Title

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Abstract

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1 Principal G-Bundles on Affine Curves

It is a consequence of a theorem of Harder [**?**, Satz 3.3] that generically trivial principal G-bundles on a smooth affine curve C over an arbitrary field k are trivial if G is a semisimple and simply connected algebraic group. When k is algebraically closed and G reductive, generic triviality, conjectured by Serre, was proved by Steinberg [**?**] and Borel-Springer [**?**].

It follows that principal bundles for simply connected semisimple groups over smooth affine curves over algebraically closed fields are trivial. This fact (and a generalization to families of bundles

[?]) plays an important role in the geometric realization of conformal blocks for smooth curves as global sections of line bundles on moduli-stacks of principal bundles on the curves (see the review [?] and the references therein).

1.1 Derived Pushforward of Admissible Complexes

Recall that a $\mathbb{C}^\times$-bundle on a nodal curve Σ is defined by a $\mathbb{C}^\times$-bundle on the normalization of Σ together with an identification of the two fibers at the preimages of each node. The stack $\mathrm{Bun}_{\mathbb{C}^\times}(g, I)$ of $\mathbb{C}^\times$-bundles over the universal stable curve fails to be complete, because the space of identifications over a given node is isomorphic to $\mathbb{C}^\times$.

Following Gieseker [?] and Caporaso [?], we add new strata which represent the limits where an identification goes to zero or infinity, by allowing projective lines carrying the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ to appear at the nodes.

The resulting stack denoted $\widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^\times])$ is complete but not separated, i.e., the limit of a family of bundles exists but may not be unique.

This stack classifies maps from marked nodal curves to the quotient stack $[pt/\mathbb{C}^\times]$. These are the moduli stacks of principal $\mathbb{C}^\times$-bundles on such curves.

My understanding is that the construction of this compactifications goes through Pablo's wonderful compactification of loop groups. But it seems this is not necessary in the $\mathbb{C}^*$ case. Then Teleman gives a modular interpretation of this compactification in terms of Gieseker bundles.

Theorem 1.1. The derived pushforward $RF_*\alpha$ of an admissible complex α along the bundle-forgetting map $F : \widetilde{\mathcal{M}}_{g,I}([pt/G]) \to \overline{\mathcal{M}_{g,I}}$ is a bounded complex of coherent sheaves.

This theorem is a relative version over varying curves of the analogous finiteness result for $\mathrm{Bun}_G(\Sigma)$ in [?, 34].

1. Section 1 reviews basic facts about nodal curves and principal $\mathbb{C}^\times$-bundles. The moduli stack $\widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^\times])$ of Gieseker bundles on stable curves is introduced with some key examples (small g and $|I|$).

2. Section 2 proves some basic facts about the geometry of our stack: it is an Artin stack, is stratified by topological type, and is complete (but not separated).

3. Section 3 gives an (étale-local) presentation of $\widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^\times])$ as a quotient A/G (where $G \simeq (\mathbb{C}^\times)^V$). We identify a stable subspace $A^\circ \subset A$ which leads to a smooth and proper quotient moduli space over $\mathcal{M}_{g,I}$.

4. Section 4 refines the stratification by topological type by tracking the nodes smoothed under

deformations. We use this to stratify A/G by distinguished spaces Z, W which are affine space bundles over their fixed-point loci under subgroups of G.

5. Section 5 reviews the admissible K-theory classes and estimates the weights of the fixed-point fibers of subgroups of G.

6. Section 6 uses a local cohomology vanishing argument to finish the proof of the main theorem.

7. Section 7 constructs a moduli stack which we expect to carry Gromov-Witten invariants for $[X/\mathbb{C}^\times]$.

We expect to recover the Gromov-Witten invariants of GIT quotients from our invariants by applying the Chern character to certain limits of our invariants. This was done for smooth curves and G-bundles in Teleman-Woodward [**?**].

1.2 Construction of $\widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^\times])$

One sees from this how $\mathbb{C}^\times$-bundles on C can become singular in families: the space of gluing isomorphisms at a node $\sigma \in C$ is a copy of $\mathbb{C}^\times$; in a family, these isomorphisms can tend to the limit points 0 and ∞. As a result, the stack $\mathrm{Bun}_{\mathbb{C}^\times}(g, I)$ of $\mathbb{C}^\times$-bundles on stable marked curves of type (g, I) fails the valuative criterion for completeness. This will be a problem for integration of cohomology or K-theory classes.

We always work over $\mathbb{C}$. In everything that follows, (C, σ_i) is a family of prestable marked curves over a finitely generated complex base scheme B. More precisely, $\pi : C \to B$ is a flat proper morphism whose fibers are connected complex projective curves of genus g with at worst nodal singularities, carrying a collection of smooth marked points $\sigma_i : B \to C$ which are indexed by an ordered set I.

A point is *special* if it is a node or a marked point. Special points are required to be pairwise disjoint. We shall always assume that any rational component of C has at least *two* special points.

We reserve the notation (Σ, σ_i) for families of *stable* marked curves. Recall that a marked curve is stable if each component of genus 0 carries at least 3 special points and each component of genus 1 carries at least 1 special point. The *stabilization morphism* $\mathrm{st} : C \to C^{st}$ blows down every unstable rational curve in C. Stabilization can be implemented by a pluricanonical embedding and thus works in families.

Definition 1.2 (Modification of curves)**.** A morphism $m : C \to \Sigma$ of prestable curves is a **modification** if

1. m is an isomorphism away from the preimage of the nodes of Σ, and

2. the preimage under m of every node in Σ is either a node or a $\mathbb{P}^1$ with two special points.

A **modification of a family** $f : \Sigma \to B$ of marked prestable curves is a morphism $m : C \to \Sigma$ such that, for each geometric $b \in B$, the induced map

$$m_b : C_b \longrightarrow \Sigma_{f(b)}$$

is a modification.

Remark 1.3. 1. Finding modifications with desirable properties—such as smoothness of the total space C—may require us to change the base B; the reader can be entrusted to write out the defining diagram.

2. Modifications of marked curves do not introduce $\mathbb{P}^1$'s at marked points, only at nodes. The marked points in a family Σ lift uniquely to the modification, and will sometimes be denoted by the same symbol.

Definition 1.4 (Gieseker bundle)**.** Let (Σ, σ_i) be a stable marked curve. A **Gieseker** $\mathbb{C}^\times$**–bundle** on (Σ, σ_i) is a pair $(m, \mathcal{P})$ consisting of

1. a modification $m : (C, \sigma_i) \to (\Sigma, \sigma_i)$, and
2. a principal $\mathbb{C}^\times$–bundle $p : \mathcal{P} \to C$,

which satisfy the **Gieseker condition**:

1. the restriction of $\mathcal{P}$ to every unstable $\mathbb{P}^1$ has degree 1.

