

Equivariant Derived Categories of Coherent Sheaves

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Abstract

Notes for a talk on equivariant derived categories of coherent sheaves, given in the low dimensional gauge theory seminar at UC Berkeley.

Contents

1	Motivation	1
2	Refresher on GIT	2
3	The standard flop	3
4	Autoequivalences from VGIT	6
4.1	Definitions	6
4.2	Summary theorem	7
4.3	Application to VGIT	8

1 Motivation

Homological mirror symmetry predicts that in certain cases, derived categories of coherent sheaves on an algebraic variety should admit twist autoequivalences corresponding to a spherical object.

1. A-model: symplectic automorphisms and generalized Dehn twists along Lagrangian spheres
2. B-model: derived autoequivalences such as spherical twists

We want to consider natural constructions of twist autoequivalences of $D^b(X)$. Techniques come from variation of GIT (VGIT) and the general theory of spherical functors and semiorthogonal decompositions. For example, this formalism makes it clear the action of the braid group on the derived category of the variety.

A remark about the general strategy: Suppose X_+ and X_- are a pair of varieties related by a flop and that both arise as GIT quotients of a larger space V by a reductive group G .

$$X_{\pm} = [V^{ss}(\chi_{\pm})/G] \subset \mathcal{X} := [V/G]$$

so X_{\pm} are open substacks of \mathcal{X} . There are exact restriction functors

$$\iota_{\pm}^* : D^b(\mathcal{X}) \longrightarrow D^b(X_{\pm})$$

Try to construct an equivalence $D^b(X_+) \simeq D^b(X_-)$ by finding a single triangulated subcategory window $\mathcal{W} \subset D^b(\mathcal{X})$ which restricts isomorphically to both sides. the functors ι_{\pm}^* induce equivalences

$$\mathcal{W} \xrightarrow{\sim} D^b(X_+), \quad \mathcal{W} \xrightarrow{\sim} D^b(X_-),$$

and composing these gives the desired derived equivalence $D^b(X_+) \xrightarrow{\sim} D^b(X_-)$. Choosing different windows and composing the resulting equivalences gives autoequivalences of $D^b(X_{\pm})$.

2 Refresher on GIT

Consider a graded noetherian algebra over \mathbb{C} :

$$R = \bigoplus_{m=0}^{\infty} R_m$$

The variety $X = \text{Proj } R$ is projective over the affine variety $\text{Spec } R_0$, and comes equipped with an ample line bundle $\mathcal{L} = \mathcal{O}_X(1)$.

Let G be a reductive algebraic group acting on R by graded algebra automorphisms. Then G acts on X and \mathcal{L} is a G -linearized ample line bundle. We can form the GIT quotient

$$X//G := \text{Proj}(R^G)$$

The invariant algebra is finitely generated (Hilbert, Nagata).

Fix an ample line bundle L on X . The space of G -linearizations of L has a wall-chamber structure; crossing a wall changes the GIT quotient by a birational map.

Theorem 2.1. The real character space $X^*(G)_{\mathbb{R}}$ is cut into finitely many chambers by rational walls; for characters in the same chamber the GIT quotient is constant, and crossing a wall induces a birational modification of the quotient.

It is also true that there are only finitely many distinct GIT quotients $X//_L G$ up to isomorphism as L ranges over all G -ample linearizations.

There are rational maps for every r

$$X//G \dashrightarrow \mathbb{P}(H^0(X, \mathcal{L}^r)^G)^*$$

Definition 2.2. A point $x \in X$ is semistable if one of the above maps is defined at x for some $r > 0$. A point $x \in X$ is stable if the orbit $G \cdot x$ is closed in X^{ss} and the stabilizer of x in G is finite. A point $x \in X$ is unstable if it is not semistable.

Restriction gives an exact dg-functor $i^* : D^b(X/G) \rightarrow D^b(X^{ss}/G)$. The meat of what I want to talk about is the construction of a functorial splitting.

3 The standard flop

We do an example following Segal. Let $V = \mathbb{C}_{x_1, x_2, y_1, y_2}^4$ and \mathbb{C}^* act on V with weight $(1, 1, -1, -1)$.

