Title

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Abstract

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In this section, we follow the treatment of [1]. Let C be a smooth projective curve. Fixing any line bundle L on C, the set of isomorphism classes of stable vector bundles of rank 2 with determinant line bundle isomorphic to L carries the structure of an algebraic variety.

$$SU_C(2, L) = \{E \text{ stable of rank } 2, \det E \cong L\}/\sim$$

We assume that L has sufficiently high degree to guarantee that E is generated by global sections, and we consider the skew-symmetric bilinear map

$$H^0(E) \times H^0(E) \to H^0(\wedge^2 E) \cong H^0(L)$$

given by wedge product of sections. This form has rank 2, and we denote by $\mathrm{Alt}^2(H^0(L))$ the affine variety which parametrises such skew-symmetric forms of rank < 2 in dimension $N = \dim H^0(E)$. We will use this wedge product to reduce our moduli problem to the quotient problem for the action of $\mathrm{GL}(N)$ on $\mathrm{Alt}^2(H^0(L))$. One encounters various difficulties that do not appear in the line bundle case of the last chapter, but it turns out that the notion of stability is the correct way to resolve these problems, and one proves the following.

Theorem 2.1 (Moduli of rank 2 vector bundles with fixed determinant)**.** Suppose that the line bundle L has degree $\geq 4g-1$.

(i) There exists a Proj quotient

$$\mathrm{Alt}^{ss}_{N,2}(H^0(L)) /\!\!/ GL(N)$$

which is a projective variety of dimension $3g-3$.

(ii) The open set

$$\mathrm{Alt}^{s}_{N,2}(H^0(L))/GL(N)$$

has an underlying set $SU_C(2, L)$. Moreover, it is nonsingular and at each point $E \in SU_C(2, L)$ its tangent space is isomorphic to $H^1(\mathfrak{sl}\, E)$.

(iii) If $\deg L$ is odd, then

$$\mathrm{Alt}^{ss}_{N,2}(H^0(L)) /\!\!/ GL(N) \;=\; \mathrm{Alt}^{s}_{N,2}(H^0(L))/GL(N) \;=\; SU_C(2, L)$$

is a smooth projective variety.

2.1 Slope stability and Pfaffians

We begin by recalling the notion of slope stability for vector bundles on curves. We will then introduce the Pfaffian of a skew-symmetric matrix, which will be the key semiinvariant we will use to study the Gieseker points associated to rank 2 vector bundles.

Definition 2.2. Let E be a vector bundle. A coherent subsheaf $F \subset E$ is a subbundle if F is a vector bundle (i.e. locally free) and the cokernel E/F is also locally free.

Remark 2.3. In particular, note that a subsheaf $F \subset E$ of a vector bundle can be a vector bundle in its own right, but not a subbundle if the cokernel E/F has torsion.

However, one can always saturate a subsheaf F to get a subbundle $\overline{F}$ defined as the kernel of the composition

$$E \to E/F \to (E/F)/\text{torsion}.$$

The saturation $\overline{F}$ is the largest subbundle of E containing F, and $\deg \overline{F} \geq \deg F$. One can check that any saturated subsheaf of a locally free sheaf is again locally free.

Definition 2.4 (Slope stability)**.** A vector bundle E on C is stable (resp. semistable) if for every proper subbundle $F \subset E$, we have

$$\frac{\deg F}{\text{rank} F} < \frac{\deg E}{\text{rank} E}$$

(resp. $\leq$). The ratio $\mu(E) = \frac{\deg E}{\text{rank} E}$ is called the slope of E, and this condition is often called slope stability (semistability).

Definition 2.5. A vector bundle E is **simple** if $\text{End}(E) = k \cdot \text{Id}_E$. A vector bundle E is decomposable if it is isomorphic to the direct sum $E_1 \oplus E_2$ of two nonzero vector bundles; otherwise, E is **indecomposable**.

If $f \in \text{End}(E)$ is an idempotent, $f^2 = f$, then f is the projection onto its image and $1 - f$ is the projection onto its kernel. Hence the natural direct–sum decomposition of E is

$$E = \text{im}\, f \oplus \ker f,$$

because $\text{im}\, f \cap \ker f = 0$ and $\text{im}\, f + \ker f = E$.

Conversely, if $E = E_1 \oplus E_2$ is a nontrivial decomposition, the projection onto the first summand along the second is an idempotent $f \in \text{End}(E)$, $f^2 = f$, whose image is E_1 and whose kernel is E_2. Thus decomposability of E is equivalent to the existence of a nontrivial idempotent $f \neq 0, 1$ in $\text{End}(E)$.

Proposition 2.6. Every vector bundle E can be uniquely decomposed as a direct sum of indecomposable vector bundles, up to isomorphism and permutation of the summands.

Proof. Omitted. □

Given an endomorphism $f : E \to E$, consider the determinant $\det f : \det E \to \det E$. This is just multiplication by a scalar because $\det E$ is a line bundle, and this scalar is nonzero if and only if f is an isomorphism. Now, for an arbitrary scalar λ consider $\det(f - \lambda \text{id})$. This is a polynomial of degree $r(E)$ in λ and is the characteristic polynomial of the endomorphism f. In particular, if α is an eigenvalue, then $f - \alpha\, \text{id}$ fails to be an isomorphism because its determinant is zero.

Lemma 2.7. If E is indecomposable, then $f \in \text{End}(E)$ has only one eigenvalue.

Proof. Suppose f has distinct eigenvalues α and β. Then its characteristic polynomial can be expressed as a product of two polynomials without common factors:

$$\det(f - \lambda \cdot \mathrm{id}) = p(\lambda)\, q(\lambda), \qquad p(\alpha) = 0, \quad q(\beta) = 0.$$

There exist polynomials $a(\lambda)$, $b(\lambda)$ satisfying

$$p(\lambda)\, a(\lambda) + q(\lambda)\, b(\lambda) = 1,$$

Let $h(f) = p(f)a(f)$. Then

$$h(1-h) = \big(p(f)a(f)\big)\big(q(f)b(f)\big) = p(f)\, q(f)\, a(f)b(f) = 0$$

by the Cayley-Hamilton theorem. This implies that E is the direct sum $\ker(h) \oplus \ker(1-h)$, and since h and $1-h$ are both nonzero, we conclude that E is decomposable. $\square$

Proposition 2.8. Consider a short exact sequence of vector bundles on a curve

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

Then

$$\mu(F) > \mu(E) \quad \iff \quad \mu(E) > \mu(G)$$

with equalities holding in the semistable case. In particular, (semi)stability of E is witnessed by slopes of quotients as well as subbundles.

Proof. One computes that

$$\mu(E) = \frac{r_F}{r_F + r_G}\, \mu(F) + \frac{r_G}{r_F + r_G}\, \mu(G)$$

In particular $\mu(E)$ is a convex combination of $\mu(F)$ and $\mu(G)$, it must lie between them. So the only possibilities are $\mu(F) < \mu(E) < \mu(G)$, or $\mu(G) < \mu(E) < \mu(F)$, or they are all equal. $\square$

Proposition 2.9. Let E, E' be semistable vector bundles of the same rank and degree, and suppose that one of them is stable. Then every nonzero homomorphism between E and E' is an isomorphism.

Proof. Let r and d be the common rank and degree of the two bundles, and let $f : E \to E'$ be a homomorphism with image $F \subset E'$.

F is not necessarily a subbundle of E', but we can consider its saturation, which is a subbundle $\overline{F} \subset E'$. Note that $\deg \overline{F} \geq \deg F$ by the short exact sequence

$$0 \longrightarrow F \longrightarrow \overline{F} \longrightarrow \text{torsion} \longrightarrow 0$$

and note that degree is additive on short exact sequences.

Since $\overline{F}$ is a subbundle of the semistable bundle E', we have

$$\mu(F) \leq \mu(\overline{F}) \leq \mu(E') = \frac{d}{r}.$$

Since F is a quotient of E, we have

$$\mu(F) = \frac{\deg F}{\mathrm{rank} F} \geq \frac{d}{r}$$

So therefore $\mu(\overline{F}) = \mu(F) = d/r$. If $\mathrm{rank}\, F < r$, then F is either a proper quotient bundle of E with the same slope (which would contradict the stability of E) or would induce a proper subbundle $F \subset E'$ with the same slope (which would contradict the stability of E'). So this contradicts the stability of E or E', and hence $\mathrm{rank}\, F = r$. In particular, this means that the induced map

$$f_{\mathrm{gen}} : E_{\mathrm{gen}} \longrightarrow E'_{\mathrm{gen}}$$

is an isomorphism of vector spaces over $k(C)$, and so $\det(f_{\mathrm{gen}})$ is also an isomorphism. This implies that $\det f : \det E \to \det E'$ is injective, and since $\deg E = \deg E'$, it follows that $\det f$ is an isomorphism. Hence f is an isomorphism. □

Corollary 2.10. Every stable vector bundle is simple.

Proof. An endomorphism $f \in \mathrm{End}\, E$ induces, at each point $p \in C$, an endomorphism of the fiber $E/E(-p) \cong k^{\oplus r}$. Let $\alpha \in k$ be an eigenvalue of this map, and consider $f - \alpha \cdot \mathrm{id} \in \mathrm{End}\, E$. This is not an isomorphism, so by the previous proposition it must be zero. □

We are going to study the semistability of Gieseker points associated to rank 2 vector bundles. For this, the central notion, with which we will build our semiinvariants, is that of the Pfaffian of a skew-symmetric matrix.

Let $\mathrm{Alt}_N(k)$ denote the space of $N \times N$ skew-symmetric matrices with entries in k. Thinking of a point $A \in \mathrm{Alt}_N(k)$ as a skew-symmetric bilinear form on k^N, there is an action of $\mathrm{GL}_N(k)$ on $\mathrm{Alt}_N(k)$ by change of basis:

$$(g, A) \mapsto gAg^t.$$

The orbits of this action are classified by the rank of the skew-symmetric form, which is always even. Because of this, the properties of the action depend in an essential way on whether N is even or odd.

Definition 2.11 (Pfaffian for even N)**.** Let N be an even integer. The Pfaffian of a skew-symmetric matrix $A \in \mathrm{Alt}_N(k)$ is defined as

$$\mathrm{Pf}(A) = \frac{1}{2^{N/2}(N/2)!} \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{i=1}^{N/2} a_{\sigma(2i-1),\sigma(2i)}$$

Proposition 2.12 (Properties of the Pfaffian)**.**

1. $\mathrm{Pf}(A)^2 = \det(A)$ for all $A \in \mathrm{Alt}_N(k)$. In particular, $\mathrm{Pf}(A) \neq 0$ if and only if A is nondegenerate.

2. $\mathrm{Pf}(gAg^t) = \det(g)\,\mathrm{Pf}(A)$ for all $g \in \mathrm{GL}_N(k)$, $A \in \mathrm{Alt}_N(k)$. In particular, the Pfaffian is a semiinvariant of weight 1 (with respect to the character $\det : \mathrm{GL}_N(k) \to k^*$).

3. For any $B \in \mathrm{Alt}_{N/2}(k)$ and $C \in \mathrm{Mat}_{N/2}(k)$, we have

$$\mathrm{Pf}\begin{pmatrix} 0 & B \\ -B^T & C \end{pmatrix} = (-1)^{N/2+1}\,\mathrm{Pf}(B)$$

For odd N, the determinant of any skew-symmetric matrix is zero, so we cannot define the Pfaffian as above. However, one can be clever and define a radical vector associated to any skew-symmetric matrix, which will play the role of the Pfaffian in this case. In particular we can define semiinvariants using the radical vector.

Definition 2.13 (Radical for odd N)**.** Let N be an odd integer. For $A \in \mathrm{Alt}_N(k)$, the radical vector $\mathrm{rad}(A) \in k^N$ is

$$\mathrm{rad}(A)_i = (-1)^{i+1}\,\mathrm{Pf}(A_{[i]}) \quad \text{for } i = 1, \dots, N$$

where $A_{[i]}$ is the $(N-1) \times (N-1)$ skew-symmetric matrix obtained by deleting the i-th row and column from A.

Proposition 2.14 (Properties of the radical)**.** Consider $A \in \mathrm{Alt}_N(k)$.

(i) $\mathrm{rank}\, A \leq N-1$, and $\mathrm{rank}\, A < N-1$ if and only if $\mathrm{rad}\, A = 0$.

(ii) $A \cdot \mathrm{rad}\, A = 0$.

(iii) If X is an $N \times N$ matrix and X^* is its matrix of cofactors, then

$$\mathrm{rad}(XAX^t) = X^{*,t}\ \mathrm{rad}\, A.$$

2.2 Gieseker points

Fix a line bundle L and consider rank 2 vector bundles E with $\det E \cong L$. Fix the number $N = \chi(E) = h^0(E) - h^1(E) = \deg L + 2 - 2g$. A set $S \subset H^0(E)$ of N linearly independent global sections is called a marking of the vector bundle E, and the pair (E, S) is called a marked vector bundle.

We will need the key properties that

1. $H^1(E) = 0$
2. E is generated by global sections, i.e. the evaluation map $H^0(E) \otimes \mathcal{O}_C \to E$ is surjective as a sheaf map.

In the moduli story of line bundles, these conditions were guaranteed by taking the degree sufficiently large. However, this is not sufficient for arbitrary rank 2 vector bundles.

However they are satisfied by semistable vector bundles of sufficiently large degree. In this case we have $N = h^0(E)$, and a marking S is a basis of $H^0(E)$. Moreover, generation by global sections means that the homomorphism

$$(s_1, \ldots, s_N) : \mathcal{O}_C^{\oplus N} \longrightarrow E, \qquad (f_1, \ldots, f_N) \longmapsto \sum_{i=1}^{N} f_i s_i, \tag{10.13}$$

is surjective. At the same time, there is a homomorphism

$$(s_1\wedge, \ldots, s_N\wedge) : E \longrightarrow (\det E)^{\oplus N}, \qquad t \longmapsto (s_1 \wedge t, \ldots, s_N \wedge t), \tag{10.14}$$

which, if E is generated by global sections, is injective. To explain this, recall that the stalk at the generic point E_{gen} is a 2-dimensional vector space over the function field $k(C)$, so there is a skew-symmetric bilinear form

$$\wedge : E_{\text{gen}} \times E_{\text{gen}} \longrightarrow \det E_{\text{gen}} \;\cong\; k(C). \tag{10.15}$$

Thus $s \wedge s = 0$ and $s \wedge s' + s' \wedge s = 0$ for $s, s' \in E_{\text{gen}}$. Moreover, if s, s' are global sections of E, then $s \wedge s'$ is a global section of $\det E$, and so restriction of (10.15) defines a skew-symmetric k-bilinear map

$$H^0(E) \times H^0(E) \;\longrightarrow\; H^0(\det E), \qquad (s, s') \longmapsto s \wedge s'.$$

The bilinear form (10.15) induces an isomorphism

$$E_{\text{gen}} \;\xrightarrow{\sim}\; \operatorname{Hom}(E_{\text{gen}}, \det E_{\text{gen}}).$$

In particular, each global section $s \in H^0(E)$ determines a homomorphism

$$s\wedge : E \longrightarrow \det E, \qquad t \longmapsto s \wedge t.$$

Definition 2.15 (Gieseker point)**.** Given a vector space V, we denote by $\mathrm{Alt}_N(V)$ the set of skew-symmetric $N \times N$ matrices whose entries belong to V. Given a marked vector bundle (E, S) with $\det E = L$, the skew-symmetric matrix

$$T_{E,S} = \begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix} \wedge (s_1, \ldots, s_N) = \begin{bmatrix} s_1 \wedge s_2 & s_1 \wedge s_3 & \cdots & s_1 \wedge s_N \\ s_2 \wedge s_3 & \cdots & & s_2 \wedge s_N \\ \vdots & & \ddots & \vdots \\ s_{N-1} \wedge s_N & & & \end{bmatrix} \in \mathrm{Alt}_N(H^0(L))$$

will be called the **Gieseker matrix**, or **Gieseker point**, of E corresponding to the marking S.

