Extending FTT to G-bundles

Songyu Ye

February 1, 2026

1 Goal

The goal of this project is to extend the results of Frenkel-Teleman-Tolland [4] on the existence of derived pushforward of admissible classes on the moduli stack of $\mathbb{C}^\times$-bundles over stable curves to the case of principal G-bundles for a general reductive group G.

2 Results from Frenkel-Teleman-Tolland

Definition 2.1 (Gieseker bundle). Let (Σ, σ_i) be a stable marked curve. A **Gieseker $\mathbb{C}^\times$–bundle** on (Σ, σ_i) is a pair $(m, \mathcal{P})$ consisting of

1. a modification $m : (C, \sigma_i) \to (\Sigma, \sigma_i)$ (in the sense that m is an isomorphism away from the nodes of Σ, and the preimage of each node is either a node or a $\mathbb{P}^1$ meeting the two branches transversely), and

2. a principal $\mathbb{C}^\times$–bundle $p : \mathcal{P} \to C$,

which satisfy the **Gieseker condition**: the restriction of $\mathcal{P}$ to every unstable $\mathbb{P}^1$ has degree 1.

Let $\widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^*])$ be the moduli stack of Gieseker $\mathbb{C}^*$-bundles over stable curves of genus g with I marked points, as constructed in Frnekel-Teleman-Tolland [4]. It carries a universal principal $\mathbb{C}^*$-bundle $\mathcal{P} \to \mathcal{C}$ over the universal curve $\pi : \mathcal{C} \to \widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^*])$.

Let $F : \widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^*]) \to \overline{\mathcal{M}_{g,I}}$ be the bundle-forgetting map. Let $\phi : C \to [pt/\mathbb{C}^*]$ be the classifying map of the universal bundle $\mathcal{P}$. Let $\pi : \mathcal{C} \to \widetilde{\mathcal{M}}_{g,I}([pt/\mathbb{C}^*])$ be the universal curve.

A line bundle $\mathcal{L}$ on $\overline{\mathcal{M}}_{g,I}([\mathrm{pt}/\mathbb{C}^\times])$ is *admissible* if

$$\mathcal{L} \simeq \left(\det R\pi_* \phi^* \mathbb{C}_1\right)^{\otimes(-q)},$$

where $\mathbb{C}_1$ is the standard representation and q is a positive rational number. An *admissible complex* is a sum of complexes of the form

$$\mathcal{L} \boxtimes \bigotimes_a \big(R\pi_* \phi^* V_a\big) \boxtimes \bigotimes_i \big(\mathrm{ev}_i^* V_i \otimes T_i^{\otimes n_i}\big).$$

The subring of $K\big(\overline{\mathcal{M}}_{g,I}([\mathrm{pt}/\mathbb{C}^\times])\big)$ generated by such products is called the *ring of admissible classes*. The main result of [4] is the following theorem.

Theorem 2.2. The derived pushforward $RF_*\alpha$ of an admissible complex α along the bundle-forgetting map $F : \widetilde{\mathcal{M}}_{g,I}([pt/G]) \to \overline{\mathcal{M}_{g,I}}$ is a bounded complex of coherent sheaves.

3 General idea

Let $S = \mathbb{C}[[s]], S^* = \mathbb{C}((s))$ and B be an S-scheme. Let $C_S \to S$ be a projective flat family of curves with generic fiber $\mathbb{C}_{S^*}$ smooth and special fiber C_0 nodal with unique node p. Let $C_B = C_S \times_S B$.

Solis [5] defines the S-stack $\mathcal{X}_G(C_S)$ whose points evaluated at a test scheme B/S are given by elements (C'_B, P_B) where C'_B is a twisted modification of C_B and P_B is an admissible G-bundle on C'_B. This stack is over a fixed curve C_S and Solis shows that it is algebraic, locally of finite type, and complete over S. It contains $M_G(C_S)$ and $M_G(C_{S^*})$ as dense open substacks, and the complement of $M_G(C_{S^*})$ is a divisor with normal crossings.

