Title

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Abstract

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1 Introduction

Theorem 1.1. *The following categories are equivalent:*

- *Compact Riemann surfaces with nonconstant holomorphic maps*
- *Smooth proper (and hence projective) algebraic curves over $\mathbb{C}$ with nonconstant morphisms*
- *Field extensions of $\mathbb{C}$ of transcendence degree 1, of finite degree over $\mathbb{C}(t)$ where t is transcendental over $\mathbb{C}$, with field homomorphisms over $\mathbb{C}$*

The correspondence in one direction is:

$$\begin{aligned} \text{Riemann surface } S &\mapsto \text{ function field } \mathbb{C}(S) \\ \text{Holomorphic map } f : S \to S' &\mapsto \text{ field homomorphism } f^* : \mathbb{C}(S') \to \mathbb{C}(S) \end{aligned}$$

Remark 1.2. *For curves, smooth and proper implies projective. This is false in higher dimensions.*

Common to both is the construction of nonconstant meromorphic functions. It suffices to find

- A map $f : R \to \mathbb{P}^1$ which realizes R as a branched cover of $\mathbb{P}^1$ (the transcendental part of the function field)

$$f^* : \mathbb{C}(z) \hookrightarrow \mathbb{C}(R)$$
$$z \mapsto f$$

- A nonconstant meromorphic function g on S which separates the sheets (the finite part of the function field)

Once you have these functions, consider the set of pairs $\{(f(p), g(p)) : p \in S\} \subset \mathbb{P}^1 \times \mathbb{P}^1$. This is an analytic curve. By a theorem of Riemann (or later by Chow's theorem), an analytic curve in projective space is algebraic. So there exists a nonzero polynomial $P(x, y)$ such that

$$P(f, g) = 0 \quad \text{on } S.$$

Thus, the image of S under (f, g) is contained in the algebraic curve $P(x, y) = 0$. Moreover, because g separates the sheets, (f, g) is generically injective, so the map is birational. Hence S and the curve $P(x, y) = 0$ have the same function field. So you've now explicitly realized $\mathbb{C}(S) = \mathbb{C}(f, g)$.

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We state Riemann's theorem which allows us to pass from the analytic setting to the algebraic setting.

Theorem 2.1. *Let R be a compact Riemann surface and $p \in R$. There exists a meromorphic function f with poles of arbitrary order n at p and holomorphic elsewhere, provided that n is sufficiently large.*

The method of proof involves constructing holomorphic differentials with poles at p, and in fact one can get them to any order of pole ≥ 2. Then if these differentials are exact, their integrals give a single valued function with pole only at p.