Complex Manifolds

Songyu Ye

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Abstract

These are notes for the course Complex Manifolds (Math 241) taught by Professor Constantin Teleman in the Fall of 2025 at UC Berkeley.

Contents

1

The classical story begins with the Weierstrass $\wp$-function, defined by

$$\wp(z; L) = \frac{1}{z^2} + \sum_{\omega \in L \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

which has the properties that it is an L-periodic meromorphic function on $\mathbb{C}$ with double poles at the lattice points, and that it satisfies the differential equation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 = 4(z-e_1)(z-e_2)(z-e_3)$$

where g_2, g_3 are constants depending on L, given explicitly by

$$g_2 = 60 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}$$
$$g_3 = 140 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}$$

and e_i are the values of $\wp$ at the half-lattice points $\omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2$. The e_i are distinct as we will show in Prop 1.2. The convergence is uniform on any compact subset $K \subset \mathbb{C}$, once the terms with poles in K are set aside.

Uniform convergence implies that the series can be differentiated term-by-term, so we get a formula for $\wp'(z)$ given by

$$\wp'(z) = -2 \sum_{\omega \in L} \frac{1}{(z - \omega)^3}$$

is an doubly periodic meromorphic function with triple poles at the lattice points. Moreover, one can see directly from the series expansion that $\wp$ is even and $\wp'$ is odd.

The oddness implies that $\wp'(z)$ vanishes at the half-lattice points. Moreover, one can check that these are simple zeros of $\wp'$, and moreover the only zeros of $\wp'$ modulo L. Thus $\wp'$ has only poles at lattice points, each of order 3. In a fundamental parallelogram there is exactly one pole (mod L), of total multiplicity 3. This implies the following proposition.

Proposition 1.1. *$\wp(z)$ and $\wp'(z)$ define holomorphic maps $\mathbb{C}/L \to \mathbb{P}^1$ of degree 2 and 3 respectively.*

We conclude that each of the half-lattice points must be a simple zero of $\wp'$ and moreover that these are all of the zeros, because any meromorphic function has divisor of degree 0.

Proposition 1.2 (Properties of the $\wp$-map)**.**

(i) The numbers e_1, e_2, e_3 are all distinct.

(ii) For any $a \in \mathbb{C}$ with $a \neq e_1, e_2, e_3$, the equation $\wp(u) = a$ has two simple roots in a fundamental period parallelogram. For the three exceptional values $a = e_i$, it has a single double root.

Proof.

(ii) General theory of meromorphic functions on a torus shows that we either have two simple roots or one double root. Since a double root corresponds to a zero of the derivative $\wp'$, the claim follows. Note that the two simple roots always differ by a sign modulo L, by the parity of $\wp$.

(i) Suppose, for contradiction, that $e_1 = e_2$. Then $\wp(u) = e_1$ would have a double root at $\frac{\omega_1}{2}$ and another double root at $\frac{\omega_2}{2}$. This would give too many roots (multiplicity 4 in a fundamental parallelogram), contradicting the fact that $\wp$ is a double covering of $\mathbb{P}^1$. Hence the e_i are distinct.

□

Remark 1.3. *Kac writes that this quadratic term which appears in the definition of t_α* **??** *"explains" the appearance of theta functions in the theory of affine algebras. This is because when you compute the characters of highest-weight representations of affine Kac-Moody algebras, you sum over the affine Weyl group:*

$$\chi(\lambda) = \sum_{w \in W} \det(w)\, e^{w(\lambda+\rho)-\rho}$$

and theta functions arise precisely when you sum exponentials of the form

$$\Theta(\tau, z) = \sum_{\alpha \in \mathit{lattice}} \exp\big(-\tfrac{1}{2}|\alpha|^2\tau + \langle \alpha, z \rangle\big).$$

Continuing with the discussion of theta functions, we have the following theorem about genus 1 Riemann surfaces.

Theorem 1.4. *Let $\theta_1, \ldots, \theta_4$ be the four Jacobi theta functions. Then there is a map*

$$E/L \to \mathbb{CP}^3, \quad z \mapsto [\theta_1(z,\tau) : \theta_2(z,\tau) : \theta_3(z,\tau) : \theta_4(z,\tau)]$$

which is a smooth embedding of the complex torus $E = \mathbb{C}/L$ into projective space. It is a degree 4 map and its image is the intersection of two quadrics.

