Title

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Abstract

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Definition 1.1. An *algebraic space* over a base scheme S is a functor $X\colon (\mathrm{Sch}/S)^{op} \to \mathrm{Sets}$ satisfying

1. X is a sheaf in the fppf topology,
2. the diagonal morphism $\Delta\colon X \to X \times_S X$ is representable by schemes,
3. there exists a surjective étale morphism $U \to X$ from a scheme U.

By definiton, if $x \in X$ is a point of an algebraic space X, then x admits an étale neighborhood $U \to X$ where U is a scheme.

Example 1.2 (An étale map need not induce an isomorphism on ordinary local rings)**.** Let

$$X = \operatorname{Spec} \mathbb{C}[t, t^{-1}] \qquad Y = \operatorname{Spec} \mathbb{C}[u, u^{-1}],$$

and let $f\colon Y \to X$ be the finite morphism given by $t = u^2$. This morphism is étale: indeed, locally it is given by adjoining a root of $g(T) = T^2 - t$, and on Y we have $g'(u) = 2u \in \mathbb{C}[u, u^{-1}]^\times$.

Consider the generic point $\eta \in X$, so $\mathcal{O}_{X,\eta} = \kappa(\eta) = \mathbb{C}(t)$. Let $\eta' \in Y$ be the generic point lying over η, so $\mathcal{O}_{Y,\eta'} = \kappa(\eta') = \mathbb{C}(u) = \mathbb{C}(t^{1/2})$. The induced map on local rings

$$\mathcal{O}_{X,\eta} = \mathbb{C}(t) \longrightarrow \mathcal{O}_{Y,\eta'} = \mathbb{C}(t^{1/2})$$

coming from $t \mapsto u^2$ is not an isomorphism.

In this example, we are making use of the following general criterion for étaleness:

Definition 1.3 (Standard étale)**.** Let R be a ring and let $g, f \in R[x]$. Assume that f is monic and that the formal derivative f' maps to a unit in the localization

$$R[x]_g/(f).$$

In this case the ring map

$$R \longrightarrow R[x]_g/(f)$$

is said to be *standard étale*.

Lemma 1.4. Let $R \to R[x]_g/(f)$ be standard étale.

(1) The ring map $R \to R[x]_g/(f)$ is étale.

(2) For any ring map $R \to R'$, the base change

$$R' \longrightarrow R'[x]_{g'}/(f') \qquad \text{with } R'[x]_{g'}/(f') \cong R' \otimes_R \big(R[x]_g/(f)\big)$$

is standard étale (where f', g' denote the images of f, g in $R'[x]$).

(3) Any principal localization of $R[x]_g/(f)$ is standard étale over R.

(4) A composition of standard étale maps need not be standard étale in general.

The example shows that the ordinary local ring $\mathcal{O}_{X,x}$ is not preserved under passing to an étale neighborhood. The correct object that is invariant under passing to an étale neighborhood is not theordinary local ring $\mathcal{O}_{X,x}$ but the *étale stalk* of the structure sheaf, which is canonically identified with a strict henselization of the ordinary local ring.

Definition 1.5. Given a site $(\mathcal{C}, J)$, a point of the associated topos $\mathbf{Sh}(\mathcal{C}, J)$ is (equivalently) a functor

$$p^{-1} : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sets}$$

that is left exact and has a right adjoint (a "geometric morphism" $\mathbf{Sets} \to \mathbf{Sh}(\mathcal{C}, J)$). The stalk of a sheaf $\mathcal{F}$ at the point p is $p^{-1}(\mathcal{F})$.

Points are not objects of $\mathcal{C}$ a priori; they are topos-theoretic. One can show a geometric point of X is a morphism

$$\bar{x}\colon \operatorname{Spec}(\Omega) \to X$$

with Ω separably closed. It lies over an underlying (ordinary) point $x \in |X|$, and choosing $\bar{x}$ is equivalent to choosing an embedding

$$\kappa(x) \hookrightarrow \Omega$$

into a separably closed field.

The subtlety is:

- The étale stalk $(\mathcal{O}_X)_{\bar{x}}$ is a stalk in the étale topology, and stalks are taken at points of the étale topos, i.e. geometric points, not just underlying Zariski points.
- The strict henselization $(\mathcal{O}_{X,x})^{\mathrm{sh}}$ is not determined by x alone: it depends on a choice of a separable closure of the residue field $\kappa(x)$ inside some separably closed field. That choice is precisely the extra datum carried by $\bar{x}$.

So if you specify only x, then $(\mathcal{O}_{X,x})^{\mathrm{sh}}$ is well-defined only up to noncanonical isomorphism; once you specify $\bar{x}$, it becomes canonical. Let X be an algebraic space.

