REPRESENTATIONS OF COMPLEX TORI AND $\mathrm{GL}(2, \mathbb{C})$

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ABSTRACT. Groups and their representations have been studied for a long time. One can extend the notion of a group by asking for the group axioms to hold in other categories. A group in the category of smooth manifolds is a Lie group, and a group in the category of algebraic varieties is an algebraic group. In this paper, we discuss the representation theory of algebraic groups, in particular complex tori and $\mathrm{GL}(2, \mathbb{C})$.

1. Introduction

The theme of this expository paper is to compare and contrast group objects in the smooth and algebraic settings. In particular, we begin by discussing the representation theory of tori in the smooth setting, and from our discussion it will become clear that some tools of Lie theory are not available to us in the algebraic setting. We remedy this by introducing different tools. One such tool we will introduce is the notion of a Hopf algebra, which axiomatizes the structure of the coordinate ring of an algebraic group. With a clear understanding of what is and is not available to us, we then discuss the representation theory of $(\mathbb{C}^*)^n$ and $\mathrm{GL}(2, \mathbb{C})$ in the algebraic setting.

2. Representations of Tori

2.1. **Real tori.** In this section we study the representations of tori in the smooth setting. A real torus T is a real Lie group which is isomorphic to the product of n circles. We say that T has rank n.

Let us first consider the case of $T = S^1$. We want to classify the finite dimensional representations of S^1. It turns out that all finite dimensional representations of S^1 are decomposable, i.e. can be written as a direct sum of irreducible representations.

Proposition 2.1. *Let K be a compact Lie group and let $\rho : K \to \mathrm{GL}(V)$ be a finite dimensional complex representation. Then ρ is completely reducible.*

Sketch of proof. The idea is to replicate the proof of Maschke's Theorem for finite groups. Choose any inner product $\langle \cdot, \cdot \rangle$ on V and average over the group action to get a K-invariant inner product on V. In particular put

$$\langle v, w \rangle_{\mathrm{avg}} = \frac{1}{|K|} \int_K \langle \rho(k)v, \rho(k)w \rangle dk$$

The existence of this inner product allows us to conclude that the orthogonal complement of a K-invariant subspace is also K-invariant. Inducting on the dimension of V allows us to completely decompose V into irreducible representations. $\square$

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We refer the reader to chapter 9 of [4] for more detailed discussion.

Thus it is enough to just consider the irreducible representations of S^1. By Schur's Lemma (in particular S^1 is abelian) they are all 1-dimensional and therefore are indexed by characters $\chi : S^1 \to \mathbb{C}^*$. Since S^1 is compact, its image in $\mathbb{C}^*$ must also be compact, and moreover it is connected and contains the identity. Therefore, the image of χ must lie in S^1.

Proposition 2.2. *All characters of S^1 are isomorphic to $\chi_n : S^1 \to S^1$ given by $z \mapsto z^n$ for $n \in \mathbb{Z}$.*

Proof. One can prove by making use of the universal covering map $\exp : \mathbb{R} \to S^1$. Given a character $\chi : S^1 \to S^1$, we can lift it to a map $\tilde{\chi} : S^1 \to \mathbb{R}$. Since χ is a group homomorphism, it carries 1 to 1, and the fiber over 1 under exp is $\mathbb{Z}$. $\square$

Since characters for $S^1 \times \cdots \times S^1$ are the same as products of characters for S^1, all characters of T are indexed by $\mathbb{Z}^n$. Explicitly if T has rank n, then a character $\chi : T \to S^1$ is given by a tuple of integers $(n_1, \ldots, n_k)$, and the character is given by

$$(z_1, \ldots, z_k) \mapsto z_1^{n_1} \cdots z_k^{n_k}$$

From our discussion above, we have the following classification statement for representations of real tori as Lie groups.

Theorem 2.3. *Let T be a real torus of rank n. Then every finite dimensional representation V of T is isomorphic to a direct sum of 1-dimensional irreducible representations with some multiplicities.*

$$V \cong \bigoplus_{\chi \in \mathbb{Z}^n} W_\chi^{\oplus n_\chi}$$

where W_χ denotes the unique 1-dimensional irreducible representation for which T acts by χ.

