Loop groups

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Abstract

These are reading notes for the book "Loop Groups" by Pressley and Segal.

Contents

1 Introduction

Definition 1.1 (Infinite dimensional Lie groups)**.** *An* ***infinite dimensional Lie group*** *is a group Γ which is at the same time an infinite dimensional smooth manifold, and is such that the composition law $\Gamma \times \Gamma \to \Gamma$ and the operation of inversion $\Gamma \to \Gamma$ are given by smooth maps. The tangent space to Γ at the identity element is its Lie algebra, the bracket being defined by identifying tangent vectors at the identity element with left-invariant vector fields on Γ. If for each element ξ of the Lie algebra there is a unique one-parameter subgroup*

$$\gamma_\xi : \mathbb{R} \to \Gamma$$

such that $\gamma'_\xi(0) = \xi$, then the exponential map is defined. This is the case in all known examples.

Example 1.2. *The simplest example of an infinite dimensional Lie group is the group $\mathrm{Map}_{\mathrm{cts}}(X; G)$ of all continuous maps from a compact space X to a finite dimensional Lie group G. (The group law, of course, is pointwise composition in G.) The natural topology on $\mathrm{Map}_{\mathrm{cts}}(X; G)$ is the topology of uniform convergence. We see that it is a smooth manifold as follows.*

If U is an open neighbourhood of the identity element in G which is homeomorphic by the exponential map to an open set $\tilde{U}$ of the Lie algebra $\mathfrak{g}$ of G, then

$$\mathcal{U} = \mathrm{Map}_{\mathrm{cts}}(X; U)$$

is an open neighbourhood of the identity in $\mathrm{Map}_{\mathrm{cts}}(X; G)$ which is homeomorphic to the open set

$$\tilde{\mathcal{U}} = \mathrm{Map}_{\mathrm{cts}}(X; \tilde{U})$$

of the Banach space $\mathrm{Map}_{\mathrm{cts}}(X; \mathfrak{g})$. If f is any element of $\mathrm{Map}_{\mathrm{cts}}(X; G)$, then

$$\mathcal{U}_f = \mathcal{U} \cdot f$$

is a neighbourhood of f which is also homeomorphic to $\tilde{\mathcal{U}}$. The sets $\mathcal{U}_f$ provide an atlas which makes $\mathrm{Map}_{\mathrm{cts}}(X; G)$ into a smooth manifold, and in fact into a Lie group: there is no difficulty at all in checking that the transition functions are smooth, or that multiplication and inversion are smooth maps.

Definition 1.3 (Loop groups)**.** *Suppose now that X is a finite dimensional compact smooth manifold, and let $\mathrm{Map}(X; G)$ denote the group of* ***smooth*** *maps $X \to G$. The case we are primarily interested in is when X is the circle S^1; then $\mathrm{Map}(X; G)$ is the* ***loop group*** *of G, which is denoted by LG. We shall think of the circle as consisting interchangeably of real numbers θ modulo 2π or of complex numbers $z = e^{i\theta}$ of modulus one.*

Fix once and for all G a compact connected Lie group. A fundamental property of the loop group LG is the existence of interesting central extensions

$$\mathbb{T} \ \to \ \widetilde{LG} \ \to \ LG$$

of LG by the circle $\mathbb{T}$. (In other words, $\widetilde{LG}$ is a group containing $\mathbb{T}$ in its centre and such that the quotient group $\widetilde{LG}/\mathbb{T}$ is LG.)

The $\widetilde{LG}$ are analogous to the finite-sheeted covering groups of a finite dimensional Lie group, in that any projective unitary representation of LG comes from a genuine representation of some $\widetilde{LG}$. We recall that a projective unitary representation of a group L on a Hilbert space H is the assignment to each $\lambda \in L$ of a unitary operator $U_\lambda : H \to H$ so that

$$U_\lambda U_{\lambda'} = c(\lambda, \lambda') U_{\lambda\lambda'}$$

holds for all $\lambda, \lambda' \in L$, where $c(\lambda, \lambda')$ is a complex number of modulus 1. $c : L \times L \to \mathbb{T}$ is called the *projective multiplier* or *cocycle* of the representation.

As topological spaces the $\widetilde{LG}$ are fibre bundles over LG with the circle as fibre. Except for the product extension $LG \times \mathbb{T}$ they are non-trivial fibre bundles: that is to say $\widetilde{LG}$ is not homeomorphic to the cartesian product $LG \times \mathbb{T}$, and there is no continuous cross-section $LG \to \widetilde{LG}$. In fact the group extension $\widetilde{LG}$ is completely determined by its topological type as a fibre bundle, and every circle bundle on LG can be made into a group extension. It is interesting that the behaviour of $\mathrm{Map}(X; G)$ when $\dim(X) > 1$ is completely different. There are often non-trivial circle bundles on $\mathrm{Map}(X; G)$, but if X is simply connected only the flat ones can be made into groups.

When G is a simple and simply connected group, there is a universal central extension among the $\widetilde{LG}$, i.e. one of which all the others are quotient groups. This is analogous to the universal covering group of a finite dimensional group. Any central extension E of LG by any abelian group A arises from the universal extension $\widetilde{LG}$ by a homomorphism $\mathbb{T} \to A$.

$\theta : \mathbb{T} \to A$, in the sense that

$$E = \widetilde{LG} \times_{\mathbb{T}} A.$$

(The last notation denotes the quotient group of $\widetilde{LG} \times A$ by the subgroup consisting of all elements

$$\{(z, -\theta(z)) : z \in \mathbb{T}\}.$$

) When G is simply connected but not simple there is still a universal central extension, but, as we shall see, it is an extension of LG by the homology group $H_3(G; \mathbb{T})$, a torus whose dimension is the number of simple factors in G.

It is worth noticing that the central extensions of LG are closely related to its natural affine action on the space of *connections* in the trivial principal G-bundle on the circle. (See (4.3.3).)

1.1 The Lie algebra extensions

On the level of Lie algebras the extensions can be defined and classified very simply: they correspond precisely to invariant symmetric bilinear forms on $\mathfrak{g}$. As a vector space

$$\widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R},$$

and the bracket is given by

$$[(\xi, \lambda), (\eta, \mu)] = ([\xi, \eta], \omega(\xi, \eta)) \tag{1}$$

for $\xi, \eta \in L\mathfrak{g}$ and $\lambda, \mu \in \mathbb{R}$, where $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ is the bilinear map

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle \, d\theta \tag{2}$$

and $\langle\ ,\ \rangle$ is a symmetric invariant form on the Lie algebra $\mathfrak{g}$. Recall that if $\mathfrak{g}$ is semisimple then every invariant bilinear form on $\mathfrak{g}$ is symmetric.

Remark 1.4. *Notice that the bracket* (1) *does not depend on the value of λ or μ. In other words, the central $\mathbb{R}$ commutes with everything in $\widetilde{L\mathfrak{g}}$.*

For the formula (1) to define a Lie algebra, ω must be skew— which is clear by integrating by parts in (2)—and must satisfy the 'cocycle condition'

$$\omega([\xi,\eta],\zeta)+\omega([\eta,\zeta],\xi)+\omega([\zeta,\xi],\eta)=0. \tag{3}$$

This condition follows from the Jacobi identity in the Lie algebra $L\mathfrak{g}$ and the fact that the inner product on $\mathfrak{g}$ is invariant:

$$\langle[\xi,\eta],\zeta\rangle=\langle\xi,[\eta,\zeta]\rangle.$$

There are essentially no other cocycles on $L\mathfrak{g}$ than the ω given by (2). To make this precise, notice that ω is invariant under conjugation by constant loops, i.e. $\omega(\xi,\eta)=\omega(g\xi,g\eta)$ for $g\in G$, where $g\xi,g\eta$ are the adjoint action of g on ξ,η.

