Coherent sheaves and exceptional collections

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Abstract

Coherent sheaves, vector bundles, and exceptional collections in algebraic geometry.

Contents

1 Preliminaries

1.1 Schemes

Definition 1.1 (Closed and Non-closed Points)**.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme.*

1. *A point $p \in X$ is called a closed point if the corresponding prime ideal $\mathfrak{p}$ is a maximal ideal of A.*
2. *A point $p \in X$ is called a non-closed point if the corresponding prime ideal $\mathfrak{p}$ is not maximal.*
3. *A generic point of an irreducible component of X corresponds to a minimal prime ideal of A.*

Example 1.2. *Consider $X = \mathrm{Spec}(\mathbb{C}[x, y])$, the affine plane over $\mathbb{C}$.*

1. *Closed points correspond to maximal ideals of the form $(x - a, y - b)$ for $a, b \in \mathbb{C}$. These are the familiar points (a, b) in the complex plane.*
2. *Prime ideals like $(x-1)$ correspond to non-closed points. Geometrically, this represents the "generic point" of the vertical line $x = 1$.*
3. *The prime ideal (0) corresponds to the generic point of the entire plane.*

Remark 1.3. *For a scheme over a field k:*

1. *If k is algebraically closed (like $\mathbb{C}$), the closed points of $\mathrm{Spec}(k[x_1, \ldots, x_n])$ correspond exactly to the n-tuples $(a_1, \ldots, a_n) \in k^n$.*
2. *If k is not algebraically closed (like $\mathbb{Q}$), there are additional closed points. For example, in $\mathrm{Spec}(\mathbb{Q}[x])$, the ideal $(x^2 + 1)$ is maximal and corresponds to a closed point, even though it does not correspond to a rational value of x.*

Proposition 1.4. *Let X be a scheme of finite type over a field k. Then:*

1. *The closed points of X are dense in X (Zariski topology).*
2. *If X is irreducible, it has a unique generic point.*
3. *The closure of any point $p \in X$ consists of p and all the specializations of p.*

Definition 1.5 (Stalk of the Structure Sheaf)**.** *Let X be a scheme and $p \in X$ a point. The stalk of* *the structure sheaf $\mathcal{O}_X$ at p, denoted $\mathcal{O}_{X,p}$, is defined as the direct limit:*

$$\mathcal{O}_{X,p} = \varinjlim_{U \ni p} \mathcal{O}_X(U)$$

where the limit is taken over all open sets U containing the point p.

Proposition 1.6. *Let $X = \operatorname{Spec}(A)$ be an affine scheme and $p \in X$ the point corresponding to a prime ideal $\mathfrak{p} \subset A$. Then:*

$$\mathcal{O}_{X,p} \cong A_{\mathfrak{p}}$$

where $A_{\mathfrak{p}}$ is the localization of A at the prime ideal $\mathfrak{p}$.

Remark 1.7. *The stalk $\mathcal{O}_{X,p}$ is always a local ring. Its unique maximal ideal, denoted $\mathfrak{m}_p$, consists of germs of functions that vanish at p.*

Example 1.8. *Let $X = \operatorname{Spec}(\mathbb{C}[x, y])$ and p the origin (corresponding to the maximal ideal (x, y)). Then:*

$$\mathcal{O}_{X,p} \cong \mathbb{C}[x, y]_{(x,y)}$$

This is the ring of rational functions in x and y that are defined at the origin.

Example 1.9. *Let $X = \operatorname{Spec}(\mathbb{C}[x, y]/(xy))$, a union of two coordinate axes, and p the origin. Then:*

$$\mathcal{O}_{X,p} \cong \mathbb{C}[x, y]_{(x,y)}/(xy)$$

This local ring has zero divisors, reflecting the fact that p is a singular point of X.

Definition 1.10 (Residue Field)**.** *Let X be a scheme and $p \in X$ a point. The residue field at p, denoted $\kappa(p)$, is defined as:*

$$\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p$$

where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$.

Proposition 1.11. *Let $X = \operatorname{Spec}(A)$ be an affine scheme and $p \in X$ the point corresponding to a prime ideal $\mathfrak{p} \subset A$. Then:*

$$\kappa(p) \cong \operatorname{Frac}(A/\mathfrak{p})$$

the fraction field of the domain $A/\mathfrak{p}$.

