# Complex Manifolds

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**Abstract**

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Let $\mathrm{Gr}(k, n)$ denote the Grassmannian of $k$-planes in $V$, a real or complex vector space of dimension $n$. We will not distinguish since the statements we make will be true for both cases. For the real Grassmannian, we take homology with $\mathbb{Z}/2$ coefficients, and for the complex Grassmannian, we take homology with $\mathbb{Z}$ coefficients.

**Theorem 1.1.** For a CW complex $X$, $\mathbb{R}, \mathbb{C}, \mathbb{H}$-vector bundles over $X$ up to isomorphism are classified by homotopy classes of maps to the infinite Grassmannian $Gr_k(\mathbb{C}^\infty) = \lim_{n\to\infty} Gr_k(\mathbb{C}^n)$.

**Remark 1.2.** $\mathrm{Sp}(n) = U(2n) \cap GL(n, \mathbb{H})$ is the compact symplectic group acts on $\mathbb{H}^n$ by left multiplication and commutes with the right $\mathbb{H}$-action which is the structure map for $\mathbb{H}$-vector bundles.

Recall that the Schubert stratification is given by row echelon form for matrices. Let $T \subset V$ be a subspace of dimension $k$. Write $T$ as the row space of a $k \times n$ matrix. We can perform row operations to put it in row echelon form, and this depends only on $T$. In particular, $T$ has a canonical basis but it does not depend continuously on $T$. It may jump whenever the pivots move. The Schubert stratification is given by the locus where the pivots are fixed. Therefore, it is enough to give an increasing sequence $\sigma_p$ of pivot positions or a sequence $s$ of 0's and 1's of length $n$ with exactly $k$ 1's.

**Example 1.3.** Let us row reduce from right to left. For example in $\mathrm{Gr}(2, 4)$

$$\begin{bmatrix} * & 0 & * & 1 \\ * & 1 & 0 & 0 \end{bmatrix}$$

corresponds to the sequence $s = (0, 1, 0, 1)$ and the pivot positions $\sigma_1 = 2, \sigma_2 = 4$. The dimension is given by

$$\dim S_\sigma = \sum_{p=1}^{k}(\sigma_p - p) = (2-1) + (4-2) = 3 = \sum_{1s\in s} \text{total number of 0s to the left} = 1 + 2$$

There are two inclusions called $i, j$ of $\mathrm{Gr}(k, n)$ into $\mathrm{Gr}(k, n+1)$ and $\mathrm{Gr}(k+1, n+1)$ given by $T \mapsto T \oplus e_{n+1}$ and $T \mapsto e_0 \oplus T$ respectively.

## 1.1 Some generalities

These cells define a CW decomposition. They give a basis of $\mathbb{Z}$ homology for $\mathbb{C}, \mathbb{H}$ or $\mathbb{Z}/2$ homology for $\mathbb{R}$. It is obvious for $\mathbb{C}, \mathbb{H}$ because there are no attaching maps, but for $\mathbb{R}$ you have to argue that the attaching maps have even degree. The closure of $S_\sigma$ is the union of cells $S_\tau$ if $\tau_p \leq \sigma_p$ for all $p$< meaning no pivot of $\sigma$ is to the left of the corresponding pivot of $\tau$. Poincare duality is implemented by reverseing the sequence of $0s$ and $1s$, reverse the order of the basis and check the intersectino of the cells. Most $\overline{S_\sigma}$ are singular but the singularities have codimension at least 2 and thusd efine fundamental classes and can use the intersection pairing to check poincare duality. In particular they are normal varieties.

## 1.2 The inclusions $i$,$j$

What are the smallest cells $i, j$ are missing? For $i$ we are missing

$$\left[F_k | M | e_0\right]$$

where $F_k$ is the antidiagonal identity, $M$ is stars in the top row and 0s elsewhere, and $e_0$ is $1, 0, \ldots, 0$ in the last column. This corresponds to the sequence $s = 11\ldots10\ldots01$ and is of dimension $n - k$.

For $j$ we are missing

$$\left[* | F_{k+1} | 0\right]$$

corresponds to the sequence $01\ldots10\ldots0$ and is of dimension $k + 1$.

