Homework 2

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Problem 1 (from RS1) Δ is the unit disk, $\Delta^\times = \Delta \setminus \{0\}$.

1. Prove that a holomorphic map $f : \Delta^\times \to \mathbb{C}$ which has an *essential* (non-pole) singularity at 0 has dense image in $\mathbb{C}$.
2. Use this to show that any map $f : \Delta^\times \to \mathbb{P}$ which is never more than N-to-1, for a fixed number N, extends holomorphically to Δ.
3. Generalize (b) to the case when the target is an arbitrary compact Riemann surface R, by invoking Riemann's theorem which guarantees the existence of meromorphic functions on R.

Remark. A much stronger (and more difficult) version of (a) says that f assumes every value infinitely often, possibly with a single exception (such as 0, for $e^{1/z}$). This is the Great Picard Theorem.

Solution:

1. Let $f : \Delta^\times \to \mathbb{C}$ have an essential (non-pole) singularity at 0. If the image is not dense, there is a disc $D(a, r) \subset \mathbb{C}$ that f misses near 0. Then $g(z) = \frac{1}{f(z)-a}$ is holomorphic and $|g(z)| \leq r^{-1}$ near 0, hence extends holomorphically to 0 (Riemann's removable singularity theorem). If $g(0) \neq 0$, then $f = a + 1/g$ extends holomorphically across 0 (removable singularity). If $g(0) = 0$, then $1/g$ has a pole at 0, so f has a pole. Either way, the singularity at 0 is not essential. Contradiction. Hence the image of every punctured neighborhood is dense in $\mathbb{C}$.

2. Assume toward a contradiction that 0 is an essential singularity. Work in the affine chart $\mathbb{C} \subset \mathbb{P}^1$, and fix a regular value $a \in \mathbb{C}$ of f (possible since the critical values are discrete). Set $g(z) := f(z) - a$.

   For $r > 0$ small with g having no zeros on $|z| = r$, define the index
   $$n(r) := \frac{1}{2\pi i} \int_{|z|=r} \frac{g'(z)}{g(z)}\, dz$$
   which equals the number of solutions of $g(z) = 0$ in $|z| < r$, counted with multiplicity (by the argument principle).

   Lemma For every $M \in \mathbb{N}$ there exists $r_M > 0$ such that $n(r_M) \geq M$.

   Because 0 is essential, Casorati-Weierstrass gives: for every $\varepsilon \in (0, 1)$ and every $r_0 > 0$ there exists $0 < r < r_0$ with $\min_{|z|=r} |g(z)| < \varepsilon$ and $\max_{|z|=r} |g(z)| > \varepsilon^{-1}$. (If not, then on all small circles $|g|$ stays in a compact annulus, and a standard maximum-minimum argument would force g to be bounded away from 0 near 0, making $1/g$ holomorphic there—contradicting that 0 is essential for g.)

Fix $\varepsilon \in (0,1)$ so small that the circle $\{|w| = \varepsilon\}$ contains no critical values of the map g from $|z| = r$ (this is possible by discreteness). Using (*) with that ε, choose r so that along the circle $|z| = r$ the continuous curve $w(t) := g(re^{it})$ intersects $|w| = \varepsilon$ transversely many times and also intersects $|w| = \varepsilon^{-1}$. By continuity, we can arrange $2M$ alternating crossings of $|w| = \varepsilon$ as t runs from 0 to 2π (inside/outside alternate because $|g|$ attains both $< \varepsilon$ and $> \varepsilon^{-1}$ values on the same circle).

Each such alternating pair forces the argument of $w(t)$ to increase by at least 2π around the origin (the curve must go from inside to outside and back, swinging around 0 once; regularity of the crossings and the fact a is a regular value ensure positive orientation). Hence the total change of $\arg g(re^{it})$ over $t \in [0, 2\pi]$ is at least $2\pi M$. Therefore the winding number of $g(|z| = r)$ about 0 is $\geq M$, i.e. $n(r) \geq M$. $\square$

With the Lemma, fix $M := N + 1$. Choose r with $n(r) \geq M$. Then $g(z) = 0$ has at least $M = N + 1$ solutions in $|z| < r$. That is, the single value a has at least $N + 1$ preimages in $\Delta^\times$, contradicting that f is never more than N-to-1.

