Soergel bimodules

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Abstract

Notes on Soergel bimodules. See "Soergel Calculus" by Elias and Williamson and notes about perverse sheaves on the flag variety by Riche.

Contents

1 Hecke algebra

Recall Bruhat decomposition. Let G be a reductive group over $\mathbb{C}$ with Borel subgroup B and maximal torus T. Let W be the Weyl group of G. Then G has a Bruhat decomposition

$$G = \coprod_{w \in W} BwB$$

Orbit closures $\overline{BwB}$ are called Schubert varieties.

The Hecke algebra is the algebra of $B \times B$-invariant functions on G under normalized convolution. It therefore has a basis $\{T_w\}_{w \in W}$ where T_w is the characteristic function of BwB. In 1979, Kazhdan and Lusztig defined another basis $\{C_w\}_{w \in W}$ which enjoys many remarkable positivity properties.

According to Grothendieck's function-sheaf dictionary, the Hecke algebra should be categorified by some category of B-biinvariant sheaves on G. The **Hecke category** is the additive subcategory of semisimple complexes

$$\mathcal{H} \subset D^b_{B\times B}(G)$$

The objects of $\mathcal{H}$ are direct sums of the various intersection complexes $IC(\overline{BwB})$. These are the simple equivariant intersection cohomology complexes. There is a monoidal structure on $D^b_{B\times B}(G)$ given by convolution which preserves $\mathcal{H}$. The Grothendieck group $[\mathcal{H}]$ of $\mathcal{H}$ is an algebra over $\mathbb{Z}[v^{\pm 1}]$ where v acts by shift.

Theorem 1.1. *$[\mathcal{H}]$ is isomorphic to the Hecke algebra. The simple objects of $\mathcal{H}$ are the intersection complexes supported on Schubert varieties. The map takes $IC(\overline{BwB})$ to the Kazhdan Lusztig basis element C_w. Explicitly, we have the Kazhdan-Lusztig polynomial $P_{x,y}(q)$ defined by*

$$P_{v,w}(q) = \sum_i (-1)^i \dim \mathcal{H}^i(IC_w)_v q^i$$

and the Kazhdan-Lusztig basis element C_w can be written in the standard basis as

$$C_w = q^{-\ell(w)/2} \sum_{v\leq w} P_{v,w}(q) T_v$$

2 Soergel bimodules

Recall that a Coexter group is a group which has a presentation of the form

$$\langle s_1, \ldots, s_n | s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $m_{ij} \geq 2$ are symmetric. Weyl groups are Coxeter groups, but so are affine Weyl groups and other cool groups.

The hypercohomology of any complex in $D^b_{B\times B}(G)$ naturally carries an action of $H^*_{B\times B} = R \otimes_C R$ where $R = \mathrm{Rep}(T)$. Since R is commutative, this extends to make the hypercohomology a bimodule over R. Therefor we get a functor from $\mathbb{H}^\cdot : D^b_{B\times B}(G) \to R$-bimod.

Theorem 2.1 (Soergel)**.** *This functor is fully faithful and monoidal.*

Therefore the Hecke category is equivalent to its essential image. We can also calculate that

$$\mathbb{H}^\cdot(IC(\overline{BsB})) = \mathcal{M}_s := R \otimes_{R^s} R$$

where R^s is the s-invariant subring of R (W acts by permuting the variables) and s is a simple reflection in W (soon to be generalized to an arbitrary Coxeter group).

Soergel proves that the essential image is the smallest full additive monoidal Karoubian subcategory of R-bimod containing the $\mathcal{M}_s$. This is the category of **Soergel bimodules**. This story works for any Coexter system. Specifically for Weyl groups, Soergel's "categorification theorem" follows from the decomposition theorem applied to Bott Samelson resolutions of Schubert varieties. It is suprising that Soergel makes it work for any Coxeter group because there is no underlying geometric context.

3 Thoughts

I don't understand precisely Soergel bimodules are related to category $\mathcal{O}$ and perverse sheaves on the flag variety.

1. Is it true that the Hecke category can also be described as the (maybe B-equivariant) perverse sheaves on the flag variety with respect to the Schubert stratification?
2. If so, can I find a quiver and relations for which the Hecke category of a general Coxeter group is the category of representations of the path algebra of that quiver modulo those relations?
3. One reason why people are interested in finding these geometric categorifications of these algebras is because the category of perverse sheaves enjoys very nice properities. In particular the simple objects carry Hodge structures, satisfy Hard Lefschetz, and have Verdier duals. People are interested in making sense of these properties purely in the context of representation theory.

4 Soergel modules from bimodules

Going from Soergel bimodules to Soergel modules morally is the same as going from $H^*_B(G/B)$ to $H^*(G/B)$.