In particular, we can decompose V into eigenspaces for the action of T.

$$V \cong \bigoplus_{\chi \in \mathbb{Z}^n} V_\chi$$

where $V_\chi = \{v \in V \mid t \cdot v = \chi(t)v \text{ for all } v \in V \text{ and } t \in T\}$. This is referred to as the **weight space decomposition** of V and we refer to the χ which appear in the decomposition as the **weights** of V. We say that $v \in V_\chi$ is a **weight vector** of weight χ.

2.2. **Complex tori.** We want an analagous story in algebraic geometry. To do so we establish the following framework.

Definition 2.4. An **algebraic group** G over $\mathbb{C}$ is an algebraic variety over $\mathbb{C}$ with a group structure so that the multipliation map $G \times G \to G$ and inversion map $G \to G$ are morphisms of algebraic varieties.

Definition 2.5. A **morphism of algebraic groups** $G \to H$ is a morphism of algebraic varieties that is also a group homomorphism.

Definition 2.6. Let G be an algebraic group. A **rational representation** of G is a morphism of algebraic groups $G \to \mathrm{GL}(V)$ for some vector space V (for us V will always be finite dimensional over $\mathbb{C}$).

We will consider complex tori $T = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$. This is an algebraic group because T is the zero locus of the polynomial equations

$$T = \operatorname{Spec} \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] := \operatorname{Spec} \left(\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1 y_1 - 1, \ldots, x_n y_n - 1) \right)$$

This is a group in the familiar way, and it is clear that the group law is indeed a morphism of algebraic varieties. The rest of this section will discuss the finite dimensional rational representations of T as an algebraic group.

Example 2.7. $\mathrm{GL}(n, \mathbb{C})$ is a familiar group which can be endowed with the structure of an algebraic group. $\mathrm{GL}(n, \mathbb{C})$ is the zero locus of the polynomial equations

$$\mathrm{GL}(n, \mathbb{C}) = \operatorname{Spec} \left(\mathbb{C}[x_{ij}, \det{}^{-1}] \right) := \operatorname{Spec} \left(\mathbb{C}[x_{ij}, t]/(\det(x_{ij})t - 1) \right)$$

This variety becomes a group in the familiar way, and it is clear that the group law is indeed a morphism of algebraic varieties.

The following theorem classifies the finite dimensional rational representations of T as an algebraic group. The story is precisely that of the smooth setting, but we introduce the language of Hopf algebras to argue the result.

Theorem 2.8. *Let T be a complex torus of rank n. Then every finite dimensional rational representation of T decomposes into the direct sum of weight spaces for the action of T.*

$$V \cong \bigoplus_{\chi \in \mathbb{Z}^n} V_\chi$$

where $V_\chi = \{v \in V \mid t \cdot v = \chi(t)v \text{ for all } v \in V\}$ and $\chi : T \to \mathrm{GL}(1, \mathbb{C})$ is identified with tuples of integers $(n_1, \ldots, n_k)$ via

$$\chi(z_1, \ldots, z_k) = z_1^{n_1} \cdots z_k^{n_k}$$

We will give an proof of this theorem after we introduce the language of Hopf algebras.

2.3. **Hopf algebras.** The notion of a Hopf algebra axiomatizes the structure of the ring of regular functions on an algebraic group. In particular let G be an algebraic group and $\mathcal{O}(G)$ its ring of regular functions. Then the multipliation, inversion, and identity maps

$$\begin{aligned} \mu : G \times G &\to G \\ \iota : G &\to G \\ e : \operatorname{Spec} \mathbb{C} &\to G \end{aligned}$$

induce maps on the coordinate rings

$$\begin{aligned} \Delta &: \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \\ \epsilon &: \mathcal{O}(G) \to \mathbb{C} \\ S &: \mathcal{O}(G) \to \mathcal{O}(G) \end{aligned}$$

where we made the identification $\mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$. Because the group axioms hold, these maps satisfy the following conditions and equip $\mathcal{O}(G)$ with the structure of a Hopf algebra.

