

# Homework 1

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**Problem Problem1** Show that the  $n$ -sheeted Riemann surface of the multi-valued function

$$w = z^{1/n}, \quad z \in \mathbb{C},$$

is topologically a sphere with 1 puncture.

*Solution:* Let  $\mathcal{R} = \{(z, w) \in \mathbb{C}^2 : w^n = z\}$ .  $\mathcal{R}$  carries the structure of a Riemann surface so that the projection  $\pi : \mathcal{R} \rightarrow \mathbb{C}$ ,  $\pi(z, w) = z$  is holomorphic. Now consider the map

$$\Phi : \mathbb{C} \longrightarrow \mathcal{R}, \quad \Phi(w) = (w^n, w)$$

$\Phi$  is bijective: given  $(z, w) \in \mathcal{R}$  we must have  $z = w^n$ , so the inverse is simply  $(z, w) \mapsto w$ .  $\Phi$  and its inverse are holomorphic because one is given by a polynomial, the other is a projection. Hence  $\Phi$  is a biholomorphism. Therefore  $\mathcal{R}$  is (as a Riemann surface, hence also topologically) just  $\mathbb{C}$ . Topologically,  $\mathbb{C}$  is a sphere with one point removed (a “punctured sphere”):  $\mathbb{C} \simeq \widehat{\mathbb{C}} \setminus \{\infty\}$ . Thus the  $n$ -sheeted Riemann surface of  $w = z^{1/n}$  is topologically a sphere with one puncture.

**Problem Problem2** Let  $f(z)$  be a polynomial of odd degree, with simple zeroes. Identify the topology of the Riemann surface of the double-valued function defined by  $w^2 = f(z)$ .

*Solution:* Consider the affine curve  $X_{\text{aff}} = \{(z, w) \in \mathbb{C}^2 : w^2 = f(z)\}$ . Its projection  $\pi_{\text{aff}} : (z, w) \mapsto z$  is a 2-sheeted branched covering of  $\mathbb{C}$  away from the zeros of  $f$ . We compactify to a projective curve  $X = \overline{X_{\text{aff}}} \subset \mathbb{P}_z^1 \times \mathbb{P}_w^1$  and extend the projection to  $\pi : X \longrightarrow \mathbb{P}_z^1$ . The map  $\pi$  has degree 2. To study the topology of  $X_{\text{aff}}$ , we will use the Riemann-Hurwitz formula to compute the genus of  $X$  and delete the point(s) over  $z = \infty$ .

If  $a$  is a simple zero of  $f$ , write locally  $f(z) = (z-a)u(z)$  with  $u(a) \neq 0$ . Then  $w^2 = (z-a)u(z)$  has a single point of  $X$  lying over  $z = a$  and the local model is  $w^2 = z - a$ , so the ramification index is  $e = 2$ . Thus each simple zero gives one branch point of ramification index 2. There are  $d$  of these in  $\mathbb{C}$ . Put  $t = 1/z$  as a coordinate near  $z = \infty$  and write

$$f(z) = z^d g(1/z) = t^{-d} g(t), \quad g(0) \neq 0$$

The equation becomes  $w^2 = t^{-d} g(t) \iff (w t^{\frac{d-1}{2}})^2 = t^{-1} g(t)$ . Let  $u = w t^{\frac{d-1}{2}}$ . Then  $u^2 = t^{-1} g(t)$ , so near  $t = 0$  we have the model  $u^2 \sim t^{-1}$ . Therefore, there is one point of  $X$  over  $z = \infty$  and it is ramified of order 2. Hence the total number of simple branch points is  $B = d + 1$ .

Apply Riemann-Hurwitz to the degree-2 map  $\pi : X \rightarrow \mathbb{P}^1$ :

$$2g(X) - 2 = 2 \cdot (-2) + \sum_{p \in X} (e_p - 1).$$

Every simple ramification contributes  $e_p - 1 = 1$ , so

$$2g(X) - 2 = -4 + B = -4 + (d + 1) = d - 3.$$

Therefore

$$g(X) = \frac{d - 1}{2}.$$

The compact Riemann surface  $X$  is a closed orientable surface of genus  $g = \frac{d-1}{2}$ . Recall that there is only one point of  $X$  over  $z = \infty$ . Therefore,  $X_{\text{aff}}$  is homeomorphic to  $X$  with one point removed. Hence  $X_{\text{aff}}$  is homeomorphic to a genus  $\frac{d-1}{2}$  surface with one puncture.