Remark 1.12. *For a closed point p corresponding to a maximal ideal $\mathfrak{m}$, we have $\kappa(p) \cong A/\mathfrak{m}$, which is already a field.*

Example 1.13. *Let* $X = \operatorname{Spec}(\mathbb{C}[x, y])$.

1. *For the closed point* p *corresponding to the maximal ideal* $(x - a, y - b)$*, the residue field is:*

$$\kappa(p) \cong \mathbb{C}[x, y]/(x - a, y - b) \cong \mathbb{C}$$

2. *For the non-closed point* q *corresponding to the prime ideal* $(x - a)$*, the residue field is:*

$$\kappa(q) \cong \operatorname{Frac}(\mathbb{C}[x, y]/(x - a)) \cong \mathbb{C}(y)$$

   the field of rational functions in one variable.

3. *For the generic point* η *corresponding to the prime ideal* (0)*, the residue field is:*

$$\kappa(\eta) \cong \operatorname{Frac}(\mathbb{C}[x, y]) \cong \mathbb{C}(x, y)$$

   the field of rational functions in two variables.

Example 1.14. *Let* $X = \operatorname{Spec}(\mathbb{Q}[x])$.

1. *For the closed point* p *corresponding to the maximal ideal* $(x - a)$ *where* $a \in \mathbb{Q}$*, the residue field is:*

$$\kappa(p) \cong \mathbb{Q}[x]/(x - a) \cong \mathbb{Q}$$

2. *For the closed point* q *corresponding to the maximal ideal* $(x^2 + 1)$*, the residue field is:*

$$\kappa(q) \cong \mathbb{Q}[x]/(x^2 + 1) \cong \mathbb{Q}(i)$$

   which is a degree 2 extension of $\mathbb{Q}$.

3. *For the generic point* η *corresponding to the prime ideal* (0)*, the residue field is:*

$$\kappa(\eta) \cong \operatorname{Frac}(\mathbb{Q}[x]) \cong \mathbb{Q}(x)$$

Definition 1.15 (Geometric Point)**.** *A geometric point of a scheme* X *is a morphism* $\operatorname{Spec}(K) \to X$*, where* K *is an algebraically closed field.*

Remark 1.16. *A geometric point can be thought of as a scheme-theoretic point together with an embedding of its residue field into an algebraically closed field.*

Proposition 1.17. *Let* X *be a scheme over a field* k*. If* k *is algebraically closed, then every closed point of* X *naturally gives rise to a geometric point. If* k *is not algebraically closed, this is not generally true.*

Example 1.18. *For $X = \operatorname{Spec}(\mathbb{Q}[x])$, the closed point corresponding to (x^2+1) has residue field $\mathbb{Q}(i)$. This gives two distinct geometric points when we consider embeddings of $\mathbb{Q}(i)$ into $\mathbb{C}$ (corresponding to i and $-i$).*

1.2 Commutative Algebra

1.2.1 Associated Primes

Definition 1.19 (Support of a module)**.** *Let A be a ring and M an A-module. The support of M, denoted $\operatorname{Supp}(M)$, is the set of prime ideals*

$$\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$$

Definition 1.20 (Annihilator of a module)**.** *Let A be a ring and M an A-module. The annihilator of M, denoted $\operatorname{Ann}(M)$, is the ideal of elements*

$$\operatorname{Ann}(M) = \{a \in A \mid a \cdot m = 0 \textit{ for all } m \in M\}$$

Proposition 1.21. *Let A be a ring and M an A-module. Then*

$$\operatorname{Supp}(M) = \mathrm{V}(\operatorname{Ann}(M)) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \operatorname{Ann}(M) \subset \mathfrak{p}\}$$

In particular, the support of M is a closed subset of $\operatorname{Spec}(A)$.

Proof. First we show that if $\operatorname{Ann}(M) \subset \mathfrak{p}$, then $M_{\mathfrak{p}} \neq 0$. Annihilators behave well with respect to localization.

$$\begin{aligned} \operatorname{Ann}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) &= \{a/s \in A_{\mathfrak{p}} \mid a/s \cdot m/s = 0 \text{ for all } m/s \in M_{\mathfrak{p}}\} \\ &= \{a/s \in A_{\mathfrak{p}} \mid a \cdot m = 0 \text{ for all } m \in M\} \\ &= \operatorname{Ann}(M)A_{\mathfrak{p}} \subset \mathfrak{p}A_{\mathfrak{p}} \end{aligned}$$

In particular, $\operatorname{Ann}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is proper. If $M_{\mathfrak{p}} = 0$, then $\operatorname{Ann}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = A_{\mathfrak{p}}$ would be the whole ring.