Definition 2.9. Let A be a $\mathbb{C}$-algebra. Then we say A is a **Hopf algebra** if there are maps

$$\begin{aligned} \text{comultiplication} &\quad \Delta : A \to A \otimes A \\ \text{counit (augmentation)} &\quad \epsilon : A \to \mathbb{C} \\ \text{coinverse (antipode)} &\quad S : A \to A \end{aligned}$$

so that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \otimes A \arrow[d, "\Delta\otimes\mathrm{id}"] \\ A \otimes A \arrow[r, "\mathrm{id}\otimes\Delta"] & A \otimes A \otimes A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"] & A \otimes A \arrow[d, "\epsilon\otimes\mathrm{id}"] \\ A \arrow[r, "\cong"] & \mathbb{C} \otimes A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\epsilon"] & A \otimes A \arrow[d, "S\otimes\mathrm{id}"] \\ \mathbb{C} \arrow[r] & A \end{tikzcd}$$

Remark 2.10. These maps can be worked out very explicitly. In particular the points of G are in correspondence with the elements of $\mathrm{Hom}_{\mathrm{kAlg}}(\mathcal{O}(G), \mathbb{C})$. The correspondence is very explicitly given by $g \mapsto \mathrm{ev}_g$ where $\mathrm{ev}_g : \mathcal{O}(G) \to \mathbb{C}$ is the evaluation map. The key idea is that for an arbitrary algebraic group G and two points x, y of G, $f, g : \mathcal{O}(G) \to \mathbb{C}$ the corresponding morphisms of $\mathbb{C}$-algebras, then the composition $(f \otimes g) \circ \Delta$ is again a map $\mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathbb{C}$. The condition that we ask from comultiplication is precisely that the composition $(f \otimes g) \circ \Delta$ corresponds to the product $xy \in G$. In particular, the group law on G uniquely determines the comultiplication map on $\mathcal{O}(G)$.

Example 2.11. Recall that

$$\mathcal{O}(\mathfrak{G}_a(\mathbb{C})) = \mathbb{C}[x]$$

where $\mathfrak{G}_a(\mathbb{C})$ is the additive group of $\mathbb{C}$. Let $f, g \in \mathrm{Hom}_{\mathrm{kAlg}}(\mathcal{O}(G), \mathbb{C})$ with $f(x) = a$ and $g(x) = b$. We want to find a map $\Delta : \mathcal{O}(\mathfrak{G}_a(\mathbb{C})) \to \mathcal{O}(\mathfrak{G}_a(\mathbb{C})) \otimes \mathcal{O}(\mathfrak{G}_a(\mathbb{C}))$ so that

$$((f \otimes g) \circ \Delta)(X) = (a + b)$$

We write down the map Δ explicitly as

$$\Delta(X) = X \otimes 1 + 1 \otimes X$$

and notice that it does the job. We see that Δ then must be the comultiplication map for $\mathfrak{G}_a(\mathbb{C})$ because such a map is unique (see the above remark), given the prescribed group law on $\mathfrak{G}_a(\mathbb{C})$.

Example 2.12. By the same token, we can work out the Hopf algebra structure for $\mathbb{C}^*$ to be

$$\begin{aligned}\Delta(x) &= x \otimes x\\ \epsilon(x) &= 1\\ S(x) &= x^{-1}\end{aligned}$$

Example 2.13. Consider the example of $\mathrm{GL}(2,\mathbb{C})$ as an algebraic group. The Hopf algebra structure is given by

$$\begin{aligned}\Delta(x_{ij}) &= \sum_{k=1}^{2} x_{ik} \otimes x_{kj}\\ \epsilon(x_{ij}) &= \delta_{ij}\\ S(x_{ij}) &= M_{ij}\end{aligned}$$

where

$$M = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}^{-1} = \frac{1}{\det(M)} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}$$

Now we want to translate the representation theory of algebraic groups G into the language of comodules over Hopf algebras.

Theorem 2.14. *Let G be an algebraic group. Then rational representations V of G correspond to linear maps $\rho : V \to V \otimes \mathcal{O}(G)$ so that the following diagrams commute:*

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"] & V \otimes \mathcal{O}(G) \arrow[d, "\rho\otimes\mathrm{id}"] \\ V \otimes \mathcal{O}(G) \arrow[r, "\mathrm{id}\otimes\Delta"] & V \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\mathrm{id}"] & V \otimes \mathcal{O}(G) \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ V \arrow[r, "\cong"] & V \otimes \mathbb{C} \end{tikzcd}$$

Sketch of proof. The first diagram says that the action of G on V is associative and the second diagram says that $e \in G$ acts by the identity transformation on V. These are precisely the conditions that say that V is a representation of G. □

We refer the reader to [7] for a more detailed discussion of this theorem.