We should learn what the Gieseker condition says. Is it a general formula or just an intuition?

Definition 1.5. The stack $\widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^\times])$ of *Gieseker* $\mathbb{C}^\times$*–bundles on stable genus* g*,* I*–marked curves* is a fibred category (over $\mathbb{C}$–schemes). Its objects are tuples $(B, C, \sigma_i, \mathcal{P})$ consisting of

1. a test scheme B;
2. a flat projective family $\pi : C \to B$ of pre-stable, genus g curves with marked points $\sigma_i : B \to C$, $i \in I$; and
3. a principal $\mathbb{C}^\times$–bundle $p : \mathcal{P} \to C$ defining a family of Gieseker bundles on the stabilization $C \to C^{st}$.

The morphisms in this category are commutative diagrams

$$\begin{tikzcd}
\mathcal{P}' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & \mathcal{P} \arrow[d, "p"] \\
C' \arrow[r, "f"] \arrow[d, "\pi'"'] & C \arrow[d, "\pi"] \\
B' \arrow[r] & B
\end{tikzcd}$$

where $\tilde{f}$ is $\mathbb{C}^\times$-equivariant and $C' = B' \times_B C$ and the morphism of curves f respects the marked points.

There is a natural forgetful morphism

$$F : \widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^\times]) \longrightarrow \overline{\mathcal{M}}_{g,I}$$

which sends a Gieseker bundle $(C, \sigma_i, \mathcal{P})$ to the stabilized curve (C^{st}, σ_i).

Example 1.6.

2 Pablo's modifications

2.1 Setup

For any torus T we have the lattice of characters $\hom(T, \mathbb{C}^\times)$ and co-characters $\hom(\mathbb{C}^\times, T)$. Further, for $(\eta, \chi) \in \hom(\mathbb{C}^\times, T) \times \hom(T, \mathbb{C}^\times)$ we set

$$\langle \eta, \chi \rangle := \chi \circ \eta \in \mathbb{Z}.$$

For $T \subset G$ a maximal torus and for $\eta \in \hom(\mathbb{C}^\times, T)$ the set

$$P(\eta) := \{g \in G \mid \lim_{t \to 0} \eta(t) g \eta(t)^{-1} \text{ exists}\}$$

is a subgroup. A **parabolic** subgroup is any subgroup $P \subset G$ conjugate to some $P(\eta)$.

We can apply the same construction for $\eta \in \hom(\mathbb{C}^\times, \mathbb{C}^\times \times T)$ to get a subgroup $P(\eta) \subset L^\times G$. A **parahoric** subgroup is any group conjugate to one of the $P(\eta)$. By abuse of notation, we use $P(\eta)$ to denote its image under the projection $L^\times G \to LG$. Parahoric subgroups of LG are any subgroups conjugate to one of the $P(\eta)$.

Parabolic and parahoric subgroups come with natural factorizations $P(\eta) = L(\eta)U(\eta)$ known as a Levi decomposition:

$$L(\eta) = \{g \in G \mid \lim_{t \to 0} \eta(t) g \eta(t)^{-1} = g\}, \qquad U(\eta) = \{g \in G \mid \lim_{t \to 0} \eta(t) g \eta(t)^{-1} = 1\}.$$

A simple example comes from $\eta_0 : \mathbb{C}^\times \to \mathbb{C}^\times \times T$ defined by $\eta_0(t) = (t, 1)$. Then $\eta_0(t)g(z)\eta_0(t)^{-1} = g(tz)$ and

$$P(\eta_0) = G[[z]] = G(\mathbb{C}[[z]]) =: L^+G.$$

The Levi factorization is $G \cdot N$ where N is the kernel of the map

$$G[[z]] \xrightarrow{z \mapsto 0} G.$$

By $\mathfrak{t}_\mathbb{Q}$ we denote $\hom(\mathbb{C}^\times, T) \otimes_\mathbb{Z} \mathbb{Q}$. The Weyl chamber is defined as

$$\mathrm{Ch} := \{\eta \in \mathfrak{t}_\mathbb{Q} \mid \langle \alpha_i, \eta \rangle \geq 0\}.$$

It is a simplicial cone whose faces are given by $\{\langle \alpha, \eta \rangle = 0 \mid \alpha \in I\}$ for subsets $I \subset \{\alpha_1, \dots, \alpha_r\}$.

Similarly, we have the affine Weyl chamber

$$\mathrm{Ch}^{\text{aff}} = \{\eta \in \mathbb{Q} \oplus \mathfrak{t}_\mathbb{Q} \mid \langle \alpha_i, \eta \rangle > 0\};$$

now the faces are in bijection with subsets $\{\alpha_0, \dots, \alpha_r\}$. It is convention to instead work with the affine Weyl alcove

$$\mathrm{Al} := \mathrm{Ch}^{\text{aff}} \cap (1 \oplus \mathfrak{t}_\mathbb{Q}) = \{\eta \in \mathfrak{t}_\mathbb{Q} \mid 0 \leq \langle \alpha_i, \eta \rangle,\ \langle \theta, \eta \rangle \leq 1\}.$$

A **face** F of Al is $F' \cap (1 \oplus \mathfrak{t}_\mathbb{Q})$ where F' is a face of Ch^{aff}.

Any $\eta \in \mathrm{Ch}$ determines a fractional co-character $\mathbb{C}^\times \to T$ but nevertheless a well-defined parabolic $P(\eta)$. Any parabolic is conjugate to some $P(\eta)$ and if η, η' are in the interior of the same face then $P(\eta) = P(\eta')$. Similarly any $\eta \in \mathrm{Al}$ determines a parahoric $P(\eta) \subset LG$. Any parahoric is conjugate either to $P(\eta)$ or to $P(-\eta)$. Let

$$\mathrm{Al}_e = \{\eta \in \mathrm{Al} \mid \langle \theta, \eta \rangle = 1\}.$$

If $\eta \in \mathrm{Al}_e$ the resulting parahoric is called **exotic**. Alternatively, the inclusion

$$\{\alpha_1, \dots, \alpha_r\} \subset \{\alpha_0, \dots, \alpha_r\}$$

defines a map from faces of Ch to those of Al. The faces missed by Ch are exactly those contained in Al_e.

The exotic parahorics give rise to moduli spaces of torsors on curves which are not isomorphic with moduli spaces of G-bundles. Informally then the exotic parahorics can be viewed as geometry only visible to LG.

