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This is a note for the talk with Mike Stillman on Feb 22, 2024.
Note: next week we are talking about Cartier divisors, line bundles, and Chern classes.

1   Introduction

The goal of intersection theory is to make sense of the interscetion product. Its existence rests on Chow's moving lemma:

Theorem 1.1 (Chow's moving lemma). *Let X be a smooth projective variety.*

- *Given $\alpha, \beta \in A(X)$ there exist representing cycles A and B intersecting transversally.*

- *The classes of the interesction of these cycles is independent of A and B.*

It seems that for a long time this theorem was folklore in the algebraic geometry community. According to Eisenbud and Harris, the first proof was given by Fulton in 1984.

There is an intersection product on $A(X)$:

Theorem 1.2. *If X is a smooth quasiprojective variety, then there exists a unique product on $A(X)$ so that if two subvarieties A, B are generically transverse then $[A] \cdot [B] = [A \cap B]$.*

If we are in a setting where Chow's moving lemma holds, then the intersection product is characterized by this property. If we cannot find generically transverse representatives, then we need to start making sense of multiplicitiies.

Theorem 1.3. *Let X be a smooth projective variety and A, B subvarieites so that each component C of the intersection has the right dimension. Then there are positive integers $m_C(A, B)$ so that*

$$[A] \cdot [B] = \sum_C m_C(A, B)[C].$$

In case A and B are Cohen–Macaulay at a general point of C, then this number is multiplicity of the component of the scheme $A \cap B$ supported on C

Serre found a formula for this number as the alternating sum of the dimension of the Tors. If A and B are not intersecting properly, what should the intersection product of their classes be?

2   Examples

Theorem 2.1. $A(\mathbb{P}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ *where x is the class of a hyperplane. If X is a irreducible subvariety of codimension r and degree d, then $[X] = dx^r$.*

Proof. Each $A^i(\mathbb{P}^n) \cong \mathbb{Z}$ as a group because of the following theorem.

Theorem 2.2. *If a scheme X admits a quasiaffine stratification, then the classes of the closed strata generate $A(X)$. If the stratification is affine, then the classes of the strata form a basis.*

Proof. Follows from excision. $\square$

Write $\mathbb{P}^n = \cup \mathbb{A}^i$ and therefore $A(\mathbb{P}^n)$ is generated by the classes of the closures $\mathbb{P}^i$ of $\mathbb{A}^i$, a copy of $\mathbb{Z}$ for each i. None of the $\mathbb{P}^i$ are equivalent to zero because it is the pushforward of the class $[\mathbb{P}^i]$ under the inclusion of $\mathbb{P}^i \hookrightarrow \mathbb{P}^n$

This statement very much reminds me of cellular homology. The CW decomposition of $\mathbb{CP}^n$ has only even dimensional cells and therefore classes of these cells generate the cohomology ring (the boundary maps go from even to odd)

To compute the ring product, note that

- a generic plane L of codimension r is the transverse intersection of r hyperplanes, so $[L] = x^r$.

- If X as above, then $[X]x^{n-r} = [X \cap L_{n-k}] = d[*] = d\zeta^n$.

$\square$

Example 2.3 (Degree of Veronese embedding)**.** *The Veronese embedding is the map $v : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ given by*

$$[x_0 : \cdots : x_n] \mapsto [x^I]$$

where z^I is a monomial of degree d in $n+1$ variables. The image is the image of the map of the complete linear system $|\mathcal{O}(d)|$. Letting H be a hyperplane in the target, $v^{-1}(H)$ comprises all hypersurfaces of degree d.

To see the degree, consider the image $\cap H_1 \cap \cdots \cap H_n$ of n hyperplanes in the target. The map is $1 : 1$ and the preimage is the intersection of the preimages of the H_i which is the intersection of n hypersurfaces of degree d. The degree of the intersection is the product of the degrees of the hypersurfaces, which is d^n.

