Homework 4

Songyu Ye

September 23, 2025

Problem Let p be a prime number, and n a positive integer greater than 1. Find an example for each of the following with brief justifications.

(1) A degree n extension of $\mathbb{Q}$ in which p is inert (i.e. the ring of integers in the extension possesses a unique prime $\mathfrak{q}$ above p, and the inertial degree $f_{\mathfrak{q}/p}$ is equal to n).

(2) A degree n extension of $\mathbb{Q}$ in which p is totally ramified.

Hint: You can apply results of Serre, I.6 after localizing at p.

Remark: There's nothing special about $\mathbb{Q}$. The same question can be answered similarly with any global field in place of $\mathbb{Q}$.

Solution:

Problem Let A be a Dedekind domain, $K = \mathrm{Frac}(A)$. Let L/K be a finite separable extension with normal closure M of L so that M is Galois over K. Let $\mathfrak{p}$ be a prime ideal of A. Fix a prime ideal $\mathfrak{t}$ of M above $\mathfrak{p}$. (By convention, this means $\mathfrak{t}$ is a nonzero prime in the integral closure of A in M such that $\mathfrak{t}$ divides $\mathfrak{p}$.) Denote by $D_{\mathfrak{t}}(M/K)$ the decomposition group of $\mathfrak{t}$ in M/K.

(i) Define a map

$$\mathrm{Gal}(M/K) \to \{\text{primes of } L \text{ above } \mathfrak{p}\}, \qquad \sigma \mapsto \sigma(\mathfrak{t}) \cap L.$$

Show that this map induces a bijection

$$\mathrm{Gal}(M/L)\backslash\mathrm{Gal}(M/K)/D_{\mathfrak{t}}(M/K) \xrightarrow{\sim} \{\text{primes of } L \text{ above } \mathfrak{p}\}.$$

(ii) Assume that $\mathrm{Gal}(M/K) \simeq S_3$, the symmetric group in 3 variables, that $D_{\mathfrak{t}}(M/K)$ and $\mathrm{Gal}(M/L)$ are order 2 subgroups of $\mathrm{Gal}(M/K)$ which are equal (not just isomorphic). Use part (i) to verify that $\mathfrak{p}$ does *not* split completely in L.

Remark: The point of (ii) is that when the decomposition group of $\mathfrak{t}$ is not normal in $\mathrm{Gal}(M/K)$, the prime $\mathfrak{t}$ need not split completely in the decomposition field, which is L here. A concrete example for (ii) can be given when

$$K = \mathbb{Q}, \quad L = \mathbb{Q}(\sqrt[3]{2}), \quad M = \mathbb{Q}(\sqrt[3]{2}, \zeta_3).$$

By the Chebotarev density theorem, or by explicit computation, you can find $\mathfrak{t}$ such that $(\mathfrak{t}, M/K)$ is the unique nontrivial element of $\mathrm{Gal}(M/L)$. Then all the conditions of (ii) are satisfied.

Solution:

Neukirch Ch. I.9, Exercise 3 Continue the general setup from Problem 2. Assume the following:

(i) L/K is solvable, meaning that $\mathrm{Gal}(M/K)$ is a solvable group. (We are not assuming $M = L$.)

(ii) $p = [L : K]$ is a prime number.

Now let $\mathfrak{p}$ be a prime of K unramified in L. If there are two primes $\mathfrak{q}$ and $\mathfrak{q}'$ of L above $\mathfrak{p}$ such that the inertial degrees $f_{\mathfrak{q}}$ and $f_{\mathfrak{q}'}$ are equal to 1, then show that $\mathfrak{p}$ splits completely in L/K.

Caveat: The extension degree p has nothing to do with the prime ideal $\mathfrak{p}$ in the problem.

Hint: Let S_p denote the symmetric group in p letters acting on $\{1, 2, \ldots, p\}$. If G is a solvable subgroup of S_p acting transitively on $\{1, 2, \ldots, p\}$ then every nontrivial element of G fixes at most one element in $\{1, 2, \ldots, p\}$. (A reference for this fact is given in Neukirch.)

Solution: