

# Homework 5

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For Questions 1 and 2, you may use the correspondence indicated in class between the representation of  $H^1$  classes by classes by principal parts versus Dolbeault distributions.

**Problem 1** For a compact Riemann surface  $R$ , verify that the Serre duality pairing

$$H^1(R; \mathcal{O}) \otimes H^0(R; \Omega^1) \longrightarrow \mathbb{C}$$

defined by principal parts and residues agrees with the one given by integration of Dolbeault representatives.

Using the relation to harmonic forms, explain how this relates to Poincaré duality on  $R$ .

*Solution:* Choose a meromorphic function  $f$  on  $R$  whose principal part at each  $p_i$  with prescribed principal parts. Let  $U_i$  be pairwise disjoint coordinate discs around  $p_i$ , and choose  $\chi \in C^\infty(R)$  such that  $\chi \equiv 1$  on smaller discs  $U'_i \subset U_i$  and  $\chi \equiv 0$  outside  $\bigcup_i U_i$ . Define a  $(0, 1)$ -current

$$T_f := \bar{\partial}(\chi f).$$

Since  $\bar{\partial}^2 = 0$ ,  $T_f$  is  $\bar{\partial}$ -closed. If we replace  $f$  by  $f + g$  for a global meromorphic function  $g$  (with poles in  $D$ ) or change  $\chi$  within the same constraints,  $T_f$  changes by a current of the form  $\bar{\partial}u$ , so the class  $[T_f]$  in

$$H_{\bar{\partial}}^{0,1}(R) \cong H^1(R, \mathcal{O})$$

depends only on the underlying principal parts.

Let  $\omega \in H^0(R, \Omega^1)$  be a holomorphic 1-form. The *Dolbeault* definition of the pairing is

$$\langle \alpha, \omega \rangle_{Dol} := \frac{1}{2\pi i} \int_R T_f \wedge \omega = \frac{1}{2\pi i} \int_R \bar{\partial}(\chi f) \wedge \omega.$$

Since  $\omega$  is of type  $(1, 0)$  and holomorphic,  $\bar{\partial}\omega = 0$ , hence

$$\bar{\partial}(\chi f) \wedge \omega = \bar{\partial}(\chi f \omega).$$

Let  $D_i \subset U'_i$  be small closed discs around  $p_i$  and set

$$R_\varepsilon := R \setminus \bigcup_i D_i(\varepsilon),$$

where  $D_i(\varepsilon)$  are concentric discs of radius  $\varepsilon$ . On  $R_\varepsilon$  the form  $\chi f \omega$  is smooth with compact support, so Stokes' theorem gives

$$\int_{R_\varepsilon} \bar{\partial}(\chi f \omega) = \int_{\partial R_\varepsilon} \chi f \omega = - \sum_i \int_{\partial D_i(\varepsilon)} f \omega,$$

the sign coming from the induced orientation on the boundary.

Letting  $\varepsilon \rightarrow 0$  and using the residue theorem,

$$\int_{\partial D_i(\varepsilon)} f\omega \longrightarrow 2\pi i \operatorname{Res}_{p_i}(f\omega),$$

we obtain

$$\frac{1}{2\pi i} \int_R \bar{\partial}(\chi f) \wedge \omega = \sum_i \operatorname{Res}_{p_i}(f\omega).$$

This is precisely the *principal parts* definition of the Serre pairing.

Now equip  $R$  with any Hermitian (necessarily Kähler) metric. Hodge theory yields the decompositions

$$H_{\mathrm{dR}}^1(R, \mathbb{C}) \cong \mathcal{H}^1(R) \cong H_{\bar{\partial}}^{1,0}(R) \oplus H_{\bar{\partial}}^{0,1}(R),$$

and every class has a unique harmonic representative. Moreover,

$$H^0(R, \Omega^1) \cong H_{\bar{\partial}}^{1,0}(R)$$

consists of harmonic  $(1,0)$ -forms, and

$$H^1(R, \mathcal{O}) \cong H_{\bar{\partial}}^{0,1}(R)$$

is represented by harmonic  $(0,1)$ -forms. Complex conjugation gives an isomorphism

$$\overline{H_{\bar{\partial}}^{1,0}(R)} \cong H_{\bar{\partial}}^{0,1}(R)$$

Poincaré duality on  $R$  is given by the nondegenerate pairing

$$H_{\mathrm{dR}}^1(R, \mathbb{C}) \times H_{\mathrm{dR}}^1(R, \mathbb{C}) \longrightarrow \mathbb{C}, \quad ([\alpha], [\beta]) \mapsto \int_R \alpha \wedge \beta.$$

It is clear that  $\alpha \wedge \beta$  is nonzero only if  $\alpha$  and  $\beta$  are of complementary types, i.e. their wedge is of type  $(1,1)$ , since  $(1,0) \wedge (1,0)$  and  $(0,1) \wedge (0,1)$  necessarily vanish. Thus the Poincaré pairing restricts to a nondegenerate pairing

$$H_{\bar{\partial}}^{0,1}(R) \otimes H_{\bar{\partial}}^{1,0}(R) \longrightarrow \mathbb{C}, \quad (\eta, \omega) \mapsto \int_R \eta \wedge \omega,$$

with  $\eta, \omega$  harmonic representatives.

Under the identifications

$$H^1(R, \mathcal{O}) \cong H_{\bar{\partial}}^{0,1}(R), \quad H^0(R, \Omega^1) \cong H_{\bar{\partial}}^{1,0}(R),$$

the Serre pairing of  $\alpha$  and  $\omega$  is

$$\langle \alpha, \omega \rangle = \frac{1}{2\pi i} \int_R \eta \wedge \omega,$$

where  $\eta$  is the harmonic  $(0,1)$ -representative of  $\alpha$ . In particular, on a compact Riemann surface the Serre duality

$$H^1(R, \mathcal{O}) \cong H^0(R, \Omega^1)^\vee$$

is nothing but Poincaré duality in degree 1 up to the constant factor  $2\pi i$ , expressed via the Hodge decomposition of  $H_{\mathrm{dR}}^1(R, \mathbb{C})$ .

**Problem 2** For a compact Riemann surface  $R$ , verify that the map

$$H^1(R; \mathbb{Z}) \longrightarrow H^1(R; \mathcal{O})$$

corresponds to the period map

$$H_1(R; \mathbb{Z}) \otimes H^0(R; \Omega^1) \longrightarrow \mathbb{C}$$

under integral Poincaré duality and Serre duality on  $R$ .

*Solution:* Let  $i : H^1(R; \mathbb{Z}) \rightarrow H^1(R; \mathcal{O})$  be the given homomorphism. We need to show for every  $c \in H^1(R; \mathbb{Z})$  and  $\omega \in H^0(R, \Omega^1)$ , the Serre pairing  $\langle i(c), \omega \rangle_{\text{Serre}}$  equals the period of  $\omega$  along the 1-cycle Poincaré dual to  $c$ .