Remark 1.5. *We elaborate a little on the invariance of ω under the adjoint action of G. Recall that for a Lie algebra $\mathfrak{a}$ with trivial coefficients, a 2-cocycle is a bilinear form $\omega:\mathfrak{a}\times\mathfrak{a}\to\mathbb{R}$ that is skew-symmetric and satisfies the cocycle condition*

$$\delta\omega(\xi,\eta,\zeta)=\omega([\xi,\eta],\zeta)+\omega([\eta,\zeta],\xi)+\omega([\zeta,\xi],\eta)=0.$$

On the loop algebra $L\mathfrak{g}$, the group G (constant loops) acts by conjugation:

$$(g\cdot\xi)(\theta)=\mathrm{Ad}_g\xi(\theta).$$

If we push forward a cocycle ω by g, we get a new cocycle

$$(g\cdot\omega)(\xi,\eta)=\omega(g^{-1}\cdot\xi,\ g^{-1}\cdot\eta).$$

To see that this transformation preserves the cohomology class, we can pass to the infinitesimal adjoint action. In particular, for ζ in the Lie algebra $\mathfrak{g}$, it is enough to show that

$$[\omega]=[\omega+(\zeta\cdot\omega)]\quad \textit{in } H^2.$$

where the infinitesimal action is given by

$$(\zeta\cdot\omega)(\xi,\eta)\ :=\ \frac{d}{dt}\bigg|_{t=0}\big(\exp(t\zeta)\cdot\omega\big)(\xi,\eta).$$

Use $\mathrm{Ad}_{\exp(-t\zeta)}=\exp(-t\ \mathrm{ad}\,\zeta)=\mathrm{id}-t\ \mathrm{ad}\,\zeta+o(t)$. *Then*

$$\begin{aligned}(\exp(t\zeta)\cdot\omega)(\xi,\eta)&=\omega\Big((\mathrm{id}-t\ \mathrm{ad}\,\zeta)\xi,\ (\mathrm{id}-t\ \mathrm{ad}_\zeta)\eta\Big)+o(t)\\&=\omega(\xi,\eta)\ -\ t\,\omega([\zeta,\xi],\eta)\ -\ t\,\omega(\xi,[\zeta,\eta])\ +\ o(t).\end{aligned}$$

Differentiating at $t=0$ gives

$$(\zeta\cdot\omega)(\xi,\eta)=-\omega([\zeta,\xi],\eta)-\omega(\xi,[\zeta,\eta])$$

Define the 1*-cochain* ϕ_ζ *by*

$$\phi_\zeta(\xi) := \omega(\zeta, \xi).$$

With trivial coefficients, the Chevalley–Eilenberg differential on a 1*-cochain is*

$$(\delta\phi_\zeta)(\xi, \eta) = -\,\phi_\zeta([\xi, \eta]) = -\,\omega(\zeta, [\xi, \eta]).$$

Now compare $(\zeta \cdot \omega)$ *with* $\delta\phi_\zeta$*:*

$$(\zeta \cdot \omega)(\xi, \eta) - (\delta\phi_\zeta)(\xi, \eta) = -\omega([\zeta, \xi], \eta) - \omega(\xi, [\zeta, \eta]) + \omega(\zeta, [\xi, \eta]).$$

Use the 2*-cocycle identity (cyclic sum zero):*

$$\omega([\zeta, \xi], \eta) + \omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) = 0.$$

Rewrite the last two terms:

$$\omega(\zeta, [\xi, \eta]) = -\,\omega([\xi, \eta], \zeta), \qquad \omega(\xi, [\zeta, \eta]) = -\,\omega([\zeta, \eta], \xi).$$

Plugging these into the difference gives exactly the negative of the cyclic sum above, hence zero:

$$-\omega([\zeta, \xi], \eta) - \omega(\xi, [\zeta, \eta]) + \omega(\zeta, [\xi, \eta]) = 0.$$

Therefore,

$$(\zeta \cdot \omega) = \delta\phi_\zeta$$

is a coboundary, and $[\omega] = [\omega + (\zeta \cdot \omega)]$ *in* $H^2(L\mathfrak{g}; \mathbb{R})$*.*

So the extension defined by α is also given by the invariant cocycle

$$\int_G g \cdot \alpha \, dg$$

obtained by averaging α over the compact group G because they are in the same cohomology class in $H^2(L\mathfrak{g}; \mathbb{R})$. Therefore, every cocycle of $L\mathfrak{g}$ is equivalent in cohomology to a G-invariant cocycle. The cocycle identity (3) expresses precisely that the cohomology class of the cocycle does not change under an infinitesimal conjugation.

Proposition 1.6 (Invariant cocycles)**.** *If* $\mathfrak{g}$ *is semisimple then the only continuous* G*-invariant cocycles on the Lie algebra* $L\mathfrak{g}$ *are those given by* (2)*.*

Proof. A cocycle $\alpha : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ extends to a complex bilinear map $\alpha : L\mathfrak{g}_{\mathbb{C}} \times L\mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$. An element $\xi \in L\mathfrak{g}_{\mathbb{C}}$ can be expanded in a Fourier series $\sum \xi_k z^k$, with $\xi_k \in \mathfrak{g}_{\mathbb{C}}$. By continuity α is completely determined by its values on elements of the form $\xi_p z^p$. Let us write

$$\alpha_{p,q}(\xi, \eta) = \alpha(\xi z^p, \eta z^q), \quad \xi, \eta \in \mathfrak{g}_{\mathbb{C}}.$$

Then $\alpha_{p,q}$ is a G-invariant bilinear map $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$, which is necessarily symmetric, and $\alpha_{p,q} = -\alpha_{q,p}$. The cocycle identity (3) translates into the statement

$$\alpha_{p+q,r} + \alpha_{q+r,p} + \alpha_{r+p,q} = 0 \tag{4}$$

for all p, q, r. Putting $q = r = 0$ we find $\alpha_{p,0} = 0$ for all p.

$r = -p - q$ we find

$$\alpha_{p+q,-p-q} = \alpha_{p,-p} + \alpha_{q,-q},$$

whence

$$\alpha_{p,-p} = p\, \alpha_{1,-1}.$$

Putting $r = n - p - q$ in (4.2.5) we find

$$\alpha_{n-p-q,p+q} = \alpha_{n-p,p} + \alpha_{n-q,q},$$

whence

$$\alpha_{n-k,k} = k\alpha_{n-1,1}.$$

This implies that $\alpha_{p,q} = 0$ if $p + q \neq 0$, for

$$n\alpha_{n-1,1} = \alpha_{0,n} = 0.$$

Returning to $\xi = \sum \xi_p z^p$ and $\eta = \sum \eta_q z^q$, we have

$$\alpha(\xi, \eta) = \sum p\, \alpha_{1,-1}(\xi_p, \eta_{-p}) = \frac{i}{2\pi} \int_0^{2\pi} \alpha_{1,-1}(\xi(\theta), \eta'(\theta))\, d\theta,$$

which is of the form (4.2.2). $\square$

Proposition 1.6 determines the universal central extension of $L\mathfrak{g}$. We can reformulate it in the following way. For any finite dimensional Lie algebra $\mathfrak{g}$ there is a universal invariant symmetric bilinear form

$$\langle\ ,\ \rangle_K : \mathfrak{g} \times \mathfrak{g} \to K \tag{5}$$

from which every $\mathbb{R}$-valued form arises by a unique linear map $K \to \mathbb{R}$.

The cocycle ω_K given by

$$\omega_K(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle_K\, d\theta \tag{6}$$

defines an extension of $L\mathfrak{g}$ by K, which by Proposition (4.2.4) is the universal central extension of $L\mathfrak{g}$ when $\mathfrak{g}$ is semisimple. For semisimple groups K can be identified with $H^3(\mathfrak{g}; \mathbb{R})$, because a bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ gives rise to an invariant skew 3-form

$$(\xi, \eta, \zeta) \mapsto \langle \xi, [\eta, \zeta] \rangle,$$

and all elements of $H^3(\mathfrak{g}; \mathbb{R})$ are so obtained. When $\mathfrak{g}$ is simple then $K = \mathbb{R}$.

Remark 1.7. *If $\mathfrak{g}$ is semisimple, then every invariant symmetric bilinear form on $\mathfrak{g}$ is a multiple of the Killing form. So in that case, the cocycle*

$$\omega(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi(\theta), \eta'(\theta)\rangle \, d\theta$$

is unique up to scalar.

