Equivariant Derived Categories of Coherent Sheaves

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Abstract

Notes for a talk I'm giving on equivariant derived categories of coherent sheaves.

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1 Preliminaries on GIT

1.1 Generalities on GIT quotients

Let $X \subset \mathbb{P}^n$ be a projective variety, and let $\widetilde{X} \subset \mathbb{C}^{n+1}$ be the corresponding affine cone. Since X is the space of lines in $\widetilde{X}$, it has a tautological line bundle

$$\mathcal{O}_X(-1) = \mathcal{O}_{\mathbb{P}^n}(-1)\big|_X$$

over it whose fibre over a point in X is the corresponding line in $\widetilde{X} \subset \mathbb{C}^{n+1}$. The total space of $\mathcal{O}_X(-1)$ therefore has a tautological map to $\widetilde{X}$ which is an isomorphism away from the zero section $X \subset \mathcal{O}_X(-1)$, which is all contracted down to the origin in $\widetilde{X}$. In fact the total space of $\mathcal{O}_X(-1)$ is the **blow up** of $\widetilde{X}$ in the origin.

Linear functions on $\mathbb{C}^{n+1}$ like x_i, restricted to $\widetilde{X}$ and pulled back to the total space of $\mathcal{O}_X(-1)$, give functions which are linear on the fibres, so correspond to sections of the **dual** line bundle $\mathcal{O}_X(1)$. Similarly degree k homogeneous polynomials on $\widetilde{X}$ define functions on the total space of $\mathcal{O}_X(-1)$ which are of degree k on the fibres, and so give sections of the kth tensor power $\mathcal{O}_X(k)$ of the dual of the line bundle $\mathcal{O}_X(-1)$.

So the grading that splits the functions on $\widetilde{X}$ into homogeneous degree (or $\mathbb{C}^*$-weight spaces) corresponds to sections of different line bundles $\mathcal{O}_X(k)$ on X. So

$$\bigoplus_{k\geq 0} H^0(\mathcal{O}_X(k))$$

considered a graded ring by tensoring sections $\mathcal{O}(k)\otimes\mathcal{O}(l)\cong\mathcal{O}(k+l)$. For the line bundle $\mathcal{O}_X(1)$ sufficiently positive, this ring will be generated in degree one. It is often called the (homogeneous) coordinate ring of the **polarised** (i.e. endowed with an ample line bundle) variety $(X, \mathcal{O}_X(1))$.

The degree one restriction is for convenience and can be dropped (by working with varieties in weighted projective spaces), or bypassed by replacing $\mathcal{O}_X(1)$ by $\mathcal{O}_X(p)$, i.e. using the ring

$$R^{(p)} = \bigoplus_{k\geq 0} R_{kp}; \qquad \text{for } p \gg 0 \text{ this will be generated by its degree one piece } R_p.$$

The choice of generators of the ring is what gives the embedding in projective space. In fact the sections of any line bundle L over X define a (rational) map

$$X \dashrightarrow \mathbb{P}\big(H^0(X,L)^*\big), \qquad x \mapsto ev_x, \quad ev_x(s) := s(x), \tag{1}$$

which in coordinates maps x to $(s_0(x) : \cdots : s_n(x)) \in \mathbb{P}^n$, where s_i form a basis for $H^0(L)$. This map is only defined for those x with $ev_x \neq 0$, i.e. for which $s(x)$ is not zero for every s.

Now suppose we are in the following situation, of G acting on a projective variety X through SL transformations of the projective space.

$$\begin{array}{ccc} G & \curvearrowright & X \\ \downarrow & & \downarrow \\ SL(n+1,\mathbb{C}) & \curvearrowright & \mathbb{P}^n \end{array}$$

Since we have assumed that G acts through $SL(n+1,\mathbb{C})$, the action lifts from X to one covering it on $\mathcal{O}_X(-1)$. In other words we don't just act on the projective space (and X therein) but on the vector space overlying it (and the cone $\widetilde{X}$ on X therein). This is called a **linearisation** of the action. Thus G acts on each $H^0(\mathcal{O}_X(r))$.

Then, just as $(X, \mathcal{O}_X(1))$ is determined by its graded ring of sections of $\mathcal{O}(r)$ (i.e. the ring of functions on $\widetilde{X}$),

$$(X, \mathcal{O}(1)) \longleftrightarrow \bigoplus_r H^0(X, \mathcal{O}(r))$$

we simply **construct** X/G (with a line bundle on it) from the ring of **invariant** sections:

$$X/G \longleftrightarrow \bigoplus_r H^0(X, \mathcal{O}(r))^G$$

This is sensible, since if there is a good quotient then functions on it pullback to give G-invariant functions on X, i.e. functions constant on the orbits, the fibres of $X \to X/G$. For it to work we need:

Lemma 1.1. *$\bigoplus_r H^0(X, \mathcal{O}(r))^G$ is finitely generated.*

Proof. Since $R := \bigoplus_r H^0(X, \mathcal{O}(r))$ is Noetherian, Hilbert's basis theorem tells us that the ideal $R \cdot \left(\bigoplus_{r>0} H^0(X, \mathcal{O}(r))^G\right)$ generated by $R_+^G := \bigoplus_{r>0} H^0(X, \mathcal{O}(r))^G$ is generated by a finite number of elements $s_0, \ldots, s_k \in R_+^G$.

