Title

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1 Principal G-Bundles on Affine Curves

It is a consequence of a theorem of Harder [**?**, Satz 3.3] that generically trivial principal G-bundles on a smooth affine curve C over an arbitrary field k are trivial if G is a semisimple and simply connected algebraic group. When k is algebraically closed and G reductive, generic triviality, conjectured by Serre, was proved by Steinberg [**?**] and Borel–Springer [**?**].

It follows that principal bundles for simply connected semisimple groups over smooth affine curves over algebraically closed fields are trivial. This fact (and a generalization to families of bundles [**?**]) plays an important role in the geometric realization of conformal blocks for smooth curves as global sections of line bundles on moduli-stacks of principal bundles on the curves (see the review [**?**] and the references therein).

1.1 Derived Pushforward of Admissible Complexes

Theorem 1.1. The derived pushforward $RF_*\alpha$ of an admissible complex α along the bundle-forgetting map $F : \mathcal{M}_{g,I}([pt/G]) \to \mathcal{M}_{g,I}$ is a bounded complex of coherent sheaves.

This theorem is a relative version over varying curves of the analogous finiteness result for $\mathrm{Bun}_G(\Sigma)$ in [**?**, 34].

1.2 Finiteness for Fixed Curves

Let G be a reductive, connected complex Lie group and $\mathcal{M}$ the moduli stack of algebraic G-bundles over a smooth projective curve Σ of genus g. We recall the finiteness theorem for this moduli stack. We recall the finiteness theorem for the moduli stack of principal bundles on a fixed smooth curve.

1.1 Admissible classes

Given a representation V of G, call E^*V the vector bundle over $\Sigma \times \mathcal{M}$ associated to the universal G–bundle. Call π the projection along Σ, the relative canonical bundle K of $\Sigma \times \mathcal{M} \to \mathcal{M}$ (so that $K|_\Sigma = K_\Sigma$), $\sqrt{K}$ its square root, $[C]$ the topological K_1–homology class of a 1–cycle C on Σ. Consider the following classes in the topological K–theory of $\mathcal{M}$:

(i) The restriction $E^*_x V \in K^0(\mathcal{M})$ of E^*V to a point $x \in \Sigma$;

(ii) The slant product $E^*_C V := E^*V/[C] \in K^{-1}(\mathcal{M})$ of E^*V with $[C]$;

(iii) The Dirac index bundle $E^*_\Sigma V := R\pi_*(E^*V \otimes \sqrt{K}) \in K^0(\mathcal{M})$ of E^*V along Σ;

(iv) The inverse determinant of cohomology,

$$D_\Sigma V := \det{}^{-1} E^*_\Sigma V.$$

We call the classes (i)–(iii) the *Atiyah–Bott generators*; they are introduced in [**?**, §2], along with their counterparts in cohomology, and can also be described from the Künneth decomposition of E^*V in

$$K^0(\Sigma \times \mathcal{M}) \;\cong\; K^0(\Sigma) \otimes K^0(\mathcal{M}) \;\oplus\; K^1(\Sigma) \otimes K^1(\mathcal{M}),$$

by contraction with the various classes in Σ. Classes (i) and (iv) are represented by algebraic vector bundles, while (iii) can be realised as a perfect complex of $\mathcal{O}$–modules. The class $E^*_C V$ in (ii) is not algebraic. Note that

$$\det E^*_\Sigma V = \det R\pi_*(E^*V)$$

when $\det V$ is trivial; an important example is the canonical bundle

$$\mathcal{K} = \det E^*_\Sigma \mathfrak{g}$$

of $\mathcal{M}$, defined from the adjoint representation $\mathfrak{g}$.

Remark 1.2. For a line bundle $\mathcal{L}$ on $\mathcal{M} = \mathrm{Bun}_G(\Sigma)$, one associates a *level* $\lambda(\mathcal{L})$, namely the invariant symmetric bilinear form on $\mathfrak{g}$ corresponding to the class $\lambda(\mathcal{L}) \in H^4(BG;\mathbb{Z})$. If $\mathcal{L}$ is a determinant line bundle $\det R\pi_*(E^*V)$ attached to a representation V of G, then $\lambda(\mathcal{L})$ is the trace form $\mathrm{Tr}_V(xy)$ on $\mathfrak{g}$. When G is not simply connected, such determinant bundles do not realise all possible integral levels. Passing from the simply connected cover $\widetilde{G}$ to $G = \widetilde{G}/Z$ cuts down the lattice of integral invariant bilinear forms by imposing congruence conditions along the finite central subgroup Z, so that only a finite–index sublattice is realised by trace forms of actual G–representations.

