

# Geometric invariant theory

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January 15, 2026

## Abstract

These are reading notes for *Geometric Invariant Theory* by Mumford, Fogarty and Kirwan.

## 1 Example

Consider the action of  $G = \mathrm{PGL}(n+1)$  on  $X = (\mathbb{P}^n)^{m+1}$  using the line bundle

$$L = \mathcal{O}_{\mathbb{P}^n}(1)^{\boxtimes(m+1)} = \mathcal{O}(1, \dots, 1) = \pi_1^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \cdots \otimes \pi_{m+1}^* \mathcal{O}_{\mathbb{P}^n}(1)$$

where  $\pi_i : X \rightarrow \mathbb{P}^n$  is the projection to the  $i$ -th factor. We need to lift the geometric action of  $G$  on  $X$  to a linear action on  $L$ . The natural group that acts linearly on  $\mathcal{O}_{\mathbb{P}^n}(1)$  is  $\mathrm{GL}(n+1)$ . There is no canonical way to make an element of  $\mathrm{PGL}(n+1)$  act linearly on the fibers of  $\mathcal{O}(1)$ , because it is only defined up to scalar. The exact sequence is

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathrm{GL}(n+1) \rightarrow \mathrm{PGL}(n+1) \rightarrow 1$$

Why don't we stay with  $\mathrm{GL}(n+1)$  instead of  $\mathrm{PGL}(n+1)$ ? Because the center  $\mathbb{G}_m$  acts trivially on  $X$  and this introduces a useless symmetry which breaks stability.

We can restrict to the subgroup  $\mathrm{SL}(n+1) \subset \mathrm{GL}(n+1)$ , which kills most of the scalars except for the finite center  $\mu_{n+1}$ . We want the linearization to descend to  $\mathrm{PGL}(n+1)$ , so we need the center  $\mu_{n+1} = \ker(\mathrm{SL}(n+1) \rightarrow \mathrm{PGL}(n+1))$  to act trivially on the fibers of  $L$ .

On  $\mathcal{O}_{\mathbb{P}^n}(1)$ , a scalar  $\zeta I \in \mu_{n+1} \subset \mathrm{SL}(n+1)$  acts as multiplication by  $\zeta$  on each fiber.

On

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)^{\boxtimes m},$$

it therefore acts as multiplication by  $\zeta^m$ .

Hence the  $\mathrm{SL}(n+1)$ -linearization of  $\mathcal{L}$  factors through  $\mathrm{PGL}(n+1)$  if and only if every  $\zeta \in \mu_{n+1}$  acts trivially on  $\mathcal{L}$ , i.e.

$$\zeta^m = 1 \quad \text{for all } \zeta \text{ with } \zeta^{n+1} = 1.$$

This holds if and only if

$$n + 1 \mid m.$$

More generally, if we consider the line bundle

$$L_i = \mathcal{O}_{\mathbb{P}^n}(a_i)$$

on the  $i$ -th factor, then the same argument shows that the  $SL(n+1)$ -linearization of

$$\mathcal{L} = \bigotimes_{i=1}^{m+1} \pi_i^* L_i = \mathcal{O}_{\mathbb{P}^n}(a_1, \dots, a_{m+1})$$

descends to  $PGL(n+1)$  if and only if

$$n + 1 \mid \sum_{i=1}^{m+1} a_i.$$

In any case, by means of these linearizations we can define invariant sections of all the sheaves  $\mathcal{L}_\alpha$ . To construct such invariant sections, let  $X_0, \dots, X_n$  be the canonical sections of  $\mathcal{O}_{\mathbb{P}^n}(1)$  on  $\mathbb{P}^n$ . Let

$$X_i^{(j)} = \pi_j^*(X_i)$$

be the induced sections of  $L_j$ .

**Definition 1.1.** For all sequences  $\alpha = (\alpha_0, \dots, \alpha_n)$  of integers such that  $0 \leq \alpha_i \leq m$ , let

$$D_{\alpha_0, \dots, \alpha_n} = \det(X_i^{(\alpha_j)})_{0 \leq i, j \leq n}$$

be the section of  $L_{\alpha_0} \otimes \cdots \otimes L_{\alpha_n}$  obtained by addition and tensor product as in the determinant.