In this section, we discuss how to generalize Solis' construction to families of curves by working over the universal curve over the moduli stack of stable curves $\overline{\mathfrak{M}}_{g,I}$. Let $\pi : \overline{\mathcal{C}_{g,I}} \to \overline{\mathfrak{M}}_{g,I}$ be the universal curve over the moduli stack of stable curves of genus g with I marked points.

Let $\pi : C \to B$ be a prestable family of nodal curves. Let

$$\Sigma := \mathrm{Sing}(C/B) \subset C$$

be the relative singular locus. It is finite étale over B after restricting to the locus where the number of nodes is constant; globally it is at least finite unramified in good situations.

Definition 3.1. A **modification of** C/B is a proper morphism $m : C' \to C$ over B such that:

1. $C' \to B$ is flat prestable curve, and m is finitely presented and projective.

2. m is an isomorphism away from the nodes:

$$m : \; C' \setminus m^{-1}(\Sigma) \xrightarrow{\sim} C \setminus \Sigma.$$

3. For every geometric point $b \to B$ and every node $p \in \Sigma_b \subset C_b$, the fiber of $m_b : C'_b \to C_b$ over p is either a point (no modification at that node) or a chain of $\mathbb{P}^1$'s meeting the two branches in the standard way, and m_b contracts that chain to p and is an isomorphism elsewhere.

A **length $\leq n$ condition** can be stated as:

- for every b and every node $p \in \Sigma_b$, the chain over p has at most n components.

Definition 3.2 (Twisted nodal curves over a base). Let B be a scheme over $\mathbb{C}$. A **twisted nodal curve over** B is a proper Deligne–Mumford stack

$$\pi : \mathcal{C} \longrightarrow B$$

such that:

1. The geometric fibers of π are connected, one–dimensional, and the coarse moduli space $\overline{\mathcal{C}}$ is a nodal curve.

2. Let $\mathcal{U} \subset \mathcal{C}$ be the complement of the relative singular locus $\mathrm{Sing}(\mathcal{C}/B)$. Then the restriction

$$\mathcal{U} \hookrightarrow \mathcal{C}$$

is an open immersion.

3. For any geometric point $p : \operatorname{Spec} k \to \mathcal{C}$ mapping to a node of the fiber over $b \in B$, there exists an integer $k \geq 1$ and an element $t \in \mathfrak{m}_{B,b}$ such that, étale-locally on B at b and strictly henselian locally on $\mathcal{C}$ at p, there is an isomorphism

$$\operatorname{Spec} \mathcal{O}^{sh}_{\mathcal{C},p} \;\cong\; \Big[\operatorname{Spec}\big(\mathcal{O}^{sh}_{B,b}[u,v]/(uv-t)\big) \;/\; \mu_k\Big],$$

where $\zeta \in \mu_k$ acts by

$$(u,v) \longmapsto (\zeta u, \zeta^{-1} v).$$

Definition 3.3. A **twisted modification of** C/B is a twisted nodal curve $\mathcal{C} \to B$ whose coarse moduli space $\overline{\mathcal{C}}$ is a modification of C/B.

Let $r = \mathrm{rk}(G)$. The ordered simple roots $\{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ determine ordered vertices $\{\eta_0, \ldots, \eta_r\}$ determined by the conditions

$$\langle \eta_i, \alpha_j \rangle = 0 \text{ for } i \neq j \quad \text{and} \quad \langle \eta_0, \alpha_0 \rangle = 1.$$

If we write $\theta = \sum_{i=1}^r n_i \alpha_i$ and set $n_0 = 1$ then one can check these conditions can be expressed as

$$\langle \alpha_i, \eta_j \rangle = \frac{1}{n_i} \delta_{i,j}. \tag{1}$$

Following [5], if C'_B is a twisted modification of length $\leq r$, then a G–bundle on C'_B is called **admissible** if the co–characters determining the equivariant structure at all nodes are linearly independent over $\mathbb{Q}$ and are given by a subset of $\{\eta_0, \ldots, \eta_r\}$.