The ordered simple roots $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$ determine ordered vertices $\{\eta_0, \dots, \eta_r\}$ determined by the conditions

$$\langle \eta_i, \alpha_j \rangle = 0 \text{ for } i \neq j \quad \text{and} \quad \langle \eta_0, \alpha_0 \rangle = 1.$$

If we write $\theta = \sum_{i=1}^{r} n_i \alpha_i$ and set $n_0 = 1$ then one can check these conditions can be expressed as

$$\langle \alpha_i, \eta_j \rangle = \frac{1}{n_i} \delta_{i,j}. \tag{1}$$

Now for each $I \subset \{0, \ldots, r\}$ we define

$$\eta_I = \sum_{i \in I} \eta_i.$$

The alcove Al is a compact convex polytope whose faces are in bijection with conjugacy classes of parahoric subgroups of LG. For each $I \subset \{0, \ldots, r\}$ the cocharacter

$$\eta_I := \sum_{i \in I} \eta_i$$

lies in the relative interior of the face of Al corresponding to the complement of I, meaning that $\langle \alpha_j, \eta_I \rangle = 0$ for $j \notin I$ and hence determines a parahoric subgroup

$$P(\eta_I) \subset LG.$$

Equivalently, $\eta_I : \mathbb{C}^\times \to \mathbb{C}^\times \times T$ is a one–parameter subgroup whose conjugation action is used to define $P(\eta_I)$ as the subgroup on which the limit $\lim_{t \to 0} \eta_I(t) g \eta_I(t)^{-1}$ exists. Note that if $I = \varnothing$ we take η_I to be the trivial co-character. Finally, we set

$$\begin{aligned} P_I &= P(\eta_I), & P_{\bar{I}} &= P(-\eta_I), \\ U_I &= U(\eta_I), & U_{\bar{I}} &= U(-\eta_I), \\ L_I &= L(\eta_I) = L(-\eta_I). \end{aligned} \tag{2}$$

Example 2.1. We work out the case $G = \mathrm{SL}_2$ in detail. Take the standard maximal torus

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \;\middle|\; t \in \mathbb{C}^\times \right\} \cong \mathbb{C}^\times.$$

A cocharacter $\eta \in \hom(\mathbb{C}^\times, T)$ is determined by an integer m:

$$\eta_m : \mathbb{C}^\times \to T, \qquad \eta_m(t) = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}.$$

So $\hom(\mathbb{C}^\times, T) \cong \mathbb{Z}$. A character $\chi \in \hom(T, \mathbb{C}^\times)$ is also determined by an integer k:

$$\chi_k \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^k.$$

The pairing $\langle \eta_m, \chi_k \rangle = \chi_k \circ \eta_m$ is

$$\langle \eta_m, \chi_k \rangle = km \in \mathbb{Z}.$$

The simple (finite) root α corresponds to the character χ_2, so if we identify $\mathfrak{t}_{\mathbb{Q}} \cong \mathbb{Q}$ using the basis η_1, then

$$\langle \alpha, \eta_m \rangle = 2m.$$

We can (and usually do) renormalize so that $\langle \alpha, \eta_1 \rangle = 1$, but the picture is the same: $\mathfrak{t}_{\mathbb{Q}}$ is a line and the Weyl chamber is the half-line $m \geq 0$.

Take $\eta(t) = \eta_1(t) = \operatorname{diag}(t, t^{-1})$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$, we have

$$\eta(t) g \eta(t)^{-1} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix}.$$

As $t \to 0$, this has a limit if and only if $c = 0$. So

$$P(\eta) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2 \right\} = \text{upper Borel } B.$$

The Levi and unipotent parts are $L(\eta)$ = diagonal torus (the copy of T), and $U(\eta)$ = strictly upper triangular unipotent matrices. Similarly $P(-\eta)$ is the lower Borel.

The Weyl chamber Ch is $\{\eta_m \mid m \geq 0\} \subset \mathbb{Q}$. All nonzero $m > 0$ lie in the interior of the same cone, so $P(\eta_m)$ is always conjugate to B. The "faces" of the cone are: the origin $\{0\}$ and the open half-line $\{m > 0\}$. At $\eta = 0$, $P(0) = G$; in the open face we get the Borel. This is the finite-type picture behind the general definition.

For the affine root system $\widehat{\mathfrak{sl}}_2$ (type $A_1^{(1)}$): there are two simple affine roots α_0, α_1, the highest finite root is $\theta = \alpha_1$, and the extended Cartan is $\mathbb{Q} \oplus \mathfrak{t}_{\mathbb{Q}}$. Restricting to the slice $1 \oplus \mathfrak{t}_{\mathbb{Q}}$ (the "height 1" slice) identifies the affine Weyl alcove

$$\mathrm{Al} = \{\eta \in \mathfrak{t}_{\mathbb{Q}} \mid 0 \leq \langle \alpha_1, \eta \rangle, \, \langle \theta, \eta \rangle \leq 1\}.$$

For $\mathfrak{sl}_2$, $\theta = \alpha_1$, so this reduces to

$$\mathrm{Al} = \{\eta \in \mathfrak{t}_{\mathbb{Q}} \mid 0 \leq \langle \alpha, \eta \rangle \leq 1\}.$$

Identifying $\mathfrak{t}_{\mathbb{Q}} \cong \mathbb{Q}$ so that $\langle \alpha, \eta \rangle$ is literally the coordinate, we get $\mathrm{Al} = [0, 1] \subset \mathbb{Q}$. You can write a cocharacter as a pair

$$\eta(t) = (t^m, \eta_T(t)),$$

where $t^m \in \mathbb{C}^\times$ and $\eta_T(t) \in T$ where first component rescales the loop parameter $(t^m \cdot g)(z) = g(t^m z)$. The vertices η_0, η_1 are the endpoints 0 and 1. The interior $0 < \langle \alpha, \eta \rangle < 1$ corresponds to the "Iwahori" parahoric (the analogue of a Borel in the loop group).

