

Algebraic \mathcal{D} -modules

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Abstract

We begin a study of algebraic \mathcal{D} -modules. The main reference is Hotta-Kashiwara, and I've supplemented with random notes along the way.

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1 Introduction

Let G act smoothly on a smooth complex algebraic variety X . Let E be a G -equivariant vector bundle on X . There is a linearized action of \mathfrak{g} on the space of sections of E . First we will explain this when $E = TX$ is the tangent bundle.

Theorem 1.1. *There is a map $\exp : \mathfrak{g} \rightarrow G$ which takes $X = [\gamma] \in \mathfrak{g}$ to $\gamma(1) \in G$ where $\gamma : \mathbb{R} \rightarrow G$ is the unique one-parameter subgroup of G with tangent vector X at 1.*

Therefore, given an element $X \in \mathfrak{g}$, we can define a vector field X_X on X by how it acts on a function $f \in C^\infty(X)$:

$$X_X(f)(p) = \frac{d}{dt} f(\exp(tX)p)|_{t=0} = \lim_{h \rightarrow 0} \frac{f(\exp(hX)p) - f(p)}{h}$$

Another way of thinking about it is the map $\gamma(t) \cdot p : \mathbb{R} \rightarrow X$ is a path in X whose tangent vector at time 0 is precisely X_X . This gives \mathfrak{g} acting on vector fields via bracket with X_X .

Now let E be a G -equivariant vector bundle on X . Given $X \in \mathfrak{g}$ we can define a section of E by

$$E_p \ni X_E(p) = \lim_{h \rightarrow 0} \frac{E_{\exp(hX)p} - E_p}{h}$$

but we crucially need to know how to transport E_p to $E_{\exp(hX)p}$. This is where the notion of a connection comes in. Basically we have a canonical choice of connection because there is an action of G on the total space of E compatible with the projection to X .

Morally an action of G on X is a map $G \rightarrow \text{Aut}(X)$ which we can differentiate to get a map $\mathfrak{g} \rightarrow \text{Der}(\mathcal{O}_X)$. Vector bundles on X correspond to \mathcal{O}_X -modules, i.e. maps $\mathcal{O}_X \rightarrow \Gamma(X, E)$, and so given $X \in \mathfrak{g}$ we get a map $\mathcal{O}_X \rightarrow \mathcal{O}_X \rightarrow \Gamma(X, E)$, i.e. we have an automorphism of the total space of E which gets an action on the sections.

What is the appropriate analog for the universal enveloping algebra for derivations of \mathcal{O}_X ?
The answer is the ring of differential operators \mathcal{D}_X on X .

1.1 Distributions

Ultimately our goal is to define the pushforward of a \mathcal{D} -module along a map of varieties. The trouble with this is that we like to think of the ring of differential operators as acting on functions, but functions don't pushforward. The correct thing to let the ring of differential operators act on is distributions. In fact, this parity is precisely the distinction between the left (functions) and right (distributions) \mathcal{D} -modules.

Example 1.2. This is uncertain. *"Distributions are things which one can integrate functions against. Therefore, to specify a distribution, it suffices to specify how it integrates functions. For us, algebraic distributions will be the sections of top exterior power of the cotangent bundle."*

Another thing that you might hear is that distributions on a smooth manifold are smooth subbundles of TM . This makes sense because if you want to integrate a function f over a manifold M , you integrate the n -form $f dx_1 \wedge \cdots \wedge dx_n$ over the n -manifold. In fact, all of the n -forms on an oriented n -manifold look like this. "The top exterior power of the cotangent bundle is trivial on an oriented manifold." This gives an explanation why Ω_X is called the "canonical bundle" of X .

There is a corresponding subbundle of TX of corank 1 given by the tangent bundle of the hypersurface $f = 0$. This is the "delta function" supported on the hypersurface $f = 0$ and is the corresponding smooth subbundle alluded to above.

1.2 Pushforward

Left \mathcal{D} -modules are about functions. Right \mathcal{D} -modules are about distributions. We can go from functions to distributions by tensoring with the top exterior power of the cotangent bundle (or its dual).

Let $X \rightarrow Y$ be a closed embedding of smooth varieties. We want to define the pushforward of a right \mathcal{D}_X -module to a right \mathcal{D}_Y -module. For simplicity we introduce an object called the transfer

module.

Definition 1.3. The *transfer \mathcal{D}_X -module* is defined by

$$\mathcal{D}_{X \rightarrow Y} = f^* \mathcal{D}_Y = \mathcal{O}_X \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{D}_Y$$

where the m_i are local coordinates, and the ∂_i are the corresponding derivations. As an \mathcal{O}_X -module, it coincides with the standard pullback of \mathcal{D}_Y as an \mathcal{O}_Y -module. It carries a left \mathcal{D}_X -module structure by the rule

$$\theta \cdot (a \otimes f^{-1}(m)) = \theta(a) \otimes f^{-1}(m) + a \sum_i \theta(f^{-1}(m_i) \otimes \partial_i m)$$

and a right $f^{-1} \mathcal{D}_Y$ -module structure as well.

We need to define the pushforward for right \mathcal{D}_X modules because distributions push forward.

Definition 1.4. There is also a *transfer $f^{-1}(\mathcal{D}_Y)$ - \mathcal{D}_X bimodule going the other way* $\mathcal{D}_{Y \leftarrow X}$ defined as

$$\mathcal{D}_{Y \leftarrow X} = \Omega_X \otimes_{\mathcal{O}_X} \otimes_{f^{-1} \mathcal{O}_Y} f^{-1} \Omega_Y^{-1}$$

This allows us to define the pushforward of a left \mathcal{D}_X -module to a left \mathcal{D}_Y -module.

Definition 1.5. For *pushforward of left \mathcal{D} -modules* we define

$$f_* M = f_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X} M)$$

Pushforward of \mathcal{D} -modules is left exact (maybe?)

Definition 1.6. Let $f : X \rightarrow Y$ be a smooth map of smooth varieties. Let M be a right \mathcal{D}_X -module. The *pushforward* $f_* M$ is defined to be the tensor product $M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \rightarrow Y}$. It is a right \mathcal{D}_Y -module.

1.3 Pullback

Definition 1.7. To define the pullback we start with the pullback of \mathcal{O} -modules.

$$f^* N = \mathcal{O}_X \otimes_{f^{-1} \mathcal{O}_Y} N$$

It turns out to carry a left \mathcal{D}_X -module structure in the same way as the transfer module. By the associativity of tensor product, we can also write it as

$$f^* N = \mathcal{D}_{X \rightarrow Y} \otimes_{\mathcal{D}_Y} N$$

and therefore we have a right exact functor

$$f^* : \mathcal{D}_Y \text{ Mod} \rightarrow \mathcal{D}_X \text{ Mod}$$

1.4 Correspondence between left and right \mathcal{D} -modules

A differential operator $P(x, \partial) = \sum_{\alpha} a_{\alpha}(x) \partial^{\alpha}$ has a formal adjoint $P^t(x, \partial)$ which is defined by

$$P^t(x, \partial) = \sum_{\alpha} (-\partial)^{\alpha} a_{\alpha}(x)$$

which satisfies $(PQ)^t = Q^t P^t$. The left action of differential operators on functions is given by differentiation. The right action of differential operators distributions is given by acting by the formal adjoint

$$(a(p) dx_1 \wedge \cdots \wedge dx_n) \cdot X := (P^t(X) a(p))(dx_1 \wedge \cdots \wedge dx_n)$$

2 Deriving things

2.1 \mathcal{D} -affineness

Definition 2.1. X is \mathcal{D} -affine if every quasicoherent \mathcal{D}_X -module is the "twiddleification" of a $\Gamma(X, \mathcal{D}_X)$ -module.

Every affine variety is \mathcal{D} -affine. Projective spaces and the flag variety for a reductive group are \mathcal{D} -affine.