Conversely, if $\operatorname{Ann}_A(M) \not\subset \mathfrak{p}$, then $M_{\mathfrak{p}} = 0$. There exists some $a \in \operatorname{Ann}_A(M)$ such that $a \notin \mathfrak{p}$. When we localize at $\mathfrak{p}$, a becomes invertible since $a \notin \mathfrak{p}$. Consider any element $m/s \in M_{\mathfrak{p}}$. We have $a \cdot m = 0$ in M, so $a \cdot m/s = 0$ in $M_{\mathfrak{p}}$. But a is invertible, so $m/s = 0$ in $M_{\mathfrak{p}}$. Therefore $M_{\mathfrak{p}} = 0$. $\square$

Remark 1.22. *There is a fundamental dictionary between commutative algebra and algebraic geometry:*

$$\mathfrak{p} \in \operatorname{Supp}(M) \iff M_{\mathfrak{p}} \neq 0 \iff \operatorname{Ann}(M) \subset \mathfrak{p} \iff \mathfrak{p} \in \mathrm{V}(\operatorname{Ann}(M))$$

Definition 1.23 (Associated Primes)**.** *Let A be a ring and M an A-module. An associated prime of M is a prime ideal of the form $\operatorname{Ann}(m)$ for some $m \in M$.*

Theorem 1.24 (Associated Primes of a Module)**.** *If I is an ideal in a Noetherian ring R and I is contained in all associated primes of a finitely generated module M, then $I^n M = 0$ for some positive integer n.*

Proof. 1. The associated primes of M are finite in number: $\operatorname{Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots \mathfrak{p}_r\}$

2. The radical of the annihilator equals the intersection of all associated primes:

$$\sqrt{\operatorname{Ann}(M)} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_r$$

If $I \subseteq \mathfrak{p}_i$ for all i, then $I \subseteq \sqrt{\operatorname{Ann}(M)}$. This means every element $f \in I$ satisfies $f^{n_f} \in \operatorname{Ann}(M)$ for some n_f. Since I is finitely generated, we can find a uniform N such that $I^N \subseteq \operatorname{Ann}(M)$. Therefore, $I^N M = 0$. □

To prove the first two statements, we need to turn to the theory of primary decompositions.

1.2.2 Primary decompositions

Primary decomposition generalizes the unique factorization of integers into prime powers. In a ring, it allows us to express ideals as intersections of simpler ideals called primary ideals.

Definition 1.25 (Prime, Primary Ideal)**.** *An ideal $\mathfrak{p}$ in a ring R is prime if:*

(i) $\mathfrak{p} \neq R$

(ii) For any $a, b \in R$, if $ab \in \mathfrak{p}$, then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$

An ideal $\mathfrak{q}$ in a ring R is primary if:

(i) $\mathfrak{q} \neq R$

(ii) For any $a, b \in R$, if $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$, then $b^n \in \mathfrak{q}$ for some positive integer n

Every primary ideal $\mathfrak{q}$ has an associated prime ideal $\mathfrak{p} = \sqrt{\mathfrak{q}}$ (the radical of $\mathfrak{q}$). We say $\mathfrak{q}$ is $\mathfrak{p}$-primary.

Theorem 1.26 (Primary Decomposition Theorem)**.** *In a Noetherian ring R, every ideal I can be* *expressed as a finite intersection of primary ideals:*

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$$

where each $\mathfrak{q}_i$ is a primary ideal with associated prime $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$.

Definition 1.27 (Minimal Primary Decomposition)**.** *A primary decomposition is called minimal if:*

(i) The associated primes $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ are all distinct

(ii) No $\mathfrak{q}_i$ can be removed from the intersection without changing the ideal I

Proposition 1.28. *In a minimal primary decomposition, the set of associated primes $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ is uniquely determined by the ideal I.*

The theory extends naturally to modules over Noetherian rings.

Theorem 1.29. *For a submodule N of a module M over a Noetherian ring R, we can find a primary decomposition:*

$$N = N_1 \cap N_2 \cap \cdots \cap N_n$$

where each N_i is a $\mathfrak{p}_i$-primary submodule of M. This means M/N_i has the property that for each $r \in R$ and $m + N_i \in M/N_i$, if $r(m + N_i) = 0 + N_i$ and $r \notin \mathfrak{p}_i$, then $(m + N_i)^n = 0 + N_i$ for some $n > 0$.