It is evident that  $D_{\alpha_0, \dots, \alpha_n}$  is an invariant section of  $L_{\alpha_0} \otimes \cdots \otimes L_{\alpha_n}$ . The non-vanishing of suitable  $D$ 's defines the open sets we are looking for. Explicitly, a point of  $(\mathbb{P}^n)^{m+1}$  is a tuple

$$(p_0, \dots, p_m), \quad p_j = [v_j], \quad v_j \in k^{n+1} \setminus \{0\}.$$

Choose homogeneous lifts  $v_j$ . Put them as columns of a matrix

$$M = [v_0 \mid v_1 \mid \cdots \mid v_m] \in \text{Mat}_{n+1, m+1}.$$

Then

$$D_{\alpha_0, \dots, \alpha_n} = \det(v_{\alpha_0}, \dots, v_{\alpha_n}).$$

The fact that  $D_{\alpha_0, \dots, \alpha_n}$  is well defined as a section of  $L_{\alpha_0} \otimes \cdots \otimes L_{\alpha_n}$  follows from the following properties:

- $D_{\alpha_0, \dots, \alpha_n} = 0$  iff the points  $p_{\alpha_0}, \dots, p_{\alpha_n}$  lie in a hyperplane.
- Under  $g \in \mathrm{GL}(n+1)$ , all minors are multiplied by  $\det(g)$ .
- Rescaling columns rescales the corresponding minors.

**Definition 1.2.** An  $R$ -partition of  $\{0, 1, \dots, n\}$  is an ordered set of subsets  $S_1, \dots, S_\nu$  of  $\{0, 1, \dots, n\}$  such that

- $S_i \cap (S_1 \cup \dots \cup S_{i-1})$  consists of exactly one integer for  $i = 2, \dots, \nu$
- $\bigcup_i S_i = \{0, 1, \dots, n\}$ .

**Definition 1.3.** Given an  $R$ -partition  $R = \{S_1, \dots, S_\nu\}$ , let  $U_R \subset (\mathbb{P}^n)^{m+1}$  be the open subset defined by

- $D_{0,1,\dots,n} \neq 0$ ,
- for all  $k$  between 1 and  $\nu$ , and for all  $i \in S_k$ ,

$$D_{0,\dots,i-1,i+1,\dots,n,n+k} \neq 0$$

Not only is  $U_R$  affine, but the whole structure of the action of  $PGL(n+1)$  on  $U_R$  can be described explicitly. On each open set  $U_R$ , a configuration of points in  $(\mathbb{P}^n)^{m+1}$  is uniquely the same thing as

1. a projective frame, and
2. a collection of free affine parameters

**Proposition 1.4.** Let  $R = \{S_1, \dots, S_\nu\}$  be an  $R$ -partition of  $\{0, 1, \dots, n\}$ . Let  $PGL(n+1)$  act on  $PGL(n+1) \times \mathbb{A}^{n\nu-n}$  by the product of left translation on itself and the trivial action on the affine space. Then there is a  $PGL(n+1)$ -linear isomorphism:

$$U_R \cong PGL(n+1) \times \mathbb{A}^{n\nu-n}.$$

Hence  $U_R$  is a globally trivial principal fibre bundle with respect to the action of  $PGL(n+1)$ , with base space  $\mathbb{A}^{n\nu-n}$ .

*Proof.* Fix an  $R$ -partition and the associated open set  $U_R \subset X$ . On  $U_R$ , define sections  $\lambda_j$  by

$$\lambda_{\mu(1)} := 1, \quad \lambda_j := \lambda_{\mu(x(j))} \frac{D_{0,1,\dots,\widehat{\mu(x(j))},\dots,n,j}}{D_{0,1,\dots,n}} \quad (0 \leq j \leq n).$$

These satisfy  $\lambda_j \in \Gamma(U_R, L_j \otimes L_{\mu(1)}^{-1})$ .

We define

$$\phi = (\phi_1, \phi_2) : U_R \longrightarrow PGL(n+1) \times A_R$$

as follows.

Identifying  $PGL(n+1)$  with the open subset  $\{\det \neq 0\} \subset \mathbb{P}^{(n+1)^2-1}$  with homogeneous coordinates  $a_{ij}$ , define  $\phi_1$  by

$$(\phi_1)^*(a_{ij}) = (-1)^j X_i^{(j)} \otimes \lambda_j^{-1}.$$

Define  $\phi_2$  by, for  $k \geq 1$ ,

$$(\phi_2)^*(x_i^{(n+k)}) = \frac{D_{0,1,\dots,\hat{i},\dots,n,n+k}}{D_{0,1,\dots,n}} \frac{\lambda_i}{\lambda_{\mu(k)}}.$$