Thus 0 cannot be essential. The remaining possibilities for a holomorphic map to $\mathbb{P}^1$ are: removable singularity or pole; in either case f extends holomorphically across 0.

3. Let $g : R \to \mathbb{P}^1$ be a nonconstant meromorphic function on the compact Riemann surface R. Let $f : \Delta^\times \to R$ be a holomorphic map which is never more than N-to-1. Then $h := g \circ f : \Delta^\times \to \mathbb{P}^1$ is also never more than Nd-to-1, where d is the degree of g. By (b), h extends holomorphically to Δ.

Problem 2 Identify successive pairs of edges of a $2n$-gon, labelled $a, a, b, b, c, c, \ldots$, by matching points on matching edge pairs in *parametric order.* (Equivalently, identify the points θ and $\theta + \pi/n$ on the boundary of the unit disk.)

Explain why the surface obtained is homeomorphic to the one obtained by sewing on n Möbius strips to an n-holed sphere, along matching boundaries.

Which of these gives a Klein bottle?

Solution: The $2n$-gon with edges $aa\,bb\,cc\cdots$ gives $\#^n\mathbb{RP}^2$. Each $\mathbb{RP}^2$ is "sphere with 1 hole + Möbius band." Taking the connected sum of n such surfaces glues the sphere pieces into a sphere with n holes, and the Möbius bands remain attached.

The case $n = 2$ gives a Klein bottle. The polygon for $\mathbb{RP}^2\#\mathbb{RP}^2$ has sides $aabb$. The polygon for the Klein bottle has sides $aba^{-1}b$. We want to show they represent the same surface. By cutting and re-gluing along the diagonal, we can transform the $aabb$ polygon into the $aba^{-1}b$ polygon, showing they are homeomorphic.

Problem 3 (from RS2) Show that any degree 2 holomorphic map $f : \mathbb{C}/L \to \mathbb{P}$ is a "Möbius transform of a shifted $\wp$–function":

$$f(u) = \frac{a\wp(u-w)+b}{c\wp(u-w)+d}, \qquad a,b,c,d,w \in \mathbb{C}.$$

Comment. You may assume standard facts about Möbius transformations.

Solution: Because $\deg f = 2$, for a generic value $y \in \mathbb{P}^1$ the fiber $f^{-1}(y) = \{u_1, u_2\}$. Define $\tau(u_1) = u_2$ and $\tau(u_2) = u_1$. Standard covering theory shows: $\tau : E \to E$ is a holomorphic involution ($\tau^2 = \mathrm{id}$), $f \circ \tau = f$, and the branch points are the fixed points of τ (there are 4 of them).

Lift τ to $\widetilde{\tau} : \mathbb{C} \to \mathbb{C}$ with $\widetilde{\tau}(z+L) \equiv \tau(z)+L$. Any holomorphic self-map of $\mathbb{C}$ that descends to the torus has the form $\widetilde{\tau}(z) = az+b$, where $aL \subseteq L$, $|a| = 1$. Since $\tau^2 = \mathrm{id}$, we have $a^2 = 1 \Rightarrow a = \pm 1$. A degree-2 branched covering must have fixed points, forcing $a = -1$. Hence $\widetilde{\tau}(z) = -z+t$ with $2t \in L$. Passing to E, τ is the map $u \mapsto -u+w$ where $2w \equiv 0$ in E.

Now translate the torus by w: define $T_w(u) = u - w$ and replace f by $g := f \circ T_w$. Then the deck involution becomes $u \mapsto -u$, so g is even: $g(u) = g(-u)$.

