Springer theory

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Abstract

Notes on Chapter 3 of Chriss-Ginzburg's book.

Contents

1 Introduction

Springer theory is about geometric constructions of representations of the Weyl group. In particular we aim to define a W-action on $H_*(B_x)$, where B_x is the Springer fiber $\mu^{-1}(x)$ for each $x \in \mathfrak{g}$.

2 The actors

Claim 2.1. *Recall that we say a subalgebra of a semisimple Lie algebra $\mathfrak{g}$ is solvable if its derived series terminates at 0. This means that*

$$\mathfrak{b} = \mathfrak{b}_0 \supset \mathfrak{b}_1 \supset \cdots \supset \mathfrak{b}_n = 0$$

where $\mathfrak{b}_{i+1} = [\mathfrak{b}_i, \mathfrak{b}_i]$. A Borel subalgebra $\mathfrak{b}$ is a maximal solvable subalgebra.

Then the key fact is when G is connective reductive (so that $\mathfrak{g}$ is semisimple), the flag manifold G/B also parametrizes the set of Borel subalgebras of $\mathfrak{g}$. G acts on the set of Borel subalgebras by conjugation (the Adjoint Ad action). This action is transitive and for a fixed Borel subground B_0 with Lie algebra $\mathfrak{b}_0$, the stabilizer of $\mathfrak{b}_0$ is preicsely B_0.

Definition 2.2. *Let $\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times G/B \mid x \in \mathfrak{b}\}$ and write $\pi : \tilde{\mathfrak{g}} \to G/B$ and $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ for the projections.*

The projection π makes $\tilde{\mathfrak{g}}$ a G-equivariant vector bundle over G/B with fiber $\mathfrak{b}$. The other projection μ is more complicated.

Recall that an element $x \in \mathfrak{g}$ is *nilpotent* if $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. The set of nilpotent elements in $\mathfrak{g}$ is denoted by $\mathcal{N}$ and is called the *nilpotent cone*. In particular it is a closed $\operatorname{Ad} G$-invariant subvariety of $\mathfrak{g}$ and is closed under dilation by $\mathbb{C}^\times$.

Denote by

$$\tilde{\mathcal{N}} = \mu^{-1}(\mathcal{N}) = \{(x, \mathfrak{b}) \in \mathcal{N} \times G/B \mid x \in \mathfrak{b}\}$$

Fix a Borel subalgebra $\mathfrak{b}_0$ and consider the fiber of the projection onto the second factor. These are the nilpotent elements of $\mathfrak{b}_0$. But it is clear that the operator $\operatorname{ad} x$ is nilpotent if and only if x has no Cartan component in the decomposition $\mathfrak{b}_0 = \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{n} := [\mathfrak{b}_0, \mathfrak{b}_0]$ is the nilradical of $\mathfrak{b}_0$. It follows that the projection $\tilde{\mathcal{N}} \to G/B$ is a vector bundle with fiber $\mathfrak{n}$. Moreover the projection makes $\tilde{\mathcal{N}}$ a G-equivariant vector bundle over G/B.

$$\tilde{\mathcal{N}} \cong G \times_B \mathfrak{n}$$

In particular $\tilde{\mathcal{N}}$ is a smooth variety, whereas $\mathcal{N}$ is singular.

Claim 2.3. *There is a natural G-equivariant isomorphism $\tilde{\mathcal{N}} \cong T^*G/B$.*

Proof. Recall that we can identify the cotangent space at the point B with $(\mathfrak{g}/\mathfrak{b})^* = \mathfrak{b}^\perp$. Therefore we have a natural isomorphism $T^*G/B \cong G \times_B \mathfrak{b}^\perp$.

Using the Killing form, we get an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ under which the annihilator $\mathfrak{b}^\perp$ gets idenitified with the annihilator of $\mathfrak{b}$ in $\mathfrak{g}$ with respect to the invariant form. The latter is equal to $\mathfrak{n}$, the nilradical of $\mathfrak{b}$.

We have previously identified $\tilde{\mathcal{N}}$ with $G \times_B \mathfrak{n}$. □

Proposition 2.4. *The projection $\mu : \tilde{\mathcal{N}} = T^*G/B \to \mathcal{N}$ is the moment map for the Hamiltonian G-action on T^*G/B arising from the G-action on G/B. Moreover μ is surjective.*

This map is known as the *Springer resolution* and is indeed a resolution of singularities.