- An underlying point $x \in |X|$ does not determine a canonical local ring. Any attempt to define $\mathcal{O}_{X,x}$ requires choosing an étale chart and a lift of x, and different choices need not agree.
- A geometric point $\bar{x} \to X$, however, does determine a canonical local ring:

  $$\mathcal{O}_{X,\bar{x}} := (\mathcal{O}_X)_{\bar{x}},$$

  which is invariant under étale pullback.

Definition 1.6 (Étale local ring)**.** Let X be an algebraic space and let $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$ be a geometric point lying over $x \in |X|$. The *étale local ring* of X at $\bar{x}$ is the stalk

$$\mathcal{O}_{X,\bar{x}} := (\mathcal{O}_X)_{\bar{x}} = \varinjlim_{(U,\bar{u})\to(X,\bar{x})} \Gamma(U, \mathcal{O}_U)$$

of the structure sheaf on the étale site X_t.

If X is a scheme (or after choosing an étale chart $U \to X$ with a lift of $\bar{x}$), the following lemma above identifies this canonically with the strict henselization

$$\mathcal{O}_{X,\bar{x}} \;\cong\; \left(\mathcal{O}_{X,x}\right)^{sh}.$$

Lemma 1.7 (Stacks Project, Lemma 59.33.1)**.** Let S be a scheme and let $\bar{s}$ be a geometric point of S lying over $s \in S$. Let $\kappa = \kappa(s)$ and let $\kappa^{\mathrm{sep}} \subset \kappa(\bar{s})$ denote the separable algebraic closure of κ inside $\kappa(\bar{s})$. Then there is a canonical identification

$$\left(\mathcal{O}_{S,s}\right)^{sh} \;\cong\; (\mathcal{O}_S)_{\bar{s}},$$

where the left-hand side is the strict henselization of the local ring at s, and the right-hand side is the stalk of the structure sheaf on the étale site S_t at the geometric point $\bar{s}$.

> Even though the etale local ring is defined as a colimit over all etale neighborhoods, it can be computed from any single etale neighborhood. This is a consequence of the following proposition about weakly etale maps, and the fact that etale maps are weakly etale.

Definition 1.8. A map $f\colon A \to B$ of rings is called weakly étale if it is flat and the diagonal map $B \otimes_A B \to B$ is also flat.

Proposition 1.9. Let $A \to B$ be a weakly étale map of local rings. Then the induced map on strict henselizations $A^{sh} \to B^{sh}$ is an isomorphism, provided you choose the strict henselizations compatibly (pick a separable closure of $\kappa(x)$, and take the separable closure of $\kappa(y)$ inside it) where x and y are the closed points of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ respectively.

Definition 1.10 (Ideal sheaves on an algebraic space)**.** Let X be an algebraic space. An *ideal sheaf* $I \subset \mathcal{O}_X$ is a subsheaf of rings of the structure sheaf on the étale site X_t such that, étale-locally, it is an ideal in the usual sense.

Concretely, choosing an étale surjection $U \to X$ with U a scheme, an ideal sheaf $I \subset \mathcal{O}_X$ is equivalent to an ideal $I_U \subset \mathcal{O}_U$ such that the pullbacks of I_U to $U \times_X U$ via the two projections agree.

Remark 1.11 (Ideal sheaves versus local rings on algebraic spaces)**.** There is an important asymmetry between ideal sheaves and local rings on an algebraic space.

(1) Ideal sheaves descend étale-locally. Ideal sheaves are sheaf-theoretic objects on the étale site. Because étale morphisms are flat and the étale topology admits effective descent for quasi-coherent sheaves, an ideal of $\mathcal{O}_X$ may be defined on any étale chart $U \to X$ and uniquely descended to X. In particular, closed subspaces and formal completions of algebraic spaces are defined étale-locally on schemes.

(2) Ordinary local rings do *not* descend étale-locally. The ordinary local ring $\mathcal{O}_{X,x}$ at a point $x \in |X|$ is defined as a filtered colimit over Zariski neighborhoods of x. Since étale morphisms need not identify Zariski neighborhoods, passing to an étale chart generally changes the ordinary local ring. Thus ordinary local rings are not invariant under étale pullback.

(3) The correct local object is the étale stalk. The local invariant preserved by étale morphisms is instead the étale stalk $(\mathcal{O}_X)_{\bar{x}}$ at a geometric point $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$, which is canonically identified with the strict henselization $(\mathcal{O}_{X,x})^{sh}$. Unlike the Zariski local ring, this object is étale-local by construction.

In summary, ideal sheaves are global objects defined by étale descent, while ordinary local rings depend on the Zariski topology and are not compatible with êtale localization. This distinction underlies the use of étale topology and formal geometry in the theory of algebraic spaces.

The étale topology over C

Even when X is a scheme of finite type over $\mathbf{C}$, the étale topology is strictly finer than the Zariski topology and leads to a genuinely different notion of locality.

