Geometric invariant theory

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January 14, 2026

Abstract

These are reading notes for *Geometric Invariant Theory* by Mumford, Fogarty and Kirwan.

1 Example

Consider the action of $G = \mathrm{PGL}(n+1)$ on $X = (\mathbb{P}^n)^{m+1}$ using the line bundle

$$L = \mathcal{O}_{\mathbb{P}^n}(1)^{\boxtimes(m+1)} = \mathcal{O}(1,\ldots,1) = \pi_1^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes \cdots \otimes \pi_{m+1}^*\mathcal{O}_{\mathbb{P}^n}(1)$$

where $\pi_i : X \to \mathbb{P}^n$ is the projection to the i-th factor. We need to lift the geometric action of G on X to a linear action on L. The natural group that acts linearly on $\mathcal{O}_{\mathbb{P}^n}(1)$ is $\mathrm{GL}(n+1)$. There is no canonical way to make an element of $\mathrm{PGL}(n+1)$ act linearly on the fibers of $\mathcal{O}(1)$, because it is only defined up to scalar. The exact sequence is

$$1 \to \mathbb{G}_m \to \mathrm{GL}(n+1) \to \mathrm{PGL}(n+1) \to 1$$

Why don't we stay with $\mathrm{GL}(n+1)$ instead of $\mathrm{PGL}(n+1)$? Because the center $\mathbb{G}_m$ acts trivially on X and this introduces a useless symmetry which breaks stability.

We can restrict to the subgroup $\mathrm{SL}(n+1) \subset \mathrm{GL}(n+1)$, which kills most of the scalars except for the finite center μ_{n+1}. We want the linearization to descend to $\mathrm{PGL}(n+1)$, so we need the center $\mu_{n+1} = \ker(\mathrm{SL}(n+1) \to \mathrm{PGL}(n+1))$ to act trivially on the fibers of L.

On $\mathcal{O}_{\mathbb{P}^n}(1)$, a scalar $\zeta I \in \mu_{n+1} \subset SL(n+1)$ acts as multiplication by ζ on each fiber.

On

$$\mathcal{L} \;=\; \mathcal{O}_{\mathbb{P}^n}(1)^{\boxtimes m},$$

it therefore acts as multiplication by ζ^m.

Hence the $SL(n+1)$–linearization of $\mathcal{L}$ factors through $PGL(n+1)$ if and only if every $\zeta \in \mu_{n+1}$ acts trivially on $\mathcal{L}$, i.e.

$$\zeta^m = 1 \quad \text{for all } \zeta \text{ with } \zeta^{n+1} = 1.$$

This holds if and only if

$$n + 1 \mid m.$$

More generally, if we consider the line bundle

$$L_i = \mathcal{O}_{\mathbb{P}^n}(a_i)$$

on the i-th factor, then the same argument shows that the $SL(n+1)$–linearization of

$$\mathcal{L} = \bigotimes_{i=1}^{m+1} \pi_i^* L_i = \mathcal{O}_{\mathbb{P}^n}(a_1, \ldots, a_{m+1})$$

descends to $PGL(n+1)$ if and only if

$$n + 1 \mid \sum_{i=1}^{m+1} a_i.$$

In any case, by means of these linearizations we can define invariant sections of all the sheaves $\mathcal{L}_{\boldsymbol{\alpha}}$. To construct such invariant sections, let $X_0, \ldots, X_n$ be the canonical sections of $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$. Let

$$X_i^{(j)} = \pi_j^*(X_i)$$

be the induced sections of L_j.

Definition 1.1. For all sequences $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n)$ of integers such that $0 \le \alpha_i \le m$, let

$$D_{\alpha_0,\ldots,\alpha_n} = \det\bigl(X_i^{(\alpha_j)}\bigr)_{0 \le i,j \le n}$$

be the section of $L_{\alpha_0} \otimes \cdots \otimes L_{\alpha_n}$ obtained by addition and tensor product as in the determinant.

It is evident that $D_{\alpha_0,\ldots,\alpha_n}$ is an invariant section of $L_{\alpha_0} \otimes \cdots \otimes L_{\alpha_n}$. The non-vanishing of suitable D's defines the open sets we are looking for. Explicitly, a point of $(\mathbb{P}^n)^{m+1}$ is a tuple

$$(p_0, \ldots, p_m), \quad p_j = [v_j], \quad v_j \in k^{n+1} \setminus \{0\}.$$

Choose homogeneous lifts v_j. Put them as columns of a matrix

$$M = [v_0 \mid v_1 \mid \cdots \mid v_m] \in \mathrm{Mat}_{n+1,m+1}.$$

Then

$$D_{\alpha_0,\ldots,\alpha_n} = \det(v_{\alpha_0}, \ldots, v_{\alpha_n}).$$

The fact that $D_{\alpha_0,\ldots,\alpha_n}$ is well defined as a section of $L_{\alpha_0} \otimes \cdots \otimes L_{\alpha_n}$ follows from the following properties:

- $D_{\alpha_0,\dots,\alpha_n} = 0$ iff the points $p_{\alpha_0}, \dots, p_{\alpha_n}$ lie in a hyperplane.
- Under $g \in \mathrm{GL}(n+1)$, all minors are multiplied by $\det(g)$.
- Rescaling columns rescales the corresponding minors.

Definition 1.2. An R-partition of $\{0, 1, \dots, n\}$ is an ordered set of subsets $S_1, \dots, S_\nu$ of $\{0, 1, \dots, n\}$ such that

(i) $S_i \cap (S_1 \cup \cdots \cup S_{i-1})$ consists of exactly one integer for $i = 2, \dots, \nu$

(ii) $\bigcup_i S_i = \{0, 1, \dots, n\}$.

Definition 1.3. Given an R-partition $R = \{S_1, \dots, S_\nu\}$, let $U_R \subset (\mathbb{P}^n)^{m+1}$ be the open subset defined by

(i) $D_{0,1,\dots,n} \neq 0$,

(ii) for all k between 1 and ν, and for all $i \in S_k$,

$$D_{0,\dots,i-1,i+1,\dots,n,n+k} \neq 0$$

Not only is U_R affine, but the whole structure of the action of $PGL(n+1)$ on U_R can be described explicitly. On each open set U_R, a configuration of points in $(\mathbb{P}^n)^{m+1}$ is uniquely the same thing as

1. a projective frame, and
2. a collection of free affine parameters

Proposition 1.4. Let $R = \{S_1, \dots, S_\nu\}$ be an R-partition of $\{0, 1, \dots, n\}$. Let $\mathrm{PGL}(n+1)$ act on $PGL(n+1) \times \mathbb{A}^{n\nu - n}$ by the product of left translation on itself and the trivial action on the affine space. Then there is a $\mathrm{PGL}(n+1)$-linear isomorphism:

$$U_R \cong \mathrm{PGL}(n+1) \times \mathbb{A}^{n\nu - n}.$$

Hence U_R is a globally trivial principal fibre bundle with respect to the action of $\mathrm{PGL}(n+1)$, with base space $\mathbb{A}^{n\nu - n}$.