Then  $\phi$  is a  $PGL(n+1)$ -equivariant isomorphism

$$U_R \cong PGL(n+1) \times A_R.$$

Unraveling over a field, a point of  $U_R$  corresponds to a tuple of points  $(p_0, \dots, p_m)$  in  $(\mathbb{P}^n)^{m+1}$  satisfying the conditions defining  $U_R$ . Writing  $p_j = [v_j]$  with  $v_j \in k^{n+1} \setminus \{0\}$  with  $\det[v_0 \mid v_1 \mid \dots \mid v_n] \neq 0$ , the map  $\phi$  sends this point to

$$A^{-1}, A^{-1}v_{n+1}, \dots, A^{-1}v_m$$

where  $A = [v_0 \mid v_1 \mid \dots \mid v_n]$ .  $\square$

What is this R-partition formalism really saying? When  $\nu = 1$ , then we are forced to take  $R = \{S_1\}$  and  $S_1 = \{0, 1, \dots, n\}$ . The definition of  $U_R$  reads  $D_{0,1,\dots,n} \neq 0$ , so  $U_R$  is those tuples of points  $(p_0, \dots, p_{n+1})$  for which the points  $p_0, \dots, p_n$  are not colinear and  $p_{n+1}$  is not in any of the coordinate hyperplanes determined by them, i.e.  $n+2$  points in  $\mathbb{P}^n$  in general position. Then there exists a unique projective transformation  $g \in PGL(n+1)$  sending  $p_i$  to  $e_i$  for  $i = 0, \dots, n$  and sending  $p_{n+1}$  to  $[1 : 1 : \dots : 1]$ . So in particular  $U_R = PGL(n+1)$  in this case.

We can also study the case when  $n = 2$ ,  $\nu = 2$  and  $S_1 = \{0, 1\}$  and  $S_2 = \{1, 2\}$ . We are then thinking about  $m = n + \nu + 1 = 5$  points in  $\mathbb{P}^2$ , say  $(p_0, p_1, p_2, p_3, p_4)$ . The set  $U_R$  is then cut out by the nonvanishing of the determinants  $D_{0,1,2}$ ,  $D_{1,2,3}$ ,  $D_{0,2,3}$ ,  $D_{0,2,4}$ , and  $D_{0,1,4}$ . Geometrically this means that  $p_0, p_1, p_2$  are not colinear,  $p_3$  does not lie on the lines  $p_1p_2$  and  $p_0p_2$ , and  $p_4$  does not lie on the lines  $p_0p_1$  and  $p_0p_2$ . Then we can uniquely normalize  $p_0, p_1, p_2$  to be the coordinate points  $[1 : 0 : 0]$ ,  $[0 : 1 : 0]$ ,  $[0 : 0 : 1]$ , and normalize  $p_3$  and  $p_4$  to be  $[1, 1, a]$  and  $[b, 1, 1]$ .

**Definition 1.5.** For fixed  $m$  let  $U_{reg} \subset (\mathbb{P}^n)^{m+1}$  be union of all  $U_R$  where  $R$  runs through all R-partitions of  $\{0, 1, \dots, n\}$  with  $\nu = m - n$ .

**Proposition 1.6 (Proposition 3.3).** Let  $x = (x^{(0)}, x^{(1)}, \dots, x^{(m)})$  be a geometric point of  $(\mathbb{P}^n)^{m+1}$ . Then the following are equivalent:

1. The stabilizer  $S(x)$  is 0-dimensional.

2. There do not exist disjoint proper linear subspaces  $L'$  and  $L''$  of  $\mathbb{P}^n$  such that every  $x^{(i)}$  lies in either  $L'$  or  $L''$ .
3.  $x$  is a geometric point of  $U_{\text{reg}}$ .

*Proof.* Let  $k$  be the algebraically closed field over which  $x$  is defined. For simplicity, we shall write  $\mathbb{P}$  for  $\mathbb{P}_k^n$ , and  $U_R$  for  $U_R \times_k \text{Spec } k$ , etc., in the course of this proof.

First, the implication (1)  $\Rightarrow$  (2) is clear; for in suitable homogeneous coordinates  $\{X_i\}$ , one may assume that

$$L' \subset \{X_0 = X_1 = \dots = X_r = 0\}, \quad L'' \subset \{X_{r+1} = \dots = X_n = 0\}.$$

Then the subgroup of transformations

$$\begin{pmatrix} \alpha I_{r+1} & 0 \\ 0 & \beta I_{n-r} \end{pmatrix} \subset PGL(n+1)$$

leaves  $x$  fixed.

Secondly, (3)  $\Rightarrow$  (1) is an immediate consequence of the equivariant trivialization  $U_R \cong PGL(n+1) \times \mathbb{A}^{n\nu-n}$  and the fact that  $PGL(n+1)$  acts freely on itself by left translation.