In the loop group $LG = \mathrm{SL}_2(\mathbb{C}((z)))$, the choice $\eta_0(t) = (t, 1)$ rescales the loop variable and gives

$$P(\eta_0) = \{g(z) \in G(\mathbb{C}((z))) \mid \lim_{t \to 0} g(tz) \text{ exists in } G(\mathbb{C}((z)))\} = \mathrm{SL}_2(\mathbb{C}[[z]]) = L^+ \mathrm{SL}_2,$$

the standard maximal parahoric corresponding to the vertex η_0. The other vertex η_1 corresponds to the cocharacter $\eta_1(t) = (t, \eta_T(t))$ where $\eta_T(t) = \mathrm{diag}(t, t^{-1})$. Then

$$(\eta_1(t) \cdot g)(z) = \begin{pmatrix} a(tz) & t^2 b(tz) \\ t^{-2} c(tz) & d(tz) \end{pmatrix}$$

so the limit as $t \to 0$ exists if and only if $c(z)$ vanishes at $z = 0$. Thus the parahoric subgroup is

$$P(\eta_1) = \left\{ \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}((z))) \mid c(z) \in z^2\mathbb{C}[[z]], z^2 b(z) \in \mathbb{C}[[z]], a(z), d(z) \in \mathbb{C}[[z]] \right\}.$$

This parahoric subgroup is not conjugate to $L^+ \mathrm{SL}_2$.

Finally any choice of point η in the interior of the interval $\mathrm{Al} = (0, 1)$ gives conjugate parahoric subgroups, so choose $\eta = \frac{1}{2}$. This gives

$$(\eta(t) \cdot g)(z) = \begin{pmatrix} a(tz) & t\, b(tz) \\ t^{-1} c(tz) & d(tz) \end{pmatrix}$$

so the limit as $t \to 0$ exists if and only if $a(z), d(z) \in \mathbb{C}[[z]]$, $zb(z) \in \mathbb{C}[[z]]$, and $c(z) \in z\mathbb{C}[[z]]$.

After intersecting with the positive loop group $\mathrm{SL}_2(\mathbb{C}[[z]])$, we get the Iwahori subgroup

$$I = \{g(z) \in \mathrm{SL}_2(\mathbb{C}[[z]]) \mid g(0) \in B\},$$

where B is the upper Borel subgroup of SL_2.

2.2 Twisted curves and admissible bundles

Generally we work over $\operatorname{Spec} \mathbb{C}$ and a scheme will mean a scheme over $\operatorname{Spec} \mathbb{C}$. Let S be a scheme. We denote a flat family of curves $C \to S$ as C_S. If B is an S–scheme then $C_B := C_S \times_S B$. For affine schemes $\operatorname{Spec} R \to S$ we write C_R for $C_{\operatorname{Spec} R}$.

Generally we work with a fixed curve over $\operatorname{Spec} \mathbb{C}$ or with a family of curves over $S = \operatorname{Spec} \mathbb{C}[[s]]$. Set $S^* = \operatorname{Spec} \mathbb{C}((s))$ and $S_0 = \operatorname{Spec} \mathbb{C} = \operatorname{Spec} \mathbb{C}[[s]]/(s)$ the closed point. Then C_S always denotes a curve with generic fiber C_{S^*} smooth and special fiber $C_0 := C_{S_0}$ nodal with unique node p. We write $C_S - p$ for the open subscheme $C_S \setminus \{p\}$. We also assume C_S is a regular surface as a scheme over $\operatorname{Spec} \mathbb{C}$.

For any closed point p in a scheme Z we denote by $\widehat{\mathcal{O}}_{Z,p}$ the completion of $\mathcal{O}_{Z,p}$ with respect to the maximal ideal. We often use D to denote a formal neighborhood of a point in a curve. The cases that will arise are:

- If $p \in C$ is a smooth curve, $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[z]]$ and we set $D = \operatorname{Spec} \mathbb{C}[[z]]$.
- If $p \in C_0$ is the node, $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x, y]]/(xy)$ and we set $D_0 = \operatorname{Spec} \mathbb{C}[[x, y]]/(xy)$.

- If $p \in C_S$ is the node, $\widehat{\mathcal{O}}_{C_S,p} \cong \mathbb{C}[[x,y,s]]/(xy-s)$ and we set $D_S = \operatorname{Spec}\mathbb{C}[[x,y,s]]/(xy-s)$.

For $k \geq 2$ and kth roots u, v of x, y we set

$$D_S^{\frac{1}{k}} = \operatorname{Spec}\mathbb{C}[[u,v]].$$

The last case arises as follows. We first notice that if we base change D_S under $s \mapsto s^k$ then D_S becomes

$$\operatorname{Spec}\mathbb{C}[[x,y,s]]/(xy-s^k).$$

If we let μ_k denote the kth roots of unity then

$$\operatorname{Spec}\mathbb{C}[[x,y,s]]/(xy-s^k) \;=\; D_S^{\frac{1}{k}}/\mu_k$$

where $\zeta \in \mu_k$ acts by $(u,v) \mapsto (\zeta u, \zeta^{-1} v)$. A basic strategy we employ is to replace the curve $\operatorname{Spec}\mathbb{C}[[x,y,s]]/(xy-s)$ with the orbifold or twisted curve

$$D_S^{\frac{1}{k}}/\mu_k.$$

Remark 2.2. Recall that a course moduli space of a stack $\mathcal{X}$ is an algebraic space X with a morphism $\pi : \mathcal{X} \to X$ satisfying the following:

1. For any algebraically closed field k, the map π induces a bijection between isomorphism classes of k-points of $\mathcal{X}$ and k-points of X.

2. The map π is universal for maps from $\mathcal{X}$ to algebraic spaces.

For example $[pt/G]$ has coarse moduli space $pt = \operatorname{Spec} k$. For

We recall the definition of a twisted curve (with no marked points) in characteristic 0.

Definition 2.3. A **twisted nodal curve** $\mathcal{C} \to S$ is a proper Deligne–Mumford stack such that

(i) the geometric fibers of $\mathcal{C} \to S$ are connected of dimension 1 and the coarse moduli space C of $\mathcal{C}$ is a nodal curve over S;

(ii) if $U \subset \mathcal{C}$ denotes the complement of the singular locus of $\mathcal{C} \to S$, then $U \to C$ is an open immersion;

(iii) let $p : \operatorname{Spec} k \to C$ be a geometric point mapping to a node and let $s \in S$ denote the image of $\operatorname{Spec} k$ under $C \to S$, and let $\mathfrak{m}_{S,s}$ denote the maximal ideal of the local ring $\mathcal{O}_{S,s}$. Then there is an integer k and an element $t \in \mathfrak{m}_{S,s}$ such that

$$\operatorname{Spec}\mathcal{O}_{C,p} \times_C \mathcal{C} \;\cong\; [D^{sh}/\mu_k],$$

where D^{sh} denotes the strict henselization of

$$D := \operatorname{Spec} \mathcal{O}_{S,s}[u,v]/(uv-t)$$

at the point $(\mathfrak{m}_{S,s}, u, v)$, and $\zeta \in \mu_k$ acts by

$$\zeta \cdot (u,v) = (\zeta u, \zeta^{-1} v).$$

We did not mention markings because largely we will not make use of them except for one exception. If C is a smooth curve we can twist at a marked point p as described below. Let $p \in C$ and $D = \operatorname{Spec} \mathbb{C}[[z]]$ as in the first bullet point above, and fix a positive integer k and a kth root w of z. We have $\operatorname{Spec} \mathbb{C}((w))/\mu_k = \operatorname{Spec} \mathbb{C}((z))$, so let $C_{[k]}$ denote

$$C_{[k]} := C \setminus \{p\} \underset{\operatorname{Spec} \mathbb{C}((z))}{\cup} [\operatorname{Spec} \mathbb{C}[[w]]/\mu_k].$$

Then $C_{[k]}$ is a twisted curve whose coarse moduli space is C.