Remark 2.2. Spaces which are \mathcal{D} -affine are not necessarily \mathcal{D}^{op} -affine. For example,

$$\begin{aligned} \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}) &= U(\mathfrak{sl}_2)/Z(U(\mathfrak{sl}_2)) \text{ would like more explicitly} \\ \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{op}) &= \Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = 0 \end{aligned}$$

since the anticanonical sheaf of \mathbb{P}^1 is negative.

Proposition 2.3. 1. \mathcal{D}_X is a coherent sheaf of rings.

2. \mathcal{D}_X -modules are coherent if and only if they are quasicoherent over \mathcal{O}_X and finitely generated over \mathcal{D}_X .
3. \mathcal{D}_X -modules are coherent if and only if they are integrable connections (need locally freeness) in which case taking Spec gets us a vector bundle with integrable connection over our variety X .

2.2 Derived pullback

Recall that $f^* : \mathcal{D}_Y \text{ Mod} \rightarrow \mathcal{D}_X \text{ Mod}$ is a right exact functor. Therefore, homological algebra tells we get left derived functors $Lf^* : D(\mathcal{D}_Y \text{ Mod}) \rightarrow D(\mathcal{D}_X \text{ Mod})$. Assume that our varieties are quasiprojective. Then this guarantees that the following story works:

Proposition 2.4. *Let M be a quasicoherent \mathcal{D}_X -module. Then there exist two resolutions*

$$\begin{aligned} \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \\ 0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0 \end{aligned}$$

where the F_i are locally free and the P_i are locally projective.

The existence of finite projective resolutions tells us that Lf^* restricts to the bounded derived category $D^b(\mathcal{D}_Y \text{ Mod}) \rightarrow D^b(\mathcal{D}_X \text{ Mod})$.

Proposition 2.5. *Lf^* moreover preserves quasicoherent cohomology sheaves. It does not preserve coherent cohomology sheaves because you can pick up infinite rank.*

Definition 2.6. *The shifted derived pullback is defined by*

$$Lf^\dagger M = Lf^*[\dim X - \dim Y]$$

is more natural in the Riemann Hilbert correspondence.

Proposition 2.7. *Let $f : X \rightarrow Y$ be a smooth map of smooth varieties and $M \in \text{Mod}(\mathcal{D}_Y)$. Then Lf^*M has no higher cohomology. If M is coherent, then the nonzero part (at most 1) Lf^*M is coherent.*

Recall that a map $i : X \rightarrow Y$ of varieties makes \mathcal{O}_X into a $i^{-1}\mathcal{O}_Y$ -module and that the inverse image functor in the category of \mathcal{O}_Y -modules is defined by

$$i^*\mathcal{G} := i^{-1}\mathcal{G} \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

because $i^{-1}\mathcal{G}$ is a sheaf of $i^{-1}\mathcal{O}_Y$ -modules and we have a ring homomorphism $i^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ coming from the adjoint to $i_*\mathcal{O}_X \rightarrow \mathcal{O}_Y$. In particular, if we have $f : \text{Spec } A \rightarrow \text{Spec } B$ then $f^*M = M \otimes_B A$.

Definition 2.8. *Define the inverse image functor for \mathcal{D} -modules by*

$$i^\natural : \text{Mod}(\mathcal{D}_Y) \rightarrow \text{Mod}(\mathcal{D}_X)$$

by the formula

$$i^\natural M = \text{Hom}_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}M)$$

Recall the shifted inverse image functor $i^\dagger = Li^*[\dim X - \dim Y]$.

Proposition 2.9. *i^\natural is left exact. For closed embeddings i we have $i^\dagger M \cong Ri^\natural M$.*

2.3 Derived pushforward

There are functors

$$D^b(\mathcal{D}_X) \rightarrow D^b(f^{-1}\mathcal{D}_Y) \rightarrow D^b(\mathcal{D}_Y \text{ Mod})$$

which take $M \mapsto \mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^L M$ and $N \mapsto Rf_* N$ respectively. Their composition is **the derived directed image** denoted

$$\int_f M : D^b(\mathcal{D}_X) \rightarrow D^b(\mathcal{D}_Y)$$

We state some of the properties of this functor.

Proposition 2.10.

1. \int_f preserves quasicoherence, and if f is proper, it preserves coherence.
2. $\int_f = Rj^*$ if j open.
3. The direct image is locally exact. In particular if $i : X \rightarrow Y$ closed and $M \in \text{Mod}(\mathcal{D}_X)$ then $\int_f^i := H^i(\int_f M)$ is zero for all $i \neq 0$ and \int_f^0 is quasicoherent if M is.
4. i^{\natural} and \int_f are adjoints.

2.4 Kashiwara's equivalence

For closed embedding $X \rightarrow Y$ we saw that \int_f^0 is exact and $\int_f^0 M$ is a \mathcal{D}_Y -module supported on X . **Let the category of such \mathcal{D}_Y -modules be denoted $\text{Mod}^X(\mathcal{D}_Y)$.**

Theorem 2.11 (Kashiwara's equivalence). *The functor \int_f^0 induces an equivalence of categories*

$$\text{Mod}_{qc,c}(\mathcal{D}_X) \rightarrow \text{Mod}_{qc,c}^X(\mathcal{D}_Y)$$

with quasiinverse i^{\natural} .

2.5 Base change theorem

The following story really captures the essence of derived categories. Let

$$Z \xrightarrow{i} X \xleftarrow{j} U$$

be complements and F an injective sheaf of X . There is a short exact sequence

$$0 \rightarrow \Gamma_Z F \rightarrow F \rightarrow j_* j^{-1} F \rightarrow 0$$

which gives a distinguished triangle in the derived category for every $F^\bullet \in D^b(\mathbb{C}_X)$

$$R\Gamma_Z(F^\bullet) \rightarrow F^\bullet \rightarrow Rj_* j^{-1} F^\bullet \rightarrow +1$$

We have the following abstract nonsense.

Proposition 2.12.

1. For $M^\bullet \in D_{qc}^b(D_X)$ we have a canonical distinguished triangle

$$R\Gamma_Z(M^\bullet) \rightarrow M^\bullet \rightarrow \int_j j^\dagger M^\bullet \rightarrow +1$$

2. Z smooth and $M \in D_{qc}^b(D_U)$ then

$$i^\dagger \int_j M = 0$$

3. Z smooth and $M^\bullet \in D_{qc}^b(D_X)$ then

$$R\Gamma_Z(M^\bullet) = \int_i i^\dagger M$$

3 Coherent \mathcal{D} -modules

Coherent \mathcal{D} -modules can be studied with the geometry of T^*X .

3.1 Characteristic cycles

If M is a coherent \mathcal{D}_X module, choose a good filtration on it and then $\text{gr}^F M$ is a coherent $\pi^* \mathcal{O}_{T^*X}$ -module.

Definition 3.1. Let $\text{gr}^{\tilde{F}} M = \text{gr}^F M \otimes_{\pi^{-1} \mathcal{O}_{T^*X}} \mathcal{O}_{T^*X}$. The support of this module is $\text{Ch}(M)$ *the characteristic variety of M* . It does not depend on the choice of good filtration.

Remark 3.2. We can also take cycle invariants, which also do not depend on the choice of good filtration. Explicitly, let C be a irreducible component of the characteristic variety of M . Then the cycle invariant comes from taking the multiplicity of the component C as follows.

Pick an open set U so that $\overline{C \cap U} = C$. *U is a rational coordinate chart.* $C \cap U$ is closed in U so take its defining ideal \mathfrak{p} . Then we have a local ring $\mathcal{O}_U(U)_{\mathfrak{p}}$ and a module over this local ring $M_{\mathfrak{p}}$. They are the stalks of \mathcal{O}_X and M at the generic point of C respectively. By commutative algebra, the length of $M_{\mathfrak{p}}$ as a module over $\mathcal{O}_U(U)_{\mathfrak{p}}$ is finite and defined as the multiplicity of C .

