

Autoequivalences from GIT

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Abstract

Notes for a talk on autoequivalences arising from variation of GIT quotients, given in the low dimensional gauge theory seminar at UC Berkeley.

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1 Motivation

Homological mirror symmetry predicts that in certain cases, derived categories of coherent sheaves on an algebraic variety should admit twist autoequivalences corresponding to a spherical object.

1. A-model: symplectic automorphisms and generalized Dehn twists along Lagrangian spheres
2. B-model: derived autoequivalences such as spherical twists

We want to consider natural constructions of twist autoequivalences of $D^b(X)$. Techniques come from variation of GIT (VGIT) and the general theory of spherical functors and semiorthogonal decompositions.

2 Refresher on GIT

Consider a graded noetherian algebra over \mathbb{C} :

$$R = \bigoplus_{m=0}^{\infty} R_m$$

The variety $X = \text{Proj } R$ is projective over the affine variety $\text{Spec } R_0$, and comes equipped with an ample line bundle $\mathcal{L} = \mathcal{O}_X(1)$.

Let G be a reductive algebraic group acting on R by graded algebra automorphisms. Then G acts on X and \mathcal{L} is a G -linearized ample line bundle. We can form the GIT quotient

$$X//G := \text{Proj}(R^G)$$

The invariant algebra is finitely generated (Hilbert, Nagata).

The inclusion $R^G \subset R$ induces a rational map

$$X \dashrightarrow X//G.$$

Definition 2.1. A point $x \in X$ is semistable if it is in the domain of definition of the above rational map. A point $x \in X$ is stable if its orbit is closed in the semistable locus and its stabilizer is finite. Denote the semistable locus by X^{ss} and the stable locus by X^s .

The map $X^{ss} \rightarrow X//G$ is the GIT quotient.

2.1 Chambers

Fix an ample line bundle L on X . The space of G -linearizations of L has a wall-chamber structure; crossing a wall changes the GIT quotient by a birational map.

Theorem 2.2. The real character space $X^*(G)_{\mathbb{R}}$ is cut into finitely many chambers by rational walls; for characters in the same chamber the GIT quotient $X^{ss}(L, \chi) \rightarrow X//_{L, \chi} G$ is constant, and crossing a wall induces a birational modification of the quotient.

It is also true that there are only finitely many distinct GIT quotients $X//_L G$ up to isomorphism as L ranges over all G -ample linearizations.

Restriction gives an exact dg-functor $i^* : D^b(X/G) \rightarrow D^b(X^{ss}/G)$. One important fact that we will see later is that there is a functorial splitting.

3 The standard flop

We do an example following Segal. Let $V = \mathbb{C}_{x_1, x_2, y_1, y_2}^4$ and \mathbb{C}^* act on V with weight $(1, 1, -1, -1)$.

There are two possible GIT quotients X_+ and X_- , depending on whether we choose a positive or negative character of \mathbb{C}^* . Both are isomorphic to the total space of the bundle $\mathcal{O}(-1)^{\oplus 2}$ over \mathbb{P}^1 .

Open substacks of the stack $\mathcal{X} = [V/\mathbb{C}^*]$ given by the semistable loci. Let $\iota_{\pm} : X_{\pm} \rightarrow \mathcal{X}$ be the open immersions. The restriction functors $\iota_{\pm}^* : D^b(\mathcal{X}) \rightarrow D^b(X_{\pm})$ are exact.

Let $\mathcal{G}_t = \langle \mathcal{O}(t), \mathcal{O}(t+1) \rangle \subset D^b(\mathcal{X})$. It is called a window subcategory.

Claim 3.1. For any $t \in \mathbb{Z}$, both ι_+^* and ι_-^* restrict to give equivalences

$$D^b(X_+) \xleftarrow{\sim} \mathcal{G}_t \xrightarrow{\sim} D^b(X_-).$$

Proof. Exactness, preserves shifts and cones, are clear. To check fully faithfulness, it is enough to show that $H_{\mathcal{X}}^{\bullet}(\mathcal{O}(i)) = H_{X_{\pm}}^{\bullet}(\mathcal{O}(i))$ for $i \in [-1, 1]$.