Thus any element $s \in H^0(X, \mathcal{O}(r))^G$, $r > 0$, may be written $s = \sum_{i=0}^k f_i s_i$ for some $f_i \in R$ of degree $< r$. To show that the s_i generate R_+^G as an algebra we must show that the f_i can be taken to lie in R^G.

We now use the fact that G is the complexification of the compact group K. Since K has an invariant metric, we can average over it and use the facts that s and s_i are invariant to give

$$s = \sum_{i=0}^k \mathrm{Av}(f_i)\, s_i,$$

where $\mathrm{Av}(f_i)$ is the (K-invariant) K-average of f_i. By complex linearity $\mathrm{Av}(f_i)$ is also G-invariant (for instance, since G has a polar decomposition $G = K \exp(i\mathfrak{t})$). The $\mathrm{Av}(f_i)$ are also of degree $< r$, and so we may assume, by an induction on r, that we have already shown that they are generated by the s_i in R_+^G. Thus s is also. $\square$

Definition 1.2 (Projective GIT quotient)**.** *Let X be a projective variety with an action of a reductive group G linearised by a line bundle $\mathcal{O}_X(1)$. We define X/G to be*

$$\mathrm{Proj} \bigoplus_r H^0(X, \mathcal{O}(r))^G.$$

If X is a variety (rather than a scheme) then so is X/G, as its graded ring sits inside that of X and so has no zero divisors.

Definition 1.3 (Affine GIT quotient)**.** *Let $X = \mathrm{Spec}\, R$ be an affine variety with an action of a reductive group G. We define the affine GIT quotient X/G to be $\mathrm{Spec}(R^G)$, where R^G is the ring of G-invariant regular functions on X.*

In some cases, this does not work so well. For instance, under the scalar action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ the only invariant polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ are the constants and this recipe for the quotient gives a single point. In the language of the next section, this is because there are no stable points in this example, and all semistable orbits' closures intersect (or equivalently, there is a unique polystable point, the origin). More generally in any affine case all points are always at least semistable (as the constants are always G-invariant functions) and so no orbits gets thrown away in making the quotient (though many may get identified with each other — those whose closures intersect which therefore cannot be separated by invariant functions). But for the scalar action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ we clearly need to remove at least the origin to get a sensible quotient.

So we should change the linearisation, from the trivial linearisation to a nontrivial one, to get a bigger quotient. This is demonstrarted in the following example.

Example 1.4 (Projective space as a GIT quotient)**.** *Consider the trivial line bundle on $\mathbb{C}^{n+1}$ but with a nontrivial linearisation, by composing the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ by a character $\lambda \mapsto \lambda^p$ of $\mathbb{C}^*$ acting on the fibres of the trivial line bundle over $\mathbb{C}^{n+1}$. The invariant sections of this no longer form a ring; we have to take the direct sum of spaces of sections of* ***all powers*** *of this linearisation, just as in the projective case, and take* Proj *of the invariants of the resulting graded ring.*

We calculate the invariant sections for general p. Look at the k-th tensor power of the linearised line bundle. Sections are homogeneous polynomials $f(x_0, \ldots, x_n)$ of some degree. Under λ, such an f transforms as

$$f(x_0, \ldots, x_n) \mapsto f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n),$$

where $d = \deg f$.

But the linearisation introduces an extra factor λ^{-pk} when we act on the fibre of the k-th tensor power. By definition, the G-action on a section s is

$$(g \cdot s)(x) \ = \ g \cdot \big(s(g^{-1} \cdot x)\big).$$

Take a polynomial f homogeneous of degree d. View the section as

$$s(x) = f(x) \cdot e$$

where e is a trivialising section of the fibre. When we apply the group action:

$$(g \cdot s)(x) = g \cdot (f(g^{-1} \cdot x) \cdot e) = \big(\lambda^{-d} f(x)\big) \cdot \lambda^{pk} e = \lambda^{-d+pk} f(x) \cdot e.$$

For invariance, we need the weight to vanish, i.e.

$$d = pk.$$

So only polynomials of degree exactly pk survive as invariants in the degree k graded piece.

If $p < 0$ then there are no invariant sections and the quotient is empty. We have seen that for $p = 0$ the quotient is a single point. For $p > 0$ the invariant sections of the kth power of the linearisation are the homogeneous polynomials on $\mathbb{C}^n$ of degree kp. So for $p = 1$ we get the quotient

$$\mathbb{C}^{n+1}/\mathbb{C}^* = \operatorname{Proj} \bigoplus_{k\geq 0} \big(\mathbb{C}[x_0, \ldots, x_n]_k\big) = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n] = \mathbb{P}^n. \tag{2}$$

For $p \geq 1$ we get the same geometric quotient but with the line bundle $\mathcal{O}(p)$ on it instead of $\mathcal{O}(1)$.