Example 3.3. When M is an integrable connection, the following are equivalent:

1. M is an integrable connection.
2. M is coherent over \mathcal{O}_X
3. $\text{Ch}(M) = \text{zero section of } T^*X$.

3.2 Holonomic \mathcal{D} -modules

Definition 3.4. A \mathcal{D}_X -module M is **holonomic** if $\dim \text{Ch}(M) = \dim X$ is as large as possible.

Character varieties of holonomic \mathcal{D} -modules are \mathbb{C}^* invariant Lagrangian subvarieties of T^*X .

Example 3.5. Integrable connections are holonomic.

Example 3.6. Let $i : X \rightarrow Y$ be closed smooth subvariety. Let

$$B_{X|Y} := \int_i^0 \mathcal{O}_X \in \text{Mod}_{qc}^X(D_Y)$$

Then $B_{X|Y}$ is holonomic and $\text{Ch}(B_{X|Y}) = T_X^*Y$ the conormal bundle of X in Y . When $X = *$ we can apply Kashiwara equivalence to get that

$$\begin{aligned} \text{Mod}_{qc}^*(D_Y) &\cong \text{Vect}_{\mathbb{C}} \\ B_{*|Y} &\iff \mathbb{C} \end{aligned}$$

which implies that objects in the left hand side are direct sums of $B_{*|Y}$.

3.3 Duality functors

We begin with a motivating example. Let $X = \mathbb{C}$ and $M = \mathcal{D}_X / \mathcal{D}_X P$ where $P \neq 0$. There is a short exact sequence

$$0 \rightarrow \mathcal{D}_X \xrightarrow{P} \mathcal{D}_X \rightarrow M \rightarrow 0$$

which gives

$$0 \rightarrow \text{Hom}_{\mathcal{D}_X}(M, \mathcal{D}_X) \rightarrow \mathcal{D}_X \xrightarrow{P} \mathcal{D}_X \rightarrow \text{Ext}_{\mathcal{D}_X}^1(M, \mathcal{D}_X) \rightarrow 0$$

and so we see that

$$\begin{aligned} \text{Ext}^0(M, \mathcal{D}_X) &= \text{Hom}_{\mathcal{D}_X}(M, \mathcal{D}_X) = 0 \\ \text{Ext}^1(M, \mathcal{D}_X) &= \mathcal{D}_X / P\mathcal{D}_X \end{aligned}$$

Sidechanging Ext^1 gives us the left \mathcal{D} -module $\mathcal{D}_X / P^* \mathcal{D}_X$.

Ext¹ is more suited to be called the dual than Ext⁰. In general, if $n = \dim X$ and M is a holonomic \mathcal{D}_X -module, then

$$\text{Ext}_{\mathcal{D}_X}^n(M, \mathcal{D}_X)$$

is the only nonzero Ext group and the left \mathcal{D}_X -module

$$\mathbb{D}M := \text{Ext}_{\mathcal{D}_X}^n(M, \mathcal{D}_X) \otimes \Omega_X^{-1}$$

is also holonomic. For a non-holonomic M , the other Ext groups are generally nonzero and therefore \mathbb{D} is defined in the derived category:

Definition 3.7. We define the *duality functors*

$$\mathbb{D} : D^-(\mathcal{D}_X) \rightarrow D^+(\mathcal{D}_X)^{op}$$

$$\mathbb{D}M := R\mathrm{Hom}_{\mathcal{D}_X}(M, \mathcal{D}_X) \otimes \Omega_X^{-1}[\dim X]$$

Example 3.8. We have the example $H^k(\mathbb{D}\mathcal{D}_X) = D_X \otimes \Omega_X^{-1}$ if $k = -\dim X$ and 0 otherwise.

Example 3.9. $\mathbb{D}B_{X|Y} = B_{Y|X}$

Example 3.10. $\mathbb{D}M = \mathrm{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X)$ if M is an integrable connection.

Proposition 3.11. If X is a smooth variety and M is a coherent \mathcal{D}_X -module, then

$$\mathrm{Ch}(M) = \bigcup_{d_X \leq i \leq 0} \mathrm{Ch}(\mathrm{Ext}_{\mathcal{D}_X}^i(M, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1})$$

In particular $\mathrm{Ch}(M) = \mathrm{Ch}(\mathbb{D}M)$ if M is holonomic.

4 Holonomic \mathcal{D} -modules

Holonomic \mathcal{D} -modules enjoy very nice properties. We learned about all of the previous functors and constructions in order to construct examples of holonomic \mathcal{D} -modules.

4.1 Basic results

Proposition 4.1. 1.

An exact sequence of coherent \mathcal{D}_X -modules

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

implies that M is holonomic if and only if M and M'' are holonomic.

Proposition 4.2. Let M be a holonomic \mathcal{D}_X -module. Then there exists an open $U \subset X$ for which $M|_U$ is coherent over \mathcal{O}_U . In other words, M is generically an integrable connection.

Definition 4.3. Let $D_h^b(\mathcal{D}_X)$ be the full subcategory of $D^b(\mathcal{D}_X)$ consisting of complexes with holonomic cohomology sheaves.

Remark 4.4. Beilinson proved that you can consider the category with holonomic \mathcal{D} -modules as objects and the answer is the same.

$$D_h^b(\mathcal{D}_X) \cong D^b(\mathrm{Mod}_h(\mathcal{D}_X))$$

Proposition 4.5. Duality preserves holonomicity. In particular,

$$\mathbb{D}_X : \mathrm{Mod}_h(\mathcal{D}_X) \rightarrow \mathrm{Mod}_h(\mathcal{D}_X)^{op}$$

$$\mathbb{D}_X : D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_X)^{op}$$

are equivalences of categories.

Recall for morphism $f : X \rightarrow Y$ of smooth varieties we have the **direct and inverse image** functors

$$\begin{aligned} \int_f : D_{qc}^b(D_X) &\rightarrow D_{qc}^b(D_Y) \\ f^\dagger : D_{qc}^b(D_Y) &\rightarrow D_{qc}^b(D_X) \end{aligned}$$

Moreover, we have proper pushforward and smooth pullback of coherent \mathcal{D} -modules (but not for general maps!) Remarkably however, holonomicity is preserved by these functors for general maps.

Theorem 4.6. *Let $f : X \rightarrow Y$ be a morphism of smooth varieties. Then*

$$\begin{aligned} \int_f : D_h^b(D_X) &\rightarrow D_h^b(D_Y) \\ f^\dagger : D_h^b(D_Y) &\rightarrow D_h^b(D_X) \end{aligned}$$

Proof. It is enough to prove that \int_f preserves holonomicity when f is the projection $\mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$. We will explain this in a bunch of steps because it is not obvious at all.

First, it's true for closed embeddings. In particular if $i : X \rightarrow Y$ is a closed embedding, then

$$M^\bullet \in D_h^b(\mathcal{D}_X) \iff \int_i M^\bullet \in D_h^b(\mathcal{D}_Y)$$

This is not hard. Then we factor f as a composition of a closed embedding and a projection by using its graph. Now we just have to show the claim for projections. Reduce to local arguments and then we see that it is enough to prove the claim for $\mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$.

Finally the claim about pushforwards implies the claim about pullbacks generally. \square

It remains to prove that the pushforward along the projection $\mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ preserves holonomicity.

4.2 Holonomicity of modules over the Weyl algebra

Let D_n be the Weyl algebra in n variables.

Definition 4.7. *Let N be a D_n -module. The **Fourier transform** of N is the D_n -module \hat{N} which is the same abelian group, but the generators $x_i, \partial_i \in D_n$ act differently.*

$$\begin{aligned} x_i \cdot f &= -\partial_i f \\ \partial_i \cdot f &= x_i f \end{aligned}$$

is an involution.

Proposition 4.8. *Let $p : \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ be the projection and $i : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^n$ the zero section. Then*

$$H^k \int_p M \cong H^k Li^* \hat{M}$$

for any k and any D_n -module M .