Definition 3.4. We define a stack $\mathcal{X}_{G,g,I}$ over $\overline{\mathfrak{M}}_{g,I}$ whose points over a test scheme $B \to \overline{\mathfrak{M}}_{g,I}$ are given by pairs (C'_B, P_B) where C'_B is a twisted modification of the pullback C_B of the universal curve $\overline{\mathfrak{C}_{g,I}}$ to B, and P_B is an admissible G-bundle on C'_B.

Let $\Sigma_0, \sigma_{0,i}$ be a fixed stable curve of genus g with I marked points. Let B be an affine etale neighborhood of the point $[\Sigma_0, \sigma_{0,i}]$ in $\overline{\mathcal{M}}_{g,I}$. Let $\mathcal{X}_{G,g,I}|_B$ be the fiber of F over the map $B \to \overline{\mathcal{M}}_{g,I}$, where $F : \mathcal{X}_{G,g,I} \to \overline{\mathcal{M}}_{g,I}$ is the natural projection.

We need to produce a local chart A for the stack $\mathcal{X}_{G,g,I}|_B$. This stack A will be a category fibered over B. We need to show that the stack A is represented by an algebraic space and then display $\mathcal{X}_{G,g,I}|_B$ as a quotient stack $[A/H]$ for some reductive group H acting on A.

Following the strategy of proof carried out for $\mathbb{C}^*$-bundles in FTT [4], we aim to carry out the following:

1. $\mathcal{X}_{G,g,I}$ is algebraic and locally of finite type over $\overline{\mathcal{M}}_{g,I}$.
2. After étale localization to $B \to \overline{\mathcal{M}}_{g,I}$, we have
$$\mathcal{X}_{G,g,I}|_B \;\simeq\; [A/H]$$
where A is a smooth algebraic space and $H = G^V$ is reductive.
3. There exists an open subspace $A^\circ \subset A$ (defined by a numerical stability condition generalizing the genus bounds) such that the quotient $[A^\circ/H]$ factors as
$$[A^\circ/H] \;\simeq\; [\mathrm{pt}\,/Z(G)_0] \times Q$$
where $Q \to B$ is a proper moduli space.
4. The complement $A \setminus A^\circ$ admits a stratification by closed H-invariant pieces $Z_\delta(\pi)$ and $W_\delta(\pi)$, each an affine bundle over a fixed-point locus $F_\delta(\pi)$ under a subgroup $H(\pi) \cong (\mathbb{C}^\times)^2$ of H.
5. For admissible K-theory classes α on $\mathcal{X}_{G,g,I}$, the weights of the fibers of α over the fixed loci $F_\delta(\pi)$ are bounded below by linear functions of the discrete defect parameters (n_+, n_-) in a way that forces vanishing of $H(\pi)$-invariants for sufficiently large defect.

These properties, combined with the local cohomology argument, imply that for any admissible class α, the H-invariant part of $R\Gamma(A, \alpha)$ is a finitely generated $\mathcal{O}_B$-module that vanishes in high degrees. Since this holds étale-locally over B and the stack structure descends, we conclude that the derived pushforward

$$RF_*\alpha : \mathcal{X}_{G,g,I} \longrightarrow \overline{\mathcal{M}}_{g,I}$$

is a bounded complex of coherent sheaves, where F is the forgetful map forgetting the bundle and the modification.

3.1 Algebraicity and finite type of $\mathcal{X}_{G,g,I}$

I have sketched a proof of the following proposition, establishing that $\mathcal{X}_{G,g,I}$ is an algebraic stack, locally of finite type over $\overline{\mathcal{M}}_{g,I}$.

Proposition 3.5. The projection

$$F : \mathcal{X}_{G,g,I} \to \overline{\mathcal{M}}_{g,I}$$

is algebraic and locally of finite type.

Proof. We decompose the definition of $\mathcal{X}_{G,g,I}$ into three pieces: twisted curves and their modifications, principal G–bundles, and the admissibility condition.