**Problem Problem3** Show that a bijective holomorphic map

$$f : R \rightarrow S$$

between Riemann surfaces is in fact bi-holomorphic (meaning, the inverse is also holomorphic). Show that two homeomorphic Riemann surfaces need not be bi-holomorphic. (*Hint: Use the unit disk  $\Delta$  and the complex plane.*) Show that no two of the following three annuli in  $\mathbb{C}$  are bi-holomorphic:

- (a)  $\{z \mid 0 < |z| < 1\}$ ,
- (b)  $\{z \mid 1 < |z| < 2\}$ ,
- (c)  $\{z \mid 0 < |z| < \infty\}$ .

*Solution:* The inverse function theorem guarantees that the inverse function  $f^{-1}$  is smooth. Moreover, it guarantees that

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

Since  $f$  is bijective, it has nonzero derivative everywhere because if it did not, it would look like  $z \mapsto z^k$  for some  $k \geq 2$  and thus it would fail to be locally bijective. Since it has nonzero derivative everywhere,  $(f^{-1})'$  is defined everywhere and is in fact a complex number. Hence  $f^{-1}$  is holomorphic.  $\Delta$  and  $\mathbb{C}$  are homeomorphic but they are not biholomorphic. An explicit homeomorphism is given by the radial map

$$\varphi : \Delta \rightarrow \mathbb{C}, \quad \varphi(re^{i\theta}) = \frac{r}{1-r} e^{i\theta}, \quad 0 \leq r < 1.$$

If there were a biholomorphism the map  $\mathbb{C} \rightarrow \Delta$  is bounded and entire, hence constant which is a contradiction. Suppose (a) and (c) are biholomorphic. Then the map from

$\{z \mid 0 < |z| < \infty\} \rightarrow \{z \mid 0 < |z| < 1\}$  could be extended across the origin because it is bounded in a neighborhood of the origin (apply Riemann's removable singularity theorem). So it extends to a bounded entire function and hence must be constant. This is a contradiction. The same argument shows that (b) and (c) are not holomorphic. Finally suppose that (a) and (b) were biholomorphic. A map  $F : \{z \mid 0 < |z| < 1\} \rightarrow \{z \mid 1 < |z| < 2\}$  again extends holomorphically to  $\tilde{F}$  across zero by Riemann's theorem. Moreover,  $\tilde{F}'(0) \neq 0$  because in a punctured neighborhood of 0,  $\tilde{F}$  is a biholomorphism. Thus  $\tilde{F} : U \rightarrow V$  admits a local holomorphic inverse  $G : V \rightarrow U$  where  $U$  is a neighborhood of 0 and  $V$  is a neighborhood of  $\tilde{F}(0) \in A(1, 2)$ . But  $F$  already has a global holomorphic inverse, call it  $F^{-1}$  and so  $F^{-1}$  and  $G$  must agree on  $V$ . But  $G$  maps  $\tilde{F}(0)$  to 0 so so must  $F^{-1}$  but this is a contradiction.

**Problem Problem4** Prove the *Weierstrass division theorem*: Given a polynomial

$$P(w, z_1, \dots, z_n) = w^n + \sum_{k=0}^{n-1} p_k(z)w^k,$$

with the functions  $p_k(z)$  holomorphic in an open set  $V \subset \mathbb{C}^n$  and satisfying  $p_k(0) = 0$ , every germ of holomorphic function  $G(w, z)$  near  $(w, z) = (0, 0)$  can be uniquely expressed as

$$G(w, z) = P(w, z) \cdot Q(w, z) + R(w, z),$$

where  $Q(w, z)$  is a holomorphic germ near 0 and  $R(w, z)$  is a polynomial in  $w$  of degree  $< n$  with coefficients germs of holomorphic functions in  $z$  near  $z = 0$ .

To do this, define

$$Q(z, w) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{G(\zeta, z)}{P(\zeta, z)(\zeta - w)} d\zeta$$

for a suitable choice of the line integral over each fixed value of  $z$ , and show that the difference

$$R(w, z) := G(w, z) - P(w, z) \cdot Q(w, z)$$

is a holomorphic function of  $(w, z)$  which is polynomial in  $w$  with degree  $< n$ . Hint: You will want to express that difference as a Cauchy integral to get your conclusion.

