Homework 1

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September 17, 2025

Problem 1 Let $K = \mathbb{Q}$, $L = \mathbb{Q}(\zeta_p)$, where ζ_p is a primitive p-th root of unity. Set $A = \mathbb{Z}$. Let B be the integral closure of A in L.

1. Prove that
$$(p) = \prod_{i=1}^{p-1}(1 - \zeta_p^i).$$

2. Show that $(p) = (1 - \zeta_p)^{p-1}$ as ideals of B. Deduce that (p) is totally ramified in $\mathbb{Q}(\zeta_p)/\mathbb{Q}$.

Problem 2 Keep using the notation from Problem 1.

1. For all positive integers i, prove that
$$B = \mathbb{Z}[\zeta_p] + (1 - \zeta_p)^i B.$$

2. Show that
$$p^m B \subset \mathbb{Z}[\zeta_p]$$
for some positive integer m.

3. Conclude from (1) and (2) that $B = \mathbb{Z}[\zeta_p]$.

Problem 3 Let d be a square-free number (positive or negative) such that $d \neq 1$ and $d \equiv 1 \pmod 4$. Give a numerical condition for each rational prime p to be split, inert, or ramified in $\mathbb{Q}(\sqrt{d})$.

Problem 4 Let A be a Dedekind domain and K its fraction field. Show that the following two sets are in bijection:

1. The set of nonzero prime ideals $\mathfrak{p}$ of A.

2. The set of discrete valuations v on K which have nonnegative values on A,

via $\mathfrak{p} \mapsto v_{\mathfrak{p}}$ and $v \mapsto \mathfrak{p}_v := \{a \in A : v(a) > 0\}$.