Definition 2.15. ρ is called a **comodule structure** on V.

Example 2.16. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by

$$t \cdot (a, b) = (ta, t^{-1}b)$$

This is a rational representation of $\mathbb{C}^*$ which we can write as

$$\begin{aligned}\tau : \mathbb{C}^* &\to \mathrm{GL}(2,\mathbb{C})\\ t &\mapsto \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\end{aligned}$$

This induces a comodule structure on $\mathbb{C}^2$ given by the map $\rho : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathcal{O}(\mathbb{C}^*)$ given by

$$\begin{aligned} \rho(a) &= a \otimes x \\ \rho(b) &= b \otimes x^{-1} \end{aligned}$$

where x is the coordinate function on $\mathbb{C}^*$.

2.4. **Weight space decomposition.** We are now ready to give a proof of Theorem 2.8 using the language of Hopf algebras.

Proof of 2.8. Let V be a finite dimensional rational representation of T and let $\rho : V \to V \otimes \mathcal{O}(T)$ be the corresponding comodule structure. Recall that

$$\mathcal{O}(T) \cong \mathbb{C}[x_1, \ldots, x_n, {x_1}^{-1}, \ldots, {x_n}^{-1}]$$

We write as a vector space decomposition

$$V \otimes \mathcal{O}(T) \cong \bigoplus_{m \in \mathbb{Z}^n} V \otimes \mathbb{C} \cdot x^m$$

Expanding $\rho(v)$ in terms of this basis, we find that

$$\begin{aligned} \rho(v) &= \sum_{m \in \mathbb{Z}^n} v_m \otimes x^m \quad \text{finitely many nonzero } m_k \\ \implies (\mathrm{id} \otimes \Delta)(\rho(v)) &= \sum_{m \in \mathbb{Z}^n} v_m \otimes x^m \otimes x^m \\ \implies (\rho \otimes \mathrm{id})(\rho(v)) &= \sum_{m \in \mathbb{Z}^n} \rho(v_m) \otimes x^m \\ \implies \rho(v_m) &= v_m \otimes x^m \quad \text{for those nonzero } v_m \end{aligned}$$

The second step comes from our computation that $\Delta(x_i) = x_i \otimes x_i$. and the fact that Δ is a coalgebra homomorphism. The claim that Δ is a morphism of coalgebras is not immediate, but it ultimately reduces to the statement that, if B is a k-algebra, then the multiplication map $B \otimes B \to B$ is a morphism of k-algebras if and only if B is commutative. We are working with (co)commutative (co)algebras, so this is not an issue. The fourth step comes from equating the second and third left hand sides.

Finally, we apply the second diagram in 2.9 to get

$$(\mathrm{id} \otimes \epsilon)(\rho(v)) = v = \sum_{m \in \mathbb{Z}^n} v_m \epsilon(x^m) = \sum_{m \in \mathbb{Z}^n} v_m$$

Thus we see that the comodule V decomposes as a direct sum of subcomodules

$$V = \bigoplus_{m \in \mathbb{Z}^n} V_m$$

where $V_m := \{v \in V \mid \rho(v) = v \otimes x^m\}$. This is precisely saying that T acts on V_m by the character $\chi_m : T \to \mathbb{C}^*$ given by $t \mapsto t^m$. $\square$

3. Representations of $\mathrm{GL}(2,\mathbb{C})$

3.1. **Reducibility.** We saw in Section 2 that every rational representation of T decomposes into a direct sum of irreducible representations, and that the irreducible representations are indexed by $\mathbb{Z}^n$.

It turns out that rational representations of $\mathrm{GL}(2,\mathbb{C})$ also decompose into a direct sum of irreducible representations. This is because we can apply Weyl's unitary trick again. We consider $U(2) \subset \mathrm{GL}(2,\mathbb{C})$, the subgroup of unitary matrices. This is a compact subgroup which is also Zariski dense in $\mathrm{GL}(2,\mathbb{C})$. We can then apply the same averaging trick to obtain an inner product on V which will actually be $\mathrm{GL}(2,\mathbb{C})$-invariant, because $U(2)$ is Zariski dense in $\mathrm{GL}(2,\mathbb{C})$.

