

Geometric invariant theory

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Abstract

These are reading notes for *Geometric Invariant Theory* by Mumford, Fogarty and Kirwan.

1 Example

Consider the action of $G = \mathrm{PGL}(n+1)$ on $X = (\mathbb{P}^n)^{m+1}$ using the line bundle

$$L = \mathcal{O}_{\mathbb{P}^n}(1)^{\boxtimes(m+1)} = \mathcal{O}(1, \dots, 1) = \pi_1^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \cdots \otimes \pi_{m+1}^* \mathcal{O}_{\mathbb{P}^n}(1)$$

where $\pi_i : X \rightarrow \mathbb{P}^n$ is the projection to the i -th factor. We need to lift the geometric action of G on X to a linear action on L . The natural group that acts linearly on $\mathcal{O}_{\mathbb{P}^n}(1)$ is $\mathrm{GL}(n+1)$. There is no canonical way to make an element of $\mathrm{PGL}(n+1)$ act linearly on the fibers of $\mathcal{O}(1)$, because it is only defined up to scalar. The exact sequence is

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathrm{GL}(n+1) \rightarrow \mathrm{PGL}(n+1) \rightarrow 1$$

Why don't we stay with $\mathrm{GL}(n+1)$ instead of $\mathrm{PGL}(n+1)$? Because the center \mathbb{G}_m acts trivially on X and this introduces a useless symmetry which breaks stability.

We can restrict to the subgroup $\mathrm{SL}(n+1) \subset \mathrm{GL}(n+1)$, which kills most of the scalars except for the finite center μ_{n+1} . We want the linearization to descend to $\mathrm{PGL}(n+1)$, so we need the center $\mu_{n+1} = \ker(\mathrm{SL}(n+1) \rightarrow \mathrm{PGL}(n+1))$ to act trivially on the fibers of L .

On $\mathcal{O}_{\mathbb{P}^n}(1)$, a scalar $\zeta I \in \mu_{n+1} \subset \mathrm{SL}(n+1)$ acts as multiplication by ζ on each fiber.

On

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)^{\boxtimes m},$$

it therefore acts as multiplication by ζ^m .

Hence the $\mathrm{SL}(n+1)$ -linearization of \mathcal{L} factors through $\mathrm{PGL}(n+1)$ if and only if every $\zeta \in \mu_{n+1}$ acts trivially on \mathcal{L} , i.e.

$$\zeta^m = 1 \quad \text{for all } \zeta \text{ with } \zeta^{n+1} = 1.$$

This holds if and only if

$$n + 1 \mid m.$$

More generally, if we consider the line bundle

$$L_i = \mathcal{O}_{\mathbb{P}^n}(a_i)$$

on the i -th factor, then the same argument shows that the $SL(n + 1)$ -linearization of

$$\mathcal{L} = \bigotimes_{i=1}^{m+1} \pi_i^* L_i = \mathcal{O}_{\mathbb{P}^n}(a_1, \dots, a_{m+1})$$

descends to $PGL(n + 1)$ if and only if

$$n + 1 \mid \sum_{i=1}^{m+1} a_i.$$

In any case, by means of these linearizations we can define invariant sections of all the sheaves \mathcal{L}_α . To construct such invariant sections, let X_0, \dots, X_n be the canonical sections of $\mathcal{O}_{\mathbb{P}^n}(1)$ on \mathbb{P}^n . Let

$$X_i^{(j)} = \pi_j^*(X_i)$$

be the induced sections of L_j .

Definition 1.1. For all sequences $\alpha = (\alpha_0, \dots, \alpha_n)$ of integers such that $0 \leq \alpha_i \leq m$, let

$$D_{\alpha_0, \dots, \alpha_n} = \det(X_i^{(\alpha_j)})_{0 \leq i, j \leq n}$$

be the section of $L_{\alpha_0} \otimes \dots \otimes L_{\alpha_n}$ obtained by addition and tensor product as in the determinant.

