Equivariant Derived Categories of Coherent Sheaves

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Abstract

Notes for a talk I'm giving on equivariant derived categories of coherent sheaves.

Contents

1 Generalities on GIT quotients

Let $X \subset \mathbb{P}^n$ be a projective variety, and let $\widetilde{X} \subset \mathbb{C}^{n+1}$ be the corresponding affine cone. Since X is the space of lines in $\widetilde{X}$, it has a tautological line bundle

$$\mathcal{O}_X(-1) = \mathcal{O}_{\mathbb{P}^n}(-1)\big|_X$$

over it whose fibre over a point in X is the corresponding line in $\widetilde{X} \subset \mathbb{C}^{n+1}$. The total space of $\mathcal{O}_X(-1)$ therefore has a tautological map to $\widetilde{X}$ which is an isomorphism away from the zero section $X \subset \mathcal{O}_X(-1)$, which is all contracted down to the origin in $\widetilde{X}$. In fact the total space of $\mathcal{O}_X(-1)$ is the **blow up** of $\widetilde{X}$ in the origin.

Linear functions on $\mathbb{C}^{n+1}$ like x_i, restricted to $\widetilde{X}$ and pulled back to the total space of $\mathcal{O}_X(-1)$, give functions which are linear on the fibres, so correspond to sections of the **dual** line bundle $\mathcal{O}_X(1)$. Similarly degree k homogeneous polynomials on $\widetilde{X}$ define functions on the total space of $\mathcal{O}_X(-1)$ which are of degree k on the fibres, and so give sections of the kth tensor power $\mathcal{O}_X(k)$ of the dual of the line bundle $\mathcal{O}_X(-1)$.

So the grading that splits the functions on $\widetilde{X}$ into homogeneous degree (or $\mathbb{C}^*$-weight spaces) corresponds to sections of different line bundles $\mathcal{O}_X(k)$ on X. So

$$\bigoplus_{k\geq 0} H^0(\mathcal{O}_X(k))$$

considered a graded ring by tensoring sections $\mathcal{O}(k)\otimes\mathcal{O}(l)\cong\mathcal{O}(k+l)$. For the line bundle $\mathcal{O}_X(1)$ sufficiently positive, this ring will be generated in degree one. It is often called the (homogeneous) coordinate ring of the **polarised** (i.e. endowed with an ample line bundle) variety $(X, \mathcal{O}_X(1))$.

The degree one restriction is for convenience and can be dropped (by working with varieties in weighted projective spaces), or bypassed by replacing $\mathcal{O}_X(1)$ by $\mathcal{O}_X(p)$, i.e. using the ring

$$R^{(p)} = \bigoplus_{k\geq 0} R_{kp}; \qquad \text{for } p \gg 0 \text{ this will be generated by its degree one piece } R_p.$$

The choice of generators of the ring is what gives the embedding in projective space. In fact the sections of any line bundle L over X define a (rational) map

$$X \dashrightarrow \mathbb{P}\big(H^0(X,L)^*\big), \qquad x \mapsto ev_x, \quad ev_x(s) := s(x), \tag{1}$$

which in coordinates maps x to $(s_0(x) : \cdots : s_n(x)) \in \mathbb{P}^n$, where s_i form a basis for $H^0(L)$. This map is only defined for those x with $ev_x \neq 0$, i.e. for which $s(x)$ is not zero for every s.

Now suppose we are in the following situation, of G acting on a projective variety X through SL transformations of the projective space.

$$\begin{array}{ccc} G & \curvearrowright & X \\ \downarrow & & \downarrow \\ SL(n+1,\mathbb{C}) & \curvearrowright & \mathbb{P}^n \end{array}$$

Since we have assumed that G acts through $SL(n+1,\mathbb{C})$, the action lifts from X to one covering it on $\mathcal{O}_X(-1)$. In other words we don't just act on the projective space (and X therein) but on the vector space overlying it (and the cone $\widetilde{X}$ on X therein). This is called a **linearisation** of the action. Thus G acts on each $H^0(\mathcal{O}_X(r))$.

Then, just as $(X, \mathcal{O}_X(1))$ is determined by its graded ring of sections of $\mathcal{O}(r)$ (i.e. the ring of functions on $\widetilde{X}$),

$$(X, \mathcal{O}(1)) \longleftrightarrow \bigoplus_r H^0(X, \mathcal{O}(r))$$

we simply **construct** X/G (with a line bundle on it) from the ring of **invariant** sections:

$$X/G \longleftrightarrow \bigoplus_r H^0(X, \mathcal{O}(r))^G$$

This is sensible, since if there is a good quotient then functions on it pullback to give G-invariant functions on X, i.e. functions constant on the orbits, the fibres of $X \to X/G$. For it to work we need:

Lemma 1.1. $\bigoplus_r H^0(X, \mathcal{O}(r))^G$ is finitely generated.

Proof. Since $R := \bigoplus_r H^0(X, \mathcal{O}(r))$ is Noetherian, Hilbert's basis theorem tells us that the ideal $R \cdot \left(\bigoplus_{r>0} H^0(X, \mathcal{O}(r))^G\right)$ generated by $R_+^G := \bigoplus_{r>0} H^0(X, \mathcal{O}(r))^G$ is generated by a finite number of elements $s_0, \ldots, s_k \in R_+^G$.

Thus any element $s \in H^0(X, \mathcal{O}(r))^G$, $r > 0$, may be written $s = \sum_{i=0}^k f_i s_i$ for some $f_i \in R$ of degree $< r$. To show that the s_i generate R_+^G as an algebra we must show that the f_i can be taken to lie in R^G.

We now use the fact that G is the complexification of the compact group K. Since K has an invariant metric, we can average over it and use the facts that s and s_i are invariant to give

$$s = \sum_{i=0}^k \mathrm{Av}(f_i)\, s_i,$$

where $\mathrm{Av}(f_i)$ is the (K-invariant) K-average of f_i. By complex linearity $\mathrm{Av}(f_i)$ is also G-invariant (for instance, since G has a polar decomposition $G = K \exp(i\mathfrak{t})$). The $\mathrm{Av}(f_i)$ are also of degree $< r$, and so we may assume, by an induction on r, that we have already shown that they are generated by the s_i in R_+^G. Thus s is also. $\square$

Definition 1.2 (Projective GIT quotient)**.** Let X be a projective variety with an action of a reductive group G linearised by a line bundle $\mathcal{O}_X(1)$. We define the projective GIT quotient X/G to be

$$X/G = \mathrm{Proj} \bigoplus_r H^0(X, \mathcal{O}(r))^G.$$

If X is a variety (rather than a scheme) then so is X/G, as its graded ring sits inside that of X and so has no zero divisors.

Definition 1.3 (Affine GIT quotient)**.** Let $X = \operatorname{Spec} R$ be an affine variety with an action of a reductive group G. We define the affine GIT quotient X/G to be $\operatorname{Spec}(R^G)$, where R^G is the ring of G-invariant regular functions on X.

> In some cases, this does not work so well. For instance, under the scalar action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ the only invariant polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ are the constants and this recipe for the quotient gives a single point. In the language of the next section, this is because there are no stable points in this example, and all semistable orbits' closures intersect (or equivalently, there is a unique polystable point, the origin). More generally in any affine case all points are always at least semistable (as the constants are always G-invariant functions) and so no orbits gets thrown away in making the quotient (though many may get identified with each other — those whose closures intersect which therefore cannot be separated by invariant functions). But for the scalar action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ we clearly need to remove at least the origin to get a sensible quotient.
>
> So we should change the linearisation, from the trivial linearisation to a nontrivial one, to get a bigger quotient. This is demonstrarted in the following example.

Example 1.4 (Projective space as a GIT quotient)**.** Consider the trivial line bundle on $\mathbb{C}^{n+1}$ but with a nontrivial linearisation, by composing the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ by a character $\lambda \mapsto \lambda^p$ of $\mathbb{C}^*$ acting on the fibres of the trivial line bundle over $\mathbb{C}^{n+1}$. The invariant sections of this no longer form a ring; we have to take the direct sum of spaces of sections of **all powers** of this linearisation, just as in the projective case, and take Proj of the invariants of the resulting graded ring.

We calculate the invariant sections for general p. Look at the k-th tensor power of the linearised line bundle. Sections are homogeneous polynomials $f(x_0, \ldots, x_n)$ of some degree. Under λ, such an f transforms as

$$f(x_0, \ldots, x_n) \mapsto f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n),$$

where $d = \deg f$.