We refer the reader to chapter 9 of [4] for more detailed discussion.

3.2. **Highest weight vectors.** To completely classify the rational representations of $\mathrm{GL}(2,\mathbb{C})$, we need to introduce highest weight vectors. Let $T \subset \mathrm{GL}(2,\mathbb{C})$ be the subgroup of diagonal matrices and $B \subset \mathrm{GL}(2,\mathbb{C})$ be the subgroup of upper triangular matrices. These ad hoc definitions will work for us, but in general T is a **maximal torus** and B is a **Borel subgroup** of $\mathrm{GL}(2,\mathbb{C})$.

Definition 3.1. Let V be a finite dimensional rational representation of $\mathrm{GL}(2,\mathbb{C})$. A **highest weight vector** $v \in V$ is a weight vector so that $B \cdot v = \mathbb{C}^* \cdot v$. A **highest weight** is a weight which corresponds to a highest weight vector.

Example 3.2. $\mathrm{GL}(2,\mathbb{C})$ has a standard representation on $\mathbb{C}^2$ given by the matrix multiplication map. This action is transitive on the nonzero vectors, so $\mathbb{C}^2$ is irreducible. If one considers the torus action $T \subset \mathrm{GL}(2,\mathbb{C})$ then we see that $\mathbb{C}^2$ decomposes into a direct sum of weight spaces

$$\mathbb{C}^2 \cong \mathbb{C} \cdot e_1 \oplus \mathbb{C} \cdot e_2$$

where e_1 and e_2 are standard basis with weights $(1,0)$ and $(0,1)$ respectively. $(1,0)$ is the unique highest weight, and the corresponding weight space is one dimensional. The standard representation of $\mathrm{GL}(2,\mathbb{C})$ is irreducible.

Example 3.3. Since $\mathrm{GL}(2,\mathbb{C})$ acts on $\mathbb{C}^2$, it also acts on $(\mathbb{C}^2)^{\otimes n}$ for $n \in \mathbb{Z}_{\geq 0}$ via

$$g \cdot (v_1 \otimes \cdots \otimes v_n) = (gv_1 \otimes \cdots \otimes gv_n)$$

This is known as the **tensor product representation** of $\mathrm{GL}(2,\mathbb{C})$. We can further quotient by the submodule generated by vectors of the form

$$v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n$$

for $1 \leq i \leq n-1$. This is known as the **symmetric power representation** of $\mathrm{GL}(2,\mathbb{C})$, denoted by $\mathrm{Sym}^n \mathbb{C}^2$. Choosing a basis e_1, e_2 for $\mathbb{C}^2$ gives a basis for $\mathrm{Sym}^n \mathbb{C}^2$ given by

$$\{e_1^k e_2^{n-k} \mid 0 \leq k \leq n\}$$

and the action of $\mathrm{GL}(2,\mathbb{C})$ on $\mathrm{Sym}^n \mathbb{C}^2$ is given by

$$g \cdot e_1^k e_2^{n-k} = (ge_1)^k (ge_2)^{n-k}$$

Setting $g \in T$ we see that $e_1^k e_2^{n-k}$ is a weight vector with weight $(k, n-k)$. One can quickly check that $\operatorname{Sym}^n \mathbb{C}^2$ is irreducible for all $n \in \mathbb{Z}_{\geq 0}$. One can also check that the highest weight vector is e_1^n and it has highest weight $(n, 0)$.

Example 3.4. We have a familiar 1-dimensional representation of $\mathrm{GL}(2, \mathbb{C})$ given by the determinant map. The determinant of a diagonal matrix is the product of its diagonal entries, and so this representaion has weight $(1, 1)$. We will denote the kth power of the determinant map by $\det^k$ for $k \in \mathbb{Z}$. This is a 1-dimensional representation with weight (k, k).

We are now ready to state the classification theorem for finite dimensional rational irreducible representations of $\mathrm{GL}(2, \mathbb{C})$.