In a similar fashion, with C_0, C_S as in the bullet points above, we can construct twisted curves $C_{0,[k]}$ and $C_{S,[k]}$ with coarse moduli spaces C_0, C_S and such that the fiber of the node is $[pt/\mu_k]$.

The motivation to consider these objects comes from the valuative criterion for completenss. Specifically it comes from the following local calculation that Pablo does in Section 4.

2.3 G-bundles on Twisted Chains

In the previous section we saw that associated to the singleton sets $\{i\} \subset \{0, r+1\}$ there is a moduli space parametrizing G-bundles on a twisted nodal curve and further the moduli space can be identified with an orbit of the wonderful embedding of the loop group. In this section we introduce a more general moduli problem which we show is isomorphic to the orbit O_I in the wonderful embedding for any $I \subset \{0, \ldots, r+1\}$.

Let R_n denote the rational chain of projective lines with n–components. There is an action of $\mathbb{C}^\times$ on R_n which scales each component. Let $p_0, \ldots, p_n$ denote the fixed points of this action.

Recall u, v are k–th roots of x, y which are our coordinates near a node. Let p', p'' be the closed points of $\operatorname{Spec} \mathbb{C}[[u]]$, $\operatorname{Spec} \mathbb{C}[[v]]$ and finally let $D_n^{\frac{1}{k}}$ be the curve obtained from

$$\operatorname{Spec} \mathbb{C}[[u]] \amalg R_n \amalg \operatorname{Spec} \mathbb{C}[[v]]$$

by identifying p' with p_0 and p'' with p_n.

The group μ_k acts on $D_n^{\frac{1}{k}}$ through its usual action on u, v and through the inclusion $\mu_k \subset \mathbb{C}^\times$ on the chain R_n. For an n–tuple $(\beta_0, \ldots, \beta_n) \in \hom(\mathbb{C}^\times, T)^n$, we can speak about the equivariant G–bundles on $D_n^{\frac{1}{k}}$ with equivariant structure at p_i determined by β_i. We refer to this equivalently as a G–bundle on $[D_n^{\frac{1}{k}}/\mu_k]$ of type $(\beta_1, \ldots, \beta_n)$.

Remark 2.4. Suppose have a space B with an action of a group Π. We have a principal H-bundle $P \to B$. We want Π to also act on P, but in a way compatible with the projection $P \to B$. An Π-equivariant H-bundle over B is equivalent to an H-bundle over the quotient stack $[B/\Pi]$. Over the point b/Π of this stack, the automorphisms are exactly Π_b. Therefore an H-bundle over $[B/\Pi]$ must specify how Π_b acts on the fiber, and that is a representation $\rho : \Pi_b \to H$.

Further, we can also glue $[D_n^{\frac{1}{k}}/\mu_k]$ to $C_0 - p_0$ to obtain a curve $C_{n,[k]}$. Let C_n denote the coarse moduli space of $C_{n,[k]}$.

We call C_n a **modification** of C_0 and $C_{n,[k]}$ a **twisted modification** of C_0.

Recall the specific co-characters $\eta_0, \dots, \eta_r$.

Definition 2.5. For $I = \{i_1, \dots, i_n\} \subset \{0, \dots, r\}$ let $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ denote the moduli space of pairs (P, τ) where P is a G–bundle on $[D_n^{\frac{1}{k}}/\mu_k]$ of type $(\eta_{i_1}, \dots, \eta_{i_n})$ and τ is a trivialization on $[\operatorname{Spec}\mathbb{C}((u)) \times C((v))]/\mu_k$.

Let $H = \operatorname{Aut}(P)$ then restriction to $\operatorname{Spec}\mathbb{C}[[u]]$ and $\operatorname{Spec}\mathbb{C}[[v]]$ realizes

$$H \subset (L_uG)^{\mu_k} \times (L_vG)^{\mu_k}.$$

Theorem 2.6. Let $I \subset \{0, \dots, r\}$ and $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ be as above. Then there is an isomorphism

$$T_{G,I}(C_{0,[k]}) \xrightarrow{\Psi^{\eta_I}} (L_uG)^{\mu_k} \times (L_vG)^{\mu_k} /\!\!/ H \xrightarrow{\eta_I^{(\cdot)}\eta_I^{-1}} \frac{L_{\text{poly}}G \times L_{\text{poly}}G}{Z(L_1) \times Z(L_1) \cdot P_I^{A,\pm}},$$

where Ψ^{η_I} is induced by the double coset construction and $\eta_I^{(\cdot)}\eta_I^{-1}$ is the map $LG \times LG/\mathcal{P}^\Delta \to (L_uG \times L_vG)^{\mu_k}/(G_{u,v}^\Delta)^{\mu_k}$ given by

$$g(z)\mathcal{P} \mapsto (\eta_I(w)g(w^k)\eta_I^{-1}(w))L_w^+G$$

on each factor.

Let $i : [D_n^{\frac{1}{k}}/\mu_k] \to C_{0,[k]}$ be the natural map. Then $i^* : T_{G,I}(C_{0,[k]}) \to T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ is an isomorphism. In particular, $T_{G,I}(C_{0,[k]})$ and $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ are isomorphic to an orbit in the wonderful embedding of $L_{\text{poly}}^\times G$.

In this section we begin with a curve C_S as in Section 2.3 and construct an algebraic S–stack $\mathcal{X}_G(C_S)$ such that $\mathcal{M}_G(C_S) \subset \mathcal{X}_G(C_S)$ is a dense open substack and the boundary is a divisor with normal crossings. Further we show the morphism $\mathcal{X}_G(C_S) \to S$ is complete.