By Hodge theory, every class in  $H^1(R; \mathbb{R})$  has a unique harmonic representative. An element  $c \in H^1(R; \mathbb{Z})$  maps to a real class  $c_{\mathbb{R}} \in H^1(R; \mathbb{R})$  whose harmonic representative we denote by  $\alpha$  so

$$[\alpha]_{\text{dR}} = c_{\mathbb{R}} \in H_{\text{dR}}^1(R; \mathbb{R}).$$

Decompose  $\alpha$

$$\alpha = \alpha^{1,0} + \alpha^{0,1}, \quad \alpha^{0,1} = \overline{\alpha^{1,0}},$$

since  $\alpha$  is real. Under the Dolbeault isomorphism and Hodge decomposition, we have

$$H^1(R, \mathcal{O}) \cong H_{\bar{\partial}}^{0,1}(R)$$

and the image  $i(c) \in H^1(R, \mathcal{O})$  is represented by the harmonic  $(0, 1)$ -form  $\alpha^{0,1}$ .

We know that the Serre pairing can be described as

$$\langle \beta, \omega \rangle_{\text{Serre}} = \frac{1}{2\pi i} \int_R \eta^{0,1} \wedge \omega$$

whenever  $\beta \in H^1(R, \mathcal{O})$  is represented by a harmonic  $(0, 1)$ -form  $\eta^{0,1}$  and  $\omega \in H^0(R, \Omega^1)$  is a holomorphic 1-form.

Applying this to  $\beta = i(c)$  and  $\eta^{0,1} = \alpha^{0,1}$  gives

$$\langle i(c), \omega \rangle_{\text{Serre}} = \frac{1}{2\pi i} \int_R \alpha^{0,1} \wedge \omega.$$

Since  $R$  has complex dimension 1, a  $(2, 0)$ -form vanishes, hence  $\alpha^{1,0} \wedge \omega = 0$ , and therefore

$$\alpha^{0,1} \wedge \omega = (\alpha^{1,0} + \alpha^{0,1}) \wedge \omega = \alpha \wedge \omega.$$

Thus

$$\langle \iota^* c, \omega \rangle_{\text{Serre}} = \frac{1}{2\pi i} \int_R \alpha \wedge \omega. \tag{1}$$

Integral Poincaré duality gives a perfect pairing

$$H^1(R; \mathbb{Z}) \times H_1(R; \mathbb{Z}) \longrightarrow \mathbb{Z},$$

and we denote by  $\gamma_c \in H_1(R; \mathbb{Z})$  the Poincaré dual of  $c$ .

The de Rham realization of this pairing is as follows. The class  $c_{\mathbb{R}} \in H^1(R; \mathbb{R})$  is represented by the closed 1-form  $\alpha$  with integral periods, i.e.

$$\int_{\gamma} \alpha \in \mathbb{Z} \quad \text{for all } \gamma \in H_1(R; \mathbb{Z}).$$

The Poincaré dual cycle  $\gamma_c$  is then characterized by

$$\int_{\gamma_c} \beta = \int_R \alpha \wedge \beta \quad \text{for all closed 1-forms } \beta,$$

Thus, if we identify

$$H^1(R; \mathbb{Z}) \xrightarrow{\text{PD}} H_1(R; \mathbb{Z}) \quad \text{and} \quad H^1(R; \mathcal{O}) \xrightarrow{\text{Serre}} H^0(R, \Omega^1)^{\vee},$$

the class  $c \in H^1(R; \mathbb{Z})$  maps to the functional

$$H^0(R, \Omega^1) \longrightarrow \mathbb{C}, \quad \omega \longmapsto \frac{1}{2\pi i} \int_{\gamma_c} \omega.$$

This is precisely the period map (up to the factor  $1/(2\pi i)$ )

$$H_1(R; \mathbb{Z}) \otimes H^0(R, \Omega^1) \longrightarrow \mathbb{C}, \quad (\gamma, \omega) \longmapsto \int_{\gamma} \omega,$$

with  $\gamma = \gamma_c$  the Poincaré dual of  $c$ .

**Problem 3** Show that the period mapping gives an isomorphism

$$H_1(R; \mathbb{Z}) \xrightarrow{\sim} H_1(J; \mathbb{Z}),$$

which can be realized geometrically by the Abel-Jacobi map

$$R \longrightarrow J_1.$$

Show that under this correspondence,  $c_1(\Theta) \in \Lambda^2 H_1(R)$  is the intersection pairing on  $R$ .

*Hints for the second part:* You can deduce it from the periodicity formulas of the Riemann  $\Theta$ -function. Alternatively, you can find this by exploiting the facts that the Poincaré dual of  $c_1(\Theta)$  in  $J_{g-1}$  is the Theta divisor, the image of  $\text{Sym}^{g-1}(R)$ . The maps

$$\text{Sym}^g(R) \longrightarrow J_g \quad \text{and} \quad \text{Sym}^{g-1}(R) \longrightarrow \text{div}(\Theta)$$

have degree 1.

*Solution:* The presentation of the Jacobian  $J$  as

$$J \cong H^1(R; \mathcal{O})/H_1(R; \mathbb{Z})$$

makes it clear that  $H_1(J; \mathbb{Z})$  is naturally identified with  $H_1(R; \mathbb{Z})$ , since the universal cover of  $J$  is the vector space  $H^1(R; \mathcal{O})$ . The period mapping

$$H_1(R; \mathbb{Z}) \rightarrow H_1(J; \mathbb{Z})$$

is injective because of the Riemann bilinear relations, and since both groups are free abelian of rank  $2g$ , it is an isomorphism. Pick a base point  $p_0 \in R$  and define the Abel-Jacobi map

$$\varphi : R \rightarrow J, \quad p \mapsto \left[ \omega \mapsto \int_{p_0}^p \omega \right].$$

precisely implements the lift of the period mapping to the universal cover and hence induces the same isomorphism on  $H_1$ .

To identify  $c_1(\Theta)$  with the intersection pairing on  $H_1(R, \mathbb{Z})$ , we first note that by the universal coefficient theorem and the fact that  $H^k(J, \mathbb{Z}) = \text{Alt}^k(H_1(J, \mathbb{Z}), \mathbb{Z})$  (the group law on  $J$  induces a map  $H_1(J, \mathbb{Z}) \otimes \cdots \otimes H_1(J, \mathbb{Z}) \rightarrow H^k(J, \mathbb{Z})$  as follows. For each  $\alpha \in H_1(J, \mathbb{Z})$  choose a loop  $\ell_\alpha : S^1 \rightarrow J$  representing  $\alpha$ . For  $\alpha_1, \dots, \alpha_k$ , consider the map  $(S^1)^k \rightarrow J$  given by  $(t_1, \dots, t_k) \mapsto \ell_{\alpha_1}(t_1) + \cdots + \ell_{\alpha_k}(t_k)$ . For orientation reasons, this map is alternating in the  $\alpha_i$ ). We have

$$\begin{aligned} H^2(J, \mathbb{Z}) &\cong \text{Hom}(H_2(J, \mathbb{Z}), \mathbb{Z}) \\ &\cong \text{Hom}(\Lambda^2 H_1(J, \mathbb{Z}), \mathbb{Z}) \\ &\cong \text{Alt}^2(H_1(J, \mathbb{Z}), \mathbb{Z}) \xrightarrow{\iota^*} \text{Alt}^2(H_1(R, \mathbb{Z}), \mathbb{Z}) \end{aligned}$$

so indeed  $c_1(\Theta)$  corresponds to an alternating bilinear form on  $H_1(R, \mathbb{Z})$ . Pick a symplectic basis  $\{a_1, \dots, a_g, b_1, \dots, b_g\}$  of  $H_1(R, \mathbb{Z})$ , i.e.