Let $\wp$ be the Weierstrass $\wp$-function for L. It is even, has a double pole at 0, and no other poles in a period parallelogram. Every even elliptic function h is a rational function of $\wp$: $h(u) = R(\wp(u))$ for some rational $R \in \mathbb{C}(x)$. This is because the poles of an even elliptic function occur in $\{\pm a_j\}$ with even principal parts. Subtract a polynomial $P(\wp)$ that matches all principal parts at $\pm a_j$; the difference is then an even elliptic function with no poles, hence constant. So $h = P(\wp) + \text{const} = R(\wp)$. Thus, for our g there is $R \in \mathbb{C}(x)$ with $g(u) = R(\wp(u))$.

The map $\wp : E \to \mathbb{P}^1$ has degree 2 so $\deg(g) = \deg(R \circ \wp) = \deg(R) \cdot \deg(\wp) = \deg(R) \cdot 2$. But $\deg(g) = \deg(f) = 2$. Therefore $\deg(R) = 1$ and so R is a Möbius transform:

$$R(x) = \frac{ax+b}{cx+d}$$

with $ad-bc \neq 0$. Undoing the translation T_w, we get $f(u) = \frac{a\wp(u-w)+b}{c\wp(u-w)+d}$ where $a,b,c,d,w \in \mathbb{C}$, $ad-bc \neq 0$.

Problem 4 (from RS2) Prove that any two meromorphic functions f, g on a compact Riemann surface are *algebraically related*: $P(f,g) \equiv 0$ for some 2-variable polynomial P.

Hint. Recall that a meromorphic function without poles must be constant, and estimate, in terms of N, the dimension of the vector space spanned by the functions $f^m g^n$, for $0 \le m, n \le N$, to conclude that a linear dependence relation must hold for large N.

Solution: Let the pole divisors of f and g be

$$(f)_\infty = \sum_{i=1}^{r} a_i p_i, \qquad (g)_\infty = \sum_{i=1}^{r} b_i p_i,$$

where $a_i, b_i \ge 0$ and the p_i are distinct points of X (allowing some a_i or b_i to be 0 if only the other function has a pole there). Set $A = \sum_i a_i$ and $B = \sum_i b_i$. If $A = 0$ or $B = 0$, the corresponding function is holomorphic on X and hence constant, so the conclusion is trivial. Thus assume $A, B > 0$.

For $m, n \ge 0$ put $h_{m,n} := f^m g^n$. Then $h_{m,n}$ has poles only at the p_i, with

$$\mathrm{ord}_{p_i}(h_{m,n}) \ge -(m a_i + n b_i), \qquad \deg(h_{m,n})_\infty = \sum_i \max\{0, -\mathrm{ord}_{p_i}(h_{m,n})\} \le mA + nB.$$

Fix $N \in \mathbb{N}$ and consider the vector space

$$V_N := \mathrm{span}_{\mathbb{C}}\{\, h_{m,n} : 0 \le m, n \le N \,\}.$$

All functions in V_N lie in the space

$$L(D_N), \qquad D_N := N \sum_{i=1}^{r} (a_i + b_i)\, p_i,$$

i.e. meromorphic functions with poles only at the p_i and of order at most $N(a_i + b_i)$ at p_i.

Lemma $\dim L(D_N) \le 1 + \deg D_N = 1 + N(A + B)$.

To see this, note that the principal parts up to order k_i at each p_i; these give at most $\sum_i k_i$ linear parameters, and adding a constant gives $+1$.