Left hand side: V affine, so $H^p(\mathcal{X}, \mathcal{O}(i)) = (\mathcal{O}_V)_i$ for $p = 0$ and 0 for $p > 0$.

Right hand side: We do the computation for X_+ . Let projection $\pi : X_+ \rightarrow \mathbb{P}^1$. Recall X_+ is the total space of the bundle $E = \mathcal{O}(-1)^{\oplus 2}$ over \mathbb{P}^1_x .

By the projection formula and affineness of π

$$H^p(X_+, \mathcal{O}_{X_+}(k)) \cong H^p(\mathbb{P}^1, \pi_* \mathcal{O}_{X_+} \otimes \mathcal{O}(k)) \cong \bigoplus_{m \geq 0} H^p(\mathbb{P}^1, \mathcal{O}(k+m))^{\oplus (m+1)}$$

When $p = 0$, this has global sections

$$\mathrm{Sym}^{k+m}(\mathbb{C}_{x_1, x_2}^2) \otimes \mathrm{Sym}^m(\mathbb{C}_{y_1, y_2}^2)$$

which is exactly the degree k piece of \mathcal{O}_V . When $p > 0$, this is zero for $k = -1, 0, 1$.

For $p = 1$, recall $k \in -1, 0, 1$ and $m \geq 0$, so $k+m \geq -1$. Thus $H^1(\mathbb{P}^1, \mathcal{O}(k+m)) = 0$. This agrees with the left hand side. Note that if the size of the window is bigger, then we would pick up some H^1 terms.

Essential surjectivity follows from exactness of

$$\pi^* : \mathrm{Coh}(\mathbb{P}^1) \rightarrow \mathrm{Coh}(X_{\pm})$$

and the fact that $D^b(\mathbb{P}^1)$ is generated by $\mathcal{O}_{\mathbb{P}^1}(t), \mathcal{O}_{\mathbb{P}^1}(t+1)$ by Beilinson's theorem. \square

So for any $t \in \mathbb{Z}$ we have a derived equivalence

$$\Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_-)$$

passing through \mathcal{G}_t . Composing these, we get auto-equivalences

$$\Phi_{t+1}^{-1} \Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_+).$$

To see what these do, we need to check them on the generating set of line-bundles $\{\mathcal{O}(t), \mathcal{O}(t+1)\}$.

Consider the Koszul resolution resolving the structure sheaf of the unstable locus $\{y_1 = y_2 = 0\}$:

$$0 \rightarrow \mathcal{O}_V(2) \xrightarrow{(y_2, -y_1)} \mathcal{O}_V(1)^{\oplus 2} \xrightarrow{(y_1, y_2)} \mathcal{O}_V \rightarrow \mathcal{O}_V/\{y_1 = y_2 = 0\} \rightarrow 0$$

Restricting the sequence to X^- , the resolution becomes exact at the end since the unstable locus $\{y_1 = y_2 = 0\}$ is removed in X^- . Thus on X^- we have a quasi-isomorphism:

$$\mathcal{O}_{X^-}(k) \simeq [\mathcal{O}_{X^-}(k+2) \xrightarrow{(y_2, -y_1)} \mathcal{O}_{X^-}(k+1)^{\oplus 2}]$$