But in general (if $\mathfrak{g}$ is not simple), the space of invariant symmetric bilinear forms on $\mathfrak{g}$ may have higher dimension. So instead of fixing one $\langle\ ,\ \rangle$, introduce the universal bilinear form:

$$\langle\ ,\ \rangle_K : \mathfrak{g}\times\mathfrak{g} \ \to\ K,$$

where K is a vector space that "records all possible invariant bilinear forms at once."

Concretely: $K =$ (space of invariant bilinear forms on $\mathfrak{g}$)*. Then for any actual $\mathbb{R}$-valued invariant form β, there is a unique linear functional $f : K \to \mathbb{R}$ such that

$$\beta(x,y) = f(\langle x,y\rangle_K).$$

Using this universal bilinear form, we define a universal cocycle:

$$\omega_K(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi(\theta), \eta'(\theta)\rangle_K \, d\theta.$$

This cocycle takes values in K, not just in $\mathbb{R}$. For any linear functional $f : K \to \mathbb{R}$, composing gives you back an $\mathbb{R}$-valued cocycle. So ω_K parametrizes all possible central extensions of $L\mathfrak{g}$ by $\mathbb{R}$.

1.2 Extensions of $\mathrm{Map}(X;\mathfrak{g})$

Before leaving the subject of Lie algebra extensions, it is worth pointing out that very little extra work is needed to determine all central extensions of $\mathrm{Map}(X;\mathfrak{g})$ for any smooth manifold X. We shall indicate briefly a proof of the following result, which is a very simple case of a general theory of Loday and Quillen relating the cohomology of Lie algebras to Connes's cohomology. We shall content ourselves with the case of a simple Lie algebra $\mathfrak{g}$. There is then an essentially unique inner product $\langle\ ,\ \rangle$.

Proposition 1.8 (4.2.8)**.** *If $\mathfrak{g}$ is simple then the kernel of universal central extension of $\mathrm{Map}(X;\mathfrak{g})$ is the space $K = \Omega^1(X)/d\Omega^0(X)$ of 1-forms on X modulo exact 1-forms. The extension is defined by the cocycle*

$$(\xi,\eta) \mapsto \langle \xi, d\eta\rangle. \tag{7}$$

Equivalently, the extensions of $\mathrm{Map}(X;\mathfrak{g})$ by $\mathbb{R}$ correspond to the one-dimensional closed currents C on X, the cocycle being given by integrating (7) over C.

Before proving this let us remark that from one point of view it is a disappointing result, as it tells us that there are no 'interesting' extensions of $\mathrm{Map}(X;\mathfrak{g})$ when $\dim(X) > 1$. More precisely, if $f : S^1 \to X$ is any smooth loop in X one can always obtain an extension of $\mathrm{Map}(X;\mathfrak{g})$ by pulling back the universal extension of $L\mathfrak{g}$ by f. Proposition (4.2.8) asserts that any extension is a weighted linear combination of extensions of this form. The first 'interesting' cohomology class of $\mathrm{Map}(X;\mathfrak{g})$, for a compact $(n-1)$-dimensional manifold X, is in dimension n, and is defined by the cocycle

$$(\xi_1, \ldots, \xi_n) \mapsto P(\xi_1, d\xi_2, \ldots, d\xi_n),$$

Proof. Let us write $\mathrm{Map}(X;\mathfrak{g})$ as $A \otimes \mathfrak{g}$, where A is the ring of smooth functions on X. Any G-invariant real-valued bilinear form on $A \otimes \mathfrak{g}$ must be of the form

$$(f \otimes \xi,\, g \otimes \eta) \mapsto \alpha(f \otimes g)\langle \xi, \eta \rangle,$$

where $\alpha : A \otimes A \to \mathbb{R}$ is linear. Such an α can be identified with a distribution with compact support on $X \times X$. The cocycle condition translates into the statement that α vanishes on functions of the form

$$fg \otimes h + gh \otimes f + hf \otimes g, \tag{8}$$

where f, g, h are smooth functions on X. This means that $\alpha(f \otimes g) = 0$ when f and g have disjoint support, for then $fg = 0$ and one can find h so that $fh = f$ and $gh = 0$. Thus the distribution α has support along the diagonal. Proposition (4.2.8) is the assertion that $\alpha(f \otimes g)$ depends only on the 1-form $f dg$. This in turn reduces to two facts:

(i) $\alpha(f \otimes 1) = 0$ for all f; and

(ii) $\alpha|_{I^2} = 0$, where I is the ideal of functions in $A \otimes A$ which vanish on the diagonal.

Put $h = 1$:

$$\alpha(fg \otimes 1) + \alpha(g \otimes f) + \alpha(f \otimes g) = 0.$$

By (skew), $\alpha(g \otimes f) = -\alpha(f \otimes g)$, so those two cancel and we get

$$\alpha(fg \otimes 1) = 0 \quad \forall f, g.$$

Since finite sums of products fg span A, it follows that

$$\alpha(f \otimes 1) = 0 \quad \forall f \in A.$$

Let $I \subset A \otimes A$ be the ideal of functions vanishing on the diagonal $\Delta = \{(x,x)\}$. It is generated (as an ideal) by the differences $a \otimes 1 - 1 \otimes a$ ($a \in A$). Thus I^2 is generated by products

$$(a \otimes 1 - 1 \otimes a)\,(b \otimes 1 - 1 \otimes b).$$

It therefore suffices to check that α vanishes on each such generator. Expand:

$$(a\otimes 1-1\otimes a)(b\otimes 1-1\otimes b)=ab\otimes 1\;-\;a\otimes b\;-\;b\otimes a\;+\;1\otimes ab.$$

Apply α and use (i) and (skew):

$$\alpha(ab\otimes 1)=0,\quad \alpha(1\otimes ab)=0,\quad \alpha(a\otimes b)+\alpha(b\otimes a)=0.$$

Hence

$$\alpha\big((a\otimes 1-1\otimes a)(b\otimes 1-1\otimes b)\big)=0.$$

By linearity,

$$\alpha|_{I^2}=0$$

Define the canonical linear map

$$\theta : A\otimes A\longrightarrow \Omega^1(X),\qquad \theta(f\otimes g)=f\,dg.$$

A quick check on the generators above shows $\theta(I^2)=0$:

$$\theta\big(ab\otimes 1-a\otimes b-b\otimes a+1\otimes ab\big)=ab\,d1-a\,db-b\,da+d(ab)=0.$$

So θ descends to a well-defined map $\bar{\theta} : I/I^2\to\Omega^1(X)$, which is the standard isomorphism $I/I^2\cong\Omega^1(X)$ (Kähler differentials).

Since α kills I^2, there is a unique linear functional

$$\Lambda : \Omega^1(X)\longrightarrow\mathbb{R}\quad\text{such that}\quad \alpha(f\otimes g)=\Lambda(f\,dg)$$

Using (), compute skew-symmetry:

$$0=\alpha(f\otimes g)+\alpha(g\otimes f)=\Lambda(f\,dg)+\Lambda(g\,df)=\Lambda\big(f\,dg+g\,df\big)=\Lambda\big(d(fg)\big).$$

Because f,g were arbitrary, the linear span of $\{d(fg)\}$ is all of $d\Omega^0(X)$. Hence

$$\Lambda|_{d\Omega^0(X)}=0.$$

Λ factors through the quotient $\Omega^1(X)/d\Omega^0(X)$, so the only data that survives is the class $[\Lambda]\in(\Omega^1/d\Omega^0)^*$. $\square$

Remark 1.9. *In the case $X=S^1$, we have $\Omega^1(S^1)/d\Omega^0(S^1)\cong H^1_{\mathrm{dR}}(S^1)\cong\mathbb{R}$ (generated by the period functional $[\alpha]\mapsto\int_{S^1}\alpha$). Taking $\Lambda(\omega)=\frac{1}{2\pi}\int_{S^1}\omega$ gives*

$$\alpha(f\otimes g)=\frac{1}{2\pi}\int_{S^1}f\,dg,\qquad c(\xi,\eta)=\frac{1}{2\pi}\int_{S^1}\langle\xi,\eta'\rangle\,d\theta,$$

the standard Kac-Moody cocycle.