But there are $(N+1)^2$ monomials $h_{m,n}$ with $0 \le m, n \le N$. For N large we have $(N+1)^2 > 1 + N(A+B)$, hence the family $\{h_{m,n}\}_{0 \le m,n \le N}$ is linearly dependent: there exist coefficients $c_{m,n}$, not all zero, such that

$$\sum_{m,n=0}^{N} c_{m,n}\, f^m g^n \equiv 0 \quad \text{on } X.$$

Problem 5

1. Specializing the period lattice to the limiting case $\omega_1 = \pi$, $\omega_2 \to i \cdot \infty$, show that

$$\wp(u) \to \cot^2(u) + \tfrac{2}{3}, \qquad \zeta(u) \to \cot(u) + u, \qquad \sigma(u) \to \sin(u) \cdot \exp(u^2/2).$$

2. Do the series expansions apply?

3. Find and check the differential equation expressing $(\wp')^2$ in terms of $\wp$ in this limit.

4. Describe the (singular) analytic set in $\mathbb{C}^2$ parametrized as $z = \wp(u), w = \wp'(u)$.

Solution:

1. Recall that we defined the Weierstrass functions ζ function and σ function by

$$\wp(u) = -\zeta'(u), \quad \zeta(u) = -\zeta(-u)$$
$$\sigma(u) = \exp\left(\int_{u_0}^{u} \zeta(t)\, dt\right), \quad \sigma'(0) = 1$$

Let the lattice be $L = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ with $\omega_1 = \pi$ and $\omega_2 = iT$ where $T \to \infty$. A generic lattice point is $\omega_{m,n} = m\pi + inT$.

Fix a compact set $K \subset \mathbb{C}$ and write $R = \sup_{u\in K} |u|$. For $n \neq 0$ and T large, $|\omega_{m,n}| \geq |n|T - |m|\pi$, so in particular $|\omega_{m,n}| \geq \frac{1}{2}|n|T$ and also $|u - \omega_{m,n}| \geq \frac{1}{2}|\omega_{m,n}|$ (since $|u| \leq R$ is bounded while $|\omega_{m,n}| \to \infty$ with T). Consider one summand in the partial–fraction expansion:

$$S_{m,n}(u) := \frac{1}{(u - \omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2}$$

We have the identity:

$$\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} = \frac{(2\omega u - u^2)}{(u-\omega)^2\, \omega^2}$$

Hence, for $u \in K$:

$$|S_{m,n}(u)| \leq \frac{2|\omega_{m,n}|\,|u| + |u|^2}{|u - \omega_{m,n}|^2\, |\omega_{m,n}|^2} \leq \frac{2R|\omega_{m,n}| + R^2}{(\frac{1}{2}|\omega_{m,n}|)^2\, |\omega_{m,n}|^2} \leq \frac{C_R}{|\omega_{m,n}|^3}$$

for a constant C_R depending only on R.

Therefore:

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ n\neq 0}} |S_{m,n}(u)| \leq C_R \sum_{n\neq 0} \sum_{m\in\mathbb{Z}} \frac{1}{|m\pi + inT|^3} = C_R \sum_{n\neq 0}\sum_{m\in\mathbb{Z}} \frac{1}{\big((m\pi)^2 + (nT)^2\big)^{3/2}}$$

For fixed $n \neq 0$, the inner sum over m is $O\big((nT)^{-2}\big)$ (compare with $\int_{\mathbb{R}} \frac{dx}{(x^2+(nT)^2)^{3/2}} = \frac{2}{(nT)^2}$). Thus:

$$\sum_{m\in\mathbb{Z}} \frac{1}{\big((m\pi)^2 + (nT)^2\big)^{3/2}} \leq \frac{C}{(nT)^2}$$

with C independent of n, T. Summing over $n \neq 0$ gives:

$$\sum_{n\neq 0}\sum_{m\in\mathbb{Z}} \frac{1}{\big((m\pi)^2+(nT)^2\big)^{3/2}} \leq \frac{C}{T^2}\sum_{n\neq 0}\frac{1}{n^2} = \frac{C'}{T^2} \xrightarrow[T\to\infty]{} 0$$

This convergence is uniform in $u \in K$ because our bound does not depend on u beyond R. Hence the total contribution to $\wp(u)$ from all terms with $n \neq 0$ tends to 0 uniformly on compact sets. The only nonvanishing terms in the partial–fraction sum are those with $n = 0$, i.e. $\omega = m\pi$ with $m \in \mathbb{Z} \setminus \{0\}$.

