Moduli in November

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Abstract

Recently there were two moduli talks at Cornell. The first talk was given by Andrés Ibáñez Núñez and the second talk was given by Rachel Webb. I am speaking with Rachel on Friday. She talked a lot of about root stacks.

Contents

1 Motivic enumerative invariants of algebraic stacks

1.1 Introduction

A stack can be thought of points with groups. Just as a scheme has a point for each subvariety of a given variety, we can do the same 'equivariantly'. A stack has a point for each subvariety of a given variety, but the points also know the stabilizer groups of the subvarieties.

A lot of my intuition for stacks comes from experts speaking about them with authority. That is how much of this note will be structured. We are also going to need to supplement with references.

Definition 1.1 (Classifying groupoid)**.** *If G is a group, the classifying groupoid BG of G is defined as the category with one object $\star$ such that $\mathrm{Aut}(\star) = \mathrm{Mor}(\star, \star) = G$.*

Example 1.2. *$B- = 0G_m$ is the classifying stack of the multiplicative group and it satisfies*

$$H_i^*(B_{G_m}) = \mathbb{Q}[t] \quad \textit{where} \quad \deg t = 2$$
$$\chi(B_{G_m}) = \infty$$

It is a 1-dimensional stack. Compare this to $\mathbb{CP}^\infty$ which is the topological classifying space of the multiplicative group. In particular it classifies homotopy classes of principal $\mathbb{C}^*$-bundles on X. The groupoid of line bundles on X is equivalent to the groupoid of $\mathbb{C}^*$-bundles on X. via the correspondence $P \mapsto P \times_{\mathbb{C}^*} \mathbb{C}$ and opposite correspondence $L \mapsto L\backslash\{0\} \to X$.

1.2 Euler characteristics

His next point was the Euler characteristics are motivic invariants. Consider the ZXU setup with X a smooth variety and Z a closed subvariety, and U the open complement.
Then Euler characteristics are additive

$$\chi(X) = \chi(U) + \chi(Z)$$

and if you take additivity along these setups as a universal property, then you get for each algebraic stack, a ring of motives, an example of which is the Grothendieck ring of varieties.
In particular, let $\mathcal{X}$ be an algebraic stack and consider the ring

$$M(\mathcal{X}) = \bigoplus \mathbb{Q}[Y \to \mathcal{X}]/\sim$$

where $Y \to \mathcal{X}$ is a representable map of stacks and we mod out by the relation generated by the relation

$$[Y \to \mathcal{X}] = [Y' \to \mathcal{X}] + [Y'' \to \mathcal{X}]$$

for $Y'YY''$ in the ZXU setup. Then there is a map

$$\int_{\chi} : M(\mathcal{X}) \to M(\mathrm{pt})[\mathbb{L}^{-1}, (\mathbb{L}^k - 1)^{-1}]$$

where $\mathbb{L}$ is the Tate motive $[\mathbb{A}^1 \to *]$ and $k \in \mathbb{Z}$.

Example 1.3.

$$[\mathrm{GL}_n] = \prod_{i=1}^{n} \mathbb{L}^n - \mathbb{L}^i$$

and the right hand side is a motive over the point that does not depend on X/GL_n as a quotient stack. He also had this formula

$$\int_{B\mathfrak{G}_m} B\mathfrak{G}_m = \frac{1}{\mathbb{L} - 1}$$

which is not defined at $\mathbb{L} = 1$, which incarnates the fact that the Euler characteristic of $\mathbb{A}^1$ is 1.

1.3 Motivic Hall algebra

If A is an abelian category, let $M(A)$ be the moduli stack of objects in A. Then there are maps $\mathrm{Ext}_A = \{0 \to A \to B \to C \to 0\} \to M(A) \times M(A)$ and $\mathrm{Ext}_A \to M(A)$ called p and q respectively. Then we can define the motivic Hall product $* = q_* p^*$ on $M(M(A))$, the ring associated to the algebraic stack $M(A)$.

1.4 The stack $\mathbb{A}^1/\mathfrak{G}_m$

Let $\theta = \mathbb{A}^1/\mathfrak{G}_m$ be the stacky line. Then

$$\mathrm{Hom}(\theta, \mathrm{Vect}) = \text{Vector bundles on } \theta = \{\text{Vector spaces with filtration}\}$$

In particular, you can pick a trivialization and look at the $t \to \infty$ action of $\mathfrak{G}_m$ and you get some power of k corresponding to the action of $\mathfrak{G}_m$, which induces a filtration (independent of the trivialization). This was explained to me by Allen and is written down in [Knutson-Sharpe 97].

Thus there are stacks

$$\mathrm{Filt}(X) := \mathrm{Hom}(\theta, X) \quad \text{and} \quad \mathrm{Gr}(X) := \mathrm{Hom}(B\mathfrak{G}_m, X)$$

1.5 More examples

Example 1.4. *$\mathbb{C}^2/\mathrm{GL}_2$ was an example that he did.*

1.6 References

References

[1] Jarod Alper, *Notes on Stacks and Moduli*, `https://sites.math.washington.edu/~jarod/moduli.pdf`

2 Abelian Orbicurves with Smooth Coarse Space