Proposition 1.5. *The function $\wp : \mathbb{C}/L \to \mathbb{P}^1$ is a degree 2 holomorphic map with branch points over e_1, e_2, e_3, ∞.*

Those of us who solved Example Sheet 1, Question 2, have seen the same picture of branching for the Riemann surface of the cubic equation

$$w^2 = (z - e_1)(z - e_2)(z - e_3);$$

in Lecture 10, we shall establish a deep connection between the two.

We will use the $\wp$-function to prove the Unique Presentation by principal parts. Uniqueness being clear on general grounds (cf. Lecture 4), we merely need to prove the existence statement; and this will emerge from the proof of the first theorem below. Remarkably, this will also allow us to describe the field of meromorphic functions over $\mathbb{C}/L$.

Theorem 1.6. *Every elliptic function is a rational function of $\wp$ and $\wp'$. Specifically, every* even *elliptic function is a rational function of $\wp$, every* odd *elliptic function is $\wp'$ times a rational function of $\wp$; and every elliptic function can be expressed uniquely as*

$$f(u) = R_0(\wp(u)) + \wp'(u)\, R_1(\wp(u)),$$

with R_0, R_1 rational functions, where the two terms are the even and odd parts of f.

Proof. It suffices to prove the statement for even elliptic functions; division by $\wp'$ reduces odd ones to even ones. Recall that

$$\wp : \mathbb{C}/L \longrightarrow \mathbb{P}^1$$

is a degree 2 holomorphic map. This map realizes $\mathbb{P}^1$ as the quotient space of the torus $\mathbb{C}/L$ under the identification of u with $-u$. Certainly the map is surjective because general theory of holomorphic maps between compact Riemann surfaces shows that any nonconstant holomorphic map is surjective. The map is injective because $\wp(u) = \wp(v)$ if and only if $u \equiv \pm v \mod L$.

A bijective holomorphic map between compact Riemann surfaces is automatically biholomorphic. Let $f : R \to S$ be such a map. The inverse function theorem guarantees that the inverse function f^{-1} is smooth. Moreover, it guarantees that

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

Since f is bijective, it has nonzero derivative everywhere because if it did not, it would look like $z \mapsto z^k$ for some $k \geq 2$ and thus it would fail to be locally bijective. Since it has nonzero derivative everywhere, $(f^{-1})'$ is defined everywhere and is in fact a complex number. Hence f^{-1} is holomorphic. Δ and $\mathbb{C}$ are homeomorphic but they are not biholomorphic.

So indeed $\mathbb{P}^1$ is the quotient of $\mathbb{C}/L$ by the involution $u \mapsto -u$. Hence, any even *continuous* map

$$f : \mathbb{C}/L \to \mathbb{P}^1$$

has the form $f = R \circ \wp$, for some continuous map $R : \mathbb{P}^1 \to \mathbb{P}^1$. Moreover, $\wp$ is a local analytic isomorphism away from the four branch points, which implies that R is holomorphic there, if $R \circ \wp$ was so. So we know that R is continuous everywhere and holomorphic away from the four branch points.

The following result shows that R is holomorphic everywhere, hence a rational function.

$$R(z) = P(z)/Q(z) \implies f(u) = P(\wp(u))/Q(\wp(u))$$

Writing every elliptic function as a sum of an even and an odd one, and the odd ones as $\wp'$ times an even one, we get the desired result. $\square$

Proposition 1.7. *Let $f : S \to R$ be a continuous map between Riemann surfaces, known to be holomorphic except at isolated points. Then f is holomorphic everywhere.*

Proof. Choosing coordinate neighbourhoods near the questionable points and their images, we are reduced to the statement that a continuous function on Δ which is holomorphic on $\Delta^\times$ is, in fact, holomorphic at 0 as well. This follows from Riemann's theorem on removable singularities. $\square$

A remarkable consequence is that the function $\wp'(u)^2$, being elliptic and even, is expressible in terms of $\wp$. Explicitly, we have the following.