But the linearisation introduces an extra factor λ^{-pk} when we act on the fibre of the k-th tensor power. By definition, the G-action on a section s is

$$(g \cdot s)(x) \;=\; g \cdot \big(s(g^{-1} \cdot x)\big).$$

Take a polynomial f homogeneous of degree d. View the section as

$$s(x) = f(x) \cdot e$$

where e is a trivialising section of the fibre. When we apply the group action:

$$(g \cdot s)(x) = g \cdot (f(g^{-1} \cdot x) \cdot e) = \big(\lambda^{-d} f(x)\big) \cdot \lambda^{pk} e = \lambda^{-d+pk} f(x) \cdot e.$$

For invariance, we need the weight to vanish, i.e.

$$d = pk.$$

So only polynomials of degree exactly pk survive as invariants in the degree k graded piece.

If $p < 0$ then there are no invariant sections and the quotient is empty. We have seen that for $p = 0$ the quotient is a single point. For $p > 0$ the invariant sections of the kth power of the linearisation are the homogeneous polynomials on $\mathbb{C}^n$ of degree kp. So for $p = 1$ we get the quotient

$$\mathbb{C}^{n+1}/\mathbb{C}^* = \operatorname{Proj} \bigoplus_{k\geq 0} \left(\mathbb{C}[x_0, \dots, x_n]_k\right) = \operatorname{Proj} \mathbb{C}[x_0, \dots, x_n] = \mathbb{P}^n. \tag{2}$$

For $p \geq 1$ we get the same geometric quotient but with the line bundle $\mathcal{O}(p)$ on it instead of $\mathcal{O}(1)$.

Another way to derive this is to embed $\mathbb{C}^{n+1}$ in $\mathbb{P}^{n+1}$ as $x_{n+1} = 1$, act by $\mathbb{C}^*$ on the latter by

$$\operatorname{diag}(\lambda, \dots, \lambda, \lambda^{-(n+1)}) \in SL(n+2, \mathbb{C})$$

and do projective GIT. This gives, on restriction to $\mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$, the $p = n+1$ linearisation above. The invariant sections of $\mathcal{O}((n+2)k)$ are of the form $x_{n+1}^k f$, where f is a homogeneous polynomial of degree $(n+1)k$ in $x_1, \dots, x_n$. Therefore the quotient is

$$\operatorname{Proj} \bigoplus_{k\geq 0} \left(\mathbb{C}[x_1, \dots, x_n]_{(n+1)k}\right) = \operatorname{Proj} \left(\mathbb{C}[x_1, \dots, x_n], \mathcal{O}(n+1)\right).$$

Definition 1.5 (Semistable points)**.** A point $x \in X$ is **semistable** iff there exists $s \in H^0(X, \mathcal{O}(r))^G$ with $r > 0$ such that $s(x) \neq 0$. Points which are not semistable are **unstable**.

So semistable points are those that the G-invariant functions "see." The map

$$\begin{aligned} X^{ss} &\to \mathbb{P}(H^0(X, \mathcal{O}(r))^G)^* \\ x &\mapsto ev_x \end{aligned}$$

is well defined on the (Zariski open, though possibly empty) locus $X^{ss} \subseteq X$ of semistable points, and it is clearly constant on G-orbits, i.e. it factors through the set-theoretic quotient X^{ss}/G. But it may contract more than just G-orbits, so we need another definition.

Definition 1.6 (Stable points)**.** A semistable point x is **stable** if and only if $\bigoplus_r H^0(X, \mathcal{O}(r))^G$ separates orbits near x and the stabiliser of x is finite.

We now come to the main example which we will study throughout these notes. Let $V = \mathbb{C}^4$ with coordinates x_1, x_2, y_1, y_2 and consider the $\mathbb{C}^*$-action given by

$$t \cdot (x_1, x_2, y_1, y_2) = (tx_1, tx_2, t^{-1}y_1, t^{-1}y_2)$$

We linearize this action by a character $\chi_m : t \mapsto t^m$ with $m \in \mathbb{Z} \setminus \{0\}$. Since V is affine, the GIT quotient for χ_m is $\operatorname{Proj} R^{(m)}$, where

$$R^{(m)} = \bigoplus_{d\geq 0} \Gamma(V, \mathcal{O}_V)^{\mathbb{C}^*, \chi_m^{\otimes d}} = \bigoplus_{d\geq 0} \{\, f \in \mathbb{C}[x_1, x_2, y_1, y_2] \mid t \cdot f = t^{md} f \,\}$$

In other words, $R_d^{(m)}$ is spanned by monomials whose total $\mathbb{C}^*$-weight is md, where the weight of a monomial $x_1^{a_1}x_2^{a_2}y_1^{b_1}y_2^{b_2}$ is $w = a_1 + a_2 - (b_1 + b_2)$.

A point $v \in V$ is χ_m-semistable iff there exists $d > 0$ and $f \in R_d^{(m)}$ with $f(v) \neq 0$.

Here $R_d^{(m)}$ consists of polynomials whose monomials have positive weight $w = md > 0$. Such a monomial must contain at least one x (indeed, more x's than y's), so it vanishes at any point with $x_1 = x_2 = 0$. Therefore no section in $R_d^{(m)}$ can be nonzero at a point with $x_1 = x_2 = 0 \Rightarrow$ those points are unstable.

Conversely, if $(x_1, x_2) \neq (0, 0)$, then pick d and the monomial $f = x_i^{md}$ with $x_i \neq 0$. It has weight md and $f(v) \neq 0$, so v is semistable.

Therefore, for $m > 0$,

$$V^{ss}(\chi_m) = V \setminus \{x_1 = x_2 = 0\}.$$

The quotient is $(V \setminus \{x_1 = x_2 = 0\})/\mathbb{C}^*$, i.e. the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1_{[x_1:x_2]}$. Similarly, for $m < 0$, we have

$$V^{ss}(\chi_m) = V \setminus \{y_1 = y_2 = 0\}.$$

The quotient is $(V \setminus \{y_1 = y_2 = 0\})/\mathbb{C}^*$, i.e. the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1_{[y_1:y_2]}$.

2 Autoequivalences from VGIT

We show that we can construct $\mathbb{Z}$ many derived equivalences between X_+ and X_-, and that the resulting autoequivalences are spherical twists. Segal's paper upgrades this equivalence to an equivalence of B-brane dg-categories. In particular, he shows that there are $\mathbb{Z}$ many quasi-equivalences between the categories of B-branes on (X_+, W) and (X_-, W). When $W = 0$, the dg-category of B-branes is just the dg-category of perfect complexes, whose homotopy category is the bounded derived category of coherent sheaves. So Segal's result recovers the derived equivalences we construct here.

2.1 Set-up

Let $V = \mathbb{C}^4$ with coordinates x_1, x_2, y_1, y_2, and let $\mathbb{C}^*$ act on V with weight 1 on each x_i and weight -1 on each y_i. There are two possible GIT quotients X_+ and X_-, depending on whether we choose a positive or negative character of $\mathbb{C}^*$. Both are isomorphic to the total space of the bundle $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{P}^1$. This is the standard "three-fold flop" situation.

Both are open substacks of the Artin quotient stack

$$\mathcal{X} = [V/\mathbb{C}^*]$$

given by the semi-stable locus for either character. Let

$$\iota_\pm : X_\pm \hookrightarrow \mathcal{X}$$

denote the inclusions.

Remark 2.1 (The quotient stack and its open substacks)**.** Recall that via the functor of points perspective, its objects are pairs (P, ϕ), where P is a principal $\mathbb{C}^*$-bundle and $\phi : P \to V$ is $\mathbb{C}^*$-equivariant.

For a given choice of character χ_m, the semistable locus $V^{ss}(\chi_m)$ is an open subset of V. It is open because it is defined by the nonvanishing of some semi-invariant sections. The corresponding GIT quotient is $[V^{ss}(\chi_m)/\mathbb{C}^*]$ as a substack. Thus:

$$X_\pm = [V^{ss}(\pm 1)/\mathbb{C}^*] \ \subset \ [V/\mathbb{C}^*] = \mathcal{X}$$

It turns out that open substacks of quotient stacks $[V/G]$ are exactly those substacks which are of the form $[U/G]$ where $U \subseteq V$ is a G-invariant open subscheme. Here $V^{ss}(\chi_m) \subset V$ is G-invariant and open, so $[V^{ss}(\chi_m)/\mathbb{C}^*] \hookrightarrow [V/\mathbb{C}^*]$ is exactly an open immersion of stacks.

