Homework 1

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February 9, 2026

Problem 1 Find the fundamental group of the complement of two Hopf–linked circles in $\mathbb{R}^3$.

Solution: The knot complement will have the same fundamental group if we pass to the one point compactification $S^3 = \mathbb{R}^3 \cup \{\infty\}$, so we may work in S^3 instead of $\mathbb{R}^3$. Let $L = K_1 \sqcup K_2 \subset \mathbb{R}^3$ be the Hopf link. Choose disjoint closed tubular neighborhoods $N_i \cong S^1 \times D^2$ of each K_i. Then $S^3 \setminus L$ deformation retracts onto

$$M := S \setminus \operatorname{int}(N_1 \cup N_2),$$

because each $N_i \setminus K_i$ deformation retracts onto ∂N_i by pushing radially in the normal disk direction. In view of the standard decomposition of S^3 into two solid tori, we can identify $S^3 \setminus \operatorname{int}(N_1)$ with a solid torus under this identification K_2 becomes the core $S^1 \times \{0\}$.

Inside a solid torus $S^1 \times D^2$, remove an open tubular neighborhood of the core $S^1 \times \{0\}$. We obtain

$$S^1 \times \left(D^2 \setminus \operatorname{int}(D^2_\varepsilon)\right) \cong S^1 \times (S^1 \times I) \cong T^2 \times I.$$

That is exactly M. Therefore

$$\mathbb{R}^3 \setminus L \simeq M \cong T^2 \times I \simeq T^2,$$

so

$$\pi_1(\mathbb{R}^3 \setminus L) \cong \pi_1(T^2) \cong \mathbb{Z}^2.$$

Problem 2 A (small) category $\mathcal{C}$ may be localized by inverting a collection M of morphisms: for each arrow $x \xrightarrow{f} y$ in M, adjoin an arrow $x \xleftarrow{f^{-1}} y$ and impose the relations $f \circ f^{-1} = \mathrm{id}_y$, $f^{-1} \circ f = \mathrm{id}_x$, together with all formal compositions of arrows and all relations between arrows forced by associativity.

1. Show that if the morphisms in M were already invertible, the natural functor $\lambda : \mathcal{C} \to \mathcal{C}_M$ to the localization is an equivalence of categories.

2. Show that the localization functor $\lambda : \mathcal{C} \to \mathcal{C}_M$ is characterized as follows, among categories where $\lambda(M)$ lands in invertible morphisms: every functor $\varphi : \mathcal{C} \to \mathcal{D}$ which takes morphisms in M to invertible morphisms in $\mathcal{D}$ factors uniquely via λ.

3. Show that in the category Top of topological spaces and continuous maps, inverting homotopy equivalences has the same effect as modding out the Hom spaces (continuous maps) by the homotopy equivalence relation.

Solution:

1. Let $\mathcal{C}_M$ be the localization obtained by formally adjoining inverses to all $m \in M$, and let $\lambda : \mathcal{C} \to \mathcal{C}_M$ be the canonical functor.

   Assume every $m \in M$ is already an isomorphism in $\mathcal{C}$. Define a functor
   $$\rho : \mathcal{C}_M \longrightarrow \mathcal{C}$$
   as follows. On objects, ρ is the identity. On morphisms, any morphism in $\mathcal{C}_M$ is represented by a finite zigzag built from morphisms in $\mathcal{C}$ and the formal inverses m^{-1} for $m \in M$. Send each genuine arrow f to f, and each formal inverse m^{-1} to the actual inverse $m^{-1} \in \mathcal{C}$. Because the defining relations in $\mathcal{C}_M$ are exactly $m \circ m^{-1} = \mathrm{id}$ and $m^{-1} \circ m = \mathrm{id}$ plus associativity, this assignment respects relations and gives a well-defined functor.

   Then $\rho \circ \lambda = \mathrm{id}_{\mathcal{C}}$ strictly. Moreover, $\lambda \circ \rho \simeq \mathrm{id}_{\mathcal{C}_M}$ strictly as well, because on generators the composite $\lambda\rho$ acts as the identity. Hence λ is an equivalence of categories.

2. Let $\mathcal{D}$ be any category and $\varphi : \mathcal{C} \to \mathcal{D}$ a functor such that $\varphi(m)$ is invertible in $\mathcal{D}$ for all $m \in M$.