Theorem 1.8 (Differential equation for $\wp$)**.**

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3,$$

where $g_2 = 60G_4$, $g_3 = 140G_6$, and

$$G_r = G_r(L) = \sum_{\omega\in L^*} \omega^{-r}.$$

Proof. Recall the Laurent expansion of the Weierstrass function

$$\wp(u) = u^{-2} + 3G_4(L)u^2 + 5G_6(L)u^4 + \cdots, \qquad \wp'(u) = -2u^{-3} + 6G_4(L)u + 20G_6(L)u^3 + \cdots.$$

For $|u| < |\omega|$ and any integer $k \geq 1$,

$$(u-\omega)^{-k} = \frac{(-1)^k}{\omega^k}\left[1 + k\frac{u}{\omega} + \frac{k(k+1)}{2!}\frac{u^2}{\omega^2} + \frac{k(k+1)(k+2)}{3!}\frac{u^3}{\omega^3} + \cdots\right].$$

Expanding each term in the defining series for $\wp$ with the above, and (for small u) interchanging sums, the odd powers in u cancel, giving

$$\wp(u) = u^{-2} + \sum_{m=1}^{\infty}\binom{-2}{2m} G_{2m+2}(L)\, u^{2m} = u^{-2} + \sum_{m=1}^{\infty}(2m+1)\, G_{2m+2}(L)\, u^{2m}.$$

Similarly,

$$\wp'(u) = -2u^{-3} + \sum_{m=0}^{\infty}(-2)\binom{-3}{2m+1} G_{2m+4}(L)\, u^{2m+1}$$

Using these expansions, the first few terms of $(\wp'(u))^2$ and $4\wp(u)^3 - g_2\wp(u) - g_3$ agree at $u = 0$; hence their difference is an elliptic function with no poles that vanishes at $u = 0$, so it is identically zero. $\square$

The two theorems immediately lead to a description of the field of meromorphic functions on $\mathbb{C}/L$.

Corollary 1.9. *The field of meromorphic functions on $\mathbb{C}/L$ is isomorphic to*

$$\mathbb{C}(z)[w]/(w^2 - 4z^3 + g_2 z + g_3),$$

the degree 2 extension of the field of rational functions $\mathbb{C}(z)$ obtained by adjoining the solutions w to the equation

$$w^2 = 4z^3 - g_2 z - g_3.$$

Theorem 1.10. *Let $z_1, \dots, z_n$ and $p_1, \dots, p_m$ denote the zeroes and poles of a non-constant elliptic function f in the period parallelogram, repeated according to multiplicity. Then:*

(i) $m = n$,

(ii) $\sum_{k=1}^{m} \mathrm{Res}_{p_k}(f) = 0$,

(iii) $\sum_{k=1}^{n} z_k = \sum_{k=1}^{m} p_k \pmod L$.

Remark 1.11. *Zeroes and poles that are on the boundary should be counted only on a single edge, or at a single vertex. In fact, we can easily avoid zeroes and poles on the boundary by shifting our parallelogram by a small complex number λ; the relations (i)–(iii) are unchanged.*

Definition 1.12. *Fix a local coordinate z at a point p. The* principal part *of a meromorphic function f at p is the part of its Laurent expansion in negative powers of $(z-p)$:*

$$\sum_{n=1}^{N} a_{-n}(z-p)^{-n}$$

Theorem 1.13 (Unique Presentation by principal parts)**.** *An elliptic function is specified uniquely, up to an additive constant, by prescribing its principal parts at all poles in the period parallelogram. The prescription is subject only to condition (ii).*

Proof. This is more computational, but also more concrete. We first show that we can realize any even assignment of principal parts on $\mathbb{C}/L$ using a suitable rational function $R(\wp(u))$. Such an assignment involves finitely many points $\lambda \in \mathbb{C}/L$ and principal parts

$$\sum_{k=1}^{n_\lambda} a_k^{(\lambda)}(u-\lambda)^{-k},$$

with the properties that:

- if $2\lambda \notin L$, then $(-\lambda)$ also appears, with assignment

$$\sum_{k=1}^{n_\lambda} (-1)^k a_k^{(\lambda)} (u+\lambda)^{-k},$$

i.e. $a_k^{(-\lambda)} = (-1)^k a_k^{(\lambda)}$;

- if $2\lambda \in L$, then only even powers of $(u-\lambda)^{-1}$ are present.

