Harder–Narasimhan filtration

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Abstract

In this lecture, we will discuss the Harder-Narasimhan filtration of vector bundles on a smooth projective curve. Then we give the generalization of the HN filtration to principal G-bundles, which gives rise to the Shatz stratification of the moduli stack of G-bundles. Finally, we will give a stack-theoretic interpretation of the numerical criterion for stability in terms of very close degenerations.

Contents

1 Vector bundles on curves

Let C be a smooth projective curve over an algebraically closed field k. Throughout, all vector bundles are assumed to be algebraic vector bundles on C.

1.1 Basic definitions

Definition 1.1 (Degree)**.** Let E be a vector bundle on C. The **degree** of E is

$$\deg(E) := \deg(\det E),$$

where $\det E = \bigwedge^{\mathrm{rk}(E)} E$ is the determinant line bundle.

Definition 1.2 (Slope)**.** The **slope** of a nonzero vector bundle E is

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Definition 1.3 (Semistable and stable bundles)**.** A vector bundle E is called

- **semistable** if for every proper nonzero subbundle $F \subset E$ one has

$$\mu(F) \leq \mu(E),$$

- **stable** if for every proper nonzero subbundle $F \subset E$ one has

$$\mu(F) < \mu(E).$$

Definition 1.4 (Maximal slope)**.** For a nonzero vector bundle E, define

$$\mu_{\max}(E) := \sup\{\, \mu(F) \mid 0 \neq F \subset E \text{ a subbundle} \,\}.$$

Theorem 1.5 (Existence of maximal destabilizing subbundle)**.** For every nonzero vector bundle E on C, there exists a unique maximal subbundle $E_1 \subset E$ such that

$$\mu(E_1) = \mu_{\max}(E),$$

and E_1 is semistable. This is called the **maximal destabilizing subbundle**.

Definition 1.6 (Harder–Narasimhan filtration)**.** Let E be a nonzero vector bundle on C. The **Harder–Narasimhan (HN) filtration** of E is the unique filtration by subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$$

such that

1. each quotient

$$\mathrm{gr}_i^{HN}(E) := E_i/E_{i-1}$$

is semistable;

2. the slopes strictly decrease:

$$\mu\big(\mathrm{gr}_1^{HN}(E)\big) > \mu\big(\mathrm{gr}_2^{HN}(E)\big) > \cdots > \mu\big(\mathrm{gr}_\ell^{HN}(E)\big)\,.$$

Theorem 1.7 (Existence and uniqueness)**.** Every vector bundle E on C admits a unique Harder–Narasimhan filtration.

If E is already semistable, then the HN filtration is trivial: $0 \subset E$. If E is not semistable, then the first step of the HN filtration is given by the maximal destabilizing subbundle E_1, and we can proceed inductively on the quotient E/E_1.

Definition 1.8 (HN slopes)**.** The numbers

$$\mu_i(E) := \mu\big(\mathrm{gr}_i^{HN}(E)\big)$$

are called the **HN slopes** of E. One writes

$$\mu_1(E) > \mu_2(E) > \cdots > \mu_\ell(E).$$

and put $\mu_{\max}(E) := \mu_1(E)$ and $\mu_{\min}(E) := \mu_\ell(E)$.

Definition 1.9 (HN type)**.** The collection of ranks and degrees of the graded pieces

$$\big(\mathrm{rk}(\mathrm{gr}_i^{HN}(E)), \deg(\mathrm{gr}_i^{HN}(E))\big)_{i=1}^{\ell}$$

(or equivalently their slopes with multiplicities) is called the **Harder–Narasimhan type** of E.

Definition 1.10 (HN polygon)**.** The **Harder–Narasimhan polygon** of E is the piecewise-linear concave polygon in $\mathbb{R}^2$ obtained by joining the points

$$(0,0),\ \big(\mathrm{rk}(E_1), \deg(E_1)\big),\ \ldots,\ \big(\mathrm{rk}(E), \deg(E)\big).$$

Its slopes are exactly the HN slopes of E.

1.2 Vector bundles on $\mathbb{P}^1$

In this section, we work out the notion of stability and the HN filtration for vector bundles on $\mathbb{P}^1$. We begin with the classification of vector bundles on $\mathbb{P}^1$.

Theorem 1.11 (Grothendieck's theorem)**.** Every vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles

$$\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

for unique integers $a_1 \geq \cdots \geq a_n$.