Another way to derive this is to embed $\mathbb{C}^{n+1}$ in $\mathbb{P}^{n+1}$ as $x_{n+1} = 1$, act by $\mathbb{C}^*$ on the latter by $\operatorname{diag}(\lambda, \ldots, \lambda, \lambda^{-(n+1)}) \in S$ and do projective GIT. This gives, on restriction to $\mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$, the $p = n+1$ linearisation above. The invariant sections of $\mathcal{O}((n+2)k)$ are of the form $x_{n+1}^k f$, where f is a homogeneous polynomial of degree $(n+1)k$ in $x_1, \ldots, x_n$. Therefore the quotient is

$$\operatorname{Proj} \bigoplus_{k\geq 0} \big(\mathbb{C}[x_1, \ldots, x_n]_{(n+1)k}\big) = \operatorname{Proj} \big(\mathbb{C}[x_1, \ldots, x_n], \mathcal{O}(n+1)\big).$$

Definition 1.5 (Semistable points)**.** *A point $x \in X$ is **semistable** iff there exists $s \in H^0(X, \mathcal{O}(r))^G$ with $r > 0$ such that $s(x) \neq 0$. Points which are not semistable are **unstable**.*

So semistable points are those that the G-invariant functions "see." The map

$$\begin{aligned} X^{ss} &\to \mathbb{P}(H^0(X, \mathcal{O}(r))^G)^* \\ x &\mapsto ev_x \end{aligned}$$

is well defined on the (Zariski open, though possibly empty) locus $X^{ss} \subseteq X$ of semistable points, and it is clearly constant on G-orbits, i.e. it factors through the set-theoretic quotient X^{ss}/G. But it may contract more than just G-orbits, so we need another definition.

Definition 1.6 (Stable points)**.** *A semistable point x is **stable** if and only if $\bigoplus_r H^0(X, \mathcal{O}(r))^G$ separates orbits near x and the stabiliser of x is finite.*

1.2 Relevant example

We recall the basic set-up of Geometric Invariant Theory (GIT) quotients relevant to our situation.

Let $V = \mathbb{C}^4$ with coordinates x_1, x_2, y_1, y_2 and consider the $\mathbb{C}^*$-action given by

$$t \cdot (x_1, x_2, y_1, y_2) = (tx_1, tx_2, t^{-1}y_1, t^{-1}y_2)$$

We linearize this action by a character $\chi_m : t \mapsto t^m$ with $m \in \mathbb{Z} \setminus \{0\}$. Since V is affine, the GIT quotient for χ_m is $\operatorname{Proj} R^{(m)}$, where

$$R^{(m)} = \bigoplus_{d\geq 0} \Gamma(V, \mathcal{O}_V)^{\mathbb{C}^*, \chi_m^{\otimes d}} = \bigoplus_{d\geq 0} \{\, f \in \mathbb{C}[x_1, x_2, y_1, y_2] \mid t \cdot f = t^{md} f \,\}$$

In other words, $R_d^{(m)}$ is spanned by monomials whose total $\mathbb{C}^*$-weight is md, where the weight of a monomial $x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}$ is $w = a_1 + a_2 - (b_1 + b_2)$.

A point $v \in V$ is χ_m-semistable iff there exists $d > 0$ and $f \in R_d^{(m)}$ with $f(v) \neq 0$.

Here $R_d^{(m)}$ consists of polynomials whose monomials have positive weight $w = md > 0$. Such a monomial must contain at least one x (indeed, more x's than y's), so it vanishes at any point with $x_1 = x_2 = 0$. Therefore no section in $R_d^{(m)}$ can be nonzero at a point with $x_1 = x_2 = 0 \Rightarrow$ those points are unstable.

Conversely, if $(x_1, x_2) \neq (0, 0)$, then pick d and the monomial $f = x_i^{md}$ with $x_i \neq 0$. It has weight md and $f(v) \neq 0$, so v is semistable.

Therefore, for $m > 0$,

$$V^{ss}(\chi_m) = V \setminus \{x_1 = x_2 = 0\}.$$

The quotient is $(V \setminus \{x_1 = x_2 = 0\})/\mathbb{C}^*$, i.e. the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1_{[x_1:x_2]}$. Similarly, for $m < 0$, we have

$$V^{ss}(\chi_m) = V \setminus \{y_1 = y_2 = 0\}.$$

The quotient is $(V \setminus \{y_1 = y_2 = 0\})/\mathbb{C}^*$, i.e. the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1_{[y_1:y_2]}$.

1.3 Quotient stack

Let $\mathcal{S}$ be a category and $p : \mathcal{X} \to \mathcal{S}$ be a functor of categories. We visualize this data as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \\ \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & b \arrow[d] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

where the lower case letters a, b are objects of $\mathcal{X}$ and the upper case letters S, T are objects of $\mathcal{S}$. We say that a is over S and that a morphism $\alpha : a \to b$ is over $f : S \to T$.

Definition 1.7 (Prestacks)**.** *A functor $p : \mathcal{X} \to \mathcal{S}$ is a* ***prestack over a category*** *$\mathcal{S}$ if*

(1) ***(pullbacks exist)*** *for every diagram*

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d] & b \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a morphism $a \to b$ over $S \to T$; and

*(2) (**universal property for pullbacks**) for every diagram*

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & b \arrow[r] \arrow[d] & c \arrow[d] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

*Prestacks are also referred to as **categories fibered in groupoids**.*

Definition 1.8 (Fiber categories)**.** *If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the **fiber category** $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over S with morphisms over id_S.*

Given an action of an algebraic group G on a scheme X, the **quotient prestack** $[X/G]^{\mathrm{pre}}$ is the prestack whose fiber category $[X/G]^{\mathrm{pre}}(S)$ over a scheme S is the quotient groupoid (or the moduli groupoid of orbits) $[X(S)/G(S)]$ This will not satisfy the gluing axioms of a stack; even when the action is free, the quotient functor $\mathrm{Sch} \to \mathrm{Sets}$ defined by $S \mapsto X(S)/G(S)$ is not a sheaf in general. Put another way, we define:

Definition 1.9 (Quotient prestacks)**.** *Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. The **quotient prestack** $[U/G]^{\mathrm{pre}}$ of an action of a smooth affine group scheme $G \to S$ on an S-scheme U is the category over Sch/S consisting of pairs (T, u) where T is an S-scheme and $u \in U(T)$. An element $g \in G(T')$ acts by $(T', u') \to (T, u)$ via the data of a map $f : T' \to T$ of S-schemes and an element $g \in G(T')$ such that $f^*u = g \cdot u'$. Note that the fiber category $[U(T)/G(T)]$ is identified with the quotient groupoid.*

It turns out that the stackification of $[U/G]^{\mathrm{pre}}$ is the quotient stack $[U/G]$, hence the name is justified.