Thus we compute:

$$\begin{aligned} \Phi_{t+1}^{-1} \Phi_t(\mathcal{O}_{X_+}(t)) &\simeq \Phi_{t+1}^{-1}(\mathcal{O}_{X_-}(t)) \\ &\simeq \Phi_{t+1}^{-1}([\mathcal{O}_{X_-}(t+2) \xrightarrow{(y_2, -y_1)} \mathcal{O}_{X_-}(t+1)^{\oplus 2}]) \\ &\simeq [\mathcal{O}_{X_+}(t+2) \xrightarrow{(y_2, -y_1)} \mathcal{O}_{X_+}(t+1)^{\oplus 2}], \\ \Phi_{t+1}^{-1} \Phi_t(\mathcal{O}_{X_+}(t+1)) &\simeq \Phi_{t+1}^{-1}(\mathcal{O}_{X_-}(t+1)) \\ &\simeq \mathcal{O}_{X_+}(t+1). \end{aligned}$$

This autoequivalence $\Phi_{t+1}^{-1} \Phi_t$ is an example of a spherical twist.

Definition 3.2. A spherical object S in a dg-category \mathcal{C} is an object such that

$$\text{Ext}(S, S) = \mathbb{C} \oplus \mathbb{C}[-n]$$

for some n . The spherical twist around S sends any object \mathcal{E} to the cone on the evaluation map

$$\text{Cone}(\text{RHom}(S, \mathcal{E}) \otimes S \rightarrow \mathcal{E})$$

Claim 3.3. The object $\mathcal{O}_{\mathbb{P}_{x_1:x_2}^1}(t)$ is spherical for the derived category $D^b(X_+)$, and the inverse twist around it sends $\mathcal{O}(t+1)$ to itself and $\mathcal{O}(t)$ to the two-term complex

$$[\mathcal{O}(t+2) \xrightarrow{(-y_2, y_1)} \mathcal{O}(t+1)^{\oplus 2}],$$

which agrees with $\Phi_{t+1}^{-1} \Phi_t$.

4 Autoequivalences from VGIT

The above example by Segal formally introduced windows to the mathematics literature and showed that window shift equivalences are given by spherical functors for gauged LG models. Note it was done for linear action of \mathfrak{G}_m , and the window was identified in an ad-hoc way.

We outline a more general treatment by Halpern-Leistner and Shipman. First a remark about the general strategy. Suppose X_+ and X_- are a pair of varieties related by a flop and that both arise as GIT quotients of a larger space V by a reductive group G .

$$X_{\pm} = [V^{ss}(\chi_{\pm})/G] \subset \mathcal{X} := [V/G]$$

so X_{\pm} are open substacks of \mathcal{X} . There are exact restriction functors

$$\iota_{\pm}^* : D^b(\mathcal{X}) \longrightarrow D^b(X_{\pm})$$

Try to construct an equivalence $D^b(X_+) \simeq D^b(X_-)$ by finding a single triangulated subcategory window $\mathcal{W} \subset D^b(\mathcal{X})$ which restricts isomorphically to both sides. the functors ι_{\pm}^* induce equivalences

$$\mathcal{W} \xrightarrow{\sim} D^b(X_+), \quad \mathcal{W} \xrightarrow{\sim} D^b(X_-),$$

and composing these gives the desired derived equivalence $D^b(X_+) \xrightarrow{\sim} D^b(X_-)$. Choosing different windows and composing the resulting equivalences gives autoequivalences of $D^b(X_{\pm})$.

The main contribution of their paper is showing that there is a functorial splitting of the restriction functor

$$i^* : D^b(X/G) \rightarrow D^b(X^{ss}/G)$$

and that the window categories arise naturally via the semiorthogonal decompositions.

4.1 Application to VGIT

X smooth projective over affine, G reductive acting on X , \mathcal{L} a G -linearized ample line bundle on X .

Pick a W invariant inner product on the cocharacter lattice of G .

Definition 4.1 (Hilbert–Mumford weight). For a point $x \in X$ and a one-parameter subgroup $\lambda : \mathfrak{G}_m \rightarrow G$ such that the limit

$$x_0 := \lim_{t \rightarrow 0} \lambda(t) \cdot x$$

exists in X , the weight of the action of λ on the fiber \mathcal{L}_{x_0} is called the **Hilbert–Mumford weight** of x with respect to λ and denoted $\mu^{\mathcal{L}}(x, \lambda)$.