It is clear that if we write E in the above form, then the vector bundle E has slope

$$\mu(E) = \frac{a_1 + \cdots + a_n}{n}$$

The Harder Narasimhan filtration of E is completely explicit. If we let $b_1 > \cdots > b_m$ be the distinct values of the a_i and write E as

$$E \cong \mathcal{O}_{\mathbb{P}^1}(b_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(b_m)^{\oplus r_m},$$

then the HN filtration of E is given by

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E,$$

where $E_i = \mathcal{O}_{\mathbb{P}^1}(b_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(b_i)^{\oplus r_i}$. Then E is semistable if and only if E is isomorphic to a direct sum of line bundles of the same degree, and E is stable if and only if E is a line bundle. If $\operatorname{rank} E \geq 2$ then $\mathcal{O}(d) \subset E$ is a proper subbundle of the same slope as E.

1.3 The moduli of bundles

Let $\operatorname{Bun}_{r,d}(\mathbb{P}^1)$ be the moduli stack of vector bundles on $\mathbb{P}^1$ of rank r and degree d. The above discussion shows that if r does not divide d then $\operatorname{Bun}^{\mathrm{ss}}_{r,d}(\mathbb{P}^1)$ is empty.

Example 1.12. If $r \mid d$ then $\operatorname{Bun}^{\mathrm{ss}}_{r,d}(\mathbb{P}^1)$ is a single point, corresponding to the unique semistable bundle $\mathcal{O}(d/r)^{\oplus r}$. This point has automorphism group GL_r, and so the open stratum in the HN stratification of $\operatorname{Bun}_{r,rm}(\mathbb{P}^1)$ is $B\,\mathrm{GL}_r$, corresponding to the splitting type $\mathcal{O}(m)^{\oplus r}$.

The boundary is the union of all other splitting types:

$$\operatorname{Bun}_{r,rm}(\mathbb{P}^1) \setminus \operatorname{Bun}^{ss}_{r,rm}(\mathbb{P}^1) \;=\; \bigcup_{\substack{a_1 \geq \cdots \geq a_r \\ \sum a_i = rm \\ (a_i) \neq (m,\ldots,m)}} \mathcal{S}_{(a_i)}$$

where $\mathcal{S}_{(a_i)}$ is the stratum corresponding to the splitting type $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$.

What do the closure relations look like? The minimal deviation from semistability is when we move one summand up and one down, i.e. the stratum corresponding to $(m+1, m, \ldots, m, m-1)$. We see that the HN polygon of this sequence lies above that of the semistable stratum.

In general, the closure of $\mathcal{S}_{(a_i)}$ consists of all strata $\mathcal{S}_{(b_i)}$ such that the HN polygon of (b_i) lies above that of (a_i). It is those sequences (b_i) so that $(b_i) - (a_i)$ is a nonnegative linear combination of the vectors $(1, -1, 0, \ldots, 0), (0, 1, -1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1, -1)$, i.e. the positive roots of GL_r.

Example 1.13. When r does not divide d, the semistable locus is empty, and the open stratum is the one which is closest to semistability, i.e. write $d = qr + s$ with $0 \leq s < r$. Then the most balanced splitting type is $(q+1)^s\, q^{r-s}$.

This type minimizes all partial sums, hence is the minimal element in the dominance order. This is the open stratum because for a fixed type τ, the corresponding locus $\mathcal{S}_\tau \subset \mathrm{Bun}_{r,d}(\mathbb{P}^1)$ is locally closed. By upper semicontinuity of the HN polygon in families, the HN polygon can only go upward (more unstable) in a family, so the open stratum is the one with the lowest HN polygon.

2 The Shatz stratification

Let G be a connected reductive group over $\mathbb{C}$. Let C be a smooth projective curve over $\mathbb{C}$, and $\mathrm{Bun}_G(C)$ be the moduli stack of G-bundles on C. Let $\mathcal{E}$ be a principal G-bundle on C.

Fix a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$. For a standard parabolic $P \supset B$ with Levi subgroup L, let $\{\chi_i\}$ denote the fundamental characters of P.

For vector bundles, the Harder Narasimhan filtration gives a filtration by subbundles. For principal G-bundles, this is replaced by a reduction of structure group to a parabolic subgroup. This data gives rise to Shatz stratification of $\mathrm{Bun}_G(C)$ is a stratification by locally closed substacks. It is useful to keep in mind the toy model of the HN stratification of $\mathrm{Bun}_{r,d}(\mathbb{P}^1)$.