This is because the local coordinates at λ and $-\lambda$ are opposite signs. Write the principal part at λ (using $v = u - \lambda$): $f(u) = \sum_{k=1}^{n_\lambda} a_k^{(\lambda)} v^{-k} + \cdots$. Near $-\lambda$ use $w = u + \lambda$. Evenness gives

$$f(-\lambda + w) = f\big(-(-\lambda + w)\big) = f(\lambda - w) = \sum_{k \geq 1} a_k^{(\lambda)} (-w)^{-k} = \sum_{k \geq 1} (-1)^k a_k^{(\lambda)} w^{-k}$$

If $2\lambda \in L$ (so $-\lambda \equiv \lambda$ on $\mathbb{C}/L$), the same calculation forces $\sum_{k\geq 1} a_k^{(\lambda)} v^{-k} = \sum_{k \geq 1} a_k^{(\lambda)} (-v)^{-k}$, hence $a_k^{(\lambda)} = 0$ for all odd k: only even powers $(u-\lambda)^{-2j}$ can appear.

Now if $2\lambda \notin L$, $(\wp(u) - \wp(\lambda))^{-1}$ has a simple pole at $u = \lambda$ and we can create any principal part there as a sum of $(\wp(u) - \wp(\lambda))^{-k}$. Evenness of $\wp$ takes care of the symmetry. If $2\lambda \in L$, then we can use either powers of $\wp$, if $\lambda \in L$, or powers of $(\wp(u) - e_{1,2,3})^{-1}$, which have double poles with no residue.

Now, onto the odd functions. Odd assignments of principal parts are of the form

$$\sum_{k=1}^{n_\lambda} a_k^{(\lambda)} (u-\lambda)^{-k},$$

with a matching term

$$-\sum_{k=1}^{n_\lambda} (-1)^k a_k^{(\lambda)} (u+\lambda)^{-k}$$

at $-\lambda$ (i.e. $a_k^{(-\lambda)} = (-1)^{k+1} a_k^{(\lambda)}$), or else with vanishing $a_k^{(\lambda)}$ (for even k) if $2\lambda \in L$.

The principal parts

$$\left(\frac{P_\lambda}{\wp'(u)} - \frac{P_{-\lambda}}{\wp'(u)} \right)$$

can be realized by a sum of powers of $(\wp(u) - \wp(\lambda))^{-1}$. If $2\lambda \in L$ but $\lambda \notin L$ (not 0), then $P_\lambda^{(u)}/\wp'(u)$ is also a well-defined even principal part, expressible via $(\wp(u) - \wp(\lambda))^{-1}$. The same goes for $P_0^{(u)}/\wp'(u)$. So there exists a function of the form $R_1(\wp(u))$ whose principal parts agree with the $P_\lambda(u)/\wp'(u)$ everywhere.

The principal parts of $R_1(\wp(u))\,\wp'(u)$ agree with the P_λ, except possibly at $\lambda = 0$, where the cubic pole of $\wp'$ could introduce unwanted or incorrect u^{-3} and u^{-1} terms. We can adjust the u^{-3} term by shifting R_1 by a constant. We have no control over the u^{-1} term, but that is determined from the condition $\sum \mathrm{Res} = 0$, which indeed must be met if a function with the prescribed principal parts is to exist. $\square$

Theorem 1.14 (Unique Presentation by zeroes and poles)**.** *An elliptic function is specified uniquely, up to a multiplicative constant, by prescribing the location of its zeroes and poles in the period parallelogram, with multiplicities. The prescription is subject to conditions (i) and (iii).*

Lemma 1.15. *$g_2^3 \neq 27g_3^2$ and e_1, e_2, e_3 are the roots of the equation*

$$4z^3 - g_2 z - g_3 = 0.$$

Proof. $\wp'$ vanishes at the half-lattice points, while $\wp$ takes the values e_1, e_2, e_3 there. The roots are distinct so the discriminant of the cubic is nonzero, i.e. $g_2^3 \neq 27g_3^2$. $\square$

Theorem 1.16 (Geometric interpretation)**.** *The map $\mathbb{C}/L \setminus \{0\} \to \mathbb{C}^2$ given by*

$$u \longmapsto (z(u), w(u)) = (\wp(u), \wp'(u))$$

gives an analytic isomorphism between the Riemann surface $\mathbb{C}/L \setminus \{0\}$ and the (concrete) Riemann surface R of the function

$$w^2 = 4z^3 - g_2 z - g_3$$

in $\mathbb{C}^2$.