It is evident that $D_{\alpha_0, \dots, \alpha_n}$ is an invariant section of $L_{\alpha_0} \otimes \dots \otimes L_{\alpha_n}$. The non-vanishing of suitable D 's defines the open sets we are looking for. Explicitly, a point of $(\mathbb{P}^n)^{m+1}$ is a tuple

$$(p_0, \dots, p_m), \quad p_j = [v_j], \quad v_j \in k^{n+1} \setminus \{0\}.$$

Choose homogeneous lifts v_j . Put them as columns of a matrix

$$M = [v_0 \mid v_1 \mid \dots \mid v_m] \in \mathbf{Mat}_{n+1, m+1}.$$

Then

$$D_{\alpha_0, \dots, \alpha_n} = \det(v_{\alpha_0}, \dots, v_{\alpha_n}).$$

The fact that $D_{\alpha_0, \dots, \alpha_n}$ is well defined as a section of $L_{\alpha_0} \otimes \dots \otimes L_{\alpha_n}$ follows from the following properties:

- $D_{\alpha_0, \dots, \alpha_n} = 0$ iff the points $p_{\alpha_0}, \dots, p_{\alpha_n}$ lie in a hyperplane.
- Under $g \in \mathrm{GL}(n+1)$, all minors are multiplied by $\det(g)$.
- Rescaling columns rescales the corresponding minors.

Definition 1.2. An R -partition of $\{0, 1, \dots, n\}$ is an ordered set of subsets S_1, \dots, S_ν of $\{0, 1, \dots, n\}$ such that

- (i) $S_i \cap (S_1 \cup \dots \cup S_{i-1})$ consists of exactly one integer for $i = 2, \dots, \nu$
- (ii) $\bigcup_i S_i = \{0, 1, \dots, n\}$.

Definition 1.3. Given an R -partition $R = \{S_1, \dots, S_\nu\}$, let $U_R \subset (\mathbb{P}^n)^{m+1}$ be the open subset defined by

- (i) $D_{0,1,\dots,n} \neq 0$,
- (ii) for all k between 1 and ν , and for all $i \in S_k$,

$$D_{0,\dots,i-1,i+1,\dots,n,n+k} \neq 0$$

Not only is U_R affine, but the whole structure of the action of $\mathrm{PGL}(n+1)$ on U_R can be described explicitly. On each open set U_R , a configuration of points in $(\mathbb{P}^n)^{m+1}$ is uniquely the same thing as

1. a projective frame, and
2. a collection of free affine parameters

Proposition 1.4. Let $R = \{S_1, \dots, S_\nu\}$ be an R -partition of $\{0, 1, \dots, n\}$. Let $\mathrm{PGL}(n+1)$ act on $\mathrm{PGL}(n+1) \times \mathbb{A}^{n\nu-n}$ by the product of left translation on itself and the trivial action on the affine space. Then there is a $\mathrm{PGL}(n+1)$ -linear isomorphism:

$$U_R \cong \mathrm{PGL}(n+1) \times \mathbb{A}^{n\nu-n}.$$

Hence U_R is a globally trivial principal fibre bundle with respect to the action of $\mathrm{PGL}(n+1)$, with base space $\mathbb{A}^{n\nu-n}$.

Proof. Fix an R -partition and the associated open set $U_R \subset X$. On U_R , define sections λ_j by

$$\lambda_{\mu(1)} := 1, \quad \lambda_j := \lambda_{\mu(x(j))} \frac{D_{0,1,\dots,\widehat{\mu(x(j))},\dots,n,j}}{D_{0,1,\dots,n}} \quad (0 \leq j \leq n).$$

These satisfy $\lambda_j \in \Gamma(U_R, L_j \otimes L_{\mu(1)}^{-1})$.

We define

$$\phi = (\phi_1, \phi_2) : U_R \longrightarrow PGL(n+1) \times A_R$$

as follows.