Hence

$$\wp(u) \longrightarrow \frac{1}{u^2} + \sum_{m\neq 0}\Big(\frac{1}{(u-m\pi)^2} - \frac{1}{(m\pi)^2}\Big)$$

Recall the classical partial fractions $\csc^2 u = \frac{1}{u^2} + \sum_{m\neq 0}\frac{1}{(u-m\pi)^2}$, $\sum_{m\neq 0}\frac{1}{(m\pi)^2} = \frac{1}{3}$, so

$$\wp(u) \longrightarrow \csc^2 u - \frac{1}{3} = \cot^2 u + \frac{2}{3}$$

Recall Weierstrass's product for σ (for the lattice $L = \langle 2\omega_1, 2\omega_2\rangle$):

$$\sigma(u) = u \prod_{\omega\in L\setminus\{0\}} \Big(1 - \frac{u}{\omega}\Big) \exp\Big(\frac{u}{\omega} + \frac{u^2}{2\omega^2}\Big).$$

Now take the trigonometric degeneration $\omega_1 = \pi$ fixed and $\omega_2 \to i\infty$. All lattice points with nonzero vertical component ($n \neq 0$) go off to infinity and their factors tend to 1. What's left is the product over the horizontal periods $\omega = m\pi$, $m \in \mathbb{Z} \setminus \{0\}$. Thus

$$\sigma(u) \;\longrightarrow\; u \prod_{m\neq 0}\Big(1 - \frac{u}{m\pi}\Big) \exp\Big(\frac{u}{m\pi} + \frac{u^2}{2m^2\pi^2}\Big).$$

Pair the terms for m and $-m$. Using the standard product for $\sin u$,

$$\sin u = u \prod_{m=1}^{\infty}\Big(1 - \frac{u^2}{m^2\pi^2}\Big),$$

and the elementary identity

$$\prod_{m=1}^{\infty} \exp\Big(\frac{u^2}{m^2\pi^2}\Big) = \exp\Big(\frac{u^2}{2}\Big) \quad \text{(telescopes after pairing } m \text{ and } -m\text{)},$$

you get (up to a nonzero constant fixed by $\sigma'(0) = 1$):

$$\sigma(u) \;\longrightarrow\; \sin u \;\exp\Big(\frac{u^2}{2}\Big).$$

Now differentiate $\log\sigma(u)$ to get $\zeta(u)$:

$$\zeta(u) = (\log\sigma)' \;\longrightarrow\; (\log\sin u)' + \Big(\frac{u^2}{2}\Big)' = \cot u + u.$$

2. Yes. We showed that the series converge uniformly on compact sets in the limit, and they converge to the series expansions with no $n \neq 0$ terms. We saw that the resulting sums are exactly the series expansions of the respective trigonometric functions.

3. Put $X = \wp(u)$ in the limit $X = \cot^2 u + \frac{2}{3}$. Then we calculate

$$\wp'(u) = \frac{d}{du}(\cot^2 u) = -2\cot u\,\csc^2 u$$

$$(\wp')^2 = 4\cot^2 u\,\csc^4 u$$

Use $\csc^2 u = \cot^2 u + 1$ to express in X:

$$\cot^2 u = X - \tfrac{2}{3}, \quad \csc^2 u = X + \tfrac{1}{3},$$

hence $(\wp')^2 = 4\,(X - \frac{2}{3})(X + \frac{1}{3})^2 = 4X^3 - \frac{4}{3}X - \frac{8}{27}$. So we find that

$$(\wp')^2 = 4\wp^3 - \frac{4}{3}\wp - \frac{8}{27}$$

4. The analytic set in $\mathbb{C}^2$ parametrized by $z = \wp(u), w = \wp'(u)$ is given by the cubic equation

$$w^2 = 4z^3 - \frac{4}{3}z - \frac{8}{27} = 4\,(z - \tfrac{2}{3})(z + \tfrac{1}{3})^2$$

Its discriminant is $g_2^3 - 27g_3^2 = 0$, so the curve is singular.