Theorem 3.5. *Every finite dimensional rational irreducible representation of* $\mathrm{GL}(2, \mathbb{C})$ *is isomorphic to*

$$\operatorname{Sym}^n \mathbb{C}^2 \otimes \det{}^k$$

for some $n \in \mathbb{Z}_{\geq 0}$ *and* $k \in \mathbb{Z}$*.*

We will prove this theorem by considering the weights that appear in the weight space decomposition of $V|_T$, where $T \subset \mathrm{GL}(2, \mathbb{C})$ is the subgroup of diagonal matrices.

In particular we appeal to the following facts from representation theory, collectively referred to as the **theorems of the highest weight**.

Theorem 3.6.

(1) *A finite dimensional rational representation V of* $\mathrm{GL}(2, \mathbb{C})$ *is irreducible if and only if it has a unique highest weight vector. In this case it makes sense to talk about the highest weight of V, defined as the weight corresponding to the highest weight vector.*

(2) *Two finite dimensional rational irreducible representations of* $\mathrm{GL}(2, \mathbb{C})$ *are isomorphic if and only if they have the same highest weight.*

(3) *Let V be a finite dimensional irreducible rational representation of* $\mathrm{GL}(2, \mathbb{C})$ *with highest weight vector v. Then the highest weight of V is contained in the set*

$$\{(a, b) \in \mathbb{Z}^2 \mid a \geq b\}$$

(4) *Every such weight above is a highest weight for some irreducible representation of* $\mathrm{GL}(2, \mathbb{C})$*.*

We will give a short discussion of the proof of these theorems in the case of $\mathrm{GL}(n, \mathbb{C})$.

Remark 3.7. This theorem holds in great generality. Analagous statements are true for other algebraic groups such as $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SO}(n)$ and as well as representations of complex semisimple Lie algebras, but in order to make sense of such a theorem, one has to find the right notion of Borel subgroups and highest weight vector.

The proof in full generality is quite technical and we refer the reader to [6] for a more detailed discussion.

The theorems of the highest weight immediately imply the classification theorem for finite dimensional rational irreducible representations of $\mathrm{GL}(2, \mathbb{C})$. In particular let V be a finite dimensional rational irreducible representation of $\mathrm{GL}(2,\mathbb{C})$ with highest weight (a,b). Then by looking at the highest weights (observe that if v is a weight vector for V with weight μ and w is a weight vector for W with weight ν, then $v \otimes w$ is a weight vector for $V \otimes W$ with weight $\mu + \nu$) , we see that $V \cong \mathrm{Sym}^{a-b}\,\mathbb{C}^2 \otimes \det{}^b$.

4. Theorems of the highest weight

In this section we will discuss some aspects of Theorem 3.6 in the case of $\mathrm{GL}(2,\mathbb{C})$, in both the smooth setting and the algebraic setting.

4.1. **The smooth setting.** The exposition in this section follows Chapter 8 of [3]. One of the main ingredients in the proof of 3.6 is considering the induced action of $\mathfrak{gl}(2,\mathbb{C})$ on V, where $\mathfrak{gl}(2,\mathbb{C})$ is the Lie algebra of $\mathrm{GL}(2,\mathbb{C})$. Recall that

$$\mathfrak{gl}(2,\mathbb{C}) = \mathrm{Mat}(2,\mathbb{C})$$

is a vector space equipped with a bracket operation given by the commutator. $\mathfrak{gl}(2,\mathbb{C})$ can be identified with the tangent space of $\mathrm{GL}(2,\mathbb{C})$ at the identity matrix. We can then consider the action of $\mathfrak{gl}(2,\mathbb{C})$ on V given by

$$X \cdot v = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot v$$

where $\exp : \mathfrak{gl}(2,\mathbb{C}) \to \mathrm{GL}(2,\mathbb{C})$ is the exponential map. In particular this map is the differential of the action of $\mathrm{GL}(2,\mathbb{C})$ on V.

Studying the action of $\mathfrak{gl}(2,\mathbb{C})$ on V is equivalent to studying the action of $\mathrm{GL}(2,\mathbb{C})$ on V because $\mathrm{GL}(2,\mathbb{C})$ is simply connected. This is a general principal which reflects the fact that any map of Lie groups $G \to H$ with G simply connected is determined by its differential at the identity. Then one can show the following lemma:

Lemma 4.1. *A subspace W of a representation of $\mathrm{GL}(2,\mathbb{C})$ is a subrepresentation if and only if W stable under the action of $\mathfrak{gl}(2,\mathbb{C})$.*

We refer the reader to Chapter 3 of [2] for a proof of this lemma. This discussion justifies our passing from the study of $\mathrm{GL}(2,\mathbb{C})$ to the study of $\mathfrak{gl}(2,\mathbb{C})$.