**Corollary 1.4** (Approximation theorem)**.** If $\dim X < n - k$ or $\dim X < 2(n-k)$ in the complex case, then a map $X \to \mathrm{Gr}_{\mathbb{R}}(k, n+1)$ or $X \to \mathrm{Gr}_{\mathbb{C}}(k, n+1)$ can be homotoped to a map $X \to \mathrm{Gr}(k, n)$. In particular $\mathrm{Gr}(k, n+1)/\mathrm{Gr}(k, n)$ is $(n-k-1)$-connected or $(2(n-k)-1)$-connected in the complex case.

If $\dim X < k + 1$ or $\dim X < 2(k+1)$ in the complex case, then a map $X \to \mathrm{Gr}_{\mathbb{R}}(k+1, n+1)$ or $X \to \mathrm{Gr}_{\mathbb{C}}(k+1, n+1)$ can be homotoped to a map $X \to \mathrm{Gr}(k, n)$. In particular $\mathrm{Gr}(k+1, n+1)/\mathrm{Gr}(k, n)$ is $k$-connected or $2k + 1$-connected in the complex case.

**Corollary 1.5** (Stability for classification of vector bundles)**.** If $\dim X < n-k-1$ or $\dim X < 2(n-k)-1$ in the complex case, then $\mathrm{Vect}^k(X) = [X, \mathrm{Gr}(k,n)]$.

Note that we lost one dimension because we need to leave room for the homotopy.

**Corollary 1.6.** If $\dim X < k+1$ or $\dim X < 2(k+1)$ in the complex case, then any vector bundle of rank $k+1$ has a nonvanishing section.

This also follows from transversality.

### 1.3 Stiefel manifodls and classifing spaces

Recall that $T^k \to \mathrm{Gr}(k,n)$ is the tautological subbundle of $\mathbb{R}^n$ or $\mathbb{C}^n$. We have the frame bundle $\mathrm{St}(k,n)$ given by basis of $k$-planes. Gram Schmidt tells us that orthonormal frames $\hookrightarrow$ frames is a homotopy equivalence so we have $\mathrm{St}(k,n) \to \mathrm{Gr}(k,n)$ is a principal $GL_k$-bundle and $\mathrm{St}^{on}(k,n) \to \mathrm{Gr}(k,n)$ is a principal $O(k)$-bundle or $U(k)$-bundle.

What is the codimension of the complement of $\mathrm{St}(k,n)$ in $\mathrm{Mat}_{k,n}$? Well $\mathrm{Mat}(k,n)$ is stratified by rank and the largest stratum in the complement of $\mathrm{St}(k,n)$ is given by rank $k-1$ matrices which has codimension $n-k+1$.

**Corollary 1.7.** If $\dim X < n-k$ or $\dim X < 2(n-k)+1$ then $[X, \mathrm{St}(k,n)] = *$. In particiular $\mathrm{St}(k,\infty)$ and $\mathrm{St}^{on}(k,\infty)$ are contractible.

Thus we have stumbled upon the fact that $\mathrm{St}^{on}(k,\infty)$ is a contractible space with a free $O(k)$ or $U(k)$ action and $\mathrm{Gr}(k,\infty)$ is the quotient.

**Definition 1.8.** Let $G$ be a topological group which is nice in the sense that it has the homotopy type of a CW complex. A CW complex with a principal $G$ bundle $EG \to BG$ with contractible total space is called a **classifing space** for $G$.

**Theorem 1.9.** *$EG \to BG$ is unique up to homology equivalence, canonical up to homotopy if you choose the base point $*$ in $BG$ and identify the fiber over $*$ with $G$. Moreover, isomorphism classes of principal $G$-bundles over a CW complex $X, *$ with trivialization over $*$ are in bijection with homotopy classes of pointed maps $[X, BG]$.*

In fact (2) implies (1) because we have identified $BG$ as a functor out of the homotopy category of pointed CW complexes, and the functor is unique up to natural isomorphism.

**Remark 1.10.** *$G$ is connected if and only if $BG$ is simply connected. Extreme cases are $G$ is discrete, then $EG$ is the universal cover of $BG$.*

**Remark 1.11.** $G$ discrete, $X$ is connected. Then $[X, BG]$ is the set of homomorphisms $\pi_1(X) \to G$ up to conjugation, whereas $[X, BG]_* = \mathrm{Hom}(\pi_1(X), G)$. The first is isomorphism classes of principal $G$-bundles, the second is isomorphism classes of principal $G$-bundles with a trivialization at the base point.

Next we are have two models of $BU(k) \to BU(k+1)$. One is the obvious inclusion, and the other model gives it as a fiber bundle with fiber $S^{2k+1}$.