Definition 2.1. A reduction of $\mathcal{E}$ to a subgroup $P \subset G$ is a principal P-bundle $\mathcal{E}_P$ on C together with an isomorphism $\mathcal{E}_P \times^P G \;\cong\; \mathcal{E}$. If $\chi : P \to \mathbb{G}_m$ is a character, there is an associated line bundle

$$\mathcal{L}_\chi(\mathcal{E}_P) := \mathcal{E}_P \times^P \mathbb{A}^1_\chi$$

and its degree $\deg \mathcal{L}_\chi(\mathcal{E}_P) \in \mathbb{Z}$.

Definition 2.2. Let us say that the bundle $\mathcal{E}$ is **Ramanathan-semistable** if for every parabolic reduction $(P, \mathcal{E}_P)$ and every dominant character χ of P,

$$\deg \mathcal{L}_\chi(\mathcal{E}_P) \leq 0.$$

It is enough to check this numerical criterion against the maximal parabolics.

In general, a P–reduction E_P determines an element $\mu(E_P) \in X_*(T)_\mathbb{Q}$ (its slope/HN type) such that for every character $\chi \in X^*(P)$ one has

$$\deg\big(E_P \times^P \chi\big) = \langle \chi, \mu(E_P) \rangle,$$

where χ is viewed as a weight of T by restriction. The degrees

$$d_i := \deg \mathcal{L}_{\chi_i}(\mathcal{E}_P)$$

determine a rational coweight

$$\mu(P, \mathcal{E}_P) \in X_*(T)^+_\mathbb{Q}$$

characterized by

$$\langle \chi_i, \mu(P, \mathcal{E}_P) \rangle = -\, d_i.$$

This coweight is called the **type** of the reduction. Using the Weyl group action, we can conjugate $\mu(P, \mathcal{E}_P)$ to a dominant coweight. Among all parabolic reductions of $\mathcal{E}$, the set of types $\mu(P, \mathcal{E}_P)$ has a unique maximal element for the dominance order. This element is denoted

$$\mu(\mathcal{E}) \in X_*(T)^+_{\mathbb{Q}}$$

and called the **Harder–Narasimhan (HN) type** of $\mathcal{E}$. The associated parabolic subgroup is

$$P_{\mathrm{HN}} = P(\mu(\mathcal{E})) = \Big\{ g \in G \;\Big|\; \lim_{t \to 0} \mu(t) g \mu(t)^{-1} \text{ exists} \Big\}$$

where μ is any integral multiple of $\mu(\mathcal{E})$.

2.1 Stability for GL_r bundles

We show that the Ramanathan semistability condition for GL_r-bundles recovers the usual slope semistability for vector bundles, and the character pairing formula gives the degree of the associated line bundle in terms of the slope of the reduction. Let C be a smooth projective curve and let E be a vector bundle of rank r, viewed as a principal GL_r–bundle. Fix an integer k with $1 \le k \le r-1$ and let $P \subset \mathrm{GL}_r$ be the maximal parabolic stabilizing a k–dimensional subspace.

$$P = \left\{ \begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \;\middle|\; A \in \mathrm{GL}_k,\ D \in \mathrm{GL}_{r-k} \right\}.$$

A reduction of E to P is equivalent to the choice of a rank k subbundle $F \subset E$, with quotient bundle $Q := E/F$. Any character of P factors through the Levi quotient $L \simeq \mathrm{GL}_k \times \mathrm{GL}_{r-k}$ and is of the form

$$\chi_{a,b}(A, D) = (\det A)^a (\det D)^b, \qquad a, b \in \mathbb{Z}.$$

The standard (fundamental) dominant character for this maximal parabolic is

$$\chi_k(A, D) = (\det A)^{r-k} (\det D)^{-k}.$$

For the reduction E_P corresponding to $F \subset E$, the associated line bundle is

$$\mathcal{L}_{\chi_k}(E_P) := E_P \times^P \mathbb{A}^1_{\chi_k} \;\cong\; (\det F)^{r-k} \otimes (\det Q)^{-k}.$$

Taking degrees and using $\deg Q = \deg E - \deg F$ yields

$$\begin{aligned} \deg \mathcal{L}_{\chi_k}(E_P) &= (r-k) \deg(\det F) - k \deg(\det Q) \\ &= (r-k) \deg F - k(\deg E - \deg F) \\ &= r \, \deg F - k \, \deg E. \end{aligned}$$

Thus the Ramanathan inequality $\deg \mathcal{L}_{\chi_k}(E_P) \le 0$ is equivalent to

$$r \deg F - k \deg E \le 0 \qquad \Longleftrightarrow \qquad \mu(F) \le \mu(E),$$

recovering the usual slope semistability for vector bundles.