1.3 Extensions of $\mathrm{Vect}(S^1)$

Another calculation that fits in very naturally at this point is that for the Lie algebra $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle, i.e. the Lie algebra of the group $\mathrm{Diff}(S^1)$. A complex-linear 2-cocycle

$$\alpha : \mathrm{Vect}_{\mathbb{C}}(S^1) \times \mathrm{Vect}_{\mathbb{C}}(S^1) \to \mathbb{C},$$

where $\mathrm{Vect}_{\mathbb{C}}(S^1) = \mathrm{Vect}(S^1) \otimes \mathbb{C}$, is determined by the numbers

$$\alpha_{p,q} = \alpha(L_p, L_q), \qquad L_n = e^{in\theta}\tfrac{d}{d\theta}.$$

Recall the Witt algebra basis

$$L_n = ie^{in\theta}\tfrac{d}{d\theta}, \quad n \in \mathbb{Z},$$

with brackets

$$[L_n, L_m] = i(m-n)L_{n+m}.$$

The bracket identity follows from the definition of the commutator of derivations:

$$[X,Y] = X(Y(h)) - Y(X(h)) \quad \text{for } h \in C^\infty(M)$$

and the general formula for brackets of vector fields in one variable:

$$[f(\theta)\tfrac{d}{d\theta}, g(\theta)\tfrac{d}{d\theta}] = \big(f(\theta)g'(\theta) - g(\theta)f'(\theta)\big)\tfrac{d}{d\theta}.$$

Now check the three vector fields:

$$L_{-1}, \quad L_0, \quad L_1.$$

The brackets close as

$$[L_1, L_{-1}] = 2iL_0, \qquad [L_0, L_{\pm 1}] = \mp i L_{\pm 1}.$$

which up to rescaling gives a copy of $\mathfrak{sl}_2(\mathbb{R})$.

The cocycle identity for (L_0, L_p, L_q) shows that the cohomology class of α is not changed by rotation, and so we can (by averaging) assume that α is itself invariant. Then $\alpha_{p,q} = 0$ unless $p+q=0$. If we write $\alpha_{p,-p} = \alpha_p$, and notice that $\alpha_{-p} = -\alpha_p$, then the cocycle identity gives

$$(p+2q)\alpha_p - (2p+q)\alpha_q = (p-q)\alpha_{p+q}.$$

This determines all the α_p in terms of α_1 and α_2. The general solution is $\alpha_p = \lambda p^3 + \mu p$. But $\alpha_p = p$ is a coboundary, so the value of μ is unimportant. We have proved

Proposition 1.10 (Virasoro cocycle)**.** *The most general central extension of* $\mathrm{Vect}(S^1)$ *by* $\mathbb{R}$ *is described by the cocycle* α*, where*

$$\alpha\big(e^{in\theta}\tfrac{d}{d\theta}, e^{im\theta}\tfrac{d}{d\theta}\big) = \begin{cases} i\lambda n(n^2-1), & \text{if } n+m=0,\\ 0, & \text{if } n+m\neq 0,\end{cases}$$

for some $\lambda \in \mathbb{R}$.

The representing cocycle given here is characterized by the fact that it is invariant under rotation and vanishes on the subalgebra $\mathfrak{sl}_2(\mathbb{R})$ of $\mathrm{Vect}(S^1)$.

1.4 Adjoint and coadjoint actions of loop groups

Proposition 1.11. *The adjoint action of $L\mathfrak{g}$ on its central extension $\widetilde{L\mathfrak{g}}$ comes from an action of LG given by*

$$(\gamma)\cdot(\xi,\lambda) = (\mathrm{Ad}_\gamma\,\xi, \lambda - \langle\gamma^{-1}\gamma', \xi\rangle)$$

Proof. We will differentiate the group action along the one-parameter subgroup $\gamma(t) = \exp(t\eta)$, where $\eta \in L\mathfrak{g}$. Differentiate at $t = 0$:

- First coordinate:
$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\gamma(t)}\,\xi = [\eta, \xi].$$
- Second coordinate: use $\gamma(t)^{-1}\gamma'(t) = t\,\eta' + O(t^2)$ (standard Maurer–Cartan expansion along the loop variable), hence
$$\frac{d}{dt}\Big|_{t=0}\big(-\big\langle\gamma(t)^{-1}\gamma'(t), \xi\big\rangle\big) = -\frac{1}{2\pi}\int_0^{2\pi}\langle\eta'(\theta), \xi(\theta)\rangle_{\mathfrak{g}}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\langle\eta(\theta), \xi'(\theta)\rangle_{\mathfrak{g}}\,d\theta,$$
where the last equality is by integration by parts (boundary term vanishes by periodicity).

Define the Kac-Moody 2-cocycle

$$\omega(\eta,\xi) \;:=\; \frac{1}{2\pi}\int_0^{2\pi}\langle\eta(\theta), \xi'(\theta)\rangle_{\mathfrak{g}}\,d\theta.$$

Thus the derivative of the action is

$$\frac{d}{dt}\Big|_{t=0}\big(\gamma(t)\cdot(\xi,\lambda)\big) = \big([\eta,\xi],\, \omega(\eta,\xi)\big).$$

This is exactly the standard adjoint action of $L\mathfrak{g}$ on the central extension $\widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R}K$:

$$\mathrm{ad}_{(\eta,0)}(\xi,\lambda) = \big([\eta,\xi],\, \omega(\eta,\xi)\big), \qquad [K,\cdot] = 0.$$

as desired. $\square$

Proposition 1.12 (Loop group coadjoint action)**.** *The coadjoint action of LG on $\widetilde{L\mathfrak{g}}^* \cong (L\mathfrak{g})^* \oplus \mathbb{R}$ is given by*

$$\gamma\cdot(\phi,\lambda) = \big(\mathrm{Ad}_\gamma\,\phi + \lambda\,\gamma'\gamma^{-1},\ \lambda\big).$$

Proof. Identify $\widetilde{L\mathfrak{g}}^* \cong (L\mathfrak{g})^* \oplus \mathbb{R}$ with pairing (note that $(\phi, \lambda) \in (\tilde{L}\mathfrak{g} \oplus \mathbb{R})^*$ and $(\xi, a) \in \tilde{L}\mathfrak{g} \oplus \mathbb{R}$)

$$\langle (\phi, \lambda), (\xi, a) \rangle \;=\; \phi(\xi) + \lambda a.$$

By definition of coadjoint action,

$$\big\langle \gamma \cdot (\phi, \lambda),\, (\xi, a) \big\rangle = \big\langle (\phi, \lambda),\, \gamma^{-1} \cdot (\xi, a) \big\rangle.$$

Insert the adjoint formula with γ^{-1}. Using $(\gamma^{-1})^{-1}(\gamma^{-1})' = \gamma(\gamma^{-1})' = -\gamma'\gamma^{-1}$,

$$\gamma^{-1} \cdot (\xi, a) = \Big(\operatorname{Ad}_{\gamma^{-1}} \xi,\; a - \big\langle (\gamma^{-1})^{-1}(\gamma^{-1})',\, \xi \big\rangle \Big) = \Big(\operatorname{Ad}_{\gamma^{-1}} \xi,\; a + \langle \gamma'\gamma^{-1},\, \xi \rangle \Big).$$

Hence

$$\begin{aligned} \big\langle \gamma \cdot (\phi, \lambda),\, (\xi, a) \big\rangle &= \phi\big(\operatorname{Ad}_{\gamma^{-1}} \xi \big) + \lambda\Big(a + \langle \gamma'\gamma^{-1},\, \xi \rangle \Big) \\ &= (\phi \circ \operatorname{Ad}_{\gamma^{-1}})(\xi) \;+\; \lambda a \;+\; \lambda \,\langle \gamma'\gamma^{-1},\, \xi \rangle. \end{aligned}$$

Since this holds for all (ξ, a), we read off

$$\gamma \cdot (\phi, \lambda) = \big(\phi \circ \operatorname{Ad}_{\gamma^{-1}} + \lambda \,\langle \gamma'\gamma^{-1},\, \cdot \rangle,\; \lambda \big).$$