Step 1: Twisted curves. Let $\mathfrak{M}^{tw}_{g,I}$ denote the stack of twisted stable curves of genus g with I markings. By Abramovich–Vistoli [2] $\mathfrak{M}^{tw}_{g,I}$ is a Deligne–Mumford stack, locally of finite type over $\mathbb{C}$, equipped with a representable morphism

$$\mathfrak{M}^{tw}_{g,I} \to \overline{\mathcal{M}}_{g,I}$$

sending a twisted curve to its coarse stable curve.

Step 2: Twisted modifications. Let $\mathsf{TwMdf}_{\leq r,g,I}$ be the stack over $\overline{\mathcal{M}}_{g,I}$ whose objects over a scheme $B \to \overline{\mathcal{M}}_{g,I}$ are twisted modifications of length $\leq r$ of the pulled–back universal curve $\Sigma_B \to B$.

Étale–locally on $\overline{\mathcal{M}}_{g,I}$, the universal curve is simultaneously versal at all nodes. Twisted modifications of bounded length are obtained by inserting chains of $\mathbb{P}^1$'s and taking balanced root stacks at the nodes. Standard results on expanded degenerations and twisted curves imply that $\mathsf{TwMdf}_{\leq r,g,I}$ is an algebraic stack, locally of finite type over $\overline{\mathcal{M}}_{g,I}$. In particular, we have the following result from [1]. They do not explicitly state their results in this manner, but the following proposition follows from their constructions.

Proposition 3.6 (twisted expansions are algebraic and lft)**.** There exist algebraic stacks

$$\mathcal{T}^{\mathrm{tw}} \qquad \text{and} \qquad \mathfrak{T}^{\mathrm{tw}}$$

called the stacks of **twisted expanded pairs** and **twisted expanded degenerations**, with the following properties.

(1) (**Moduli interpretation**) For any scheme S, objects of $\mathcal{T}^{\mathrm{tw}}(S)$ (resp. $\mathfrak{T}^{\mathrm{tw}}(S)$) are flat families over S whose geometric fibers are **standard r–twisted expansions** of the universal pair $(\mathbb{A}^1, 0)$ (resp. of the universal degeneration $\mathbb{A}^2 \to \mathbb{A}^1$), in the sense of [1, Def. 2.4.2]. In particular, these families are obtained from the basic local node $xy = t$ by inserting chains of $\mathbb{P}^1$'s and equipping the resulting nodes with balanced stabilizers (equivalently, by iterated balanced root constructions at the boundary).

(2) (**Algebraicity and finiteness**) The stacks $\mathcal{T}^{\mathrm{tw}}$ and $\mathfrak{T}^{\mathrm{tw}}$ are algebraic (indeed Deligne–Mumford) and locally of finite type (equivalently, locally of finite presentation) over $\operatorname{Spec}\mathbb{Z}$. Moreover, the locus of expansions of length $\leq r$ is an open substack

$$\mathcal{T}^{\mathrm{tw}}_{\leq r} \subset \mathcal{T}^{\mathrm{tw}}, \qquad \mathfrak{T}^{\mathrm{tw}}_{\leq r} \subset \mathfrak{T}^{\mathrm{tw}},$$

and is locally of finite type. This follows from [1, Lem. 3.1.3] (and the discussion surrounding it).

(3) (**Base change**) For any morphism $S' \to S$, formation of twisted expansions commutes with base change: the pullback of a twisted expansion family over S is a twisted expansion family over S'. Equivalently, $\mathcal{T}^{\mathrm{tw}}$ and $\mathfrak{T}^{\mathrm{tw}}$ define categories fibered in groupoids over schemes.

Step 3: G–bundles on twisted curves. Let $\mathrm{Bun}_G^{tw} \to \mathfrak{M}_{g,I}^{tw}$ be the stack assigning to a twisted curve $\mathcal{C} \to B$ the groupoid of principal G–bundles on $\mathcal{C}$. For reductive G, this stack is algebraic and locally of finite presentation over $\mathfrak{M}_{g,I}^{tw}$.