When considering the zero submodule (0) in M, and its minimal primary decomposition

$$(0) = N_1 \cap N_2 \cap \cdots \cap N_n$$

the associated primes of this decomposition are exactly the associated primes of M.

Theorem 1.30. *In a Noetherian ring, the set of associated primes of a finitely generated module M is finite.*

Proof. The finiteness follows directly from:

(1) The Noetherian property ensures every ideal has a finite primary decomposition

(2) The associated primes of a module M correspond to the primes in the primary decomposition of the zero submodule (0) in M

(3) Therefore, $\mathrm{Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$ is finite

□

Theorem 1.31. *For a finitely generated module M over a Noetherian ring R with associated primes $\mathrm{Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$, the following equality holds:*

$$\sqrt{\mathrm{Ann}(M)} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$$

Proof. This equality follows from:

(1) The primary decomposition of (0) in M as $(0) = N_1 \cap N_2 \cap \cdots \cap N_r$ where each N_i is $\mathfrak{p}_i$-primary

(2) An element $a \in R$ annihilates M completely if and only if $aM \subseteq N_1 \cap N_2 \cap \cdots \cap N_r = (0)$

(3) If $a \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$, then some power of a lies in each $\mathfrak{q}_i$ (the primary ideals associated with the N_i), which means some power of a annihilates M, so $a \in \sqrt{\mathrm{Ann}(M)}$

(4) Conversely, if $a \in \sqrt{\mathrm{Ann}(M)}$, then some power of a annihilates M, which means a belongs to each $\mathfrak{p}_i$

This establishes the equality $\sqrt{\mathrm{Ann}(M)} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$. □

1.3 Sheaves

Definition 1.32 (Quasi-Coherent Sheaf)**.** *Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called quasi-coherent if for every open subset $U \subset X$, there exists a covering $\{U_i\}$ of U and a family of $\mathcal{O}_{U_i}$-modules $\mathcal{F}_i$ such that for each i, there exists an isomorphism $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$.*

Definition 1.33 (Coherent Sheaf)**.** *Let X be a scheme. A quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called coherent if:*

1. *$\mathcal{F}$ is of finite type, i.e., for every open subset $U \subset X$, there exists a surjection $\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$ for some integer n.*

2. *For any open set $U \subset X$ and any morphism $\varphi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$ of $\mathcal{O}_U$-modules, the kernel $\ker \varphi$ is of finite type.*

Definition 1.34 (Support of a Sheaf)**.** *Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. The* *support of $\mathcal{F}$, denoted $\mathrm{Supp}(\mathcal{F})$, is the set of points $x \in X$ where the stalk $\mathcal{F}_x$ is non-zero:*

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$$

Proposition 1.35. *For a coherent sheaf $\mathcal{F}$ on a scheme X:*

1. $\mathrm{Supp}(\mathcal{F})$ *is a closed subset of X.*
2. *If X is Noetherian, then $\mathrm{Supp}(\mathcal{F})$ equals the set of points where the fiber $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is non-zero.*
3. *For an affine scheme $X = \mathrm{Spec}(A)$ and $\mathcal{F} = \widetilde{M}$ corresponding to an A-module M, the support of $\mathcal{F}$ corresponds to $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$.*

Remark 1.36. *On a noetherian scheme, a sheaf of $\mathcal{O}_X$-modules is coherent if and only if it is of finite type.*

Definition 1.37 (Vector Bundle)**.** *A vector bundle of rank r on a scheme X is a coherent sheaf $\mathcal{E}$ on X that is locally free of rank r, i.e., for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$.*

Definition 1.38 (Torsion Sheaf)**.** *A coherent sheaf $\mathcal{F}$ on a scheme X is called a torsion sheaf if for any open affine subset $\mathrm{Spec}(A) \subset X$, the corresponding A-module $\Gamma(\mathrm{Spec}(A), \mathcal{F})$ is a torsion A-module. If X is locally noetherian, then equivalently $\mathcal{F}$ is a torsion sheaf if its support is a proper closed subset of X.*

Proof. Suppose $\mathcal{F}$ is a coherent sheaf on X with support that is a proper closed subset of X. Let Spec$(A) \subset X$ be an affine open subset. We want to show that $\Gamma(\text{Spec}(A), \mathcal{F})$ is a torsion A-module. Since Supp$(\mathcal{F})$ is a proper closed subset of X, its intersection with Spec(A) is a proper closed subset of Spec(A). This means there exists a non-zero ideal $I \subset A$ such that $V(I)$ contains Supp$(\mathcal{F}) \cap$ Spec(A).