Let $T \subset \mathrm{GL}_r$ be the diagonal torus with character lattice $X^*(T) = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_r$ (where $\varepsilon_i(\mathrm{diag}(t_1, \dots, t_r)) = t_i$) and cocharacter lattice $X_*(T) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r$ (where $e_i : \mathbb{G}_m \to T$ is $t \mapsto \mathrm{diag}(1, \dots, 1, t, 1, \dots, 1)$). The pairing is $\langle \varepsilon_i, e_j \rangle = \delta_{ij}$.

For the reduction E_P coming from $F \subset E$, define the associated (rational) coweight

$$\mu(E_P) := \big(\underbrace{\mu(F), \dots, \mu(F)}_{k \text{ times}}, \underbrace{\mu(Q), \dots, \mu(Q)}_{r-k \text{ times}} \big) \in X_*(T)_{\mathbb{Q}}.$$

The restriction of χ_k to T is the weight

$$\chi_k|_T = (r-k)(\varepsilon_1 + \cdots + \varepsilon_k) - k(\varepsilon_{k+1} + \cdots + \varepsilon_r).$$

Therefore

$$\begin{aligned} \langle \chi_k, \mu(E_P) \rangle &= \Big\langle (r-k) \sum_{i=1}^{k} \varepsilon_i - k \sum_{i=k+1}^{r} \varepsilon_i, \;\; (\mu(F)^k, \mu(Q)^{r-k}) \Big\rangle \\ &= (r-k) \cdot k\, \mu(F) - k \cdot (r-k)\, \mu(Q) \\ &= k(r-k)\big(\mu(F) - \mu(Q)\big). \end{aligned}$$

Since $\mu(Q) = \deg Q/(r-k)$ and $\mu(F) = \deg F/k$, a short computation shows

$$k(r-k)\big(\mu(F) - \mu(Q)\big) = r \,\deg F - k\, \deg E.$$

Combining with the earlier degree computation gives the pairing identity

$$\deg\big(E_P \times^P \chi_k\big) = \langle \chi_k, \mu(E_P) \rangle.$$

Theorem 2.3 (Harder–Narasimhan filtration for G-bundles)**.** Let $\mathcal{E}$ be a principal G-bundle on C. Then $\mathcal{E}$ determines a unique parabolic reduction $(P_{\mathrm{HN}}, \mathcal{E}_{P_{\mathrm{HN}}})$ of $\mathcal{E}$ satisfying the following properties:

(i) **Prescribed type:**

$$\mu(P_{\mathrm{HN}}, \mathcal{E}_{P_{\mathrm{HN}}}) = \mu(\mathcal{E}).$$

(ii) **Semistable Levi quotient:** if L_{HN} is a Levi subgroup of P_{HN}, then the induced principal L_{HN}–bundle

$$\mathcal{E}_{L_{\mathrm{HN}}} = \mathcal{E}_{P_{\mathrm{HN}}}/U_{\mathrm{HN}}$$

is semistable.

(iii) **Maximal destabilizing property:** for every other reduction $(Q, \mathcal{E}_Q)$,

$$\mu(Q, \mathcal{E}_Q) \leq \mu(\mathcal{E}).$$

Equality holds only when the reduction is isomorphic to $\mathcal{E}_{P_{\mathrm{HN}}}$.

The pair

$$(P_{\mathrm{HN}}, \mathcal{E}_{P_{\mathrm{HN}}})$$

is called the **Harder–Narasimhan reduction** of $\mathcal{E}$. It is characterized entirely by the bundle $\mathcal{E}$ itself and does not depend on a choice of representation of G. Note that choosing a representation $\rho : G \to \mathrm{GL}(V)$ and applying the associated bundle construction to the HN reduction gives the HN filtration of the associated vector bundle $\mathcal{E}(V)$.

2.2 Very close degenerations

In this section, we give a stack-theoretic interpretation of semistability in terms of very close degenerations. The quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ has two geometric points 1 and 0 which are the images of the points of the same name in $\mathbb{A}^1$. For any algebraic stack $\mathcal{M}$ and $f : [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{M}$ we will write $f(0), f(1) \in \mathcal{M}(k)$ for the points given by the images of $0, 1 \in \mathbb{A}^1(k)$.