Remark 1.3 (Smoothness and the relative canonical bundle)**.** Let $\mathcal{M} = \mathrm{Bun}_G(\Sigma)$ and let

$$\pi : \Sigma \times \mathcal{M} \longrightarrow \mathcal{M}$$

be the projection. Although the coarse moduli space of semistable G–bundles may be singular, the *stack* $\mathcal{M}$ is a smooth Artin stack of dimension $(g-1)\dim G$. Indeed, for a bundle P one has

$$T_{[P]}\mathcal{M} \simeq H^1(\Sigma, \mathrm{Ad}\, P)$$

and $H^2(\Sigma, \mathrm{Ad}\, P) = 0$ because $\dim \Sigma = 1$, so deformations are unobstructed.

The relative canonical bundle $K := K_{\Sigma\times\mathcal{M}/\mathcal{M}}$ is defined purely from the morphism π, which is smooth of relative dimension 1; no smoothness of the base is required. In fact,

$$K_{\Sigma\times\mathcal{M}/\mathcal{M}} \;\cong\; \mathrm{pr}_\Sigma^* K_\Sigma,$$

the pullback of the ordinary canonical bundle of the curve.

Remark 1.4. By contrast, the "canonical bundle" of the moduli stack itself is

$$\mathcal{K} := \det R\pi_*(E^*\mathfrak{g}),$$

the determinant of the cotangent complex of $\mathcal{M}$, and Laszlo–Sorger construct a canonical Pfaffian square root $\mathcal{K}^{1/2}$ of this line bundle. In particular, for semi-simple, not necessarily simply connected G and for every theta characteristic $K_\Sigma^{1/2}$ on Σ, one has a square root

$$\mathcal{K}^{1/2} := \det R\pi_*(E^*\mathfrak{g} \otimes \mathrm{pr}_\Sigma^* K_\Sigma^{1/2}).$$

This gives rise to a natural "reference level" $\lambda(\mathcal{K}^{1/2}) = \frac{1}{2}\lambda(\mathcal{K})$. We call a line bundle $\mathcal{L}$ on $\mathcal{M}$ *admissible* if its level exceeds that of $\mathcal{K}^{1/2}$, in the sense that $\lambda(\mathcal{L}) - \lambda(\mathcal{K}^{1/2})$ is positive definite on every simple factor of $\mathfrak{g}$.

Such positivity plays the role of an ampleness condition, and admissible line bundles provide the appropriate class of twistings needed for the K–theoretic index and Verlinde formulas. Products of an admissible line bundle and any number of Atiyah-Bott generators span the ring of *admissible classes*.

Remark 1.5. We have defined a level by an integral invariant symmetric bilinear form on $\mathfrak{g}$ and simultaneously identified with central extensions of the loop group LG. The latter is completely determined by the action of the central scalar, which is to be an integer by the integrality condition. Abstractly, the Chern-Weil homomorphism identifies the cohomology ring $H^*(BG;\mathbb{R})$ of the classifying space BG with the ring of invariant polynomials on the Lie algebra $\mathfrak{g}$ of G:

$$H^*(BG;\mathbb{R}) \cong \mathrm{Inv}(\mathfrak{g}) := \mathrm{Sym}(\mathfrak{g}^*)^G$$

and in degree four, we have

$$H^4(BG;\mathbb{R}) \cong \text{Inv}^2(\mathfrak{g})$$

the space of invariant symmetric bilinear forms on $\mathfrak{g}$. In particular $H^4(BG;\mathbb{R}) \cong H^3(\mathfrak{g})$ via the isomorphism we have just discussed. There is a transgression map arising from the fibration $G \to EG \to BG$:

$$\tau : H^4(BG;\mathbb{R}) \to H^3(G;\mathbb{R})$$

which is an isomorphism when G is compact, simple, and simply connected. Thus we have the chain of isomorphisms

$$H^4(BG;\mathbb{R}) \;\cong\; H^3(\mathfrak{g}) \;\cong\; H^3(G;\mathbb{R}) \;\cong\; H^2(L\mathfrak{g})$$

which identifies the level defined via $H^4(BG;\mathbb{R})$ with the level defined via central extensions of the loop group LG, all of which are classified by invariant symmetric bilinear forms on $\mathfrak{g}$.