Algebraically, this means I is contained in all prime ideals $P \in$ Supp$(\mathcal{F}) \cap$ Spec(A). The associated primes of $\Gamma(\text{Spec}(A), \mathcal{F})$ are contained in Supp$(\mathcal{F}) \cap$ Spec(A), so I is contained in all associated primes of $\Gamma(\text{Spec}(A), \mathcal{F})$. Since A is Noetherian (X has to be locally noetherian), and I is contained in all associated primes of the finitely generated A-module $\Gamma(\text{Spec}(A), \mathcal{F})$, by Theorem 1.24, there exists some $n > 0$ such that:

$$I^n \cdot \Gamma(\text{Spec}(A), \mathcal{F}) = 0$$

This shows that $\Gamma(\text{Spec}(A), \mathcal{F})$ is a torsion A-module.

Conversely, suppose for every affine open Spec$(A) \subset X$, the A-module $\Gamma(\text{Spec}(A), \mathcal{F})$ is a torsion module. For each $m \in \Gamma(\text{Spec}(A), \mathcal{F})$, there exists a non-zero element $a_m \in A$ such that $a_m \cdot m = 0$. Let I be the ideal generated by all such a_m for all $m \in \Gamma(\text{Spec}(A), \mathcal{F})$. By construction, I is non-zero and $I \cdot \Gamma(\text{Spec}(A), \mathcal{F}) = 0$. This means $V(I)$ contains Supp$(\mathcal{F}) \cap$ Spec(A). Since I is non-zero, $V(I)$ is a proper closed subset of Spec(A). Therefore, the support of $\mathcal{F}$ restricted to Spec(A) is a proper closed subset. Since this holds for all affine opens covering X, we conclude that Supp$(\mathcal{F})$ is a proper closed subset of X, making $\mathcal{F}$ a torsion sheaf. □

Definition 1.39 (Points of a Scheme)**.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. The points of X are in one-to-one correspondence with the prime ideals of A. Given a prime ideal $\mathfrak{p} \subset A$, we denote the corresponding point by $p_\mathfrak{p}$, or simply p when the context is clear.*

To unpack this example, we recall how to work with the tangent sheaf of an affine variety.

Definition 1.40. *Let X be a variety over a field k. The* ***tangent sheaf*** *of X, denoted $\mathcal{T}_X$, is defined as the dual of the sheaf of differentials:*

$$\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \tag{1}$$

For an affine variety $X = $ Spec(R) where $R = k[x_1, \ldots, x_n]/I$, the tangent sheaf has a concrete description in terms of derivations.

Proposition 1.41. *Let $X =$ Spec(R) be an affine variety. Then the global sections of the tangent sheaf correspond to the R-module of k-derivations of R:*

$$\Gamma(X, \mathcal{T}_X) \cong \textit{Der}_k(R, R) \tag{2}$$

Definition 1.42. *A k-derivation on a k-algebra R is a k-linear map $D : R \to R$ satisfying the Leibniz rule:*

$$D(fg) = fD(g) + gD(f) \quad \textit{for all } f, g \in R \tag{3}$$

Proposition 1.43. *The tangent sheaf $\mathcal{T}_X$ of any variety X is a coherent sheaf of $\mathcal{O}_X$-modules.*

Proposition 1.44. *For a smooth variety X, the tangent sheaf $\mathcal{T}_X$ is locally free of rank equal to the dimension of X.*

1.4 Dual Perspective: The Cotangent Sheaf

For completeness, we briefly discuss the dual perspective in terms of the cotangent sheaf.