There are two possible GIT quotients X_+ and X_- , depending on whether we choose a positive or negative character of \mathbb{C}^* . Both are isomorphic to the total space of the bundle $\mathcal{O}(-1)^{\oplus 2}$ over \mathbb{P}^1 .

Open substacks of the stack $\mathcal{X} = [V/\mathbb{C}^*]$ given by the semistable loci. Let $\iota_{\pm} : X_{\pm} \rightarrow \mathcal{X}$ be the open immersions. The restriction functors $\iota_{\pm}^* : D^b(\mathcal{X}) \rightarrow D^b(X_{\pm})$ are exact.

Let \mathcal{G}_t be the full subcategory of $D^b(\mathcal{X})$ generated by the line bundles $\mathcal{O}(t), \mathcal{O}(t+1)$ for some integer t .

Claim 3.1. For any $t \in \mathbb{Z}$, both ι_+^* and ι_-^* restrict to give equivalences

$$D^b(X_+) \xleftarrow{\sim} \mathcal{G}_t \xrightarrow{\sim} D^b(X_-).$$

for

Proof. Exactness, preserves shifts and cones, are clear. To check fully faithfulness, it is enough to show that $H_{\mathcal{X}}^{\bullet}(\mathcal{O}(i)) = H_{X_{\pm}}^{\bullet}(\mathcal{O}(i))$ for $i \in [-1, 1]$.

Left hand side: V affine, so $H^p(\mathcal{X}, \mathcal{O}(i)) = (\mathcal{O}_V)_i$ for $p = 0$ and 0 for $p > 0$.

Right hand side: We do the computation for X_+ . Let projection $\pi : X_+ \rightarrow \mathbb{P}^1$. Recall X_+ is the total space of the bundle $E = \mathcal{O}(-1)^{\oplus 2}$ over \mathbb{P}^1 . Then

Then

$$\pi_* \mathcal{O}_{X_+} \cong \text{Sym}^{\bullet}(E^{\vee}) = \text{Sym}^{\bullet}(\mathcal{O}(1)^{\oplus 2}) \cong \bigoplus_{m \geq 0} \text{Sym}^m(\mathcal{O}(1)^{\oplus 2}) \cong \bigoplus_{m \geq 0} \mathcal{O}(m)^{\oplus (m+1)}$$

has global sections $\bigoplus_{m \geq 0} \text{Span}\{x_1^a x_2^{m-a}\}^{\oplus (m+1)}$, one for each monomial in y_1, y_2 of degree m .

Write $\mathcal{O}_{X_+}(k) = i^* \mathcal{O}_{\mathcal{X}}(k)$. We can identify

$$i^* \mathcal{O}_V(k) \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(k) \cong \mathcal{O}_{X_+} \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(k).$$

By the projection formula and affineness of π

$$H^p(X^+, \mathcal{O}_{X^+}(k)) \cong H^p(\mathbb{P}^1, \pi_* \mathcal{O}_{X^+} \otimes \mathcal{O}(k)) \cong \bigoplus_{m \geq 0} H^p(\mathbb{P}^1, \mathcal{O}(k+m))^{\oplus(m+1)}$$

When $p = 0$, this has global sections

$$\mathrm{Sym}^{k+m}(\mathbb{C}_{x_1, x_2}^2) \otimes \mathrm{Sym}^m(\mathbb{C}_{y_1, y_2}^2)$$

which is exactly the degree k piece of \mathcal{O}_V . When $p > 0$, this is zero for $k = -1, 0, 1$.

For $p = 1$, recall $k \in -1, 0, 1$ and $m \geq 0$, so $k + m \geq -1$. Thus $H^1(\mathbb{P}^1, \mathcal{O}(k+m)) = 0$. This agrees with the left hand side. Note that if the size of the window is bigger, then we would pick up some H^1 terms.