Definition 1.10 (Quotient stacks)**.** *The **quotient stack** $[U/G]$ is the prestack over Sch/S consisting of diagrams*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of S-schemes.

A morphism

$$(T' \longleftarrow P' \longrightarrow U) \longrightarrow (T \longleftarrow P \longrightarrow U)$$

consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the *diagram*

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and the left square is cartesian.

A stack over a site $\mathcal{S}$ is a prestack $\mathcal{X}$ where the objects and morphisms glue uniquely in the Grothendieck topology of $\mathcal{S}$.

Definition 1.11 (Stack). *A* ***stack*** *$\mathcal{X}$ over a site $\mathcal{C}$ is a prestack over $\mathcal{C}$ satisfying the following descent conditions:*

- *(Descent for morphisms) For any $U \in \mathcal{C}$, any covering $\{f_i : U_i \to U\}$, and any $x, y \in \mathcal{X}(U)$, the presheaf*
$$\underline{\mathrm{Hom}}(x, y) : (V \to U) \mapsto \mathrm{Hom}_{\mathcal{X}(V)}(f^*x, f^*y)$$
is a sheaf on $\mathcal{C}/U$.

- *(Descent for objects) For any $U \in \mathcal{C}$, any covering $\{f_i : U_i \to U\}$, and any descent datum (x_i, ϕ_{ij}) relative to $\{f_i : U_i \to U\}$, there exists an object $x \in \mathcal{X}(U)$ and isomorphisms $\psi_i : f_i^*x \xrightarrow{\sim} x_i$ such that $\phi_{ij} \circ f_j^*\psi_j = f_i^*\psi_i$.*

Definition 1.12 (Substack). *A* ***substack*** *$\mathcal{Y} \subseteq \mathcal{X}$ is given by:*

- *For each $U \in \mathcal{C}$, a full subcategory $\mathcal{Y}(U) \subseteq \mathcal{X}(U)$.*

- *Stability under restriction: If $y \in \mathcal{Y}(U)$ and $f : V \to U$ is a morphism in the site, then the pullback $f^*y \in \mathcal{X}(V)$ must lie in $\mathcal{Y}(V)$.*

- *Stack condition: The collection $\mathcal{Y}$ is itself a stack (i.e. satisfies descent for objects and morphisms).*

Definition 1.13 (Open and closed substacks). *A substack $\mathcal{T} \subseteq \mathcal{X}$ of a stack over $\mathrm{Sch}_{\text{ét}}$ is called an* ***open substack*** *(resp.* ***closed substack****) if the inclusion $\mathcal{T} \to \mathcal{X}$ is representable by schemes and an open immersion (resp. closed immersion).*

2 Introduction

We show that we can construct $\mathbb{Z}$ many derived equivalences between X_+ and X_-, and that the resulting autoequivalences are spherical twists. Segal's paper upgrades this equivalence to an equivalence of B-brane dg-categories. In particular, he shows that there are $\mathbb{Z}$ many quasi-equivalences between the categories of B-branes on (X_+, W) and (X_-, W). When $W = 0$, the dg-category of B-branes is just the dg-category of perfect complexes, whose homotopy category is the bounded derived category of coherent sheaves. So Segal's result recovers the derived equivalences we construct here.

Can you shed any intuition on what the B-brane dg-category is?

I don't understand how VGIT, derived categories, and B-branes fit into the TQFT picture and what we are talking about in seminar.

What is so interesting about the fact that these derived equivalences are spherical twists? Do you get something nice?

2.1 Set-up

Let $V = \mathbb{C}^4$ with co-ordinates x_1, x_2, y_1, y_2, and let $\mathbb{C}^*$ act on V with weight 1 on each x_i and weight -1 on each y_i. There are two possible GIT quotients X_+ and X_-, depending on whether we choose a positive or negative character of $\mathbb{C}^*$. Both are isomorphic to the total space of the bundle $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{P}^1$. This is the standard "three-fold flop" situation.

Both are open substacks of the Artin quotient stack

$$\mathcal{X} = [V/\mathbb{C}^*]$$

given by the semi-stable locus for either character. Let

$$\iota_\pm : X_\pm \hookrightarrow \mathcal{X}$$

denote the inclusions.