This stacky point of view makes it clear that there are (exact) restriction functors

$$\iota_\pm^* : D^b(\mathcal{X}) \ \to \ D^b(X_\pm).$$

By $D^b(\mathcal{X})$ we mean the derived category of the category of $\mathbb{C}^*$-equivariant sheaves on V. This contains the obvious equivariant line bundles $\mathcal{O}(i)$ associated to the characters of $\mathbb{C}^*$.

Remark 2.2 (General fact about open immersions)**.** If $j : U \hookrightarrow X$ is an open immersion of schemes, then there is an exact restriction functor $j^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$. This is because $j^*\mathcal{F}$ has the same stalk as $\mathcal{F}$ at points of U.

Althernatively, exactness comes from the fact that restricting a quasi-coherent sheaf to an open set is just tensoring with $\mathcal{O}_U$, which is flat (in general localisation is flat).

Passing to derived categories, you still have $j^* : D^b(\mathrm{QCoh}(X)) \to D^b(\mathrm{QCoh}(U))$ which has no higher derived functors since j^* is exact. The exact same holds in the stack setting: if $\iota : \mathcal{U} \hookrightarrow \mathcal{X}$ is an open immersion of stacks, you get $\iota^* : D^b(\mathcal{X}) \to D^b(\mathcal{U})$.

Remark 2.3 (General dictionary for quotient stacks and equivariant geometry)**.** There is a general dictionary relating the stack-theoretic concepts and the equivariant geometry of X. Here G is a reductive algebraic group acting on a scheme X and $[X/G]$ is the quotient stack.

Geometry of $[X/G]$	G**-equivariant geometry of** X
$\mathbb{C}$-point $\bar{x} \in [X/G]$	orbit Gx of $\mathbb{C}$-point $x \in X$ (with $\bar{x}$ the image of x under $X \to [X/G]$)
automorphism group $\mathrm{Aut}(\bar{x})$	stabilizer G_x
function $f \in \Gamma([X/G], \mathcal{O}_{[X/G]})$	G-equivariant function $f \in \Gamma(X, \mathcal{O}_X)^G$
map $[X/G] \to Y$ to a scheme Y	G-equivariant map $X \to Y$
line bundle	G-equivariant line bundle (or G-linearization)
quasi-coherent sheaf	G-equivariant quasi-coherent sheaf
tangent space $T_{[X/G],\bar{x}}$	normal space $T_{X,x}/T_{Gx,x}$ to the orbit
coarse moduli space $[X/G] \to Y$	geometric quotient $X \to Y$
good moduli space $[X/G] \to Y$	good GIT quotient $X \to Y$

The unstable locus for the negative character is the set $\{y_1 = y_2 = 0\} \subset V$. Consider the Koszul resolution of the associated sky-scraper sheaf:

$$K_- = \mathcal{O}(2) \xrightarrow{(y_2,-y_1)} \mathcal{O}(1)^{\oplus 2} \xrightarrow{(y_1,y_2)} \mathcal{O}.$$

Then $\iota_- K_-$ is exact, it is the pull-up of the Euler sequence from $\mathbb{P}^1_{y_1:y_2}$. On the other hand $\iota_+ K_-$ is a resolution of the sky-scraper sheaf $\mathcal{O}_{\mathbb{P}^1_{x_1:x_2}}$ along the zero section. Similar comments apply for the Koszul resolution K_+ of the set $\{x_1 = x_2 = 0\}$.

Let

$$\mathcal{G}_t \subset D^b(\mathcal{X})$$

be the triangulated subcategory generated by the line bundles $\mathcal{O}(t)$ and $\mathcal{O}(t+1)$. This is the smallest thick triangulated subcategory generated by these two objects. This is the **grade restriction rule** of Hori-Herbst-Page, which informally says if you restrict this window to either quotient $X^\pm$, you recover the derived category $D^b(X^\pm)$.

Claim 2.4. For any $t \in \mathbb{Z}$, both ι_+^* and ι_-^* restrict to give equivalences

$$D^b(X_+) \xleftarrow{\sim} \mathcal{G}_t \xrightarrow{\sim} D^b(X_-).$$

Proof. The restriction functors

$$\iota_\pm^* : \ D^b(\mathcal{X}) \longrightarrow D^b(X^\pm)$$

are exact and preserve shifts and cones. To prove that the restrictions

$$\iota_\pm^* : \ \mathcal{G}_t \xrightarrow{\sim} D^b(X^\pm)$$

are equivalences, we need:

1. Fully faithfulness: On $\mathcal{G}_t$, the restriction maps induce isomorphisms

$$\mathrm{Hom}_{D^b(\mathcal{X})}(E,F) \cong \mathrm{Hom}_{D^b(X^\pm)}(\iota_\pm^* E, \iota_\pm^* F)$$

Since $\mathcal{G}_t$ is generated by $\{\mathcal{O}(t), \mathcal{O}(t+1)\}$, it suffices to check this on these generators. Concretely, we need to compute $\mathrm{Ext}^\bullet_{\mathcal{X}}(\mathcal{O}(t+k), \mathcal{O}(t+l))$ for $k,l \in \{0,1\}$, and show it matches the Ext groups in $X^\pm$.

2. Essential surjectivity: Every object in $D^b(X^\pm)$ should be built out of $\iota_\pm^* \mathcal{G}_t$. In other words, the images of $\mathcal{O}(t), \mathcal{O}(t+1)$ generate $D^b(X^\pm)$.

To see that these functors are fully-faithful it suffices to check what they do to the maps between the generating line-bundles, so we just need to check that

$$\mathrm{Ext}^\bullet_{\mathcal{X}}(\mathcal{O}(t+k), \mathcal{O}(t+l)) = \mathrm{Ext}^\bullet_{X_\pm}(\mathcal{O}(t+k), \mathcal{O}(t+l))$$

for $k,l \in [0,1]$. For line bundles, $\mathrm{Ext}^\bullet(\mathcal{O}(a), \mathcal{O}(b)) \cong H^\bullet(\cdot\,,\ \mathcal{O}(b-a))$. Thus we need to verify that $H^\bullet_{\mathcal{X}}(\mathcal{O}(i)) = H^\bullet_{X_\pm}(\mathcal{O}(i))$ for $i \in [-1,1]$.

$\mathcal{X}$ is an affine quotient stack (with V affine), so for any equivariant coherent sheaf, higher cohomology on $\mathcal{X}$ vanishes; taking global sections means "equivariant global sections" on V. Hence $H^p(\mathcal{X}, \mathcal{O}(i)) = (\mathcal{O}_V)_i$ for $p=0$ and 0 for $p>0$, where $(\mathcal{O}_V)_i$ is the weight-i subspace of the polynomial ring $\mathcal{O}_V = \mathbb{C}[V]$.

To compute $H^\bullet(X_\pm, \mathcal{O}(i))$, we use the projection $\pi : X_\pm \to \mathbb{P}^1$ and the fact that $X_\pm$ is the total space of the bundle $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{P}^1$. We do the computation for X^+; the case of X^- is similar. Let $\pi : X^+ \to \mathbb{P}^1$ be the projection and $E = \mathcal{O}(-1)^{\oplus 2}$. Then

$$\pi_* \mathcal{O}_{X^+} \cong \mathrm{Sym}^\bullet(E^\vee) = \mathrm{Sym}^\bullet(\mathcal{O}(1)^{\oplus 2}) \cong \bigoplus_{m \geq 0} \mathrm{Sym}^m(\mathcal{O}(1)^{\oplus 2}) \cong \bigoplus_{m\geq 0} \mathcal{O}(m)^{\oplus(m+1)}.$$

Remark 2.5. Recall that the total space of a vector bundle $E \to X$ is $\underline{\mathrm{Spec}}_X(\mathrm{Sym}^\bullet(E^\vee))$ where we take the relative Spec over X. Associated to any sheaf of algebras $\mathcal{A}$ over a base scheme B is the relative Spec, which is a scheme $Y, \mathcal{O}_Y$ equipped with a morphism $\pi : Y \to B$. It has the property that $\pi_* \mathcal{O}_Y = \mathcal{A}$ and $\pi : Y \to B$ is affine. In our case, the sheaf of algebras is $\mathrm{Sym}^\bullet(E^\vee)$, which is the symmetric algebra on the dual bundle $E^\vee$.