Using the invariant inner product to identify $(L\mathfrak{g})^* \cong L\mathfrak{g}$, write $\phi(\,\cdot\,) = \langle \phi,\, \cdot \rangle$. Then

$$\phi \circ \operatorname{Ad}_{\gamma^{-1}} = \langle \operatorname{Ad}_\gamma \phi,\, \cdot \rangle, \qquad \langle \gamma'\gamma^{-1},\, \cdot \rangle \;\leftrightarrow\; \gamma'\gamma^{-1},$$

so the coadjoint action becomes

$$\gamma \cdot (\phi, \lambda) = \big(\operatorname{Ad}_\gamma \phi + \lambda\, \gamma'\gamma^{-1},\; \lambda \big).$$

as desired. □

Remark 1.13 (Reminder about the infinitesimal adjoint and coadjoint actions)**.** *Let G be compact, connected, and simply connected, with Lie algebra $\mathfrak{g}$ and an Ad–invariant inner product $\langle\,,\,\rangle_{\mathfrak{g}}$. For loops we use the L^2–pairing*

$$\langle \xi, \eta \rangle_{L\mathfrak{g}} := \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta(\theta) \rangle_{\mathfrak{g}}\, d\theta, \qquad \xi, \eta \in L\mathfrak{g}.$$

Define the 2–cocycle

$$\omega(\eta, \xi) := \frac{1}{2\pi} \int_0^{2\pi} \langle \eta(\theta), \xi'(\theta) \rangle_{\mathfrak{g}}\, d\theta.$$

The (Lie–algebra) central extension is

$$\widetilde{L\mathfrak{g}} \;=\; L\mathfrak{g} \oplus \mathbb{R}K, \qquad [(\eta, aK), (\xi, bK)] = \big([\eta, \xi],\; \omega(\eta, \xi)\, K \big).$$

Here K is central, $[K, \cdot] = 0$. We identify $\widetilde{L\mathfrak{g}}^* \cong (L\mathfrak{g})^* \oplus \mathbb{R}$ and, via the L^2–pairing, $(L\mathfrak{g})^* \cong L\mathfrak{g}$. We write the dual pairing as

$$\big\langle (\phi, \lambda), (\xi, aK) \big\rangle = \langle \phi, \xi \rangle_{L\mathfrak{g}} + \lambda a.$$

Adjoint (Lie–algebra) action. *By definition,* $\mathrm{ad}_{(\eta,aK)}(\xi,bK) = [(\eta,aK),(\xi,bK)]$*; since* K *is central,*

$$\mathrm{ad}_{(\eta,0)}(\xi,bK) = \big([\eta,\xi],\, \omega(\eta,\xi)\,K\big), \qquad \mathrm{ad}_{(0,aK)}(\xi,bK) = 0.$$

Equivalently, the adjoint representation of $\widetilde{L\mathfrak{g}}$ *on itself is*

$$\mathrm{ad}_{(\eta,aK)} \begin{pmatrix} \xi \\ bK \end{pmatrix} = \begin{pmatrix} [\eta,\xi] \\ \omega(\eta,\xi)\,K \end{pmatrix}.$$

Coadjoint (Lie–algebra) action. *Recall the coadjoint action is defined by*

$$\big\langle\, \mathrm{ad}^*_X(\Phi),\, Y \big\rangle = \big\langle \Phi,\, [Y,X] \big\rangle \qquad (X,Y \in \widetilde{L\mathfrak{g}},\ \Phi \in \widetilde{L\mathfrak{g}}^{\,*}),$$

which matches the sign convention that integrates to the group formula in Pressley–Segal. Take $X = (\eta,0)$*,* $Y = (\xi,aK)$*,* $\Phi = (\phi,\lambda)$*:*

$$\begin{aligned} \big\langle\, \mathrm{ad}^*_{(\eta,0)}(\phi,\lambda),\, (\xi,aK) \big\rangle &= \big\langle (\phi,\lambda),\, [(\xi,aK),(\eta,0)] \big\rangle \\ &= \big\langle (\phi,\lambda),\, \big([\xi,\eta],\, \omega(\xi,\eta)K\big) \big\rangle \\ &= \langle \phi,\, [\xi,\eta] \rangle_{L\mathfrak{g}} + \lambda\,\omega(\xi,\eta). \end{aligned}$$

Use Ad*–invariance of* $\langle\ ,\ \rangle_{\mathfrak{g}}$*:* $\langle \phi,[\xi,\eta]\rangle = \langle [\phi,\eta],\xi\rangle$*, and integrate by parts (using periodicity) for the cocycle term*

$$\omega(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi,\eta' \rangle\, d\theta.$$

Thus

$$\big\langle\, \mathrm{ad}^*_{(\eta,0)}(\phi,\lambda),\, (\xi,aK) \big\rangle = \langle [\eta,\phi] + \lambda\,\eta',\, \xi \rangle_{L\mathfrak{g}},$$

which identifies

$$\mathrm{ad}^*_{(\eta,0)}(\phi,\lambda) \ = \ \big([\eta,\phi] + \lambda\,\eta',\ 0\big).$$

Since K *is central,* $\mathrm{ad}^*_{(0,aK)} = 0$*. Therefore, for general* (η,aK)*:*

$$\mathrm{ad}^*_{(\eta,aK)}(\phi,\lambda) \ = \ \big([\eta,\phi] + \lambda\,\eta',\ 0\big).$$

Consistency with the group formula. *Let* $\gamma(t) = \exp(t\eta) \in LG$*. The group–level coadjoint action is*

$$\gamma \cdot (\phi,\lambda) = \big(\,\mathrm{Ad}_\gamma\,\phi + \lambda\,\gamma'\gamma^{-1},\ \lambda\big).$$

Differentiating at $t = 0$ *gives*

$$\frac{d}{dt}\Big|_0 \big(\,\mathrm{Ad}_{\gamma(t)}\,\phi\big) = [\eta,\phi], \qquad \frac{d}{dt}\Big|_0 \big(\gamma'(t)\gamma(t)^{-1}\big) = \eta',$$

hence

$$\frac{d}{dt}\Big|_0 \big(\gamma(t)\cdot(\phi,\lambda)\big) = \big([\eta,\phi] + \lambda\,\eta',\ 0\big) = \mathrm{ad}^*_{(\eta,0)}(\phi,\lambda),$$

as derived above.

Let us assume that the inner product on $\mathfrak{g}$ is positive-definite. Then $L\mathfrak{g}$ is identified with a dense subspace of $(L\mathfrak{g})^*$ which we shall call the 'smooth part' of the dual. We can describe the orbits of the action of LG on this in the following way.

For each smooth element $(\phi, \lambda) \in (\widetilde{L\mathfrak{g}})^*$ with $\lambda \neq 0$ we can find a unique smooth path $f : \mathbb{R} \to G$ by solving the differential equation

$$f' f^{-1} = \lambda^{-1} \phi \tag{9}$$

with the initial condition $f(0) = 1$.

Definition 1.14 (Parallel transport ODE)**.** *Define $f : \mathbb{R} \to G$ by the first-order ODE*

$$f'(\theta)\, f(\theta)^{-1} \;=\; \lambda^{-1}\, \phi(\theta), \qquad f(0) = \mathbf{1}. \tag{10}$$

Lemma 1.15 (Existence and uniqueness)**.** *For any smooth ϕ and $\lambda \neq 0$, the initial value problem* (10) *has a unique smooth solution on all of $\mathbb{R}$. Moreover,*

$$f(\theta) \;=\; \mathcal{P}\exp\left(\lambda^{-1} \int_0^\theta \phi(s)\, ds\right), \tag{11}$$

where $\mathcal{P}\exp$ denotes the path-ordered exponential.

Because ϕ is periodic in θ we have

$$f(\theta + 2\pi) = f(\theta) \cdot M_\phi,$$

where $M_\phi = f(2\pi)$. If (ϕ, λ) is transformed by $\gamma \in LG$ then f is changed to $\tilde{f}$, where

$$\tilde{f}(\theta) = \gamma(\theta) f(\theta) \gamma(0)^{-1}. \tag{12}$$

Thus M_ϕ is changed to $\gamma(0) M_\phi \gamma(0)^{-1}$. In fact (9) defines a bijection between $L\mathfrak{g} \times \{\lambda\}$ and the space of maps f such that $f(0) = 1$ and $f(\theta + 2\pi) = f(\theta) \cdot M$ for some $M \in G$.