$$a_i \cdot a_j = 0, \quad b_i \cdot b_j = 0, \quad a_i \cdot b_j = \delta_{ij}.$$

Under the identification  $H_1(R, \mathbb{Z}) \xrightarrow{\sim} \Lambda \cong H_1(J, \mathbb{Z})$  coming from the period map and the Abel-Jacobi embedding, a homology class  $\gamma \in H_1(R, \mathbb{Z})$  corresponds to an integral vector  $(m, n) \in \mathbb{Z}^{2g}$ . The intersection pairing on  $H_1(R, \mathbb{Z})$  is given in these coordinates by

$$(m, n) \cdot (m', n') = m^T n' - m'^T n.$$

The Riemann theta function with period matrix  $\tau$  is

$$\theta(z \mid \tau) := \sum_{k \in \mathbb{Z}^g} \exp(\pi i k^T \tau k + 2\pi i k^T z), \quad z \in \mathbb{C}^g.$$

The Riemann theta function satisfies the quasi-periodicity property.

$$\theta(z + m + \tau n \mid \tau) = \exp(-\pi i n^T \tau n - 2\pi i n^T z) \theta(z \mid \tau)$$

In particular, the Riemann theta function defines a holomorphic section of the line bundle  $\mathcal{O}_J(\Theta)$ .

Hence, identifying  $H^2(U, \mathbb{Z})$  and  $H^2(X, \mathbb{Z})$  by the above isomorphism, the Chern class of  $L$  is simply  $\delta(\text{cl}\{e_u\})$ . Write  $e_u(z) = e^{2\pi i f_u(z)}$  with  $f_u$  holomorphic in  $V$ . Then by definition,  $\delta(\text{cl}\{e_u\}) \in H^2(U, \mathbb{Z})$  is given by the 2-cocycle  $F(u_1, u_2)$  on  $U$  with coefficients in  $\mathbb{Z}$  defined by

$$F(u_1, u_2) = f_{u_2}(z + u_1) - f_{u_1+u_2}(z) + f_{u_1}(z) \in \mathbb{Z}. \quad (*)$$

**Lemma 1 (Mumford)** Let  $U \subset V$  be a lattice in a complex vector space  $V$ . The map which associates to any map  $F : U \times U \rightarrow \mathbb{Z}$  the map  $AF : U \times U \rightarrow \mathbb{Z}$  defined by

$$AF(u_1, u_2) = F(u_1, u_2) - F(u_2, u_1)$$

maps the group of 2-cocycles  $Z^2(U, \mathbb{Z})$  into the space of alternating linear maps  $U \times U \rightarrow \mathbb{Z}$ , and induces an isomorphism

$$A : H^2(U, \mathbb{Z}) \xrightarrow{\sim} \text{Hom}(\Lambda^2 U, \mathbb{Z}) \cong \Lambda^2 \text{Hom}(U, \mathbb{Z}).$$

Furthermore for  $\xi, \eta \in \text{Hom}(U, \mathbb{Z}) = H^1(U, \mathbb{Z})$ , we have  $A(\xi \smile \eta) = \xi \wedge \eta$ .

**Proposition 2 (Mumford)** The Chern class of the line bundle corresponding to  $\{e_u\} \in Z^1(U, H^*)$  is the alternating 2-form on  $U$  with values in  $\mathbb{Z}$  given by

$$E(u_1, u_2) = f_{u_2}(z + u_1) + f_{u_1}(z) - f_{u_1}(z + u_2) - f_{u_2}(z), \quad (z \text{ arbitrary in } V), \quad (**)$$

where

$$e_u(z) = e^{2\pi i f_u(z)}.$$

Moreover if we extend  $E$   $\mathbb{R}$ -linearly to a map  $V \times V \rightarrow \mathbb{R}$ , then  $E$  satisfies the identity

$$E(ix, iy) = E(x, y) \quad \text{for } x, y \in V.$$

**Problem 4** Prove the following generalized Cauchy formula for a smooth function  $f$  defined in the unit disk  $\Delta$ :

$$f(z, \bar{z}) = \frac{1}{2\pi i} \oint_{|\zeta - z| = r} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \iint_{\Delta'} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z},$$

where  $\Delta' \subset \Delta$  is the subdisk of radius  $r < 1$ .

*Solution:* We prove the two lemmas below at the end.

**Lemma 1** Let  $X \subset \mathbb{C}$  be a bounded domain with smooth boundary  $\partial X$ . Then

$$\frac{\partial \chi_X}{\partial \bar{z}} = \frac{i}{2} \oint_{\partial X} dz,$$

where the distribution on the right denotes contour integration along  $\partial X$ .

We can now deduce the generalized Cauchy integral formula. Let

$$u = \frac{\chi_X}{\pi(z - z_0)} \in L^1_{\text{loc}}(X), \quad z_0 \in X.$$

where  $L^1_{\text{loc}}(X)$  is the space of locally integrable functions on  $X$ . We may apply the Leibniz rule to compute

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\pi(z - z_0)} \right) \chi_X + \frac{1}{\pi(z - z_0)} \frac{\partial \chi_X}{\partial \bar{z}}.$$

**Lemma 2**

$$\frac{\partial}{\partial \bar{z}} \left( \frac{1}{\pi(z - z_0)} \right) = \delta_{z_0}$$

It follows from the lemma and the identity above that

$$\frac{\partial u}{\partial \bar{z}} = \delta_{z_0} + \frac{1}{\pi(z - z_0)} \frac{\partial \chi_X}{\partial \bar{z}}.$$

Applying both sides to  $\varphi \in C_c^\infty(X)$  gives

$$\left\langle \frac{\partial u}{\partial \bar{z}}, \varphi \right\rangle = \varphi(z_0) + \left\langle \frac{\partial \chi_X}{\partial \bar{z}}, \frac{\varphi}{\pi(z - z_0)} \right\rangle = \varphi(z_0) + \frac{i}{2} \oint_{\partial X} \frac{\varphi(z)}{\pi(z - z_0)} dz.$$

Rearranging yields the desired generalized Cauchy formula:

$$\varphi(z_0) = \frac{1}{2\pi i} \oint_{\partial X} \frac{\varphi(z)}{z - z_0} dz + \frac{1}{2\pi i} \iint_X \frac{\partial \varphi}{\partial \bar{z}}(z) \frac{dx \wedge dy}{z - z_0}.$$

*Proof of Lemma 1.* For  $\varphi \in C_c^\infty(X)$ , compute

$$\left\langle \frac{\partial}{\partial \bar{z}} \chi_X, \varphi \right\rangle = - \int_X \frac{\partial \varphi}{\partial \bar{z}} dx dy = - \frac{1}{2} \int_X (\partial_x \varphi + i \partial_y \varphi) dx dy.$$