For the remainder of this section we fix a simple group G and further fix an integer $k = k_G$ defined as follows. Let η_i be the vertices of Al. Define k_i as the minimum integer such that

$k_i \cdot \eta_i \in \hom(\mathbb{C}^\times, T)$ and set $k_G = \mathrm{lcm}(k_i)$. The η_i correspond to the maximal parahorics P_i of LG and further any parahoric P is conjugate to a subgroup of some P_i. It follows readily that $k = k_G$ is the minimum value of k for which the statement of Corollary 4.3 holds for any particular parahoric P.

We recall some of the notation from 2.3. Namely, $S = \operatorname{Spec}\mathbb{C}[[s]]$, $S^* = \operatorname{Spec}\mathbb{C}((s))$, $S_0 = \operatorname{Spec}\mathbb{C}[[s]]/(s) = \operatorname{Spe}$ $C_0 = C_{S_0}$. For B an S–scheme we set $B^* = B \times_S S^*$, $B_0 = B \times_S S_0$. We also have $D_S = \operatorname{Spec}\mathbb{C}[[x,y]]$ considered as an S–scheme via $s \mapsto xy$ and $D_0 = \operatorname{Spec}\mathbb{C}[[x,y]]/(xy)$. Further, we set $D_S^{1/k} := \mathbb{C}[[u,v]]$ where $u^k = x$ and $v^k = y$. Then $D_{S,[k]} = [D_S^{1/k}/\mu_k]$; the coarse moduli space of $D_{S,[k]}$ is $\operatorname{Spec}\mathbb{C}[[x,y,s]]/(xy - s^k)$. We further fix $p \in C_S$ to be the node.

To define $\mathcal{X}_G(C_S)$ we need to define twisted modifications of C_S; this is a relative version of (11). Then in Subsection 5.2 we define $\mathcal{X}_G(C_S)$ to be the moduli stack parametrizing G–bundles on twisted modifications. There we prove the main theorem which shows that $\mathcal{X}_G(C_S)$ satisfies the valuative criterion for completeness.

2.4 Twisted modifications

Let C_S be a nodal curve. A **modification of length** $\leq n$ **of** C_S **over** B is a curve C'_B over B with a morphism $C'_B \xrightarrow{\pi} C_B$ such that

- C'_B is flat over B and π is finitely presented and projective;
- $C'_B \xrightarrow{\pi} C^*_B$ is an isomorphism;
- for $b \in B_0$ the map of curves $C'_b \xrightarrow{\pi} C_b$ is a modification; that is, the fiber $\pi^{-1}(p_b)$ over the unique node $p_b \in C_b$ is a rational chain of $\mathbb{P}^1$s with at most n components and there is $b \in B_0$ such that $\pi^{-1}(p_b)$ has exactly n components.

Let $(g_1, \ldots, g_n) \in (\mathbb{C}^\times)^n$ act on $\mathbb{C}[[t_1, \ldots, t_{n+1}]]$ by

$$(t_1, \ldots, t_n) \xmapsto{(g_1,\ldots,g_n)} (g_1 t_1, \tfrac{g_2}{g_1} t_2, \ldots, \tfrac{g_n}{g_{n-1}} t_n, \tfrac{1}{g_n} t_{n-1}).$$

This action extends to $C'_{[[t_1,\ldots,t_{n+1}]]}$ such that for every closed point $q \in \operatorname{Spec}\mathbb{C}[[t_1, \ldots, t_{n+1}]]$ the stabilizer of q in $(\mathbb{C}^\times)^n$ coincides with $\operatorname{Aut}(C'_q/C_q)$. We set

$$\mathsf{Mdf}_n = [\mathbb{C}[[t_1, \ldots, t_{n+1}]]/(\mathbb{C}^\times)^n].$$

This is an algebraic S–stack that comes equipped with a curve $[C'_S/(\mathbb{C}^\times)^n]$ and the modifications of C_S over B that arise from S–maps $B \to \mathsf{Mdf}_n$ we call **local modifications of length** $\leq n$.

A **twisted modification of length** $\leq n$ **of** C_S **over** B is a twisted curve C'_B such that its coarse moduli space $\overline{C'_B}$ is a modification of length $\leq n$ of C_S over B. A twisted modification is **of order** k if the order of the stabilizer group of every twisted point has order exactly k. Similarly, a twisted modification is **of order** $\leq k$ if the order of the stabilizer of every twisted point has order $\leq k$. A **local twisted modification** C'_B is a twisted modification whose coarse moduli space $\overline{C'_B}$ is a local modification.

Let $\mathsf{Mdf}_n^{\mathrm{tw}}$ denote the functor that assigns to $B \to S$ the groupoid of twisted local modifications of C_S over B of length $\leq n$. Let $\mathsf{Mdf}_n^{\mathrm{tw},k} \subset \mathsf{Mdf}_n^{\mathrm{tw},\leq k}$ be the functors of twisted local modifications of order k and order $\leq k$, respectively.

2.5 Construction of the algebraic stack

Let $r = \mathrm{rk}(G)$. If C'_B is a twisted modification of length $\leq r$, then a G–bundle on C'_B is called **admissible** if the co–characters determining the equivariant structure at all nodes are linearly independent over $\mathbb{Q}$ and are given by a subset of $\{\eta_0, \ldots, \eta_r\}$.

Let B be an S–scheme. Define a groupoid $\mathcal{X}_G(C_S)$ over S–schemes by the assignment

$$\mathcal{X}(C_S)(B) = \Big\langle\; P_B \longrightarrow C'_B \longrightarrow C_B \;\Big\rangle$$

where C'_B is a twisted local modification of C_B and P_B is an admissible G–bundle on C_B. Isomorphisms are commutative diagrams

$$\begin{tikzcd}
P_B \arrow[r, "\cong"] \arrow[d] & Q_B \arrow[d] \\
C'_B \arrow[r, "\cong"] \arrow[dr] & C''_B \arrow[d] \\
 & C_B
\end{tikzcd}$$

For notational convenience we abbreviate $\mathcal{X}_G(C_S)(B)$ as $\mathcal{X}_G(B)$.

Theorem 2.7. The functor $\mathcal{X}_G = \mathcal{X}_G(C_S)$ is an algebraic stack locally of finite type. It contains $\mathcal{M}_G(C_S)$ and $\mathcal{M}_G(C_{S^*})$ as dense open substacks, and the complement of $\mathcal{M}_G(C_{S^*})$ is a divisor with normal crossings.