Just as we obtained a decomposition of V as a $\mathrm{GL}(2,\mathbb{C})$ into eigenspaces for the action of T, there is an analagous decomposition for the action of $\mathfrak{gl}(2,\mathbb{C})$. The object which replaces our maximal torus $T \subset \mathrm{GL}(2,\mathbb{C})$ is the **Cartan subalgebra** $\mathfrak{h} \subset \mathfrak{gl}(2,\mathbb{C})$. For us, $\mathfrak{h}$ will be the subspace of diagonal matrices in $\mathfrak{gl}(2,\mathbb{C})$. In general, $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{gl}(2,\mathbb{C})$.

We can obtain a decomposition of V into eigenspaces for the action of $\mathfrak{h}$:

$$V = \bigoplus_{\chi \in \mathfrak{h}^*} V_\chi$$

where $V_\chi = \{v \in V \mid X \cdot v = \chi(X)v \text{ for all } X \in \mathfrak{h}\}$. Moreover $\mathfrak{gl}(2,\mathbb{C})$ acts on itself via the bracket (adjoint representation) and we can decompose this action

$$\begin{aligned} \mathfrak{gl}(2,\mathbb{C}) &\cong \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \\ &\cong \mathfrak{h} \oplus \mathbb{C}e \oplus \mathbb{C}f \end{aligned}$$

where $\alpha(\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}) = d_1 - d_2$ and

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

One can check that for all $h \in \mathfrak{h}$ we have

$$[h, e] = \alpha(h)e$$
$$[h, f] = -\alpha(h)f$$

The weights which appear in the adjoint representation of $\mathfrak{gl}(2, \mathbb{C})$ are called the **roots**. We will say α is a **positive root** and $-\alpha$ is a **negative root**, $\mathbb{C}e$ and $\mathbb{C}f$ the corresponding positive and negative root spaces. Then a weight vector $v \in V$ is a highest weight vector if and only if $e \cdot v = 0$.

We care about roots of the adjoint representation for the following reason. Let V be a finite dimensional rational representation of $\mathfrak{gl}(2, \mathbb{C})$. Decompose V into eigenspaces for the action of $\mathfrak{h}$ as before:

$$V = \bigoplus_{\chi \in \mathfrak{h}^*} V_\chi$$

Knowing how $\mathfrak{h}$ acts on V, we now need to investigate the action of e and f. As it turns out, e and f are operators which translate between the weight spaces. Specifically, let v be an weight vector for the action of $\mathfrak{h}$ with weight χ. Then $e \cdot v$ is a weight vector with weight $\chi + \alpha$. Indeed for $X \in \mathfrak{h}$ we have (recall the action of the Lie algebra respects brackets)

$$\begin{aligned} X \cdot ev &= e \cdot Xv + [X, e] \cdot v \\ &= \chi(X)ev + \alpha(X)ev \end{aligned}$$

A priori, we know nothing about the weights of V. Now we know that all of the weights of V are translates of each other by the roots of $\mathfrak{gl}(2, \mathbb{C})$. Now let μ be any weight which appears in the decomposition of V. Then we can consider the translates

$$\mu + \mathbb{Z}\alpha$$

and since V is finite dimensional, only finitely many of the weight spaces of V corresponding to these weights are nonzero. Recall we picked a positive system, so now it makes to talk about the highest weight (it is the weight χ so that all of the weights $\chi + \mathbb{N}\alpha$ correspond to empty weight spaces).

If V is an irreducible representation then a highest weight vector must span its root space. This is because if v is a highest weight vector, then one can show that the subspace generated by v, $f \cdot v$, $f^2 \cdot v$, $\ldots$ is a subrepresentation. It follows that an irreducible representation can have only one highest weight vector (up to scale).

4.2. **The algebraic setting.** In order to justify the passage from GL$(2, \mathbb{C})$ to $\mathfrak{gl}(2, \mathbb{C})$, we made use of the exponential map. This is not available in the category of varieties. However we can still make sense of the Lie algebra of an algebraic group and the induced action of the Lie algebra on a vector space. To do so we need to pass to the Zariski tangent space of a variety.