Proposition 4.9. *A coherent D_n module is holonomic if and only if its Fourier transform is.*

Proposition 4.10. *Let $j : \mathbb{C}^* \times \mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ the inclusion. If M is a holonomic D_n -module then so is*

$$H^0 \int_j j^\dagger M$$

These propositions, which we will prove next, immediately imply the theorem. The distinguished triangle

$$\int_i i^\dagger M \rightarrow M \rightarrow \int_j j^\dagger M \rightarrow +1$$

gives an exact sequence

$$0 \rightarrow H^0 \int_i i^\dagger M \rightarrow M \rightarrow H^0 \int_j j^\dagger M \rightarrow H^1 \int_i i^\dagger M \rightarrow 0$$

and note that $H^0 \int_j j^\dagger M$ is holonomic by the proposition. Therefore, $i^\dagger M$ is holonomic because H^0 and H^1 are (holonomicity behaves really well under extension).

4.3 Hilbert series

This section is devoted to the proof of the above propositions. Fix the Bernstein filtration on D_n (filtered by degree of symbols) and we define a notion of a **good filtration** on a D_n -module M with respect to the Bernstein filtration.

Any finitely generated D_n module has a good filtration with respect to the Bernstein filtration. **The advantage is that for a good filtration on M with respect to the Bernstein filtration, every $F^i M$ is finite dimensional over \mathbb{C} .** Therefore we can apply results on Hilbert polynomials to the associated graded $\text{gr}^B D_n$ module.

Proposition 4.11. *The function*

$$i \mapsto \dim_{\mathbb{C}} F^i M$$

is eventually polynomial and will be denoted $\chi(M, F)$. The highest degree term is d and the leading coefficient is $m/d!$. The integers d and m do not depend on the choice of the good filtration F .

d is the dimension and m is the multiplicity.

d is also the dimension of the characterstic variety of M in this case.

Proposition 4.12. *If you find an integer c so that*

$$\dim_C F^i M \leq c/n!i^n + O(i^{n-1})$$

then M is holonomic and $m(M) \leq c$.

Proof of Proposition 4.9. N and \hat{N} have the same Hilbert polynomial. \square

Proof of Proposition 4.10. Let $N = \Gamma(\mathbb{C}^n, M)$ and note that $\Gamma(\mathbb{C}^n, H^0 \int_j j^\dagger M)$ is isomorphic to the localization $N_{x_i} = \mathbb{C}[x, x_i^{-1}] \otimes_{\mathbb{C}[x]} N$. Hence it is sufficient to show that the localization is holonomic. Kashiwara does this by picking a filtration on N_{x_i} and then finding an integer c as in the proposition above. \square

4.4 Exceptional functors

Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties.

Definition 4.13. *We have the **exceptional pushforward and pullback***

$$\begin{aligned} \int_{f!} &:= \mathbb{D}_Y \circ \int_f \circ \mathbb{D}_X : D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_Y) \\ f^* &:= \mathbb{D}_X \circ f^\dagger \circ \mathbb{D}_Y : D_h^b(\mathcal{D}_Y) \rightarrow D_h^b(\mathcal{D}_X) \end{aligned}$$

Theorem 4.14. *There exists a morphism of functors*

$$\int_{f!} \rightarrow \int_f : D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_Y)$$

which is an isomorphism if f is proper. This is some compactly supported thing in the spirit of Borel Moore homology and Poincare duality.

4.5 Stratifications and minimal extensions

Theorem 4.15. *The following conditions on $M^\bullet \in D_c^b(\mathcal{D}_X)$ are equivalent:*

1. M^\bullet is holonomic.
2. There eexists a decreasing sequence

$$X = X_0 \supset X_1 \supset \cdots \supset X_m \subset X_{m+1} = \emptyset$$

of closed subsets of X so that $X_r \setminus X_{r+1}$ is smooth and all of the cohomology sheaves $H^k(i_r^\dagger M^\bullet)$ are integrable connections, where $i_r : X_r \setminus X_{r+1} \rightarrow X$ is the inclusion.

3. For any $x \in X$ all of the cohomology groups $H^k(i_x^\dagger M^\bullet)$ are finite dimensional where $i_x : \{x\} \rightarrow X$ is the inclusion.

M coherent is called **simple** if it contains no coherent \mathcal{D} -submodules other than M or 0 . Since the holonomic \mathcal{D} -module form an Artinian category, we have a Jordan Holder series for each coherent M . We will construct simple holonomic \mathcal{D} -modules from integrable connections on locally closed subvarieties using functors introduced in the previous section, and all of them are of this type. This construction corresponds under RH correspondence to the minimal DGM extension in the category of perverse sheaves.

Definition 4.16. Let $f : X \rightarrow Y$ be the inclusion of a locally closed smooth subvariety and assume that f is an affine map. Recall that there is a map

$$\int_{f!} M \rightarrow \int_f M$$

which is an isomorphism if f is proper. We call the image of this map the **minimal extension** $L(Y, M)$ of M .

Theorem 4.17.

1. Let Y be a locally closed smooth connected subvariety of X so that $i : Y \rightarrow X$ is an affine map. Let M be a simple holonomic \mathcal{D}_Y -module. Then $L(Y, M)$ is a simple holonomic \mathcal{D}_X -module and it is the unique simple submodule of $\int_i M$.
2. Any simple holonomic \mathcal{D}_X -module is isomorphic to $L(Y, M)$ for some locally closed smooth connected subvariety Y of X and some simple holonomic \mathcal{D}_Y -module M .
3. Two such pairs (Y, M) and (Y', M') give isomorphic simple holonomic \mathcal{D}_X -modules if and only if $\overline{Y} = \overline{Y'}$ and $M \cong M'$ on an open dense subset of $Y \cap Y'$.

5 Analytic \mathcal{D} -modules

Let $f : X \rightarrow Y$ be a morphism of complex manifolds. All the previous theory can be carried out in the analytic category.

5.1 Solution complex and de Rham functors

For $M^\bullet \in D^b(\mathcal{D}_X)$ we have the **solution complex**

$$\text{Sol}(M^\bullet) = R\text{Hom}_{\mathcal{D}_X}(M^\bullet, \mathcal{O}_X)$$

which gives a functor

$$D^b(\mathcal{D}_X) \rightarrow D^b(\mathbb{C}_X)^{op}$$

We have the **de Rham complex**

$$\mathrm{DR}(M^\bullet) = \mathcal{O}_X \otimes_{\mathcal{D}_X} M^\bullet$$

which gives a functor

$$D^b(\mathcal{D}_X) \rightarrow D^b(\mathbb{C}_X)$$

The motivation for introducing the solutino complex comes from linear PDEs. For a coherent \mathcal{D}_X -module M , the sheaf $\mathrm{Hom}_{\mathcal{D}_X}(M, \mathcal{O}_X)$ is the sheaf of solutions to the system of linear PDEs defined by M .

Proposition 5.1. *For $M^\bullet \in D_c^b(\mathcal{D}_X)$ we have*

$$\mathrm{DR}_X(M^\bullet) \cong \mathrm{Sol}_X(\mathbb{D}_X M^\bullet)[\dim X]$$

The two complexes are related via duality but the functor DR_X has the advantage that it can be computer using a resoplution of the right \mathcal{D}_X -module Ω_X . We have a locally free resolution

$$0 \rightarrow \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \cdots \rightarrow \Omega_X^{\dim X} \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \Omega_X \rightarrow 0$$

For $M \in \mathrm{Mod}(\mathcal{D}_X)$ the object $\mathrm{DR}_X(M)[- \dim X]$ is represented in the derived category by the complex

$$\Omega_X^0 \otimes_{\mathcal{O}_X} M \rightarrow \cdots \rightarrow \Omega_X^n \otimes_{\mathcal{O}_X} M$$

has differential

$$\begin{aligned} d^p &: \Omega_X^p \otimes_{\mathcal{O}_X} M \rightarrow \Omega_X^{p+1} \otimes_{\mathcal{O}_X} M \\ d^p(\omega \otimes s) &= d\omega \otimes s + \sum_i dx_i \wedge \omega \otimes \partial_i s \end{aligned}$$

where x_i, ∂_i are local coordinates on X .

