# Homework 3

Songyu Ye

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**Problem 1** Which of the following is a Galois cover of the complex $z$-plane?

(a) $w^2 = 4z^3 - g_2 z - g_3$;

(b) $w^n - z^n = 1$;

(c) $w^3 + z + z^2 = w^2 + wz$; *Hint: look at the fiber over* $0$.

(d) $w^2 - 2zw + z^3 = 1$.

*Solution:*

**Problem 2** Let $V$ be a rank 2 (for simplicity) vector bundle over a Riemann surface $R$. Assume that $V$ has two meromorphic sections $s_1, s_2$ which, at some point, are holomorphic and span the fiber.

(a) Show that this will be the case everywhere except at a set of isolated points.

(b) At an exceptional point, show that we can modify $V$ by a finite sequence of elementary transformations so that $s_1$ and $s_2$ form a holomorphic frame of the new bundle.

*Suggestion:* First make the sections holomorphic, then find some numerical measure for their failure to give a basis. Then find a way to reduce that number.
*Remark:* The argument generalizes to any dimension. If $R$ is compact, it follows that we can trivialize $V$ by a finite number of elementary transformations. If $R$ is non-compact, one can show that every vector bundle is in fact trivial.

*Solution:* Let $s_1, s_2$ be two meromorphic sections of a rank 2 vector bundle $V$ over a Riemann surface $R$. Since $V$ is a holomorphic vector bundle, there exists a local trivialization of $V$ around $p$.

$$V|_U \cong \mathcal{O}_U e_1 \oplus \mathcal{O}_U e_2$$

and we can write

$$s_1 = f_1 e_1 + f_2 e_2, \quad s_2 = g_1 e_1 + g_2 e_2$$

where $f_i, g_i$ are meromorphic functions on $U$. The failure of $s_1, s_2$ to span the fiber at a point $q \in U$ is given by the vanishing of the determinant

$$D(q) = f_1(q) g_2(q) - f_2(q) g_1(q).$$

which is a meromorphic function on $U$. The zeroes of a meromorphic function are isolated unless the function is identically zero. Since $s_1, s_2$ span the fiber at $p$, $D$ is not identically zero. Therefore, the set of points where $s_1, s_2$ fail to be holomorphic or fail to span the fiber is a discrete set of isolated points in $R$, because meromorphic functions can only have isolated singularities and the determinant $D$ is meromorphic.

Let $D$ be the effective divisor of the poles of $s_1, s_2$. We can make $s_1, s_2$ holomorphic by twisting $V$ with the line bundle $\mathcal{O}(D)$, i.e. consider the new vector bundle

$$V(D) = V \otimes \mathcal{O}(D)$$

Then $s_1, s_2$ are holomorphic sections of $V(D)$. Now consider a point $p$ where $s_1, s_2$ fail to span the fiber of $V(D)$. If $s_1(p)$ and $s_2(p)$ both vanish, then twist by an appropriate power of $\mathcal{O}(-p)$ to make at least one of them non-vanishing at $p$, say $s_1(p) \neq 0$. There is a 1 dimensional subspace $L$ of $V(D)_p$ such that $s_1(p), s_2(p)$ span $L$. We can perform an elementary transformation of $V(D)$ at $p$ with respect to $L$ to obtain a new vector bundle $V'$ which fits into the short exact sequence of coherent sheaves

$$0 \to V' \to V(D) \to (V(D)_p/L) \otimes \mathcal{O}_p \to 0. \tag{1}$$

In a chart near $V(D)$ we have a local trivialization $V(D)|_U \cong \mathcal{O}_U e_1 \oplus \mathcal{O}_U e_2$ so that $s_1 = e_1$ and $s_2 = f(z)e_1 + g(z)e_2$ for some holomorphic functions $f(z), g(z)$. Their wedge product is given by

$$s_1 \wedge s_2 = g(z) e_1 \wedge e_2.$$

Since $s_1, s_2$ fail to span the fiber at $p$, we have $g(0) = 0$, so we can write $g(z) = z^n h(z)$ for some $n \geq 1$ and unit $h(0) \neq 0$. After absorbing the unit $h(z)$ into $e_2$, we can assume $g(z) = z^n$. Then we have in local coordinates sections $s_1 = e_1$ and $s_2 = f(z)e_1 + z^n e_2$. The

**Problem 3** (a) Consider the vector bundle $V$ with sheaf of sections $\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_k)$ over $\mathbb{P}^1$, with $n_1 \leq \cdots \leq n_k$. Show that the sequence of integers $n_i$ is uniquely determined by $V$.
(b) In contrast with (a), show that $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O}$ are isomorphic as topological vector bundles.
(c) Show that there is a holomorphic automorphism of $V$ which takes the vector $[1, 0, \dots, 0]$ in the fiber over 0 to $[1, 1, \dots, 1]$.
(d) Assuming the fact that every rank $k$ holomorphic vector bundle on $\mathbb{P}^1$ can be constructed from $\mathcal{O}^{\oplus k}$ by elementary transformations, show that it must be isomorphic to one of the form in (a).

*Solution:*

**Problem Problem**4 Show that on a compact Riemann surface $R$ of genus $g$ and a line bundle $L$ of degree $> 2g - 2$, we have $H^1(R; \mathcal{O}(L)) = 0$. Find a counterexample to this if $L$ is a vector bundle instead.

*Remark:* For noncompact Riemann surfaces, $H^1$ vanishes for any vector bundle.

*Solution:*

**Problem Problem**5 Prove that every compact Riemann surface of genus 2 is *hyperelliptic*, meaning that it can be realized as a double (branched) cover of $\mathbb{P}^1$.
*Hint:* Use differentials.

*Solution:*