Definition 2.4 (Very close degenerations)**.** Let $\mathcal{M}$ be an algebraic stack over k and $x \in \mathcal{M}(K)$ a geometric point for some algebraically closed field K/k. A **very close degeneration** of x is a morphism $f : [\mathbb{A}^1_K/\mathbb{G}_{m,K}] \to \mathcal{M}$ with $f(1) \simeq x$ and $f(0) \not\simeq x$.

We textbfasize that $f(0)$ is an object that lies in the closure of a K point of $\mathcal{M}_K$, which only happens for stacks and orbit spaces, but if $X = \mathcal{M}$ is a scheme, then there are no very close degenerations.

Definition 2.5 ($\mathcal{L}$-stability)**.** Let $\mathcal{M}$ be an algebraic stack over k, locally of finite type with affine diagonal and $\mathcal{L}$ a line bundle on $\mathcal{M}$. A geometric point $x \in \mathcal{M}(K)$ is called **$\mathcal{L}$-stable** if

1. for all very close degenerations $f : [\mathbb{A}^1_K/\mathfrak{G}_{m,K}] \to \mathcal{M}$ of x we have

$$\mathrm{wt}(f^*\mathcal{L}) < 0$$

and

2. $\dim_K(\mathrm{Aut}_{\mathcal{M}}(x)) = 0$.

We can also introduce the notion of $\mathcal{L}$-semistable points, by requiring only $\leq$ in (1) and dropping condition (2).

Proposition 2.6. A very close degeneration

$$f : [\mathbb{A}^1/\mathbb{G}_m] \to \mathrm{Bun}_G(C)$$

corresponding to a family $\mathcal{E}$ of G-bundles on $X \times [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to the following data:

1. a cocharacter $\lambda : \mathbb{G}_m \to G$, canonical up to conjugation,
2. a reduction $\mathcal{E}_\lambda$ of the bundle $\mathcal{E}$ to P_λ,
3. an isomorphism
$$\mathcal{E} \cong \mathrm{Rees}(\mathcal{E}_\lambda|_{X\times 1}, \lambda).$$

Proof. **First we describe how to start with Lie algebra data and produce a very close degeneration.** For a cocharacter

$$\lambda : \mathbb{G}_m \to G$$

we denote by P_λ, U_λ, L_λ the corresponding parabolic subgroup, its unipotent radical and the Levi subgroup.

The source of degenerations is the following analog of the Rees construction. Given $\lambda : \mathbb{G}_m \to G$ we obtain a homomorphism of group schemes over $\mathbb{G}_m$:

$$\mathrm{conj}_\lambda : P_\lambda \times \mathbb{G}_m \longrightarrow P_\lambda \times \mathbb{G}_m, \qquad (p,t) \longmapsto (\lambda(t)p\lambda(t)^{-1}, t).$$

This homomorphism extends to a morphism of group schemes over $\mathbb{A}^1$:

$$\mathrm{gr}_\lambda : P_\lambda \times \mathbb{A}^1 \longrightarrow P_\lambda \times \mathbb{A}^1$$

in such a way that

$$\mathrm{gr}_\lambda(p, 0) = \lim_{t\to 0} \lambda(t)p\lambda(t)^{-1} \in L_\lambda \times 0.$$

These morphisms are $\mathbb{G}_m$-equivariant with respect to the action $(\mathrm{conj}_\lambda, \mathrm{act})$ on $P_\lambda \times \mathbb{A}^1$. Over $t = 1$, gr_λ is just the identity automorphism, so nothing changes:

$$\mathrm{Rees}(E_\lambda, \lambda)|_{t=1} \cong E_\lambda.$$

Over $t = 0$, the twisting morphism becomes project to the Levi $P_\lambda = L_\lambda \ltimes U_\lambda \to L_\lambda$. The bundle is given by the formula

$$\mathrm{Rees}(E_\lambda, \lambda)|_{t=0} \cong (E_\lambda/U_\lambda) \times_{L_\lambda} P_\lambda.$$

where E_λ/U_λ is the quotient L_λ-bundle given by dividing the P_λ-bundle E_λ by the unipotent radical U_λ, and then we extend structure group back to P_λ via the inclusion $L_\lambda \subset P_\lambda$.