5.2 Integrable connections and local systems

Suppose M is an integrable connection of rank m . Consider the 0th cohomology sheaf L of the solution complex $\mathrm{Sol}(M)$. Then L coincides with the kernel of the sheaf homomorphism

$$d^0 = \nabla : M = \Omega_X^0 \otimes_{\mathcal{O}_X} M \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} M$$

which is the sheaf of horizontal sections of M .

$$M^\nabla = \{s \in M \mid \nabla s = 0\} = \{s \in M \mid \Theta_X s = 0\}$$

It is a locally free \mathbb{C}_X module of rank m by the classical Frobenius theorem.

Definition 5.2. A locally free \mathbb{C}_X module of finite rank is a **local system** on X . Let $\text{Loc}(X)$ be the category of local systems on X .

Using the local system L , we can define an integrable connection M by $M = \mathcal{O}_X \otimes_{\mathbb{C}_X} L$. I still have to tell you what the operator is:

$$\nabla = d \otimes \text{id}_L : M = \mathcal{O}_X \otimes_{\mathbb{C}_X} L \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X \otimes_{\mathbb{C}_X} L = \Omega_X^1 \otimes_{\mathcal{O}_X} M$$

Passing to Ω_X^\bullet with similarly defined differentials in the tensor product, we study the higher cohomology groups $H^i(\Omega_X^\bullet \otimes_{\mathcal{O}_X} M)$. All of the higher ones vanish by the **holomorphic Poincare lemma** and therefore we get an isomorphism

$$\Omega_X^\bullet \otimes_{\mathcal{O}_X} M \cong M^\nabla$$

This gives a functor

$$H^{-\dim X}(\text{DR}_X(\cdot)) : \text{Conn}(X) \rightarrow \text{Loc}(X)$$

which is an equivalence of categories.

Remark 5.3. Given an integrable connection M , pick coordinates locally. Then ∇ looks like the standard derivative operator, and so $\nabla s = 0$ is the condition that s is constant function. Locally there is a vector space of constant functions of dimension $\text{rank } m$. This is precisely the correspondence between integrable connections and local systems. ∇ is flat meaning it has no curvature and therefore, the transport of sections along paths depends only on the homotopy class of the path. Therefore, you can study how sections of M vary along homotopy classes of loops in X .

This is called the **monodromy representation** of M .

$$\pi_1(X) \rightarrow \text{Aut}(L)$$

5.3 Kashiwara constructibility theorem

Theorem 5.4 (Kashiwara). The solution complex $\text{Sol}_X(M)$ of a holonomic \mathcal{D}_X -module M is a constructible complex, i.e. all of its cohomology sheaves are constructible.

6 Meromorphic connections

We begin a study of meromorphic connections and Deligne's Riemann-Hilbert correspondence.

6.1 Meromorphic connections

We start from the classical theory of ODEs. We are considering open neighborhoods of $0 \in \mathbb{C}$ as a complex manifold. Let

$$\begin{aligned}\mathcal{O} &= (\mathcal{O}_{\mathbb{C}})_0 = \text{convergent power series at } 0 = \mathbb{C}\{x\} \\ K &= \text{quotient field of } \mathcal{O} = \mathbb{C}\{x\}[x^{-1}]\end{aligned}$$

Let $A(x) \in M_n(K)$ and consider the system of ODEs

$$\frac{d}{dx}\hat{u}(x) = A(x)\hat{u}(x)$$

where $\hat{u}(x) \in M_n(K)$ is a column vector. Given an invertible $T \in \text{GL}_n(K)$ we can change the basis of A and rewrite the system as

$$\frac{d}{dx}\hat{v}(x) = (T^{-1}A(x)T - T^{-1}\frac{d}{dx}T)\hat{v}(x)$$

and so we say that two systems are equivalent if they are related by a such a transformation.

Let \tilde{K} denote the ring of possibly multivalued holomorphic functions defined on a punctured disk B_ε^* for sufficiently small ε . We say that \hat{u} is a solution of the system if $\hat{u} \in \tilde{K}^n$.

Definition 6.1. Let M be a finite dimensional K -vector space with a \mathbb{C} -linear map $\nabla : M \rightarrow M$. Then M, ∇ is a **meromorphic connection** if

$$\nabla(fu) = \frac{df}{dx}u + f\nabla u$$

for all $f \in K$ and $u \in M$. It is enough to check that this holds for $f \in \mathcal{O}$. A morphism of meromorphic connections is a K -linear map $M \rightarrow N$ commuting with ∇ .

Meromorphic connections naturally form an abelian category. Given a meromorphic connection M, ∇ the vector space M is a left $(\mathcal{D}_{\mathbb{C}})_0$ -module by the action $\partial u = \nabla u$. Note that ∇ also extends uniquely to an element of $\text{End}_{\mathbb{C}}(\tilde{K} \otimes_K M)$. We say that $u \in \tilde{K} \otimes_K M$ is a **horizontal section** if $\nabla u = 0$.

Let M be a meromorphic connection and choose a K -basis e_1, \dots, e_n of M . Then the change of basis matrix with respect to ∇ is:

$$\nabla e_j = \sum_i A_{ij} e_i$$

"the connection matrix" The condition $\nabla u = 0$ for $u = \sum_i u_i e_i \in \tilde{K} \otimes_K M$ is equivalent to the system of ODEs

$$\frac{d}{dx}u = A(x)u$$

for the connection matrix A .

6.2 Regular singularities

Set $S_{a,b}^\varepsilon = \{x \in \mathbb{C} \mid |x| < \varepsilon, a < \theta < b\}$ an open angular sector of the universal cover of \mathbb{C}^* . We say that a function $f \in \tilde{K}$ has moderate growth (also called Nilsson class) at $x = 0$ if it satisfies the following condition.

For every a, b and ε so that f is defined on $S_{a,b}^\varepsilon$,
there is a constant $C > 0$ and $N \gg 0$ so that $|f(x)| \leq C|x|^{-N}$ for all $x \in S_{a,b}^\varepsilon$.

Denote such functions by \tilde{K}_{mod} .

6.2.1 Important example

The system of ODEs

$$\frac{d}{dx}u = \frac{A}{x}u$$

for $A \in M_n(K)$ has a set of solutions $u \in \tilde{K}^n$ which is in fact a \mathbb{C} -vector space of dimension n . We can take n linearly indepdent solutions u_1, \dots, u_n and call the matrix $S = (u_1, \dots, u_n)$ the **fundamental matrix solution** of the system. **Since the analytic continuation of $S(x)$ along a circlce around 0 is again a solution matrix** there exists some $G \in \text{GL}_n(\mathbb{C})$ so that $S(x) = S(0)G$ for all $x \in \mathbb{C}^*$. G is called the **monodromy matrix** of the system.

Theorem 6.2. *The following three conditions on the ODE system are equivalent:*

1. *The system is equivalent to a system like*

$$\frac{d}{dx}u = \frac{A(x)}{x}u$$

for $A(x) \in M_n(\mathcal{O})$.

2. *The system is equivalent to a system like*

$$\frac{d}{dx}u = \frac{A}{x}u$$

with \mathbb{C} -entries.

3. *All solutions in \tilde{K}^n belong to \tilde{K}_{mod}^n .*

We expand on what it means for the analytic continuation to introduce monodromy.

Example 6.3. *The most basic example is*

$$A = 1/x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The solutions are $u_1 = x$ and $u_2 = x$ and so the fundamental matrix solution is

$$S(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

The monodromy matrix is $G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ because nothing happens when you go around 0. See the next example for a more interesting case with the details that I didn't write here.