*Solution:* Using Cauchy's integral formula we write

$$G(w, z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{G(\zeta, z)}{\zeta - w} d\zeta$$

from which we can write

$$R(w, z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{G(\zeta, z)P(\zeta, z)}{P(\zeta, z)(\zeta - w)} - \frac{G(\zeta, z)P(w, z)}{P(\zeta, z)(\zeta - w)} d\zeta$$

Now write

$$P(\zeta, z) - P(w, z) = (\zeta^k - w^k) + \sum_{i=1}^{k-1} p_k(z)(\zeta^i - w^i)$$

which is divisible by  $\zeta - w$ , and the quotient

$$\frac{P(\zeta, z) - P(w, z)}{\zeta - w}$$

is a polynomial in  $w$  of degree  $n - 1$ . If for a fixed value of  $z$ , we pick a contour  $\Gamma$  in the  $w$  plane for which  $P(\zeta, z)$  does not vanish on  $\Gamma$ , then the function  $R$  is holomorphic in  $w$  and  $z$  because it is the contour integral of an integrand, holomorphic in both  $w$  and  $z$ . Since  $R$  is holomorphic, we may differentiate  $n$  times with respect to  $w$  under the integral and we find that the integrand becomes zero, since the integrand is a polynomial in  $w$  of degree  $n - 1$ . Therefore,

$$\frac{d^n R}{dw^n} = 0$$

so  $R$  is indeed polynomial of degree  $n - 1$ .

**Problem Problem5** A *Reinhardt domain*  $R \subset \mathbb{C}^n$  is an open set such that

$$(z_1, \dots, z_n) \in R \Rightarrow (qz_1, \dots, qz_n) \in R, \quad \forall q \in \mathbb{C} \text{ with } |q| < 1.$$

- (a) Show that the intersection of finitely many Reinhardt domains is Reinhardt.
- (b) Show that if a multi-variable power series centered at 0 in some neighborhood of some point  $(z_1, \dots, z_n) \in \mathbb{C}^n$ , then it converges uniformly in some Reinhardt domain containing  $z$ .
- (c) Prove that the *domain of convergence* of an  $n$ -variable Taylor series centered at 0 — defined as the interior of the set of points where the series converges — is a Reinhardt domain.

*Solution:*

1. If  $x \in R_i$  then  $qx \in R_i$  and in particular  $x \in \bigcap R_i$  implies  $qx \in \bigcap R_i$  so  $\bigcap R_i$  is Reinhardt. The intersection of finitely many open sets is open.
2. Let  $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$  be a power series which converges in a neighborhood of some point  $z = (z_1, \dots, z_n)$ . We want to show that it converges uniformly in some Reinhardt domain containing  $z$ . Define  $b_m = \sum_{|\alpha|=m} a_\alpha z^\alpha$  and group the series by total degree:

$$f(z) = \sum_{m=0}^{\infty} b_m$$

For a scalar  $q \in \mathbb{C}$ , consider  $F(q) = \sum_{\alpha} a_{\alpha}(qz)^{\alpha}$ . Observe that

$$F(q) = \sum_{m=0}^{\infty} \left( \sum_{|\alpha|=m} a_{\alpha} z^{\alpha} \right) q^m = \sum_{m=0}^{\infty} b_m q^m$$

By assumption, the original series converges at  $z$ . That means  $F(1) = \sum_{m=0}^{\infty} b_m$  converges. For any  $r < 1$ , the series  $\sum b_m q^m$  converges absolutely and uniformly on  $|q| \leq r$  by the one-variable Weierstrass M-test, since  $\sum |b_m| r^m < \infty$ . Fix  $r < 1$ . Then for all  $|q| \leq r$ ,  $F(q) = \sum_{\alpha} a_{\alpha}(qz)^{\alpha}$  converges uniformly. In other words, the original  $n$ -variable series converges uniformly on the set  $\Omega_r(z) = \{qz : |q| \leq r\}$ . This set  $\Omega_r(z)$  is a Reinhardt domain containing  $z$ .

3. Let  $X \subset \mathbb{C}^n$  be the set of points where  $\sum_{\alpha} a_{\alpha} z^{\alpha}$  converges. Let  $Y := \text{int } X$  be its domain of convergence. We claim that  $X$  is Reinhardt. Indeed, if  $z \in X$ , then as in part (b) the function  $F(q) = \sum_{m=0}^{\infty} b_m q^m$  where  $b_m = \sum_{|\alpha|=m} a_{\alpha} z^{\alpha}$ , converges at  $q = 1$ , hence has one-variable radius of convergence  $R \geq 1$ . Therefore  $F(q)$  converges for all  $|q| < 1$ , i.e.  $|q| < 1 \Rightarrow qz \in X$ . Thus  $X$  is stable under common scalings by  $|q| < 1$ . The interior  $Y$  is both open and stable under such scalings, hence is a Reinhardt domain as desired.