To check essential surjectivity, note that general theory says on a quasi-projective variety, an ample line bundle and its twists generate the derived category. Thus it is enough to show that $\mathcal{O}(t), \mathcal{O}(t+1)$ generate all powers of $\mathcal{O}(1)$ on X_{\pm} . This follows from the exactness of

$$\pi^* : \mathrm{Coh}(\mathbb{P}^1) \rightarrow \mathrm{Coh}(X_{\pm})$$

and the fact that $D^b(\mathbb{P}^1)$ is generated by $\mathcal{O}_{\mathbb{P}^1}(t), \mathcal{O}_{\mathbb{P}^1}(t+1)$ by Beilinson's theorem. \square

So for any $t \in \mathbb{Z}$ we have a derived equivalence

$$\Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_-)$$

passing through \mathcal{G}_t . Composing these, we get auto-equivalences

$$\Phi_{t+1}^{-1} \Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_+).$$

To see what these do, we need to check them on the generating set of line-bundles $\{\mathcal{O}(t), \mathcal{O}(t+1)\}$.

Consider the Koszul resolution resolving the structure sheaf of the unstable locus $\{y_1 = y_2 = 0\}$:

$$0 \rightarrow \mathcal{O}_V(2) \xrightarrow{(y_2, -y_1)} \mathcal{O}_V(1)^{\oplus 2} \xrightarrow{(y_1, y_2)} \mathcal{O}_V \rightarrow \mathcal{O}_V/\{y_1 = y_2 = 0\} \rightarrow 0$$

Restricting the sequence to X^- , the resolution becomes exact at the end since the unstable locus $\{y_1 = y_2 = 0\}$ is removed in X^- . Thus on X^- we have a quasi-isomorphism:

$$\mathcal{O}_{X^-}(k) \simeq [\mathcal{O}_{X^-}(k+2) \xrightarrow{(y_2, -y_1)} \mathcal{O}_{X^-}(k+1)^{\oplus 2}]$$

Thus we compute:

$$\begin{aligned}
\Phi_{t+1}^{-1} \Phi_t(\mathcal{O}_{X_+}(t)) &\simeq \Phi_{t+1}^{-1}(\mathcal{O}_{X_-}(t)) \\
&\simeq \Phi_{t+1}^{-1}\left([\mathcal{O}_{X_-}(t+2) \xrightarrow{(y_2, -y_1)} \mathcal{O}_{X_-}(t+1)^{\oplus 2}]\right) \\
&\simeq [\mathcal{O}_{X_+}(t+2) \xrightarrow{(y_2, -y_1)} \mathcal{O}_{X_+}(t+1)^{\oplus 2}], \\
\Phi_{t+1}^{-1} \Phi_t(\mathcal{O}_{X_+}(t+1)) &\simeq \Phi_{t+1}^{-1}(\mathcal{O}_{X_-}(t+1)) \\
&\simeq \mathcal{O}_{X_+}(t+1).
\end{aligned}$$

This autoequivalence $\Phi_{t+1}^{-1} \Phi_t$ is an example of a spherical twist.

Definition 3.2. A **spherical twist** is an autoequivalence associated to any spherical object in the derived category, i.e. an object $S \in D^b(X)$ such that

$$\text{Ext}(S, S) = \mathbb{C} \oplus \mathbb{C}[-n]$$

for some n (i.e. the homology of the n -sphere). It sends any object \mathcal{E} to the cone on the evaluation map

$$\text{Cone}(\text{RHom}(S, \mathcal{E}) \otimes S \longrightarrow \mathcal{E})$$

The inverse twist sends \mathcal{E} to the cone on the dual evaluation map

$$\text{Cone}(\mathcal{E} \longrightarrow \text{RHom}(\mathcal{E}, S)^\vee \otimes S)$$

Claim 3.3. The object $\mathcal{O}_{\mathbb{P}^1_{x_1, x_2}}(t)$ is spherical for the derived category $D^b(X_+)$, and the inverse twist around it sends $\mathcal{O}(t+1)$ to itself and $\mathcal{O}(t)$ to the two-term complex

$$[\mathcal{O}(t+2) \xrightarrow{(-y_2, y_1)} \mathcal{O}(t+1)^{\oplus 2}],$$

which agrees with $\Phi_{t+1}^{-1} \Phi_t$.