Definition 1.16 (Monodromy / holonomy)**.** *Because ϕ is 2π-periodic, there exists a unique $M_\phi \in G$ (the* monodromy*) such that*

$$f(\theta + 2\pi) \;=\; f(\theta)\, M_\phi \qquad (\theta \in \mathbb{R}), \tag{13}$$

equivalently $M_\phi = f(2\pi)$. In terms of (11)*,*

$$M_\phi \;=\; \mathcal{P}\exp\left(\lambda^{-1} \int_0^{2\pi} \phi(\theta)\, d\theta\right). \tag{14}$$

Proof of (13). Let $g(\theta) := f(\theta + 2\pi)$. Then

$$\begin{aligned} g' g^{-1} &= f'(\theta + 2\pi) f(\theta + 2\pi)^{-1} \\ &= \lambda^{-1} \phi(\theta + 2\pi) \\ &= \lambda^{-1} \phi(\theta), \end{aligned}$$

and $g(0) = f(2\pi)$. By uniqueness for (10), $g(\theta) = f(\theta)\, f(2\pi)$, giving $f(\theta + 2\pi) = f(\theta) M_\phi$ with $M_\phi = f(2\pi)$. $\square$

Proposition 1.17 (Transformation under the LG–coadjoint action)**.** *Let $\gamma \in LG$. The LG–coadjoint action on $(\widetilde{L\mathfrak{g}})^*$ is*

$$\gamma \cdot (\phi, \lambda) \;=\; (\mathrm{Ad}_\gamma \phi + \lambda\, \gamma' \gamma^{-1},\, \lambda).$$

If f solves (10) *for (ϕ, λ), then*

$$\tilde{f}(\theta) \;:=\; \gamma(\theta)\, f(\theta)\, \gamma(0)^{-1} \tag{15}$$

solves (10) *for $(\mathrm{Ad}_\gamma \phi + \lambda\, \gamma'\gamma^{-1},\, \lambda)$ and satisfies $\tilde{f}(0) = \mathbf{1}$. Consequently the monodromy transforms by conjugation:*

$$M_{\gamma \cdot \phi} \;=\; \gamma(0)\, M_\phi\, \gamma(0)^{-1}. \tag{16}$$

Proof. Let $\gamma \in LG$, and suppose $f : \mathbb{R} \to G$ solves

$$f'(\theta)\, f(\theta)^{-1} \;=\; \lambda^{-1}\, \phi(\theta), \qquad f(0) = \mathbf{1}.$$

Define

$$\tilde{f}(\theta) \;:=\; \gamma(\theta)\, f(\theta)\, \gamma(0)^{-1}.$$

We claim that

$$\tilde{f}'(\theta)\, \tilde{f}(\theta)^{-1} \;=\; \lambda^{-1}\Big(\mathrm{Ad}_{\gamma(\theta)}\, \phi(\theta) \;+\; \lambda\, \gamma'(\theta)\gamma(\theta)^{-1} \Big),$$

so $\tilde{f}$ solves the ODE corresponding to $(\mathrm{Ad}_\gamma \phi + \lambda\, \gamma'\gamma^{-1}, \lambda)$ and satisfies $\tilde{f}(0) = \mathbf{1}$.

Proof. First compute the derivative:

$$\tilde{f}'(\theta) = \gamma'(\theta)\, f(\theta)\, \gamma(0)^{-1} + \gamma(\theta)\, f'(\theta)\, \gamma(0)^{-1}.$$

Next note that

$$\tilde{f}(\theta)^{-1} = \gamma(0)\, f(\theta)^{-1}\, \gamma(\theta)^{-1}.$$

Hence

$$\begin{aligned} \tilde{f}'(\theta)\, \tilde{f}(\theta)^{-1} &= \Big(\gamma' f \gamma(0)^{-1} + \gamma f' \gamma(0)^{-1} \Big)\Big(\gamma(0) f^{-1} \gamma^{-1} \Big) \\ &= \gamma' f f^{-1} \gamma^{-1} \;+\; \gamma f' f^{-1} \gamma^{-1} \\ &= \gamma' \gamma^{-1} \;+\; \gamma\big(f' f^{-1}\big) \gamma^{-1}. \end{aligned}$$

Insert the original ODE $f' f^{-1} = \lambda^{-1}\phi$:

$$\tilde{f}' \tilde{f}^{-1} = \gamma'\gamma^{-1} + \lambda^{-1}\gamma\phi\gamma^{-1} = \lambda^{-1}\Big(\mathrm{Ad}_\gamma\,\phi \;+\; \lambda\,\gamma'\gamma^{-1}\Big).$$

Finally, $\tilde{f}(0) = \gamma(0)\,f(0)\,\gamma(0)^{-1} = \mathbf{1}$, as required. Also $\tilde{f}(0) = \gamma(0)\mathbf{1}\gamma(0)^{-1} = \mathbf{1}$. Evaluating at $\theta = 2\pi$ and using $\gamma(2\pi) = \gamma(0)$ (loop),

$$M_{\gamma\cdot\phi} = \tilde{f}(2\pi) = \gamma(0)\,f(2\pi)\,\gamma(0)^{-1} = \gamma(0)\,M_\phi\,\gamma(0)^{-1}.$$

as desired. $\square$

Remark 1.18. *Equation* (11) *identifies f as the parallel transport for the connection one-form $A = \lambda^{-1}\phi(\theta)\,d\theta$ on the trivial G-bundle over S^1, and M_ϕ as its holonomy around the circle.*

The following proposition follows from the previous discussion.

Proposition 1.19 (Coadjoint orbits and conjugacy classes)**.** *(i) If G is simply connected and $\lambda \neq 0$ then the orbits of LG on the smooth part of $(L\mathfrak{g})^* \times \{\lambda\} \subset (\widetilde{L\mathfrak{g}})^*$ correspond precisely to the conjugacy classes of G under the map $(\phi, \lambda) \mapsto M_\phi$.*

(ii) The stabilizer of (ϕ, λ) in LG is isomorphic to the centralizer Z_ϕ of M_ϕ in G by the map $\gamma \mapsto \gamma(0)$; and γ stabilizes (ϕ, λ) if and only if

$$\gamma(\theta) = f(\theta)\gamma(0)f(\theta)^{-1}.$$

Proof. The relation $M_{\gamma\cdot(\phi,\lambda)} = \gamma(0)\,M_\phi\,\gamma(0)^{-1}$ shows that (ϕ, λ) and (ϕ', λ) are in the same LG-orbit implies M_ϕ and $M_{\phi'}$ are conjugate in G. Conversely, if $M_{\phi'} = gM_\phi g^{-1}$ for some $g \in G$, there exists a loop $\gamma \in LG$ with $\gamma(0) = g$. Then by the previous proposition, $\gamma \cdot (\phi, \lambda)$ has monodromy $M_{\phi'}$.

The map $(\phi, \lambda) \mapsto M_\phi$ is surjective onto conjugacy classes. Let $C \subset G$ be a conjugacy class. Choose $g \in C$. Pick $X \in \mathfrak{g}$ with $\exp(2\pi X) = g$ (for compact connected G this is always possible since every element lies in a maximal torus and $\exp : \mathfrak{t} \to T$ is surjective). Take $\phi(\theta) \equiv -\lambda X$ (constant). Then the solution is $f(\theta) = \exp(\theta X)$, hence $M_\phi = g \in C$.

we can find ϕ'' such that $M_{\phi''} = M_{\phi'}$. Thus (ϕ', λ) and (ϕ'', λ) have the same monodromy and hence are in the same LG-orbit. This establishes the bijection between LG-orbits and conjugacy classes in G.