**Problem Problem6** Let  $C_1$  and  $C_2$  be two circles in the  $w$ - and  $z$ -planes in  $\mathbb{C}^2$ , and  $\Delta_{1,2}$  the disks that they bound. Show that a holomorphic function defined in an open set containing

$$C_1 \times \Delta_2 \cup \Delta_1 \times C_2$$

has a unique holomorphic extension over  $\Delta_1 \times \Delta_2$ . *Hint:* Use Cauchy's formula in a way very similar to the one exploited above.

*Solution:* Suppose  $f$  is holomorphic on a neighborhood of  $X = (\partial\Delta \times \Delta) \cup (\Delta \times \partial\Delta) \subset \mathbb{C}^2$ . For each fixed  $z_2 \in \Delta$ , the map  $\zeta_1 \mapsto f(\zeta_1, z_2)$  is holomorphic on a neighborhood of  $\partial\Delta$ . Define

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_1|=1} \frac{f(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1, \quad (z_1, z_2) \in \Delta \times \Delta.$$

$F$  is holomorphic on  $\Delta \times \Delta$ . Moreover, on a neighborhood of  $\partial\Delta \times \Delta$  (where  $f$  is defined in a full annulus in  $\zeta_1$ ), Cauchy's formula gives  $F = f$ . Similarly, for each fixed  $z_1 \in \Delta$  the map  $\zeta_2 \mapsto f(z_1, \zeta_2)$  is holomorphic near  $\partial\Delta$ . Define

$$G(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_2|=1} \frac{f(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2.$$

Then  $G$  is holomorphic on  $\Delta \times \Delta$  and  $G = f$  on a neighborhood of  $\Delta \times \partial\Delta$ . On a neighborhood of the torus  $\partial\Delta \times \partial\Delta \subset X$ , both representations are valid and equal  $f$ , hence  $F = G$  there. By the identity theorem for holomorphic functions on  $\Delta \times \Delta$ ,  $F \equiv G$  on all of  $\Delta \times \Delta$ . Thus this common function extends  $f$  holomorphically to the full interior.

The extension is unique because if  $H$  were another holomorphic extension, then  $H = F$  on  $\partial\Delta \times \Delta$  by the identity theorem applied to the first variable, hence  $H = F$  on all of  $\Delta \times \Delta$  by the identity theorem applied to the second variable.

**Problem Problem7** Let  $F, G$  be two irreducible holomorphic functions in  $n > 1$  variables defined on an open set  $U$ , and call their common zero-set  $Z$ . Using the Weierstrass Preparation Theorem (twice) and Q6, show that any holomorphic function defined on  $U \setminus Z$  extends holomorphically over  $Z$ .

**Remark 1.** This is a version of *Hartogs' theorem* for holomorphic functions of several variables; somewhat loosely, the singular set of a holomorphic function defined on “most of” an open  $U \subset \mathbb{C}^n$  cannot lie in an analytic subset of co-dimension 2, unless it's empty. Contrast that with the real function  $1/(x^2 + y^2)$  on  $\mathbb{R}^2$ .

*Solution:* Let  $F, G$  be irreducible holomorphic functions on  $U \subset \mathbb{C}^n$ ,  $n > 1$ , and set  $Z = \{F = 0\} \cap \{G = 0\}$ . Fix  $p \in Z$  and change coordinates so that  $p = 0$ , with  $F$  regular in  $w$  and  $G$  regular in  $z$ . By Weierstrass Preparation we may write  $F = U_F \cdot P(w; z, t)$  and  $G = U_G \cdot Q(z; w, t)$  where  $P$  is a Weierstrass polynomial in  $w$ ,  $Q$  one in  $z$ , and  $t$  denotes the other coordinates. Thus for small polydisks, the zeros of  $F$  in  $w$  and of  $G$  in  $z$  form finite sets of roots varying holomorphically with the parameters. Choosing circles  $C_1 = \{|w| = r_1\}$  and  $C_2 = \{|z| = r_2\}$  that avoid these roots (uniformly in  $t$ ), we see that  $(C_1 \times \Delta_2) \cup (\Delta_1 \times C_2)$  is contained in  $U \setminus Z$ , so  $f$  is holomorphic there. By Question 6,  $f$  extends holomorphically to  $\Delta_1 \times \Delta_2$  for each fixed  $t$ , and the extension depends holomorphically on  $t$ . Hence  $f$  extends to a neighborhood of  $p$ , and by uniqueness these local extensions glue to give a holomorphic extension of  $f$  to all of  $U$ .