Identifying $PGL(n+1)$ with the open subset $\{\det \neq 0\} \subset \mathbb{P}^{(n+1)^2-1}$ with homogeneous coordinates a_{ij} , define ϕ_1 by

$$(\phi_1)^*(a_{ij}) = (-1)^j X_i^{(j)} \otimes \lambda_j^{-1}.$$

Define ϕ_2 by, for $k \geq 1$,

$$(\phi_2)^*(x_i^{(n+k)}) = \frac{D_{0,1,\dots,\hat{i},\dots,n,n+k}}{D_{0,1,\dots,n}} \frac{\lambda_i}{\lambda_{\mu(k)}}.$$

Then ϕ is a $PGL(n+1)$ -equivariant isomorphism

$$U_R \cong PGL(n+1) \times A_R.$$

Unraveling over a field, a point of U_R corresponds to a tuple of points (p_0, \dots, p_m) in $(\mathbb{P}^n)^{m+1}$ satisfying the conditions defining U_R . Writing $p_j = [v_j]$ with $v_j \in k^{n+1} \setminus \{0\}$ with $\det[v_0 \mid v_1 \mid \dots \mid v_n] \neq 0$, the map ϕ sends this point to

$$A^{-1}, A^{-1}v_{n+1}, \dots, A^{-1}v_m$$

where $A = [v_0 \mid v_1 \mid \dots \mid v_n]$. \square

What is this R-partition formalism really saying? When $\nu = 1$, then we are forced to take $R = \{S_1\}$ and $S_1 = \{0, 1, \dots, n\}$. The definition of U_R reads $D_{0,1,\dots,n} \neq 0$, so U_R is those tuples of points (p_0, \dots, p_{n+1}) for which the points p_0, \dots, p_n are not colinear and p_{n+1} is not in any of the coordinate hyperplanes determined by them, i.e. $n+2$ points in \mathbb{P}^n in general position. Then there exists a unique projective transformation $g \in PGL(n+1)$ sending p_i to e_i for $i = 0, \dots, n$ and sending p_{n+1} to $[1 : 1 : \dots : 1]$. So in particular $U_R = PGL(n+1)$ in this case.

We can also study the case when $n = 2$, $\nu = 2$ and $S_1 = \{0, 1\}$ and $S_2 = \{1, 2\}$. We are then thinking about $m = n + \nu + 1 = 5$ points in \mathbb{P}^2 , say $(p_0, p_1, p_2, p_3, p_4)$. The set U_R is then cut out by the nonvanishing of the determinants $D_{0,1,2}$, $D_{1,2,3}$, $D_{0,2,3}$, $D_{0,2,4}$, and $D_{0,1,4}$. Geometrically this means that p_0, p_1, p_2 are not colinear, p_3 does not lie on the lines p_1p_2 and p_0p_2 , and p_4 does not lie on the lines p_0p_1 and p_0p_2 . Then we can uniquely normalize p_0, p_1, p_2 to be the coordinate points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, and normalize p_3 and p_4 to be $[1, 1, a]$ and $[b, 1, 1]$.

Definition 1.5. For fixed m let $U_{reg} \subset (\mathbb{P}^n)^{m+1}$ be union of all U_R where R runs through all R-partitions of $\{0, 1, \dots, n\}$ with $\nu = m - n$.

Proposition 1.6 (Proposition 3.3). Let $x = (x^{(0)}, x^{(1)}, \dots, x^{(m)})$ be a geometric point of $(\mathbb{P}^n)^{m+1}$. Then the following are equivalent:

1. The stabilizer $S(x)$ is 0-dimensional.

2. There do not exist disjoint proper linear subspaces L' and L'' of \mathbb{P}^n such that every $x^{(i)}$ lies in either L' or L'' .
3. x is a geometric point of U_{reg} .

Proof. Let k be the algebraically closed field over which x is defined. For simplicity, we shall write \mathbb{P} for \mathbb{P}_k^n , and U_R for $U_R \times_k \text{Spec } k$, etc., in the course of this proof.

First, the implication (1) \Rightarrow (2) is clear; for in suitable homogeneous coordinates $\{X_i\}$, one may assume that

$$L' \subset \{X_0 = X_1 = \cdots = X_r = 0\}, \quad L'' \subset \{X_{r+1} = \cdots = X_n = 0\}.$$

Then the subgroup of transformations

$$\begin{pmatrix} \alpha I_{r+1} & 0 \\ 0 & \beta I_{n-r} \end{pmatrix} \in PGL(n+1)$$

leaves x fixed.