Definition 1.45. *The* ***cotangent sheaf*** *or* ***sheaf of differentials*** Ω^1_X *of a variety* X *over* k *is the sheaf associated to the presheaf* $U \mapsto \Omega^1_{\mathcal{O}_X(U)/k}$*, where* $\Omega^1_{A/k}$ *denotes the module of Kähler differentials of a* k*-algebra* A*.*

Proposition 1.46. *For an affine variety* $X = Spec(R)$ *where* $R = k[x_1, \ldots, x_n]/I$*, the module of Kähler differentials can be described as:*

$$\Omega^1_{R/k} \cong \frac{R \cdot dx_1 \oplus \cdots \oplus R \cdot dx_n}{\{df \mid f \in I\}} \tag{4}$$

2 Examples of Non-Vector Bundle Coherent Sheaves

2.1 Skyscraper Sheaf

Example 2.1 (Skyscraper Sheaf)**.** *Let* X *be a scheme and* $p \in X$ *a point. The skyscraper sheaf* $\mathcal{O}_p$ *is a coherent sheaf defined as:*

$$\mathcal{O}_p(U) = \begin{cases} \kappa(p) & \textit{if } p \in U \\ 0 & \textit{if } p \notin U \end{cases}$$

The residue field $\kappa(p)$ *is a module over several rings. In particular, we can see that it is coherent because it is generated by a single element over the ring at hand.*

- $\mathcal{O}_{X,p}$*-module via the natural quotient map* $\mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/\mathfrak{m}_p$
- $\mathcal{O}_X(U)$*-module for any open set* U *containing* p
- *For affine opens* $U = \mathrm{Spec}(A)$ *containing* p*, it's an* A*-module*
  - *If* p *corresponds to the prime ideal* $\mathfrak{p} \subset A$
  - *Then* $\kappa(p) \cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong \mathrm{Frac}(A/\mathfrak{p})$
  - *The action is via* $A \to A/\mathfrak{p} \to \mathrm{Frac}(A/\mathfrak{p})$

It is not a vector bundle because:

- *It fails to be locally free at all points. It is a torsion sheaf: any function vanishing at* p *annihilates the entire sheaf.*
- *Its support is just the single point* $\{p\}$*, whereas vector bundles have support equal to* X*.*

2.2 Tangent sheaf of nodal curve

Example 2.2 (Tangent Sheaf of a Singular Variety)**.** *For a singular variety X, the tangent sheaf $\mathcal{T}_X$ is coherent but not a vector bundle because:*

- *At smooth points $x \in X$, the sheaf is locally free of rank $\dim X$.*
- *At singular points, the stalk $(\mathcal{T}_X)_x$ fails to be a free $\mathcal{O}_{X,x}$-module.*
- *For example, on a nodal curve, the tangent sheaf at the node has torsion.*

Consider the nodal curve $X = \operatorname{Spec} R = \operatorname{Spec} k[x, y]/(y^2 - x^2)$. The curve X has a singularity (node) at the origin, where the two branches (given by $y = x$ and $y = -x$) intersect. We work with derivations on the quotient ring $R = k[x, y]/(y^2 - x^2)$.

Lemma 2.3. *Any k-derivation D on the polynomial ring $k[x, y]$ can be written uniquely as*

$$D = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \tag{5}$$

for some $a, b \in k[x, y]$, where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the standard partial derivatives.

Proof. A derivation D on $k[x, y]$ is completely determined by its values on the generators x and y. Setting $a = D(x)$ and $b = D(y)$, we can verify that $D = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ by the Leibniz rule and linearity of D. $\square$

For a derivation on $k[x, y]$ to induce a well-defined derivation on $R = k[x, y]/(y^2 - x^2)$, it must preserve the ideal $(y^2 - x^2)$. This constrains the possible derivations.

Proposition 2.4. *Let $\bar{x}$ and $\bar{y}$ denote the images of x and y in the quotient ring $R = k[x, y]/(y^2 - x^2)$. A k-derivation D on R is determined by its values $D(\bar{x}) = \bar{a}$ and $D(\bar{y}) = \bar{b}$, where $\bar{a}, \bar{b} \in R$ are subject to the constraint:*

$$\bar{y} \cdot \bar{b} = \bar{x} \cdot \bar{a} \tag{6}$$

Proof. Since $\bar{y}^2 = \bar{x}^2$ in R, applying the derivation D to both sides and using the Leibniz rule, we get:

$$D(\bar{y}^2) = D(\bar{x}^2) \tag{7}$$

$$2\bar{y} \cdot D(\bar{y}) = 2\bar{x} \cdot D(\bar{x}) \tag{8}$$

$$2\bar{y} \cdot \bar{b} = 2\bar{x} \cdot \bar{a} \tag{9}$$

Simplifying, we obtain $\bar{y} \cdot \bar{b} = \bar{x} \cdot \bar{a}$. $\square$