   Existence of the factorization. Define $\widetilde{\varphi} : \mathcal{C}_M \to \mathcal{D}$ by:

   - On objects: $\widetilde{\varphi}(x) = \varphi(x)$.
   - On arrows: on a generating arrow f from $\mathcal{C}$, set $\widetilde{\varphi}(\lambda(f)) = \varphi(f)$; on a formal inverse m^{-1} in $\mathcal{C}_M$, set
   $$\widetilde{\varphi}(m^{-1}) := \varphi(m)^{-1}.$$

   Extend multiplicatively to composites. The relations in $\mathcal{C}_M$ are satisfied because $\varphi(m)\varphi(m)^{-1} = \mathrm{id}$, etc., so this is well-defined and yields a functor $\widetilde{\varphi}$ with $\widetilde{\varphi} \circ \lambda = \varphi$.

   Uniqueness. Any functor $\psi : \mathcal{C}_M \to \mathcal{D}$ with $\psi \circ \lambda = \varphi$ must agree with φ on all arrows of $\mathcal{C}$, and must send the formal inverse m^{-1} to $\psi(m)^{-1} = \varphi(m)^{-1}$. Since $\mathcal{C}_M$ is generated by these under composition, $\psi = \widetilde{\varphi}$. Hence the factorization is unique.

3. Let W be the class of homotopy equivalences in Top. Consider the localization
   $$\lambda : \mathrm{Top} \longrightarrow \mathrm{Top}[W^{-1}].$$

   Let hTop denote the homotopy category. If $f, g : X \to Y$ are homotopic, choose a homotopy $H : X \times I \to Y$. Let $i_0, i_1 : X \to X \times I$ be the inclusions at $0, 1$. Then $H \circ i_0 = f$ and $H \circ i_1 = g$.

   But i_0 and i_1 are homotopy equivalences (with projection $p : X \times I \to X$ as a homotopy inverse). In the localized category, $\lambda(i_0)$ and $\lambda(i_1)$ become isomorphisms. Therefore
   $$\lambda(f) = \lambda(H) \circ \lambda(i_0) = \lambda(H) \circ \lambda(i_1) = \lambda(g).$$

where $\lambda(i_0) = \lambda(i_1)$ because i_0 and i_1 are themselves homotopic. Hence λ factors through the quotient by homotopy:

$$\mathrm{Top} \xrightarrow{q} \mathrm{hTop} \xrightarrow{\overline{\lambda}} \mathrm{Top}[W^{-1}].$$

Every morphism in the localization is represented by an honest map $X \to Y$. A morphism in $\mathrm{Top}[W^{-1}]$ is represented by a zigzag

$$X \xleftarrow{w_1} X_1 \xrightarrow{f_1} X_2 \xleftarrow{w_2} \cdots \xrightarrow{f_n} Y,$$

where $w_i \in W$. Because each w_i is a homotopy equivalence, choose a homotopy inverse w_i^{-1}. In the localization, $\lambda(w_i)^{-1} = \lambda(w_i^{-1})$. Thus the above zigzag is represented by a single continuous map $X \to Y$ well-defined up to homotopy.

So the induced map

$$[X, Y] \longrightarrow \mathrm{Hom}_{\mathrm{Top}[W^{-1}]}(X, Y)$$

is surjective.

It is also injective. If $\lambda(f) = \lambda(g)$ in $\mathrm{Top}[W^{-1}]$, then there is a zigzag of homotopy equivalences relating them. This shows f and g are equal precisely when they are homotopic. Hence $\overline{\lambda} : \mathrm{hTop} \to \mathrm{Top}[W^{-1}]$ is fully faithful.

Therefore $\overline{\lambda}$ is an isomorphism of categories:

$$\mathrm{Top}[W^{-1}] \;\cong\; \mathrm{hTop}.$$

Problem 3 Which of the following spaces are homotopy equivalent? Prove it, or explain why they are not.

1. The standard solid torus $\{(r-2)^2 + z^2 \leq 1\} \subset \mathbb{R}^3$ and its complement in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.
2. $\mathbb{CP}^2$ and the quotient $S^2 \times S^2/(\mathbb{Z}/2)$ by the swapping symmetry.
3. A torus (surface) and the sphere S^2 plus two arcs joining the North and South poles (disjoint except at the poles).

Solution:

1. **Yes.** The standard solid torus $V \subset \mathbb{R}^3 \subset S^3$ is homeomorphic to $S^1 \times D^2$, hence deformation retracts onto its core circle $S^1 \times \{0\}$, so $V \simeq S^1$.

