Coherent sheaves and exceptional collections

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Abstract

Coherent sheaves, vector bundles, and exceptional collections in algebraic geometry.

Contents

1 Preliminaries

1.1 Schemes

Definition 1.1 (Closed and Non-closed Points)**.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme.*

1. *A point $p \in X$ is called a closed point if the corresponding prime ideal $\mathfrak{p}$ is a maximal ideal of A.*

2. *A point $p \in X$ is called a non-closed point if the corresponding prime ideal $\mathfrak{p}$ is not maximal.*

3. *A generic point of an irreducible component of X corresponds to a minimal prime ideal of A.*

Example 1.2. *Consider $X = \mathrm{Spec}(\mathbb{C}[x, y])$, the affine plane over $\mathbb{C}$.*

1. *Closed points correspond to maximal ideals of the form $(x - a, y - b)$ for $a, b \in \mathbb{C}$. These are the familiar points (a, b) in the complex plane.*

2. *Prime ideals like $(x - 1)$ correspond to non-closed points. Geometrically, this represents the "generic point" of the vertical line $x = 1$.*

3. *The prime ideal (0) corresponds to the generic point of the entire plane.*

Remark 1.3. *For a scheme over a field k:*

1. *If k is algebraically closed (like $\mathbb{C}$), the closed points of $\mathrm{Spec}(k[x_1, \ldots, x_n])$ correspond exactly to the n-tuples $(a_1, \ldots, a_n) \in k^n$.*

2. *If k is not algebraically closed (like $\mathbb{Q}$), there are additional closed points. For example, in $\mathrm{Spec}(\mathbb{Q}[x])$, the ideal $(x^2 + 1)$ is maximal and corresponds to a closed point, even though it does not correspond to a rational value of x.*

Proposition 1.4. *Let X be a scheme of finite type over a field k. Then:*

1. *The closed points of X are dense in X (Zariski topology).*

2. *If X is irreducible, it has a unique generic point.*

3. *The closure of any point $p \in X$ consists of p and all the specializations of p.*

Definition 1.5 (Stalk of the Structure Sheaf)**.** *Let X be a scheme and $p \in X$ a point. The stalk of the structure sheaf $\mathcal{O}_X$ at p, denoted $\mathcal{O}_{X,p}$, is defined as the direct limit:*

$$\mathcal{O}_{X,p} = \varinjlim_{U \ni p} \mathcal{O}_X(U)$$

where the limit is taken over all open sets U containing the point p.

Proposition 1.6. *Let $X = \mathrm{Spec}(A)$ be an affine scheme and $p \in X$ the point corresponding to a prime ideal $\mathfrak{p} \subset A$. Then:*

$$\mathcal{O}_{X,p} \cong A_{\mathfrak{p}}$$

where $A_{\mathfrak{p}}$ is the localization of A at the prime ideal $\mathfrak{p}$.

Remark 1.7. *The stalk $\mathcal{O}_{X,p}$ is always a local ring. Its unique maximal ideal, denoted $\mathfrak{m}_p$, consists of germs of functions that vanish at p.*

Example 1.8. *Let $X = \mathrm{Spec}(\mathbb{C}[x, y])$ and p the origin (corresponding to the maximal ideal (x, y)). Then:*

$$\mathcal{O}_{X,p} \cong \mathbb{C}[x, y]_{(x,y)}$$

This is the ring of rational functions in x and y that are defined at the origin.

Example 1.9. *Let $X = \mathrm{Spec}(\mathbb{C}[x, y]/(xy))$, a union of two coordinate axes, and p the origin. Then:*

$$\mathcal{O}_{X,p} \cong \mathbb{C}[x, y]_{(x,y)}/(xy)$$

This local ring has zero divisors, reflecting the fact that p is a singular point of X.

Definition 1.10 (Residue Field)**.** *Let X be a scheme and $p \in X$ a point. The residue field at p, denoted $\kappa(p)$, is defined as:*

$$\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p$$

where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$.