Let  $\partial X$  be oriented counterclockwise, and parametrize it by arc length  $s \mapsto (x(s), y(s))$ . Denote by  $\tau = (x'(s), y'(s))$  the unit tangent vector and  $\nu = (-y'(s), x'(s))$  the unit outward normal. Define  $V = (\varphi, i\varphi) \in C_c^\infty(X)^2$ , so that

$$\text{div } V = \partial_x \varphi + i \partial_y \varphi.$$

By the divergence theorem,

$$-\frac{1}{2} \int_X (\partial_x \varphi + i \partial_y \varphi) dx dy = -\frac{1}{2} \int_{\partial X} V \cdot \nu ds = -\frac{1}{2} \int_0^\ell (\varphi \nu_1 + i \varphi \nu_2) ds.$$

Since  $\nu = (-y', x')$ , we get

$$V \cdot \nu = \varphi(-y' + ix') = i \varphi(x' + iy').$$

Thus

$$-\frac{1}{2} \int_X (\partial_x \varphi + i \partial_y \varphi) dx dy = \frac{i}{2} \int_0^\ell \varphi(x(s), y(s)) (x'(s) + iy'(s)) ds = \frac{i}{2} \oint_{\partial X} \varphi dz.$$

Hence

$$\frac{\partial \chi_X}{\partial \bar{z}} = \frac{i}{2} \oint_{\partial X} dz.$$

□

*Proof of Lemma 2.* Let  $\varphi \in C_c^\infty(X)$  be a test function. By definition,

$$\left\langle \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\pi(z - z_0)} \right), \varphi \right\rangle := -\frac{1}{\pi} \iint_{X \setminus \{z_0\}} \frac{1}{z - z_0} \frac{\partial \varphi}{\partial \bar{z}} dx dy.$$

Remove a small disk  $D_\varepsilon$  of radius  $\varepsilon$  centered at  $z_0$ , and write  $A_\varepsilon = X \setminus D_\varepsilon$ . Then

$$\left\langle \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\pi(z - z_0)} \right), \varphi \right\rangle = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \iint_{A_\varepsilon} \frac{1}{z - z_0} \frac{\partial \varphi}{\partial \bar{z}} dx dy.$$

We know that

$$d\left(\frac{\varphi}{z - z_0} dz\right) = d\left(\frac{\varphi}{z - z_0}\right) \wedge dz = \frac{1}{z - z_0} d\varphi \wedge dz + \varphi d\left(\frac{1}{z - z_0}\right) \wedge dz.$$

But

$$d\left(\frac{1}{z - z_0}\right) \wedge dz = g(z) dz \wedge dz = 0 \implies d\left(\frac{\varphi}{z - z_0} dz\right) = \frac{1}{z - z_0} d(\varphi dz).$$

This, together with the fact that

$$\frac{\partial \varphi}{\partial \bar{z}} dx dy = \frac{1}{2i} d(\varphi dz).$$

gives upon application of Stokes' theorem,

$$\iint_{A_\varepsilon} \frac{1}{z - z_0} \frac{\partial \varphi}{\partial \bar{z}} dx dy = \frac{1}{2i} \int_{\partial A_\varepsilon} \frac{\varphi(z)}{z - z_0} dz.$$

The boundary  $\partial A_\varepsilon$  consists of the small circle  $\partial D_\varepsilon$ , oriented negatively. Parametrizing  $\partial D_\varepsilon$  positively gives an extra minus sign, hence

$$\iint_{A_\varepsilon} \frac{1}{z - z_0} \frac{\partial \varphi}{\partial \bar{z}} dx dy = -\frac{1}{2i} \int_{|z-z_0|=\varepsilon} \frac{\varphi(z)}{z - z_0} dz.$$

Expand  $\varphi$  near  $z_0$ :

$$\varphi(z) = \varphi(z_0) + O(\varepsilon).$$

Thus

$$\int_{|z-z_0|=\varepsilon} \frac{\varphi(z)}{z - z_0} dz = \varphi(z_0) \int_{|z-z_0|=\varepsilon} \frac{dz}{z - z_0} + O(\varepsilon).$$

But

$$\int_{|z-z_0|=\varepsilon} \frac{dz}{z - z_0} = 2\pi i.$$

Hence letting  $\varepsilon \rightarrow 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_{|z-z_0|=\varepsilon} \frac{\varphi(z)}{z - z_0} dz = 2\pi i \varphi(z_0).$$

Combining everything,

$$\left\langle \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\pi(z - z_0)} \right), \varphi \right\rangle = -\frac{1}{\pi} \left( -\frac{1}{2i} \right) (2\pi i) \varphi(z_0) = \varphi(z_0).$$

This is precisely the defining property of the Dirac delta at  $z_0$ .  $\square$

**Problem 5** Let  $L \rightarrow X$  be a holomorphic line bundle on a complex manifold, and let  $\alpha \in \mathcal{E}^{0,1}$  be a  $\bar{\partial}$ -closed form. Show that the re-defined operator

$$\tilde{\bar{\partial}} = \bar{\partial} + \alpha$$

on sections of  $L$  defines a new holomorphic structure  $L'$  on the same underlying bundle, where local holomorphic sections are defined as those killed by  $\tilde{\bar{\partial}}$ . Show that  $L \simeq L'$  if  $\alpha$  is  $\bar{\partial}$ -exact. Relate this to the exponential sequence.

*Remark:* For vector bundles, the same applies with an  $\alpha \in \mathcal{E}^{0,1}(\text{End}(V))$  satisfying the non-linear equation

$$\bar{\partial}\alpha + \alpha \wedge \alpha = 0.$$

The new bundle is isomorphic to the old one if  $\alpha = a^{-1}\bar{\partial}a$ , for some smooth section  $a$  of  $\text{Aut}(V)$ .

*Solution:* From the given holomorphic structure, we have a  $\mathbb{C}$ -linear map

$$\bar{\partial}_L: \mathcal{E}^0(L) \longrightarrow \mathcal{E}^{0,1}(L)$$

satisfying the Leibniz rule and the condition  $\bar{\partial}_L^2 = 0$ . We have the new operator

$$\tilde{\bar{\partial}}s := \bar{\partial}s + \alpha \wedge s \in \mathcal{E}^{0,1}(L).$$

First we check that  $\tilde{\bar{\partial}}$  is a  $\bar{\partial}$ -operator. For  $f \in C^\infty(X)$  and  $s \in \mathcal{E}^0(L)$ ,

$$\tilde{\bar{\partial}}(fs) = \bar{\partial}(fs) + \alpha fs = (\bar{\partial}f)s + f\bar{\partial}s + f\alpha s = (\bar{\partial}f)s + f\tilde{\bar{\partial}}s,$$

so the Leibniz rule holds.