Theorem 2.8. Let $R = \mathbb{C}[[s]]$ and $K = \mathbb{C}((s))$; for a finite extension $K \to K'$, let R' denote the integral closure of R in K'. Given the right commutative square below, there is a finite extension $K \to K'$ and a dotted arrow making the entire diagram commute:

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r, "h^*"] \arrow[d] & \mathcal{X}_G(C_S) \arrow[d] \\
\operatorname{Spec} R' \arrow[r] \arrow[ur, dashed, "h"] & \operatorname{Spec} R \arrow[r, "f"'] & S
\end{tikzcd}$$

In particular, $\mathcal{X}_G(C_S)$ is complete over S.

3 Finiteness for Fixed Curves

Let G be a reductive, connected complex Lie group and $\mathcal{M}$ the moduli stack of algebraic G-bundles over a smooth projective curve Σ of genus g. We recall the finiteness theorem for this moduli stack. We recall the finiteness theorem for the moduli stack of principal bundles on a fixed smooth curve.

3.1 Admissible classes

Given a representation V of G, call E^*V the vector bundle over $\Sigma \times \mathcal{M}$ associated to the universal G-bundle. Call π the projection along Σ, the relative canonical bundle K of $\Sigma \times \mathcal{M} \to \mathcal{M}$ (so that $K|_\Sigma = K_\Sigma$), $\sqrt{K}$ its square root, $[C]$ the topological K_1-homology class of a 1-cycle C on Σ. Consider the following classes in the topological K-theory of $\mathcal{M}$:

(i) The restriction $E^*_x V \in K^0(\mathcal{M})$ of E^*V to a point $x \in \Sigma$;

(ii) The slant product $E^*_C V := E^*V/[C] \in K^{-1}(\mathcal{M})$ of E^*V with $[C]$;

(iii) The Dirac index bundle $E^*_\Sigma V := R\pi_*(E^*V \otimes \sqrt{K}) \in K^0(\mathcal{M})$ of E^*V along Σ;

(iv) The inverse determinant of cohomology,

$$D_\Sigma V := \det{}^{-1} E^*_\Sigma V.$$

We call the classes (i)-(iii) the **Atiyah-Bott generators**; they are introduced in [**?**, §2], along with their counterparts in cohomology, and can also be described from the Künneth decomposition of E^*V in

$$K^0(\Sigma \times \mathcal{M}) \;\cong\; K^0(\Sigma) \otimes K^0(\mathcal{M}) \;\oplus\; K^1(\Sigma) \otimes K^1(\mathcal{M}),$$

by contraction with the various classes in Σ. Classes (i) and (iv) are represented by algebraic vector bundles, while (iii) can be realised as a perfect complex of $\mathcal{O}$-modules. The class $E^*_C V$ in (ii) is not algebraic. Note that

$$\det E^*_\Sigma V = \det R\pi_*(E^*V)$$

when $\det V$ is trivial; an important example is the canonical bundle

$$\mathcal{K} = \det E^*_\Sigma \mathfrak{g}$$

of $\mathcal{M}$, defined from the adjoint representation $\mathfrak{g}$.

Remark 3.1. For a line bundle $\mathcal{L}$ on $\mathcal{M} = \mathrm{Bun}_G(\Sigma)$, one associates a **level** $\lambda(\mathcal{L})$, namely the invariant symmetric bilinear form on $\mathfrak{g}$ corresponding to the class $\lambda(\mathcal{L}) \in H^4(BG; \mathbb{Z})$. If $\mathcal{L}$ is a determinant line bundle $\det R\pi_*(E^*V)$ attached to a representation V of G, then $\lambda(\mathcal{L})$ is the trace form $\mathrm{Tr}_V(xy)$ on $\mathfrak{g}$. When G is not simply connected, such determinant bundles do not realise all possible integral levels. Passing from the simply connected cover $\widetilde{G}$ to $G = \widetilde{G}/Z$ cuts down the lattice of integral invariant bilinear forms by imposing congruence conditions along the finite central subgroup Z, so that only a finite–index sublattice is realised by trace forms of actual G–representations.

Remark 3.2 (Smoothness and the relative canonical bundle). Let $\mathcal{M} = \mathrm{Bun}_G(\Sigma)$ and let

$$\pi : \Sigma \times \mathcal{M} \longrightarrow \mathcal{M}$$

be the projection. Although the coarse moduli space of semistable G-bundles may be singular, the **stack** $\mathcal{M}$ is a smooth Artin stack of dimension $(g-1)\dim G$. Indeed, for a bundle P one has

$$T_{[P]}\mathcal{M} \simeq H^1(\Sigma, \mathrm{Ad}\, P)$$

and $H^2(\Sigma, \mathrm{Ad}\, P) = 0$ because $\dim \Sigma = 1$, so deformations are unobstructed.

The relative canonical bundle $K := K_{\Sigma\times\mathcal{M}/\mathcal{M}}$ is defined purely from the morphism π, which is smooth of relative dimension 1; no smoothness of the base is required. In fact,

$$K_{\Sigma\times\mathcal{M}/\mathcal{M}} \;\cong\; \mathrm{pr}_\Sigma^* K_\Sigma,$$

the pullback of the ordinary canonical bundle of the curve.

Remark 3.3. By contrast, the "canonical bundle" of the moduli stack itself is

$$\mathcal{K} := \det R\pi_*(E^*\mathfrak{g}),$$

the determinant of the cotangent complex of $\mathcal{M}$, and Laszlo–Sorger construct a canonical Pfaffian square root $\mathcal{K}^{1/2}$ of this line bundle. In particular, for semi-simple, not necessarily simply connected G and for every theta characteristic $K_\Sigma^{1/2}$ on Σ, one has a square root

$$\mathcal{K}^{1/2} := \det R\pi_*(E^*\mathfrak{g} \otimes \mathrm{pr}_\Sigma^* K_\Sigma^{1/2}).$$

This gives rise to a natural "reference level" $\lambda(\mathcal{K}^{1/2}) = \frac{1}{2}\,\lambda(\mathcal{K})$. We call a line bundle $\mathcal{L}$ on $\mathcal{M}$ **admissible** if its level exceeds that of $\mathcal{K}^{1/2}$, in the sense that $\lambda(\mathcal{L}) - \lambda(\mathcal{K}^{1/2})$ is positive definite on every simple factor of $\mathfrak{g}$.

Such positivity plays the role of an ampleness condition, and admissible line bundles provide the appropriate class of twistings needed for the K–theoretic index and Verlinde formulas. Products of an admissible line bundle and any number of Atiyah-Bott generators span the ring of **admissible classes**.