Remark 2.1 (The quotient stack and its open substacks)**.** *Recall that via the functor of points perspective, its objects are pairs (P, ϕ), where P is a principal $\mathbb{C}^*$-bundle and $\phi : P \to V$ is $\mathbb{C}^*$-equivariant.*

For a given choice of character χ_m, the semistable locus $V^{ss}(\chi_m)$ is an open subset of V. It is open because it is defined by the nonvanishing of some semi-invariant sections. The corresponding GIT quotient is $[V^{ss}(\chi_m)/\mathbb{C}^*]$ as a substack. Thus:

$$X_\pm = [V^{ss}(\pm 1)/\mathbb{C}^*] \ \subset \ [V/\mathbb{C}^*] = \mathcal{X}$$

It turns out that open substacks of quotient stacks $[V/G]$ are exactly those substacks which are of the form $[U/G]$ where $U \subseteq V$ is a G-invariant open subscheme. Here $V^{ss}(\chi_m) \subset V$ is G-invariant and open, so $[V^{ss}(\chi_m)/\mathbb{C}^*] \hookrightarrow [V/\mathbb{C}^*]$ is exactly an open immersion of stacks.

This stacky point of view makes it clear that there are (exact) restriction functors

$$\iota_\pm^* : D^b(\mathcal{X}) \ \to \ D^b(X_\pm).$$

By $D^b(\mathcal{X})$ we mean the derived category of the category of $\mathbb{C}^*$-equivariant sheaves on V. This contains the obvious equivariant line bundles $\mathcal{O}(i)$ associated to the characters of $\mathbb{C}^*$.

Remark 2.2 (General fact about open immersions)**.** *If $j : U \hookrightarrow X$ is an open immersion of schemes, then there is an exact restriction functor $j^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$. This is because $j^*\mathcal{F}$ has the same stalk as $\mathcal{F}$ at points of U.*

Althernatively, exactness comes from the fact that restricting a quasi-coherent sheaf to an open set is just tensoring with $\mathcal{O}_U$, which is flat (in general localisation is flat).

Passing to derived categories, you still have $j^* : D^b(\mathrm{QCoh}(X)) \to D^b(\mathrm{QCoh}(U))$ which has no higher derived functors since j^* is exact. The exact same holds in the stack setting: if $\iota : \mathcal{U} \hookrightarrow \mathcal{X}$ is an open immersion of stacks, you get $\iota^* : D^b(\mathcal{X}) \to D^b(\mathcal{U})$.

Remark 2.3 (General dictionary for quotient stacks and equivariant geometry)**.** *There is a general dictionary relating the stack-theoretic concepts and the equivariant geometry of X. Here G is a reductive algebraic group acting on a scheme X and $[X/G]$ is the quotient stack.*

Geometry of $[X/G]$	G***-equivariant geometry of*** X
$\mathbb{C}$*-point* $\bar{x} \in [X/G]$	*orbit* Gx *of* $\mathbb{C}$*-point* $x \in X$ *(with* $\bar{x}$ *the image of* x *under* $X \to [X/G]$*)*
automorphism group $\mathrm{Aut}(\bar{x})$	*stabilizer* G_x
function $f \in \Gamma([X/G], \mathcal{O}_{[X/G]})$	G*-equivariant function* $f \in \Gamma(X, \mathcal{O}_X)^G$
map $[X/G] \to Y$ *to a scheme* Y	G*-equivariant map* $X \to Y$
line bundle	G*-equivariant line bundle (or* G*-linearization)*
quasi-coherent sheaf	G*-equivariant quasi-coherent sheaf*
tangent space $T_{[X/G],\bar{x}}$	*normal space* $T_{X,x}/T_{Gx,x}$ *to the orbit*
coarse moduli space $[X/G] \to Y$	*geometric quotient* $X \to Y$
good moduli space $[X/G] \to Y$	*good GIT quotient* $X \to Y$

The unstable locus for the negative character is the set $\{y_1 = y_2 = 0\} \subset V$. Consider the Koszul resolution of the associated sky-scraper sheaf:

$$K_- = \mathcal{O}(2) \xrightarrow{(y_2,-y_1)} \mathcal{O}(1)^{\oplus 2} \xrightarrow{(y_1,y_2)} \mathcal{O}.$$

Then $\iota_- K_-$ is exact, it is the pull-up of the Euler sequence from $\mathbb{P}^1_{y_1:y_2}$. On the other hand $\iota_+ K_-$ is a resolution of the sky-scraper sheaf $\mathcal{O}_{\mathbb{P}^1_{x_1:x_2}}$ along the zero section. Similar comments apply for the Koszul resolution K_+ of the set $\{x_1 = x_2 = 0\}$.

Let

$$\mathcal{G}_t \subset D^b(\mathcal{X})$$

be the triangulated subcategory generated by the line bundles $\mathcal{O}(t)$ and $\mathcal{O}(t+1)$. This is the smallest thick triangulated subcategory generated by these two objects. This is the **grade restriction rule** of Hori-Herbst-Page, which informally says if you restrict this window to either quotient $X^{\pm}$, you recover the derived category $D^b(X^{\pm})$.

Claim 2.4. *For any $t \in \mathbb{Z}$, both ι_+^* and ι_-^* restrict to give equivalences*

$$D^b(X_+) \xleftarrow{\sim} \mathcal{G}_t \xrightarrow{\sim} D^b(X_-).$$

Proof. The restriction functors

$$\iota_\pm^* : \ D^b(\mathcal{X}) \longrightarrow D^b(X^{\pm})$$

are exact and preserve shifts and cones. To prove that the restrictions

$$\iota_\pm^* : \ \mathcal{G}_t \xrightarrow{\ \sim\ } D^b(X^{\pm})$$

are equivalences, we need:

1. Fully faithfulness: On $\mathcal{G}_t$, the restriction maps induce isomorphisms

$$\mathrm{Hom}_{D^b(\mathcal{X})}(E, F) \cong \mathrm{Hom}_{D^b(X^{\pm})}(\iota_\pm^* E, \iota_\pm^* F)$$

   Since $\mathcal{G}_t$ is generated by $\{\mathcal{O}(t), \mathcal{O}(t+1)\}$, it suffices to check this on these generators. Concretely, we need to compute $\mathrm{Ext}^\bullet_{\mathcal{X}}(\mathcal{O}(t+k), \mathcal{O}(t+l))$ for $k, l \in \{0, 1\}$, and show it matches the Ext groups in $X^{\pm}$.