Given a P_λ-bundle $\mathcal{E}_\lambda$ on a scheme X, this morphism defines a P_λ-bundle on $X \times [\mathbb{A}^1/\mathbb{G}_m]$ by

$$\mathrm{Rees}(\mathcal{E}_\lambda, \lambda) := \left[((\mathcal{E}_\lambda \times \mathbb{A}^1) \times^{\mathrm{gr}_\lambda}_{\mathbb{A}^1} (P_\lambda \times \mathbb{A}^1))/\mathbb{G}_m\right],$$

where $\times_{\mathbb{A}^1}^{\mathrm{gr}_\lambda}$ denotes the bundle induced via the morphism gr_λ, i.e. we take the product over $\mathbb{A}^1$ and divide by the diagonal action of the group scheme $P_\lambda \times \mathbb{A}^1/\mathbb{A}^1$, which acts on the right factor via gr_λ.

By construction this bundle satisfies

$$\mathrm{Rees}(\mathcal{E}_\lambda, \lambda)|_{X\times 1} \cong \mathcal{E}_\lambda$$

and

$$\mathrm{Rees}(\mathcal{E}_\lambda, \lambda)|_{X\times 0} \cong \mathcal{E}_\lambda/U_\lambda \times_{L_\lambda} P_\lambda,$$

which is the analog of the associated graded bundle.

Now we describe how to start with a very close degeneration and extract the Lie algebra data. A map

$$f : [\mathbb{A}^1/\mathbb{G}_m] \to \mathrm{Bun}_G(X)$$

is a $\mathbb{G}_m$-equivariant G-bundle $\mathcal{E}$ on $X \times \mathbb{A}^1$. Consider special fiber $X \times [0/\mathbb{G}_m] \subset X \times [\mathbb{A}^1/\mathbb{G}_m]$. Because 0 is fixed by scaling, the restriction $\mathcal{E}_0 := \mathcal{E}|_{X\times[0/\mathbb{G}_m]}$ is a G-bundle together with a $\mathbb{G}_m$-action.

Trivializing $\mathcal{E}_0$ at a point gives a homomorphism $\mathbb{G}_m \to \mathrm{Aut}(\mathcal{E}_0) \cong G$ and changing trivialization changes this homomorphism by conjugation, so we get a well-defined cocharacter $\lambda : \mathbb{G}_m \to G$ up to conjugation.

With λ, we recover the parabolic reduction as attractor subbundle. Write $E := \mathcal{E}|_{X\times\{1\}}$ for the general fiber. Choose a local trivialization of $\mathcal{E}$ in the fpqc topology over $X \times \mathbb{A}^1$, so that over such a trivializing open the $\mathbb{G}_m$-action is given by λ up to G-conjugacy. In this local model, a point of the fiber E_x may be written as a frame $g \in G$, and the $\mathbb{G}_m$-action transports it by conjugation, so the condition that the orbit has a limit as $t \to 0$ is exactly

$$\lim_{t\to 0} \lambda(t)\, g\, \lambda(t)^{-1} \text{ exists.}$$

By definition this is equivalent to $g \in P_\lambda$. Therefore the subset of points of E whose $\mathbb{G}_m$-orbit admits a limit is stable under the right action of P_λ and defines a principal P_λ-subbundle

$$E_\lambda \subset E.$$

Equivalently, E_λ is the reduction of E corresponding to the canonical $\mathbb{G}_m$-fixed section of the associated bundle $E \times^G (G/P_\lambda)$ coming from the special fiber at $t = 0$.

Finally, applying the Rees construction to (E_λ, λ) yields a $\mathbb{G}_m$-equivariant family of G-bundles on $X \times \mathbb{A}^1$, and by construction it agrees with $\mathcal{E}$ over $\mathbb{A}^1 \setminus \{0\}$; the $\mathbb{G}_m$-equivariant extension across 0 is unique, hence the Rees family recovers $\mathcal{E}$. $\square$

2.3 Determinant of cohomology

The determinant of cohomology line bundle $\mathcal{L}_{\det}$ on $\mathrm{Bun}_G(C)$ is defined by

$$\mathcal{L}_{\det}|_E = \det H^*(C,\, E \times^G \mathrm{Lie}(G))$$

where $\mathrm{Lie}(G)$ is the adjoint representation of G on its Lie algebra. The following result gives a numerical criterion for $\mathcal{L}_{\det}$-semistability in terms of the degrees of the associated line bundles for reductions to maximal parabolics, which is equivalent to Ramanathan semistability.