Proposition 1.11. *Let $X = \mathrm{Spec}(A)$ be an affine scheme and $p \in X$ the point corresponding to a prime ideal $\mathfrak{p} \subset A$. Then:*

$$\kappa(p) \cong \mathrm{Frac}(A/\mathfrak{p})$$

the fraction field of the domain $A/\mathfrak{p}$.

Remark 1.12. *For a closed point p corresponding to a maximal ideal $\mathfrak{m}$, we have $\kappa(p) \cong A/\mathfrak{m}$, which is already a field.*

Example 1.13. *Let $X = \mathrm{Spec}(\mathbb{C}[x, y])$.*

1. *For the closed point p corresponding to the maximal ideal $(x - a, y - b)$, the residue field is:*

$$\kappa(p) \cong \mathbb{C}[x, y]/(x - a, y - b) \cong \mathbb{C}$$

2. *For the non-closed point q corresponding to the prime ideal $(x - a)$, the residue field is:*

$$\kappa(q) \cong \mathrm{Frac}(\mathbb{C}[x, y]/(x - a)) \cong \mathbb{C}(y)$$

the field of rational functions in one variable.

3. *For the generic point η corresponding to the prime ideal (0), the residue field is:*

$$\kappa(\eta) \cong \mathrm{Frac}(\mathbb{C}[x, y]) \cong \mathbb{C}(x, y)$$

the field of rational functions in two variables.

Example 1.14. *Let $X = \mathrm{Spec}(\mathbb{Q}[x])$.*

1. *For the closed point p corresponding to the maximal ideal $(x - a)$ where $a \in \mathbb{Q}$, the residue field is:*

$$\kappa(p) \cong \mathbb{Q}[x]/(x - a) \cong \mathbb{Q}$$

2. *For the closed point q corresponding to the maximal ideal $(x^2 + 1)$, the residue field is:*

$$\kappa(q) \cong \mathbb{Q}[x]/(x^2 + 1) \cong \mathbb{Q}(i)$$

which is a degree 2 extension of $\mathbb{Q}$.

3. *For the generic point η corresponding to the prime ideal (0), the residue field is:*

$$\kappa(\eta) \cong \mathrm{Frac}(\mathbb{Q}[x]) \cong \mathbb{Q}(x)$$

Definition 1.15 (Geometric Point)**.** *A geometric point of a scheme X is a morphism $\mathrm{Spec}(K) \to X$, where K is an algebraically closed field.*

Remark 1.16. *A geometric point can be thought of as a scheme-theoretic point together with an embedding of its residue field into an algebraically closed field.*

Proposition 1.17. *Let X be a scheme over a field k. If k is algebraically closed, then every closed point of X naturally gives rise to a geometric point. If k is not algebraically closed, this is not generally true.*

Example 1.18. *For $X = \mathrm{Spec}(\mathbb{Q}[x])$, the closed point corresponding to $(x^2 + 1)$ has residue field $\mathbb{Q}(i)$. This gives two distinct geometric points when we consider embeddings of $\mathbb{Q}(i)$ into $\mathbb{C}$ (corresponding to i and $-i$).*

1.2 Commutative Algebra

Definition 1.19 (Support of a module)**.** *Let A be a ring and M an A-module. The support of M, denoted $\mathrm{Supp}(M)$, is the set of prime ideals*

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$$

Definition 1.20 (Annihilator of a module)**.** *Let A be a ring and M an A-module. The annihilator of M, denoted $\operatorname{Ann}(M)$, is the ideal of elements*

$$\operatorname{Ann}(M) = \{a \in A \mid a \cdot m = 0 \text{ for all } m \in M\}$$

Proposition 1.21. *Let A be a ring and M an A-module. Then*

$$\operatorname{Supp}(M) = \operatorname{V}(\operatorname{Ann}(M)) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \operatorname{Ann}(M) \subset \mathfrak{p}\}$$

In particular, the support of M is a closed subset of $\operatorname{Spec}(A)$.

