Explaining the Identification $H^1(X, \mathbb{Z}_p) \cong T_pJ(-1)$

Let X be a smooth projective curve over an algebraically closed field k of characteristic different from p, and let $J = \mathrm{Pic}^0(X) \cong \mathrm{Jac}(X)$ be its Jacobian.

We aim to explain the canonical isomorphism:

$$H^1(X, \mathbb{Z}_p) \cong T_pJ(-1)$$

in a way that avoids defining one side in terms of the other.

1. Étale Cohomology and the Kummer Sequence

Consider the Kummer exact sequence of étale sheaves:

$$1 \to \mu_{p^r} \to \mathcal{O}_X^* \xrightarrow{(\cdot)^{p^r}} \mathcal{O}_X^* \to 1$$

This induces a long exact sequence in étale cohomology:

$$H^0(X, \mathcal{O}_X^*) \xrightarrow{(\cdot)^{p^r}} H^0(X, \mathcal{O}_X^*) \to H^1(X, \mu_{p^r}) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{(\cdot)^{p^r}} H^1(X, \mathcal{O}_X^*)$$

If k is algebraically closed and X is proper and connected, then $H^0(X, \mathcal{O}_X^*) = k^*$, and the map $(\cdot)^{p^r} : k^* \to k^*$ is surjective. Hence, the connecting homomorphism

$$\delta \colon H^1(X, \mu_{p^r}) \to \mathrm{Pic}(X)$$

is injective with image equal to $\mathrm{Pic}(X)[p^r]$, giving an isomorphism:

$$H^1(X, \mu_{p^r}) \cong \mathrm{Pic}(X)[p^r] \cong J[p^r](k)$$

2. Passing to the Limit

Taking the inverse limit over r, we obtain:

$$\varprojlim_r H^1(X, \mu_{p^r}) \cong \varprojlim_r J[p^r](k) = T_p J$$

This gives an isomorphism:

$$H^1(X, \mathbb{Z}_p(1)) := \varprojlim_r H^1(X, \mu_{p^r}) \cong T_p J$$

Note: here we define $\mathbb{Z}_p(1) := \varprojlim_r \mu_{p^r}$, and view this as a sheaf or Galois module with the natural action of $\mathrm{Gal}(\overline{k}/k)$ (if applicable).

3. The Tate Twist

The Tate twist (-1) corresponds to tensoring with the sheaf:

$$\mathbb{Z}_p(-1) := \underline{\mathrm{Hom}}(\mathbb{Z}_p(1), \mathbb{Z}_p)$$

This is the dual Galois module to $\mathbb{Z}_p(1)$. Therefore, we get:

$$H^1(X, \mathbb{Z}_p) \cong H^1(X, \mathbb{Z}_p(1)) \otimes \mathbb{Z}_p(-1)$$

Combining with the earlier identification:

$$H^1(X, \mathbb{Z}_p) \cong T_p J(-1)$$

4. Conclusion

We have constructed the isomorphism:

$$H^1_{\mathrm{et}}(X, \mathbb{Z}_p) \cong T_p J(-1)$$

by:

- Using the Kummer sequence to identify $H^1(X, \mu_{p^r}) \cong \mathrm{Pic}(X)[p^r]$
- Taking inverse limits to define $H^1(X, \mathbb{Z}_p(1)) \cong T_p J$
- Interpreting the Tate twist as dualizing the cyclotomic Galois module

This route avoids any circularity and builds the identification from geometric input.