The Hilbert–Mumford numerical criterion states that x is L -semistable if and only if $\mu^L(x, \lambda) \leq 0$ for every 1-PS λ of G . Thus if x is unstable, there exists some λ with $\mu^L(x, \lambda) > 0$.

For an unstable point x consider the *normalized instability*

$$M(x) := \sup_{\lambda \neq 0} \frac{\mu^L(x, \lambda)}{\|\lambda\|}.$$

Theorem 4.2 (Kempf, Kempf–Ness). For every unstable point $x \in X^{us}$, the supremum $M(x)$ is attained by some 1-PS λ_x , unique up to positive rescaling of λ_x . conjugation by the associated parabolic subgroup

$$P(\lambda_x) := \{ g \in G \mid \lim_{t \rightarrow 0} \lambda_x(t) g \lambda_x(t)^{-1} \text{ exists in } G \}$$

Let β run over the set of such equivalence classes of optimal 1-PS's. Define

$$S_\beta := \{ x \in X^{us} \mid \text{the optimal 1-PS for } x \text{ is of type } \beta \}.$$

Theorem 4.3 (Kirwan–Ness stratification). Each S_β is a locally closed G -invariant subvariety of X , and the unstable locus decomposes as a disjoint union

$$X^{us} = \bigsqcup_{\beta} S_\beta.$$

Moreover, one can order the indices so that

$$\overline{S_i} \subset \bigcup_{j \geq i} S_j,$$

The subvarieties S_i are called the **Kirwan–Ness strata**. They provide a canonical G -equivariant stratification of the unstable locus X^{us} .

Each stratum comes with a distinguished one-parameter subgroup $\lambda_i : \mathbb{C}^* \rightarrow G$ and S_i fits into the diagram

$$Z_i \begin{array}{c} \xleftarrow{\sigma_i} \\ \xrightarrow{\pi_i} \end{array} Y_i \subset S_i := G \cdot Y_i \xrightarrow{j_i} X \quad (1)$$

where Z_i is an open subvariety of $X^{\lambda_i\text{-fixed}}$, characterized by

$$Z = \{ z \in X^\lambda \mid \text{KN type of } z \text{ is } [\lambda], \text{ and } z \notin \overline{S'} \text{ for any more unstable } S' \}.$$

and Y_i is the “attracting slice” of the stratum S_i

$$Y_i = \left\{ x \in X - \bigcup_{j > i} S_j \mid \lim_{t \rightarrow 0} \lambda_i(t) \cdot x \in Z_i \right\}.$$

The maps σ_i and j_i are the inclusions and π_i is taking the limit under the flow of λ_i as $t \rightarrow 0$. We denote the immersion $Z_i \rightarrow X$ by σ_i as well.

Example 4.4 (Kirwan–Ness stratification for binary quartics). Let $X = \mathbb{P}(\text{Sym}^4 \mathbb{C}^2)$ be the space of binary quartic forms with the standard SL_2 -action. A quartic is unstable precisely when it has a root

of multiplicity ≥ 3 , and the unstable locus decomposes into two Kirwan–Ness strata, the *most unstable* stratum

$$\begin{aligned} S_{(4)} &= \{ \text{quartics with a quadruple root} \} = G \cdot Y_{(4)} \\ &\supset Y_{(4)} = \{ \text{quartics with quadruple root at } x \text{ or } y \} \supset Z_{(4)} = \{x^4, y^4\} \end{aligned}$$

and the *next most unstable* stratum

$$\begin{aligned} S_{(3,1)} &= \{ \text{quartics with a triple root} \} = G \cdot Y_{(3,1)} \\ &\supset Y_{(3,1)} = \{ \text{quartics with triple root at } x \text{ or } y \} \supset Z_{(3,1)} = \{x^3y, xy^3\} \end{aligned}$$

They both have the one-parameter subgroup

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \lambda(t) \cdot (x^{4-i}y^i) = t^{4-2i} x^{4-i}y^i,$$

We are now ready to relate the equivariant derived category of X to that of the GIT quotient.