Example 6.4. Consider taking

$$A(x) = 1/x \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The off-diagonal entries introduce log terms into the fundamental matrix solution and therefore we pick up monodromy. In particular, we can solve this ODE with separation of variables to get

$$\begin{aligned} u_1 &= C_1 x + C_2 x \log x \\ u_2 &= C_2 x \end{aligned}$$

so we have a fundamental solution matrix

$$S(x) = \begin{pmatrix} x & x \log x \\ 0 & x \end{pmatrix}$$

Now we will be very explicit about the analytic continuation. Write $x = |x|e^{i\theta}$ and let θ vary from 0 to 2π . We have in general

$$S(x) = \begin{pmatrix} |x|e^{i\theta} & |x|e^{i\theta} \log |x| + i\theta |x|e^{i\theta} \\ 0 & |x|e^{i\theta} \end{pmatrix}$$

Going from 0 to 2π , the solution matrix goes from

$$S(x) = \begin{pmatrix} x & x \log x \\ 0 & x \end{pmatrix}$$

to

$$S(x) = \begin{pmatrix} x & x \log x + 2\pi i x \\ 0 & x \end{pmatrix}$$

which tells us the monodromy matrix is

$$G = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$$

6.2.2 Definition of regularity

Let $\theta = x\nabla = x\partial$.

Definition 6.5. We say that a meromorphic connection M, ∇ is *regular singular* at $x = 0$ if there exists a finitely generated \mathcal{O} -submodule $L \subset M$ which is stable under θ and generates M over K . Such an L is called an \mathcal{O} -lattice.

Lemma 6.6. Any \mathcal{O} -lattice L is a free \mathcal{O} -module of rank $\dim_K M$.

In particular, a meromorphic connection is regular if and only if there exists a K -basis e_1, \dots, e_n of M so that the associated ODE is of the form

$$\frac{d}{dx}u = \frac{A(x)}{x}u$$

for $A(x) \in M_n(\mathcal{O})$. Equivalently if

$$\frac{d}{dx}u = \frac{A}{x}u$$

for $A \in M_n(\mathbb{C})$.

Remark 6.7. Recall that the affine Grassmannian can be thought of the space of lattices where $c = k[x_1, \dots, x_n]$ and $K\mathbb{C}((x_1, \dots, x_n))$. In this case we are thinking about regular singularities for the divisor which is the union of all coordinate hyperplanes.

Note that the set of regular weights in \mathfrak{t}^* is precisely corresponding to ripping out the union of coordinate hyperplanes in \mathfrak{t}^* .

Proposition 6.8. A meromorphic connection is regular iff all of its horizontal sections are in $\tilde{K}_{mod} \otimes_K M$.

Proposition 6.9. For a meromorphic connection M, ∇ at $x = 0$, the following are equivalent:

1. M, ∇ is regular.
2. For any $u \in M$ there exists $L \ni u$ finitely generated over \mathcal{O} so that $\theta L \subset L$.
3. For any $u \in M$ there exists a polynomial

$$F(t) = t^m + a_{m-1}t^{m-1} + \dots + a_0$$

so that $F(\theta)u = 0$.

Remark 6.10. Regularity is a notion which comes from the theory of linear ODEs. In particular if we consider a second-order linear differential equation of the form

$$\frac{d^2}{dx^2}u + p(x)\frac{d}{dx}u + q(x)u = 0$$

then the point $x = 0$ is called a regular singular point if $xp(x)$ and $x^2q(x)$ are holomorphic at 0. The solutions of the ODE are then well-behaved and can be written as power series or generalized series involving logarithms.

A regular \mathcal{D} -module is then a generalization of this, which includes the \mathcal{D} -modules which arise from linear ODEs with regular singularities.

Proposition 6.11. *An exact sequence of meromorphic connections*

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

implies that M is regular if and only if M' and M'' are regular. If M, N are regular, then so is $M \otimes_K N$ and $\text{Hom}(M, N)$.

6.3 For a general algebraic curve

The ring $\mathcal{O} = \mathbb{C}\{x\}$ is replaced by the stalk $\mathcal{O}_{C,p}$ where C is a smooth algebraic curve and $p \in C$. The field K is replaced by the quotient field of $\mathcal{O}_{C,p}$. \mathcal{O}_p is a DVR and in particular a PID.

Definition 6.12.

1. Let M be a finite dimensional $K_{C,p}$ -vector space with a \mathbb{C} -linear map

$$\nabla : M \rightarrow \Omega_{C,p}^1 \otimes_{\mathcal{O}_{C,p}} M \cong (K_{C,p} \otimes_{\mathcal{O}_{C,p}} \Omega_{C,p}^1) \otimes_{K_{C,p}} M$$

The pair M, ∇ is a **algebraic meromorphic connection** at $p \in C$ if

$$\nabla(fu) = df \otimes u + f\nabla u \quad (\text{for all } f \in K_{C,p}, u \in M)$$

2. A morphism is a $K_{C,p}$ -linear map $\phi : M \rightarrow N$ so that $\nabla \circ \phi = (\text{id} \otimes \phi) \circ \nabla$.

Algebraic meromorphic connections at $p \in C$ naturally form an abelian category. Choose a local coordinate $x \in \mathcal{O}_{C,p}$ so that $K_{C,p} = \mathcal{O}_{C,p}[x^{-1}]$. Then we can identify $\Omega_{C,p}^1$ with $\mathcal{O}_{C,p}$ via $f \iff fdx$ and an algebraic meromorphic connection at $p \in C$ is a finite dimensional $K_{C,p}$ -vector space endowed with a \mathbb{C} -linear map

$$\nabla : M \rightarrow M$$

which satisfies

$$\nabla(fu) = \frac{df}{dx}u + f\nabla u \quad (\text{for all } f \in K_{C,p}, u \in M)$$

Definition 6.13. *An algebraic meromorphic connection M, ∇ at $p \in C$ is **regular** if there exists a finitely generated $\mathcal{O}_{C,p}$ -submodule $L \subset M$ which is stable under $\theta = x\nabla$ for any (and all) local coordinate $x \in \mathcal{O}_{C,p}$ at p . L is called an $\mathcal{O}_{C,p}$ -lattice.*

This definition is about being regular at a point $p \in C$. We now globalize this definition. Let M be an integrable connection on an algebraic curve C . Take a smooth completion $j : C \rightarrow \overline{C}$, unique up to isomorphism because C is a curve. j is an affine map so pushforward is exact and we have j_*M . Since M is locally free over \mathcal{O}_C it is free on a nontrivial Zariski open $U = C \setminus V$ where V is finitely many points. Hence $j_*(M)|_{\overline{C} \setminus V}$ is a locally free $j_*\mathcal{O}_C|_{\overline{C} \setminus V}$ -module. In particular j_*M is locally free over $j_*\mathcal{O}_C$. Let $p \in \overline{C} \setminus C$ and then the stalk $(j_*M)_p$ is a $K_{\overline{C},p}$ -vector space. The stalk is a $\mathcal{D}_{\overline{C},p}$ -module and is naturally endowed with a structure of an algebraic meromorphic connection at $p \in \overline{C}$ via $\nabla(m) = dx \otimes \partial m$ where x is a local coordinate at p . We call this $\mathcal{D}_{\overline{C}}$ -module **the algebraic meromorphic extension of M** .

Definition 6.14. Let M be an integrable connection on a smooth algebraic curve C . For a boundary point $p \in \overline{C} \setminus C$ we say that M has regular singularity at p if the algebraic meromorphic connection $(j_*M)_p$ is regular.

An integrable connection M on C is **regular** if it has regular singularities at all boundary points.

Recall that a holonomic \mathcal{D} -module is generically an integrable connection. In dimension 1 the converse is also true. Therefore we define regularity for holonomic \mathcal{D} -modules specifically for C a smooth algebraic curve.