   Its complement in S^3 is also a solid torus: S^3 admits a genus-1 Heegaard splitting

   $$S^3 = V \cup_{\partial V} V',$$

   where both V and V' are solid tori with common boundary a torus. Hence $S^3 \backslash \mathrm{int}(V) = V'$ is again homeomorphic to $S^1 \times D^2$, so it also retracts onto S^1. Therefore $V \simeq V'$.

2. **Yes.** Unordered pairs of points on $\mathbb{CP}^1$ correspond to effective degree-2 divisors, hence to lines in $H^0(\mathbb{CP}^1, \mathcal{O}(2)) \cong \mathbb{C}^3$. Thus the quotient is $\mathbb{CP}^2$, so it is certainly homotopy equivalent to $\mathbb{CP}^2$. The map sending an unordered pair $\{p, q\} \subset \mathbb{CP}^1$ to the quadratic form vanishing at p and q defines a continuous bijection

$$(S^2 \times S^2)/(\mathbb{Z}/2) = \mathrm{Sym}^2(\mathbb{CP}^1) \longrightarrow \mathbb{CP}^2.$$

   Since both spaces are compact Hausdorff, this is a homeomorphism.

3. **No.** Let Y be the space obtained from S^2 by attaching two arcs between the north and south poles, disjoint except at endpoints. Choose an embedded path γ in S^2 from north to south and collapse γ to a point. This is a deformation retraction of S^2 onto $S^2 \vee S^1$ relative to the poles, and after attaching the two arcs it shows

$$Y \simeq S^2 \vee S^1 \vee S^1.$$

   Hence

$$\pi_1(Y) \cong F_2$$

   (the free group on two generators). But the torus T^2 has

$$\pi_1(T^2) \cong \mathbb{Z}^2,$$

   which is abelian. Since fundamental groups are homotopy invariants and $F_2 \not\cong \mathbb{Z}^2$, the torus is not homotopy equivalent to Y.

Definition (CW–approximation) Let Y be a topological space. A *CW–approximation* of Y is a CW complex X together with a map $f : X \to Y$ such that for every CW complex Z, the induced map $f_* : [Z, X] \to [Z, Y]$ is a bijection, where $[-, -]$ denotes the set of homotopy classes of maps.

Definition (Homotopy equivalence) A map $f : X \to Y$ is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Problem 4 Consider the raviolo X obtained gluing two copies of the closed unit disk D together everywhere except at the origin. (So there are two points where 0 used to be.)

Show that the natural projection $S^2 \to X$ (collapsing along the z–axis) is a CW–approximation but not a homotopy equivalence.

Suggestion: Replace S^2 with the homotopy equivalent "genuine raviolo" which identifies the exteriors of the open ε–disks $D_\varepsilon \subset D$ in two copies of D. This should handle the lifting of maps from finite CW–complexes. For general ones, you need the homotopy extension lemmas in Hatcher, Chapter 0.

Solution: Fix $0 < \varepsilon < 1$ and form the *genuine raviolo* R_ε by taking two copies D_+, D_- of the closed unit disk and identifying the exteriors $D_\pm \setminus D_\varepsilon$ by the identity. This space is homotopy equivalent to S^2. There is a natural collapse map

$$c : R_\varepsilon \longrightarrow X$$

which is the identity on the common exterior $D \setminus D_\varepsilon$ and collapses each inner disk $D_\varepsilon \subset D_\pm$ to the points $0_\pm \in X$. On the open set $X \setminus \{0_\pm\} \cong D \setminus \{0\}$ the map c is a homeomorphism.

Let Z be a finite CW complex and $f : Z \to X$ a map. Define the *bad set*

$$A = f^{-1}(\{0_+, 0_-\}).$$

Because Z has finitely many cells, A is contained in a finite subcomplex $K \subset Z$. The lifting problem only occurs on A, since over $X \setminus \{0_\pm\}$ the map c is invertible.

For each cell e of K choose a small closed neighborhood $N(e) \supset A \cap e$ inside that cell. Using the contractibility of cells, homotope f rel ∂e so that on $N(e)$ the map takes on the constant value 0_+ or 0_-, and on $e \setminus N(e)$ it avoids $0_\pm$. After performing this modification for all cells of K we obtain a map $f' \simeq f$ with the properties

- f' is constant with value $0_\pm$ on a neighborhood $N(A) = \bigcup N(e)$,
- $f'(Z \setminus N(A)) \subset X \setminus \{0_\pm\} \cong D \setminus \{0\}$.