This means that if locally on B we have $E \cong \mathcal{O}_B^{\oplus r}$ is trivial of rank r, then $\underline{\mathrm{Spec}}_B(\mathrm{Sym}^\bullet(E^\vee))$ means we glue together the affine scheme $\mathrm{Spec}(\mathcal{O}_B[t_1, \ldots, t_r])$ fiberwise over B. Thus

$$\mathrm{Sym}^\bullet(E^\vee) \cong \mathcal{O}_B[t_1, \ldots, t_r]$$

The last isomorphism above can be seen from the general fact that if L is a line bundle and V is a vector space, then $\mathrm{Sym}^m(L \otimes V) \cong L^{\otimes m} \otimes \mathrm{Sym}^m(V)$. Locally trivialize L. Then $\mathrm{Sym}^m(L \otimes V)$ is generated by monomials $(\ell \otimes v_1) \cdots (\ell \otimes v_m) = \ell^m \otimes (v_1 \cdots v_m)$, which shows the factorization.

By the projection formula and affineness of π

$$H^p(X^+, \mathcal{O}(k)) \cong H^p\Big(\mathbb{P}^1,\ \pi_*\mathcal{O}_{X^+} \otimes \mathcal{O}(k)\Big) \cong \bigoplus_{m\geq 0} H^p(\mathbb{P}^1,\ \mathcal{O}(k+m))^{\oplus(m+1)}.$$

Remark 2.6. Recall that for a morphism $\pi : X \to B$ and a sheaf F on X, there is a spectral sequence (Leray)

$$E_2^{p,q} = H^p(B, R^q\pi_* F) \Longrightarrow H^{p+q}(X, F)$$

Since π is affine, $R^p\pi_* = 0$ for $p > 0$. So in the Leray spectral sequence, all rows with $q > 0$ are zero. That means already on the E_2-page, only the bottom row q=0 survives. No differentials are possible, so $E_2 = E_\infty$. Thus

$$H^p(X, F) \cong H^p(B, \pi_* F)$$

Therefore we need to compute $\pi_*\mathcal{O}_{X^+} \otimes \mathcal{O}(k)$. The projection formula says: for any quasi-coherent sheaf F on X and any sheaf G on B, $\pi_*(F \otimes \pi^* G) \cong \pi_* F \otimes G$. Take $F = \mathcal{O}_X$ and $G = \mathcal{O}_B(k)$. Then: $\pi_*(\mathcal{O}_X \otimes \pi^*\mathcal{O}_B(k)) \cong \pi_*\mathcal{O}_X \otimes \mathcal{O}_B(k)$. But $\mathcal{O}_X \otimes \pi^*\mathcal{O}_B(k)$ is exactly $\mathcal{O}_X(k)$ so

$$\pi_*\mathcal{O}_X(k) \cong \pi_*\mathcal{O}_X \otimes \mathcal{O}_B(k)$$

Now use the standard $\mathbb{P}^1$ cohomology:

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1} & n \geq 0 \\ 0 & n < 0 \end{cases}, \qquad H^1(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} 0 & n \geq -1 \\ \mathbb{C}^{(-n-1)} & n \leq -2 \end{cases}.$$

So for $p = 0$, we get that

$$\begin{aligned} H^0(X^+, \mathcal{O}(k)) &\cong \bigoplus_{m\geq 0} H^0(\mathbb{P}^1, \mathcal{O}(k+m))^{\oplus(m+1)} \\ &\cong \mathrm{Sym}^{k+m}(\mathbb{C}^2_{x_1,x_2}) \otimes \mathrm{Sym}^m(\mathbb{C}^2_{y_1,y_2}) \end{aligned}$$

which is exactly the weight-k part of $\mathcal{O}_V = \mathbb{C}[x_1, x_2, y_1, y_2]$.

For $p = 1$, we have

$$H^1(X^+, \mathcal{O}(k)) \cong \bigoplus_{m\geq 0} H^1(\mathbb{P}^1, \mathcal{O}(k+m))^{\oplus(m+1)}$$

On $\mathbb{P}^1$, $H^1(\mathcal{O}(n)) = 0$ for $n \geq -1$. So if $k \geq -1$ (the window for the flop), then $k + m \geq -1$ for all $m \geq 0$, hence $H^1(X^+, \mathcal{O}(k)) = 0$. This matches the stack, where all higher H^p vanish because $[V/\mathbb{C}^*]$ is (relatively) affine. This proves fully faithfulness of ι_+^* on the window.

Remark 2.7. Note that if $k \leq -2$, then at least the $m = 0$ term contributes $H^1(\mathbb{P}^1, \mathcal{O}(k)) \neq 0$. Here H^1 on X^+ is nonzero, while on the stack it is zero - this is exactly where agreement fails outside the window.

To see that they are essentially surjective we need to know that the two given line bundles generate $D^b(X_\pm)$. That is, every object of $D^b(X^\pm)$ should be quasi-isomorphic to a complex built out of $\iota_\pm^* \mathcal{O}(t)$ and $\iota_\pm^* \mathcal{O}(t+1)$. $\square$

Remark 2.8. Essential surjectivity follows from a general theorem which says that on quasi-projective varieties, an ample line bundle and its twists generate the derived category. The intuition behind this statement is Serre's theorem which says that for any coherent sheaf $\mathcal{F}$, $\mathcal{F}(n)$ is globally generated for $n \gg 0$.

Pick an ample line bundle L on X. Serre vanishing gives, for $m \gg 0$ that $H^i(X, F \otimes L^{\otimes m}) = 0$ for all $i > 0$ and any coherent F, and $F \otimes L^{\otimes m}$ is globally generated. For m large, the evaluation map is surjective:

$$H^0(X, F(m)) \otimes \mathcal{O}_X \twoheadrightarrow F(m).$$

Twist down by L^{-m}:

$$H^0(X, F(m)) \otimes L^{-m} \twoheadrightarrow F.$$

So F is a quotient of a finite direct sum of a power of L^{-1}. Let $K_1 := \ker(1)$. Then K_1 is coherent. Apply Serre vanishing again to K_1: choose $m_1 \gg 0$ so that $K_1(m_1)$ is globally generated and $K_1(m_1))0$ has no higher cohomology. Again we get a surjection

$$H^0(X, K_1(m_1)) \otimes L^{-m_1} \twoheadrightarrow K_1,$$

with kernel K_2. Continuing this way and using Castelnuovo-Mumford regularity, you can choose $m, m_1, \dots$ so this iteration stops in at most $\dim X + 1$ steps, giving a finite resolution:

$$0 \to \bigoplus L^{-m_r} \to \cdots \to \bigoplus L^{-m_1} \to \bigoplus L^{-m} \to F \to 0.$$

Thus every coherent F has a finite resolution by direct sums of powers of L^{-1}. Passing to derived categories, this means the triangulated subcategory generated by the line bundles $\{L^{\otimes n} \mid n \in \mathbb{Z}\}$ contains every object of $D^b(\mathrm{Coh}(X))$.

It remains to see that on X_+, the two line bundles $\mathcal{O}(t)$ and $\mathcal{O}(t+1)$ generate all powers of $\mathcal{O}(1)$. This follows quickly from Beilinson's theorem on $\mathbb{P}^1$ as follows. The projection $p : X_+ \to \mathbb{P}^1$ is affine so $p_* : \mathrm{Coh}(X_+) \to \mathrm{Coh}(\mathbb{P}^1)$ is exact, and p^* gives an equivalence

$$\mathrm{Coh}(X_+) \simeq \mathrm{Coh}(\mathbb{P}^1, \mathrm{Sym}(E^\vee)).$$

So every coherent sheaf (or complex) on X_+ is a module over the quasi-coherent algebra $\mathrm{Sym}(E^\vee)$ on $\mathbb{P}^1$ where $X_+ = \underline{\mathrm{Spec}}_{\mathbb{P}^1}(\mathrm{Sym}(E^\vee))$. By Beilinson's theorem on $\mathbb{P}^1$, we have:

$$D^b(\mathbb{P}^1) = \langle \mathcal{O}_{\mathbb{P}^1}(t), \mathcal{O}_{\mathbb{P}^1}(t+1) \rangle.$$