Definition 6.15. A holonomic \mathcal{D} -module M on a smooth algebraic curve C is **regular** if there is an open dense C_0 so that $M|_{C_0}$ is a regular integrable connection. An object $M^\bullet \in D_h^b(\mathcal{D}_C)$ is regular if all of its cohomology sheaves are regular.

Lemma 6.16. Let $f : C \rightarrow C'$ dominant (i.e. the image is dense) and M a holonomic \mathcal{D}_C -module. Then $f^!M$ is regular if and only if M is regular. Then $\int_f N$ is regular if N is regular.

6.4 Deligne's (classical) Riemann-Hilbert correspondence

6.4.1 Regularity in general

Let X be a complex manifold and D divisor. Let $\mathcal{O}_X(D)$ be the sheaf of meromorphic functions on X which are holomorphic on $Y = X \setminus D$ and have poles along D . Let $h \in \mathcal{O}_X$ be a local defining equation for D so that $\mathcal{O}_X(D) = \mathcal{O}_X(h^{-1})$ is a coherent sheaf of rings.

Definition 6.17. Let M be a coherent $\mathcal{O}_X(D)$ -module with a \mathbb{C} -linear map

$$\nabla : M \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} M$$

so that

$$\nabla(fu) = df \otimes u + f\nabla u \quad (\text{for all } f \in \mathcal{O}_X(D), u \in M) \quad [\nabla_\theta, \nabla'_\theta] = [\nabla_\theta, \nabla_{\theta'}] \quad (\text{for all } \theta, \theta' \in \Theta_X)$$

The pair M, ∇ is a **meromorphic connection** on X with poles along D .

Let $\text{Conn}(X, D)$ be the category of meromorphic connections on X with poles along D . Set $B =$ open unit disk in \mathbb{C} . For a morphism $i : B \rightarrow X$ so that $i^{-1}D = \{0\}$, the stalk $(i^*M)_0$ is a meromorphic connection in one variable as studied in the \mathbb{A}^1 case.

Definition 6.18. A meromorphic connection M, ∇ on X with poles along D is *regular* if $(i^*M)_0$ is regular for all $i : B \rightarrow X$ so that $i^{-1}D = \{0\}$.

Let $\text{Conn}^{\text{reg}}(X, D)$ be the category of regular meromorphic connections along D .

Definition 6.19. A meromorphic connection on X along D is *effective* if it is generated as an $\mathcal{O}_X(D)$ -module by a coherent \mathcal{O}_X -submodule.

Any regular meromorphic connection is effective.

6.4.2 Logarithmic poles

Let D be a normal crossing divisor. Say that D is locally defined by $x_1 \cdots x_r = 0$ in local coordinates x_1, \dots, x_n . Let $p \in D$ and denote D_k the irreducible components of D defined by $x_k = 0$.

Let M be a meromorphic connection on X with poles along D . We also assume that there exists a holomorphic vector bundle L on X so that $M = L \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ as a $\mathcal{O}_X(D)$ -module. Therefore, in an appropriate basis e_1, \dots, e_n for L we have

$$\nabla e_i = \sum_j A_{ij}^k e_j \otimes dx_k$$

where $A_{ij}^k \in \mathcal{O}_X(D) = \mathcal{O}_X[x_1^{-1}, \dots, x_n^{-1}]$. Then we further assume that the functions $x_k A_{ij}^k$ are *holomorphic*. In this case we say that M along D has a logarithmic pole with respect to the lattice L at p . If this is the case at all points of D then we say that M has logarithmic poles along D with respect to L .

Let M be a meromorphic connection on X with poles along D , with logarithmic poles with respect to L along D . Take basis e_i of L and consider the A^k matrices from before. Let $B^k = x_k A^k$. By hypothesis B^k are holomorphic and so we consider its restriction $B^k|_{D_k} \in M_n(\mathcal{O}_{D_k})$. Then $B^k|_{D_k}$ defines a canonical section $\text{Res}_{D_k}^L \nabla$ of the vector bundle $\text{End}_{\mathcal{O}_{D_k}}(L|_{D_k})$. This is called the *residue* of ∇ along D_k with respect to L .

We care about logarithmic poles from a regularity standpoint.

Proposition 6.20. Let M be a meromorphic connection which has a logarithmic pole along D with respect to a lattice L . Then M is regular.

Remark 6.21. What is the difference between an integrable connection and a meromorphic connection? It is worth considering this question because in the next section, we will be extending integrable connections along divisors at the cost of them becoming meromorphic connections.

Recall that integrable connections are about parallel transport of sections of a vector bundle. The requirement that the connection is flat means that the transport of sections along paths is independent of the path. This is in fact a hallmark. Given an integrable connection, its horizontal sections are the solutions to a system of linear ODEs. The corresponding system of ODEs is nonsingular. For a meromorphic connection, the corresponding system of ODEs is allowed to have singularities. Regularity is about the singularities being "finite dimensional" in some sense.

Remark 6.22. Meromorphic Connections: Away from the Divisor

- Away from the divisor, the meromorphic connection behaves similarly to a regular connection. That is, it smoothly defines how the sections of the bundle evolve as you move along different directions on the manifold.
- For a vector bundle E over a complex manifold X with coordinates z_1, z_2, \dots, z_n , the connection locally looks like:

$$\nabla = d + A(z),$$

where $A(z)$ is a matrix of holomorphic functions, and d is the exterior derivative that captures how sections change infinitesimally.

- In this region, the connection is holomorphic and behaves like a flat or integrable connection if the curvature vanishes.

Meromorphic Connections: Near the Divisor

- Geometrically, near a divisor (which is typically defined by $D = \{z_1 = 0\}$ locally), the connection takes on a different character. The presence of a pole along the divisor means that sections of the bundle can exhibit singular behavior as you approach D . However, this singularity is controlled or regular in the case of a regular meromorphic connection.
- The local form of a meromorphic connection near a divisor might look like:

$$\nabla = d + \frac{A(z)}{z_1} dz_1 + B(z) dz_2 + \dots,$$

where $A(z)$ is a matrix of holomorphic functions. The term $\frac{A(z)}{z_1} dz_1$ indicates the presence of a pole along $z_1 = 0$ (the divisor D).

6.4.3 What does ODEs and \mathcal{D} -module language really mean?

Recall that earlier I told you the following theorem:

Theorem 6.23. *Holonomic \mathcal{D} -modules are generically integrable connections.*

What does this really mean? Recall that given a linear ODE

$$P(x, \partial)u = 0$$

we can form the corresponding \mathcal{D} -module $M = \mathcal{D}/\mathcal{D}P$.

We say that a linear ODE is holonomic if its solution space is finite dimensional. The point is that this means that the ODE is highly constrained and therefore so should the corresponding characteristic variety.

In the context of linear ODEs, a regular singular point is defined as a point where the ODE has specific controlled behavior. For a second-order linear ODE of the form:

$$y'' + p(x)y' + q(x)y = 0,$$

a point x_0 is a regular singular point if the coefficients $p(x)$ and $q(x)$ have certain pole behaviors:

- The function $p(x)$ can have a pole of order at most 1 at x_0 .
- The function $q(x)$ can have a pole of order at most 2 at x_0 .

Recall that we defined regularity as follows: A meromorphic connection ∇ on a sheaf \mathcal{M} is said to be regular at $x = 0$ if there exists a finitely generated \mathcal{O} -submodule L of \mathcal{M} such that L is stable under the action of the Euler operator θ and generates \mathcal{M} over the field K (the field of fractions of \mathcal{O}). **Therefore, being regular is about having solutions exhibit controlled growth near singular points.**

Reminder! We also proved an equivalent criterion for regularity in the context of meromorphic connections over \mathbb{C} :

Theorem 6.24. *A meromorphic connection M, ∇ is regular if and only if there exists a finite K -basis e_i for M the associated ODE is of the form*

$$\frac{d}{dx}u = \frac{A}{x}u$$

for $A(x) \in M_n(\mathbb{C})$.