Remark 3.4. Tedious but straightforward computation. I checked it was spherical using the following fact. For a regular embedding $i : \Sigma \hookrightarrow X_+$ of codimension 2 there is a well-known identity:

$$\text{Ext}_{X_+}^i(i_*F, i_*G) \cong \bigoplus_{p=0}^2 \text{Ext}_{\Sigma}^{i-p}(F, G \otimes \wedge^p N_{\Sigma/X_+}).$$

The normal bundle of a zero section in the total space of a vector bundle $E \rightarrow B$ is canonically identified with E itself.

4 Autoequivalences from VGIT

The above example by Segal formally introduced windows to the mathematics literature and showed that window shift equivalences are given by spherical functors for gauged LG models. Note it was done for linear action of \mathfrak{G}_m , and the window was identified in an ad-hoc way.

We outline a more general treatment by Halpern-Leistner and Shipman. The main contribution of their paper is showing that there is a functorial splitting of the restriction functor

$$i^* : D^b(X/G) \rightarrow D^b(X^{ss}/G)$$

and that the window categories arise naturally via the semiorthogonal decompositions.

4.1 Definitions

Definition 4.1. A **semiorthogonal decomposition** of a triangulated category \mathcal{D} is a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ of \mathcal{D} such that:

1. For all $1 \leq i < j \leq n$, we have

$$\mathrm{Hom}_{\mathcal{D}}(A_j, A_i) = 0 \quad \text{for all } A_i \in \mathcal{A}_i, A_j \in \mathcal{A}_j.$$

2. The smallest triangulated subcategory of \mathcal{D} containing $\mathcal{A}_1, \dots, \mathcal{A}_n$ coincides with \mathcal{D} . This is equivalent (under the orthogonality hypothesis) to the condition that for every object $D \in \mathcal{D}$, there exists a sequence of morphisms

$$0 = D_n \rightarrow D_{n-1} \rightarrow \dots \rightarrow D_1 \rightarrow D_0 = D$$

such that the cone of the morphism $D_i \rightarrow D_{i-1}$ is an object of \mathcal{A}_i for each $1 \leq i \leq n$.

We denote such a semiorthogonal decomposition by

$$\mathcal{D} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle.$$

Definition 4.2. An object E is **exceptional** if

$$\mathrm{Hom}(E, E) = k \quad \text{and} \quad \mathrm{Hom}(E, E[t]) = 0 \text{ for } t \neq 0.$$

An exceptional collection is a collection of exceptional objects E_1, E_2, \dots, E_m such that

$$\mathrm{Hom}(E_i, E_j[t]) = 0 \quad \text{for all } i > j \text{ and all } t \in \mathbb{Z}.$$

Definition 4.3. Let (E, F) be an exceptional pair in a triangulated category \mathcal{D} . The **left mutation** of F through E is the object $L_E F$ defined by the distinguished triangle

$$L_E F \rightarrow \mathrm{Hom}^\bullet(E, F) \otimes E \xrightarrow{\mathrm{ev}} F \rightarrow L_E F[1],$$

where ev is the evaluation map. Similarly, the **right mutation** of E through F is the object $R_F E$ defined by the distinguished triangle

$$R_F E[-1] \rightarrow E \xrightarrow{\mathrm{coev}} \mathrm{Hom}^\bullet(E, F)^* \otimes F \rightarrow R_F E,$$

where coev is the coevaluation map.

A mutation of an exceptional collection $\sigma = (E_0, \dots, E_n)$ is defined by applying left or right mutations to adjacent pairs of objects in the collection.

$$R_i \sigma = (E_0, \dots, E_{i-1}, E_{i+1}, R_{E_{i+1}} E_i, E_{i+2}, \dots, E_n),$$

$$L_i \sigma = (E_0, \dots, E_{i-1}, L_{E_i} E_{i+1}, E_i, E_{i+2}, \dots, E_n).$$

Theorem 4.4 (Properties of mutations).

