Homework 5

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Problem 1 For a compact Riemann surface R, verify that the Serre duality pairing

$$H^1(R;\mathcal{O}) \otimes H^0(R;\Omega^1) \longrightarrow \mathbb{C}$$

defined by principal parts and residues agrees with the one given by integration of Dolbeault representatives.

Using the relation to harmonic forms, explain how this relates to Poincaré duality on R.

Problem 2 For a compact Riemann surface R, verify that the map

$$H^1(R;\mathbb{Z}) \longrightarrow H^1(R;\mathcal{O})$$

corresponds to the period map

$$H_1(R;\mathbb{Z}) \otimes H^0(R;\Omega^1) \longrightarrow \mathbb{C}$$

under integral Poincaré duality and Serre duality on R.

Problem 3 Show that the period mapping gives an isomorphism

$$H_1(R;\mathbb{Z}) \xrightarrow{\sim} H_1(J;\mathbb{Z}),$$

which can be realized geometrically by the Abel–Jacobi map

$$R \longrightarrow J_1.$$

Show that under this correspondence, $c_1(\Theta) \in \Lambda^2 H_1(R)$ is the intersection pairing on R.

Hints for the second part: You can deduce it from the periodicity formulas of the Riemann Θ-function. Alternatively, you can find this by exploiting the facts that the Poincaré dual of $c_1(\Theta)$ in J_{g-1} is the Theta divisor, the image of $\mathrm{Sym}^{g-1}(R)$. The maps

$$\mathrm{Sym}^g(R) \longrightarrow J_g \qquad \text{and} \qquad \mathrm{Sym}^{g-1}(R) \longrightarrow \mathrm{div}(\Theta)$$

have degree 1.

Problem 4 Prove the following generalized Cauchy formula for a smooth function f defined in the unit disk Δ:

$$f(z,\bar{z}) = \frac{1}{2\pi i}\oint_{|\zeta - z| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\iint_{\Delta'} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z},$$

where $\Delta' \subset \Delta$ is the subdisk of radius $r < 1$.

Remark: When f is holomorphic, you recover Cauchy's formula.

Problem 5 Let $L \to X$ be a holomorphic line bundle on a complex manifold, and let $\alpha \in \mathcal{E}^{0,1}$ be a $\bar{\partial}$-closed form. Show that the re-defined operator

$$\tilde{\bar{\partial}} = \bar{\partial} + \alpha$$

on sections of L defines a new holomorphic structure L' on the same underlying bundle, where local holomorphic sections are defined as those killed by $\tilde{\bar{\partial}}$. Show that $L \simeq L'$ if α is $\bar{\partial}$-exact. Relate this to the exponential sequence.

Remark: For vector bundles, the same applies with an $\alpha \in \mathcal{E}^{0,1}(\mathrm{End}(V))$ satisfying the non-linear equation

$$\bar{\partial}\alpha + \alpha \wedge \alpha = 0.$$

The new bundle is isomorphic to the old one if $\alpha = a^{-1}\bar{\partial}a$, for some smooth section a of $\mathrm{Aut}(V)$.

Problem 6 Let V be a complex g-dimensional vector space and $L \simeq \mathbb{Z}^{2g} \subset V$ a lattice. Let $A = V/L$.

1. Using harmonic theory, compute the Dolbeault cohomology $H^*(A;\mathcal{O})$.
2. Show that the moduli space of holomorphic line bundles on A with zero Chern class is naturally identified with

   $$A^\vee := V^\vee / L^\vee.$$

3. Show that the moduli space of holomorphic line bundles on $A^\vee$ is naturally identified with A.
4. Define a line bundle

   $$\mathcal{P} \longrightarrow A \times A^\vee$$

   from the trivial line bundle over $V \times V^\vee$ with connection

   $$\nabla = d + i(x\, d\xi + \xi\, dx),$$

by quotienting out the $L \times L^\vee$-action as follows: identify the fiber $\mathbb{C}$ over $(x, \xi) \in V \times V^\vee$ with that over $(x + \ell, \xi + \lambda)$ by multiplication by

$$\exp\bigl(2\pi i(\lambda(x) + \xi(\ell))\bigr).$$

Show that $\mathcal{P}$ is holomorphic, that $\mathcal{P}|_{A\times\{a^\vee\}}$ is the line bundle over A classified by $a^\vee \in A^\vee$, and prove the corresponding statement for $\{a\} \times A^\vee$.

Problem 7 Show that, in the case of the Jacobian J of a Riemann surface R, one has a natural isomorphism $J \simeq J^\vee$.

Hint: Remember the natural Hilbert space structure on holomorphic differentials.

Remark: This self-duality is a property of principally polarized Abelian varieties, those A equipped with a positive line bundle having a single holomorphic section (the Θ-function).

Problem 8 Given a holomorphic line bundle $\mathcal{L}$ on a complex manifold and a smooth real closed 2-form ω in the cohomology class of $c_1(\mathcal{L})$, prove that there exists a Hermitian metric on $\mathcal{L}$ whose holomorphic connection has curvature $-2\pi i\,\omega$.

Conclude (from Kodaira vanishing) that the holomorphic line bundles on a compact Riemann surface R which carry metrics of positive curvature are precisely those of positive degree.

Show also that for every holomorphic vector bundle V on R, there exists a d so that the twisted bundle $V(D)$ has no H^1 for any $D > d$.

Problem 9 Show that isomorphism classes of *flat unitary* line bundles on a manifold X are classified by $H^1(X; U(1))$, with the constant sheaf $U(1)$ associated to the unit circle group in $\mathbb{C}^\times$.

When X is compact Kähler, compare the constant and holomorphic exponential sequences to conclude that the map

$$H^1(X; U(1)) \longrightarrow H^1(X; \mathcal{O}^\times)$$

induces a bijection from isomorphism classes of flat unitary line bundles to those of holomorphic line bundles with zero Chern class.

Remark: You probably need the Hodge decomposition theorem for the second part.

Problem 10 Prove the global $\partial\bar{\partial}$-Lemma on a compact Kähler manifold X: for any d-exact form $\varphi \in \mathcal{E}^{p,q}$, there exists $\psi \in \mathcal{E}^{p-1,q-1}$ with

$$\partial\bar{\partial}\psi = \varphi.$$

Hint: Show that

$$\varphi = \partial\bar{\partial}^{*}\Box\varphi$$

and use this and similar identities to find ψ.