Next, compute  $\tilde{\bar{\partial}}^2$ . View  $\bar{\partial}$  as a derivation of degree  $(0, 1)$  on  $\mathcal{E}^{0,\bullet}(L)$ ; then for  $\beta \in \mathcal{E}^{0,1}$  and  $\eta \in \mathcal{E}^{0,q}(L)$ ,

$$\bar{\partial}(\beta \wedge \eta) = (\bar{\partial}\beta) \wedge \eta - \beta \wedge \bar{\partial}\eta.$$

Hence, for a section  $s \in \mathcal{E}^0(L)$ ,

$$\begin{aligned} \tilde{\bar{\partial}}^2 s &= \tilde{\bar{\partial}}(\bar{\partial}s + \alpha \wedge s) \\ &= \bar{\partial}(\bar{\partial}s + \alpha \wedge s) + \alpha \wedge (\bar{\partial}s + \alpha \wedge s) \\ &= \bar{\partial}^2 s + \bar{\partial}(\alpha \wedge s) + \alpha \wedge \bar{\partial}s + \alpha \wedge \alpha \wedge s \\ &= 0 + (\bar{\partial}\alpha) \wedge s - \alpha \wedge \bar{\partial}s + \alpha \wedge \bar{\partial}s + \alpha \wedge \alpha \wedge s \\ &= (\bar{\partial}\alpha) \wedge s + \alpha \wedge \alpha \wedge s. \end{aligned}$$

By assumption  $\bar{\partial}\alpha = 0$ , and since  $\alpha$  is a 1-form,  $\alpha \wedge \alpha = 0$ . Thus  $\tilde{\bar{\partial}}^2 s = 0$  for all  $s$ , so  $\tilde{\bar{\partial}}^2 = 0$  and  $\tilde{\bar{\partial}}$  is a  $\bar{\partial}$ -operator. It therefore defines a new holomorphic structure  $L'$  on the same underlying smooth bundle, whose local holomorphic sections are those killed by  $\tilde{\bar{\partial}}$ .

Now we check that if  $\alpha$  is  $\bar{\partial}$ -exact, then  $L' \simeq L$ . Suppose  $\alpha = \bar{\partial}\phi$  for some smooth complex-valued function  $\phi$ . Define an automorphism of the  $C^\infty$  line bundle  $L$  by multiplication with  $e^\phi$ :

$$F: L \longrightarrow L, \quad s \longmapsto e^\phi s.$$

We claim that  $F$  is an isomorphism of holomorphic line bundles  $L' \rightarrow L$ , i.e.

$$\bar{\partial}(Fs) = F(\tilde{\bar{\partial}}s) \quad \text{for all } s.$$

Indeed,

$$\bar{\partial}(Fs) = \bar{\partial}(e^\phi s) = e^\phi (\bar{\partial}\phi \wedge s + \bar{\partial}s) = e^\phi (\alpha \wedge s + \bar{\partial}s) = F(\tilde{\bar{\partial}}s).$$

Thus  $F$  is holomorphic with respect to  $\tilde{\bar{\partial}}$  on the domain and  $\bar{\partial}$  on the target, so  $L' \simeq L$ .

The  $(0, 1)$ -form  $\alpha$  is  $\bar{\partial}$ -closed, so it defines a Dolbeault cohomology class

$$[\alpha] \in H_{\bar{\partial}}^{0,1}(X) \cong H^1(X; \mathcal{O}).$$

Changing  $\alpha$  by a  $\bar{\partial}$ -exact form does not change this class, and by the computation above such a change yields an isomorphic holomorphic structure. Thus the isomorphism class of the new line bundle  $L'$  depends only on  $[\alpha]$ .

Recall the holomorphic exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}^\times \longrightarrow 1,$$

whose long exact cohomology sequence contains

$$H^1(X; \mathcal{O}) \xrightarrow{\exp} H^1(X; \mathcal{O}^\times),$$

and  $H^1(X; \mathcal{O}^\times) \cong \text{Pic}(X)$  classifies holomorphic line bundles. The class  $[\alpha] \in H^1(X; \mathcal{O})$  maps under the exponential to the class of the holomorphic line bundle  $L' \otimes L^{-1}$ .

**Problem 6** Let  $V$  be a complex  $g$ -dimensional vector space and  $L \simeq \mathbb{Z}^{2g} \subset V$  a lattice. Let  $A = V/L$ .

1. Using harmonic theory, compute the Dolbeault cohomology  $H^*(A; \mathcal{O})$ .
2. Show that the moduli space of holomorphic line bundles on  $A$  with zero Chern class is naturally identified with

$$A^\vee := \bar{V}^\vee / L^\vee.$$

3. Show that the moduli space of holomorphic line bundles on  $A^\vee$  with zero Chern class is naturally identified with  $A$ .
4. Define a line bundle

$$\mathcal{P} \longrightarrow A \times A^\vee$$

from the trivial line bundle over  $V \times V^\vee$  with connection

$$\nabla = d + i(x d\xi + \xi dx),$$

by dividing the  $L \times L^\vee$ -action as follows: identify the fiber  $\mathbb{C}$  over  $(x, \xi) \in V \times V^\vee$  with that over  $(x + \ell, \xi + \lambda)$  by multiplication by

$$\exp(2\pi i(\lambda(x) + \xi(\ell))).$$

Show that  $\mathcal{P}$  is holomorphic, that  $\mathcal{P}|_{A \times \{a^\vee\}}$  is the line bundle over  $A$  classified by  $a^\vee \in A^\vee$ , and prove the corresponding statement for  $\{a\} \times A^\vee$ .

*Solution:* Write  $g = \dim_{\mathbb{C}} V$ . Choose a Hermitian inner product on  $V$  which is  $L$ -invariant. By translation invariance and the locality of Kahler geometry, this induces a flat Kahler metric on  $A$ .

1. The Dolbeault Laplacian  $\square_{\bar{\partial}}$  on  $(0, q)$ -forms is translation invariant. On a flat torus, a  $(0, q)$ -form is harmonic iff it has constant coefficients. This is because in a global parallel frame, the Laplacian on forms acts coefficientwise as the usual scalar Laplacian

and the that any harmonic function on a compact manifold is constant by the maximum principle.

A translation-invariant  $(0, 1)$ -form on  $A$  is determined by its value at a single point (say 0), and conversely any linear functional on  $T_0^{0,1}A$  extends uniquely to a translation-invariant  $(0, 1)$ -form. Thus

$$\mathcal{H}^{0,q}(A) \cong \Lambda^q(T_0^{0,1}A)^\vee \cong \Lambda^q \overline{V}^\vee,$$

where  $\overline{V}$  is  $V$  with the conjugate complex structure. By Hodge theory,

$$H^{0,q}(A) \cong \mathcal{H}^{0,q}(A),$$

and since  $H^q(A; \mathcal{O}) \cong H^{0,q}(A)$  we obtain

$$H^q(A; \mathcal{O}) \cong \Lambda^q \overline{V}^\vee \cong \Lambda^q V^\vee, \quad 0 \leq q \leq g.$$

More generally, the same reasoning shows

$$H^{p,q}(A) \cong \Lambda^p V^\vee \otimes \Lambda^q \overline{V}^\vee$$

2. Isomorphism classes of holomorphic line bundles on  $A$  are classified by

$$\text{Pic}(A) \cong H^1(A; \mathcal{O}^\times).$$

Consider the holomorphic exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}^\times \longrightarrow 1,$$

with long exact cohomology sequence

$$H^1(A; \mathcal{O}) \longrightarrow H^1(A; \mathcal{O}^\times) \xrightarrow{c_1} H^2(A; \mathbb{Z}).$$

