

# Quintic Threelfolds

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## Abstract

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Take a smooth degree-5 hypersurface  $X \subset \mathbb{P}^4$ . For a generic one **hodge numbers?**

$$h^{1,1}(X) = 1, h^{2,1}(X) = 101, h^{3,0}(X) = h^{0,3}(X) = 1$$

and all other  $h^{p,q}$  vanish. (The lone  $h^{1,1}$  is the Kähler class;  $h^{2,1}$  is complex-structure deformations.)

There is a standard 1-parameter family (the Dwork pencil)

$$X_\psi : x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 5\psi x_1 x_2 x_3 x_4 x_5 \subset \mathbb{P}^4$$

1. Picard–Fuchs for periods. Let  $z = (5\psi)^{-5}$ ,  $\theta = z \frac{d}{dz}$ . Periods  $\Pi(z)$  of the holomorphic 3-form satisfy  $\mathcal{L}\Pi = 0$ , where  $\mathcal{L} = \theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4)$ . Solve near the large complex structure limit  $z = 0$ :  $\Pi_0(z) = \sum_{n \geq 0} \frac{(5n)!}{(n!)^5} z^n$ ,  $\Pi_1 = \Pi_0 \log z + \dots$  Extract the mirror map  $q = \exp(\Pi_1/\Pi_0)$ .
2. Yukawa coupling & GW invariants. Compute the Yukawa coupling  $C_{zzz}$  from the PF system, convert to  $C_{ttt}$  in the flat coordinate  $t = \frac{1}{2\pi i} \log q + \dots$ , expand  $C_{ttt} = 5 + \sum_{d \geq 1} \frac{n_d d^3 q^d}{1-q^d}$ , and read off the genus-0 instanton numbers  $n_d$  (curve counts on  $X$ ):  $n_1 = 2875$  lines,  $n_2 = 609, 250$ ,  $n_3 = 317, 206, 375$ , etc.
3. Monodromy. Compute monodromies around  $z = 0$  (maximally unipotent), the conifold point  $z = 5^{-5}$ , and the Gepner point  $\psi = 0$ . Check that one monodromy is maximally unipotent (mirror criterion).

4. Kähler vs complex moduli. Identify the complex moduli of  $Y$  with the 1-parameter Dwork modulus, and the Kähler moduli of  $X$  with the  $q$ -coordinate you built. This is the mirror map statement in practice.
5. (Optional) Toric re-derivation. Rebuild the whole story via Batyrev's reflexive polytopes for the quintic and its polar dual; compute Hodge numbers from lattice point counts to see the  $(1, 101) \leftrightarrow (101, 1)$  swap without period theory.

## 1 Basic setup

Let  $Y = \mathbb{C}^n$  and  $f : Y \rightarrow \mathbb{C}$  a holomorphic function with an isolated critical point at 0.

Define the local algebra  $H_f = \mathbb{C}[y_1, \dots, y_n]/(\partial f / \partial y_i)$ , called the Milnor ring or Jacobian algebra. It's a finite-dimensional vector space of dimension  $\mu$  (the Milnor number).

Choose a monomial basis  $a_1, \dots, a_\mu$  representing classes in  $H_f$ . Then consider a versal deformation (a general perturbation)

$$f_\lambda(y) = f(y) + \lambda_1 a_1(y) + \cdots + \lambda_\mu a_\mu(y)$$

This gives a  $\mu$ -dimensional parameter space with coordinates  $\lambda = (\lambda_1, \dots, \lambda_\mu)$ .

Define  $I_i(\lambda) = \int e^{f_\lambda(y)/h} a_i(y) \omega$  where  $\omega = dy_1 \wedge \cdots \wedge dy_n$ . As  $h \rightarrow 0$ , the integral is dominated by critical points of  $f_\lambda$ ; so by stationary phase

$$I_i(\lambda) \sim \sum_{y_*(\lambda)} \frac{a_i(y_*(\lambda))}{\sqrt{J_\lambda(y_*(\lambda))}} e^{f_\lambda(y_*(\lambda))/h},$$

where  $J_\lambda = \det(\partial^2 f_\lambda / \partial y_i \partial y_j)$ . Each critical point contributes an exponential term with phase  $f_\lambda(y_*)/h$ .

The key observation is that the functions  $I_i(\lambda)$  satisfy a system of differential equations in the parameters

$$\lambda_j : h \frac{\partial I_i}{\partial \lambda_j} = \sum_k c_{ij}^k(\lambda) I_k$$

where the  $c_{ij}^k(\lambda)$  are the structure constants of the algebra  $a_i a_j = \sum_k c_{ij}^k(\lambda) a_k$  in  $H_{f_\lambda}$ .

This has something to do with the Gauss Manin connection. Formally, you can think of the family of vector spaces  $\mathcal{H}_\lambda = H_n(\mathbb{C}^n, \Re f_\lambda = -\infty)$  as forming a flat vector bundle over the parameter space of  $\lambda$ .

The integrals  $I_i(\lambda)$  can be viewed as flat sections of the dual bundle  $\mathcal{H}_\lambda^*$ . The differential equations satisfied by the  $I_i(\lambda)$  reflect the flatness of this connection.

The family of quintic-mirrors  $Y_\lambda$  is one of the examples for which one can construct flat coordinates on moduli spaces of complex structures.

## 2 Givental Hori Vafa

Let  $X$  be a compact toric Fano variety. Let  $\mathcal{F}(X)$  be its **Fukaya category** which is  $D(\mathbb{Z}/2c_1(X)\mathbb{Z})$ -graded. Then conjecturally

$$HH^*(\mathcal{F}(X)) \cong QH^*(X)$$

Write  $X = \mathbb{C}^n // H$  where

$$\begin{aligned} 0 &\rightarrow H \rightarrow (\mathbb{C}^*)^n \rightarrow T_{\mathbb{C}} \rightarrow 0 \\ 0 &\rightarrow T_{\mathbb{C}}^\vee \rightarrow (\mathbb{C}^*)^{n^\vee} \rightarrow H^\vee \rightarrow 0 \end{aligned}$$

Consider the fiber  $T_h^\vee$  of  $h \in H^\vee$ . Take the **superpotential** function

$$W = x_1 + \cdots + x_n : T_h^\vee \rightarrow \mathbb{C}$$

**Remark 2.1.** Joe made the remark that if you try to make the naive statement that there are two derived categories on the  $A$  and  $B$  side of mirror symmetry which are equivalent, then this cannot possibly work and one needs to introduce extra structures, such as the superpotential  $W$  here.

Then by homological mirror symmetry this defines a matrix factorization category  $MF(T_h^\vee, W)$  with a "map"

$$MF(T_h^\vee, W) \rightarrow T_h^\vee$$

Then we have the following theorem:

**Theorem 2.2.**

1. There is an equivalence of categories

$$MF(T_h^\vee, W) \cong \mathcal{F}(X, \mathfrak{h})$$

2.  $MF(T_h^\vee, W)$  is a module category over  $\mathbb{C}[T_h^\vee]$  and the Fourier modes are the Seidel shift operators.

3. There is an isomorphism of algebras

$$HH^*(MF(T_h^\vee, W)) \cong Jac(W) \cong QH^*(X, \mathfrak{h})$$