Proposition 2.16. Given $S = \{s_1, \ldots, s_N\} \subset H^0(E)$, the composition of (10.13) and (10.14)

$$\mathcal{O}_C^{\oplus N} \xrightarrow{(s_1, \ldots, s_N)} E \xrightarrow{(s_1 \wedge, \ldots, s_N \wedge)} L^{\oplus N}$$

is given by the matrix $T_{E,S} \in \mathrm{Alt}_N(H^0(L))$.

Note that any matrix $T \in \mathrm{Alt}_N(H^0(L))$ determines a vector bundle map

$$\langle T \rangle : \mathcal{O}_C^{\oplus N} \longrightarrow L^{\oplus N},$$

and this is skew-symmetric in the sense that the dual map

$$\langle T \rangle^t : (L^{-1})^{\oplus N} \longrightarrow \mathcal{O}_C^{\oplus N},$$

after tensoring with L, is equal to $-\langle T \rangle$.

Proposition 2.17 (Reconstruction from Gieseker point)**.** Suppose that $H^1(E) = 0$ and that E is generated by global sections. Then, for any marking S, the bundle E is isomorphic to the image of the homomorphism

$$\langle T_{E,S} \rangle : \mathcal{O}_C^{\oplus N} \longrightarrow L^{\oplus N}$$

defined by its Gieseker point.

Proof. Consider the sequence

$$\mathcal{O}_C^{\oplus N} \xrightarrow{\mathrm{ev}_S} E \xrightarrow{i_S} L^{\oplus N}$$

where $\alpha = \mathrm{ev}_S$ is surjective (since E is generated by global sections) and $\beta = i_S$ is injective (as explained above). Since i_S is injective, we have $\ker(i_S \circ \mathrm{ev}_S) = \ker(\mathrm{ev}_S)$. Therefore,

$$\mathrm{im}\langle T_{E,S} \rangle = \mathrm{im}(i_S \circ \mathrm{ev}_S) \cong \mathcal{O}_C^{\oplus N} / \ker(\mathrm{ev}_S) \cong \mathrm{im}(\mathrm{ev}_S) = E.$$

□

We now consider the action

$$GL(N) \curvearrowright \mathrm{Alt}_N(H^0(L)), \qquad T \longmapsto XTX^t, \quad T \in \mathrm{Alt}_N(H^0(L)),\ X \in GL(N),$$

where we view $\mathrm{Alt}_N(H^0(L))$ as an affine space $\mathbb{A}^n$, with $n = h^0(L)\, N(N-1)/2$.

If we assume $H^1(E) = 0$, so that the marking S is a basis of $H^0(E)$, then the $GL(N)$-orbit of its Gieseker points depends only on the isomorphism class of E and not on the choice of S. Conversely, the vector bundle E can be recovered from any Gieseker point by Proposition 2.17, so we have the following.

Corollary 2.18. The mapping

$$\left\{ \begin{array}{c} \text{isomorphism classes of vector bundles } E \text{ with } H^1(E) = 0 \\ \text{and generated by global sections} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} GL(N)\text{-orbits} \\ \text{in } \mathrm{Alt}_N(H^0(L)) \end{array} \right\}$$

sending E to the orbit of its Gieseker points $T_{E,S}$ is injective.

2.3 Semistability of Gieseker points

We now need to consider the question of (semi)stability of a point $T \in \mathrm{Alt}_N(H^0(L))$ under the action of $GL(N)$, with respect to the determinant character $g \mapsto \det g$. We will show that if E is a rank 2 vector bundle with $H^1(E) = 0$ and $\deg E \geq 4g-2$, then the Gieseker points $T_{E,S}$ are semistable if and only if E is slope-semistable as a vector bundle. Conversely, we will see that if $\deg L \geq 4g-2$, then every semistable $T \in \mathrm{Alt}_N(H^0(L))$ is a Gieseker point of a semistable vector bundle.

Definition 2.19 (Gieseker semistability)**.** A **semiinvariant** of weight w is a polynomial function

$$F = F(T) \in k[\mathrm{Alt}_N(H^0(L))]$$

with the property

$$F(g \cdot T) = (\det g)^w F(T), \qquad \text{for all } g \in GL(N),$$

and the unstable set in $\mathrm{Alt}_N(H^0(L))$ is the common zero-set of all semiinvariants of positive weight. In particular, if there exists a semiinvariant F of positive weight with $F(T) \neq 0$, then T is semistable.

Recall that a point T is unstable if and only if the closure of its $SL(N)$-orbit contains the origin. A "Gieseker point" $\Psi(\xi, S, T)$ of a line bundle ξ is always stable. However, for vector bundles this is no longer the case. For rank greater than 1 the following phenomenon appears.

Proposition 2.20. Let S be a marking and $M \subset E$ a line subbundle of the vector bundle E, and consider the vector subspaces $\langle S \rangle \subset H^0(E)$ (of dimension N) and $H^0(M) \subset H^0(E)$.

(i) If there exists $M \subset E$ such that

$$\dim\bigl(H^0(M) \cap \langle S \rangle\bigr) > \frac{N}{2},$$

then the Gieseker point $T_{E,S} \in \mathrm{Alt}_N(H^0(L))$ is unstable.

(ii) If there exists $M \subset E$ such that

$$\dim\bigl(H^0(M) \cap \langle S \rangle\bigr) \geq \frac{N}{2},$$

then $T_{E,S} \in \mathrm{Alt}_N(H^0(L))$ fails to be stable.

Proof. The strategy goes to pick a basis aligned with the subspace $H^0(M) \subset H^0(E)$, then use the Hilbert-Mumford criterion.

Let

$$a = \dim\left(H^0(M) \cap \langle S \rangle\right), \qquad b = N - a.$$

Reorder $S = (s_1, \dots, s_N)$ so that $s_1, \dots, s_a \in H^0(M)$. Since M is a line bundle, we have $s_i \wedge s_j = 0$ for $1 \leq i, j \leq a$. Therefore, the Gieseker matrix $T_{E,S}$ has the block form

$$T_{E,S} = \begin{pmatrix} 0 & B \\ -B^t & C \end{pmatrix}$$

where the blocks have sizes $a \times a$, $a \times b$, and $b \times b$ respectively. Take

$$g(t) = \mathrm{diag}(t^{-b} I_a,\ t^a I_b) \in SL(N) \quad (\det g(t) = t^{-ab} \cdot t^{ab} = 1).$$

Since the action is $T \mapsto gTg^t$, we have

$$g(t) T_{E,S} g(t)^t = \begin{pmatrix} 0 & t^{a-b} B \\ -\, t^{a-b} B^t & t^{2a} C \end{pmatrix}.$$

If $a > b$, the exponents $a - b > 0$ and $2a > 0$, so as $t \to 0$ the right-hand side tends to the zero matrix. Thus, 0 lies in the closure of the $\mathrm{SL}(N)$-orbit of $T_{E,S}$. By the standard GIT criterion (the orbit closure contains the origin), $T_{E,S}$ is unstable.

If $a = b$, then the right hand side tends to

$$T_0 = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix}.$$

The 1-parameter subgroup

$$\lambda \mapsto \mathrm{diag}(\lambda I_a,\ \lambda^{-1} I_b) \subset \mathrm{SL}(N)$$

stabilizes T_0, because

$$\operatorname{diag}(\lambda I_a,\ \lambda^{-1} I_b)\, T_0\, \operatorname{diag}(\lambda I_a,\ \lambda^{-1} I_b)^t = T_0,$$

Recall that a point $x \in X$ is stable iff x is semistable, the orbit $G \cdot x$ is closed in X^{ss}, and $\operatorname{Stab}_G(x)$ is finite. Hence, $T_{E,S}$ is not stable because either its orbit is not closed (if T_0 is not in the orbit of $T_{E,S}$), or if T_0 is in the orbit of $T_{E,S}$, in which case $T_0 = T_{E,S}$ has a positive-dimensional stabilizer (the 1-parameter subgroup above). □

This phenomenon motivates the following definition.

Definition 2.21. Let E be a rank 2 vector bundle. If

$$h^0(M) \leq \tfrac{1}{2} h^0(E) \quad (\text{resp. } <)$$

for every line subbundle $M \subset E$, then we say that E is H^0**-semistable** (resp. H^0**-stable**).

The following corollary is an immediate consequence of the previous proposition.

Corollary 2.22 (Gieseker semistability implies H^0-semistability)**.** Suppose that $H^1(E) = 0$. Then $N = h^0(E)$ and S is a basis of $H^0(E)$. Let $T = T_{E,S}$ be any Gieseker point of E. Then:

(i) If T is $GL(N)$-semistable, then E is H^0-semistable;

(ii) If T is $GL(N)$-stable, then E is H^0-stable.

Proposition 2.23 (Equivalence of H^0-semistability and slope semistability)**.** Suppose that $H^1(E) = 0$ and $\deg E \geq 4g - 2$. Then E is H^0-semistable if and only if it is slope-semistable.

Proof. First observe that by Riemann–Roch any line bundle M satisfies

$$h^0(M) - h^1(M) - \frac{h^0(E) - h^1(E)}{2} = \deg M - \frac{\deg E}{2}.$$

Since $H^1(E) = 0$, this implies

$$\frac{h^0(E)}{2} - h^0(M) \leq \left(\frac{h^0(E)}{2} - h^0(M)\right) + h^1(M) = \frac{\deg E}{2} - \deg M. \qquad \text{(stability-inequality)}$$

Letting M run through the line subbundles of E, this shows at once that H^0-semistability of E implies slope-semistability (H^0-semistability means that the left-hand side of (stability-inequality) is nonnegative for all M).

For the converse, suppose that there exists a line subbundle $M \subset E$ for which the left-hand side of (stability-inequality) is negative. Note that, by hypothesis,

$$h^0(E) = \deg E + 2 - 2g \ \geq \ 2g,$$

and therefore $h^0(M) > \frac{1}{2}h^0(E) \geq g$.

This implies that $H^1(M) = 0$. Indeed, suppose $h^1(M) > 0$. By Serre duality, this means there exists a nonzero section $s \in H^0(K \otimes M^{-1})$, where K is the canonical bundle of C. Multiplication by s gives an injective sheaf map $M \hookrightarrow K$, and thus an injective map on global sections $H^0(M) \hookrightarrow H^0(K)$. Therefore, $h^0(M) \leq h^0(K) = g$. Now take the contrapositive to conclude that $h^0(M) > g$ implies $h^1(M) = 0$.

Thus we get the equality

$$\frac{h^0(E)}{2} - h^0(M) = \left(\tfrac{h^0(E)}{2} - h^0(M)\right) + h^1(M) = \frac{\deg E}{2} - \deg M$$

and the left hand side is negative and so we get the failure of slope-semistability. □

Proposition 2.24 (H^0-semistability implies Gieseker semistability)**.** Suppose that $H^1(E) = 0$. Then, if the vector bundle E is H^0-semistable, its Gieseker points $T_{E,S} \in \mathrm{Alt}_N(H^0(L))$ are semistable for the action of $GL(N)$.

A quotient line bundle $Q = E/M$ of an H^0-semistable vector bundle E satisfies

$$h^0(Q) \geq h^0(E) - h^0(M) \geq \tfrac{1}{2}h^0(E).$$

Lemma 2.25 (Global generation)**.** If E is H^0-semistable and $h^0(E) \geq 2$, then E is generated by global sections at a general point $p \in C$. In particular,

$$h^0(E(-p)) = h^0(E) - 2$$

at the general point.

Proof. Consider the evaluation homomorphism

$$H^0(E) \otimes \mathcal{O}_C \longrightarrow E.$$

The image sheaf has $h^0(E)$ linearly independent sections; if it had rank 1, then its saturation would be a line bundle violating H^0-semistability. So the image has rank 2.

The image sheaf cannot have rank 1. If $\mathrm{rk}\, I = 1$, then I is a torsion-free sheaf of rank 1, so its saturation $M := I^{\mathrm{sat}} \subset E$ is a line subbundle. Since $I \subset M$, the map $H^0(I) \hookrightarrow H^0(M)$ is injective, so

$$h^0(M) \geq h^0(I) = h^0(E).$$

But H^0-semistability requires $h^0(M) \leq \frac{1}{2}h^0(E)$ for every line subbundle $M \subset E$, so this is a contradiction. Therefore, $\mathrm{rk}\, I \neq 1$. It follows that $\mathrm{rk}\, I = 2$, so $I = E$ at the generic point; that is, ev is generically surjective. For a general point $p \in C$, the evaluation map

$$\mathrm{ev}_p : H^0(E) \to E|_p$$

is surjective. From the exact sequence

$$0 \to E(-p) \to E \to E|_p \to 0$$

we obtain

$$0 \to H^0(E(-p)) \to H^0(E) \xrightarrow{\mathrm{ev}_p} E|_p \to 0,$$

so $\dim E|_p = 2$ implies $h^0(E(-p)) = h^0(E) - 2$. $\square$

Remark 2.26 (Saturation)**.** The image of the evaluation map

$$\mathrm{ev} : H^0(E) \otimes \mathcal{O}_C \longrightarrow E$$

is a coherent subsheaf $I \subset E$. It need not be a subbundle because a subbundle means locally free subsheaf with torsion-free (equivalently locally free on a curve) quotient. The quotient E/I can have torsion at the base locus of the sections.

On a smooth curve, I is torsion-free (subsheaf of a vector bundle), hence locally free. But E/I may have zero-dimensional torsion, so I is not saturated, hence not a subbundle.