The subgroup

$$\text{Pic}^0(A) := \ker(c_1 : \text{Pic}(A) \rightarrow H^2(A; \mathbb{Z}))$$

is precisely those bundles with zero Chern class. Exactness shows

$$\text{Pic}^0(A) \cong H^1(A; \mathcal{O}) / \text{im}(H^1(A; \mathbb{Z}))$$

We have already computed

$$H^1(A; \mathcal{O}) \cong H^{0,1}(A) \cong \overline{V}^\vee$$

On the other hand,  $H^1(A; \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong L^\vee$ , and this sits inside  $H^1(A; \mathcal{O})$  as a lattice (one way to see this is via  $H^1(A; \mathbb{Z}) \subset H^1(A; \mathbb{R}) \subset H^1(A; \mathbb{C})$  and the Hodge decomposition). Thus

$$\text{Pic}^0(A) \cong H^1(A; \mathcal{O}) / H^1(A; \mathbb{Z}) \cong \overline{V}^\vee / L^\vee = A^\vee.$$

So the moduli space of holomorphic line bundles on  $A$  with  $c_1 = 0$  is canonically identified with  $A^\vee$ .

3. Apply the same argument to  $A^\vee = V^\vee/L^\vee$ .

4. Consider the trivial line bundle

$$V \times V^\vee \times \mathbb{C} \longrightarrow V \times V^\vee$$

equipped with the connection

$$\nabla = d + i(x d\xi + \xi dx),$$

where  $x \in V$ ,  $\xi \in V^\vee$ , and  $x d\xi + \xi dx$  denotes the tautological pairing (so in coordinates it is linear in  $x$  and  $\xi$ ).

The group  $L \times L^\vee$  acts on  $V \times V^\vee \times \mathbb{C}$  by

$$(\ell, \lambda) \cdot (x, \xi, z) = (x + \ell, \xi + \lambda, e^{2\pi i(\lambda(x) + \xi(\ell))} z).$$

The multiplier  $e^{2\pi i(\lambda(x) + \xi(\ell))}$  is holomorphic in  $(x, \xi)$  (it is the exponential of a holomorphic linear function), so this action is by holomorphic bundle automorphisms of the trivial holomorphic line bundle  $V \times V^\vee \times \mathbb{C}$ .

The quotient

$$\mathcal{P} := (V \times V^\vee \times \mathbb{C}) / (L \times L^\vee) \longrightarrow (V/L) \times (V^\vee/L^\vee) = A \times A^\vee$$

is therefore a holomorphic line bundle, the *Poincaré bundle*. The connection  $\nabla$  is invariant under the  $L \times L^\vee$ -action, hence descends to a connection on  $\mathcal{P}$ .

Now fix  $a^\vee \in A^\vee$  and choose a lift  $\xi_0 \in V^\vee$ . The restriction  $\mathcal{P}|_{A \times \{a^\vee\}}$  is obtained as the quotient of  $V \times \mathbb{C}$  by the following  $L$ -action: an element  $\ell \in L$  sends  $(x, z)$  to

$$(x + \ell, e^{2\pi i \xi_0(\ell)} z),$$

because along the slice  $\xi = \xi_0$  the factor  $\lambda(x)$  vanishes (we take  $\lambda = 0$  to stay in the same fiber over  $a^\vee$ ), while  $\xi(\ell) = \xi_0(\ell)$  is constant in  $x$ . Thus the monodromy of the resulting line bundle around  $\ell \in L$  is exactly

$$\chi_{\xi_0}(\ell) = e^{2\pi i \xi_0(\ell)}.$$

By part (2), a holomorphic line bundle on  $A$  with zero Chern class is classified precisely by such a character  $L \rightarrow U(1)$ , and changing  $\xi_0$  by an element of  $L^\vee$  does not change the character  $\chi_{\xi_0}$ . Hence the isomorphism class of  $\mathcal{P}|_{A \times \{a^\vee\}}$  depends only on the class  $a^\vee = [\xi_0] \in A^\vee$  and is exactly the line bundle over  $A$  classified by  $a^\vee$ .

The argument for the restriction to  $\{a\} \times A^\vee$  is symmetric. Fix  $a \in A$  with lift  $x_0 \in V$ . Then the effective  $L^\vee$ -action on  $V^\vee \times \mathbb{C}$  along the slice  $x = x_0$  is

$$\lambda \cdot (\xi, z) = (\xi + \lambda, e^{2\pi i \lambda(x_0)} z),$$

so the monodromy around  $\lambda \in L^\vee$  is

$$\chi_{x_0}(\lambda) = e^{2\pi i \lambda(x_0)}.$$

This is precisely the character of  $L^\vee$  corresponding to the point  $a = [x_0] \in A$ , and hence  $\mathcal{P}|_{\{a\} \times A^\vee}$  is the line bundle on  $A^\vee$  classified by  $a$ .

**Remark 1** A unitary character is a homomorphism  $\chi : L \longrightarrow U(1)$ .

Any  $\chi$  can be written as  $\chi(\ell) = \exp(2\pi i \xi(\ell))$  for some real-valued group homomorphism  $\xi : L \rightarrow \mathbb{R}$ . Conversely, any such  $\xi$  defines a character this way. So

$$\mathrm{Hom}(L, U(1)) \cong \mathrm{Hom}(L, \mathbb{R}) / \mathrm{Hom}(L, \mathbb{Z}).$$

But

$$\mathrm{Hom}(L, \mathbb{R}) \cong L^\vee \otimes_{\mathbb{Z}} \mathbb{R} \cong \bar{V}^\vee$$

(using the inclusion  $L \subset V$  and identifying  $V \cong L \otimes_{\mathbb{Z}} \mathbb{R}$ ), and

$$\mathrm{Hom}(L, \mathbb{Z}) \cong L^\vee.$$

Therefore

$$\mathrm{Hom}(L, U(1)) \cong \mathrm{Hom}(L, \mathbb{R}) / \mathrm{Hom}(L, \mathbb{Z}) \cong \bar{V}^\vee / L^\vee = A^\vee.$$

**Problem 7** Show that, in the case of the Jacobian  $J$  of a Riemann surface  $R$ , one has a natural isomorphism  $J \simeq J^\vee$ .

*Hint:* Remember the natural Hilbert space structure on holomorphic differentials.

*Remark:* This self-duality is a property of principally polarized Abelian varieties, those  $A$  equipped with a positive line bundle having a single holomorphic section (the  $\Theta$ -function).