That is, any bounded complex of coherent sheaves on $\mathbb{P}^1$ can be built out of just these two line bundles by taking cones, shifts, and summands. Note that $p^*\mathcal{O}_{\mathbb{P}^1}(t) = \mathcal{O}_{X_+}(t)$. Given $F \in D^b(\operatorname{Coh} X_+)$, you can write $p_*F \in D^b(\operatorname{Coh}\mathbb{P}^1)$ as a complex built from $\mathcal{O}_{\mathbb{P}^1}(t)$ and $\mathcal{O}_{\mathbb{P}^1}(t+1)$ by Beilinson. Applying p^* to that construction gives you a complex built from their pullbacks $\mathcal{O}_{X_+}(t)$ and $\mathcal{O}_{X_+}(t+1)$. Hence

$$D^b(X_+) = \langle \mathcal{O}_{X_+}(t),\ \mathcal{O}_{X_+}(t+1)\rangle.$$

So for any $t \in \mathbb{Z}$ we have a derived equivalence

$$\Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_-)$$

passing through $\mathcal{G}_t$. Composing these, we get auto-equivalences

$$\Phi_{t+1}^{-1}\Phi_t : D^b(X_+) \xrightarrow{\sim} D^b(X_+).$$

To see what these do, we need to check them on the generating set of line-bundles $\{\mathcal{O}(t), \mathcal{O}(t+1)\}$.

Remark 2.9. Φ_t identifies $D^b(X_+)$ and $D^b(X_-)$ through the common window $\mathcal{G}_t$. Thus:

$$\Phi_t(\mathcal{O}(t)) = \mathcal{O}(t)_{X_-}, \qquad \Phi_t(\mathcal{O}(t+1)) = \mathcal{O}(t+1)_{X_-}.$$

So Φ_t just sends the line bundles to the same ones on the other phase. Now, when we apply Φ_{t+1}^{-1} (the inverse equivalence for the next window) to these line bundles on X_-, we have to interpret them as objects of the new window $\mathcal{G}_{t+1} = \langle \mathcal{O}(t+1), \mathcal{O}(t+2)\rangle$.

But $\mathcal{O}(t)$ is not in that window. So we must rewrite $\mathcal{O}(t)$ in terms of $\mathcal{O}(t+1)$ and $\mathcal{O}(t+2)$. Consider the Koszul resolution resolving the structure sheaf of the unstable locus $\{y_1 = y_2 = 0\}$ on the stack $\mathcal{X} = [V/\mathbb{C}^*]$:

$$0 \to \mathcal{O}_V(2) \xrightarrow{(y_2,-y_1)} \mathcal{O}_V(1)^{\oplus 2} \xrightarrow{(y_1,y_2)} \mathcal{O}_V \to \mathcal{O}_V/\{y_1 = y_2 = 0\} \to 0$$

Restricting to X^+, this resolution restricts to a resolution of the structure sheaf of the zero section $\Sigma = \mathbb{P}^1_{x_1:x_2} \subset X^+$, i.e. the subvariety $\{y_1 = y_2 = 0\}$ inside X^+ where $\mathcal{O}_{X^+}(k)$ denotes the restriction of $\mathcal{O}_V(k)$ to X^+:

$$0 \to \mathcal{O}_{X^+}(k+2) \to \mathcal{O}_{X^+}(k+1)^{\oplus 2} \to \mathcal{O}_{X^+}(k) \to \mathcal{O}_\Sigma(k) \to 0$$

Restricting the same sequence to X^-, the resolution becomes exact at the end since the unstable locus $\{y_1 = y_2 = 0\}$ is removed in X^-. Thus on X^- we have a quasi-isomorphism:

$$\mathcal{O}_{X^-}(k) \simeq \left[\mathcal{O}_{X^-}(k+2) \xrightarrow{(y_2,-y_1)} \mathcal{O}_{X^-}(k+1)^{\oplus 2}\right]$$

Thus we have shown that

$$\begin{aligned}
\Phi_{t+1}^{-1}\Phi_t(\mathcal{O}(t)) &\simeq \Phi_{t+1}^{-1}(\mathcal{O}(t)_{X_-}) \\
&\simeq \Phi_{t+1}^{-1}\Big(\big[\mathcal{O}(t+2)_{X_-} \xrightarrow{(y_2,-y_1)} \mathcal{O}(t+1)_{X_-}^{\oplus 2}\big]\Big) \\
&\simeq \big[\mathcal{O}(t+2) \xrightarrow{(y_2,-y_1)} \mathcal{O}(t+1)^{\oplus 2}\big], \\
\Phi_{t+1}^{-1}\Phi_t(\mathcal{O}(t+1)) &\simeq \Phi_{t+1}^{-1}(\mathcal{O}(t+1)_{X_-}) \\
&\simeq \mathcal{O}(t+1).
\end{aligned}$$

This autoequivalence $\Phi_{t+1}^{-1}\Phi_t$ is an example of a spherical twist.

Definition 2.10. A **spherical twist** is an autoequivalence discovered by [**?**] associated to any spherical object in the derived category, i.e. an object S such that

$$\mathrm{Ext}(S,S) = \mathbb{C} \oplus \mathbb{C}[-n]$$

for some n (i.e. the homology of the n-sphere). It sends any object $\mathcal{E}$ to the cone on the evaluation map

$$\big[\,\mathrm{RHom}(S,\mathcal{E}) \otimes S \;\longrightarrow\; \mathcal{E}\big].$$

The inverse twist sends $\mathcal{E}$ to the cone on the dual evaluation map

$$\big[\mathcal{E} \;\longrightarrow\; \mathrm{RHom}(\mathcal{E},S)^\vee \otimes S\big].$$

Claim 2.11. The object $\mathcal{O}_{\mathbb{P}^1_{x_1:x_2}}(t) \simeq \iota_+ K_-(t)$ is spherical for the derived category $D^b(X_+)$, and the inverse twist around it sends $\mathcal{O}(t+1)$ to itself and $\mathcal{O}(t)$ to the cone

$$\big[\mathcal{O}(t) \longrightarrow \iota_+ K_-(t)\big] \simeq \big[\mathcal{O}(t+2) \xrightarrow{(-y_2,y_1)} \mathcal{O}(t+1)^{\oplus 2}\big],$$

which agrees with $\Phi_{t+1}^{-1}\Phi_t$.

Remark 2.12. Let $\Sigma = \mathbb{P}^1_{x_1:x_2} \subset X_+$ be the zero section. Then $\mathcal{O}_\Sigma(t)$ is supported on a 1-dimensional subvariety, and we will show that it is spherical, i.e. that

$$\mathrm{Ext}^i_{X_+}(\mathcal{O}_\Sigma(t),\mathcal{O}_\Sigma(t)) \cong H^i(\Sigma,\mathcal{O}_\Sigma) \oplus H^{i-2}(\Sigma,\mathcal{O}_\Sigma) \cong \begin{cases} \mathbb{C} & i=0, \\ \mathbb{C} & i=2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $i : \Sigma \hookrightarrow X_+$ be the zero section. Then $\mathcal{O}_\Sigma(t) = i_*\mathcal{O}_\Sigma(t)$. We need to compute

$$\mathrm{Ext}^i_{X_+}(i_*\mathcal{O}_\Sigma(t), i_*\mathcal{O}_\Sigma(t)).$$

For a regular embedding $i : \Sigma \hookrightarrow X_+$ of codimension 2 there is a well-known identity (Proposition 3.3):

$$\operatorname{Ext}^i_{X_+}(i_*F, i_*G) \cong \bigoplus_{p=0}^{2} \operatorname{Ext}^{i-p}_{\Sigma}(F, G \otimes \wedge^p N_{\Sigma/X_+}).$$

The normal bundle of a zero section in the total space of a vector bundle $E \to B$ is canonically identified with E itself:

$$N_{\Sigma/X_+} = \mathcal{O}_\Sigma(-1)^{\oplus 2}, \quad \wedge^0 N = \mathcal{O}, \quad \wedge^1 N = \mathcal{O}(-1)^{\oplus 2}, \quad \wedge^2 N = \mathcal{O}(-2).$$

Therefore,

$$\operatorname{Ext}^i_{X_+}(\mathcal{O}_\Sigma(t), \mathcal{O}_\Sigma(t)) \cong H^i(\Sigma, \mathcal{O}_\Sigma) \oplus H^{i-1}(\Sigma, \mathcal{O}_\Sigma(-1))^{\oplus 2} \oplus H^{i-2}(\Sigma, \mathcal{O}_\Sigma(-2)).$$

Now we can compute these cohomology groups on $\Sigma = \mathbb{P}^1$ using:

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1}, & n \geq 0 \\ 0, & n < 0 \end{cases}, \quad H^1(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} 0, & n \geq -1 \\ \mathbb{C}^{-n-1}, & n \leq -2 \end{cases}$$

Substituting into our formula above, we see that the only nonzero contributions occur at $i = 0$ from $H^0(\mathcal{O}) \cong \mathbb{C}$, and at $i = 2$ from $H^0(\mathcal{O}(-2))[2]$ shifting to degree 2 via the $i-2$ term.