Now we show injectivity of fixed monodromy. Suppose (ϕ, λ) and (ϕ', λ) have the same monodromy: $M_\phi = M_{\phi'} = M$. Let f, f' be their ODE solutions with $f(0) = f'(0) = \mathbf{1}$. Define $\gamma(\theta) := f'(\theta)\,f(\theta)^{-1}$. Then $\gamma(0) = \mathbf{1}$ and, using $f(\theta + 2\pi) = f(\theta)M$, $f'(\theta + 2\pi) = f'(\theta)M$,

$\gamma(\theta + 2\pi) = f'(\theta)M\,(f(\theta)M)^{-1} = \gamma(\theta)$, so $\gamma \in LG$. A direct calculation gives (with the "+" convention)

$$\gamma \cdot \phi = \mathrm{Ad}_\gamma\,\phi + \lambda\,\gamma'\gamma^{-1} = \phi'.$$

Hence points with the same monodromy lie in the same LG-orbit.

As for the second claim, suppose $\gamma \in LG$ stabilizes (ϕ, λ). Then by definition of the action, $(\phi, \lambda) = \gamma \cdot (\phi, \lambda)$. This means the transformed ODE solution $\tilde{f}(\theta) = \gamma(\theta)f(\theta)\gamma(0)^{-1}$ equals the original $f(\theta)$ (since both solve the same ODE with same initial condition). So we must have $f(\theta) = \gamma(\theta)f(\theta)\gamma(0)^{-1}$, or equivalently, $\gamma(\theta) = f(\theta)\gamma(0)f(\theta)^{-1}$. In particular, at $\theta = 2\pi$, $\gamma(2\pi) = f(2\pi)\gamma(0)f(2\pi)^{-1}$. But since γ is a loop, $\gamma(2\pi) = \gamma(0)$. This forces $\gamma(0) \in Z_G(M_\phi)$, i.e. $\gamma(0)$ lies in the centralizer of M_ϕ.

So the stabilizer subgroup of LG maps isomorphically to the centralizer Z_ϕ under the map $\gamma \mapsto \gamma(0)$.
$\square$

Remark 1.20. *In general, the coadjoint action only integrates to an action of the component containing γ. To guarantee there's no component obstruction (and to integrate the infinitesimal formulas globally), we want LG to be connected. For connected G, $\pi_0(LG) \cong \pi_1(G)$. Thus if G is simply connected, then LG is connected, and the coadjoint action integrates on all of LG with no ambiguity.*

According to Kirillov's idea, the irreducible unitary representations of a group Γ correspond to the coadjoint orbits Ω with the property

> (C) if the stabilizer of $\Phi \in \Omega$ is the subgroup H of Γ then Φ is the derivative of a character of the identity component of H.

The group–level central extension is $1 \to \mathbb{T} \to \widetilde{LG} \to LG \to 1$, where $\mathbb{T} = U(1)$ is the circle. The Lie algebra of this circle is just $\mathbb{R}K$ with basis element K. In the dual $(\widetilde{L\mathfrak{g}})^*$, the functional (ϕ, λ) evaluates to $\langle(\phi, \lambda), K\rangle = \lambda$.

Kirillov's condition (C) says: If H is the stabilizer of a coadjoint point Φ, then the restriction $\Phi|_{\mathfrak{h}}$ must equal the differential of a unitary character of H^0. For every (ϕ, λ), the central subgroup $\mathbb{T} \subset \widetilde{LG}$ is contained in its stabilizer (since it's central, it fixes everything). So H^0 contains $\mathbb{T}$, and we must check condition (C) on that subgroup.

So we need: the restriction of Φ to the Lie algebra of $\mathbb{T}$ (spanned by K) must be the differential of some unitary character $\chi : \mathbb{T} \to U(1)$. The circle group $\mathbb{T} = \{e^{i\theta} : \theta \in \mathbb{R}\}$ has all unitary characters given by $\chi_n(e^{i\theta}) = e^{in\theta}$ for $n \in \mathbb{Z}$. Differentiate at the identity ($\theta = 0$): $\chi_n'(0) = in$. By definition of (ϕ, λ), $\langle(\phi, \lambda), K\rangle = \lambda$. Condition (C) requires this number λ to equal the derivative of some unitary character of $\mathbb{T}$. Therefore we have $\lambda \in \mathbb{Z}$.

By the above argument, if $(L\mathfrak{g})^* \times \{\lambda\}$ is allowable then λ must be an integer. Then an orbit in the smooth part of the dual corresponds to the conjugacy class of an element $g \in G$, which we can assume to belong to a given maximal torus T. If we choose

$$\xi \in \mathfrak{t} \subset \mathfrak{g} \subset L\mathfrak{g} \subset (L\mathfrak{g})^*$$

so that $\exp(\lambda^{-1}\xi) = g$, then (ξ, λ) belongs to the orbit. This is because one can check that the solution of (10) is $f(\theta) = \exp(\lambda^{-1}\theta\xi)$, which has monodromy $M_\phi = \exp(2\pi\lambda^{-1}\xi) = g$.

If g is sufficiently generic then its centralizer in G is T. Recall that we say an element $g \in G$ (or equivalently $X \in \mathfrak{g}$) is regular if its centralizer has minimal possible dimension. The minimal possible centralizer in a compact Lie group is precisely a maximal torus T. Concretely, if $X \in \mathfrak{t}$, then $Z_G(X)$ consists of the torus T plus all root subgroups $\mathfrak{g}_\alpha$ for which $\alpha(X) = 0$. If $\alpha(X) = 0$ for some root, then the centralizer strictly contains T. So for X regular (i.e. $\alpha(X) \neq 0$ for all roots), $Z_G(X) = T$. Therefore, if $g = \exp(X)$ with X regular in $\mathfrak{t}$, then $Z_G(g) = T$.

And the condition (C) amounts to the requirement that $\xi \in \mathfrak{t} \subset \mathfrak{t}^*$ belongs to the lattice $\hat{T}$.

Recall that the stabilizer H of (ϕ, λ) in LG is isomorphic to $Z_G(g) = T$ by $\gamma \mapsto \gamma(0)$; more concretely, after conjugating by the associated f, one may (and we will) work at the representative (ξ, λ) with $\xi \in \mathfrak{t}$ constant. Then $H^0 \cong T$ and $\mathfrak{h} \cong \mathfrak{t}$. Condition (C) says that the restriction of (ξ, λ) to $\mathfrak{h} \cong \mathfrak{t}$ must be the differential of a character of T. The restriction is simply $Y \in \mathfrak{t} \mapsto \langle \xi, Y \rangle \in \mathbb{R}$ (using the fixed invariant inner product to identify $\mathfrak{t} \cong \mathfrak{t}$). This linear form exponentiates to a character of T if and only if it takes integral values on the period lattice $\widehat{T} := \ker(\exp : \mathfrak{t} \to T)$. That is, $\langle \xi, \eta \rangle \in 2\pi\mathbb{Z}$ for all $\eta \in \widehat{T}$. With our normalization (Pressley-Segal identify $\mathfrak{t} \simeq \mathfrak{t}^*$ using the basic inner product and absorb the 2π in the definition of $\widehat{T}$), this is precisely the statement $\xi \in \widehat{T}$.

On the other hand, (ξ, λ) and $(\tilde{\xi}, \lambda)$ belong to the same orbit if $\tilde{\xi} = w \cdot \xi + \lambda\eta$ for some $\eta \in \hat{T}$ and some w in the Weyl group W of G.