On $\mathbb{P}^1$ consider the vector bundle $E = \mathcal{O} \oplus \mathcal{O}(1)$, consider a single section $s \in H^0(\mathcal{O}(1))$ with $\mathrm{div}(s) = p$. The evaluation by the two sections $(1,0)$ and $(0,s)$ gives a map

$$\mathcal{O}_C^2 \longrightarrow \mathcal{O} \oplus \mathcal{O}(1), \quad (f,g) \mapsto (f,\, gs).$$

Hence the image is

$$I = \mathcal{O} \oplus (s \cdot \mathcal{O}_C) \subset \mathcal{O} \oplus \mathcal{O}(1).$$

Locally at p, trivialize $\mathcal{O}(1)$ so that $s = t\,e$ with $t \in m_p$ a uniformizer. Then

$$s \cdot \mathcal{O}_{C,p} = t\,\mathcal{O}_{C,p} \cdot e = m_p\, e,$$

so globally $s \cdot \mathcal{O}_C = \mathcal{I}_p \otimes \mathcal{O}(1) = \mathcal{O}(1)(-p)$. Thus,

$$I = \mathcal{O} \oplus \mathcal{O}(1)(-p),$$

and the quotient is

$$E/I \cong \mathcal{O}(1)/\mathcal{O}(1)(-p) \cong \mathcal{O}_p \cong k(p),$$

a torsion skyscraper sheaf. The inclusion $I \hookrightarrow E$ is therefore not a subbundle inclusion (the quotient is not locally free), even though I itself is locally free.

Lemma 2.27 (Semistability after twist)**.** If E is H^0-semistable and $h^0(E) \geq 4$ then there exists a point $p \in C$ such that the vector bundle $E(-p)$ is H^0-semistable.

Proof. Let $h^0(E) = n$. At a general point $p \in C$ we have $h^0(E(-p)) = n-2$ by Lemma 2.25. We suppose that at every point the bundle $E(-p)$ is H^0 unstable and therefore contains some line subbundle, which we denote by $M^p(-p) \subset E(-p)$, with

$$h^0(M^p(-p)) > \frac{n}{2} - 1.$$

Claim. The line subbundle $M^p \subset E$ is independent of the choice of the general point $p \in C$.

Granted the claim, we have a line subbundle $M (= M^p) \subset E$ which satisfies

$$h^0(M(-p)) > \frac{n}{2} - 1$$

at a general point of the curve. But this implies $h^0(M) > n/2$ (using the generality of p since every line bundle M has finitely many points where every global section vanishes). This contradicts the H^0-semistability of E, and we are done.

To prove the claim, let $q \in C$ be another, distinct, point. We first consider the case $n \geq 5$. Then

$$h^0(E(-p)) = n - 2 > \frac{n}{2},$$

and this implies that $E(-p)$ is generically generated by global sections (otherwise we would get a line subbundle of $E(-p) \subset E$ violating the H^0-semistability of E). Hence $h^0(E(-p-q)) = n-4$. On the other hand,

$$h^0(M^p(-p-q)) + h^0(M^q(-p-q)) = h^0(M^p(-p)) - 1 + h^0(M^q(-q)) > n - 4.$$

This implies that

$$0 \neq H^0(M^p(-p-q)) \cap H^0(M^q(-p-q)) \subset H^0(E(-p-q)),$$

and hence the line subbundles $M^p(-p-q)$ and $M^q(-p-q) \subset E(-p-q)$ coincide. Hence $M^p = M^q$.

Now consider the case $n = 4$. We have $h^0(E(-p)) = 2$ and $h^0(M^p(-p)) \geq 2$, so $H^0(E(-p)) = H^0(M^p(-p))$. In particular,

$$H^0(E(-p-q)) = H^0(M^p(-p-q)) \cong k.$$

Similarly,

$$H^0(E(-p-q)) = H^0(M^q(-p-q)) \cong k.$$

So again the two line subbundles $M^p(-p-q)$ and $M^q(-p-q) \subset E(-p-q)$ have a common global section, and they therefore coincide. $\square$

Proof of Proposition 2.24. To show semistability of a Gieseker point $T_{E,S}$ we have to exhibit a semiinvariant of positive weight which is nonzero at $T_{E,S}$. We consider separately the cases when N is even or odd (note that $N \equiv \deg L \pmod 2$).

When N is even we can construct semiinvariants as follows. For any linear form $f : H^0(L) \to k$, we can evaluate f on the entries of a matrix $T \in \mathrm{Alt}_N(H^0(L))$ to obtain a skew–symmetric matrix $f(T) \in \mathrm{Alt}_N(k)$. The function

$$\mathrm{Alt}_N(H^0(L)) \longrightarrow k, \qquad T \longmapsto \mathrm{Pfaff}\,\big(f(T)\big)$$

is a semiinvariant of weight 1.

By repeated use of Lemmas 2.27 and 2.25 we can find points $p_1, \dots, p_{N/2} \in C$ such that

$$H^0\big(E(-p_1 - \cdots - p_{N/2})\big) = 0. \tag{10.17}$$

If we let $\mathrm{ev}_i = \mathrm{ev}_{p_i} : H^0(L) \to k$ be evaluation at the i-th point, then the above equation says that the linear map of N–dimensional vector spaces

$$g := (\mathrm{ev}_1, \dots, \mathrm{ev}_{N/2}) : H^0(E) \longrightarrow \bigoplus_{i=1}^{N/2} E/E(-p_i)$$

is an isomorphism. Now consider the skew–symmetric pairing

$$H^0(E) \times H^0(E) \xrightarrow{\wedge} H^0(L) \xrightarrow{f} k,$$

where $f := \mathrm{ev}_1 + \cdots + \mathrm{ev}_{N/2} : H^0(L) \to k$. This pairing has matrix $f(T_{E,S})$ and transforms, via the isomorphism g, to a skew–pairing $k^N \times k^N \to k$ with matrix

$$\begin{pmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{pmatrix}.$$

In other words, there is a commutative diagram

$$\begin{tikzcd} H^0(E) \times H^0(E) \arrow[r, "\wedge"] \arrow[d, "g\times g"'] & H^0(L) \arrow[d, "f"] \\ k^N \times k^N \arrow[r] & k \end{tikzcd} \tag{10.18}$$

General theory of Pfaffians now shows that $\mathrm{Pfaff}\,\big(f(T_{E,S})\big)$ is equal to $\det g \neq 0$. Hence the Gieseker point $T_{E,S} \in \mathrm{Alt}_N(H^0(L))$ is semistable.

We turn now to the case when N is odd. In this case the strategy for producing semiinvariants is to use triples of linear forms

$$f,\, f',\, h : H^0(L) \longrightarrow k.$$

From these and from $T \in \mathrm{Alt}_N(H^0(L))$ we get vectors $\mathrm{rad}\, f(T)$, $\mathrm{rad}\, f'(T) \in k^N$ and a skew–symmetric matrix $h(T) \in \mathrm{Alt}_N(k)$ by applying the linear functionals entrywise. We then form the scalar product

$$\mathrm{Alt}_N(H^0(L)) \longrightarrow k, \qquad T \longmapsto (\mathrm{rad}\, f(T))^t\, h(T)\ \mathrm{rad}\, f'(T).$$

This is a semiinvariant of weight 2.

Pick N distinct points $p_1, \ldots, p_N \in C$ and let $f_i : H^0(L) \to k$ be evaluation at the point p_i. Now take $n = (N-1)/2$ and let $f = f_1 + \cdots + f_n$ and $f' = f_{n+1} + \cdots + f_{2n}$ and $h = f_N$.

Lemma 2.28. If E is H^0–semistable, then there exist points $p_1, \ldots, p_N \in C$ such that, for any marking $S \subset H^0(E)$,

$$(\mathrm{rad}\, f(T_{E,S}))^t\, h(T_{E,S})\ \mathrm{rad}\, f'(T_{E,S}) \neq 0,$$

where f, f', h are defined as above.

Proof. Let $n := (N-1)/2$. The function $f : H^0(L) \to k$ is the sum of the evaluation maps at the points $p_1, \ldots, p_n \in C$, and moreover

$$\mathrm{rad}\, f(T_{S,E}) \neq 0 \quad \Longleftrightarrow \quad h^0\big(E(-p_1 - \cdots - p_n)\big) = 1.$$

To see this let $V := H^0(E)$ and choose a basis S. For points $p_1, \ldots, p_n$ set

$$g = (\mathrm{ev}_{p_1}, \ldots, \mathrm{ev}_{p_n}) : V \longrightarrow W := \bigoplus_{k=1}^{n} E|_{p_k},$$

where ev_{p_k} is evaluation at p_k and $E|_{p_k}$ denotes the fiber of E at p_k. On each fiber $E|_{p_k}$ the wedge pairing $E|_{p_k} \wedge E|_{p_k} \to L|_{p_k}$ gives a nondegenerate skew form ω_k after identifying $L|_{p_k} \cong k$. Write $\omega := \omega_1 \oplus \cdots \oplus \omega_n$ for the induced skew form on W. Define a skew form on V by

$$\beta_f(u, v) := f(u \wedge v) = \sum_{k=1}^{n} \omega_k\big(u(p_k), v(p_k)\big) = \omega\big(g(u), g(v)\big).$$

Relative to the basis S, the matrix of β_f is exactly $f(T_{S,E})$.

Since ω is nondegenerate on W, we have $\ker \beta_f = \ker g$ and

$$\ker g = H^0\big(E(-p_1 - \cdots - p_n)\big),$$

the space of sections vanishing at all the p_k. Hence

$$\dim \ker \beta_f = h^0\big(E(-p_1 - \cdots - p_n)\big).$$

By Proposition 2.14 for an odd-size skew matrix A the radical vector $\operatorname{rad} A$ is nonzero precisely when $\ker A$ is one-dimensional. Therefore

$$\operatorname{rad} f(T_{S,E}) \neq 0 \quad \Longleftrightarrow \quad \dim\ker\beta_f = 1 \quad \Longleftrightarrow \quad h^0\big(E(-p_1 - \cdots - p_n)\big) = 1.$$

Moreover if these equivalent conditions hold, then the vector $\operatorname{rad} f(T_{S,E})$ spans the 1–dimensional space

$$H^0\big(E(-p_1 - \cdots - p_n)\big) = \ker\big((\mathrm{ev}_1, \ldots, \mathrm{ev}_n) : H^0(E) \to k^{2n}\big),$$

relative to the basis $S \subset H^0(E)$.

Now by repeated use of Lemma 2.27 we can find points $p_1, \ldots, p_{n-1} \in C$ such that $E(-p_1 - \cdots - p_{n-1})$ is H^0–semistable and $h^0(E(-p_1 - \cdots - p_{n-1})) = 3$.

For a general point $q \in C$ the evaluation map $\mathrm{ev}_q : H^0(E') \to E'|_q$ is surjective. Since $\dim H^0(E') = 3$ and $\dim E'|_q = 2$, the kernel is one–dimensional, namely

$$\ker(\mathrm{ev}_q) = H^0\big(E'(-q)\big).$$

Choose a general p_n. Then $H^0\big(E'(-p_n)\big)$ is a 1-dimensional space; take a generator $s \neq 0$, so $s(p_n) = 0$. The zero locus of s is finite, hence we may choose a general point p_{n+1} with $s(p_{n+1}) \neq 0$. For such p_{n+1} the space $H^0\big(E'(-p_{n+1})\big)$ is again one-dimensional; choose a generator $t \neq 0$, so $t(p_{n+1}) = 0$. Because $s(p_{n+1}) \neq 0$, the sections s and t are not proportional.

Thus we obtain sections $s, t \in H^0(E')$ with $s(p_n) = 0$, $t(p_{n+1}) = 0$, and s, t linearly independent. Moreover, by H^0-semistability, two independent global sections cannot land in a rank-1 subsheaf (otherwise its saturation would be a line subbundle with $h^0 \geq 2 > \frac{1}{2}h^0(E')$), so s and t generate a rank-2 subsheaf; in particular, at a general point the fibre is spanned by values of global sections.

These sections are necessarily linearly independent and, by H^0–semistability, generate a subsheaf of rank 2. Thus if p_n is general, the fibre at this point will be generated by global sections.

Finally we check that with respect to the N points

$$p_1, \ldots, p_{n-1}, p_n, p_{n+1}, p_1, \ldots, p_{n-1}, p_n,$$

the scalar product of the lemma is nonzero. By our choice of $h = \mathrm{ev}_{p_N} = \mathrm{ev}_{p_n}$, we have

$$(\operatorname{rad} f)^t\, h(T_{E,S})\ \operatorname{rad} f' = \beta_h(s,t) = (s \wedge t)(p_n) \in k.$$

But $s(p_n)$ and $t(p_n)$ are linearly independent in $E|_{p_n}$, so $(s \wedge t)(p_n) \neq 0$. This concludes the proof of the lemma.

□

We have shown that for both even and odd N there exist semiinvariants of positive weight which are nonzero at the Gieseker point $T_{E,S}$. Hence $T_{E,S}$ is semistable. This concludes the proof of Proposition 2.24. $\square$

2.4 Construction of the moduli space

We are now prepared to prove Theorem 2.1 For this we need to study the $GL(N)$–orbits in the affine space $\mathrm{Alt}_N(H^0(L))$ coming from vector bundles via Corollary 10.62.

By identifying $L \cong \mathcal{O}_C(D)$ for some divisor $D \in \mathrm{Div}\, C$ we can view elements $T \in \mathrm{Alt}_N(H^0(L))$ as skew–symmetric matrices with entries in the function field $k(C)$; we then observe that the Gieseker points $T_{E,S}$, as matrices over $k(C)$, have rank 2. This is because T is given by the composition $\mathcal{O}_C^{\oplus N} \to E \to L^{\oplus N}$, and passing to the generic point η of C, we see that the matrix $T : \mathcal{O}_{C,\eta}^{\oplus N} \cong k(C)^N \to L_\eta \cong k(C)$ has rank 2 over $k(C)$ since E_η is a 2-dimensional vector space over $k(C)$. Geometrically, this is saying that the wedge product $E \otimes E \to \det E$ is a nondegenerate alternating form on a 2-dimensional space.

Definition 2.29. The set of matrices $T \in \mathrm{Alt}_N(H^0(L))$ of rank ≤ 2 over $k(C)$ is a closed subvariety which we denote by $\mathrm{Alt}_{N,2}(H^0(L)) \subset \mathrm{Alt}_N(H^0(L))$

Let $x_{ij}^{(\alpha)}$, for $1 \leq i, j \leq N$ and $1 \leq \alpha \leq h^0(L)$, be coordinates in the affine space $\mathrm{Alt}_N(H^0(L))$. Then

$$\mathrm{Alt}_{N,2}(H^0(L)) \subset \mathrm{Alt}_N(H^0(L))$$

is defined by

$$\binom{N}{4} h^0(L^2)$$

equations determined by the vanishing of global sections

$$\mathrm{Pfaff}\begin{bmatrix} x_{ij} & x_{ik} & x_{il} \\ x_{jk} & x_{jl} & \\ & x_{kl} & \end{bmatrix} = x_{ij} \circ x_{kl} - x_{ik} \circ x_{jl} + x_{il} \circ x_{jk} \in H^0(L^2),$$

for $1 \leq i < j < k < l \leq N$, and where

$$x_{ij} := (x_{ij}^{(\alpha)})_{1 \leq \alpha \leq h^0(L)} \in H^0(L),$$

and $\circ : H^0(L) \times H^0(L) \to H^0(L^2)$ is the natural multiplication map.