Theorem 4.5 (HL15). Let η_i be the weight of $\det(N_{S_i}^\vee X)|_{Z_i}$ with respect to λ_i . Choose an integer w_i for each stratum and define the full subcategory

$$\mathcal{G}_w := \{ F^\bullet \in D^b(X/G) \mid \forall i, \sigma_i^* F^\bullet \text{ has weights in } [w_i, w_i + \eta_i) \text{ w.r.t. } \lambda_i \}.$$

Then the restriction functor

$$r : \mathcal{G}_w \longrightarrow D^b(X^{ss}/G)$$

is an equivalence of dg-categories.

Now we introduced balanced GIT wall crossings. Suppose L is on a wall and $L_\pm = L \pm \epsilon L'$ are linearizations on either side of the wall for sufficiently small $\epsilon > 0$ and some ample line bundle L' .

In this case, $X^{ss}(L_0) - X^{ss}(L_\pm)$ is a union of KN strata for L_\pm .

Definition 4.6. The wall crossing is *balanced* if the strata S_i^+ and S_i^- lying in $X^{ss}(L_0)$ are indexed by the same set, with $Z_i^+ = Z_i^-$ and $\lambda_i^+ = (\lambda_i^-)^{-1}$.

If G is abelian and there is some linearization with a stable point, then all codimension one wall crossings are balanced.

In this situation, HL15 implies that the restriction functors

$$r_\pm : \mathcal{G}_w \longrightarrow D^b(X_\pm^{ss}/G)$$

are equivalences. In particular we get a family of derived equivalences

$$\Phi_w := r_- \circ r_+^{-1} : D^b(X_+^{ss}/G) \longrightarrow D^b(X_-^{ss}/G)$$

and the dependence on the choice of integer w gives a family of autoequivalences (the window shift functors)

$$\Phi_{w+1}^{-1} \circ \Phi_w : D^b(X_+^{ss}/G) \longrightarrow D^b(X_+^{ss}/G).$$

which are induced by the mutations

$$\begin{array}{ccc} \mathcal{G}_{w+1}^+ & \xrightarrow{\mathbb{L}_{\mathcal{A}_w^+}} & \mathcal{G}_w^+ \\ \uparrow r_-^R = r_-^{-1} & & \downarrow r_- \\ D^b(X_-^{ss}/G) & \xrightarrow{\Phi_w} & D^b(X_-^{ss}/G) \end{array}$$

Finally, Halpern-Leistner shows that the window shift autoequivalence is a twist corresponding to a spherical functor.

By describing window shifts both in terms of mutations and as spherical twists, we see why these two operations have the “same formula” in this setting.

Theorem 4.7 (HL-S16). If \mathcal{C} is a pretriangulated dg category and

$$\mathcal{C} = \langle \mathcal{A}, \mathcal{G} \rangle$$

is fixed by the braid action acting by mutation, then the autoequivalence of \mathcal{G} induced by mutation is the twist T_S corresponding to a spherical functor $S : \mathcal{A} \rightarrow \mathcal{G}$. Conversely, if $S : \mathcal{A} \rightarrow \mathcal{B}$ is a spherical functor, then there is a larger category \mathcal{C} admitting a semiorthogonal decomposition fixed by this braid which recovers S and T_S .

In the context of a balanced GIT wall crossing, the category \mathcal{C} arises naturally as a subcategory of the equivariant category $D^b(X/G)$, defined in terms of a “grade restriction rule.” The resulting autoequivalence agrees with the window shift Φ_w and corresponds to a spherical functor

$$f_w : D^b(Z/L)_w \longrightarrow D^b(X^{ss}/G),$$

where Z/L is the “critical locus” of the VGIT, which is unstable in both quotients.