2. Essential surjectivity: Every object in $D^b(X^{\pm})$ should be built out of $\iota_\pm^* \mathcal{G}_t$. In other words, the images of $\mathcal{O}(t), \mathcal{O}(t+1)$ generate $D^b(X^{\pm})$.

To see that these functors are fully-faithful it suffices to check what they do to the maps between the generating line-bundles, so we just need to check that

$$\mathrm{Ext}^\bullet_{\mathcal{X}}(\mathcal{O}(t+k), \mathcal{O}(t+l)) = \mathrm{Ext}^\bullet_{X_\pm}(\mathcal{O}(t+k), \mathcal{O}(t+l))$$

for $k, l \in [0, 1]$. For line bundles, $\mathrm{Ext}^\bullet(\mathcal{O}(a), \mathcal{O}(b)) \cong H^\bullet(\,\cdot\,,\ \mathcal{O}(b-a))$. Thus we need to verify that $H^\bullet_{\mathcal{X}}(\mathcal{O}(i)) = H^\bullet_{X_\pm}(\mathcal{O}(i))$ for $i \in [-1, 1]$.

$\mathcal{X}$ is an affine quotient stack (with V affine), so for any equivariant coherent sheaf, higher cohomology on $\mathcal{X}$ vanishes; taking global sections means "equivariant global sections" on V. Hence $H^p(\mathcal{X}, \mathcal{O}(i)) = (\mathcal{O}_V)_i$ for $p = 0$ and 0 for $p > 0$, where $(\mathcal{O}_V)_i$ is the weight-i subspace of the polynomial ring $\mathcal{O}_V = \mathbb{C}[V]$.

To compute $H^\bullet(X_\pm, \mathcal{O}(i))$, we use the projection $\pi : X_\pm \to \mathbb{P}^1$ and the fact that $X_\pm$ is the total space of the bundle $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{P}^1$. We do the computation for X^+; the case of X^- is similar. Let $\pi : X^+ \to \mathbb{P}^1$ be the projection and $E = \mathcal{O}(-1)^{\oplus 2}$. Then

$$\pi_*\mathcal{O}_{X^+} \cong \mathrm{Sym}^\bullet(E^\vee) = \mathrm{Sym}^\bullet(\mathcal{O}(1)^{\oplus 2}) \cong \bigoplus_{m\geq 0} \mathrm{Sym}^m(\mathcal{O}(1)^{\oplus 2}) \cong \bigoplus_{m\geq 0} \mathcal{O}(m)^{\oplus(m+1)}.$$

Remark 2.5. *Recall that the total space of a vector bundle $E \to X$ is $\underline{\mathrm{Spec}}_X(\mathrm{Sym}^\bullet(E^\vee))$ where we take the relative Spec over X. Associated to any sheaf of algebras $\mathcal{A}$ over a base scheme B is the relative Spec, which is a scheme $Y, \mathcal{O}_Y$ equipped with a morphism $\pi : Y \to B$. It has the property that $\pi_*\mathcal{O}_Y = \mathcal{A}$ and $\pi : Y \to B$ is affine. In our case, the sheaf of algebras is $\mathrm{Sym}^\bullet(E^\vee)$, which is the symmetric algebra on the dual bundle $E^\vee$.*

This means that if locally on B we have $E \cong \mathcal{O}_B^{\oplus r}$ is trivial of rank r, then $\underline{\mathrm{Spec}}_B(\mathrm{Sym}^\bullet(E^\vee))$ means we glue together the affine scheme $\mathrm{Spec}(\mathcal{O}_B[t_1, \dots, t_r])$ fiberwise over B. Thus

$$\mathrm{Sym}^\bullet(E^\vee) \cong \mathcal{O}_B[t_1, \dots, t_r]$$

The last isomorphism above can be seen from the general fact that if L is a line bundle and V is a vector space, then $\mathrm{Sym}^m(L \otimes V) \cong L^{\otimes m} \otimes \mathrm{Sym}^m(V)$. Locally trivialize L. Then $\mathrm{Sym}^m(L \otimes V)$ is generated by monomials $(\ell \otimes v_1) \cdots (\ell \otimes v_m) = \ell^m \otimes (v_1 \cdots v_m)$, which shows the factorization.