Secondly, (3) \Rightarrow (1) is an immediate consequence of the equivariant trivialization $U_R \cong PGL(n+1) \times \mathbb{A}^{n\nu-n}$ and the fact that $PGL(n+1)$ acts freely on itself by left translation.

Thirdly, we will prove that (2) \Rightarrow (3). By virtue of (2), all the points $x^{(i)}$ cannot lie in one hyperplane, hence we can choose $n+1$ of the $x^{(i)}$ which are not in one hyperplane, say $x^{(0)}, x^{(1)}, \dots, x^{(n)}$. Without loss of generality, we may assume that these have homogeneous coordinates $x_i^{(j)} = \delta_{ij}$.

Now for each $n+k$ between $n+1$ and m , let S_k be the set of integers i such that

$$D_{0,1,\dots,\widehat{i},\dots,n,n+k} \neq 0,$$

i.e. $x^{(n+k)}$ is not in the hyperplane spanned by $x^{(0)}, \dots, \widehat{x^{(i)}}, \dots, x^{(n)}$.

Then I claim that there is no partition of the set $\{0, 1, \dots, n\}$ into two disjoint subsets T' and T'' such that every S_k is contained in either T' or T'' . For if there were, and if one let L' (resp. L'') be the linear subspace defined by $X_i = 0$ for all $i \in T'$ (resp. $i \in T''$), then every point $x^{(k)}$ would lie in $L' \cup L''$, contradicting (2).

It follows immediately from a combinatorial argument that a suitable set of subsets $S_i \subset S_j$ is an R -partition R and that $x \in U_R$. \square

It thus follows that U_{reg} is the locus of prestable points in the following sense. Let G be a reductive algebraic group acting via σ , on X scheme of finite type over a field k . Now suppose that L is an invertible sheaf on X and that ϕ is a G -linearization of L . The key concepts are the following.

Definition 1.7 (Mumford, Definition 1.7). Let x be a geometric point of X .

- (a) x is *pre-stable* (with respect to σ) if there exists an invariant affine open subset $U \subset X$ such that $x \in U$ and every G -orbit in U is closed in U
- (b) x is *semi-stable* (with respect to σ, L, ϕ) if there exists a section $s \in H^0(X, L^{\otimes n})$ for some $n > 0$ such that $s(x) \neq 0$, the open subset X_s is affine, and s is invariant. Equivalently, if $\phi_n : \sigma^*(L^{\otimes n}) \rightarrow p_2^*(L^{\otimes n})$ is the induced linearization, then

$$\phi_n(\sigma^* s) = p_2^*(s).$$

- (c) x is *stable* (with respect to σ, L, ϕ) if there exists a section $s \in H^0(X, L^{\otimes n})$ for some $n > 0$ such that $s(x) \neq 0$, the open subset X_s is affine, s is invariant, and the action of G on X_s is closed.

A single closed orbit can sit inside a region where nearby orbits are wildly non-closed. That gives bad quotient behavior. You cannot form a reasonable local quotient around such a point.

For example, consider the action of \mathfrak{G}_m on \mathbb{A}^2 by

$$t \cdot (x, y) = (tx, t^{-1}y).$$

The origin is a closed orbit, but any neighborhood of the origin contains points whose orbits are not closed (e.g. points on the hyperbolas $xy = c \neq 0$). Thus the origin is not prestable.

Prestable points are exactly those that sit inside a region where no orbit collapses onto another. This is the weakest hypothesis under which local quotients look like honest orbit spaces.