Example 2.4 (Product of projective spaces)**.** $A(\mathbb{P}^r \times \mathbb{P}^s) \cong A(\mathbb{P}^r) \otimes A(\mathbb{P}^s)$. *If α, β denote pullbacks via projection of hyperplane classes on $\mathbb{P}^r$ and $\mathbb{P}^s$, then*

$$A(\mathbb{P}^r \times \mathbb{P}^s) = \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1})$$

Moreover the class of hypersurface $V(f)$ where f is a form of bidegree (d, e) is $d\alpha + e\beta$.

There is an affine stratification given by the product of the affine stratifications of the factors so the classes $\phi_{a,b} = [\mathbb{P}^{r-a} \times \mathbb{P}^{s-b}]$ generates $A(\mathbb{P}^r \times \mathbb{P}^s)$. We have

$$\phi_{a,b} = \alpha^a \beta^b$$

because $\mathbb{P}^{r-a} \times \mathbb{P}^{s-b}$ is the transverse intersection of a hyperplanes in the first factor and b hyperplanes in the second factor. Moreover it is clear that $\alpha^{r+1} = 0$ and $\beta^{s+1} = 0$. Therefore, there is an onto map

$$\mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1}) \to A(\mathbb{P}^r \times \mathbb{P}^s)$$

and it is injective because there is a pairing

$$A^{p+q}(\mathbb{P}^r \times \mathbb{P}^s) \times A^{r+s-p-q}(\mathbb{P}^r \times \mathbb{P}^s) \to \mathbb{Z}$$

given by the intersection product which sends $(\alpha^p \beta^q, \alpha^m \beta^n)$ to $\delta_{p+m,r}\delta_{q+n,s}$.

Finally if F is bihomogeneous of bidegree (d,e), then $F/X_0^d Y_0^e$ is a rational function on $\mathbb{P}^r \times \mathbb{P}^s$ and its divisor is $[F] - d\alpha + e\beta$.

Example 2.5. *We can use this to immediately calculate the degree of the Segre embedding $s : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{rs+r+s}$*

$$[x_0 : \cdots : x_r] \times [y_0 : \cdots : y_s] \mapsto [x_i y_j]$$

We consider the intersection with the image of $r + s$ hyperplanes. We pullback to $\mathbb{P}^r \times \mathbb{P}^s$ because the map is an embedding and intersect with the preimage of the hyperplanes. The function $X_i Y_j$ is bihomogeneous of bidegree $(1,1)$ so

$$[S_{r,s}] = ([\alpha] + [\beta])^{r+s}$$
$$= \binom{r+s}{r} \alpha^r \beta^s$$

Example 2.6 (Blowup of $\mathbb{P}^n$ at a point). *Let $B \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ be the blowup of $\mathbb{P}^n$ at a point p. I like to think of this as the incidence variety of lines in $\mathbb{P}^n$ and points in $\mathbb{P}^{n-1}$.*

$$B = \{(x, L) \in \mathbb{P}^n \times \mathbb{P}^{n-1} \mid x \in L\}$$

We want to write down the Chow ring of B. We need to write down an affine stratification of B.

Pick $\Lambda' \subset \mathbb{P}^n$ hyperplane not containing p and consider $\Lambda \subset B$ the preimage of Λ'. Then $\Lambda \cong \mathbb{P}^{n-1}$ is those (q,l) so that $q \in \Lambda'$ and l is a line through q and p.

Pick $\Gamma_0' \subset \Gamma_1' \cdots \subset \Gamma_{n-1}' = \mathbb{P}^{n-1}$ and set $\Gamma_k = \pi_l^{-1}(\Gamma_k') \subset B$ where $\pi_l : B \to \mathbb{P}^{n-1}$ is the projection and put $\Lambda_k = \Gamma_{k+1} \cap \Lambda$. The open strata are all affine.

Take Γ_k and rip out Γ_{k-1} and I have affine space, but with $\mathbb{P}^1$ fiber over each point. Once I rip out Λ_k, I have an $\mathbb{A}^1$ fiber over each point is the product of affine spaces. *All bundles over affine spaces are trivial.*

This shows that the classes $\lambda_k = [\Lambda_k]$ and $\gamma_k = [\Gamma_k]$ generate $A(B)$.