Step 4: The ambient stack. Form the fiber product

$$\mathcal{Y} := \mathsf{TwMdf}_{\leq r,g,I} \times_{\mathfrak{M}_{g,I}^{tw}} \mathrm{Bun}_G^{tw}\,.$$

An object of $\mathcal{Y}(B)$ is a twisted modification $\mathcal{C}'_B \to \Sigma_B$ together with a principal G–bundle on $\mathcal{C}'_B$. By Steps 2 and 3, $\mathcal{Y}$ is an algebraic stack, locally of finite type over $\overline{\mathcal{M}}_{g,I}$.

Step 5: Admissibility. The admissibility condition on G-bundles is a restriction on the local monodromy homomorphisms $\mu_k \to G$ at the twisted nodes, requiring the associated rational cocharacters to lie in a fixed finite set and satisfy a linear independence condition. After restricting to strata where the set of nodes is locally constant, these local types are discrete invariants and are locally constant in families. Since only finitely many types are allowed, the admissible locus defines an open-and-closed substack

$$\mathcal{X}_{G,g,I} \subset \mathcal{Y}.$$

Being an open-and-closed substack of the algebraic stack $\mathcal{Y}$, $\mathcal{X}_{G,g,I}$ is algebraic and locally of finite type over $\overline{\mathcal{M}}_{g,I}$. This proves the proposition. □

Lemma 3.7 (Simultaneous versal smoothing of all nodes)**.** Let $(\Sigma_0, \sigma_{0,i})$ be a stable marked curve over an algebraically closed field of characteristic 0, and let $p_1, \ldots, p_m \in \Sigma_0$ be its nodes. Let $\pi : \overline{\mathcal{C}}_{g,I} \to \overline{\mathcal{M}}_{g,I}$ be the universal curve and let $x := [\Sigma_0, \sigma_{0,i}] \in \overline{\mathcal{M}}_{g,I}$.

Then there exists an affine étale neighborhood $B \to \overline{\mathcal{M}}_{g,I}$ of x with pulled-back family $\Sigma_B := \overline{\mathcal{C}}_{g,I} \times_{\overline{\mathcal{M}}_{g,I}} B \to B$ such that the following hold.

(1) (*Simultaneous standard form at the nodes*) For each $i = 1, \ldots, m$ there exists an étale neighborhood $U_i \subset \Sigma_B$ of the relative node lying over p_i and an element $t_i \in \mathcal{O}_B(B)$ such that there is an isomorphism of B–schemes

$$U_i \;\cong\; \mathrm{Spec}\big(\mathcal{O}_B[u_i, v_i]/(u_i v_i - t_i)\big)$$

sending the relative node to the locus $(u_i = v_i = t_i = 0)$.

(2) (*Independence of smoothing parameters*) After possibly shrinking B, the functions $t_1, \ldots, t_m$ define a morphism

$$t = (t_1, \ldots, t_m) : B \longrightarrow \mathbb{A}^m$$

which is smooth (equivalently, étale after completing at x), and whose coordinates t_i cut out the boundary divisors corresponding to the nodes: the locus in B where the ith node persists is exactly $\{t_i = 0\}$.

Proof sketch of Proposition 3.5 via reduction to Solis. Fix an affine étale chart $B \to \overline{\mathcal{M}}_{g,I}$ as above, and write

$$\Sigma_B := \overline{\mathcal{C}}_{g,I} \times_{\overline{\mathcal{M}}_{g,I}} B, \qquad \Sigma_B^{\mathrm{sm}} := \Sigma_B \setminus \mathrm{Sing}(\Sigma_B/B).$$

Let $p_1, \ldots, p_m$ be the relative nodes, and let $\Sigma_{B,i}^{\mathrm{loc}} \subset \Sigma_B$ be small étale neighborhoods of p_i identified with the standard smoothings

$$\mathrm{Spec}\, \mathcal{O}_B[u_i, v_i]/(u_i v_i - t_i) \to B.$$

Write $\Sigma_{B,i}^{\mathrm{punct}} := \Sigma_{B,i}^{\mathrm{loc}} \setminus \{p_i\}$.