If $T \neq 0$, then the rank condition is equivalent to saying that the image E of the sheaf homomorphism

$$\langle T \rangle : \mathcal{O}_C^{\oplus N} \longrightarrow L^{\oplus N}$$

is a rank 2 vector bundle.

Proposition 2.30 (Smoothness at Gieseker points)**.** Let E be a rank 2 vector bundle with $\det E = L$ and $H^1(E) = 0$. Then:

(i) $\mathrm{Alt}_{N,2}(H^0(L))$ is smooth at each Gieseker point $T_{E,S}$.

(ii) If E is simple, then the quotient of the tangent space to $\mathrm{Alt}_{N,2}(H^0(L))$ at a Gieseker point $T_{E,S}$ by the Lie space $\mathfrak{gl}(N)$ is isomorphic to $H^1(\mathfrak{sl}\, E)$:

$$T_{T_{E,S}} \mathrm{Alt}_{N,2}(H^0(L))/\mathfrak{gl}(N) \;\cong\; H^1(\mathfrak{sl}\, E).$$

Given vector spaces U, V, the space $\mathrm{Hom}(U, V)$ of linear maps $f : U \to V$ can be viewed as an affine space. For each natural number r, there is then a subset $\mathrm{Hom}_r(U, V) \subset \mathrm{Hom}(U, V)$ consisting of linear maps of rank $\leq r$, defined as a closed subvariety by the vanishing of all the $(r + 1) \times (r + 1)$ minors.

Lemma 2.31 (Tangent space to rank varieties)**.** Suppose that $f \in \mathrm{Hom}(U, V)$ has rank exactly equal to r. Then the tangent space to $\mathrm{Hom}_r(U, V)$ at f is equal to

$$S_f := \{\, h \mid h(\ker f) \subset \operatorname{im} f \,\} \subset \mathrm{Hom}(U, V).$$

Proof. Choose bases of U and V so that the matrix representing $f : U \to V$ is in canonical form

$$\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0).$$

If $h : U \to V$ is another linear map, then $f + \varepsilon h$, where $\varepsilon^2 = 0$, $\varepsilon \neq 0$, is represented by a matrix

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Since $\varepsilon^2 = 0$, the only possible nonzero $(r + 1) \times (r + 1)$ minors in this matrix are the entries of D (concatenated with the block I_r). Hence the condition that all $(r + 1) \times (r + 1)$ minors vanish is equivalent to $D = 0$. But this is the case if and only if $h(\ker f) \subset \operatorname{im} f$. □

In the tangent vector space S_f there are two vector subspaces to consider. One consists of h satisfying $h(\ker f) = 0$, which is equivalent to factoring through an element of $\mathrm{Hom}(\operatorname{im} f, V)$. The other consists of h satisfying $h(U) \subset \operatorname{im} f$, or, in other words, h comes from an element of $\mathrm{Hom}(U, \operatorname{im} f)$. The intersection consists of endomorphisms of $\operatorname{im} f$, and in this way we obtain an exact sequence of vector spaces:

$$0 \longrightarrow \mathrm{End}(\operatorname{im} f) \longrightarrow \mathrm{Hom}(\operatorname{im} f, V) \oplus \mathrm{Hom}(U, \operatorname{im} f) \longrightarrow S_f \longrightarrow 0. \tag{10.19}$$

Now suppose that $V = U^\vee$, and consider the subset $\mathrm{Hom}^-(U, U^\vee)$ of skew–symmetric linear maps: those $f : U \to U^\vee$ that is, equal to minus their transpose (dual) map. Suppose that

$f \in \mathrm{Hom}^-(U, U^\vee)$ has rank $\leq r$. This means that all its $(r+2) \times (r+2)$ Pfaffian minors vanish, and these Pfaffians define a closed subvariety $\mathrm{Hom}_r^-(U, U^\vee) \subset \mathrm{Hom}^-(U, U^\vee)$. The same argument as above gives the following lemma:

Lemma 2.32 (Tangent space to skew–symmetric rank varieties)**.** Suppose that $f : U \to U^\vee$ is skew–symmetric and has rank equal to r. Then the tangent space to $\mathrm{Hom}_r^-(U, U^\vee)$ at f is equal to

$$S_f^- := \{ h \mid h(\ker f) \subset \operatorname{im} f \} \subset \mathrm{Hom}^-(U, U^\vee).$$

The two subspaces $\{ h \mid h(\ker f) = 0 \}$ and $\{ h \mid h(U) \subset \operatorname{im} f \}$, when the maps f, h are skew–symmetric, are exchanged by taking the transpose; moreover, the intersection

$$\{ h \mid h(\ker f) = 0 \} \ \cap \ \{ h \mid h(U) \subset \operatorname{im} f \} \ \cap \ \mathrm{Hom}^-(U, U^\vee)$$

is exactly the space of endomorphisms of $\operatorname{im} f$ which preserve a skew– symmetric form. We will denote this space by $\mathrm{End}^-(\operatorname{im} f)$. From (10.19) we obtain an exact sequence:

$$0 \longrightarrow \mathrm{End}^-(\operatorname{im} f) \longrightarrow \mathrm{Hom}(U, \operatorname{im} f) \longrightarrow S_f^- \longrightarrow 0. \tag{10.20}$$

Remark 2.33. When $r = 2$, the space of endomorphisms preserving a skew-symmetric form is isomorphic to the special linear Lie algebra $\mathfrak{sl}_2(k)$.

Suppose $g \in \mathrm{GL}(V)$ preserves a nondegenerate skew-symmetric form $\omega : V \times V \to k$. Then $g \in \mathrm{Sp}(V, \omega) = \{g \in \mathrm{GL}(V) \mid g^t J g = J\}$ where J is the matrix of ω in some basis.

Taking the derivative at the identity, we find that the Lie algebra

$$\mathfrak{sp}(V, \omega) = \{X \in \mathrm{End}(V) \mid X^t J + JX = 0\}$$

When $\dim V = 2$, we can take $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the condition becomes $X^t J + JX = 0$ which is equivalent to $\mathrm{tr}(X) = 0$. Thus $\mathfrak{sp}(2, k) \cong \mathfrak{sl}_2(k)$.

We will need the following functorial characterization of smoothness.

Definition 2.34. An Artin ring over k is a finitely generated ring containing k which satisfies the following equivalent conditions.

1. R is finite-dimensional as a vector space over k.
2. R has only finitely many maximal ideals, and these are all nilpotent.

It turns out that every Artin ring is a finite sum of local Artin rings,

Lemma 2.35. For a variety X the following properties are equivalent.

1. X is nonsingular.

2. For any surjective homomorphism of Artin local rings $f : A' \to A$ the map
$$X(A') \to X(A)$$
is surjective.

Proof. Come back to this later. $\square$

Proof of Proposition 2.30.

(ii) Let E be a simple rank 2 vector bundle with Gieseker point $T = T_{E,S} \in \mathrm{Alt}_{N,2}\big(H^0(L)\big)$. We apply Lemma 2.32 to the map
$$\langle T\rangle : \mathcal{O}_C^{\oplus N} \longrightarrow L^{\oplus N}$$
on stalks at the generic point, whose image is E. This determines a subbundle
$$S_T^- := \{\, h \mid h(\ker\langle T\rangle) \subset E \,\} \ \subset\ \mathrm{Hom}^-\big(\mathcal{O}_C^{\oplus N}, L^{\oplus N}\big) \ \cong\ L^{\oplus N(N-1)/2}.$$

The tangent space to $\mathrm{Alt}_{N,2}(L_{\text{gen}})$ at $\langle T\rangle_{\text{gen}}$ is the space of rational sections of S_T^-, and that of $\mathrm{Alt}_{N,2}(H^0(L))$ is $H^0(S_T^-)$. Corresponding to (10.20), we have an exact sequence of vector bundles on C:
$$0 \longrightarrow \mathfrak{sl}(E) \longrightarrow \mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E) \longrightarrow S_T^- \longrightarrow 0.$$

But $\mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E) \cong E^{\oplus N}$ while $H^1(E) = 0$ by hypothesis, and so taking global sections gives an exact sequence
$$0 \longrightarrow H^0(\mathfrak{sl}E) \longrightarrow \mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E) \longrightarrow H^0(S_T^-) \longrightarrow H^1(\mathfrak{sl}E) \longrightarrow 0.$$

The term $\mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E)$ is the tangent space to the $GL(N)$–orbit of the Gieseker point $T_{E,S}$ and identifies with the Lie algebra $\mathfrak{gl}(N)$.

This is because the $GL(N)$–action on the space of marked matrices is given by changing the ordered basis S; if $g \in GL(N)$ then $S' = g \cdot S$ and $T_{E,S'} = g \cdot T_{E,S}$. Thus we get the orbit map
$$\alpha : GL(N) \longrightarrow \mathrm{Alt}_{N,2}(H^0(L)), \qquad \alpha(g) = g \cdot T_{E,S},$$
and the orbit $\mathcal{O} = GL(N) \cdot T_{E,S}$ equals $\mathrm{Im}(\alpha)$. The tangent space to the orbit at $T_{E,S}$ is the image of the differential at the identity,
$$T_{T_{E,S}}\mathcal{O} = \mathrm{Im}\left(d\alpha_{\text{id}}\right) \subset T_{T_{E,S}}\,\mathrm{Alt}_{N,2}(H^0(L)).$$

Infinitesimally, an element $A \in \mathfrak{gl}(N) = T_{\mathrm{id}}GL(N)$ acts by $g_\varepsilon = I + \varepsilon A$ ($\varepsilon^2 = 0$) sending the basis $S = (s_1, \dots, s_N)$ to

$$S_\varepsilon = \big(s_1 + \varepsilon \sum_j A_{j1} s_j, \dots, s_N + \varepsilon \sum_j A_{jN} s_j\big).$$

Hence the first–order variation of the ith basis vector is

$$\delta s_i = \sum_j A_{ji} s_j,$$

and the collection $(\delta s_1, \dots, \delta s_N)$ defines an element of $H^0(E)^{\oplus N} \cong \mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E)$. In this way the differential

$$d\alpha_{\mathrm{id}} : \mathfrak{gl}(N) \longrightarrow T_{T_{E,S}} \mathrm{Alt}_{N,2}(H^0(L))$$

factors through the natural map

$$\mathfrak{gl}(N) \longrightarrow \mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E) \longrightarrow H^0(S_T^-) = T_{T_{E,S}} \mathrm{Alt}_{N,2}(H^0(L)),$$

so that $T_{T_{E,S}}\mathcal{O}$ is exactly the image of $\mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E)$ inside $H^0(S_T^-)$ coming from the infinitesimal change of basis.

Putting this together with the exact sequence (10.20) on global sections, we obtain the long exact sequence

$$0 \longrightarrow H^0(\mathfrak{sl}E) \longrightarrow \mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E) \longrightarrow H^0(S_T^-) \longrightarrow H^1(\mathfrak{sl}E) \longrightarrow 0.$$

Finally, if E is simple then $H^0(\mathfrak{sl}E) = 0$. Identifying $H^0(S_T^-) = T_{T_{E,S}} \mathrm{Alt}_{N,2}(H^0(L))$ and observing that the image of $\mathfrak{gl}(N)$ in $H^0(S_T^-)$ equals the image of $\mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E)$ (the infinitesimal orbit), we conclude

$$T_{T_{E,S}} \mathrm{Alt}_{N,2}(H^0(L))/T_{T_{E,S}}\mathcal{O} \cong H^1(\mathfrak{sl}E).$$

as desired.

(i) We use the functorial characterization of smoothness as in the lemma above. Let $A' \to A$ be a surjective homomorphism of Artin local rings with maximal ideals $\mathfrak{n}', \mathfrak{n}$. To show that $\mathrm{Alt}_{N,2}(H^0(L))$ is smooth at the Gieseker matrix T, it is enough to show any deformation of T over an Artinian A lifts along a small extension $A' \twoheadrightarrow A$. We may assume $\dim_k \ker f = 1$ so let ε span $\ker f$.

Let T be an A-point of $\mathrm{Alt}_{N,2}(H^0(L))$ whose reduction modulo $\mathfrak{n}$ is the Gieseker point $T_{E,S}$. The matrix T can be expressed as

$$T = \begin{bmatrix} s_1 \wedge s_2 & s_1 \wedge s_3 & \cdots & s_1 \wedge s_N \\ s_2 \wedge s_3 & \cdots & & \vdots \\ \vdots & & \ddots & \\ & & & s_{N-1} \wedge s_N \end{bmatrix}$$

for some rational sections $s_i \in E_{\text{gen}} \otimes_k A$. Since this is an A-valued point of $\mathrm{Alt}_{N,2}(H^0(L))$, the entries $a_{ij} := s_i \wedge s_j$ belong to $H^0(L \otimes_k A)$. Since f is surjective, we can lift each s_i to an element $s'_i \in E_{\text{gen}} \otimes_k A'$ and each a_{ij} to an element $a'_{ij} \in H^0(L \otimes_k A')$ since tensoring by a finite dimensional vector space is exact, preserving the skew–symmetry since if the lift for a_{ij} is chosen arbitrarily then we can set $a'_{ji} = -a'_{ij}$.

The matrix

$$\left(s'_i \wedge s'_j \; - \; a'_{ij}\right)_{1 \le i,j \le N} \tag{10.21}$$

measures the failure of the lifts a'_{ij} to equal the wedge products of the lifts s'_i. It determines a rational section of $\mathrm{Hom}^-(\mathcal{O}_C^{\oplus N}, L^{\oplus N}) \otimes_k A'$, and since this section vanishes when we apply f, every entry actually lies in the subspace $L \otimes_k \ker f \; = \; L \otimes_k k\varepsilon$. In other words, it is a rational section of

$$\mathrm{Hom}^-(\mathcal{O}_C^{\oplus N}, L^{\oplus N}) \otimes_k \ker f \;\; = \;\; \mathrm{Hom}^-(\mathcal{O}_C^{\oplus N}, L^{\oplus N})_{\varepsilon}.$$

We want this section to be everywhere regular so that the lifts a'_{ij} equal the wedge products $s'_i \wedge s'_j$. We arrange this as follows.