In particular, central extensions of the loop algebra $L\mathfrak{g}$ are classified by invariant symmetric bilinear forms on $\mathfrak{g}$, which are classified by $H^3(\mathfrak{g})$ defined by the Chevalley-Eilenberg complex. Given such a form $\langle\ ,\ \rangle$, the associated 3-cocycle is

$$\sigma(\xi,\eta,\zeta) = \langle[\xi,\eta],\zeta\rangle.$$

Conversely, given a 3-cocycle σ on $\mathfrak{g}$, one can define an invariant symmetric bilinear form by

$$\langle\xi,\eta\rangle := \sigma(\xi,[\eta_1,\eta_2]),$$

where η_1, η_2 are any elements satisfying $\eta = [\eta_1,\eta_2]$ (such elements exist since $\mathfrak{g}$ is semisimple, and the definition is independent of the choice because σ is a cocycle). We have seen that invariant symmetric bilinear forms on $\mathfrak{g}$ classify central extensions of the loop algebra $L\mathfrak{g}$ via the construction which takes $\langle\ ,\ \rangle$ to the cocycle

$$\omega(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \langle\xi(\theta),\eta'(\theta)\rangle\, d\theta.$$

Moreover we have seen that any such cocycle ω arises from such a bilinear form. Thus we have an isomorphism

$$H^3(\mathfrak{g}) \;\xrightarrow{\cong}\; H^2(L\mathfrak{g})$$

On the other hand, if G is compact, then the de Rham cohomology $H^3(G)$ is isomorphic to the Lie algebra cohomology $H^3(\mathfrak{g})$. This is because every de Rham cohomology class has a unique left invariant representative form given by averaging, and therefore the cohomology of G can be calculated from the cochain complex of the Lie algebra $\mathfrak{g}$.

1.2 Line bundles with a level

To certain line bundles on $\mathcal{M}$ we now associate a *level*, a quadratic form on the Lie algebra $\mathfrak{g}$. Briefly, for any representation V, the level of $\det E^*_\Sigma V$ is the trace form $\xi,\eta \mapsto \mathrm{Tr}_V(\xi\eta)$, and we wish to extend this definition by linearity in the first Chern class of the line bundle.

Riemann–Roch along Σ expresses $c_1(E^*_\Sigma V)$ as the image of $\mathrm{ch}_2(V) = \frac{1}{2}c_1(V)^2 - c_2(V)$ under *transgression along* Σ,

$$\tau : H^4(BG; \mathbb{Q}) \longrightarrow H^2(\mathcal{M}; \mathbb{Q}) \qquad \text{(construction (1.1.iii) in cohomology).}$$

It is important that τ is injective (Remark 4.11). We now identify $H^4(BG; \mathbb{R})$ with the space of invariant symmetric bilinear forms on $\mathfrak{g}_\kappa$ so that Tr_V corresponds to $\mathrm{ch}_2(V)$. We say that the line bundle $\mathcal{L}$ *has a level* if its Chern class $c_1(\mathcal{L})$ agrees with some $\tau(h)$ in $H^2(\mathcal{M}; \mathbb{Q})$; the form h, called the *level* of $\mathcal{L}$, is then unique.

For SL_n, the level of the positive generator of $\mathrm{Pic}(\mathcal{M})$ is $-\mathrm{Tr}_{\mathbb{C}^n}$ in the standard representation; the calculation is due to Quillen. For another example, the level of $\mathcal{K}^{-1/2}$ is $c := -\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}}$. Positivity of a level refers to the quadratic form on $\mathfrak{g}_\kappa$; thus $D_\Sigma V$ has positive level iff V is $\mathfrak{g}$–faithful. Finally, $\mathcal{L}$, with level h, is *admissible* iff $h > -c$ as a quadratic form.

1.3 Remark

(i) When G is simply connected, the map $\tau : H^4(BG; \mathbb{Z}) \to H^2(\mathcal{M}; \mathbb{Z})$ is an isomorphism, but this fails (even rationally) as soon as $\pi_1(G) \neq 0$. Line bundles with a level satisfy a prescribed relation between their Chern classes over the different components of $\mathcal{M}$; cf. (4.8).

(ii) The trace forms span the negative semi–definite cone in $H^4(BG; \mathbb{R})$; so $\mathcal{L}$ has positive level iff $c_1(\mathcal{L})$ lies in the $\mathbb{Q}_+$–span of the $c_1(D_\Sigma V)$'s for $\mathfrak{g}$–faithful V.

(iii) For semi–simple G, the line bundle $\mathcal{K}$ has negative level, and so $\mathcal{O}$ is admissible. This fails for a torus, but positive–level line bundles are admissible for any G.

(iv) For $g > 1$ and simply connected G, positivity of the level is equivalent to ampleness on the moduli space. (It suffices to check this for simple G: recall then that $\mathrm{Pic}(\mathcal{M}) = \mathbb{Z}$ and that $\mathcal{K}^{-1}$ is ample.) When $\pi_1(G) \neq 0$, the positive–level condition is much more restrictive.