Description of objects. By definition, an object of $\mathcal{X}_{G,g,I}(B)$ is a pair $(\mathcal{C}'_B, P_B)$ where $\mathcal{C}'_B \to \Sigma_B$ is a twisted modification and P_B is an admissible G–bundle on $\mathcal{C}'_B$. Since $\mathcal{C}'_B \to \Sigma_B$ is an isomorphism over Σ_B^{sm}, this is equivalently the data of:

(i) a principal G–bundle P^{sm} on Σ_B^{sm};

(ii) for each i, a twisted modification $\mathcal{C}'_{B,i} \to \Sigma_{B,i}^{\mathrm{loc}}$ together with a G–bundle P_i on $\mathcal{C}'_{B,i}$;

(iii) isomorphisms

$$P^{\mathrm{sm}}|_{\Sigma_{B,i}^{\mathrm{punct}}} \;\xrightarrow{\sim}\; P_i|_{\Sigma_{B,i}^{\mathrm{punct}}}$$

for each i, satisfying the evident compatibility on overlaps;

(iv) such that each $(\mathcal{C}'_{B,i}, P_i)$ is admissible at the twisted nodes.

The equivalence follows because twisted modifications and principal G–bundles satisfy étale descent, and the admissibility condition is local at twisted nodes.

Identification with the fiber product. For each i, Solis's stack $\mathcal{X}_G(C_{S_i}) \times_{S_i} B$ classifies exactly the local admissible data $(\mathcal{C}'_{B,i}, P_i)$, and carries a natural restriction map

$$\rho_i : \mathcal{X}_G(C_{S_i}) \times_{S_i} B \longrightarrow \mathrm{Bun}_G(\Sigma_{B,i}^{\mathrm{punct}})$$

given by restricting P_i to the punctured neighborhood. Likewise, restriction defines a morphism

$$\rho : \mathrm{Bun}_G(\Sigma_B^{\mathrm{sm}}) \longrightarrow \prod_i \mathrm{Bun}_G(\Sigma_{B,i}^{\mathrm{punct}}).$$

By the description above, giving $(\mathcal{C}'_B, P_B) \in \mathcal{X}_{G,g,I}(B)$ is equivalent to giving objects of $\mathcal{X}_G(C_{S_i}) \times_{S_i} B$ for each i and an object of $\mathrm{Bun}_G(\Sigma_B^{\mathrm{sm}})$ whose images under the maps ρ_i and ρ agree. This is precisely the groupoid of B–points of the 2–fiber product

$$\Big(\prod_{i=1}^{m} \mathcal{X}_G(C_{S_i}) \times_{S_i} B\Big) \times_{\prod_i \mathrm{Bun}_G(\Sigma_{B,i}^{\mathrm{punct}})} \mathrm{Bun}_G(\Sigma_B^{\mathrm{sm}}).$$

Functoriality in B shows this identification is compatible with pullback, hence defines an equivalence of stacks over B.

Conclusion. Since each $\mathcal{X}_G(C_{S_i})$ is algebraic and locally of finite type by Solis, and $\mathrm{Bun}_G(\Sigma_B^{\mathrm{sm}})$ is algebraic and locally of finite type, the same holds for $\mathcal{X}_{G,g,I}|_B$. As algebraicity and local finite type are étale–local on the base, the result follows globally. $\square$

3.2 Etale presentation by an algebraic space

Fix a geometric point $[\Sigma_0, \sigma_{0,i}] \in \overline{\mathcal{M}}_{g,I}$ and let $B \to \overline{\mathcal{M}}_{g,I}$ be an affine étale neighborhood. Let

$$\Sigma := \overline{\mathcal{C}}_{g,I} \times_{\overline{\mathcal{M}}_{g,I}} B \to B$$

be the pulled-back universal curve. Let V be the set of stable components of Σ_0 (equivalently, the stable vertices of its dual graph). After possibly refining B étale-locally, choose sections

$$\sigma_v : B \to \Sigma \qquad (v \in V)$$

landing in the smooth locus of $\Sigma \to B$ such that every stable component of every geometric fiber meets at least one σ_v.

Let $\mathcal{X}|_B := \mathcal{X}_{G,g,I} \times_{\overline{\mathcal{M}}_{g,I}} B$.