Definition 1.8 (Mumford, Definition 0.6). Given an action of G/S on X/S , a pair (Y, ϕ) consisting of a pre-scheme Y over S and an S -morphism $\phi : X \rightarrow Y$ is called a *geometric quotient* (of X by G) if the following conditions are satisfied:

- (i) $\phi \circ \sigma = \phi \circ p_2$ (as in Definition 0.5).
- (ii) ϕ is surjective, and the image of Ψ is $X \times_Y X$ (cf. Definition 0.4). Equivalently, the geometric fibres of ϕ are precisely the orbits of the geometric points of X (for geometric points over an algebraically closed field of sufficiently high transcendence degree).
- (iii) ϕ is submersive, i.e. a subset $U \subset Y$ is open if and only if $\phi^{-1}(U)$ is open in X . Likewise, $U' \subset Y'$ is open if and only if $\phi^{-1}(U')$ is open in X' .
- (iv) The fundamental sheaf \mathcal{O}_Y is the subsheaf of $\phi_*(\mathcal{O}_X)$ consisting of invariant functions. That

is, if $f \in \Gamma(U, \phi_*(\mathcal{O}_X)) = \Gamma(\phi^{-1}(U), \mathcal{O}_X)$, then $f \in \Gamma(U, \mathcal{O}_Y)$ if and only if the diagram

$$\begin{array}{ccc} G \times \phi^{-1}(U) & \xrightarrow{\sigma} & \phi^{-1}(U) \\ p_2 \downarrow & & \downarrow F \\ \phi^{-1}(U) & \xrightarrow{F} & \mathbb{A}^1 \end{array}$$

commutes, where F is the morphism defined by f .

Our next step is to construct a geometric quotient of U_{reg} by $PGL(n+1)$. Let U_1, \dots, U_N be the open subsets U_R of U_{reg} and the subsets obtained from these by permuting the coordinates. Let (Z_i, ϕ_i) be the geometric quotient of U_i by $PGL(n+1)$. For all pairs i, j , $U_i \cap U_j$ is an invariant open subset of U_i and U_j . Therefore, by Corollary 3.2, if $\sigma_i : Z_i \rightarrow U_i$ is the global section of ϕ_i , we know:

$$PGL(n+1) \times \sigma_i^{-1}(U_i \cap U_j) \cong U_i \cap U_j \cong PGL(n+1) \times \sigma_j^{-1}(U_i \cap U_j).$$

In other words, both $\sigma_i^{-1}(U_i \cap U_j)$ and $\sigma_j^{-1}(U_i \cap U_j)$ are geometric quotients of $U_i \cap U_j$ by $PGL(n+1)$; therefore, they are canonically isomorphic. We use this isomorphism to glue together Z_i and Z_j . For any three of the quotients Z_i, Z_j, Z_k , these identifications are obviously compatible. Therefore, we have defined a pre-scheme Z and a morphism $\phi : U_{\text{reg}} \rightarrow Z$. Clearly, U_{reg} is a locally trivial principal fibre bundle over Z ; a fortiori, (Z, ϕ) is a geometric quotient of U_{reg} by $PGL(n+1)$. However, Z is very far from being a scheme, let alone being quasi-projective.

Although Z is not quasi-projective, it carries various invertible sheaves. To investigate these, we make use of the theory of descent: by SGA 8, §1, the set of invertible sheaves on Z is isomorphic to the set of invertible sheaves on U_{reg} plus descent data for ϕ . But ϕ -descent data is precisely the same as a $PGL(n+1)$ -linearization, since:

$$U_{\text{reg}} \times_Z U_{\text{reg}} \cong PGL(n+1) \times U_{\text{reg}}.$$

But L_i^{n+1} admits a $PGL(n+1)$ -linearization. Therefore, there is an invertible sheaf M_i on Z such that

$$L_i^{n+1} \cong \phi^*(M_i).$$

Moreover, the section

$$(D_{\alpha_0, \dots, \alpha_n})^{n+1} \in \Gamma(U_{\text{reg}}, L_{\alpha_0}^{n+1} \otimes \dots \otimes L_{\alpha_n}^{n+1})$$

is invariant in the $SL(n+1)$ -linearization of this sheaf, hence in the $PGL(n+1)$ -linearization of this sheaf. Therefore, according to SGA 8, §1, there is a section $E_{\alpha_0, \dots, \alpha_n}$ of

$$M_{\alpha_0} \otimes \dots \otimes M_{\alpha_n}$$

such that

$$(D_{\alpha_0, \dots, \alpha_n})^{n+1} = \phi^*(E_{\alpha_0, \dots, \alpha_n}).$$