Definition 4.2. Let A be a local ring and $\mathfrak{m}$ its maximal ideal. The residue field k of A is the field $A/\mathfrak{m}$ and the **Zariski cotangent space** of A is the k-vector space $\mathfrak{m}/\mathfrak{m}^2$. The **Zariski tangent space** of A is the dual vector space $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. If X is a variety and $p \in X$, then we define the **Zariski tangent space of** X at p to be the Zariski tangent space of $\mathcal{O}_{X,p}$.

To make sense of this definition, we need to borrow a little motivation from the theory of differentiable manifolds. If M is a smooth manifold, tangent vectors at a point $p \in M$ are in one-to-one correspondence with derivations of the ring of germs of smooth functions at p, i.e. $\mathbb{R}$-linear maps $\mathcal{O}_{M,p} \to \mathbb{R}$ which satisfy the Leibniz rule

$$D(fg) = f(p)Dg + g(p)Df$$

for all $f, g \in \mathcal{O}_{M,p}$. We refer to chapter 3 of [5] for a more detailed discussion of this point of view.

Proposition 4.3. *Let X be a variety over a field k and let $p \in X$. Consider the local ring $\mathcal{O}_{X,x}$ and its maximal ideal $\mathfrak{m}$. Let $k(p)$ be the residue field of $\mathcal{O}_{X,p}$. It coincides with k. There is an isomorphism*

$$\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) \cong \mathrm{Der}_k(\mathcal{O}_{X,p}, k(x))$$

Proof. A derivation is precisely the data of a k-linear map $\mathfrak{m} \to k$ which satisfies the Leibniz rule. This extends to a k-linear map $\mathcal{O}_{X,p} \to k$ by precomposing with $f \mapsto f - f(p)$. Moreover $\mathfrak{m}^2$ maps to zero because if $f(p) = g(p) = 0$ then

$$D(fg) = f(p)Dg + g(p)Df = 0$$

Therefore, a derivation induces an element of the tangent space of X at p.

Conversely if we have a k-linear map $\mathfrak{m}/\mathfrak{m}^2 \to k$, precompose with the quotient map to get $D : \mathfrak{m} \to k$. Then we have to show that D satisfies the Leibniz rule. This is a straightforward computation. Let $f, g \in \mathcal{O}_{X,p}$. Then $(f - f(p))(g - g(p)) \in \mathfrak{m}^2$ and so

$$\begin{aligned} 0 = D((f - f(p))(g - g(p))) &= D(fg - f(p)g - fg(p) + f(p)g(p)) \\ \implies D(fg) &= f(p)Dg + g(p)Df \end{aligned}$$

since constants derive to zero and so D is a derivation. It is clear that these two maps are inverses of each other. □

Now we can make sense of the Lie algebra of an algebraic group.

Definition 4.4. Let G be an algebraic group. The **Lie algebra** of G, denoted $\mathfrak{g}$ is the Zariski tangent space of G at the identity.

Note that if $\alpha : G \to W$ is a morphism of varieties, then there is an induced map $\mathcal{O}(W) \to \mathcal{O}(G)$ on coordinate rings, and this map is local in the sense that $\mathcal{O}_{W,\alpha(p)} \to \mathcal{O}_{G,p}$ is a local ring homomorphism for all $p \in G$. Geometrically this is saying that if a regular function on W vanishes at a point $\alpha(p)$, then its pullback to G vanishes at p.

In particular we see that a morphism of varieties α has a differential $d\alpha$ which takes a derivation $D : \mathcal{O}_{W,\alpha(p)} \to k$ to a derivation $d\alpha(D) : \mathcal{O}_{G,p} \to k$. Letting $W = \mathrm{GL}(V)$ we see that the differential of the action of G on V gives us a Lie algebra representation (in the sense that it respects the bracket) of $\mathfrak{g}$ on V.

Then again one proves that Lemma 4.1 holds in the algebraic setting and so we have reduced to the study of the action of $\mathfrak{g}$ on V. A reference for this proof can be found in Chapter 1 of [1]. We can then proceed as in the smooth setting to prove Theorem 3.6.

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