On $Z \setminus N(A)$ the lift is uniquely determined by the inverse of c. On each component of $N(A)$ the map f' is constant, so we may choose an arbitrary constant lift into the disk fiber $D_\varepsilon \subset R_\varepsilon$. These choices glue continuously along the boundary because the forced lift on $Z \setminus N(A)$ approaches the boundary circle ∂D_ε. Thus f' admits a continuous lift $\tilde{f} : Z \to R_\varepsilon$ with $c \circ \tilde{f} \simeq f$.

The argument for injectivity is similar. Let $H : Z \times I \to X$ be a homotopy between $c \circ \tilde{f}_0$ and $c \circ \tilde{f}_1$. Let $B = H^{-1}(\{0_+, 0_-\})$ be the bad set of the homotopy. For finite Z, B lies in a finite subcomplex of $Z \times I$. Modify H rel boundary so that on a neighborhood $N(B)$ it is constant at $0_\pm$, outside it avoids $0_\pm$. Then on the complement, H lifts uniquely to R_ε and on $N(B)$ we can choose constant lifts.

This produces a lifted homotopy

$$\widetilde{H} : Z \times I \to R_\varepsilon$$

from $\tilde{f}_0$ to $\tilde{f}_1$.

If Z has infinitely many cells, the set $A = f^{-1}(\{0_\pm\})$ may meet infinitely many of them, so the above cell–by–cell modification cannot be carried out directly. However, Hatcher shows that every CW pair (Z, K) with K a subcomplex has the Homotopy Extension Property.

Choose an increasing sequence of finite subcomplexes

$$K_1 \subset K_2 \subset \cdots \subset Z, \qquad \bigcup_n K_n = Z.$$

Restrict f to K_n:

$$f_n := f|_{K_n} : K_n \to X.$$

Now the bad set $A_n = f_n^{-1}(\{0_\pm\})$ meets only finitely many cells (since K_n is finite), so the finite argument applies and we get lifts on each K_n. To make the lifts agree, we use the Homotopy Extension Property for the CW pair (K_{n+1}, K_n). Taking the union of the lifts on K_n gives a lift on all of Z.

The space X is not locally contractible at the points $0_\pm$: any small neighborhood of 0_+ deformation retracts onto a punctured disk, which has fundamental group $\mathbb{Z}$. Every space having the homotopy type of a CW complex is locally contractible; hence X cannot be homotopy equivalent to S^2. Therefore the projection $S^2 \to X$ is a CW–approximation but not a homotopy equivalence.

Problem 5 Compute the groups $H_*(\mathbb{RP}^n;\mathbb{Z})$, $H_*(\mathbb{RP}^n;\mathbb{Z}/2)$, $H^*(\mathbb{RP}^n;\mathbb{Z})$, $H^*(\mathbb{RP}^n;\mathbb{Z}/2)$ in two ways:

- From the chain/cochain complexes,
- By universal coefficient formulas, starting from $H_*(\mathbb{RP}^n;\mathbb{Z})$.

Solution: The CW structure of $\mathbb{RP}^n$ has one cell e^k in each dimension $0 \le k \le n$. With $\mathbb{Z}$–coefficients the cellular boundary maps are

$$d_k : C_k \cong \mathbb{Z} \longrightarrow C_{k-1} \cong \mathbb{Z}, \qquad d_k = \begin{cases} 0, & k \text{ odd}, \\ 2, & k \text{ even}, \end{cases} \qquad (1 \le k \le n).$$

From the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0,$$

one obtains $H_0(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}$. For $0 < k < n$,

$$H_k(\mathbb{RP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

In top degree,

$$H_n(\mathbb{RP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n \text{ odd (orientable case)}, \\ 0, & n \text{ even}. \end{cases}$$

Over $\mathbb{Z}/2$ the multiplication by 2 becomes 0, so all differentials vanish. Hence

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2 \qquad \text{for all } 0 \le k \le n.$$

The cellular cochain groups are $\mathrm{Hom}(C_k,\mathbb{Z}) \cong \mathbb{Z}$, and the coboundaries $d^k = \mathrm{Hom}(d_{k+1},\mathbb{Z})$ satisfy

$$d^k = \begin{cases} 0, & k \text{ even},\\ 2, & k \text{ odd}, \end{cases} \qquad (d^k : C^k \to C^{k+1}).$$

Computing cohomology gives $H^0(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}$. For $0 < k < n$,

$$H^k(\mathbb{RP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2, & k \text{ even},\\ 0, & k \text{ odd}. \end{cases}$$

In top degree,

$$H^n(\mathbb{RP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n \text{ odd},\\ \mathbb{Z}/2, & n \text{ even}. \end{cases}$$

The $\mathbb{Z}/2$ in the even case follows from the universal coefficient theorem since $H_{n-1}(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}/2$.