1.3 Sheaves

Definition 1.22 (Quasi-Coherent Sheaf)**.** *Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called quasi-coherent if for every open subset $U \subset X$, there exists a covering $\{U_i\}$ of U and a family of $\mathcal{O}_{U_i}$-modules $\mathcal{F}_i$ such that for each i, there exists an isomorphism $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$.*

Definition 1.23 (Coherent Sheaf)**.** *Let X be a scheme. A quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called coherent if:*

1. *$\mathcal{F}$ is of finite type, i.e., for every open subset $U \subset X$, there exists a surjection $\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$ for some integer n.*
2. *For any open set $U \subset X$ and any morphism $\varphi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$ of $\mathcal{O}_U$-modules, the kernel $\ker \varphi$ is of finite type.*

Definition 1.24 (Support of a Sheaf)**.** *Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. The support of $\mathcal{F}$, denoted $\operatorname{Supp}(\mathcal{F})$, is the set of points $x \in X$ where the stalk $\mathcal{F}_x$ is non-zero:*

$$\operatorname{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$$

Proposition 1.25. *For a coherent sheaf $\mathcal{F}$ on a scheme X:*

1. *$\operatorname{Supp}(\mathcal{F})$ is a closed subset of X.*
2. *If X is Noetherian, then $\operatorname{Supp}(\mathcal{F})$ equals the set of points where the fiber $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is non-zero.*
3. *For an affine scheme $X = \operatorname{Spec}(A)$ and $\mathcal{F} = \widetilde{M}$ corresponding to an A-module M, the support of $\mathcal{F}$ corresponds to $\{\mathfrak{p} \in \operatorname{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$.*

Remark 1.26. *On a noetherian scheme, a sheaf of $\mathcal{O}_X$-modules is coherent if and only if it is of finite type.*

Definition 1.27 (Vector Bundle)**.** *A vector bundle of rank r on a scheme X is a coherent sheaf $\mathcal{E}$ on X that is locally free of rank r, i.e., for every point $x \in X$, there exists an open neighborhood U of x such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$.*

Definition 1.28 (Torsion Sheaf)**.** *A coherent sheaf $\mathcal{F}$ on a scheme X is called a torsion sheaf if its support is a proper closed subset of X. Equivalently, for any open affine subset $\mathrm{Spec}(A) \subset X$, the corresponding A-module $\Gamma(\mathrm{Spec}(A), \mathcal{F})$ is a torsion A-module.*

Definition 1.29 (Points of a Scheme)**.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme. The points of X are in one-to-one correspondence with the prime ideals of A. Given a prime ideal $\mathfrak{p} \subset A$, we denote the corresponding point by $p_{\mathfrak{p}}$, or simply p when the context is clear.*

2 Examples of Non-Vector Bundle Coherent Sheaves

Example 2.1 (Skyscraper Sheaf)**.** *Let X be a scheme and $p \in X$ a point. The skyscraper sheaf $\mathcal{O}_p$ is a coherent sheaf defined as:*

$$\mathcal{O}_p(U) = \begin{cases} \kappa(p) & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

The residue field $\kappa(p)$ is a module over several rings. In particular, we can see that it is coherent because it is generated by a single element over the ring at hand.

- *It's an $\mathcal{O}_{X,p}$-module via the natural quotient map $\mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/\mathfrak{m}_p$*
  - *Any function germ in $\mathcal{O}_{X,p}$ acts on elements of $\kappa(p)$*
  - *Elements in $\mathfrak{m}_p$ act by zero*
  - *Elements outside $\mathfrak{m}_p$ act as non-zero scalars*
- *It's an $\mathcal{O}_X(U)$-module for any open set U containing p*
  - *The action is via the composition $\mathcal{O}_X(U) \to \mathcal{O}_{X,p} \to \kappa(p)$*
  - *This allows functions defined on U to act on the residue field*
- *For affine opens $U = \mathrm{Spec}(A)$ containing p, it's an A-module*