Remark 2.13. We now want to see what the inverse spherical twist T_S^{-1} does to the generators $\mathcal{O}(t)$ and $\mathcal{O}(t+1)$ where $S = \mathcal{O}_\Sigma(t)$ is our spherical object.

For $\mathcal{O}(t+1)$, we have $R\operatorname{Hom}(\mathcal{O}(t+1), S) = \operatorname{Hom}_{\mathsf{Coh}(X_+)}(\mathcal{O}_{X_+}(t+1), \mathcal{O}_\Sigma(t))$ because we are dealing with sheaves both sitting in degree zero. But f any sheaf E on X_+, we have

$$\operatorname{Hom}_{X_+}(E, i_*F) \cong \operatorname{Hom}_\Sigma(i^*E, F)$$

because i_* is fully faithful on the abelian subcategory of sheaves supported on Σ. Thus

$$\operatorname{Hom}_{X_+}\big(\mathcal{O}(t+1), i_*\mathcal{O}_\Sigma(t)\big) \cong \operatorname{Hom}_\Sigma\big(i^*\mathcal{O}(t+1), \mathcal{O}_\Sigma(t)\big).$$

The restriction of $\mathcal{O}(t+1)$ to the zero section is $i^*\mathcal{O}(t+1) = \mathcal{O}_\Sigma(t+1)$, since $\mathcal{O}(k)$ on X_+ is pulled back from the base Σ with the same twisting character. So we see that

$$\operatorname{Hom}_\Sigma\big(\mathcal{O}_\Sigma(t+1), \mathcal{O}_\Sigma(t)\big) = H^0\big(\Sigma, \mathcal{O}_\Sigma(t-(t+1))\big) = H^0\big(\Sigma, \mathcal{O}_\Sigma(-1)\big) = 0$$

For $E = \mathcal{O}_{X_+}(t)$, we have a nontrivial morphism $\mathcal{O}_{X_+}(t) \to S = \mathcal{O}_\Sigma(t)$, namely the restriction map to the zero section. By the inverse twist definition:

$$T_S^{-1}(E) = \operatorname{Cone}\big(E \xrightarrow{\text{coev}} R\operatorname{Hom}(E, S)^\vee \otimes S\big)[-1].$$

$$R\mathrm{Hom}(E,S)^\vee \otimes S \simeq \bigoplus_i \mathrm{Ext}^i(E,S)^\vee \otimes S[-i].$$

$$\mathrm{Ext}^\bullet(E,S) = \mathrm{Ext}^\bullet_{X_+}((t), i_{*\Sigma}(t)) \cong H^\bullet(\Sigma,(1)) = \mathbb{C}^2[-1].$$

But $R\,\mathrm{Hom}(E,S)^\vee \otimes S$ is just a rank-2 copy of $\mathcal{O}_\Sigma(t)$ (corresponding to the two fiber coordinates y_1, y_2), and the map $E \to \mathcal{O}_\Sigma(t)^{\oplus 2}$ is encoded by the same pair (y_1, y_2). Hence

$$T_S^{-1}(\mathcal{O}(t)) \simeq [\mathcal{O}(t+2) \xrightarrow{(-y_2,y_1)} \mathcal{O}(t+1)^{\oplus 2}].$$

That's the same two-term complex we found from the Koszul resolution earlier

$$T_S^{-1}(\mathcal{O}(t)) = \Phi_{t+1}^{-1}\Phi_t(\mathcal{O}(t))$$

The wall-crossing functor $\Phi_t : D^b(X_+) \to D^b(X_-)$ was constructed via the window $\mathcal{G}_t$, and shifting the window by one changes how the same line bundle is expressed:

$$\mathcal{O}(t) \mapsto [\mathcal{O}(t+2) \to \mathcal{O}(t+1)^{\oplus 2}].$$

$X_+ = \mathrm{Tot}_\Sigma(\mathcal{O}_\Sigma(-1)^{\oplus 2})$, $\Sigma = \mathbb{P}^1$, $i : \Sigma \hookrightarrow X_+$ the zero section, and $S := i_*\mathcal{O}_\Sigma(t)$ is the spherical object. On the stack $\mathcal{X} = [V/\mathbb{C}^*]$ with weights $(+1,+1,-1,-1)$ on (x_1,x_2,y_1,y_2), the zero section is cut out by the regular sequence y_1, y_2. Let $E := \mathcal{O}(t)$.

On $\mathcal{X}$ there is the Koszul exact sequence resolving the structure sheaf of the zero section:

$$0 \to \mathcal{O}(t+2) \xrightarrow{(y_2,-y_1)} \mathcal{O}(t+1)^{\oplus 2} \xrightarrow{(y_1,y_2)} \mathcal{O}(t) \to \mathcal{O}_\Sigma(t) \to 0$$

Passing to $D^b(\mathcal{X})$, this gives a boundary morphism $\delta : \mathcal{O}(t) \to \mathcal{O}_\Sigma(t)[1]$, which is the extension class of the short exact sequence. In other words, δ is a canonical element of $\mathrm{Ext}^1_{\mathcal{X}}(\mathcal{O}(t), \mathcal{O}_\Sigma(t))$.

The conormal along the zero section is spanned (equivariantly) by the weight $+1$ functions y_1, y_2. Concretely, the Koszul class δ is obtained by contracting with (y_1, y_2); the space of such extension classes is exactly the weight-$+1$ piece of the conormal:

$$\mathrm{Ext}^1_{\mathcal{X}}(\mathcal{O}(t), \mathcal{O}_\Sigma(t)) \cong (H^0(\Sigma, \mathcal{O}_\Sigma(1)))_{\text{fiber coords}} \cong \mathbb{C}^2$$

with a basis represented by the two fiber coordinates y_1, y_2. If you compute without the $\mathbb{C}^*$-grading, you see a larger space; the window/equivariant grading cuts it down to the two degree-$+1$ generators coming from y_1, y_2.

Equivalently: take $R\,\mathrm{Hom}_{\mathcal{X}}(\mathcal{O}(t), \mathcal{O}_\Sigma(t))$. In degrees, the only nonzero piece inside the chosen window sits in degree 1, and it is spanned by y_1, y_2. Thus:

$$R\,\mathrm{Hom}_{\mathcal{X}}(\mathcal{O}(t), \mathcal{O}_\Sigma(t)) \cong \mathbb{C}^2[-1], \quad \text{so} \quad R\,\mathrm{Hom}_{\mathcal{X}}(\mathcal{O}(t), S)^\vee \cong \mathbb{C}^2[+1]$$

The inverse spherical twist is:

$$T_S^{-1}(E) = \operatorname{Cone}(E \xrightarrow{\text{coev}} R\operatorname{Hom}(E,S)^\vee \otimes S)[-1]$$

With $R\operatorname{Hom}(E,S)^\vee \cong \mathbb{C}^2$ (in the relevant degree), this becomes $E \xrightarrow{(y_1,y_2)} S^{\oplus 2}$. Here "$(y_1, y_2)$" means precisely that the two basis elements of $R\operatorname{Hom}(E,S)^\vee$ are the classes corresponding to the two fiber coordinates; evaluating them gives the two components of the coevaluation.

Unwinding the cone gives exactly the familiar two-term complex on X_+ (after restricting the window equivalence):

$$T_S^{-1}(\mathcal{O}(t)) \simeq [\mathcal{O}(t+2) \xrightarrow{(-y_2,y_1)} \mathcal{O}(t+1)^{\oplus 2}]$$

because this cone is just the truncation of the Koszul resolution governed by the same pair (y_1, y_2).

To complete the proof of the claim we would just need to check that the two functors also agree on the Hom-sets between $\mathcal{O}(t)$ and $\mathcal{O}(t+1)$.