Its principal part is $(s'_i \wedge s'_j)_{1 \le i,j \le N}$ and is contained in $S_T^- \otimes_k A'$. It follows that at each point $p \in C$ this matrix determines a principal part in the vector bundle $S_T^- \otimes_k \ker f$. One checks that $H^1(S_T^-) = 0$ from the exact sequence

$$0 \longrightarrow \mathfrak{sl}(E) \longrightarrow \mathrm{Hom}(\mathcal{O}_C^{\oplus N}, E) \longrightarrow S_T^- \longrightarrow 0$$

and the corresponding long exact sequence in cohomology, together with the fact that $\mathcal{H}om(\mathcal{O}_C^{\oplus N}, E) \cong E^{\oplus N}$ has vanishing H^1 using that $H^1(E) = 0$ by hypothesis.

Therefore, these principal parts come from a global rational section. In other words, there exist

$$s''_1, \dots, s''_N \in E_{\text{gen}}$$

such that (10.21) is everywhere the principal part of

$$\left((\bar{s}_i + s''_i\varepsilon) \wedge (\bar{s}_j + s''_j\varepsilon) \; - \; (\bar{s}_i \wedge \bar{s}_j)\right)_{1 \le i,j \le N},$$

where $\bar{s}_i$ is the reduction of s_i modulo $\mathfrak{n}$. Hence, if we set

$$T' \;\; = \;\; \left((s'_i + s''_i\varepsilon) \wedge (s'_j + s''_j\varepsilon)\right)_{1 \le i,j \le N},$$

then T' has entries in $H^0(L \otimes_k A')$, in particular everywhere regular, and T' is an A'-valued point of $\mathrm{Alt}_{N,2}(H^0(L))$ lifting T.

□

Proposition 2.36 (Vanishing of H^1)**.** If E is a semistable vector bundle with $\mu(E) > 2g - 2$, or if E is stable and $\mu(E) \ge 2g - 2$, then $H^1(E) = 0$.

Proof. By Serre duality, it suffices to show that there is no nonzero homomorphism

$$E \longrightarrow \Omega_C,$$

where Ω_C is the canonical line bundle on C. If $\phi : E \to \Omega_C$ is nonzero, then $\mathrm{im}(\phi)$ is a nonzero subsheaf of Ω_C.

Since Ω_C is a line bundle, the image is a line subbundle, so

$$\mathrm{im}(\phi) \cong \Omega_C(-D) \quad \text{for some effective divisor } D \geq 0.$$

Thus ϕ factors as

$$E \twoheadrightarrow Q \hookrightarrow \Omega_C,$$

where $Q := \mathrm{im}(\phi)$ is a quotient line bundle of E. Moreover

$$\deg Q = \deg\big(\Omega_C(-D)\big) = 2g - 2 - \deg D \leq 2g - 2,$$

which contradicts the hypothesis $\mu(E) > 2g - 2$. Hence no nonzero map ϕ exists. □

Proposition 2.37 (Generation by global sections)**.** If E is semistable and $\mu(E) > 2g - 1$, or if E is stable and $\mu(E) \geq 2g - 1$, then E is generated by global sections.

Proof. By the previous proposition, $H^1(E(-p)) = 0$ for every point $p \in C$. It follows that, for every positive divisor $D \geq 0$, the restricted principal part map

$$H^0(E(D - p)) \longrightarrow E(D - p)/E(-p)$$

is surjective. In particular, taking $D = p$ shows that the evaluation map

$$H^0(E) \longrightarrow E/E(-p)$$

is surjective at every point $p \in C$. □

2.5 Proof of the main theorem

We now take our fixed line bundle L to have degree $\geq 4g - 1$, and we consider the action of $\mathrm{GL}(N)$ on $\mathrm{Alt}_{N,2}\big(H^0(L)\big)$.

Suppose that $E \in SU_C(2, L)$. Then by Proposition 2.36 we have $H^1(E) = 0$, so the orbit $GL(N) \cdot T_{E,S}$ of a Gieseker point depends only on E and not on the marking S. By Proposition 2.37, moreover, E is generated by global sections and is therefore recovered up to isomorphism from its Gieseker points by Proposition 2.17.

And by Propositions **??** and 2.24, the Gieseker points of E are semistable for the action of $GL(N)$.

Conversely, suppose that $T \in \mathrm{Alt}_{N,2}(H^0(L))$ is a semistable point for the $GL(N)$-action. The columns of T are vectors in $H^0(L)^{\oplus N}$, and as in Proposition 9.63 in the line bundle case we can show the following.

Lemma 2.38 (Linear independence of columns)**.** If $T \in \mathrm{Alt}_{N,2}(H^0(L))$ is semistable, then the N columns of T are linearly independent vectors in $H^0(L)^{\oplus N}$ over k.

Proof. Suppose not. Then by a suitable change of basis (that is, by moving within the $GL(N)$-orbit) we can assume that the first row and column of T are zero:

$$T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

Consider the action of the 1-parameter subgroup

$$t \longmapsto g(t) := \begin{pmatrix} t^{-N+1} & 0 & \cdots & 0 \\ 0 & t & & \\ \vdots & & \ddots & \\ 0 & & & t \end{pmatrix} \in \mathrm{SL}(N).$$

Then

$$g(t)Tg(t)^t = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & t^2* & \cdots & t^2* \\ \vdots & \vdots & \ddots & \vdots \\ 0 & t^2* & \cdots & t^2* \end{pmatrix}.$$

Letting $t \to 0$ shows that the origin lies in the closure of the $SL(N)$-orbit of T. Therefore T is unstable. $\square$

Proposition 2.39 (Properties of bundles from semistable Gieseker points)**.** Suppose that $\deg L \geq 4g-2$ and that $T \in \mathrm{Alt}_{N,2}(H^0(L))$ is semistable for the action of $GL(N)$. Then $E := \mathrm{Im}\langle T\rangle \subset L^{\oplus N}$ satisfies:

(i) $H^1(E) = 0$;

(ii) $\det E \cong L$;

(iii) E is semistable.

Proof.

1. Let $V \subset H^0(E)$ be the space of global sections coming from the surjection $\mathcal{O}_C^{\oplus N} \to E$. Lemma 10.80 implies that $\dim V = N$, and, in particular, that $h^0(E) \geq N$. By Serre duality, the vanishing of $H^1(E)$ implies that there is a nonzero homomorphism $f : E \to \Omega_C$, and this induces a linear map
$$V \longrightarrow H^0(\Omega_C).$$
Since $\dim H^0(\Omega_C) = g$, the kernel of this map then has dimension at least
$$N - g \geq g,$$
and so, letting $M := \ker(f) \subset E$, we have
$$\dim\bigl(H^0(M) \cap V\bigr) \;\geq\; \frac{N}{2}.$$
Recall that we showed if for some line subbundle $M \subset E$,
$$\dim\bigl(H^0(M) \cap V\bigr) > \frac{N}{2},$$
then the Gieseker matrix T is unstable under the $GL(N)$-action. Here S is the chosen basis giving the map $\mathcal{O}_C^{\oplus N} \to E$, and V is the image of $H^0(\mathcal{O}_C^{\oplus N})$ in $H^0(E)$, and $M = \ker(f)$ is a line subbundle of E.

   But we assumed at the beginning that T is semistable under $GL(N)$. Therefore it is impossible that $H^1(E) \neq 0$.

2. Consider the bilinear pairing
$$\mathcal{O}_C^{\oplus N} \times \mathcal{O}_C^{\oplus N} \;\longrightarrow\; L, \qquad (u, v) \longmapsto u^t T v.$$
This is skew–symmetric and vanishes if u or $v \in \ker\langle T \rangle$, and hence defines a sheaf homomorphism
$$\wedge^2 E \;\longrightarrow\; L.$$
Now recall there are only maps of line bundles from lower to higher degree. Also, note that any nonzero map of line bundles $A \to B$ of the same degree is an isomorphism. This is because
$$0 \to A \to B \to A/B$$
where A/B is torsion sheaf whose degree is equal to the sum of the lengths at the support points, so if $\deg A = \deg B$ then $A/B = 0$ and the map is an isomorphism.

   Thus to show that it is an isomorphism, it is enough to check that $\deg L \leq \deg E$.
$$\deg L - 2g + 2 \;=\; N \;\leq\; h^0(E),$$
while by part (i) we have $H^1(E) = 0$, so that
$$h^0(E) = \deg E - 2g + 2,$$
and we are done.

3. By construction T is a Gieseker point of the vector bundle E, and so semistability follows from Propositions 2.23 and 2.24. $\square$

Lemma 2.40 (Finite stabilizers of stable bundles)**.** Suppose that $H^1(E) = 0$, that E is generated by global sections, and that E is simple. Given a marking S and a matrix $X \in GL(N)$,

$$X\, T_{E,S}\, X^t = T_{E,S} \qquad \text{if and only if} \qquad X = \pm I_N.$$

Proof. The hypothesis $X\, T_{E,S}\, X^t = T_{E,S}$ is equivalent to the commutativity of the diagram

$$\begin{tikzcd} \mathcal{O}_C^{\oplus N} \arrow[r, "\langle T_{E,S}\rangle"] \arrow[d, "X^t"'] & L^{\oplus N} \arrow[d, "X"] \\ \mathcal{O}_C^{\oplus N} \arrow[r, "\langle T_{E,S}\rangle"] & L^{\oplus N}. \end{tikzcd}$$

This diagram determines an endomorphism ϕ of E, and the assumption that E is simple implies that $\phi = c\,\mathrm{id}_E$ for some $c \in k$. But then $X = X^t = c \cdot I_N$, and in particular $c^2 = 1$. Thus $X = \pm I_N$. $\square$

Proof of Theorem **??**. To construct the moduli space

$$\mathrm{Alt}^{ss}_{N,2}(H^0(L))//GL(N)$$

as a projective GIT quotient we consider the graded semiinvariant ring with respect to the character $\chi : GL(N) \to k^*$, choosing $\chi = \det$.

$$R = \bigoplus_{m \geq 0} H^0(\mathrm{Alt}_{N,2}(H^0(L)), \mathcal{O}(m))^{GL(N),\chi^m}.$$

However, this argument is a little subtle since the ring R might not be an integral domain, because strictly semistable points create multiple components. So one needs a result guaranteeing that Proj still gives an algebraic variety (or a disjoint union of such).

Lemma 2.41. If X^{ss} is smooth, then the Proj quotient exists as a disjoint union of varieties. We do not need R to be integrally closed or irreducible.

Proof. Follows from the general theory of GIT quotients. $\square$

The smoothness follows from Proposition 2.39, which guarantees the condition $H^1(E) = 0$, together with Proposition 2.30.

Now consider the open set

$$\mathrm{Alt}^{\mathrm{s}}_{N,2}(H^0(L))/GL(N)$$

of stable orbits. First note that, for each stable Gieseker point T, the vector bundle $E = \mathrm{im}\langle T\rangle$ is stable. This follows from Corollary 2.22 and the proof of Proposition 2.23.

Conversely, if E is stable as a vector bundle, then it is simple, and so by Lemma 2.40 its Gieseker points T have a finite stabiliser and hence are stable for the $GL(N)$–action. We therefore arrive at a bijection:

$$SU_C(2, L) \xrightarrow{\sim} \mathrm{Alt}^{\mathrm{s}}_{N,2}(H^0(L))/GL(N).$$

By Lemma 2.40, moreover under the action

$$GL(N)/\{\pm I_N\} \curvearrowright \mathrm{Alt}^{\mathrm{s}}_{N,2}(H^0(L)),$$

all orbits are free and closed. The stabilizer of any point is exactly $\{\pm I_N\}$, and so dividing by this subgroup makes the action free. Orbits are closed because stable points have closed orbits in GIT.

Proposition 2.42. If an affine variety X is nonsingular at every point of a free closed orbit $G \cdot x$, then the affine quotient $X//G$ is nonsingular at the image point, with dimension = $\dim X - \dim G$.

Proof. Follows from the general theory of GIT quotients. $\square$

Applying the above proposition to the open set $\mathrm{Alt}^{\mathrm{s}}_{N,2}(H^0(L))$ with the free action of

$$G' = GL(N)/\{\pm I_N\}$$

, we see that the quotient

$$\mathrm{Alt}^{\mathrm{s}}_{N,2}(H^0(L))/G'$$

is nonsingular since $\mathrm{Alt}^{\mathrm{s}}_{N,2}(H^0(L))$ is nonsingular by Proposition 2.30. But note that

$$\mathrm{Alt}^{\mathrm{s}}_{N,2}(H^0(L))/G' \cong \mathrm{Alt}^{\mathrm{s}}_{N,2}(H^0(L))/GL(N)$$

since $\{\pm I_N\}$ acts trivially. Thus $\mathrm{SU}_C(2, L)$ is nonsingular.

Moreover, when E is stable,

$$\dim H^1(\mathfrak{sl}(E)) \;=\; 3g - 3$$

This follows from Riemann-Roch for vector bundles and the fact that $H^0(\mathfrak{sl}(E)) = 0$ for stable E. More generally, if E is simple, then the only endomorphisms of E are scalars, so $H^0(\mathfrak{sl}(E)) = 0$ and $\dim H^0(\mathrm{End}(E)) = 1$, and in particular every stable bundle is simple.

This proves parts (i) and (ii). For part (iii) we note that when $\deg L$ is odd, stability and semistability of E are equivalent. $\square$

3 The Picard functor

Definition 3.1. A functor $\mathcal{F} : (\text{Schemes}/k)^{op} \to \text{Sets}$ is representable if there exists a scheme M over k and a natural isomorphism of functors

$$\mathcal{F} \cong \operatorname{Hom}_k(-, M)$$

The scheme M is called a **fine moduli space** for the functor $\mathcal{F}$.

For many moduli problems, a fine moduli space does not exist. Mumford introduced the more relaxed notion of a best approximation to a moduli functor, and a more refined version of this called the coarse moduli space.

Definition 3.2. Given a functor $\mathcal{F} : (\text{Schemes}/k)^{op} \to \text{Sets}$, a scheme M over k is called a **best approximation** to $\mathcal{F}$ if there exists a natural transformation of functors

$$\phi : \mathcal{F} \to \operatorname{Hom}_k(-, M)$$

such that for any scheme N over k and any natural transformation $\psi : \mathcal{F} \to \operatorname{Hom}_k(-, N)$, there exists a unique morphism $f : M \to N$ such that $\psi = f \circ \phi$.

If in addition for any algebraically closed field extension K/k, the map

$$\phi(K) : \mathcal{F}(K) \to \operatorname{Hom}_k(\operatorname{Spec} K, M)$$

is a bijection, then M is called a **coarse moduli space** for the functor $\mathcal{F}$.

It is clear that fine implies coarse, and best approximation implies unique.

Remark 3.3. The second condition is meant to test bijectivity on the level of geometric points. If you only checked k-points for a non-algebraically-closed k, you could get false failures of surjectivity just because some polynomial has no k-rational root, not because the moduli problem is wrong. To show a scheme is not a coarse moduli space, it suffices to exhibit one algebraically closed extension K where bijection fails. In practice you take $K = \bar{k}$.