Proof. We have the commutative diagram:

$$\begin{tikzcd} \mathbb{C}/L \setminus \{0\} \arrow[r, "(\wp,\wp')"] \arrow[dr, "\wp"'] & R \arrow[d, "\pi"] \\ & \mathbb{C} \end{tikzcd}$$

and we know that:

- π is proper and 2-to-1 except at the branch points e_1, e_2, e_3, which are the roots of $4z^3 - g_2 z - g_3$.
- $\wp$ is proper and 2-to-1 except at the half-period points $\omega_1/2, \omega_2/2, \omega_1/2 + \omega_2/2$, which map to the roots e_1, e_2, e_3.
- $\wp(u) = \wp(-u)$ and $\wp'(u) = -\wp'(-u)$: this means that, unless u is a half-period, $\wp'$ takes both values $\pm w = \pm\wp'(u)$ at the two points $\pm u$ mapping to the same $z = \wp(u)$ of $\mathbb{C}$.

Together, these three properties show that the map we just constructed is bijective. Note further that, at no point $u \in \mathbb{C}/L \setminus \{0\}$, is $\wp'(u) = \wp''(u) = 0$, because $\wp'$ has simple zeros only (there are three of them); this means that for every $u \in \mathbb{C}/L \setminus \{0\}$, either the map $\wp$ or the map $\wp'$ gives an analytic isomorphism of a neighbourhood of u with a small disc in the z-plane or in the w-plane.

Since the Riemann surface structure on the (concrete, non-singular) Riemann surface R is defined by the projections to the z- and w-planes, appropriately, we conclude that $(\wp, \wp')$ gives an analytic isomorphism

$$\mathbb{C}/L \longrightarrow R.$$

□

2 Introduction

Theorem 2.1. *The following categories are equivalent:*

- *Compact Riemann surfaces with nonconstant holomorphic maps*
- *Smooth proper (and hence projective) algebraic curves over $\mathbb{C}$ with nonconstant morphisms*
- *Field extensions of $\mathbb{C}$ of transcendence degree 1, of finite degree over $\mathbb{C}(t)$ where t is transcendental over $\mathbb{C}$, with field homomorphisms over $\mathbb{C}$*

The correspondence in one direction is:

$$\begin{aligned} \textit{Riemann surface } S &\mapsto \textit{ function field } \mathbb{C}(S) \\ \textit{Holomorphic map } f : S \to S' &\mapsto \textit{ field homomorphism } f^* : \mathbb{C}(S') \to \mathbb{C}(S) \end{aligned}$$

Remark 2.2. *For curves, smooth and proper implies projective. This is false in higher dimensions.*

Common to both is the construction of nonconstant meromorphic functions. It suffices to find

- A map $f : R \to \mathbb{P}^1$ which realizes R as a branched cover of $\mathbb{P}^1$ (the transcendental part of the function field)

$$\begin{aligned} f^* : \mathbb{C}(z) &\hookrightarrow \mathbb{C}(R) \\ z &\mapsto f \end{aligned}$$

- A nonconstant meromorphic function g on S which separates the sheets (the finite part of the function field)

Once you have these functions, consider the set of pairs $\{(f(p), g(p)) : p \in S\} \subset \mathbb{P}^1 \times \mathbb{P}^1$. This is an analytic curve. By a theorem of Riemann (or later by Chow's theorem), an analytic curve in projective space is algebraic. So there exists a nonzero polynomial $P(x, y)$ such that

$$P(f, g) = 0 \quad \text{on } S.$$

Thus, the image of S under (f, g) is contained in the algebraic curve $P(x, y) = 0$. Moreover, because g separates the sheets, (f, g) is generically injective, so the map is birational. Hence S and the curve $P(x, y) = 0$ have the same function field. So you've now explicitly realized $\mathbb{C}(S) = \mathbb{C}(f, g)$.

3

We state Riemann's theorem which allows us to pass from the analytic setting to the algebraic setting.