Theorem 2.7. A G-bundle E is $\mathcal{L}_{\det}$-semistable if and only if it is Ramanathan-semistable, i.e. if and only if for all reductions E_P to maximal parabolic subgroups $P \subset G$ we have $\deg(E_P \times_P \mathrm{Lie}(P)) \leq 0$

Proof. We have to compute the weight of $\mathcal{L}_{\det}$ on very close degenerations. Choose $T \subset B \subset G$ a maximal torus and a Borel subgroup and a dominant cocharacter $\lambda : \mathbb{G}_m \to G$.

Let us denote by I the set of positive simple roots with respect to (T, B) and by

$$I_P := \{\alpha_i \in I \mid \lambda(\alpha_i) = 0\}$$

the simple roots α_i for which $-\alpha_i$ is also a root of P_λ. For $j \in I$ let us denote by

$$\tilde{\omega}_j \in X_*(T)_{\mathbb{R}}$$

the cocharacter defined by

$$\tilde{\omega}_j(\alpha_i) = \delta_{ij},$$

and by P_j the corresponding maximal parabolic subgroup.

Then

$$\lambda : \mathbb{G}_m \to Z(L_\lambda) \subset L_\lambda \subset P_\lambda.$$

Thus for any very close degeneration $f : [\mathbb{A}^1/\mathbb{G}_m] \to \mathrm{Bun}_G$ given by $\mathrm{Rees}(\mathcal{E}_\lambda, \lambda)$ the bundle $\mathcal{L}_{\det}$ defines a morphism

$$\mathrm{wt}_{\mathcal{L}} : X_*(Z_\lambda) \subset \mathrm{Aut}_{\mathrm{Bun}_G}(f(0)) \to \mathbb{Z}.$$

Then the weight function is additive in the cocharacter so it is enough to compute for one fundamental direction at a time. Write $\lambda = \sum_{j \in I - I_P} a_j \tilde{\omega}_j$ for some $a_j > 0$. Then

$$\mathrm{wt}(\mathcal{L}_{\det}|_{f(0)}) = \mathrm{wt}_{\mathcal{L}}(\lambda) = \sum_{j \in I - I_P} a_j \, \mathrm{wt}_{\mathcal{L}}(\tilde{\omega}_j).$$

For each j we get a decomposition

$$\mathrm{Lie}(G) = \bigoplus_i \mathrm{Lie}(G)_i,$$

where $\mathrm{Lie}(G)_i$ is the subspace of the Lie algebra on which $\tilde{\omega}_j$ acts with weight i. Each of these spaces is a representation of L_λ and also of the Levi subgroups L_j of P_j. Using this decomposition we find as in the case of vector bundles:

$$\begin{aligned}
\mathrm{wt}_{\mathcal{L}}(\tilde{\omega}_j) &= -\mathrm{wt}_{\mathbb{G}_m}\big(\det H^*(C,\, \mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i)\big) \\
&= \sum_i i \cdot \chi\big(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i\big) \\
&= \sum_i i\Big(\deg(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i) + \dim(\mathrm{Lie}(G)_i)(1-g)\Big) \\
&= 2\sum_{i>0} i \, \deg(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i).
\end{aligned}$$

because the decomposition is symmetric with respect to $i \mapsto -i$ and so the terms with $1-g$ cancel out. Now

$$\deg(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i) = \deg\big(\det(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i)\big)\,.$$

Since the Levi subgroups of maximal parabolics have only a one-dimensional space of characters, all of these degrees are positive multiples of $\det(\mathrm{Lie}(P_j))$. $\square$

2.4 Stratification of $\mathrm{Bun}_G(C)$

The stratification of $\mathrm{Bun}_G(C)$ is indexed by the HN type $\mu(\mathcal{E})$, which is a dominant rational coweight. The open stratum corresponds to the semistable locus, and the closed stratum corresponds to the most unstable bundles. The closure relations are given by the dominance order on coweights: the closure of the stratum corresponding to μ consists of all strata corresponding to ν such that $\nu \leq \mu$ in the dominance order.

2.5 The Shatz stratification for SL_2 bundles in high genus

Let $G = \mathrm{SL}_2$ and C be a smooth projective curve of genus $g \geq 2$. An SL_2-bundle is a rank-2 vector bundle E with $\det E \simeq \mathcal{O}_C$.

- E is semistable iff every line subbundle $L \subset E$ satisfies $\deg L \leq 0$.
- If E is unstable, it has a unique maximal destabilizing line subbundle $L \subset E$ with $\deg L > 0$. Then automatically $E/L \simeq L^{-1}$ because $\det E \simeq \mathcal{O}$.