Example 3.4. Take $G = \mathbf{G}_m$ acting on $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ by $t \cdot (x, y) = (tx, t^{-1}y)$. Then

$$k[x, y]^G = k[xy], \quad \text{so} \quad X//G \cong \operatorname{Spec} k[xy] \cong \mathbb{A}^1,$$

with quotient map $(x, y) \mapsto xy$. The failure of this map to be a coarse moduli space is non-injectivity of

$$X(K)/G(K) \to (X//G)(K), \qquad (x, y) \mapsto xy$$

because many distinct orbits land on the same invariant value $xy = 0$. Note that upon choosing a linearization of $\mathcal{O}_X$ (in particular the trivial character $\chi = 1$), the semistable locus is $X^{ss} = D(xy)$

and since all orbits in $D(xy)$ are closed and free, $X^{ss} = X^s$. Thus the stable locus is $D(xy)$, and the map $X^s/G \to X^s//G$ is given by $(x, y) \mapsto xy \neq 0$, which is now a bijection on K-points for any algebraically closed K/k. Thus the stable quotient is a coarse moduli space for the stable locus.

Example 3.5. Suppose that a reductive group G acts on an affine variety $X = \operatorname{Spec} R$. Then the functor of points

$$\mathcal{F} : (\text{Schemes}/k)^{op} \to \text{Sets}, \quad S \mapsto X(S)/G(S)$$

is best approximated by the affine GIT quotient $X \to X//G$ induced by the inclusion of invariant rings $R^G \hookrightarrow R$. Moreover, the stable quotient X^s/G is a coarse moduli space for the quotient functor X^s/G.

Remark 3.6. The coarse quotient X^s/G is a fine moduli space if and only if the G-action on X^s is free. In general, finite stabilizers give a Deligne–Mumford stack $[X^s/G]$ whose coarse moduli space is X^s/G; the obstruction to the existence of a universal family is exactly the nontrivial stabilizer groups.

3.1 Cohomology modules and direct images

Let A be a finitely generated algebra over k. We shall consider the extension $A \otimes_k k(C)$ of A, extending coefficients from k to the function field $k(C)$. Also, we denote by $A \otimes_k \mathcal{O}_C$ the elementary sheaf on C defined by the ring extensions

$$U \longmapsto A \otimes_k \mathcal{O}_C(U).$$

The pair of topological space C and the elementary sheaf $A \otimes_k \mathcal{O}_C$ we denote by C_A.

Definition 3.7. A **vector bundle on** C_A is an elementary sheaf $\mathcal{E}$ of $A \otimes_k \mathcal{O}_C$-modules satisfying the following conditions:

(i) The total set (denoted $\mathcal{E}_{\text{gen}}$) is a locally free $A \otimes_k k(C)$-module.

(ii) If $U \subset C$ is an affine open set, then $\mathcal{E}(U)$ is a locally free $A \otimes_k \mathcal{O}_C(U)$-module.

In the case $A = k$, of course, $\mathcal{E}$ is nothing but a vector bundle on the curve C. In what follows we will only consider A-modules of finite rank. Since the rank of a locally free module is locally constant, this number depends only on the connected component of $\operatorname{Spec} A$.

A vector bundle $\mathcal{E}$ on C_A associates to each point $t \in \operatorname{Spm} A$ (corresponding to a maximal ideal $\mathfrak{m} \subset A$) a vector bundle $\mathcal{E}_t$ on C (by pull-back via $\operatorname{Spm} A/\mathfrak{m} \hookrightarrow \operatorname{Spm} A$), and we can observe that this correspondence does not change if $\mathcal{E}$ is replaced with $\mathcal{E} \otimes_A M$ for any invertible A-module M.

Definition 3.8.

(i) Two vector bundles $\mathcal{E}, \mathcal{E}'$ on C_A are **equivalent** if

$$\mathcal{E}' \cong \mathcal{E} \otimes_A M$$

for some invertible A-module M.

(ii) By an **algebraic family of vector bundles on** C **parametrised by** $\operatorname{Spm} A$ we mean an equivalence class (in the sense of (i)) of vector bundles on C_A.

The set of families of line bundles parametrised by $\operatorname{Spm} A$ becomes a group under a tensor product, and this group is just

$$\operatorname{Pic} C_A / \operatorname{Pic} A.$$

Furthermore, given a ring homomorphism $f : A \to A'$, the pullback of a family via $\operatorname{Spm} A' \to \operatorname{Spm} A$ is well defined (if $\mathcal{E}$ and $\mathcal{E}'$ are equivalent, then so are $\mathcal{E} \otimes_A A'$ and $\mathcal{E}' \otimes_A A'$), and the pullback of families of line bundles is a group homomorphism

$$\otimes f : \operatorname{Pic} C_A / \operatorname{Pic} A \longrightarrow \operatorname{Pic} C_{A'} / \operatorname{Pic} A', \qquad \mathcal{L} \longmapsto \mathcal{L} \otimes_A A'.$$

If $g : A' \to A''$ is another ring homomorphism, then this operation satisfies

$$(\otimes g)(\otimes f) = \otimes(gf).$$

Definition 3.9. The covariant functor

$$\operatorname{Pic}_C : \left\{ \begin{array}{c} \text{finitely generated} \\ \text{rings over } k \end{array} \right\} \longrightarrow \{\text{groups}\}$$

which assigns

$$A \longmapsto \operatorname{Pic} C_A / \operatorname{Pic} A$$

is called the **Picard functor** for the curve C.

Given a family of line bundles $\mathcal{L} \in \operatorname{Pic} C_A / \operatorname{Pic} A$, the degree of $\mathcal{L}|_{C \times t}$ is constant on connected components of $\operatorname{Spm} A$ (Corollary 11.20). We will denote by $\operatorname{Pic}^d_C \subset \operatorname{Pic}_C$ the subfunctor which assigns families of line bundles of degree d.

3.2 Construction of the Jacobian

Fix a curve C of genus g and an integer $d \in \mathbb{Z}$. We are going to construct in this section a g-dimensional nonsingular projective variety, the Jacobian of C, whose underlying set is $\operatorname{Pic}^d C$, the set of isomorphism classes of line bundles on C of degree d.

(a) Some preliminaries

We will assume throughout this section that $d \geq 2g$, and we fix a line bundle $L \in \operatorname{Pic}^{2d} C$. We note that every line bundle $\xi \in \operatorname{Pic}^d C$ has the following properties:

(i) $H^1(\xi) = 0$.

(ii) ξ is generated by global sections.

(iii) $\dim H^0(\xi) = d + 1 - g =: N > g$.

We set $\widehat{\xi} := L \otimes \xi^{-1} \in \operatorname{Pic}^d C$ and note that $\widehat{\xi}$ also has all of the properties (i)–(iii). The key tool in the algebraic construction of the Jacobian is the multiplication map

$$H^0(\xi) \times H^0(\widehat{\xi}) \longrightarrow H^0(L), \qquad (s,t) \longmapsto st.$$

Definition 3.10. Given a line bundle $\xi \in \operatorname{Pic}^d C$, a pair (S,T) consisting of a basis $S = \{s_1, \dots, s_N\}$ of $H^0(\xi)$ and a basis $T = \{t_1, \dots, t_N\}$ of $H^0(\widehat{\xi})$ is called a **double marking** of ξ.

Given a line bundle $\xi \in \operatorname{Pic}^d C$ and a double marking (S,T), we introduce the following $N \times N$ matrix with entries in $H^0(L)$:

$$\Psi(\xi, S, T) := \begin{pmatrix} s_1 t_1 & \cdots & s_1 t_N \\ \vdots & \ddots & \vdots \\ s_N t_1 & \cdots & s_N t_N \end{pmatrix}.$$

If we fix a rational section of L, then $\Psi(\xi, S, T)$ can be viewed as a matrix of rank 1 over the function field $k(C)$.

Definition 3.11. We denote by $\operatorname{Mat}_N\big(H^0(L)\big)$ the set of $N \times N$ matrices with entries in $H^0(L)$. The subset of matrices of rank 1 over $k(C)$, or equivalently, those for which all 2×2 minors vanish, is denoted by

$$\operatorname{Mat}_{N,1}\big(H^0(L)\big).$$

Proposition 3.12. Given a matrix $\Psi \in \operatorname{Mat}_{N,1}\big(H^0(L)\big)$, the following two conditions are equivalent.

(1) The N rows and the N columns of Ψ are linearly independent over k.

(2) $\Psi = \Psi(\xi, S, T)$ for some $\xi \in \operatorname{Pic}^d C$ and some double marking (S,T).

Moreover, the line bundle ξ is the image $\xi \subset L^{\oplus N}$ of the sheaf homomorphism determined by Ψ,

$$\langle \Psi \rangle : \mathcal{O}_C^{\oplus N} \longrightarrow L^{\oplus N}.$$

Proof. (2) $\Rightarrow$ (1) is clear, so we will show (1) $\Rightarrow$ (2).

Given a matrix
$$\Psi = (\psi_{ij}) \in \mathrm{Mat}_{N,1}(H^0(L)).$$
This means:

1. Each entry ψ_{ij} is a global section of L.

2. After restricting to the generic point η of C, $\psi_{ij,\eta} \in L_\eta \cong k(C)$, so $\Psi_\eta \in \mathrm{Mat}_N(k(C))$ and $\mathrm{rank}_{k(C)}(\Psi_\eta) = 1$.

Separately, hypothesis (1) of the proposition says the N rows and N columns are k-linearly independent as vectors in $H^0(L)^{\oplus N}$. That is a condition in the k-vector space of global sections, not over $k(C)$.

Step 1. Construct ξ as an image sheaf, and show it is a line bundle

Let
$$\langle\Psi\rangle : \mathcal{O}_C^{\oplus N} \longrightarrow L^{\oplus N}$$
be the sheaf morphism whose matrix of global sections is Ψ. Define
$$\xi := \mathrm{im}\langle\Psi\rangle \subset L^{\oplus N}.$$

Now look at the generic fiber. At η,
$$\langle\Psi\rangle_\eta : k(C)^N \to k(C)^N$$
has rank 1, so ξ_η is a 1-dimensional $k(C)$-subspace of $k(C)^N$. Thus ξ is a torsion-free sheaf of rank 1. On a smooth curve, torsion-free rank 1 sheaves are line bundles. So ξ is a line bundle on C.

Step 2. Use row/column independence to get $h^0(\xi) \geq N$ and $h^0(\widehat{\xi}) \geq N$

You have a surjection of sheaves
$$\mathcal{O}_C^{\oplus N} \twoheadrightarrow \xi.$$
Taking global sections gives a k-linear map
$$k^N = H^0(\mathcal{O}_C^{\oplus N}) \longrightarrow H^0(\xi).$$

Concretely, the standard basis vector e_i maps to the i-th column of Ψ, viewed as an N-tuple of global sections of L, then projected into $\xi \subset L^{\oplus N}$.

If the columns of Ψ are k-linearly independent in $H^0(L)^{\oplus N}$, then these N sections of ξ are k-linearly independent. Hence

$$h^0(\xi) \geq N.$$

Similarly, considering the transpose picture (or dually the map $(L^{-1})^{\oplus N} \to \mathcal{O}_C^{\oplus N}$) and using row independence gives

$$h^0(\widehat{\xi}) \geq N$$

for $\widehat{\xi} := L \otimes \xi^{-1}$.

Step 3. Deduce $H^1(\xi) = 0$ and the degree bound $\deg \xi \geq d$

From Step 2: $h^0(\xi) \geq N > g$. The standard fact which follows from Riemann–Roch and Serre duality is if a line bundle M has $h^0(M) > g$, then $H^1(M) = 0$.

So $H^1(\xi) = 0$. Then Riemann–Roch gives

$$h^0(\xi) = \deg \xi + 1 - g.$$

Thus

$$\deg \xi = h^0(\xi) + g - 1 \ \geq \ N + g - 1 \ = \ d.$$

Apply the same argument to $\widehat{\xi} := L \otimes \xi^{-1}$. You get $h^0(\widehat{\xi}) \geq N > g$, hence $H^1(\widehat{\xi}) = 0$, hence

$$\deg(\widehat{\xi}) \geq N + g - 1 = d.$$

But $\deg(\widehat{\xi}) = \deg L - \deg \xi = 2d - \deg \xi$. So

$$2d - \deg \xi \geq d \quad \Rightarrow \quad \deg \xi \leq d.$$

It follows that $\deg \xi = d$ and thus $\xi \in \mathrm{Pic}^d(C)$.

Step 5. Reconstruct the double marking (S, T) and the equality $\Psi = \Psi(\xi, S, T)$

Now you know $\xi \in \mathrm{Pic}^d$ and $\widehat{\xi} \in \mathrm{Pic}^d$, so by the preliminary facts,

$$h^0(\xi) = h^0(\widehat{\xi}) = N, \quad \text{and both are generated by global sections.}$$

At the generic point, rank 1 implies a factorization

$$\Psi_\eta = f \cdot g^t$$

with $f, g \in k(C)^N$. The f_i, g_j are only defined up to $f \mapsto uf$, $g \mapsto u^{-1}g$ with $u \in k(C)^\times$, and they can have poles.

You then choose ξ (equivalently an effective divisor D with $\xi = \mathcal{O}_C(D)$) so that the poles of the f_i's are absorbed into ξ:

$$s_i := f_i \cdot 1_D \in H^0(\xi) \quad \text{(regular everywhere).}$$

With that choice, the complementary twist is forced:

$$\widehat{\xi} := L \otimes \xi^{-1},$$

and then the g_j's become regular sections of $\widehat{\xi}$ (after the same normalization), so that

$$\Psi_{ij} = s_i t_j \in H^0(L).$$

Finally, the k-linear independence of rows/columns forces these N sections to be linearly independent in $H^0(\xi)$ and $H^0(\widehat{\xi})$, hence they are bases S, T. Then by construction

$$\Psi_{ij} = s_i t_j,$$

so $\Psi = \Psi(\xi, S, T)$. $\square$

The space of matrices $\mathrm{Mat}_N\big(H^0(L)\big)$ is a vector space over k isomorphic to the direct sum of N^2 copies of $H^0(L)$, and the general linear group $GL(N)$ acts on this space by left and right multiplication. In particular, this gives an action of the direct product $GL(N) \times GL(N)$, under which the image of the group homomorphism

$$\mathbb{G}_m \longrightarrow GL(N) \times GL(N), \qquad t \longmapsto (tI_N, t^{-1}I_N)$$

acts trivially. We therefore consider the cokernel

$$GL(N,N) \ := \ GL(N) \times GL(N) \big/ \mathbb{G}_m.$$

Note that, since $GL(N)$ is linearly reductive, so is $GL(N,N)$. As a representation of $GL(N,N)$ the space $\mathrm{Mat}_N\big(H^0(L)\big)$ is isomorphic to a direct sum of $\dim H^0(L)$ copies of the space $\mathrm{Mat}_N(k)$ of square matrices over k. This can be viewed as an affine space $\mathbb{A}^n$, where

$$n = N^2 \dim H^0(L),$$

and $\mathrm{Mat}_{N,1}\big(H^0(L)\big)$ as a closed subvariety. In particular, $\mathrm{Mat}_{N,1}\big(H^0(L)\big)$ is an affine variety (or, more precisely, each irreducible component is an affine variety, and the discussion below applies to each irreducible component) and is preserved by the action of $GL(N,N)$. This action is of ray type.