1. The mutation of an exceptional collection is again an exceptional collection generating the same subcategory.
2. For an adjacent pair (E_i, E_{i+1}) the left and right mutation functors L_i and R_i are inverse to each other on the subcategory generated by that pair.
3. The mutations satisfy the braid relations

Consequently the operators R_i (resp. L_i) induce an action of the braid group on the set of exceptional collections.

4.2 Summary theorem

Recall that if B is an object in a dg-category, then we can define the twist functor

$$T_B : \mathcal{C} \longrightarrow \mathcal{C}$$

$$T_B(A) := \mathrm{Cone}(\mathrm{RHom}_{\mathcal{C}}(B, A) \otimes B \longrightarrow A)$$

If B is a spherical object, then T_B is by definition the spherical twist autoequivalence defined by B . If B were instead an exceptional object, then T_B is the formula for the left mutation equivalence ${}^\perp B \rightarrow B^\perp$ coming from a pair of semiorthogonal decompositions $\langle B^\perp, B \rangle = \langle B, {}^\perp B \rangle$.

If \mathcal{C} is a pretriangulated dg-category, then the braid group on n strands acts by left and right mutation on the set of length- n semiorthogonal decompositions

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle,$$

with each \mathcal{A}_i admissible.

The left and right mutation functors satisfy the braid relations, for example

$$R_i R_{i+1} R_i \cong R_{i+1} R_i R_{i+1},$$

so the assignment of mutations defines a genuine braid group action on the collection of admissible decompositions.

I don't have time to define spherical functors, but they generalize spherical objects. In particular, a spherical object S is a spherical functor with source $D^b(\text{pt})$.

Theorem 4.5 (HL-S16). If \mathcal{C} is a pretriangulated dg category and

$$\mathcal{C} = \langle \mathcal{A}, \mathcal{G} \rangle$$

fixed by the braid action acting by mutation, then the autoequivalence of \mathcal{G} induced by mutation is the twist T_S corresponding to a spherical functor $S : \mathcal{A} \rightarrow \mathcal{G}$.

Conversely, if $S : \mathcal{A} \rightarrow \mathcal{B}$ is a spherical functor, then there is a larger category \mathcal{C} admitting a semiorthogonal decomposition fixed by this braid which recovers S and T_S .

4.3 Application to VGIT

X smooth projective over affine, G reductive acting on X , \mathcal{L} a G -linearized ample line bundle on X .

Pick a W invariant inner product on the cocharacter lattice of G .

Definition 4.6 (Hilbert–Mumford weight). For a point $x \in X$ and a one-parameter subgroup $\lambda : \mathfrak{G}_m \rightarrow G$ such that the limit

$$x_0 := \lim_{t \rightarrow 0} \lambda(t) \cdot x$$

exists in X , the weight of the action of λ on the fiber \mathcal{L}_{x_0} is called the **Hilbert–Mumford weight** of x with respect to λ and denoted $\mu^{\mathcal{L}}(x, \lambda)$.

The Hilbert–Mumford numerical criterion states that x is L -semistable if and only if $\mu^L(x, \lambda) \leq 0$ for every 1-PS λ of G . Thus if x is unstable, there exists some λ with $\mu^L(x, \lambda) > 0$.

For an unstable point x consider the *normalized instability*

$$M(x) := \sup_{\lambda \neq 0} \frac{\mu^L(x, \lambda)}{\|\lambda\|}.$$

Theorem 4.7 (Kempf, Kempf–Ness). For every unstable point $x \in X^{us}$, the supremum $M(x)$ is a maximum, attained by some 1-PS λ_x . The 1-PS λ_x is unique up to conjugation by the associated parabolic subgroup

$$P(\lambda_x) := \{ g \in G \mid \lim_{t \rightarrow 0} \lambda_x(t) g \lambda_x(t)^{-1} \text{ exists in } G \}$$

and up to positive rescaling of λ_x .

