. for each i, a gluing class

   $$[\phi_i] \in \mathcal{P}_i \backslash G((z_i)) / \mathcal{P}_i,$$

   giving an identification of the punctured restrictions $E_{p_i}|_{D^{\times}} \cong E_{q_i}|_{D^{\times}}$.

If the node is stacky with stabilizer μ_k, one must additionally specify homomorphisms $\rho_i : \mu_k \to \mathcal{P}_i$ and require the gluing to be μ_k–equivariant.

How the compactification includes ordinary bundles

How does this compactify G-bundles on a nodal curve? Ordinary bundles sit inside as the trivial parahoric structure.

(1) Ordinary bundles on nodal curves. Choose trivializations of E on formal discs at p, q. Then a G-bundle on C is the same as

- a principal G-bundle E on $\widetilde{C}$, and
- an identification of fibers $E_p \simeq E_q$.

After choosing framings, that identification is an element $g \in G$. So the gluing parameter space for bundles on the fixed nodal curve is G.

Equivalently (using discs), the gluing is a loop $\gamma(z) \in G((z))$ which actually lies in $G[[z]]$ and has the same value at $z = 0$ on both branches, so it descends across the node. Concretely, the descent condition forces you into the "diagonal" subgroup

$$\Delta(G[[z]]) \subset G[[z]] \times G[[z]].$$

(2) Parahoric bundles enlarge the allowed local data. Now pick a parahoric $\mathcal{P} \subset G((z))$. A $\mathcal{P}$-parahoric torsor on C is (after choosing local trivializations) given by a double coset

$$[\gamma] \in \mathcal{P}\backslash G((z))/\mathcal{P},$$

i.e. you allow gluing by any loop $\gamma(z)$, but you declare two loops equivalent if they differ by "integral" gauge transformations on each side lying in $\mathcal{P}$.

This enlarges the moduli because $\mathcal{P}\backslash G((z))/\mathcal{P}$ contains more than the constant loops G.

(3) Where ordinary bundles sit inside: the hyperspecial case. There is a distinguished maximal parahoric, the *hyperspecial*

$$\mathcal{P}_0 := G[[z]] \subset G((z)).$$

Then

$$\mathcal{P}_0\backslash G((z))/\mathcal{P}_0$$

contains the constant loops G as an open subset (the big cell) in the Bruhat decomposition

$$G((z)) = \bigsqcup_{\lambda \in X_*(T)_+} \mathcal{P}_0 z^\lambda \mathcal{P}_0$$

where $X_*(T)_+$ are the dominant coweights. The constant loops correspond to $\lambda = 0$.

In words:

- trivial parahoric structure means: at the node you are not allowing any degeneration of the local group scheme; you keep the standard integral model $G[[z]]$;
- ordinary G-bundles on C correspond to those parahoric torsors whose gluing class can be represented by a constant loop (equivalently, whose local modification at the node is trivial).

G–bundles on twisted chains

In the previous section we saw that associated to the singleton sets $\{i\} \subset \{0, \ldots, r+1\}$ there is a moduli space parametrizing G–bundles on a twisted nodal curve, and further the moduli space can be identified with an orbit of the wonderful embedding of the loop group. In this section we introduce a more general moduli problem which we show is isomorphic to the orbit O_I in the wonderful embedding for any $I \subset \{0, \ldots, r+1\}$.

Let R_n denote the rational chain of projective lines with n components; Figure 3.1 in the introduction depicts a chain of length 3. There is an action of $\mathbb{C}^\times$ on R_n which scales each component. Let $p_0, \ldots, p_n$ denote the fixed points of this action.

Recall that u, v are kth roots of x, y which are our coordinates near a node. Let p', p'' denote the closed points of $\operatorname{Spec} \mathbb{C}[[u]]$ and $\operatorname{Spec} \mathbb{C}[[v]]$. Finally, let $D_n^{1/k}$ be the curve obtained from

$$\operatorname{Spec} \mathbb{C}[[u]] \,\sqcup\, R_n \,\sqcup\, \operatorname{Spec} \mathbb{C}[[v]]$$

by identifying p' with p_0 and p'' with p_n.

The group μ_k acts on $D_n^{1/k}$ through its usual action on u, v and through the inclusion $\mu_k \subset \mathbb{C}^\times$ on the chain R_n. For an n–tuple $(\beta_0, \ldots, \beta_n) \in \operatorname{Hom}(\mathbb{C}^\times, T)^n$, we can speak about the equivariant G–bundles on $D_n^{1/k}$ with equivariant structure at p_i determined by β_i. We refer to this equivalently as G–bundles on

$$\big[D_n^{1/k}/\mu_k\big]$$

of type $(\beta_0, \ldots, \beta_n)$.

Further, we can also glue $\big[D_n^{1/k}/\mu_k\big]$ to $C_0 - p_0$ to obtain a curve $C_{n,[k]}$. Let C_n denote the coarse moduli space of $C_{n,[k]}$. We call C_n a *modification* of C_0 and $C_{n,[k]}$ a *twisted modification* of C_0.

Recall the specific cocharacters $\eta_0, \ldots, \eta_r$ defined in *(3.1)* in §3.2. For $I = \{i_1, \ldots, i_n\} \subset \{0, \ldots, r\}$, let $T_{G,I}\big([D_n^{1/k}/\mu_k]\big)$ denote the moduli space of pairs (P, τ) where P is a G–bundle on $\big[D_n^{1/k}/\mu_k\big]$ of type $(\eta_{i_1}, \ldots, \eta_{i_n})$ and τ is a trivialization on

$$\big[\operatorname{Spec} \mathbb{C}((u)) \times \mathbb{C}((v))/\mu_k\big].$$

Let $H = \mathrm{Aut}(P)$; then restriction to $\mathrm{Spec}\, \mathbb{C}[[u]]$ and $\mathrm{Spec}\, \mathbb{C}[[v]]$ realizes

$$H \;\subset\; (L_u G)^{\mu_k} \times (L_u G)^{\mu_k}.$$

Theorem 3.9 ([**?**, Thm. 3.4.7])**.** Let $I \subset \{0, \ldots, r\}$ and $T_{G,I}\big([D_n^{1/k}/\mu_k]\big)$ be as above. Then there is an isomorphism

$$T_{G,I}(C_{0,[k]}) \xrightarrow{\Psi^{\eta_I}} (L_u G)^{\mu_k} \times (L_u G)^{\mu_k}/H \xrightarrow{\eta_I^{-1}(\ )\eta_I} \frac{L_{\mathrm{poly}}G \times L_{\mathrm{poly}}G}{Z(L_I) \times Z(L_I) \cdot P_I^{\Delta,\pm}}.$$

Here Ψ^{η_I} is as in *(3.10)* and $\eta_I^{-1}(\ )\eta_I$ is described in Proposition 3.4.5. Let

$$i :\ [D_n^{1/k}/\mu_k] \;\longrightarrow\; C_{0,[k]}$$

be the natural map. Then

$$i^* :\ T_{G,I}(C_{0,[k]}) \;\longrightarrow\; [D_n^{1/k}/\mu_k]$$

is an isomorphism. In particular, $T_{G,I}(C_{0,[k]})$ and $T_{G,I}\big([D_n^{1/k}/\mu_k]\big)$ are isomorphic to an orbit in the wonderful embedding of $L^{\times}_{\mathrm{poly}}G$.

4 References

1. Teleman, C., & Woodward, C. (2012). The index formula on the moduli of G-bundles. Annals of Mathematics, 176(2), 601-77.