The set of matrices Ψ satisfying the linear independence condition (1) in Proposition 9.56 forms an open set

$$\mathcal{U}(L) \subset \mathrm{Mat}_{N,1}\big(H^0(L)\big),$$

which is therefore a parameter space for double-marked line bundles (ξ, S, T) of degree d. Moreover, the open set $\mathcal{U}(L)$ is preserved by the action of $GL(N,N)$.

Proposition 3.13 (Proposition 9.58)**.** Matrices $\Psi, \Psi' \in \mathcal{U}(L)$ give isomorphic line bundles ξ, ξ' if and only if they belong to the same $GL(N,N)$-orbit.

This identifies the set $\operatorname{Pic}^d C$ with the space of $GL(N,N)$-orbits in $\mathcal{U}(L) \subset \operatorname{Mat}_{N,1}\big(H^0(L)\big)$. We are going to take the projective GIT quotient of $\operatorname{Mat}_{N,1}\big(H^0(L)\big)$ by $GL(N,N)$ to construct a coarse moduli space for the Picard functor Pic^d_C. We need to spell out a linearization of the action. We fix the character $\chi : \mathrm{GL}(N,N) \to k^*$ induced by

$$(A,B) \mapsto \det(A)\det(B)$$

Moreover let $\mathrm{SL}(N,N)$ be the kernel of this character. It turns out that with respect to this character, we have

$$\operatorname{Mat}^s_{N,1}\big(H^0(L)\big) = \operatorname{Mat}^{ss}_{N,1}\big(H^0(L)\big) = \mathcal{U}(L)$$

Moreover, one can identify the tangent space at each point of $\mathcal{U}(L)$ as the space of first order deformations of the corresponding line bundle, controlled by $H^1(\mathcal{O}_C) = k^g$. In particular, $\mathcal{U}(L)$ is nonsingular of dimension g.

The Proj GIT quotient is built from a graded ring of semiinvariants. That graded ring can fail to be an integral domain (several components, strictly semistable behavior). However, when the semistable locus is smooth, the Proj quotient still exists as a disjoint union of varieties.

A smooth scheme is regular, hence reduced. It also implies that each local ring is a domain. From that you get: distinct irreducible components cannot meet. Hence a regular scheme is a disjoint union of its irreducible components. Therefore, if a connected component is regular, it is integral because it cannot contain two disjoint nonempty opens.

In GIT the quotient is built affine-locally from invariants, and for reductive G invariants behave well with reducedness. If A is reduced and G is linearly reductive (in characteristic 0, reductive suffices), then A^G is reduced because the Reynolds operator splits $A^G \hookrightarrow A$. So you do not accidentally create nilpotents in the quotient ring.

Therefore the projective GIT quotient is a projective variety. In this case, since all points are stable, the quotient is a good quotient in the sense that its points correspond one-to-one to the orbits of the group action.

Theorem 3.14. The projective GIT quotient

$$\operatorname{Mat}^s_{N,1}\big(H^0(L)\big)/GL(N,N) = \operatorname{Proj} k[\operatorname{Mat}_{N,1}\big(H^0(L)\big)]^{SL(N,N)}$$

is a projective variety whose underlying set is in natural bijection with $\operatorname{Pic}^d C$.

Remark 3.15. Note that in general for any reductive G and character χ,

$$\bigoplus_{m \geq 0} k[X]^{G,\chi^m} \;\cong\; k[X]^{\ker \chi}$$

as a graded ring, and the projective quotient can be written using either description.

Remark 3.16 (The stable quotient as a geometric quotient)**.** Let G be a reductive group acting on a projective variety X with a fixed linearization. Then there are open subsets

$$X^s \subset X^{ss} \subset X$$

of stable and semistable points. A fundamental theorem of GIT asserts that the quotient

$$\pi : X^s \longrightarrow X^s/G$$

exists and is a *geometric quotient*. Concretely, this means:

(i) (**Orbit fibers**) $\pi(x) = \pi(x')$ if and only if x' lies in the G–orbit of x. In particular, the points of X^s/G are in bijection with G–orbits in X^s.

(ii) (**Topological quotient**) The map π is surjective and open, and X^s/G carries the quotient topology.

(iii) (**Invariant functions**) The structure sheaf is the sheaf of invariants:

$$\mathcal{O}_{X^s/G} = (\pi_* \mathcal{O}_{X^s})^G.$$

(iv) (**Closed orbits and finite stabilizers**) Every G–orbit in X^s is closed in X^s and has finite stabilizer. Consequently, X^s/G is a genuine orbit space in the sense of algebraic geometry.

In particular, X^s/G is always quasi–projective, and it is projective if and only if $X^s = X^{ss}$, in which case $X^s/G = X^{ss}//G$. By contrast, the projective GIT quotient $X^{ss}//G$ is only a categorical quotient in general: distinct orbits in X^{ss} may be identified, and points correspond to closed orbits in orbit closures (S–equivalence classes). Thus the stable quotient X^s/G is the part of the GIT quotient that behaves as a true moduli space of objects. In the example of the Picard functor above, since all semistable points are stable, the projective GIT quotient turns out to be a geometric quotient.

Remark 3.17 (Set-theoretic vs. functorial moduli)**.** Saying that a variety M has underlying set $\mathrm{Pic}^d(C)$ only means that its k-points classify isomorphism classes of degree d line bundles on C. This is a purely set-theoretic statement: it records which objects exist, but not how they vary in algebraic families.

By contrast, to say that M represents the Picard functor Pic^d_C means that for every scheme S there is a natural bijection

$$\mathrm{Hom}(S, M) \;\cong\; \mathrm{Pic}^d_C(S),$$

identifying morphisms $S \to M$ with families of degree d line bundles on $C \times S$, up to the standard equivalence. Equivalently, M carries a universal (Poincaré) line bundle $\mathcal{P}$ on $C \times M$ such that every family over S is obtained uniquely, up to twisting by a line bundle from S, by pullback:

$$\mathcal{L} \;\cong\; (\mathrm{id}_C \times f)^* \mathcal{P} \quad \text{for a unique } f : S \to M.$$

Thus representability encodes not only the set of isomorphism classes, but the entire functorial behavior of line bundles in families and under base change. This is the extra structure that makes M a genuine moduli space rather than just a parameter set.

Theorem 3.18. The quotient variety

$$\mathrm{Mat}^s_{N,1}\big(H^0(L)\big)/GL(N,N)$$

represents the Picard functor Pic^d_C. In particular, it is a fine moduli space for line bundles of degree d on C.

In order to prove this theorem, we will first establish that the moduli problem admits a coarse moduli space structure. Then we will upgrade this to a fine moduli space structure by constructing a universal family, classically known as a Poincaré line bundle.

Proposition 3.19. Let $L \in \mathrm{Pic}^{2d} C$, where $d \geq 2g$, and let $N = d+1-g$. Then the projective quotient

$$\mathrm{Mat}^s_{N,1}\big(H^0(L)\big)/GL(N,N)$$

is a coarse moduli space for the Picard functor Pic^d_C.

Corollary 3.20. The isomorphism class of the variety

$$J_d \;:=\; \mathrm{Mat}^s_{N,1}\big(H^0(L)\big)/GL(N,N)$$

depends only on C and d, and not on the line bundle $L \in \mathrm{Pic}^{2d} C$.

Proof of Proposition 3.19. Since line bundles satisfy Zariski descent and morphisms into a fixed scheme form a Zariski sheaf, it suffices to construct the correspondence functorially for affine schemes $S = \mathrm{Spec}\, A$. Compatibility on overlaps ensures the maps glue uniquely to arbitrary base schemes. Therefore, for each finitely generated k-algebra A, our aim is to find a natural bijection between line bundles Ξ on C_A such that $\deg \Xi_t = d$ at every $t \in \mathrm{Spm}\, A$, up to equivalence, and morphisms $\mathrm{Spm}\, A \to J_d$.

Let $\widetilde{\Xi} := L_A \otimes \Xi$. Then both of $H^0(\Xi)$ and $H^0(\widetilde{\Xi})$ are locally free A-modules of rank N, and their fibres at a point $t \in \mathrm{Spm}\, A$ are the spaces $H^0(C, \Xi_t)$ and $H^0(C, \widetilde{\Xi}_t)$.

Remark 3.21. Here we are invoking the following fact. Suppose $\mathcal{E}$ is a vector bundle on C_A with $H^1(C_A, \mathcal{E}) = 0$. Then $H^0(C_A, \mathcal{E})$ is a locally free A-module, and for every ring homomorphism $f : A \to A'$, the base change homomorphism

$$H^0(C_A, \mathcal{E}) \otimes_A A' \longrightarrow H^0(C_{A'}, \mathcal{E} \otimes_A A')$$

is an isomorphism. This is an incarnation of the more general theory of cohomology and base change. For the sake of completeness, we state the relevant result here without proof.

Theorem 3.22 (Grauert; cohomology and base change, curve case)**.** Let $\pi : X \to S$ be a proper flat morphism, with S reduced and locally Noetherian, and let $\mathcal{F}$ be a coherent sheaf on X flat over S. Suppose that for some $p \geq 0$ the function

$$s \longmapsto h^p(X_s, \mathcal{F}_s)$$

is locally constant on S. Then:

(i) $R^p\pi_*\mathcal{F}$ is locally free on S;

(ii) for every morphism $f : T \to S$, the natural base change map

$$f^*(R^p\pi_*\mathcal{F}) \longrightarrow R^p\pi'_*(\mathcal{F}_T)$$

is an isomorphism, where $\pi' : X_T := X \times_S T \to T$.

In particular, when $\dim X_s = 1$ and $H^1(X_s, \mathcal{F}_s) = 0$ for all s, then $\pi_*\mathcal{F}$ is locally free and commutes with arbitrary base change.

In topology and complex geometry, a map can have all fibers homeomorphic (or diffeomorphic) and still not be a fiber bundle; extra conditions are needed to guarantee local triviality. Similarly, in algebraic geometry, given a proper morphism $\pi : X \to S$ and a coherent sheaf $\mathcal{F}$, you can have all fibers $H^0(X_s, \mathcal{F}_s)$ with the same dimension, but $\pi_*\mathcal{F}$ may not be locally free, and base change can fail. Grauert's theorem upgrades a set of vector spaces depending on s into a vector bundle on S, providing the crucial finiteness and continuity conditions that allow sheaf cohomology to behave well under base change.

Remark 3.23 (How Grauert is used in the Jacobian construction)**.** We apply this with

$$\pi : C_A = C \times \operatorname{Spec} A \longrightarrow \operatorname{Spec} A, \qquad \mathcal{F} = \Xi \text{ or } \widetilde{\widetilde{\Xi}} = L_A \otimes \Xi^{-1},$$

where Ξ is a family of line bundles of degree $d \geq 2g$.

For every $t \in \operatorname{Spec} A$ we have, by Riemann–Roch,

$$H^1(C, \Xi_t) = 0, \qquad h^0(C, \Xi_t) = d + 1 - g =: N,$$

and the same for $\widetilde{\widetilde{\Xi}}$. Hence the hypotheses of Grauert are satisfied for $p = 0$.

Therefore

$$\pi_*\Xi \;\cong\; H^0(C_A, \Xi), \qquad \pi_*\widetilde{\widetilde{\Xi}} \;\cong\; H^0(C_A, \widetilde{\widetilde{\Xi}})$$

are locally free A-modules of rank N, and for every $A \to A'$,

$$H^0(C_A, \Xi) \otimes_A A' \;\xrightarrow{\sim}\; H^0(C_{A'}, \Xi \otimes_A A'),$$

$$H^0(C_A, \widetilde{\Xi}) \otimes_A A' \xrightarrow{\sim} H^0(C_{A'}, \widetilde{\Xi} \otimes_A A').$$

This is the key input that upgrades the fiberwise vector spaces $H^0(C, \Xi_t)$ into vector bundles on $\operatorname{Spec} A$, allowing one to choose local bases and construct the matrix-valued morphism $\operatorname{Spec} A \to \operatorname{Mat}_{N,1}(H^0(L))$ functorially.

Example 3.24 (What Grauert is saying in elementary terms)**.** Let $\pi : \mathbb{P}^1_S \to S$ be the projection and take $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1_S}(n)$.

For every $s \in S$,

$$H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0 \quad (n \geq 0), \qquad h^0(\mathbb{P}^1, \mathcal{O}(n)) = n + 1.$$

Grauert's theorem implies

$$\pi_* \mathcal{O}_{\mathbb{P}^1_S}(n) \;\cong\; \mathcal{O}_S^{\oplus(n+1)},$$

and for all $T \to S$,

$$H^0(\mathbb{P}^1_S, \mathcal{O}(n)) \otimes_{\mathcal{O}_S} \mathcal{O}_T \;\cong\; H^0(\mathbb{P}^1_T, \mathcal{O}(n)).$$

The intuition is that the vector spaces $H^0(\mathbb{P}^1, \mathcal{O}(n))$ fit together into a vector bundle over S with constant fiber dimension, and global sections commute with base change.

There then exists a bilinear homomorphism of A-modules:

$$H^0(\Xi) \times H^0(\widetilde{\Xi}) \longrightarrow H^0(L) \otimes_k A. \tag{1}$$

Step 1. We first consider the case when both of $H^0(\Xi)$ and $H^0(\widetilde{\Xi})$ are free A-modules. Let $S, \widetilde{S}$ be free bases. Via (1), these determine an $N \times N$ matrix with entries in $H^0(L) \otimes_k A$, and so we get a morphism to an (affine) space of matrices,

$$\operatorname{Spm} A \longrightarrow \operatorname{Mat}_N\big(H^0(L)\big).$$

This maps into the closed subvariety

$$\operatorname{Mat}_{N,1}\big(H^0(L)\big) \subset \operatorname{Mat}_N\big(H^0(L)\big)$$

defined by the vanishing of the 2×2 minors because each fiber matrix factors as column vector of $H^0(\Xi_t)$ tensor row vector of $H^0(\widetilde{\Xi}_t)$.