Remark 3.4. We have defined a level by an integral invariant symmetric bilinear form on $\mathfrak{g}$ and simultaneously identified with central extensions of the loop group LG. The latter is completely determined by the action of the central scalar, which is to be an integer by the integrality condition. Abstractly, the Chern-Weil homomorphism identifies the cohomology ring $H^*(BG;\mathbb{R})$ of the classifying space BG with the ring of invariant polynomials on the Lie algebra $\mathfrak{g}$ of G:

$$H^*(BG;\mathbb{R}) \cong \mathrm{Inv}(\mathfrak{g}) := \mathrm{Sym}(\mathfrak{g}^*)^G$$

and in degree four, we have

$$H^4(BG;\mathbb{R}) \cong \text{Inv}^2(\mathfrak{g})$$

the space of invariant symmetric bilinear forms on $\mathfrak{g}$. In particular $H^4(BG;\mathbb{R}) \cong H^3(\mathfrak{g})$ via the isomorphism we have just discussed. There is a transgression map arising from the fibration $G \to EG \to BG$:

$$\tau : H^4(BG;\mathbb{R}) \to H^3(G;\mathbb{R})$$

which is an isomorphism when G is compact, simple, and simply connected. Thus we have the chain of isomorphisms

$$H^4(BG;\mathbb{R}) \;\cong\; H^3(\mathfrak{g}) \;\cong\; H^3(G;\mathbb{R}) \;\cong\; H^2(L\mathfrak{g})$$

which identifies the level defined via $H^4(BG;\mathbb{R})$ with the level defined via central extensions of the loop group LG, all of which are classified by invariant symmetric bilinear forms on $\mathfrak{g}$.

In particular, central extensions of the loop algebra $L\mathfrak{g}$ are classified by invariant symmetric bilinear forms on $\mathfrak{g}$, which are classified by $H^3(\mathfrak{g})$ defined by the Chevalley-Eilenberg complex. Given such a form $\langle\ ,\ \rangle$, the associated 3-cocycle is

$$\sigma(\xi,\eta,\zeta) = \langle[\xi,\eta],\zeta\rangle.$$

Conversely, given a 3-cocycle σ on $\mathfrak{g}$, one can define an invariant symmetric bilinear form by

$$\langle\xi,\eta\rangle := \sigma(\xi,[\eta_1,\eta_2]),$$

where η_1,η_2 are any elements satisfying $\eta = [\eta_1,\eta_2]$ (such elements exist since $\mathfrak{g}$ is semisimple, and the definition is independent of the choice because σ is a cocycle). We have seen that invariant symmetric bilinear forms on $\mathfrak{g}$ classify central extensions of the loop algebra $L\mathfrak{g}$ via the construction which takes $\langle\ ,\ \rangle$ to the cocycle

$$\omega(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi}\langle\xi(\theta),\eta'(\theta)\rangle\,d\theta.$$

Moreover we have seen that any such cocycle ω arises from such a bilinear form. Thus we have an isomorphism

$$H^3(\mathfrak{g}) \;\stackrel{\cong}{\to}\; H^2(L\mathfrak{g})$$

On the other hand, if G is compact, then the de Rham cohomology $H^3(G)$ is isomorphic to the Lie algebra cohomology $H^3(\mathfrak{g})$. This is because every de Rham cohomology class has a unique left invariant representative form given by averaging, and therefore the cohomology of G can be calculated from the cochain complex of the Lie algebra $\mathfrak{g}$.

3.2 Levels of Line Bundles

To certain line bundles on $\mathcal{M}$ we now associate a **level**, a quadratic form on the Lie algebra $\mathfrak{g}$. Briefly, for any representation V, the level of $\det E^*_\Sigma V$ is the trace form $\xi,\eta \mapsto \mathrm{Tr}_V(\xi\eta)$, and we wish to extend this definition by linearity in the first Chern class of the line bundle.

Riemann–Roch along Σ expresses $c_1(E_\Sigma^* V)$ as the image of $\mathrm{ch}_2(V) = \frac{1}{2}c_1(V)^2 - c_2(V)$ under **transgression along** Σ,

$$\tau : H^4(BG;\mathbb{Q}) \longrightarrow H^2(\mathcal{M};\mathbb{Q}) \qquad \text{(construction (1.1.iii) in cohomology).}$$

It is important that τ is injective (Remark 4.11). We now identify $H^4(BG;\mathbb{R})$ with the space of invariant symmetric bilinear forms on $\mathfrak{g}_\kappa$ so that Tr_V corresponds to $\mathrm{ch}_2(V)$. We say that the line bundle $\mathcal{L}$ **has a level** if its Chern class $c_1(\mathcal{L})$ agrees with some $\tau(h)$ in $H^2(\mathcal{M};\mathbb{Q})$; the form h, called the **level** of $\mathcal{L}$, is then unique.

For SL_n, the level of the positive generator of $\mathrm{Pic}(\mathcal{M})$ is $-\mathrm{Tr}_{\mathbb{C}^n}$ in the standard representation; the calculation is due to Quillen. For another example, the level of $\mathcal{K}^{-1/2}$ is $c := -\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}}$. Positivity of a level refers to the quadratic form on $\mathfrak{g}_\kappa$; thus $D_\Sigma V$ has positive level iff V is $\mathfrak{g}$–faithful. Finally, $\mathcal{L}$, with level h, is **admissible** iff $h > -c$ as a quadratic form.

Remark 3.5 (Properties of levels). (i) When G is simply connected, the map $\tau : H^4(BG;\mathbb{Z}) \to H^2(\mathcal{M};\mathbb{Z})$ is an isomorphism, but this fails (even rationally) as soon as $\pi_1(G) \neq 0$. Line bundles with a level satisfy a prescribed relation between their Chern classes over the different components of $\mathcal{M}$; cf. (4.8).

(ii) The trace forms span the negative semi–definite cone in $H^4(BG;\mathbb{R})$; so $\mathcal{L}$ has positive level iff $c_1(\mathcal{L})$ lies in the $\mathbb{Q}_+$–span of the $c_1(D_\Sigma V)$'s for $\mathfrak{g}$–faithful V.

(iii) For semi–simple G, the line bundle $\mathcal{K}$ has negative level, and so $\mathcal{O}$ is admissible. This fails for a torus, but positive–level line bundles are admissible for any G.

(iv) For $g > 1$ and simply connected G, positivity of the level is equivalent to ampleness on the moduli space. (It suffices to check this for simple G: recall then that $\mathrm{Pic}(\mathcal{M}) = \mathbb{Z}$ and that $\mathcal{K}^{-1}$ is ample.) When $\pi_1(G) \neq 0$, the positive–level condition is much more restrictive.