Thirdly, we will prove that (2)  $\Rightarrow$  (3). By virtue of (2), all the points  $x^{(i)}$  cannot lie in one hyperplane, hence we can choose  $n+1$  of the  $x^{(i)}$  which are not in one hyperplane, say  $x^{(0)}, x^{(1)}, \dots, x^{(n)}$ . Without loss of generality, we may assume that these have homogeneous coordinates  $x_i^{(j)} = \delta_{ij}$ .

Now for each  $n+k$  between  $n+1$  and  $m$ , let  $S_k$  be the set of integers  $i$  such that

$$D_{0,1,\dots,\widehat{i},\dots,n,n+k} \neq 0,$$

i.e.  $x^{(n+k)}$  is not in the hyperplane spanned by  $x^{(0)}, \dots, \widehat{x^{(i)}}, \dots, x^{(n)}$ .

Then I claim that there is no partition of the set  $\{0, 1, \dots, n\}$  into two disjoint subsets  $T'$  and  $T''$  such that every  $S_k$  is contained in either  $T'$  or  $T''$ . For if there were, and if one let  $L'$  (resp.  $L''$ ) be the linear subspace defined by  $X_i = 0$  for all  $i \in T'$  (resp.  $i \in T''$ ), then every point  $x^{(k)}$  would lie in  $L' \cup L''$ , contradicting (2).

It follows immediately from a combinatorial argument that a suitable set of subsets  $S_i \subset S_j$  is an  $R$ -partition  $R$  and that  $x \in U_R$ .  $\square$

It thus follows that  $U_{\text{reg}}$  is the locus of prestable points in the following sense. Let  $G$  be a reductive algebraic group acting via  $\sigma$ , on  $X$  scheme of finite type over a field  $k$ . Now suppose that  $L$  is an invertible sheaf on  $X$  and that  $\phi$  is a  $G$ -linearization of  $L$ . The key concepts are the following.

**Definition 1.7 (Mumford, Definition 1.7).** Let  $x$  be a geometric point of  $X$ .

- (a)  $x$  is *pre-stable* (with respect to  $\sigma$ ) if there exists an invariant affine open subset  $U \subset X$  such that  $x \in U$  and every  $G$ -orbit in  $U$  is closed in  $U$
- (b)  $x$  is *semi-stable* (with respect to  $\sigma, L, \phi$ ) if there exists a section  $s \in H^0(X, L^{\otimes n})$  for some  $n > 0$  such that  $s(x) \neq 0$ , the open subset  $X_s$  is affine, and  $s$  is invariant. Equivalently, if  $\phi_n : \sigma^*(L^{\otimes n}) \rightarrow p_2^*(L^{\otimes n})$  is the induced linearization, then

$$\phi_n(\sigma^* s) = p_2^*(s).$$

- (c)  $x$  is *stable* (with respect to  $\sigma, L, \phi$ ) if there exists a section  $s \in H^0(X, L^{\otimes n})$  for some  $n > 0$  such that  $s(x) \neq 0$ , the open subset  $X_s$  is affine,  $s$  is invariant, and the action of  $G$  on  $X_s$  is closed.

A single closed orbit can sit inside a region where nearby orbits are wildly non-closed. That gives bad quotient behavior. You cannot form a reasonable local quotient around such a point.

For example, consider the action of  $\mathbb{G}_m$  on  $\mathbb{A}^2$  by

$$t \cdot (x, y) = (tx, t^{-1}y).$$

The origin is a closed orbit, but any neighborhood of the origin contains points whose orbits are not closed (e.g. points on the hyperbolas  $xy = c \neq 0$ ). Thus the origin is not prestable.

Prestable points are exactly those that sit inside a region where no orbit collapses onto another. This is the weakest hypothesis under which local quotients look like honest orbit spaces.