Moreover, the image lies in theopen set $\operatorname{Mat}^s_{N,1}\big(H^0(L)\big)$, since for all $t \in \operatorname{Spm} A$ the line bundles Ξ_t and $\widetilde{\Xi}_t$ are generated by global sections. We will denote this map by

$$\widetilde{\varphi} : \operatorname{Spm} A \longrightarrow \operatorname{Mat}^s_{N,1}\big(H^0(L)\big),$$

and the composition of $\widetilde{\varphi}$ with the quotient map by

$$\varphi : \operatorname{Spm} A \longrightarrow J_d = \operatorname{Mat}^s_{N,1}\big(H^0(L)\big)/GL(N,N).$$

The map φ depends only on the equivalence class of Ξ (in the sense of Definition 3.8) and not on the choice of $S, \widetilde{S}$.

We now take an affine open cover

$$\operatorname{Spm} A = U_1 \cup \cdots \cup U_n$$

such that the A-modules $H^0(\Xi)$ and $H^0(\widetilde{\Xi})$ restrict to free modules on each U_i. For each i, by choosing free bases of $H^0(\Xi)|_{U_i}$ and $H^0(\widetilde{\Xi})|_{U_i}$ we obtain a map

$$\widetilde{\varphi}_i : U_i \longrightarrow \operatorname{Mat}^s_{N,1}\big(H^0(L)\big)$$

and on intersections $U_i \cap U_j$ the maps $\widetilde{\varphi}_i$ and $\widetilde{\varphi}_j$ differ only by the choice of free bases of $H^0(\Xi)|_{U_i \cap U_j}$ and $H^0(\widetilde{\Xi})|_{U_i \cap U_j}$. It follows that the corresponding maps

$$\varphi_i : U_i \longrightarrow J_d \quad \text{and} \quad \varphi_j : U_j \longrightarrow J_d$$

agree on the intersection $U_i \cap U_j$, and by gluing we therefore obtain a morphism

$$\varphi : \operatorname{Spm} A \longrightarrow J_d.$$

This is called the *classifying map* for the family of line bundles Ξ. Let Ξ, Ξ' be two line bundles on C_A which are locally equivalent as families of line bundles on C. By this we mean that there is an open cover of $\operatorname{Spm} A$ as above, such that on each open set the restrictions $\Xi|_{U_i}$ and $\Xi'|_{U_i}$ are equivalent.

For these line bundles the classifying maps

$$\varphi, \varphi' : \operatorname{Spm} A \longrightarrow J_d$$

are the same, and in particular we see that φ depends only on the equivalence class of Ξ (Definition 11.22). This verifies the first requirement for J_d to be a coarse moduli space: we have constructed a natural transformation of functors

$$\operatorname{Pic}^d_C \longrightarrow J_d.$$

Over an algebraically closed field k this is bijective essentially because over an algebraically closed field, geometric points correspond to k-points. Over non-closed fields, points correspond to Galois orbits of geometric points, and bijectivity on k-points is false in general. (Recall that a geometric point of a k-scheme X is a morphism

$$\operatorname{Spec} \Omega \to X$$

where Ω is an algebraically closed field. Equivalently: a point of $X(\bar{k})$ together with an embedding $k \hookrightarrow \bar{k}$.)

Finally, we have to show universality. Suppose that we have a natural transformation

$$\psi : \mathrm{Pic}^d_C \longrightarrow Y$$

for some variety Y. Over the product $C \times \mathrm{Mat}_{N,1}\big(H^0(L)\big)$ there is a tautological homomorphism of vector bundles

$$\mathcal{O}^{\oplus N}_{C\times \mathrm{Mat}} \longrightarrow \big(L \otimes \mathcal{O}_{\mathrm{Mat}}\big)^{\oplus N}.$$

Restricted to the open set $C \times \mathrm{Mat}^s_{N,1}\big(H^0(L)\big)$, this map has rank 1 at each point, and its image is a line bundle. We denote this line bundle on $C \times \mathrm{Mat}^s_{N,1}\big(H^0(L)\big)$ by $\mathcal{Q}$, called the *universal line bundle*. Composing the natural transformation

$$\mathrm{Mat}^s_{N,1}\big(H^0(L)\big) \longrightarrow \mathrm{Pic}^d_C,$$

defined in the obvious way by the pullback of $\mathcal{Q}$, with ψ gives a natural transformation of functors

$$\mathrm{Mat}^s_{N,1}\big(H^0(L)\big) \longrightarrow \mathrm{Pic}^d_C \xrightarrow{\psi} Y.$$

Since the line bundle $\mathcal{Q}$ is trivial on $GL(N,N)$–orbits, it follows that the corresponding morphism

$$\mathrm{Mat}^s_{N,1}\big(H^0(L)\big) \longrightarrow Y$$

descends to the quotient, and so we obtain a morphism

$$J_d \longrightarrow Y$$

with the required properties. Here we are invoking the fact that J_d is a geometric quotient, which is true here because

$$\mathrm{Mat}^s_{N,1}(H^0(L)) = \mathrm{Mat}^{ss}_{N,1}(H^0(L)) \quad \square$$

Now we are ready to upgrade the coarse moduli space structure of J_d to a fine moduli space structure by constructing a universal family of line bundles on $C \times J_d$. We recall some preliminaries which we will apply to the universal line bundle $\mathcal{Q}$.

Let G be a linearly reductive algebraic group acting on an affine variety $X = \mathrm{Spm}\, R$, and let M be an R-module with a G-linearisation, i.e. G-action on M as a k-vector space, compatible with the G-action on R in the sense that

$$g \cdot (rm) = (g \cdot r)(g \cdot m)$$

Thus M is also a representation of G and has a subset of invariants $M^G \subset M$ which, by definition of a linearisation, is an R^G-module.

Lemma 3.25 (11.26)**.** If M is a finitely generated module over a Noetherian ring R, then M^G is finitely generated as an R^G-module.

Proof. The idea of the proof is the same as that of Hilbert's Theorem 4.51. Denote by $M' \subset M$ the submodule generated by M^G. Since R (and M) are Noetherian, it follows that M' is finitely generated. Letting $m_1, \ldots, m_n \in M'$ be generators, the map

$$R \oplus \cdots \oplus R \longrightarrow M', \qquad (a_1, \ldots, a_n) \longmapsto a_1 m_1 + \cdots + a_n m_n$$

is surjective, and hence by linear reductivity the induced map

$$R^G \oplus \cdots \oplus R^G \longrightarrow (M')^G = M^G$$

is surjective, so that M^G is generated as an R^G-module by $m_1, \ldots, m_n$. $\square$

Proposition 3.26 (11.27)**.** Suppose that all orbits of G on $\operatorname{Spm} R$ are free closed orbits and that M is a locally free R-module. Then M^G is a locally free R^G-module and

$$M \cong M^G \otimes_{R^G} R.$$

Proof. Let $I \subset R$ be the ideal of an orbit, with corresponding maximal ideal $\mathfrak{m} = I \cap R^G \subset R^G$. Then M/IM is a $k[G]$-module with a G-linearisation and so is a free $k[G]$-module by Lemma 9.49. By linear reductivity we can find $m_1, \ldots, m_r \in M^G$ whose residue classes $\overline{m}_1, \ldots, \overline{m}_r \in M/IM$ are a free basis, and hence the natural homomorphism of R-modules

$$M^G \otimes_{R^G} R \longrightarrow M$$

is an isomorphism along each orbit. Since, by Lemma 11.26, M^G is finitely generated, it follows from Nakayama's Lemma that this homomorphism is surjective. But it is also injective because M is locally free. $\square$

Remark 3.27. The proposition establishes the existence and uniqueness of descent data for locally free sheaves under the hypotheses stated. More precisely:

You have an affine geometric quotient

$$\pi : \; X = \operatorname{Spec} R \; \to \; Y = \operatorname{Spec} R^G$$

with all orbits free and closed.

A G-linearised vector bundle on X is the same as a locally free R-module M with a compatible G-action.

To say that M descends means: There exists a vector bundle E_0 on Y such that $M \cong \pi^* E_0$. To say it descends uniquely means if $M \cong \pi^* E_0 \cong \pi^* E_1$, then $E_0 \cong E_1$.

The proposition constructs $E_0 \leftrightarrow M^G$ and proves:

- M^G is locally free over R^G,
- the natural map $M^G \otimes_{R^G} R \longrightarrow M$ is an isomorphism.

This is exactly:

$M \cong \pi^*(M^G)$.

So the descended bundle exists. For uniqueness, suppose $M \cong N \otimes_{R^G} R$ for some locally free R^G-module N.

Take invariants:

$M^G \cong (N \otimes_{R^G} R)^G$.

But because the orbits are free and closed and G is linearly reductive, one proves:

$(N \otimes_{R^G} R)^G \cong N$.

So:

$N \cong M^G$.

Therefore any descent must equal M^G. No choice.

So uniqueness is automatic once the invariant functor behaves well, which is exactly what linear reductivity + freeness give you.

Let X be affine with a G-linearisation by a character χ. Then

$$X//_\chi G = \mathrm{Proj} \bigoplus_{n \geq 0} R^{G,\chi^n}.$$

A basic fact from GIT:

For any homogeneous invariant $f \in R^{G,\chi^n}$ with $n > 0$,

$$D_+(f) \subset X//_\chi G$$

is affine, and

$$\Phi^{-1}(D_+(f)) = X_f^{ss}$$

is affine, with

$$\Phi^{-1}(D_+(f)) \to D_+(f)$$

equal to the affine quotient map

$$\operatorname{Spec} R_{(f)} \longrightarrow \operatorname{Spec}(R_{(f)})^G.$$

In particular, the projective quotient map in the direction of some character $\chi \in \operatorname{Hom}(G, \mathbb{G}_m)$,

$$\Phi = \Phi_\chi : X^{ss} \longrightarrow X//_\chi G,$$

is locally an affine quotient map.

Corollary 3.28 (11.28)**.** Suppose that all semistable points are stable and that every orbit of the action G on $X^{ss} = X^s$ is a free closed orbit. Then, given a vector bundle E on X with a G-linearisation, there exists a vector bundle E_0 on X^s/G such that

$$E \cong \Phi^* E_0.$$

Proof. The proof reduces to the affine case: 1. Cover $X//_\chi G$ by affines $U_i = D_+(f_i)$. 2. On each U_i, $\Phi^{-1}(U_i) \to U_i$ is an affine quotient $\operatorname{Spec} R_i \to \operatorname{Spec} R_i^G$. 3. Over each affine piece, Proposition 11.27 applies: equivariant locally free modules descend uniquely. 4. The descended bundles glue because everything was G-equivariant and the quotient is geometric. □

$\mathcal{Q}$ had the property that

$$(1 \times \tilde{\varphi})^* \mathcal{Q} \cong \Xi$$

under

$$1 \times \tilde{\varphi} : C \times \operatorname{Spm} A \longrightarrow C \times \operatorname{Mat}^s_{N,1}\big(H^0(L)\big).$$

Let R be the coordinate ring of the affine variety $\operatorname{Mat}_{N,1}\big(H^0(L)\big)$. There is a tautological homomorphism of R-modules

$$\tau : R^{\oplus N} \longrightarrow R^{\oplus N} \otimes_k H^0(L)$$

given in the obvious way by matrix multiplication. Given a linear map $f : H^0(L) \to k$, we then get a homomorphism of R-modules as the composition

$$\phi_f : R^{\oplus N} \xrightarrow{\tau} R^{\oplus N} \otimes_k H^0(L) \xrightarrow{1 \otimes f} R^{\oplus N} \otimes_k k = R^{\oplus N}.$$

When f is the evaluation map at a point $p \in C$ this homomorphism has rank ≤ 1 everywhere, and on the open set $\mathrm{Mat}^s_{N,1}\big(H^0(L)\big)$ its image is precisely the line bundle

$$\mathcal{Q}_p := \mathcal{Q}|_{p\times \mathrm{Mat}}.$$

The group $GL(N) \times GL(N)$ acts on R, and using this action we let it act on the source and target $R^{\oplus N}$ of the homomorphism ϕ_f, respectively, by

$$\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} \longmapsto A \begin{pmatrix} g \cdot f_1 \\ \vdots \\ g \cdot f_N \end{pmatrix}, \qquad \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} \longmapsto \begin{pmatrix} g \cdot f_1 \\ \vdots \\ g \cdot f_N \end{pmatrix} B^{t,-1}.$$

The map ϕ_f is then a $GL(N)\times GL(N)$-homomorphism, and in particular $GL(N)\times GL(N)$ acts on the R-module $\mathcal{Q}_p$. Similarly, the universal line bundle $\mathcal{Q}$ carries a $GL(N) \times GL(N)$-linearisation, under which $(t, t^{-1}) \in GL(N) \times GL(N)$ acts nontrivially. However, this element acts trivially on the line bundle $\mathcal{Q} \otimes \mathcal{Q}_p^{-1}$ with its induced linearisation, and so this line bundle possesses a $GL(N, N)$-linearisation. Applying Corollary 11.28, we conclude that $\mathcal{Q} \otimes \mathcal{Q}_p^{-1}$ is the pullback of some line bundle

$$\mathcal{P}_d \in \mathrm{Pic}(C \times J_d).$$

This is called the *Poincaré line bundle*.

Lemma 3.29 (11.30)**.** Let Ξ be a line bundle on C_A with classifying map $\varphi : \mathrm{Spm}\, A \to J_d$. Then Ξ is equivalent to the pullback $(1 \times \varphi)^*\mathcal{P}_d$ via $1 \times \varphi : C_A \to C \times J_d$.

Proof. Let $\mathcal{L} = (1\times\varphi)^*\mathcal{P}_d$. By construction of the classifying map, Ξ is already locally isomorphic to $\mathcal{L}$. In other words,

$$\Xi|_{C\times U_i} \cong \mathcal{L}|_{C\times U_i}$$

over some affine open cover $\mathrm{Spm}\, A = U_1 \cup \cdots \cup U_n$. Thus

$$M := H^0(\Xi \otimes \mathcal{L}^{-1})$$

is an invertible A-module, and the natural homomorphism

$$\mathcal{L} \otimes_A M \longrightarrow \Xi$$

is an isomorphism. Hence Ξ and $\mathcal{L}$ are equivalent. $\square$

This completes the proof that J_d is a fine moduli space for the Picard functor Pic^d_C.

Remark 3.30. The reason the Picard moduli problem admits a fine moduli space is that every line bundle has the same automorphism group $\mathbb{G}_m$, acting centrally. Thus the moduli stack of line bundles is a $\mathbb{G}_m$–gerbe over its coarse space. One can rigidify globally by this universal $\mathbb{G}_m$, killing all stabilizers at once. After rigidification the inertia is trivial, and the resulting moduli problem is representable by a scheme, namely the Jacobian. This uniformity of automorphism groups is special to rank one and fails in higher rank, where stabilizers vary and obstruct the existence of a fine moduli space.

4 References

1. Mukai, S., *An Introduction to Invariants and Moduli*, Cambridge Studies in Advanced Mathematics, vol. 81, Cambridge University Press, Cambridge, 2003.