Homework 4

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Problem 1 Prove the Weierstraß gap theorem. (See notes for lecture 10/23 for the statement.)

Problem 2 When $d > 2g-2$, show that the Abel-Jacobi map

$$\operatorname{Sym}^d R \longrightarrow J^d$$

is a bundle of projective spaces. Equivalently, the holomorphic sections over R of a varying line bundle of degree d are the fibers of a vector bundle over the Jacobian of degree d.

Suggestion: Check that the differential of the map is onto at every point, and use this to find local sections that give a local framing of the bundle.

Suggestion: Prove the vanishing of the sheaf cohomologies of $\mathcal{M}(V)$ and $\mathcal{P}(V)$.

Problem 3 Kodaira's theorem implies that for any vector bundle $V \to R$ on a compact Riemann surface, there exists a (bundle-dependent) degree $d \in \mathbb{Z}$ such that

$$H^1(R; \mathcal{O}(V \otimes L)) = 0$$

for all line bundles L of degree $> d$.

Use this fact, the long exact sequence for cohomology of

$$\mathcal{O} \to \mathcal{M} \to \mathcal{P},$$

and the result of Question 4 below to show that the ad hoc definition of cohomology of vector bundles via principal parts computes the genuine sheaf cohomology of vector bundles.

Problem 4 Show that, on a compact Hausdorff space X, sheaf cohomology commutes with filtered colimits (formerly known as direct limits): if $\mathcal{S} = \varinjlim_{n\to\infty} \mathcal{S}_n$, then

$$\varinjlim_{n\to\infty} H^q(X; \mathcal{S}_n) = H^q(X; \mathcal{S}).$$

(Here the indexing set can be any filtered set, not just $\mathbb{N}$.)

Use this to prove that the sheaf of principal parts of meromorphic sections of a vector bundle on a Riemann surface has no H^1 or higher cohomology.

Problem 5 A *nodal Riemann surface* S is one which is smooth except for finitely many singularities that look locally like

$$\{(x,y) \mid xy = 0\} \subset \mathbb{C}^2.$$

It is obtained from a smooth Riemann surface $\widetilde{S}$ by identifying pairs of points; $\widetilde{S} \to S$ is called the *normalization*.

Define the canonical bundle K_S of S to be the sheaf of differentials holomorphic on $\widetilde{S}$, except for simple poles at the nodes, with opposite residues on the two branches of S.

Show that K_S is a line bundle. When S is compact, prove the residue theorem for a meromorphic section of K_S that is holomorphic at the nodes. Prove Serre duality for vector bundles on S.

Hint: Reduce to $\widetilde{S}$. (Use the Čech definition to show that you can compute H^1 on $\widetilde{S}$ instead of S.)

Problem 6 Let S be a chain of elliptic Riemann surfaces $E_1, \ldots, E_g$, connected by nodes.

- Describe the space of holomorphic differentials (sections of K_S) and the period lattice for S.
- Show that $\mathrm{Pic}(S) \cong \mathbb{Z}^g \times E_1 \times \cdots \times E_g$.
- Collapse this to $\mathbb{Z} \times E_1 \times \cdots \times E_g$, "remembering only the total degree," by allowing a divisor to skip from one component to another at the node. That is, if p', p'' are the two points on $\widetilde{S}$ over a node, identify the line bundles corresponding to $\mathcal{O}(p') \times \mathcal{O}$ and $\mathcal{O} \times \mathcal{O}(p'')$ on adjacent elliptic curves. Show that you can now define an Abel–Jacobi map
$$\mathrm{Sym}^k(S) \longrightarrow \mathrm{Pic}\,.$$
- What is the image of S in Pic^1? What is the canonical Theta-divisor W_{g-1} in Pic^{g-1}?
- Check that the Riemann Theta function for S is the product of the Jacobi θ_3 functions for the E_i:
$$\theta_3(u \mid \tau) = \sum_{n=0}^{\infty} \exp(2\pi i n u + \pi i n^2 \tau),$$
and identify the Riemann Theta divisor. Compare with W_{g-1}.

Note: θ_3 has a unique simple zero at the center of the period parallelogram. *Caution:* The instructor has not checked any of this!