Thus each unstable point x determines a distinguished "optimal" 1-PS λ_x up to this equivalence. Let β run over the set of such equivalence classes of optimal 1-PS's. Define

$$S_\beta := \{ x \in X^{us} \mid \text{the optimal 1-PS for } x \text{ is of type } \beta \}.$$

Theorem 4.8 (Kirwan–Ness stratification). Each S_β is a locally closed G -invariant subvariety of X , and the unstable locus decomposes as a disjoint union

$$X^{us} = \bigsqcup_{\beta} S_\beta.$$

Moreover, one can order the indices so that

$$\overline{S_i} \subset \bigcup_{j \geq i} S_j,$$

The subvarieties S_i are called the **Kirwan–Ness strata**. They provide a canonical G -equivariant stratification of the unstable locus X^{us} .

Each stratum comes with a distinguished one-parameter subgroup $\lambda_i : \mathbb{C}^* \rightarrow G$ and S_i fits into the diagram

$$Z_i \begin{array}{c} \xrightarrow{\sigma_i} \\ \xleftarrow{\pi_i} \end{array} Y_i \subset S_i := G \cdot Y_i \xrightarrow{j_i} X \quad (1)$$

where Z_i is an open subvariety of $X^{\lambda_i\text{-fixed}}$, characterized by

$$Z = \{ z \in X^\lambda \mid \text{KN type of } z \text{ is } [\lambda], \text{ and } z \notin \overline{S'} \text{ for any more unstable } S' \}.$$

and

$$Y_i = \left\{ x \in X - \bigcup_{j > i} S_j \mid \lim_{t \rightarrow 0} \lambda_i(t) \cdot x \in Z_i \right\}.$$

The maps σ_i and j_i are the inclusions and π_i is taking the limit under the flow of λ_i as $t \rightarrow 0$. We denote the immersion $Z_i \rightarrow X$ by σ_i as well.

We are now ready to relate the equivariant derived category of X to that of the GIT quotient.

Theorem 4.9 (HL15). Let η_i be the weight of $\det(N_{S_i}^\vee X)|_{Z_i}$ with respect to λ_i . Choose an integer w_i for each stratum and define the full subcategory

$$\mathcal{G}_w := \{F^\bullet \in D^b(X/G) \mid \forall i, \sigma_i^* F^\bullet \text{ has weights in } [w_i, w_i + \eta_i) \text{ w.r.t. } \lambda_i\}.$$

Then the restriction functor

$$r : \mathcal{G}_w \longrightarrow D^b(X^{ss}/G)$$

is an equivalence of dg-categories.

Now we introduced balanced GIT wall crossings. Let L_0 be a G -ample line bundle such that $X^{ss} - X^s$ is nonempty, and let L' be another G -equivariant line bundle. We assume that $X^{ss} = X^s$ for the linearizations $L_\pm = L_0 \pm \epsilon L'$ for sufficiently small ϵ , and we denote $X_\pm^{ss} = X^{ss}(L_\pm)$.

In this case, $X^{ss}(L_0) - X^{ss}(L_\pm)$ is a union of KN strata for L_\pm .

Definition 4.10. The wall crossing is *balanced* if the strata S_i^+ and S_i^- lying in $X^{ss}(L_0)$ are indexed by the same set, with $Z_i^+ = Z_i^-$ and $\lambda_i^+ = (\lambda_i^-)^{-1}$.

If G is abelian and there is some linearization with a stable point, then all codimension one wall crossings are balanced.

In this situation, HL15 implies that the restriction functors

$$r_\pm : \mathcal{G}_w \longrightarrow D^b(X_\pm^{ss}/G)$$

are equivalences. In particular we get a family of derived equivalences

$$\Phi_w := r_- \circ r_+^{-1} : D^b(X_+^{ss}/G) \longrightarrow D^b(X_-^{ss}/G)$$

and the dependence on the choice of integer w gives a family of autoequivalences

$$\Phi_{w+1}^{-1} \circ \Phi_w : D^b(X_+^{ss}/G) \longrightarrow D^b(X_+^{ss}/G).$$

Finally, Halpern-Leistner shows that the window shift autoequivalence is a twist corresponding to a spherical functor.