*Solution:* Let  $R$  be a compact Riemann surface of genus  $g$  and  $J$  its Jacobian. Recall that

$$J \simeq V/\Lambda, \quad V := H^0(R, \Omega^1)^\vee,$$

where the lattice  $\Lambda$  is the image of  $H_1(R; \mathbb{Z})$  under the period map

$$H_1(R; \mathbb{Z}) \longrightarrow H^0(R, \Omega^1)^\vee, \quad \gamma \longmapsto (\omega \mapsto \int_\gamma \omega).$$

There is a natural Hermitian inner product on the space of holomorphic differentials

$$\langle \omega, \eta \rangle := \frac{i}{2} \int_R \omega \wedge \bar{\eta}, \quad \omega, \eta \in H^0(R, \Omega^1).$$

This is positive definite and defines a Hilbert space structure on  $H^0(R, \Omega^1)$ . This inner product gives a linear isomorphism

$$\bar{\rho}: \overline{H^0(R, \Omega^1)} \xrightarrow{\sim} V.$$

Dualizing, we obtain an antilinear isomorphism

$$\overline{V}^\vee \xrightarrow{\sim} H^0(R, \Omega^1).$$

The intersection pairing on  $H_1(R; \mathbb{Z})$  is unimodular, so it induces an isomorphism

$$H_1(R; \mathbb{Z}) \xrightarrow{\sim} H^1(R; \mathbb{Z}) \cong \text{Hom}(H_1(R; \mathbb{Z}), \mathbb{Z}).$$

Translating this through the identification  $\Lambda \cong H_1(R; \mathbb{Z})$ , we obtain a canonical isomorphism of lattices

$$\Lambda \xrightarrow{\sim} \Lambda^\vee.$$

Recall that the first Chern class  $c_1(\Theta) \in H^2(J; \mathbb{Z})$  of the theta line bundle corresponds, under the isomorphism  $H^2(J; \mathbb{Z}) \cong \text{Alt}^2(H_1(J; \mathbb{Z}), \mathbb{Z})$ , to the intersection form

$$H_1(R; \mathbb{Z}) \times H_1(R; \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

This implies that the inner product on holomorphic differentials actually refines the intersection pairing on  $H_1(R; \mathbb{Z})$ . Therefore the isomorphism  $\overline{V}^\vee \xrightarrow{\sim} H^0(R, \Omega^1)$  actually descends to an isomorphism of complex tori

$$J^\vee = \overline{V}^\vee / \Lambda^\vee \xrightarrow{\sim} H^0(R, \Omega^1) / \Lambda \xrightarrow{\sim} J.$$

### Problem 8

1. Given a holomorphic line bundle  $\mathcal{L}$  on a complex manifold and a smooth real closed 2-form  $\omega$  in the cohomology class of  $c_1(\mathcal{L})$ , prove that there exists a Hermitian metric on  $\mathcal{L}$  whose holomorphic connection has curvature  $-2\pi i \omega$ .
2. Conclude (from Kodaira vanishing) that the holomorphic line bundles on a compact Riemann surface  $R$  which carry metrics of positive curvature are precisely those of positive degree.
3. Show also that for every holomorphic vector bundle  $V$  on  $R$ , there exists a  $d$  so that the twisted bundle  $V(D)$  has no  $H^1$  for any  $D > d$ .

*Solution:*

1. Let  $L \rightarrow X$  be a holomorphic line bundle on a complex manifold and let  $\omega$  be a smooth real closed 2-form representing  $c_1(L) \in H^2(X; \mathbb{R})$ . Choose any Hermitian metric  $h_0$  on  $L$  and let  $F_0$  denote the curvature of its Chern connection. Then

$$\frac{i}{2\pi} F_0 \in \Omega^{1,1}(X, \mathbb{R}) \quad \text{represents } c_1(L).$$

Since  $\omega$  also represents  $c_1(L)$ , the difference

$$\omega - \frac{i}{2\pi} F_0$$

is an exact real  $(1, 1)$ -form. By the  $\partial\bar{\partial}$ -lemma (which holds on every complex curve and more generally on Kähler manifolds), there exists a real-valued smooth function  $\varphi$  such that

$$\omega - \frac{i}{2\pi} F_0 = \frac{i}{2\pi} \partial\bar{\partial}\varphi.$$

Define a new Hermitian metric  $h$  by  $h = e^{-\varphi} h_0$ . In a local holomorphic frame  $l$ , if  $h(l, l) = e^{-\phi}$  the curvature is  $F_h = \partial\bar{\partial}\phi$ . Thus

$$F_h = F_0 + \partial\bar{\partial}\varphi = -2\pi i \omega.$$

Hence  $h$  is a Hermitian metric whose Chern connection has curvature  $-2\pi i \omega$ .

2. For a Hermitian metric  $h$  on a holomorphic line bundle  $L \rightarrow R$ , the degree is

$$\deg(L) = \int_R c_1(L) = \int_R \frac{i}{2\pi} F_h.$$

If  $h$  has positive curvature, then the form  $\frac{i}{2\pi} F_h$  is positive on  $R$ , hence its integral is strictly positive and  $\deg(L) > 0$ .

The hyperplane bundle  $\mathcal{O}_{\mathbb{P}^N}(1)$  carries the Fubini–Study Hermitian metric  $h_{\text{FS}}$  whose curvature form  $F_{\text{FS}}$  is a positive  $(1, 1)$ -form. Pulling back gives a metric  $h_m = \Phi_m^* h_{\text{FS}}$  on  $L^{\otimes m}$  with positive curvature

$$F_{h_m} = \Phi_m^* F_{\text{FS}} > 0.$$

Now define a Hermitian metric  $h$  on  $L$  by taking an  $m$ th root locally: in a local holomorphic frame  $e$  of  $L$ , write

$$h_m(e^{\otimes m}, e^{\otimes m}) = e^{-\phi_m}$$

and set

$$h(e, e) := e^{-\phi_m/m}.$$

Then the curvature satisfies

$$F_h = \frac{1}{m} F_{h_m},$$

which is still a positive  $(1, 1)$ -form. Thus  $L$  admits a Hermitian metric of positive curvature. Therefore the holomorphic line bundles on  $R$  which carry metrics of positive curvature are precisely those of positive degree.

3. This follows immediately from Serre duality and the Riemann–Roch theorem. For a holomorphic vector bundle  $V$  on  $R$  and a divisor  $D$ , Serre duality gives

$$H^1(R, V(D)) \cong H^0(R, K_R \otimes V^\vee \otimes \mathcal{O}(-D))^\vee.$$

By Riemann–Roch, for  $\deg(D)$  sufficiently large, the degree of the bundle  $K_R \otimes V^\vee \otimes \mathcal{O}(-D)$  becomes negative, and hence

$$H^0(R, K_R \otimes V^\vee \otimes \mathcal{O}(-D)) = 0.$$

Thus, for such  $D$ , we have  $H^1(R, V(D)) = 0$ .

**Problem 9** Show that isomorphism classes of *flat unitary* line bundles on a manifold  $X$  are classified by  $H^1(X; U(1))$ , with the constant sheaf  $U(1)$  associated to the unit circle group in  $\mathbb{C}^\times$ .

When  $X$  is compact Kähler, compare the constant and holomorphic exponential sequences to conclude that the map

$$H^1(X; U(1)) \longrightarrow H^1(X; \mathcal{O}^\times)$$

induces a bijection from isomorphism classes of flat unitary line bundles to those of holomorphic line bundles with zero Chern class.

*Remark:* You probably need the Hodge decomposition theorem for the second part.