The HN type is determined by the integer $n = \deg L \in \mathbb{Z}_{>0}$ corresponding to the dominant rational coweight proportional to $n\alpha^\vee$. Thus the Shatz stratification is:

$$\mathrm{Bun}_{\mathrm{SL}_2}(C) = \bigsqcup_{n\geq 0} \mathrm{Shatz}_n$$

where Shatz_n is the locus of unstable bundles with $\deg(L_{\max}) = n$ and there is an open stratum $\mathrm{Bun}^{ss}_{\mathrm{SL}_2} = \mathrm{Shatz}_0$ of semistable bundles.

Fix $n \geq 1$. For each line bundle $L \in \mathrm{Pic}^n(C)$, consider extensions

$$0 \to L \to E \to L^{-1} \to 0$$

These extensions are classified by

$$\mathrm{Ext}^1(L^{-1}, L) \cong H^1(C, L^2).$$

There is a natural stack

$$\mathcal{E}_n \to \mathrm{Pic}^n(C)$$

whose fiber over L is the quotient stack $[H^1(C, L^2)/\mathbb{G}_m]$ where $\mathbb{G}_m = \mathrm{Aut}(L)$ acts by scaling the extension class, i.e. scalar automorphisms of $L \hookrightarrow E$. Then Shatz_n is obtained from $\mathcal{E}_n$ by removing the sublocus where the resulting E admits a line subbundle of degree $> n$ as that would mean the maximal destabilizing degree is larger than n.

Let $\deg(L^2) = 2n$. By Riemann-Roch and Serre duality, $h^1(L^2) = h^0(K \otimes L^{-2})$.

- If $2n > 2g - 2$ (i.e. $n \geq g$), then $\deg(K \otimes L^{-2}) < 0$, so $h^1(L^2) = 0$. Hence every extension splits, and we get a map of stacks $\pi : \mathrm{Shatz}_n \to \mathrm{Pic}^n$ taking E to its maximal destabilizing line subbundle $L_{\max}$. The automorphism group of E as an $\mathrm{SL}(2)$-bundle is those upper trinagular shears

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \qquad a \in \mathbb{G}_m, b \in H^0(C, L^2)$$

  so the fiber of π over a point L is the classifying stack $B\big(\mathbb{G}_m \ltimes H^0(C, L^2)\big)$.

- If $1 \leq n \leq g - 1$, then $h^1(L^2)$ can be positive, and one has

$$h^1(L^2) = h^0(K \otimes L^{-2}) = g - 1 - 2n + h^0(L^2)$$

  with $h^0(L^2)$ varying (it jumps precisely on Brill-Noether loci). So the fibers of $\mathrm{Shatz}_n \to \mathrm{Pic}^n$ have varying dimension, but there is a large open subset where the dimension is constant.

For SL_2, dominance order is just order on the integer n, so the closure of Shatz_n contains Shatz_m for all $m \geq n$. and in particular $\mathrm{Bun}^{\mathrm{ss}}_{\mathrm{SL}_2}$ is the open stratum.

For an unstable SL_2-bundle, the canonical reduction is always to a Borel $B \subset \mathrm{SL}_2$ (stabilizer of a line), i.e. it is exactly the canonical line subbundle $L_{\max} \subset E$.

The associated Levi is $T \simeq \mathbb{G}_m$, and the induced T-bundle is essentially $L_{\max}$ (up to the SL_2 determinant constraint), which is automatically semistable because T is a torus.

Remark 2.8. For $n \geq g$, the structure of the map $\pi : \mathrm{Shatz}_n \to \mathrm{Pic}^n$ is best captured by the language of inertia stacks. Recall that for any algebraic stack $\mathcal{X}$, the inertia stack $I\mathcal{X}$ is defined as the fiber product

$$I\mathcal{X} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$$

where the maps to $\mathcal{X} \times \mathcal{X}$ are given by the diagonal and the identity. The inertia stack parametrizes pairs (x, g) where x is a point of $\mathcal{X}$ and g is an automorphism of x. In our case, the fiber of π over a point L in Pic^n can be identified with the classifying stack of the automorphism group of the corresponding unstable bundle, which is precisely captured by the inertia stack of Shatz_n. In particular, we see that the inertia of Shatz_n over Pic^n is given by the group $\mathbb{G}_m \ltimes H^0(C, L^2)$.