Now instead let $V = \mathbb{C}^{p+q}$ with co-ordinates $x_1, \ldots, x_p, y_1, \ldots, y_q$. Let $\mathbb{C}^*$ act linearly on V with positive weights on each x_i and negative weights on each y_i. The two GIT quotients X_+ and X_- are both the total spaces of orbi-vector bundles over weighted projective spaces.

We must assume the Calabi-Yau condition that $\mathbb{C}^*$ acts through $SL(V)$. Let d be the sum of the positive weights, so the sum of the negative weights is $-d$. The above argument goes through word-for-word, where now

$$\mathcal{G}_t = \langle \mathcal{O}(t), \ldots, \mathcal{O}(t+d-1) \rangle.$$

3 Appendix: Algebraic Geometry

We collect some definitions and facts from algebraic geometry that are used in the main text.

3.1 Cohomology and affineness

If $X = \operatorname{Spec} A$ is an affine scheme, then every quasi-coherent sheaf $\mathcal{F}$ on X has no higher cohomology

$$H^p(X, \mathcal{F}) = 0 \quad \text{for } p > 0.$$

This is because quasi-coherent sheaves on affine schemes correspond to A-modules, and taking global sections corresponds to taking the module itself, which is an exact functor. In general, whenever a quasiseparated scheme X has an open cover by affine schemes U_i, the Čech complex associated to this cover can be used to compute the cohomology of quasi-coherent sheaves on X. In particular, if X can be covered by m open affine sets then

$$H^p(X, \mathcal{F}) = 0 \quad \text{for } p \geq m.$$

It turns out that the vanishing of higher cohomology for all quasi-coherent sheaves characterizes affineness. This is known as Serre's affineness criterion.

Theorem 3.1 (Serre's affineness criterion)**.** Let X be a scheme. Assume that

1. X is quasi-compact, and
2. for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we have $H^1(X, \mathcal{I}) = 0$.

Then X is affine.

Proof. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x. Write $U = \operatorname{Spec}(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. There are quasi-coherent sheaves of ideals $\mathcal{I}, \mathcal{I}'$ cutting out the reduced closed subschemes Z and Z' respectively. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}' \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}/\mathcal{I}' \longrightarrow 0.$$

Since x is a closed point of X and $x \notin Z$, we see that $\mathcal{I}/\mathcal{I}'$ is supported at x. In fact, the restriction of $\mathcal{I}/\mathcal{I}'$ to U corresponds to the A-module $A/\mathfrak{m}$. Hence

$$\Gamma(X, \mathcal{I}/\mathcal{I}') = A/\mathfrak{m}.$$

Since by assumption $H^1(X, \mathcal{I}') = 0$, there exists a global section $f \in \Gamma(X, \mathcal{I})$ mapping to the element $1 \in A/\mathfrak{m}$ as a section of $\mathcal{I}/\mathcal{I}'$.

Let $X_f = D_X(f)$ be the open subset of X where f is invertible. Since the image of f in $A/\mathfrak{m}$ equals 1, we have $f(x) \notin \mathfrak{m}_x$, equivalently, f is invertible in the local ring $\mathcal{O}_{X,x}$ and so $x \in X_f$.

Moreover $X_f \subset U$ because on $Z = X \setminus U$, the section sheaf $\mathcal{I}$ vanishes because it cuts out Z. So $f|_Z = 0$, and hence f is not invertible on Z. Thus $X_f \subset U$. This clearly implies that $X_f = D(f_A)$ where f_A is the image of f in A.

Consider the union

$$W = \bigcup_{f \in \Gamma(X, \mathcal{O}_X)} X_f$$

over all f such that X_f is affine. Obviously W is open in X. By the arguments above, every closed point of X is contained in W. The closed subset $X \setminus W$ of X is also quasi-compact and so it has a closed point if it is nonempty. This would contradict the fact that all closed points are in W. Hence we conclude $X = W$.

Choose finitely many $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that

$$X = X_{f_1} \cup \cdots \cup X_{f_n},$$

and such that each X_{f_i} is affine. The finite cover above exists because X is quasi-compact. First we argue that it suffices to show that $f_1, \ldots, f_n$ generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$.

Suppose $X = \bigcup_i X_{f_i}$ and each X_{f_i} affine, and $(f_1, \ldots, f_n) = \Gamma(X, \mathcal{O}_X)$. Let $A := \Gamma(X, \mathcal{O}_X)$ and let $\varphi : X \to \operatorname{Spec} A$ be the canonical map. For any $f \in A$, $\varphi^{-1}(D(f)) = X_f$.

If $(f_1, \ldots, f_n) = A$, then $\{D(f_i)\}$ covers $\operatorname{Spec} A$. Since $\{X_{f_i}\}$ covers X and each X_{f_i} is affine, the restrictions $A_{f_i} \to \Gamma(X_{f_i}, \mathcal{O}_X)$ are isomorphisms and they agree on overlaps $X_{f_i f_j}$ (compatibility comes from functoriality of restriction). Therefore φ is an isomorphism Zariski-locally on the cover $\{X_{f_i}\}$ and on the target cover $\{D(f_i)\}$. Since these cover X and $\operatorname{Spec} A$, φ is an isomorphism globally. Hence $X \simeq \operatorname{Spec} A$ is affine.

Now we show that $f_1, \ldots, f_n$ generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus n} \xrightarrow{(f_1, \ldots, f_n)} \mathcal{O}_X \longrightarrow 0.$$

The arrow defined by $f_1, \ldots, f_n$ is surjective since the opens X_{f_i} cover X. Let $\mathcal{F}$ be the kernel of this surjective map. Observe that $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

such that each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to a quasi-coherent sheaf of ideals. Namely, we can take $\mathcal{F}_i$ to be the intersection of $\mathcal{F}$ with the first i direct summands of $\mathcal{O}_X^{\oplus n}$. The assumption of the lemma implies that $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for all i. This implies $H^1(X, \mathcal{F}_2) = 0$, because it is sandwiched between $H^1(X, \mathcal{F}_1)$ and $H^1(X, \mathcal{F}_2/\mathcal{F}_1)$. Continuing in this way, we deduce that $H^1(X, \mathcal{F}) = 0$. Therefore, we conclude that the map

$$\bigoplus_{i=1}^{n} \Gamma(X, \mathcal{O}_X) \xrightarrow{(f_1, \ldots, f_n)} \Gamma(X, \mathcal{O}_X)$$

is surjective, as desired. $\square$

The statement can actually be upgraded to a relative affineness criterion. Recall that a morphism of schemes $f : X \to Y$ is **affine** if for every affine open subset $V \subset Y$, the preimage $f^{-1}(V)$ is an affine scheme. Equivalently, f is affine if and only if the direct image sheaf $f_* \mathcal{O}_X$ is a quasi-coherent sheaf of algebras on Y and X is isomorphic to the relative Spec $\underline{\operatorname{Spec}}_Y(f_* \mathcal{O}_X)$.

Theorem 3.2 (Relative affineness criterion)**.** Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. Then the following are equivalent:

1. The morphism f is affine.

2. For every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, we have $R^1 f_* \mathcal{I} = 0$.

3.2 Koszul resolutions and Ext along a closed immersion

Let $i : Z \hookrightarrow X$ be a closed immersion of smooth varieties of codimension c. If $N_{Z/X}$ denotes the normal bundle, then for any coherent sheaves F, G on Z, there is a natural isomorphism

$$\mathrm{Ext}^i_X(i_*F,\, i_*G) \;\cong\; \bigoplus_{p=0}^{c} \mathrm{Ext}^{i-p}_Z\left(F,\, G \otimes \wedge^p N_{Z/X}\right).$$

We check this Zariski locally. Assume $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/I$ where $I = (f_1, \ldots, f_c)$ is a regular sequence since Z is a smooth subvariety of codimension c. The conormal module is I/I^2, and $N^\vee \simeq I/I^2$, so $N \simeq (I/I^2)^\vee$. The Koszul complex $K(f_\bullet)$ is a free A-resolution of A/I:

$$0 \to \wedge^c A^{\oplus c} \xrightarrow{d} \cdots \xrightarrow{d} A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} A \to A/I \to 0.$$

If F, G are coherent on Z (i.e. A/I-modules), then i_*F, i_*G are the same modules regarded as A-modules with I acting trivially.