Theorem 3.1. *Let R be a compact Riemann surface and $p \in R$. There exists a meromorphic function f with poles of arbitrary order n at p and holomorphic elsewhere, provided that n is sufficiently large.*

The method of proof involves constructing holomorphic differentials with poles at p, and in fact one can get them to any order of pole ≥ 2. Then if these differentials are exact, their integrals give a single valued function with pole only at p.

4

If f is a nonconstant meromorphic function on a compact Riemann surface R, then we defined the divisor of f to be

$$(f) = \sum_{p \in R} \operatorname{ord}_p(f) p$$

where $\operatorname{ord}_p(f)$ is the order of vanishing of f at p (negative if f has a pole at p).

We defined the following sets:

$$\begin{aligned} \operatorname{Div}(R) &= \{\text{formal finite sums } \textstyle\sum n_p p, n_p \in \mathbb{Z}\} \\ \operatorname{PDiv}(R) &= \{\text{divisors of meromorphic functions}\} \\ \operatorname{Cl}(R) &= \operatorname{Div}(R) / \operatorname{PDiv}(R) \end{aligned}$$

and there is a map

$$\begin{aligned} \operatorname{Div}(R) &\to \operatorname{Pic}(R) \\ D &\mapsto \mathcal{O}(D) \end{aligned}$$

where

$$\mathcal{O}(D)(U) = \{f \text{ meromorphic on } U : (f)|_U + D|_U \geq 0\}$$

is an invertible sheaf. More precisely, from D one gets an invertible sheaf $\mathcal{O}(D)$ along with a meromorphic section s_D such that $(s_D) = D$.

One can think of s_D as the constant function 1. In particular, recall that $\mathcal{O}(D)$ is locally isomorphic to $\mathcal{O}_R$ by picking local defining equations η_α for D on an open cover U_α. Recall that on a smooth variety there is an equivalence between Cartier divisors and Weil divisors. Then the isomorphism $\mathcal{O}(D)|_{U_\alpha} \to \mathcal{O}_R|_{U_\alpha}$ is given by multiplication by η_α. Then the canonical meromorphic section s_D, when restricted to U_α, is given by η_α which has divisor $D|_{U_\alpha}$.

Therefore, there is an isomorphism of abelian groups

$$\begin{aligned} \mathrm{Cl}(R) &\to \text{subgroup of } \mathrm{Pic}(R) \text{ consisting of invertible sheaves admitting meromorphic sections} \\ D &\mapsto (\mathcal{O}(D), 1) \end{aligned}$$

and this is in fact an isomorphism of groups because of the following theorem.

Theorem 4.1. *Every $\mathcal{L}$ on a Riemann surface has a nonzero meromorphic section. More generally, every vector bundle admits a global meromorphic frame.*

Remark 4.2. *The compact case follows from the Kodaira vanishing theorem. In the noncompact case, all holomorphicq vector bundles on noncompact R are trivializable and therefore admit a global holomorphic frame.*

Recall that the multiplicative Cousin problem is the problem of finding a global meromorphic function with prescribed zeroes and poles. The additive Cousin problem is the problem of finding a global meromorphic function with prescribed principal parts. The above theorem shows that both problems are always solvable on a noncompact Riemann surface.

Theorem 4.3. *On a noncompact Riemann surface, the multiplicative and additive Cousin problems are always solvable.*

All holomorphic vector bundles on a noncompact Riemann surface are trivializable.

Definition 4.4 (Degree of a line/vector bundle)**.** *The degree of a line bundle $\mathcal{L}$ on a compact Riemann surface R is defined to be the degree of any meromorphic section of $\mathcal{L}$. This is well defined because if s, s' are two meromorphic sections of $\mathcal{L}$, then s/s' is a meromorphic function on R and has degree 0.*

The degree of a vector bundle $\mathcal{E}$ is defined to be the degree of its determinant line bundle $\det \mathcal{E} = \wedge^{\mathrm{rank}\,\mathcal{E}} \mathcal{E}$.

Fact 4.5. *On a compact Riemann surface, the degree and dimension of a vector bundle completely determine the topology of the bundle.*

Proposition 4.6. *Every holomorphic line bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(n)$ for some integer n.*

Proof. We can solve the multiplicative Cousin problem on $\mathbb{P}^1$ for degree zero divisors. □

Proposition 4.7. *Let $E = \mathbb{C}/L$ be an elliptic curve. Then*

$$0 \to E \to \mathrm{Pic}(E) \to \mathbb{Z} \to 0$$

is a short exact sequence of abelian groups. It splits, so $\mathrm{Pic}(E) \cong E \times \mathbb{Z}$.