Again all coboundaries vanish over $\mathbb{Z}/2$, so

$$H^k(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2 \qquad \text{for all } 0 \le k \le n.$$

The universal coefficient theorem for cohomology gives

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(X;\mathbb{Z}),\mathbb{Z}) \longrightarrow H^k(X;\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_k(X;\mathbb{Z}),\mathbb{Z}) \longrightarrow 0.$$

Using

$$\mathrm{Hom}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}, \qquad \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}) = 0, \qquad \mathrm{Ext}^1(\mathbb{Z}/2,\mathbb{Z}) \cong \mathbb{Z}/2,$$

and the homology from part (1) reproduces the cohomology groups above.

UCT for homology gives

$$0 \longrightarrow H_k(X;\mathbb{Z}) \otimes \mathbb{Z}/2 \longrightarrow H_k(X;\mathbb{Z}/2) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{k-1}(X;\mathbb{Z}),\mathbb{Z}/2) \longrightarrow 0.$$

Since $\mathrm{Tor}_1(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2$, inserting the groups from (1) yields $H_k(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for all $0 \le k \le n$, in agreement with part (2).

Problem 6 Determine the ring structure on $H^*(\mathbb{RP}^3 \times \mathbb{RP}^3;\mathbb{Z}/2)$ and $H^*(\mathbb{RP}^3 \times \mathbb{RP}^3;\mathbb{Z})$.

Solution: Write $H^*(\mathbb{RP}^3;\mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha]/(\alpha^4)$ with $|\alpha| = 1$. Write $\alpha_1, \alpha_2 \in H^1(\mathbb{RP}^3 \times \mathbb{RP}^3;\mathbb{Z}/2)$ be the pullbacks from the two factors. By Künneth (over a field) and naturality of cup product,

$$H^*(\mathbb{RP}^3 \times \mathbb{RP}^3;\mathbb{Z}/2) \;\cong\; (\mathbb{Z}/2)[\alpha_1,\alpha_2]/(\alpha_1^4,\alpha_2^4), \qquad |\alpha_1| = |\alpha_2| = 1,$$

with graded-commutativity (over $\mathbb{Z}/2$ this is just commutativity). Recall that the cohomology ring of $\mathbb{RP}^3$ with $\mathbb{Z}$–coefficients is

$$H^*(\mathbb{RP}^3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & * = 0, \\ \mathbb{Z}/2, & * = 2, \\ \mathbb{Z}, & * = 3, \\ 0, & \text{otherwise}, \end{cases}$$

and the only nontrivial products are those forced by the unit (all products of positive-degree classes vanish for degree reasons).

Let $u \in H^2(\mathbb{RP}^3;\mathbb{Z}) \cong \mathbb{Z}/2$ be the torsion generator and $\omega \in H^3(\mathbb{RP}^3;\mathbb{Z}) \cong \mathbb{Z}$ the orientation class. On $X := \mathbb{RP}^3 \times \mathbb{RP}^3$, write

$$u_1 = p_1^*(u), \quad \omega_1 = p_1^*(\omega), \qquad u_2 = p_2^*(u), \quad \omega_2 = p_2^*(\omega).$$

Then the external product classes generate the "tensor part" of cohomology, and Künneth gives the additive groups:

$$H^k(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ (\mathbb{Z}/2)\{u_1, u_2\}, & k = 2, \\ \mathbb{Z}\{\omega_1, \omega_2\} \oplus \mathbb{Z}/2\{\tau\}, & k = 3, \\ \mathbb{Z}/2\{u_1u_2\}, & k = 4, \\ (\mathbb{Z}/2)\{u_1\omega_2,\ \omega_1u_2\}, & k = 5, \\ \mathbb{Z}\{\omega_1\omega_2\}, & k = 6, \\ 0, & \text{otherwise}. \end{cases}$$

Here $\tau \in H^3(X;\mathbb{Z})$ is the extra $\mathbb{Z}/2$ coming from the Tor-term in the integral Künneth theorem.

Cup products among u_i, ω_i are determined by: graded-commutativity, $u_i^2 = 0$ and $u_i\omega_i = 0$ (degree reasons on each factor), and the nonzero cross products

$$u_1u_2 \in H^4, \qquad u_1\omega_2,\ \omega_1u_2 \in H^5, \qquad \omega_1\omega_2 \in H^6$$