Theorem 2.5. *The tangent sheaf $\mathcal{T}_X$ of the nodal curve X defined by $y^2 = x^2$ can be described as:*

$$\mathcal{T}_X \cong \{(\bar{a}, \bar{b}) \in R \times R \mid \bar{y} \cdot \bar{b} = \bar{x} \cdot \bar{a}\} \tag{10}$$

where each pair $(\bar{a}, \bar{b})$ corresponds to the derivation $D = \bar{a}\frac{\partial}{\partial \bar{x}} + \bar{b}\frac{\partial}{\partial \bar{y}}$.

Remark 2.6. *This description shows that $\mathcal{T}_X$ is a submodule of the free R-module $R \times R$, defined by a single linear constraint.*

To show that the tangent sheaf is not locally free at the node, we need to identify torsion elements.

Proposition 2.7. *The tangent sheaf $\mathcal{T}_X$ has torsion at the node, proving it is not a free R-module.*

Proof. Consider the element $v = (\bar{y} + \bar{x}, \bar{y} + \bar{x}) \in \mathcal{T}_X$. This corresponds to the derivation $(\bar{y} + \bar{x})\frac{\partial}{\partial \bar{x}} + (\bar{y} + \bar{x})\frac{\partial}{\partial \bar{y}}$.

First, let's verify that $v \in \mathcal{T}_X$ by checking the constraint:

$$\bar{y}(\bar{y} + \bar{x}) = \bar{x}(\bar{y} + \bar{x}) \tag{11}$$

$$\bar{y}^2 + \bar{x}\bar{y} = \bar{x}\bar{y} + \bar{x}^2 \tag{12}$$

Using the relation $\bar{y}^2 = \bar{x}^2$, this equation is satisfied.

Now, let's multiply v by the element $f = \bar{y} - \bar{x} \in R$:

$$\begin{aligned} f \cdot v &= (\bar{y} - \bar{x}) \cdot (\bar{y} + \bar{x}, \bar{y} + \bar{x}) && (13) \\ &= ((\bar{y} - \bar{x})(\bar{y} + \bar{x}), (\bar{y} - \bar{x})(\bar{y} + \bar{x})) && (14) \\ &= (\bar{y}^2 - \bar{x}^2, \bar{y}^2 - \bar{x}^2) && (15) \end{aligned}$$

Since $\bar{y}^2 = \bar{x}^2$ in R, we have $f \cdot v = (0, 0)$. This demonstrates that v is a non-zero element of the tangent sheaf that is annihilated by the non-zero element f, confirming that $\mathcal{T}_X$ has torsion at the node. $\square$

Remark 2.8 (Geometric Interpretation)**.** *The element $v = (\bar{y} + \bar{x}, \bar{y} + \bar{x})$ corresponds to a derivation that acts as the standard tangent vector along the component $y = x$ but vanishes along the component $y = -x$. The element $f = \bar{y} - \bar{x}$ that annihilates this derivation is precisely the function that defines the component where the derivation doesn't vanish.*

This reflects the fact that at the node, we have two distinct tangent directions (corresponding to the two branches of the curve), and a vector field can act along one branch while vanishing along the other. Such vector fields cannot be part of a free basis for the tangent sheaf, as they are annihilated by functions that distinguish between the branches.

3 Exceptional Collections in Derived Categories

Definition 3.1 (Exceptional Object)**.** *An object E in a derived category $D^b(X)$ is called exceptional if:*

1. $\mathrm{Hom}(E, E) \cong k$ *(the base field)*
2. $\mathrm{Hom}(E, E[n]) = 0$ *for all* $n \neq 0$

Definition 3.2 (Exceptional Collection)**.** *An exceptional collection in $D^b(X)$ is an ordered sequence of exceptional objects $\{E_1, E_2, \ldots, E_n\}$ such that:*

$$\mathrm{Hom}(E_j, E_i[m]) = 0 \quad \textit{for all } j > i \textit{ and all } m \in \mathbb{Z}$$

Definition 3.3 (Full Exceptional Collection)**.** *An exceptional collection $\{E_1, E_2, \ldots, E_n\}$ in $D^b(X)$ is called full if the objects generate the derived category. Formally, this means the smallest triangulated subcategory of $D^b(X)$ containing the collection and closed under direct sums and direct summands is $D^b(X)$ itself.*

Equivalently, for any object $Y \in D^b(X)$, if $\mathrm{Hom}(E_i[m], Y) = 0$ for all $i = 1, 2, \ldots, n$ and all $m \in \mathbb{Z}$, then $Y \cong 0$.