Example 1.12. Over $\mathbf{C}$, finite étale morphisms are exactly finite unbranched coverings in the analytic topology. Consider the map

$$\mathbb{G}_m \to \mathbb{G}_m, \quad z \mapsto z^n$$

is finite étale but not Zariski locally trivial.

$$X = \operatorname{Spec} \mathbf{C}[t, t^{-1}], \qquad Y = \operatorname{Spec} \mathbf{C}[u, u^{-1}], \qquad t = u^n.$$

Then

$$\mathbf{C}[u, u^{-1}] \cong \mathbf{C}[t, t^{-1}][T]/(T^n - t).$$

So $Y \to X$ is finite (monic polynomial).

For étale: use the standard criterion for $A[T]/(f)$ with f monic. Here $f(T) = T^n - t$, so $f'(T) = nT^{n-1}$. In $B = \mathbf{C}[u, u^{-1}]$, $f'(u) = nu^{n-1}$, which is a unit since u is invertible on $\mathbb{G}_m$. Hence the map is étale everywhere.

However, this map is not Zariski locally trivial: there is no Zariski open cover of X over which the map splits as a disjoint union of copies of the base. Equivalently, on rings this says: for each i, the A_i-algebra $A_i[T]/(T^n - t) \quad$ with $A_i = \Gamma(U_i, \mathcal{O}_X)$ splits as a product $A_i \times \cdots \times A_i$. That happens iff $T^n - t$ factors into linear factors over A_i, i.e. iff t admits an n-th root in A_i: $t = s_i^n \quad$ in A_i.

But $X = \operatorname{Spec} \mathbf{C}[t, t^{-1}]$ is irreducible, so at least one U_i contains the generic point. For that U_i, the inclusion of rings $\mathbf{C}(t) \hookrightarrow \operatorname{Frac}(A_i)$ holds, and $t = s_i^n$ in A_i would imply t is an n-th power in the function field $\mathbf{C}(t)$. That is false: in $\mathbf{C}(t)$, the element t has valuation 1 at $t = 0$, so it cannot be an n-th power (an n-th power would have valuation divisible by n). Contradiction.

Let $x \in X$ be a closed point and let $\bar{x}$ be a geometric point lying over x. Since $\kappa(x) = \mathbf{C}$ is already separably closed, the étale local ring is given by

$$\mathcal{O}_{X,\bar{x}}^{\text{ét}} = (\mathcal{O}_{X,x})^{sh} = (\mathcal{O}_{X,x})^h,$$

the henselization of the Zariski local ring. In general this ring is strictly larger than $\mathcal{O}_{X,x}$, although the two have the same completion:

$$\widehat{\mathcal{O}_{X,x}} \;\cong\; \widehat{\mathcal{O}^{\acute{e}t}_{X,\bar{x}}}.$$

Thus completion only captures the formal neighborhood of x, while the étale local ring retains the minimal henselian enlargement required to solve étale equations (via Hensel's lemma). In this sense, étale locality is strictly stronger than Zariski locality even over $\mathbf{C}$.

2 Elliptic curves and contractions

A smooth elliptic curve in $\mathbb{P}^2$ is a smooth plane cubic $E \subset \mathbb{P}^2$. As a divisor $E \sim 3H$, so

$$E^2 = (3H)^2 = 9,$$

so it is not (-1).

You can produce an elliptic curve of self-intersection -1 on a blowup of $\mathbb{P}^2$: blow up n distinct points $p_1, \ldots, p_n$ on a smooth cubic E, and let $\widetilde{E}$ be the strict transform. Then

$$\widetilde{E}^2 = E^2 - \sum_{i=1}^{n} (\operatorname{mult}_{p_i} E)^2 = 9 - n,$$

so choosing $n = 10$ gives $\widetilde{E}^2 = -1$.

For a smooth projective surface X, the classical Castelnuovo contraction theorem says that a curve that can be contracted by a morphism $X \to Y$ to a smooth surface is necessarily a smooth rational curve with self-intersection -1.

More generally, if a proper birational morphism from a smooth surface contracts a single irreducible curve to a point and the target is a scheme, then the exceptional curve has to be $\mathbb{P}^1$ (in particular, genus 0). An elliptic curve can't be the exceptional locus of such a contraction to a scheme.

Artin developed criteria ensuring that certain negative-definite curves on surfaces admit contractions in the category of algebraic spaces (even when no contraction exists as a scheme).

Definition 2.1. Let X be a surface (algebraic space or scheme). A proper curve $C \subset X$ is *contractible in the sense of Artin* if there exists a proper surjective morphism

$$f\colon X \to Y$$

to an algebraic space Y such that

1. f is an isomorphism on $X \setminus C$,
2. $f(C) = \{p\}$ is a single point,
3. Y is separated and of finite type over the base field,
4. (usually imposed) $f_*\mathcal{O}_X = \mathcal{O}_Y$ (so you are not introducing extra connected components in the fibers; this is the "Stein" condition).