– *If p corresponds to the prime ideal $\mathfrak{p} \subset A$*

– *Then $\kappa(p) \cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong \mathrm{Frac}(A/\mathfrak{p})$*

– *The action is via $A \to A/\mathfrak{p} \to \mathrm{Frac}(A/\mathfrak{p})$*

It is not a vector bundle because:

- *It fails to be locally free at all points. It is a torsion sheaf: any function vanishing at p annihilates the entire sheaf.*
- *Its support is just the single point $\{p\}$, whereas vector bundles have support equal to X.*

Example 2.2 (Ideal Sheaf of a Subvariety)**.** *Let $L \subset \mathbb{P}^n$ be a line with ideal sheaf $\mathcal{I}_L$. This is a coherent sheaf that fails to be a vector bundle because:*

- *Its rank is not constant: $\mathrm{rank}(\mathcal{I}_L) = 1$ on $\mathbb{P}^n \setminus L$ but $\mathrm{rank}(\mathcal{I}_L) = 0$ along L.*
- *The dimension of $(\mathcal{I}_L)_p \otimes \kappa(p)$ varies: it equals 1 for $p \notin L$ (as the stalk $(\mathcal{I}_L)_p \cong \mathcal{O}_{\mathbb{P}^n,p}$) but equals 0 for $p \in L$ (as all functions in the ideal vanish at points on L).*

The exact sequence $0 \to \mathcal{I}_L \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_L \to 0$ illustrates this behavior.

Example 2.3 (Tangent Sheaf of a Singular Variety)**.** *For a singular variety X, the tangent sheaf $\mathcal{T}_X$ is coherent but not a vector bundle because:*

- *At smooth points $x \in X$, the sheaf is locally free of rank $\dim X$.*
- *At singular points, the stalk $(\mathcal{T}_X)_x$ fails to be a free $\mathcal{O}_{X,x}$-module.*
- *For example, on a nodal curve, the tangent sheaf at the node has torsion.*

Example 2.4 (Structure Sheaf of a Singular Variety)**.** *Let X be a singular variety with structure sheaf $\mathcal{O}_X$. Though $\mathcal{O}_X$ is always coherent, it fails to be locally free at singular points:*

- *At a singular point $p \in X$, the stalk $\mathcal{O}_{X,p}$ is not a regular local ring.*
- *For instance, if $X = \{xy = 0\} \subset \mathbb{A}^2$, then at the origin, $\mathcal{O}_{X,(0,0)} \cong k[x,y]/(xy)$, which is not a free module over itself.*

3 Exceptional Collections in Derived Categories

Definition 3.1 (Exceptional Object)**.** *An object E in a derived category $D^b(X)$ is called exceptional if:*

1. $\mathrm{Hom}(E, E) \cong k$ *(the base field)*
2. $\mathrm{Hom}(E, E[n]) = 0$ *for all* $n \neq 0$

Definition 3.2 (Exceptional Collection)**.** *An exceptional collection in $D^b(X)$ is an ordered sequence of exceptional objects $\{E_1, E_2, \ldots, E_n\}$ such that:*

$$\mathrm{Hom}(E_j, E_i[m]) = 0 \quad \textit{for all } j > i \textit{ and all } m \in \mathbb{Z}$$

Definition 3.3 (Full Exceptional Collection)**.** *An exceptional collection $\{E_1, E_2, \ldots, E_n\}$ in $D^b(X)$ is called full if the objects generate the derived category. Formally, this means the smallest triangulated subcategory of $D^b(X)$ containing the collection and closed under direct sums and direct summands is $D^b(X)$ itself.*

Equivalently, for any object $Y \in D^b(X)$, if $\mathrm{Hom}(E_i[m], Y) = 0$ for all $i = 1, 2, \ldots, n$ and all $m \in \mathbb{Z}$, then $Y \cong 0$.