Definition 3.4 (Strong Exceptional Collection)**.** *An exceptional collection $\{E_1, E_2, \ldots, E_n\}$ is called strong if:*

$$\mathrm{Hom}(E_i, E_j[m]) = 0 \quad \textit{for all } i, j \textit{ and all } m \neq 0$$

4 Semiorthogonal Decompositions

Definition 4.1 (Semiorthogonal Decomposition)**.** *A semiorthogonal decomposition of a triangulated category $\mathcal{T}$ is a sequence of full triangulated subcategories $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ such that:*

1. *For any objects $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ with $i > j$, we have* $\mathrm{Hom}(A_i, A_j) = 0$.
2. *For any object $T \in \mathcal{T}$, there exists a unique sequence of morphisms:*

$$0 = T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 = T$$

such that the cone of each morphism $T_i \to T_{i-1}$ lies in $\mathcal{A}_i$ for $i = 1, 2, \ldots, n$.

We denote this by $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \rangle$.

Proposition 4.2. *A full exceptional collection $\{E_1, E_2, \ldots, E_n\}$ in $D^b(X)$ gives rise to a semiorthogonal decomposition:*

$$D^b(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \rangle$$

where $\mathcal{A}_i$ is the triangulated subcategory generated by E_i.

5 The Splitting Problem and Beilinson's Exceptional Collection

5.1 The Splitting Problem

The splitting problem in algebraic geometry asks: When is a vector bundle on a variety isomorphic to a direct sum of line bundles?

Theorem 5.1 (Grothendieck)**.** *Every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles:*

$$\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}(a_i)$$

for some integers $a_1, a_2, \ldots, a_r$.

However, for projective spaces of higher dimension, the situation is different:

Theorem 5.2. *For $n \geq 2$, there exist vector bundles on $\mathbb{P}^n$ that do not split as direct sums of line bundles.*

Example 5.3. *The tangent bundle $T\mathbb{P}^n$ is a non-split vector bundle on $\mathbb{P}^n$ for $n \geq 2$. It fits into the Euler sequence:*

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T\mathbb{P}^n \to 0$$

This sequence is non-split, as it represents a non-zero element in the group $\mathrm{Ext}^1(T\mathbb{P}^n, \mathcal{O})$.

5.2 Beilinson's Exceptional Collection

Theorem 5.4 (Beilinson)**.** *The collection $\{\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots, \mathcal{O}(n)\}$ is a full exceptional collection in $D^b(\mathbb{P}^n)$.*

This has several important consequences:

1. Every coherent sheaf (or complex of coherent sheaves) on $\mathbb{P}^n$ can be reconstructed from its "cohomological information" with respect to this collection.

2. The Grothendieck group $K_0(\mathbb{P}^n)$ is a free abelian group of rank $n+1$ with basis given by the classes $[\mathcal{O}], [\mathcal{O}(1)], \ldots, [\mathcal{O}(n)]$. This means that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, its class in $K_0(\mathbb{P}^n)$ can be written uniquely as a integer linear combination of these classes:

$$[\mathcal{F}] = a_0[\mathcal{O}] + a_1[\mathcal{O}(1)] + \cdots + a_n[\mathcal{O}(n)]$$

Remark 5.5. *The fact that any coherent sheaf can be written as a linear combination of $[\mathcal{O}(i)]$ in $K_0(\mathbb{P}^n)$ does not imply that every vector bundle on $\mathbb{P}^n$ splits as a direct sum of line bundles.*

In particular the tangent bundle $T\mathbb{P}^n$ has class:

$$[T\mathbb{P}^n] = (n+1)[\mathcal{O}(1)] - [\mathcal{O}]$$

in $K_0(\mathbb{P}^n)$, but this does not mean $T\mathbb{P}^n \cong \mathcal{O}(1)^{\oplus(n+1)} \oplus \mathcal{O}^{\oplus(-1)}$, which is not even meaningful for a negative exponent.

The obstruction to a vector bundle splitting is measured by extension groups Ext^1, which precisely capture the non-splitting of exact sequences.