*Solution:* A flat unitary line bundle on  $X$  is a complex line bundle with structure group  $U(1)$  and a flat unitary connection. Choosing a good open cover  $\{U_i\}$ , such a bundle is given by locally constant transition functions  $g_{ij}: U_{ij} \rightarrow U(1)$  satisfying the cocycle condition  $g_{ij}g_{jk}g_{ki} = 1$  on  $U_{ijk}$ , and two such bundles are isomorphic if their cocycles differ by a coboundary  $g'_{ij} = h_i^{-1}g_{ij}h_j$  with  $h_i: U_i \rightarrow U(1)$  locally constant. This is exactly the description of Čech 1-cocycles and coboundaries for the constant sheaf  $U(1)$ , so isomorphism classes of flat unitary line bundles are classified by

$$H^1(X; U(1)).$$

Now suppose  $X$  is compact Kähler. Consider the *constant exponential sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\exp(2\pi i \cdot)} U(1) \longrightarrow 1$$

and the *holomorphic exponential sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}^\times \longrightarrow 1.$$

The inclusions  $\mathbb{R} \hookrightarrow \mathcal{O}$  and  $U(1) \hookrightarrow \mathcal{O}^\times$  give a morphism of short exact sequences and therefore a natural map

$$H^1(X; U(1)) \longrightarrow H^1(X; \mathcal{O}^\times) \cong \text{Pic}(X).$$

The connecting homomorphism in the holomorphic sequence

$$c_1^{\text{hol}}: H^1(X; \mathcal{O}^\times) \longrightarrow H^2(X; \mathbb{Z})$$

is the first Chern class of the corresponding holomorphic line bundle; let

$$\text{Pic}^0(X) := \ker(c_1^{\text{hol}})$$

be the subgroup of holomorphic line bundles with  $c_1 = 0$ .

The long exact sequence of the constant exponential sequence yields

$$H^1(X; \mathbb{R}) \longrightarrow H^1(X; U(1)) \longrightarrow H^2(X; \mathbb{Z}) \longrightarrow H^2(X; \mathbb{R}),$$

so  $H^1(X; U(1))$  is an extension of  $H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$  by the torsion subgroup of  $H^2(X; \mathbb{Z})$ . Similarly, the long exact sequence of the holomorphic exponential sequence gives

$$H^1(X; \mathcal{O}) \longrightarrow H^1(X; \mathcal{O}^\times) \xrightarrow{c_1^{\text{hol}}} H^2(X; \mathbb{Z}) \longrightarrow H^2(X; \mathcal{O}),$$

and exactness shows

$$\text{Pic}^0(X) \cong H^1(X; \mathcal{O})/H^1(X; \mathbb{Z}).$$

By Hodge decomposition, on a compact Kähler manifold

$$H^1(X; \mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X),$$

and  $H^1(X; \mathcal{O}) \cong H^{0,1}(X)$ . The inclusion  $\mathbb{R} \hookrightarrow \mathcal{O}$  induces a map

$$H^1(X; \mathbb{R}) \longrightarrow H^1(X; \mathcal{O}) \cong H^{0,1}(X)$$

which, under Hodge decomposition, is the projection  $H^{1,0}(X) \oplus H^{0,1}(X) \rightarrow H^{0,1}(X)$ .

Passing to the quotients by  $H^1(X; \mathbb{Z})$ , we obtain an isomorphism of real tori

$$\frac{H^1(X; \mathbb{R})}{H^1(X; \mathbb{Z})} \xrightarrow{\sim} \frac{H^1(X; \mathcal{O})}{H^1(X; \mathbb{Z})} \cong \text{Pic}^0(X).$$

Thus we see that the image of  $H^1(X; U(1))$  is precisely  $\text{Pic}^0(X)$  and that the induced map

$$H^1(X; U(1)) \xrightarrow{\sim} \text{Pic}^0(X)$$

is a bijection.

**Problem 10** Prove the global  $\partial\bar{\partial}$ -Lemma on a compact Kähler manifold  $X$ : for any  $d$ -exact form  $\varphi \in \mathcal{E}^{p,q}$ , there exists  $\psi \in \mathcal{E}^{p-1,q-1}$  with

$$\partial\bar{\partial}\psi = \varphi.$$

*Hint:* Show that

$$\varphi = \partial\bar{\partial}^* \square \varphi$$

and use this and similar identities to find  $\psi$ .

*Solution:* Fix a Kähler metric on  $X$  and let  $\Delta_d$ ,  $\Delta_{\partial}$  and  $\Delta_{\bar{\partial}}$  denote the  $d$ -,  $\partial$ - and  $\bar{\partial}$ -Laplacians. On a Kähler manifold we have the identities

$$\Delta_d = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}$$

acting on each  $\mathcal{E}^{p,q}$ , and these Laplacians commute with  $\partial$  and  $\bar{\partial}$ .

We define the Green operator  $G_{\bar{\partial}} : \mathcal{E}^{p,q}(X) \rightarrow \mathcal{E}^{p,q}(X)$  as the inverse of  $\Delta_{\bar{\partial}}$  on the orthogonal complement of the space of  $\bar{\partial}$ -harmonic forms, and zero on the harmonic forms.

Regard  $\varphi$  as a  $\bar{\partial}$ -exact form. Since its Dolbeault class in  $H_{\bar{\partial}}^{p,q}(X)$  is zero, its  $\bar{\partial}$ -harmonic part vanishes and we can write

$$\varphi = \Delta_{\bar{\partial}} G_{\bar{\partial}} \varphi = (\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) G_{\bar{\partial}} \varphi,$$

Using that  $\partial$  commutes with  $\Delta_{\bar{\partial}}$  on a Kähler manifold, we have

$$0 = \partial \varphi = \partial \Delta_{\bar{\partial}} G_{\bar{\partial}} \varphi = \Delta_{\bar{\partial}} \partial G_{\bar{\partial}} \varphi,$$

so  $\partial G_{\bar{\partial}} \varphi$  is  $\bar{\partial}$ -harmonic and  $\bar{\partial}$ -exact, hence  $\partial G_{\bar{\partial}} \varphi = 0$ . It follows that  $\bar{\partial}^* \bar{\partial} G_{\bar{\partial}} \varphi = 0$  and

$$\varphi = \bar{\partial} \bar{\partial}^* G_{\bar{\partial}} \varphi.$$

Now use the Kähler identity  $\bar{\partial}^* = -i[\Lambda, \partial]$ , where  $\Lambda$  is contraction with the Kähler form. Then

$$\varphi = -i \bar{\partial} [\Lambda, \partial] G_{\bar{\partial}} \varphi = -i (\bar{\partial} \Lambda \partial - \bar{\partial} \partial \Lambda) G_{\bar{\partial}} \varphi.$$

Since  $\partial G_{\bar{\partial}} \varphi = 0$ , the first term vanishes, and using  $\bar{\partial} \partial = -\partial \bar{\partial}$  we obtain

$$\varphi = i \partial \bar{\partial} (\Lambda G_{\bar{\partial}} \varphi).$$

Thus, if we set

$$\psi := i \Lambda G_{\bar{\partial}} \varphi \in \mathcal{E}^{p-1, q-1}(X),$$

then

$$\partial \bar{\partial} \psi = \varphi.$$