Definition 3.8 (Framed chart)**.** Define a category fibered in groupoids $A \to (\mathrm{Sch}\,/B)$ by the following assignment. For a B–scheme $T \to B$, an object of $A(T)$ is a triple

$$(\mathcal{C}'_T \xrightarrow{m} \Sigma_T,\ \mathcal{P}_T,\ (t_v)_{v\in V})$$

where:

(i) $\Sigma_T := \Sigma \times_B T$;

(ii) $m : \mathcal{C}'_T \to \Sigma_T$ is a twisted modification (with the chosen length bound);

(iii) $\mathcal{P}_T$ is an admissible principal G–bundle on $\mathcal{C}'_T$;

(iv) for each $v \in V$, a **framing** is an isomorphism of G–torsors

$$t_v : (\sigma_{v,T})^*\mathcal{P}_T \xrightarrow{\sim} G_T.$$

Morphisms are isomorphisms of $(\mathcal{C}'_T, m, \mathcal{P}_T)$ compatible with all t_v.

Let

$$H := G^V$$

act on A by changing framings: for $h = (h_v) \in H(T)$ send t_v to $h_v \circ t_v$. There is a forgetful morphism

$$\pi : A \longrightarrow \mathcal{X}|_B$$

forgetting the framings.

However unlike the case of line bundles, framings do not necessarily rigidify G–bundles, as the following example shows.

Example 3.9 (Why a single framing does not rigidify G–bundles)**.** Let $C = \mathbb{P}^1$ and $G = \mathrm{SL}_2$. Consider the rank–2 vector bundle

$$E \ = \ \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \qquad \det(E) \cong \mathcal{O}_{\mathbb{P}^1},$$

and let $\mathcal{P}$ be the associated principal SL_2–bundle.

Automorphisms of $\mathcal{P}$ are the same as determinant–1 automorphisms of E. There is a unipotent subgroup

$$U \ = \ \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \ \middle| \ s \in H^0\big(\mathbb{P}^1, \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(1))\big) \right\} \ \subset \ \mathrm{Aut}(\mathcal{P}).$$

Since $\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(1)) \cong \mathcal{O}(2)$ and $H^0(\mathbb{P}^1, \mathcal{O}(2)) \cong \mathbb{C}^3$, we obtain

$$U \cong \mathfrak{G}_a^3.$$

In particular, $\mathrm{Aut}(\mathcal{P})$ is positive–dimensional and far larger than the center of SL_2.

Now fix a point $p \in \mathbb{P}^1$ and choose a framing $t : \mathcal{P}|_p \cong \mathrm{SL}_2$, equivalently a basis of E_p. An element of U preserves this framing if and only if it acts trivially on the fiber E_p, i.e. if $s(p) = 0$. The subspace of sections vanishing at p has dimension 2:

$$\{s \in H^0(\mathbb{P}^1, \mathcal{O}(2)) \mid s(p) = 0\} \cong \mathbb{C}^2.$$

Therefore,

$$\mathrm{Aut}(\mathcal{P}, t) \supset \mathfrak{G}_a^2,$$

and the framed bundle still has a positive–dimensional automorphism group.

This shows that, unlike the case $G = \mathbb{C}^\times$, a single trivialization point does **not** rigidify a G–bundle for general reductive G. Non–central infinitesimal automorphisms coming from $H^0(C, \mathrm{ad}(\mathcal{P}))$ may vanish at a point while remaining nonzero globally.

In the $\mathbb{C}^\times$ case, after étale localization $B \to \overline{\mathcal{M}}_{g,I}$, one presents

$$\widetilde{\mathcal{M}}|_B \simeq [A/\mathcal{G}], \qquad \mathcal{G} \cong (\mathbb{C}^\times)^V,$$

where A is an algebraic space obtained by rigidifying bundles with trivializations at points σ_v on each stable component. The key input is that once you trivialize at one point per stable component, every automorphism dies, so A has no stabilizers.

For general reductive G, this step fails on the full stack $\mathcal{X}_{G,g,I}$. There are non-central automorphisms of principal G-bundles on projective curves which vanish at a point but are nontrivial globally.