By projection formula and affineness of π

$$H^p(X^+, \mathcal{O}(k)) \cong H^p\Big(\mathbb{P}^1,\ \pi_*\mathcal{O}_{X^+} \otimes \mathcal{O}(k)\Big) \cong \bigoplus_{m\geq 0} H^p(\mathbb{P}^1,\ \mathcal{O}(k+m))^{\oplus(m+1)}.$$

Remark 2.6. *Recall that for a morphism $\pi : X \to B$ and a sheaf F on X, there is a spectral sequence (Leray)*

$$E_2^{p,q} = H^p(B, R^q\pi_*F) \Longrightarrow H^{p+q}(X, F)$$

Since π is affine, $R^p\pi_* = 0$ for $p > 0$. So in the Leray spectral sequence, all rows with $q > 0$ are zero. That means already on the E_2-page, only the bottom row q=0 survives. No differentials are possible, so $E_2 = E_\infty$. Thus

$$H^p(X, F) \cong H^p(B, \pi_*F)$$

Therefore we need to compute $\pi_*\mathcal{O}_{X^+} \otimes \mathcal{O}(k)$. The projection formula says: for any quasi-coherent sheaf F on X and any sheaf G on B, $\pi_*(F \otimes \pi^*G) \cong \pi_*F \otimes G$. Take $F = \mathcal{O}_X$ and $G = \mathcal{O}_B(k)$. Then: $\pi_*(\mathcal{O}_X \otimes \pi^*\mathcal{O}_B(k)) \cong \pi_*\mathcal{O}_X \otimes \mathcal{O}_B(k)$. But $\mathcal{O}_X \otimes \pi^*\mathcal{O}_B(k)$ is exactly $\mathcal{O}_X(k)$ so

$$\pi_*\mathcal{O}_X(k) \cong \pi_*\mathcal{O}_X \otimes \mathcal{O}_B(k)$$

Now use the standard $\mathbb{P}^1$ cohomology:

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1} & n \geq 0 \\ 0 & n < 0 \end{cases}, \qquad H^1(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} 0 & n \geq -1 \\ \mathbb{C}^{(-n-1)} & n \leq -2 \end{cases}.$$

So for $p = 0$, we get that

$$\begin{aligned} H^0(X^+, \mathcal{O}(k)) &\cong \bigoplus_{m \geq 0} H^0(\mathbb{P}^1, \mathcal{O}(k+m))^{\oplus(m+1)} \\ &\cong \operatorname{Sym}^{k+m}(\mathbb{C}^2_{x_1,x_2}) \otimes \operatorname{Sym}^m(\mathbb{C}^2_{y_1,y_2}) \end{aligned}$$

which is exactly the weight-k part of $\mathcal{O}_V = \mathbb{C}[x_1, x_2, y_1, y_2]$.

For $p = 1$, we have

$$H^1(X^+, \mathcal{O}(k)) \cong \bigoplus_{m \geq 0} H^1(\mathbb{P}^1, \mathcal{O}(k+m))^{\oplus(m+1)}$$

On $\mathbb{P}^1$, $H^1(\mathcal{O}(n)) = 0$ for $n \geq -1$. So if $k \geq -1$ (the window for the flop), then $k + m \geq -1$ for all $m \geq 0$, hence $H^1(X^+, \mathcal{O}(k)) = 0$. This matches the stack, where all higher H^p vanish because $[V/\mathbb{C}^*]$ is (relatively) affine. This proves fully faithfulness of ι_+^* on the window.

Remark 2.7. *Note that if $k \leq -2$, then at least the $m = 0$ term contributes $H^1(\mathbb{P}^1, \mathcal{O}(k)) \neq 0$. Here H^1 on X^+ is nonzero, while on the stack it is zero - this is exactly where agreement fails outside the window.*

To see that they are essentially surjective we need to know that the two given line bundles generate $D^b(X_\pm)$. That is, every object of $D^b(X^\pm)$ should be quasi-isomorphic to a complex built out of $\iota_\pm^*\mathcal{O}(t)$ and $\iota_\pm^*\mathcal{O}(t+1)$.

This is essentially a corollary of Beilinson's theorem. I think I understand the statement of Beilinson's theorem, but what exactly is the content? □

Remark 2.8. *Essential surjectivity follows from a general theorem which says that on quasi-projective varieties, an ample line bundle and its twists generate the derived category. The intuition behind this statement is Serre's theorem which says that for any coherent sheaf $\mathcal{F}$, $\mathcal{F}(n)$ is globally generated for $n \gg 0$.*

Pick an ample line bundle L on X. Serre vanishing gives, for $m \gg 0$: (1) $H^i(X, F \otimes L^{\otimes m}) = 0$ for all $i > 0$ and any coherent F, and (2) $F \otimes L^{\otimes m}$ is globally generated.

For m large, the evaluation map is surjective:

$$H^0(X, F(m)) \otimes \mathcal{O}_X \twoheadrightarrow F(m).$$

Twist down by L^{-m}:

$$H^0(X, F(m)) \otimes L^{-m} \twoheadrightarrow F.$$

So F is a quotient of a finite direct sum of a power of L^{-1}. Let $K_1 := \ker(1)$. Then K_1 is coherent.

Apply Serre vanishing again to K_1: choose $m_1 \gg 0$ so that $K_1(m_1)$ is globally generated and $H^{>0}(K_1(m_1)) = 0$. Get a surjection

$$H^0(X, K_1(m_1)) \otimes L^{-m_1} \twoheadrightarrow K_1,$$

with kernel K_2. Repeat. Using Castelnuovo–Mumford regularity, you can choose $m, m_1, \ldots$ so this iteration stops in at most $\dim X + 1$ steps, giving a finite resolution:

$$0 \to \bigoplus L^{-m_r} \to \cdots \to \bigoplus L^{-m_1} \to \bigoplus L^{-m} \to F \to 0.$$

Thus every coherent F has a finite resolution by direct sums of powers of L^{-1}. Passing to derived categories, this means the triangulated subcategory generated by the line bundles $\{L^{\otimes n} \mid n \in \mathbb{Z}\}$ contains every object of $D^b(\mathrm{Coh}(X))$.