Definition 3.4 (Strong Exceptional Collection)**.** *An exceptional collection $\{E_1, E_2, \ldots, E_n\}$ is called strong if:*

$$\mathrm{Hom}(E_i, E_j[m]) = 0 \quad \textit{for all } i, j \textit{ and all } m \neq 0$$

4 Semiorthogonal Decompositions

Definition 4.1 (Semiorthogonal Decomposition)**.** *A semiorthogonal decomposition of a triangulated category $\mathcal{T}$ is a sequence of full triangulated subcategories $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ such that:*

1. *For any objects $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ with $i > j$, we have $\mathrm{Hom}(A_i, A_j) = 0$.*
2. *For any object $T \in \mathcal{T}$, there exists a unique sequence of morphisms:*

$$0 = T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 = T$$

such that the cone of each morphism $T_i \to T_{i-1}$ lies in $\mathcal{A}_i$ for $i = 1, 2, \ldots, n$.

We denote this by $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \rangle$.

Proposition 4.2. *A full exceptional collection $\{E_1, E_2, \ldots, E_n\}$ in $D^b(X)$ gives rise to a semiorthogonal decomposition:*

$$D^b(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \rangle$$

where $\mathcal{A}_i$ is the triangulated subcategory generated by E_i.

5 The Splitting Problem and Beilinson's Exceptional Collection

5.1 The Splitting Problem

The splitting problem in algebraic geometry asks: When is a vector bundle on a variety isomorphic to a direct sum of line bundles?

Theorem 5.1 (Grothendieck)**.** *Every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles:*

$$\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}(a_i)$$

for some integers $a_1, a_2, \ldots, a_r$.

However, for projective spaces of higher dimension, the situation is different:

Theorem 5.2. *For $n \geq 2$, there exist vector bundles on $\mathbb{P}^n$ that do not split as direct sums of line bundles.*

Example 5.3. *The tangent bundle $T\mathbb{P}^n$ is a non-split vector bundle on $\mathbb{P}^n$ for $n \geq 2$. It fits into the Euler sequence:*

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T\mathbb{P}^n \to 0$$

This sequence is non-split, as it represents a non-zero element in the group $\mathrm{Ext}^1(T\mathbb{P}^n, \mathcal{O})$.

5.2 Beilinson's Exceptional Collection

Theorem 5.4 (Beilinson)**.** *The collection $\{\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots, \mathcal{O}(n)\}$ is a full exceptional collection in $D^b(\mathbb{P}^n)$.*

This has several important consequences:

1. Every coherent sheaf (or complex of coherent sheaves) on $\mathbb{P}^n$ can be reconstructed from its "cohomological information" with respect to this collection.

2. The Grothendieck group $K_0(\mathbb{P}^n)$ is a free abelian group of rank $n+1$ with basis given by the classes $[\mathcal{O}]$, $[\mathcal{O}(1)], \ldots, [\mathcal{O}(n)]$. This means that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, its class in $K_0(\mathbb{P}^n)$ can be written uniquely as a integer linear combination of these classes:

$$[\mathcal{F}] = a_0[\mathcal{O}] + a_1[\mathcal{O}(1)] + \cdots + a_n[\mathcal{O}(n)]$$

Remark 5.5. *The fact that any coherent sheaf can be written as a linear combination of $[\mathcal{O}(i)]$ in $K_0(\mathbb{P}^n)$ does not imply that every vector bundle on $\mathbb{P}^n$ splits as a direct sum of line bundles.*

In particular the tangent bundle $T\mathbb{P}^n$ has class:

$$[T\mathbb{P}^n] = (n+1)[\mathcal{O}(1)] - [\mathcal{O}]$$

in $K_0(\mathbb{P}^n)$, but this does not mean $T\mathbb{P}^n \cong \mathcal{O}(1)^{\oplus(n+1)} \oplus \mathcal{O}^{\oplus(-1)}$, which is not even meaningful for a negative exponent.

The obstruction to a vector bundle splitting is measured by extension groups Ext^1, which precisely capture the non-splitting of exact sequences.