If C is an elliptic curve with $C^2 < 0$ and it is "contractible" in Artin's sense, then:

- A contraction $f\colon X \to Y$ exists in the category of algebraic spaces.
- The target Y is an algebraic space (of finite type, separated) which is a scheme away from the point p.

Let S be a finite type scheme over a field k.

Theorem 2.2 (Existence of contractions, Artin)**.** Let $Y' \subset X'$ be a closed subset of an algebraic space X' of finite type over a base scheme S, and let $\mathfrak{X}'$ be the formal completion of X' along Y'. Given a formal modification

$$f_1\colon \mathfrak{X}' \longrightarrow \mathfrak{X}_1,$$

there exists a modification

$$f\colon X' \longrightarrow X, \qquad Y \subset X,$$

together with an isomorphism φ between the completion of f along Y and the given formal modification f_1. The pair (f, φ) is determined up to unique isomorphism.

Theorem 2.3 (Artin, Formal Moduli II, Thm. 6.2)**.** Let $\mathfrak{X}'$ be a formal algebraic space, let $Y' = V(I')$ be the closed formal subspace defined by a defining ideal $I' \subset \mathcal{O}_{\mathfrak{X}'}$, and let $f\colon Y' \longrightarrow Y$ be a proper morphism.

Suppose the following conditions hold:

(i) For every coherent sheaf $\mathcal{F}$ on $\mathfrak{X}'$, we have

$$R^1 f_*\big(I'^n\mathcal{F}/I'^{n+1}\mathcal{F}\big) = 0 \qquad \text{for } n \gg 0.$$

(ii) For every $n \geq 0$, the natural morphism of sheaves on $Y_0 := V(I')_{\mathrm{red}}$

$$f_*\big(\mathcal{O}_{\mathfrak{X}'}/I'^n\big) \times_{f_*\mathcal{O}_{Y'}} \mathcal{O}_Y \;\longrightarrow\; \mathcal{O}_Y$$

is surjective.

Then there exists a formal modification $\widehat{f}\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ and a defining ideal $I \subset \mathcal{O}_{\mathfrak{X}}$ such that $V(I) = Y$ and such that $\widehat{f}$ induces the given morphism $f\colon Y' \to Y$.

Theorem 2.4 (Knutson, Thm. III.6.2)**.** Let X' be a nonsingular algebraic space of finite type over a field k, and let $Y' \subset X'$ be a closed nonsingular subspace. Assume that $\dim X' = d$ and $\dim Y' = d-1$, and that Y' is proper over k. Let

$$\mathcal{L} := N^{\vee}_{Y'/X'}$$

denote the conormal bundle of Y' in X', and assume that $\mathcal{L}$ is ample on Y'.

Then there exists a proper birational morphism

$$f\colon X' \longrightarrow X$$

to an algebraic space X such that:

1. f is an isomorphism over $X \setminus \{p\}$,
2. $f(Y') = \{p\}$ is a single point,
3. $f^{-1}(p) = Y'$.

In other words, Y' can be contracted to a point in the category of algebraic spaces.

Proof. The proof proceeds by reducing the problem to Artin's formal algebraization theorem (Theorem 2.3).

Let $\mathfrak{X}' := \widehat{X'}_{Y'}$ denote the formal completion of X' along Y', with defining ideal $I' \subset \mathcal{O}_{\mathfrak{X}'}$. Since Y' is a smooth divisor in the smooth space X', the graded pieces of the I'-adic filtration are given by

$$I'^n/I'^{n+1} \;\cong\; \mathrm{Sym}^n(N^{\vee}_{Y'/X'}).$$

Because Y' is proper and the conormal bundle $N^{\vee}_{Y'/X'}$ is ample, Serre vanishing implies that for any coherent sheaf $\mathcal{F}$ on X',

$$H^1\big(Y', \mathrm{Sym}^n(N^{\vee}_{Y'/X'}) \otimes \mathcal{F}|_{Y'}\big) = 0 \quad \text{for } n \gg 0.$$

This verifies condition (i) of Theorem 2.3.

As for condition (ii), we have $Y = \{p\}$ a point, so $\mathcal{O}_Y = k$. Writing $A_n := \Gamma\big(Y', \mathcal{O}_{\mathfrak{X}'}/I'^n\big)$ and $A_0 := \Gamma\big(Y', \mathcal{O}_{Y'}\big)$, Artin's map in (ii) becomes the ring map

$$A_n \times_{A_0} k \longrightarrow k,$$

where $k \to A_0$ is the structure map. This map is the projection to the second factor, hence is automatically surjective. Thus, in this special case, Artin's condition (ii) is automatic. □