Remark 6.25. *The following is speculation. From the above theorem and our characterization of regular holonomic ODEs, I suspect that all of the regular holonomic \mathcal{D} -modules on $\mathbb{A}_{\mathbb{C}}^1$ are of the form*

$$\mathcal{D}/\mathcal{D}(x\partial - \lambda)$$

Remark 6.26. *An ODE system corresponds to an integrable connection if it is flat and has no monodromy. This is different than a meromorphic connection! A meromorphic connection can have singularities without specific restrictions.*

6.4.4 Extending integrable connections along normal crossing divisors

Theorem 6.27. *Let D be a normal crossing divisor on a complex manifold X . Let $Y = X \setminus D$ and fix a section $\tau : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}$ of the projection $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$. Let M be an integrable connection on Y . There exists an extension L_τ of M as a vector bundle on X satisfying the following conditions:*

- *The \mathbb{C} -linear morphism $\nabla_M : M \rightarrow \Omega_Y^1 \otimes M$ can be uniquely extended to a \mathbb{C} -linear morphism $\nabla : \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} L_\tau \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} L_\tau$ so that $(\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} L_\tau, \nabla)$ is a meromorphic connection with a logarithmic pole along D with respect to L_τ .*
- *For any irreducible component D' of D the eigenvalues of the residue $\text{Res}_{D'}^{L_\tau} \nabla$ of $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} L_\tau$ along D' are contained in $\tau(\mathbb{C}/\mathbb{Z}) \subset \mathbb{C}$.*

Moreover L_τ is unique up to isomorphism.

Now we want to extend to a general divisor D . We do this and I skip a lot of things and then we get the main result:

Theorem 6.28 (Deligne). *Let X be a complex manifold and D a divisor. Let $Y = X \setminus D$. Then the restriction functor*

$$\text{Conn}^{\text{reg}}(X, D) \rightarrow \text{Conn}(Y) \cong \text{Loc}(Y)$$

is an equivalence of categories.

For a general algebraic variety we have the following definition:

Definition 6.29. *An integrable connection M on X is called **regular** if for every morphism $i_C : C \rightarrow X$ from a smooth algebraic curve C the pullback $i_C^* M$ is regular.*

Theorem 6.30 (Deligne). *Let X be a smooth algebraic variety. Then the functor*

$$M \mapsto M^{\text{an}}$$

is an equivalence of categories

$$\text{Conn}^{\text{reg}}(X) \rightarrow \text{Conn}(X^{\text{an}})$$

It turns out that when X is a smooth projective variety

$$\text{Conn}(X) \cong \text{Conn}(X^{\text{an}})$$

These statements are all about Serre GAGA.

7 The Riemann-Hilbert correspondence

7.1 Regularity in full generality

Definition 7.1. Let X be a smooth algebraic variety. A holonomic \mathcal{D} -module M on X is called **regular** if any composition factor of M is isomorphic to the minimal extension $L(Y, N)$ of some regular integrable connection N on a locally closed smooth subvariety $Y \subset X$ so that the inclusion $Y \hookrightarrow X$ is affine.

We denote $\text{Mod}_{rh}(\mathcal{D}_X)$ the full subcategory of $\text{Mod}_h(\mathcal{D}_X)$ consisting of regular holonomic \mathcal{D} -modules. We denote $D_{rh}^b(\mathcal{D}_X)$ the full subcategory of $D_h^b(\mathcal{D}_X)$ consisting of complexes with regular holonomic cohomology sheaves.

This agrees with our definition for curves. Any simple holonomic \mathcal{D}_C -module is of the form $L(Y, N)$ where Y is a single point or a connected nonempty open subset of C and N is a regular integrable connection on Y .

We come to some big results.

Theorem 7.2. Let X smooth algebraic variety.

1. The duality functor \mathbb{D}_X preserves $D_{rh}^b(\mathcal{D}_X)$.
2. Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties. Then $\int_f, \int_{f!}, f^\dagger, f^\star$ preserve $D_{rh}^b(\mathcal{D}_X)$.

Theorem 7.3 (Curve testing criterion). Let X be a smooth algebraic variety. The following about $M^\bullet \in D_h^b(\mathcal{D}_X)$ are equivalent:

1. M^\bullet is regular.
2. For any locally closed embedding $i_C : C \rightarrow X$ from a smooth algebraic curve C the pullback $i_C^\dagger M^\bullet \in D_h^b(\mathcal{D}_C)$ is regular.
3. For any morphism $k : C \rightarrow X$ from a smooth algebraic curve C $k^\dagger M^\bullet \in D_h^b(\mathcal{D}_C)$ is regular.

7.2 The Riemann-Hilbert correspondence

Recall that for a smooth algebraic variety X we have the duality functors

$$\begin{aligned} \mathbb{D}_X &: D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_X)^{op} \\ \mathbb{D}_X(M^\bullet) &= R\text{Hom}_{\mathcal{D}_X}(M^\bullet, \mathcal{D}_X) \\ \mathbf{D}_X &: D_c^b(X) \rightarrow D_c^b(X)^{op} \\ \mathbf{D}_X(M^\bullet) &= R\text{Hom}_{\mathbb{C}_X}(M^\bullet, \omega_X^\bullet) \end{aligned}$$

where $D_c^b(X)$ is the bounded derived category of constructible sheaves on X , \mathbf{D} is Verdier duality. All of the functors $\mathbb{D}_X, \int_f, \int_{f!}, f^\dagger, f^*$ preserve regular holonomic \mathcal{D} -modules. We also know that

$$\mathbf{D}_X \mathrm{DR}_X(M^\bullet) = \mathrm{DR}_X(\mathbb{D}_X(M^\bullet)) \quad \text{for } M^\bullet \in D_h^b(\mathcal{D}_X)$$

i.e. DR commutes with duality. It turns out that it also commutes with the inverse and direct image functors.

Theorem 7.4. *Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties. Then*

$$\begin{aligned} \mathrm{DR}_Y \circ \int_f &\cong Rf_* \circ \mathrm{DR}_X : D_{rh}^b(\mathcal{D}_X) \rightarrow D_c^b(Y) \\ \mathrm{DR}_Y \circ \int_{f!} &\cong Rf_! \circ \mathrm{DR}_X : D_{rh}^b(\mathcal{D}_X) \rightarrow D_c^b(Y) \\ \mathrm{DR}_X \circ f^\dagger &\cong f^! \circ \mathrm{DR}_Y : D_{rh}^b(\mathcal{D}_Y) \rightarrow D_c^b(X) \\ \mathrm{DR}_X \circ f^* &\cong f^{-1} \circ \mathrm{DR}_Y : D_{rh}^b(\mathcal{D}_Y) \rightarrow D_c^b(X) \end{aligned}$$

Theorem 7.5 (Riemann-Hilbert correspondence). *Let X be a smooth algebraic variety. The DR functor is an equivalence of categories*

$$\mathrm{DR}_X : D_{rh}^b(\mathcal{D}_X) \rightarrow D_c^b(X)$$

The solution functor

$$\mathrm{Sol}_X : D_{rh}^b(X) \rightarrow D_c^b(X)^{op}$$

is an equivalence of categories as well. Passing to the full subcategory $\mathrm{Mod}_{rh}(\mathcal{D}_X)$ we get perverse sheaves which receive the image of the DR functor.

Theorem 7.6. *The DR functor induces an equivalence of categories*

$$\mathrm{DR}_X : \mathrm{Mod}_{rh}(\mathcal{D}_X) \rightarrow \mathrm{Perv}(\mathbb{C}_X)$$

Remark 7.7. *Let $Y \subset X$ algebraic subvariety and consider a local system L on U^{an} for open dense subset U of the regular part of Y . Then we can associate to it an intersection complex $IC_X(L)^\bullet$ on X which is a perverse sheaf which is irreducible and whose support is contained in Y . Consider the regular integrable connection M on U which corresponds to L . Then*

$$\mathrm{DR}_X L(U, M) \cong IC_L$$

8 Perverse sheaves