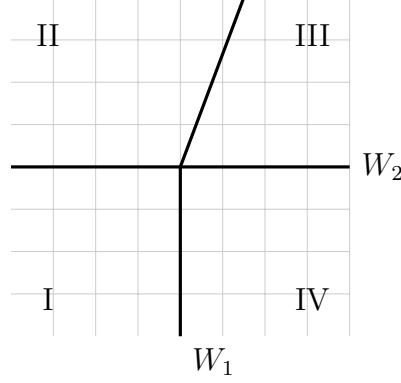
Remark 4.8. In the standard flop example, the window $\mathcal{G}_t \subset D^b(\mathcal{X})$ is generated by the exceptional collection $\{\mathcal{O}(t), \mathcal{O}(t+1)\}$, and the spherical functor is given by the spherical object $\mathcal{O}_{\mathbb{P}^1}(t)$, and the stack Z/L is just $[\text{pt}/\mathbb{C}^*]$.

Example 4.9 (Window shifts on a K3 surface). Let

$$X \subset \mathbb{P}_x^2 \times \mathbb{P}_y^2$$

be a K3 surface cut out by a divisor of bidegree $(2, 0)$ and a divisor of bidegree $(1, 3)$. Line bundles on a K3 surface are spherical objects, so any autoequivalence which can be written in terms of such line bundles is automatically a composition of spherical twists.

The example is realized inside a VGIT picture as follows.



- Let

$$\mathcal{V} = \mathcal{O}_{\mathbb{P}_x^2 \times \mathbb{P}_y^2}(-2, 0) \oplus \mathcal{O}_{\mathbb{P}_x^2 \times \mathbb{P}_y^2}(-1, -3),$$

and consider $\text{tot}(\mathcal{V})$ as a toric variety given as a GIT quotient of \mathbb{A}^8 by a torus $T \cong (\mathbb{C}^*)^2$ with weight matrix $(t, s) \mapsto (t, t, t, s, s, s, t^{-2}, t^{-1}s^{-3})$.

- One introduces a Landau-Ginzburg potential

$$W = pf + qg \in \mathbb{C}[x_i, y_j, p, q]_{\deg=2},$$

where f has bidegree $(2, 0)$ and g has bidegree $(1, 3)$, and f cuts out a smooth rational curve on \mathbb{P}_x^2 . The LG pair (\mathcal{V}, W) carries a second \mathbb{C}^* -grading (the LG grading, "R-charge") so that the variables p, q have weight 2 and the x_i, y_j have weight 0, so that W has weight 2; the associated category $D^b(\mathcal{V}, W)$ is equivalent to $D^b(X)$.

The key point is that, although Z_i/L_i is *non-compact* as a usual GIT quotient, the restriction $W|_{Z_i}$ makes the LG quotient $(Z_i/L_i, W|_{Z_i})$ effectively compact. Concretely:

- Near W_1 one has $Z_1/L_1 \cong \mathbb{P}^2/\mathbb{C}^*$; the LG category $D^b(Z_1/L_1, W|_{Z_1})$ is equivalent to $D^b(\mathbb{P}^2)$, which admits the full exceptional collection $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$.
- Near W_2 one has $Z_2/L_2 \cong \text{tot } \mathcal{O}_{\mathbb{P}^2}(-2)/\mathbb{C}^*$; the potential restricts to pf . Then $D^b(Z_2/L_2, W|_{Z_2}) \simeq D^b(C)$, and for each window this is equivalent to $D^b(\mathbb{P}^1)$ with its exceptional pair $\langle \mathcal{O}, \mathcal{O}(1) \rangle$.

- One finds:

- the window shift across W_1 is a composition of spherical twists around the line bundles

$$\mathcal{O}_X(0, i), \mathcal{O}_X(0, i+1), \mathcal{O}_X(0, i+2),$$

- the window shift across W_2 is a composition of spherical twists around

$$\mathcal{O}_X(i, 0), \mathcal{O}_X(i+1, 0),$$

for suitable integers i (depending on the chosen windows).

References

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