What seems to fix this is to restrict to a "regularly stable" locus: require that on every stable component the restricted bundle satisfies

$$\mathrm{Aut}(P|_X) = Z(G).$$

On this open locus, any automorphism is central, and a single framing point forces it to be trivial. Then the same construction gives a chart A^{rs} which is an algebraic space, with

$$\mathcal{X}^{\mathrm{rs}}|_B \simeq [A^{\mathrm{rs}}/G^V].$$

So the difficulty is that on the full stack $\mathcal{X}$ there is no way to uniformly kill stabilizers with finitely many framings, whereas on the regularly stable locus the FTT mechanism goes through unchanged.

I wanted to ask your thoughts on the best way to proceed conceptually: either (1) work on $\mathcal{X}^{\mathrm{rs}}$ and then try to extend results from this open substack, or (2) attempt a genuinely stacky version of the local chart and redo the local cohomology argument with stabilizers present.

4 Questions Jan 27

1. I also wanted to ask about the moduli problem I've set up and the basic algebraicity statement. I defined the stack $\mathcal{X}_{G,g,I}$ of admissible G-bundles on twisted modifications of the universal curve, and I wrote a proof sketch that the projection

   $$F : \mathcal{X}_{G,g,I} \to \overline{\mathcal{M}}_{g,I}$$

   is algebraic and locally of finite type, by decomposing it into: twisted curves and expansions, principal G-bundles on twisted curves, and then cutting out the admissible locus as an open-and-closed substack.

   I'd like to sketch this argument for you and ask whether it sounds essentially correct, or whether there is something subtle I'm missing — for example about boundedness, effectivity of descent, or whether admissibility really behaves as an open-and-closed condition in families.

2. In the FTT argument, after étale localization one presents the moduli stack as $[A/H]$ with A an algebraic space, by rigidifying bundles with framings at points. In the G-bundle case, this fails on the full stack because framings do not kill non-central automorphisms, so the natural "framed" chart is only an Artin stack, not an algebraic space.

   Conceptually, how does one usually overcome this kind of obstruction? Is the right approach to restrict to a regularly stable locus where framings rigidify, and then try to extend results from that open substack, or is there a standard way to work directly with stacky local charts in a situation like this?

3. I think I now understand formally what admissibility says — it restricts the $\mu_k \to G$ data at twisted nodes to a finite set of rational cocharacters and a linear independence condition — but I still don't really have a geometric or representation-theoretic picture in my head.

   Could you explain how you think about admissibility conceptually? For example, how it relates to Atiyah-Bott, and what geometric behavior it is really controlling?

4. I feel like I'm still reasoning about these objects in a very hands-off way. Is there a good toy model of this whole setup — a simpler moduli problem or a lower-rank/group case — where the roles of twisted curves, admissibility, and the boundary are completely explicit and one can really see what is going on?

5 Answer 1/27

Teleman basically said that there is no way to get an atlas of the full stack $\mathrm{Bun}_G(C)$ (or your $\mathcal{X}_{G,g,I}$) by an algebraic space modulo a fixed reductive group. This is because even for smooth curves, the stabilizers are already too big. He said that the correct approach can be found in Teleman-Woodward, that he expects there to be an explicit formula as that which he works out in that paper, and his suggestion is to do it for $G = \mathrm{GL}_2$ or perhaps $G = E_8$ and to do the case with two components meeting at a node.

6 References

1. D. Abramovich, C. Cadman, B. Fantechi, and J. Wise, *Expanded degenerations and pairs*, preprint, arXiv:1208.6322.

2. D. Abramovich and A. Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. 15 (2002), no. 1, 27–75.

3. K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. 128 (1997), no. 1, 45–88.

4. Edward Frenkel, Constantin Teleman, and A. J. Tolland, *Gromov-Witten Gauge Theory I.*

5. P. Solis, *Wonderful Loop Group Embeddings and Applications to the Moduli of G-bundles on Curves*, PhD thesis, ProQuest Dissertations and Theses (2014).