Example 4.8 (Doubled lattice)**.** *Recall that every ellptic curve $E = \mathbb{C}/L$ has a degree four cover by $\tilde{E} = \mathbb{C}/2L$. We defined four θ functions on E, let $\mathcal{L}_i$ be the corresponding line bundles. Then $\pi^*\mathcal{L}_i$ are all isomorphic on $\tilde{E}$ because the beriodicity conditions all become the same after doubling the lattice. Moreover recall that there is a map*

$$E \to \mathbb{P}^3, \quad z \mapsto [\theta_1(z,\tau) : \theta_2(z,\tau) : \theta_3(z,\tau) : \theta_4(z,\tau)]$$

which is in fact a projective embedding by a line bundle.

Recall that in general if one has $\mathcal{L}$ a line bundle on X, then we can consider the evaluation map $X \to \mathbb{P}(H^0(X,\mathcal{L})^*)$ given by $x \mapsto \{s \in H^0(X,\mathcal{L}) : s(x) = 0\}$ when $\mathcal{L}$ has enough sections. For example, if $\mathcal{L}$ has negative degree than it has no sections. If $\mathcal{L}$ has degree 0 then it has a section if and only if it is trivial.

The analog of $\otimes\mathcal{O}(D)$ for vector bundles is called an elementary transformation. Let V be a vector bundle on R and choose a subspace $S \subset V_p$.

Define $\mathrm{elm}(V, p, S)$ to be the sheaf of sections of V whose value at p lies in S. This is a vector bundle whose degree is $\deg V - \operatorname{codim} S$.

Proposition 4.9. *Every vector bundle is obtained from a trivial vector bundle by a finite sequence of elementary transformations.*

Exercise 4.10. *Let V be a rank 2 (for simplicity) vector bundle over a Riemann surface R. Assume that V has two meromorphic sections s_1, s_2 which, at some point, are holomorphic and span the fiber.*

(a) Show that this will be the case everywhere except at a set of isolated points.

(b) At an exceptional point, show that we can modify V by a finite sequence of elementary transformations so that s_1 and s_2 form a holomorphic frame of the new bundle.

Suggestion. *First make the sections holomorphic, then find some numerical measure for their failure to give a basis. Then find a way to reduce that number.*

Solution 4.11. *Let V be rank 2 over a Riemann surface R. Let s_1, s_2 be meromorphic sections that at some point are holomorphic and span V there.*

First clear poles once and for all. Pick an effective divisor D dominating all poles of s_1, s_2. Then $\tilde{s}_i := s_i \otimes 1 \in H^0\big(R, V(D)\big)$ are holomorphic sections of $V(D) := V \otimes \mathcal{O}(D)$, and agree with the original s_i on $R \setminus \operatorname{supp}(D)$.

Write $V' := V(D)$ and still denote the sections by s_1, s_2. Consider the wedge $\sigma = s_1 \wedge s_2 \in H^0\big(R, \det V'\big)$. At your original point it's nonzero, so $\sigma \not\equiv 0$. On a Riemann surface any nonzero holomorphic section of a line bundle has discrete zero set. Hence the locus where s_1, s_2 fail to span (i.e. $\sigma = 0$) is a finite/locally finite set of isolated points. Everywhere else they are a holomorphic frame.

Remark 4.12. *The argument generalizes to any dimension. If R is compact, it follows that we can trivialize V by a finite number of elementary transformations. If R is non-compact, one can show that every vector bundle is in fact trivial.*

Theorem 4.13. *Every vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles.*

$$V \cong \bigoplus_{i=1}^{\operatorname{rank} V} \mathcal{O}_{\mathbb{P}^1}(n_i)$$

where $n_i \geq n_{i+1}$. Moreover, the n_i are uniquely determined by V.

The degree of V is $\sum n_i$.

Example 4.14. *On $\mathbb{P}^1$, we have homeomorphic but not biholomorphic vector bundles $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O}$. They both have degreee zero and the same number of sections, but the sections sit inside the bundles differently.*