There's a standard Koszul resolution on the stack $\mathcal{X} = [V/\mathbb{C}^*]$:

$$0 \to \mathcal{O}(2) \xrightarrow{(y_2, -y_1)} \mathcal{O}(1)^{\oplus 2} \xrightarrow{(y_1, y_2)} \mathcal{O} \to 0.$$

Peng said something about a simliar sequence on $\mathcal{X}$ with the skyscraper sheaf at 0 which restricts to 0 on X^+. This is exact because it's the Koszul complex resolving the unstable locus $\{y_1 = y_2 = 0\}$. Restricting to X^+, it's still exact.

Now if you tensor by $\mathcal{O}(k)$, you get

$$0 \to \mathcal{O}(k+2) \to \mathcal{O}(k+1)^{\oplus 2} \to \mathcal{O}(k) \to 0.$$

Because $\mathcal{O}(k)$ is a line bundle (locally free), tensoring preserves exactness. Then this sequence says that $\mathcal{O}(k)$ sits in a short exact sequence whose terms are in degrees $k+1$ and $k+2$.

Translated into the derived category: $\mathcal{O}(k)$ is quasi-isomorphic to a 2-term complex built from $\mathcal{O}(k+1)$ and $\mathcal{O}(k+2)$. So inductively, you can move any line bundle "into the window range" by expressing it as a cone of maps between line bundles shifted up or down. In other words, two consecutive line bundles $\mathcal{O}(t), \mathcal{O}(t+1)$ generate all line bundles. And since line bundles generate the whole $D^b(X^+)$, you're done.

So for any $t \in \mathbb{Z}$ we have a derived equivalence

$$\Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_-)$$

passing through $\mathcal{G}_t$. Composing these, we get auto-equivalences

$$\Phi_{t+1}^{-1}\Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_+).$$

To see what these do, we need to check them on the generating set of line-bundles $\{\mathcal{O}(t), \mathcal{O}(t+1)\}$. Applying Φ_t to this set is easy, it just sends them to the same line-bundles on X_-. To apply Φ_{t+1}^{-1} however, we first have to resolve $\mathcal{O}(t)$ in terms of $\mathcal{O}(t+1)$ and $\mathcal{O}(t+2)$. We do this using the exact sequence $\iota_- K_-(t)$. The result is that $\Phi_{t+1}^{-1}\Phi_t$ sends

$$\mathcal{O}(t) \;\mapsto\; \big[\mathcal{O}(t+2) \xrightarrow{(-y_2, y_1)} \mathcal{O}(t+1)^{\oplus 2}\big], \qquad \mathcal{O}(t+1) \;\mapsto\; \mathcal{O}(t+1).$$

Claim 2.9. *$\Phi_{t+1}^{-1}\Phi_t$ is an inverse spherical twist around $\mathcal{O}_{\mathbb{P}^1_{x_1:x_2}}(t)$.*

A spherical twist is an autoequivalence discovered by [**?**] associated to any spherical object in the derived category, i.e. an object S such that

$$\mathrm{Ext}(S, S) = \mathbb{C} \oplus \mathbb{C}[-n]$$

for some n (i.e. the homology of the n-sphere). It sends any object $\mathcal{E}$ to the cone on the evaluation map

$$\big[\,\mathrm{RHom}(S, \mathcal{E}) \otimes S \;\longrightarrow\; \mathcal{E}\big].$$

The inverse twist sends $\mathcal{E}$ to the cone on the dual evaluation map

$$\big[\mathcal{E} \;\longrightarrow\; \mathrm{RHom}(\mathcal{E}, S)^\vee \otimes S\big].$$

The object $\mathcal{O}_{\mathbb{P}^1_{x_1:x_2}}(t) \simeq \iota_+ K_-(t)$ is spherical, and the inverse twist around it sends $\mathcal{O}(t+1)$ to itself and $\mathcal{O}(t)$ to the cone

$$\big[\mathcal{O}(t) \longrightarrow \iota_+ K_-(t)\big] \simeq \big[\mathcal{O}(t+2) \xrightarrow{(-y_2, y_1)} \mathcal{O}(t+1)^{\oplus 2}\big],$$

which agrees with $\Phi_{t+1}^{-1}\Phi_t$. To complete the proof of the claim we would just need to check that the two functors also agree on the Hom-sets between $\mathcal{O}(t)$ and $\mathcal{O}(t+1)$.

Now instead let $V = \mathbb{C}^{p+q}$ with co-ordinates $x_1, \ldots, x_p, y_1, \ldots, y_q$. Let $\mathbb{C}^*$ act linearly on V with positive weights on each x_i and negative weights on each y_i. The two GIT quotients X_+ and X_- are both the total spaces of orbi-vector bundles over weighted projective spaces.

We must assume the Calabi-Yau condition that $\mathbb{C}^*$ acts through $SL(V)$. Let d be the sum of the positive weights, so the sum of the negative weights is $-d$. The above argument goes through word-for-word, where now

$$\mathcal{G}_t = \langle \mathcal{O}(t), \ldots, \mathcal{O}(t+d-1) \rangle.$$

3 References

NOTES ON GIT AND SYMPLECTIC REDUCTION FOR BUNDLES AND VARIETIES by R. P. THOMAS

Alper Moduli

Segal paper