**Definition 1.8 (Mumford, Definition 0.6).** Given an action of  $G/S$  on  $X/S$ , a pair  $(Y, \phi)$  consisting of a pre-scheme  $Y$  over  $S$  and an  $S$ -morphism  $\phi : X \rightarrow Y$  is called a *geometric quotient* (of  $X$  by  $G$ ) if the following conditions are satisfied:

- (i)  $\phi \circ \sigma = \phi \circ p_2$  (as in Definition 0.5).
- (ii)  $\phi$  is surjective, and the image of  $\Psi$  is  $X \times_Y X$  (cf. Definition 0.4). Equivalently, the geometric fibres of  $\phi$  are precisely the orbits of the geometric points of  $X$  (for geometric points over an algebraically closed field of sufficiently high transcendence degree).
- (iii)  $\phi$  is submersive, i.e. a subset  $U \subset Y$  is open if and only if  $\phi^{-1}(U)$  is open in  $X$ . Likewise,  $U' \subset Y'$  is open if and only if  $\phi^{-1}(U')$  is open in  $X'$ .
- (iv) The fundamental sheaf  $\mathcal{O}_Y$  is the subsheaf of  $\phi_*(\mathcal{O}_X)$  consisting of invariant functions. That

is, if  $f \in \Gamma(U, \phi_*(\mathcal{O}_X)) = \Gamma(\phi^{-1}(U), \mathcal{O}_X)$ , then  $f \in \Gamma(U, \mathcal{O}_Y)$  if and only if the diagram

$$\begin{array}{ccc} G \times \phi^{-1}(U) & \xrightarrow{\sigma} & \phi^{-1}(U) \\ p_2 \downarrow & & \downarrow F \\ \phi^{-1}(U) & \xrightarrow{F} & \mathbb{A}^1 \end{array}$$

commutes, where  $F$  is the morphism defined by  $f$ .

Our next step is to construct a geometric quotient of  $U_{\text{reg}}$  by  $PGL(n+1)$ . Let  $U_1, \dots, U_N$  be the open subsets  $U_R$  of  $U_{\text{reg}}$  and the subsets obtained from these by permuting the coordinates. Let  $(Z_i, \phi_i)$  be the geometric quotient of  $U_i$  by  $PGL(n+1)$ . For all pairs  $i, j$ ,  $U_i \cap U_j$  is an invariant open subset of  $U_i$  and  $U_j$ . Therefore, by Corollary 3.2, if  $\sigma_i : Z_i \rightarrow U_i$  is the global section of  $\phi_i$ , we know:

$$PGL(n+1) \times \sigma_i^{-1}(U_i \cap U_j) \cong U_i \cap U_j \cong PGL(n+1) \times \sigma_j^{-1}(U_i \cap U_j).$$

In other words, both  $\sigma_i^{-1}(U_i \cap U_j)$  and  $\sigma_j^{-1}(U_i \cap U_j)$  are geometric quotients of  $U_i \cap U_j$  by  $PGL(n+1)$ ; therefore, they are canonically isomorphic. We use this isomorphism to glue together  $Z_i$  and  $Z_j$ . For any three of the quotients  $Z_i, Z_j, Z_k$ , these identifications are obviously compatible. Therefore, we have defined a pre-scheme  $Z$  and a morphism  $\phi : U_{\text{reg}} \rightarrow Z$ . Clearly,  $U_{\text{reg}}$  is a locally trivial principal fibre bundle over  $Z$ ; a fortiori,  $(Z, \phi)$  is a geometric quotient of  $U_{\text{reg}}$  by  $PGL(n+1)$ . However,  $Z$  is very far from being a scheme, let alone being quasi-projective.

Although  $Z$  is not quasi-projective, it carries various invertible sheaves. To investigate these, we make use of the theory of descent: by SGA 8, §1, the set of invertible sheaves on  $Z$  is isomorphic to the set of invertible sheaves on  $U_{\text{reg}}$  plus descent data for  $\phi$ . But  $\phi$ -descent data is precisely the same as a  $PGL(n+1)$ -linearization, since:

$$U_{\text{reg}} \times_Z U_{\text{reg}} \cong PGL(n+1) \times U_{\text{reg}}.$$

But  $L_i^{n+1}$  admits a  $PGL(n+1)$ -linearization. Therefore, there is an invertible sheaf  $M_i$  on  $Z$  such that

$$L_i^{n+1} \cong \phi^*(M_i).$$

Moreover, the section

$$(D_{\alpha_0, \dots, \alpha_n})^{n+1} \in \Gamma(U_{\text{reg}}, L_{\alpha_0}^{n+1} \otimes \cdots \otimes L_{\alpha_n}^{n+1})$$

is invariant in the  $SL(n+1)$ -linearization of this sheaf, hence in the  $PGL(n+1)$ -linearization of this sheaf. Therefore, according to SGA 8, §1, there is a section  $E_{\alpha_0, \dots, \alpha_n}$  of

$$M_{\alpha_0} \otimes \cdots \otimes M_{\alpha_n}$$

such that

$$(D_{\alpha_0, \dots, \alpha_n})^{n+1} = \phi^*(E_{\alpha_0, \dots, \alpha_n}).$$