Now we make heavy use of the moving lemma. Since Λ_k is the preimage of k-plane in $\mathbb{P}^n$ not containing p and all such planes are rationally equivalent in $\mathbb{P}^n$, the classes of their pullbacks are all λ_k.

Similarly the class of the proper transform (meaning rip out p, take preimage, and take closure) of any k-plane in $\mathbb{P}^n$ is γ_k. *The preimage of hyperplane $H \subset \mathbb{P}^n$ is the sum of irreducible divisors E and $\tilde{H}$ where E is the exceptional divisor and $\tilde{H}$ is the proper transform. $\tilde{H}$ looks like all the lines (points of $\mathbb{P}^{n-1}$) in H.*

With the representatives we can compute

$$\lambda_k \lambda_l = \lambda_{k+l} \quad \text{if } k + l \geq n$$
$$\gamma_k \lambda_l = \lambda{k+l} \quad \text{if } k + l \geq n$$
$$\gamma_k \gamma_l = \gamma_{k+l} \quad \text{if } k + l \geq n + 1$$

Note that if we have $k + l = n$ in the last case, then two general representatives do not meet. We see that the pairing

$$\lambda_k \cdot \lambda_{n-k} = \lambda_k \cdot \gamma_{n-k} = 1$$
$$\gamma_k \cdot \gamma_{n-k} = 0$$

and therefore the classes $\lambda_0, \ldots, \lambda_{n-1}, \gamma_1, \ldots, \gamma_n$ form a basis for $A(B)$.

Finally we consider the exceptional divisor E. It has $\dim E = n - 1$ and therefore we should be able to write it as a combination of γ_{n-1} and λ_{n-1}.

Λ'_{n-1} is linearly equivalent in $\mathbb{P}^n$ to hyperplane $\Sigma \subset \mathbb{P}^n$ containing p. The pullback of Σ is linearly equivalent to $E + D$ some divisor D *pullback along the blowup map preserves linear equivalence? but the blowup map is not flat since it doesn't have equidimensional fibers.*

D projects to a hyperplane in $\mathbb{P}^{n-1}$ and consider Y the preimage of such hyperplane. Y is irreducible since it is a $\mathbb{P}^1$-bundle over its image and therefore $Y = D$. Therefore $D \sim \gamma_{n-1}$ since they are both linearly equivalent to the preimage of a hyperplane in $\mathbb{P}^{n-1}$.

We get ahold of the formula

$$E = \lambda_{n-1} - \gamma_{n-1}$$

There are relations

$$\lambda \cdot E = 0 \quad \text{since they don't meet} \tag{1}$$
$$\gamma_k = \gamma_{n-1}^{n-k} = (\lambda_{n-1} - E)^{n-k} = \lambda_{n-1}^{n-k} + (-1)^{n-k} E^{n-k} \tag{2}$$
$$0 = \lambda_{n-1}^n + (-1)^n E^n \tag{3}$$

so there is a map

$$\mathbb{Z}[\lambda_{n-1}, E]/(\lambda_{n-1}^n + (-1)^n E^n) \to A(B)$$

and it is surjective because λ, e generate $A(B)$. It is injective injective $A^m(B)$ is a free $\mathbb{Z}$-module of rank 2 for each $0 < m < n$.

One reason we compute Chow groups is because they are supposed to tell us about the geometry of the variety. Let's compute E^2. We see that

$$\begin{aligned}
E^2 &= E \cdot \lambda_{n-1} - E \cdot \gamma_{n-1} \\
&= [E \cap (\Lambda - \Gamma)] \\
&= -[E \cap \Gamma_{n-1}]
\end{aligned}$$

is minus the class of a hyperplane in E. How do I make sense of this? What exactly happened in the second equality? In what sense is sum about union and difference is about set difference? The class of the union of two irreducible subvarieties is equivalent to the sum of the classes of the subvarieties. This follows from the defintion of rational equivalence. One must be more careful when there are coefficents involved.

3   References

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