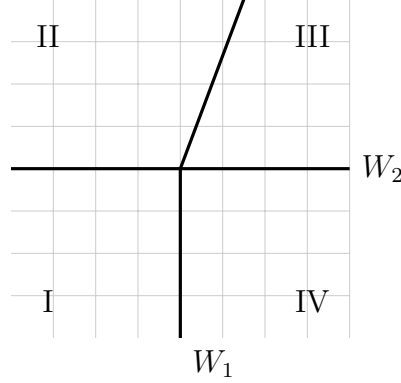
By describing window shifts both in terms of mutations and as spherical twists, we see why these two operations have the “same formula” in this setting.

Example 4.11 (Window shifts on a K3 surface). Following [?, Ex. 4.19], let

$$X \subset \mathbb{P}_x^2 \times \mathbb{P}_y^2$$

be a K3 surface cut out by a divisor of bidegree $(2, 0)$ and a divisor of bidegree $(1, 3)$. Line bundles on a K3 surface are spherical objects, so any autoequivalence which can be written in terms of such line bundles is automatically a composition of spherical twists.

The example is realized inside a VGIT picture as follows.



- Let

$$\mathcal{V} = \mathcal{O}_{\mathbb{P}_x^2 \times \mathbb{P}_y^2}(-2, 0) \oplus \mathcal{O}_{\mathbb{P}_x^2 \times \mathbb{P}_y^2}(-1, -3),$$

and consider $\text{tot}(\mathcal{V})$ as a toric variety given as a GIT quotient of \mathbb{A}^8 by a torus $T \cong (\mathbb{C}^*)^2$ with weight matrix $(t, s) \mapsto (t, t, t, s, s, s, t^{-2}, t^{-1}s^{-3})$.

- For each wall W_i there is a Kirwan-Ness stratification near W_i (Table 1 in [?]). The least unstable stratum has fixed locus Z_i and Levi quotient L_i , so that the local GIT quotient for the wall is Z_i/L_i .
- One introduces a Landau–Ginzburg potential

$$W = pf + qg \in \mathbb{C}[x_i, y_j, p, q]_{\deg=2},$$

where f has bidegree $(2, 0)$ and g has bidegree $(1, 3)$, and f cuts out a smooth rational curve on \mathbb{P}_x^2 . The LG pair (\mathcal{V}, W) carries a second \mathbb{C}^* -grading (the LG grading, "R-charge") so that the variables p, q have weight 2 and the x_i, y_j have weight 0, so that W has weight 2; the associated category $D^b(\mathcal{V}, W)$ is equivalent to $D^b(X)$.

The key point is that, although Z_i/L_i is *non-compact* as a usual GIT quotient, the restriction $W|_{Z_i}$ makes the LG quotient $(Z_i/L_i, W|_{Z_i})$ effectively compact. Concretely:

- Near W_1 one has $Z_1/L_1 \cong \mathbb{P}^2/\mathbb{C}^*$; the LG category $D^b(Z_1/L_1, W|_{Z_1})$ is equivalent to $D^b(\mathbb{P}^2)$, which admits the full exceptional collection $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$.
- Near W_2 one has $Z_2/L_2 \cong \text{tot } \mathcal{O}_{\mathbb{P}^2}(-2)/\mathbb{C}^*$; the potential restricts to pf . Then $D^b(Z_2/L_2, W|_{Z_2}) \simeq D^b(C)$, and for each window this is equivalent to $D^b(\mathbb{P}^1)$ with its exceptional pair $\langle \mathcal{O}, \mathcal{O}(1) \rangle$.

- One finds:

- the window shift across W_1 is a composition of spherical twists around the line bundles

$$\mathcal{O}_X(0, i), \mathcal{O}_X(0, i + 1), \mathcal{O}_X(0, i + 2),$$

- the window shift across W_2 is a composition of spherical twists around

$$\mathcal{O}_X(i, 0), \mathcal{O}_X(i + 1, 0),$$

for suitable integers i (depending on the chosen windows).

Thus, in this example the abstract window-shift autoequivalences arising from VGIT of (\mathcal{V}, W) are identified explicitly with compositions of spherical twists by line bundles on the K3 surface X .