Proposition 1.21 (Coadjoint orbits satisfying (C))**.** *If λ is a non-zero integer then the coadjoint orbits in the smooth part of $(L\mathfrak{g})^* \times \{\lambda\}$ which satisfy the condition (C) correspond to the orbits of the affine Weyl group $W_{\text{aff}} = W \ltimes \hat{T}$ on the lattice $\hat{T}$, where $(w, \eta) \in W_{\text{aff}}$ acts on $\hat{T}$ by*

$$\xi \mapsto w \cdot \xi + \lambda\eta.$$

Proof. Every orbit contains a representative (ξ, λ) with $\xi \in \mathfrak{t}$, since any conjugacy class of G meets T and the monodromy of (ξ, λ) is $M_\xi = \exp(2\pi\xi/\lambda) \in T$. Suppose (ξ, λ) and $(\tilde{\xi}, \lambda)$ are in the same orbit. Then there exists $\gamma \in LG$ such that

$$\tilde{\xi} = \operatorname{Ad}_\gamma \xi + \lambda\, \gamma' \gamma^{-1}.$$

Both M_ξ and $M_{\tilde\xi}$ lie in T. Since $M_{\tilde\xi} = \gamma(0)M_\xi\gamma(0)^{-1}$, the endpoint $\gamma(0)$ normalizes T. This is because if (ξ, λ) and $(\tilde\xi, \lambda)$ are in the same orbit, then their monodromies are conjugate by $\gamma(0)$, but since they both lie in T, they are in fact equal and therefore

$$M_{\tilde\xi} = M_\xi \implies \gamma(0)M_\xi\gamma(0)^{-1} = M_\xi$$

So $\gamma(0) \in N_G(T)$. Modulo T this determines an element $w \in W$, and the constant loop $\gamma(\theta) \equiv n$ with $n \in N_G(T)$ representing w acts by

$$\gamma \cdot (\xi, \lambda) = (w \cdot \xi, \lambda).$$

There are a second class of loops $\gamma(\theta)$ in T act trivially on ξ but contribute through the cocycle term:

$$\mathrm{Ad}_\gamma \xi = \xi, \qquad \gamma'\gamma^{-1} \in \widehat{T}.$$

Concretely, if $\gamma(\theta) = \exp(\theta\eta)$ with $\eta \in \widehat{T}$, then $\gamma'\gamma^{-1} = \eta$ and

$$\gamma \cdot (\xi, \lambda) = (\xi + \lambda\eta, \lambda).$$

because of the general formula for the coadjoint action.

$$\gamma \cdot (\phi, \lambda) \;=\; (\mathrm{Ad}_\gamma \phi + \lambda\, \gamma'\gamma^{-1},\, \lambda).$$

This shows that the orbit contains all points of the form $(w \cdot \xi + \lambda\eta, \lambda)$ with $w \in W$ and $\eta \in \widehat{T}$.

The converse will be treated in the following lemma. This establishes a bijection between orbits and W_{aff}-orbits in $\hat{T}$. $\square$

Lemma 1.22 (Exhaustion by Weyl and lattice moves)**.** *Let G be compact, connected and simply connected, $T \subset G$ a maximal torus with Lie algebra $\mathfrak{t}$, Weyl group $W = N_G(T)/T$, and $\widehat{T} = \ker(\exp : \mathfrak{t} \to T)$. Fix $\lambda \in \mathbb{Z} \setminus \{0\}$. If (ξ, λ) and $(\tilde\xi, \lambda)$ with $\xi, \tilde\xi \in \mathfrak{t}$ lie in the same LG–orbit, then there exist $w \in W$ and $\eta \in \widehat{T}$ such that*

$$\tilde\xi \;=\; w\cdot \xi \;+\; \lambda\,\eta.$$

Proof. Assume $(\tilde\xi, \lambda) = \gamma \cdot (\xi, \lambda)$ for some $\gamma \in LG$.

Let $M_\xi := \exp(2\pi\xi/\lambda) \in T$ and $M_{\tilde\xi} := \exp(2\pi\tilde\xi/\lambda) \in T$ be their monodromies. The general monodromy formula gives

$$M_{\tilde\xi} \;=\; \gamma(0)\, M_\xi\, \gamma(0)^{-1}.$$

Since $M_\xi, M_{\tilde\xi} \in T$, by the standard conjugacy theorem ("G–conjugacy on T is W–conjugacy"), there exists $n \in N_G(T)$ with

$$M_{\tilde\xi} \;=\; n\, M_\xi\, n^{-1}.$$

Let $w \in W$ be the class of n. Replace γ by

$$\gamma_1 \; := \; n^{-1}\gamma \in LG, \qquad \text{and set} \qquad \hat{\xi} \; := \; \mathrm{Ad}_{n^{-1}} \tilde{\xi} \; = \; w^{-1} \cdot \tilde{\xi} \in \mathfrak{t}.$$

Then $\gamma_1 \cdot (\xi, \lambda) = (\hat{\xi}, \lambda)$ and

$$M_{\hat{\xi}} = \gamma_1(0)\, M_\xi\, \gamma_1(0)^{-1} = n^{-1}\gamma(0)\, M_\xi\, \gamma(0)^{-1} n = n^{-1} M_{\tilde{\xi}} n = M_\xi.$$

Thus $\xi, \hat{\xi} \in \mathfrak{t}$ have *equal* monodromy: $\exp(2\pi\hat{\xi}/\lambda) = \exp(2\pi\xi/\lambda)$.

Consider the solutions

$$f(\theta) = \exp\Big(\tfrac{\theta}{\lambda}\xi\Big), \qquad \hat{f}(\theta) = \exp\Big(\tfrac{\theta}{\lambda}\hat{\xi}\Big) \qquad (\in T),$$

and define the T–valued loop

$$\delta(\theta) \; := \; \hat{f}(\theta)\, f(\theta)^{-1} \in T.$$

Since T is abelian, we have

$$\delta(\theta) = \exp\Big(\tfrac{\theta}{\lambda}(\hat{\xi} - \xi)\Big), \qquad \delta'(\theta)\, \delta(\theta)^{-1} = \tfrac{1}{\lambda}(\hat{\xi} - \xi) \in \mathfrak{t}.$$

Moreover $\delta(2\pi) = 1$ because $M_{\hat{\xi}} = M_\xi$. Hence

$$\eta \; := \; \tfrac{1}{\lambda}(\hat{\xi} - \xi) \; \in \; \widehat{T}, \qquad \text{and} \qquad \hat{\xi} = \xi + \lambda\eta.$$

By the coadjoint action formula,

$$\delta \cdot (\xi, \lambda) = (\xi + \lambda\eta, \lambda) = (\hat{\xi}, \lambda).$$

Finally, undoing the n^{-1}–conjugation gives

$$(\tilde{\xi}, \lambda) = (n \cdot \delta) \cdot (\xi, \lambda) = \Big(w\cdot (\xi + \lambda\eta),\, \lambda\Big) = \Big(w\cdot \xi + \lambda\, w\cdot \eta,\, \lambda\Big).$$

Since $\widehat{T}$ is W–stable, $w\cdot \eta \in \widehat{T}$; renaming $\eta \leftarrow w\cdot \eta$ yields the claimed form $\tilde{\xi} = w\cdot \xi + \lambda\eta$. $\square$

Remark 1.23 (Rotation action)**.** *We can rotate the loop parameter:* $(R_\alpha\phi)(\theta) := \phi(\theta + \alpha)$*, where* $\alpha \in \mathbb{T} = S^1$*. This gives an action of the rotation group* $\mathbb{T}$ *on* $L\mathfrak{g}$*, hence also on* $(\widetilde{L\mathfrak{g}})^*$*.*

An orbit $\mathcal{O}$ *is in the smooth part if it is stable under circle rotations. In other words, if rotating the loop parameter can be undone by some* LG*-coadjoint action.*

At every point $(\phi, \lambda) \in \mathcal{O}$*, the vector field generating rotations is tangent to the orbit. Equivalently: the infinitesimal variation* $\delta_{\mathrm{rot}}\phi = \phi'$ *must lie in the tangent space of the orbit. The tangent space at* (ϕ, λ) *to the coadjoint orbit is spanned by infinitesimal coadjoint actions:*

$$T_{(\phi,\lambda)}(\mathcal{O}) = \{([\eta, \phi] + \lambda\eta',\, 0) \; : \; \eta \in L\mathfrak{g}\}$$

So for stability we require $\phi' \in \{[\eta, \phi] + \lambda\eta' : \eta \in L\mathfrak{g}\}$. Thus there must exist some $\eta \in L\mathfrak{g}$ such that $\phi'(\theta) = [\eta(\theta), \phi(\theta)] + \lambda\eta'(\theta)$.

If $\eta \in L\mathfrak{g}$ is smooth, then both $[\eta, \phi]$ and η' are smooth in θ. Hence ϕ' is smooth, which forces ϕ to be smooth. Representation-theoretically, this matches the fact that positive energy representations (the ones stable under rotations) correspond to smooth coadjoint orbits.