We want to compute

$$\mathrm{Ext}^i_A(i_*F, i_*G).$$

First we need to resolve i_*F by a free A-resolution using the Koszul complex. The Koszul complex for $f_1, \ldots, f_c$ is:

$$K(f_\bullet): \qquad 0 \to \wedge^c A^c \xrightarrow{d_c} \cdots \xrightarrow{d_1} A \to 0,$$

where d_p acts by contraction with $f_1 e_1 + \cdots + f_c e_c$.

Tensor it with i_*F (which is killed by I):

$$K(f_\bullet) \otimes_A i_*F: \quad 0 \to i_*F \otimes \wedge^c A^c \to \cdots \to i_*F \to 0.$$

This is a projective resolution of i_*F as an A-module. Now we apply $\mathrm{Hom}_A(-,\, i_*G)$.

Compute the cochain complex:

$$\mathrm{Hom}_A(K(f_\bullet) \otimes i_*F,\; i_*G).$$

whose p-th term is

$$\mathrm{Hom}_A(i_*F \otimes \wedge^p A^c,\; i_*G) \cong \mathrm{Hom}_{A/I}\left(F,\; G \otimes (\wedge^p A^c)^\vee\right)$$

because I acts trivially on both sides, so we can reduce mod I. Here $(\wedge^p A^c)^\vee \cong \wedge^p (A^c)^\vee$, which geometrically is $\wedge^p N_{Z/X}$.

So we have constructed a cochain complex $C^\bullet$ with terms

$$C^p = \mathrm{Hom}_{A/I}\left(F,\, G \otimes \wedge^p N\right), \qquad N = (I/I^2)^\vee.$$

The differential d in the Koszul complex $K(f_\bullet) : \wedge^p A^c \to \wedge^{p-1} A^c$ induces, after applying Hom, a map $d^* : C^{p-1} \to C^p$. Now $d_p \otimes 1$ itself is "multiplication by the f_i" acting on the $\wedge^p A^c$-factor. But both i_*F and i_*G are annihilated by $I = (f_1, \ldots, f_c)$, so multiplying by any f_i on their modules gives zero. Hence $d_p \otimes 1$ is zero after applying $\mathrm{Hom}_A(-, i_*G)$ and so in fact this differential d^* is zero.

So the complex $C^\bullet$ has zero differential, i.e. it is just a direct sum of its terms:

$$C^\bullet \cong \bigoplus_{p=0}^{c} C^p[-p]$$

Replacing F, G by injective (or projective) resolutions over A/I, you can promote this chain-level equality to an equality of derived objects:

$$R\,\mathrm{Hom}_A(i_*F, i_*G) \;\simeq\; \bigoplus_{p=0}^{c} R\,\mathrm{Hom}_{A/I}\big(F,\, G \otimes \wedge^p N\big)[-p].$$

Taking H^i of both sides gives the desired formula:

Proposition 3.3 (Ext along a closed immersion)**.** Let $i : Z \hookrightarrow X$ be a closed immersion of smooth varieties of codimension c, and let $N_{Z/X}$ be the normal bundle. For any coherent sheaves F, G on Z, there is a natural isomorphism

$$\mathrm{Ext}^i_X(i_*F, i_*G) \;\cong\; \bigoplus_{p=0}^{c} \mathrm{Ext}^{i-p}_Z\big(F,\, G \otimes \wedge^p N_{Z/X}\big).$$

Remark 3.4 (Spectral sequence version)**.** In general, each C^p can have its own internal derived functor $\mathrm{Ext}^q_{A/I}(F, G \otimes \wedge^p N)$ if we replace F or G by injective resolutions over A/I. Hence we really have a double complex

$$C^{p,q} = \mathrm{Ext}^q_{A/I}(F, G \otimes \wedge^p N),$$

with horizontal differential (Koszul) and vertical differential (Exts). There is a spectral sequence of a double complex:

$$E_1^{p,q} = \mathrm{Ext}^q_{A/I}(F,\, G \otimes \wedge^p N) \implies \mathrm{Ext}^{p+q}_A(i_*F, i_*G).$$

However, in our case the horizontal differential is zero, so the spectral sequence degenerates at E_1 and we get the direct sum formula above.

3.3 Quotient stack

Let $\mathcal{S}$ be a category and $p : \mathcal{X} \to \mathcal{S}$ be a functor of categories. We visualize this data as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \\ \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & b \arrow[d] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

where the lower case letters a, b are objects of $\mathcal{X}$ and the upper case letters S, T are objects of $\mathcal{S}$. We say that a is over S and that a morphism $\alpha : a \to b$ is over $f : S \to T$.

Definition 3.5 (Prestacks)**.** A functor $p : \mathcal{X} \to \mathcal{S}$ is a **prestack over a category** $\mathcal{S}$ if

(1) (**pullbacks exist**) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d] & b \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a morphism $a \to b$ over $S \to T$; and

(2) (**universal property for pullbacks**) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & b \arrow[r] \arrow[d] & c \arrow[d] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

Prestacks are also referred to as **categories fibered in groupoids**.

Definition 3.6 (Fiber categories)**.** If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the **fiber category** $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over S with morphisms over id_S.

Given an action of an algebraic group G on a scheme X, the **quotient prestack** $[X/G]^{\mathrm{pre}}$ is the prestack whose fiber category $[X/G]^{\mathrm{pre}}(S)$ over a scheme S is the quotient groupoid (or the moduli groupoid of orbits) $[X(S)/G(S)]$ This will not satisfy the gluing axioms of a stack; even when the action is free, the quotient functor $\mathrm{Sch} \to \mathrm{Sets}$ defined by $S \mapsto X(S)/G(S)$ is not a sheaf in general. Put another way, we define:

Definition 3.7 (Quotient prestacks)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. The **quotient prestack** $[U/G]^{\mathrm{pre}}$ of an action of a smooth affine group scheme $G \to S$ on an S-scheme U is the category over Sch/S consisting of pairs (T, u) where T is an S-scheme and $u \in U(T)$. An element $g \in G(T')$ acts by $(T', u') \to (T, u)$ via the data of a map $f : T' \to T$ of S-schemes and an element $g \in G(T')$ such that $f^* u = g \cdot u'$. Note that the fiber category $[U(T)/G(T)]$ is identified with the quotient groupoid.

It turns out that the stackification of $[U/G]^{\mathrm{pre}}$ is the quotient stack $[U/G]$, hence the name is justified.

Definition 3.8 (Quotient stacks)**.** The **quotient stack** $[U/G]$ is the prestack over Sch/S consisting of diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of S-schemes.

A morphism

$$(T' \longleftarrow P' \longrightarrow U) \longrightarrow (T \longleftarrow P \longrightarrow U)$$

consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and the left square is cartesian.

> A stack over a site $\mathcal{S}$ is a prestack $\mathcal{X}$ where the objects and morphisms glue uniquely in the Grothendieck topology of $\mathcal{S}$.

Definition 3.9 (Stack)**.** A **stack** $\mathcal{X}$ over a site $\mathcal{C}$ is a prestack over $\mathcal{C}$ satisfying the following descent conditions:

- (Descent for morphisms) For any $U \in \mathcal{C}$, any covering $\{f_i : U_i \to U\}$, and any $x, y \in \mathcal{X}(U)$, the presheaf
$$\underline{\mathrm{Hom}}(x, y) : (V \to U) \mapsto \mathrm{Hom}_{\mathcal{X}(V)}(f^*x, f^*y)$$
is a sheaf on $\mathcal{C}/U$.

- (Descent for objects) For any $U \in \mathcal{C}$, any covering $\{f_i : U_i \to U\}$, and any descent datum (x_i, ϕ_{ij}) relative to $\{f_i : U_i \to U\}$, there exists an object $x \in \mathcal{X}(U)$ and isomorphisms $\psi_i : f_i^* x \xrightarrow{\sim} x_i$ such that $\phi_{ij} \circ f_j^* \psi_j = f_i^* \psi_i$.

Definition 3.10 (Substack)**.** A **substack** $\mathcal{Y} \subseteq \mathcal{X}$ is given by:

- For each $U \in \mathcal{C}$, a full subcategory $\mathcal{Y}(U) \subseteq \mathcal{X}(U)$.

- Stability under restriction: If $y \in \mathcal{Y}(U)$ and $f : V \to U$ is a morphism in the site, then the pullback $f^*y \in \mathcal{X}(V)$ must lie in $\mathcal{Y}(V)$.

- Stack condition: The collection $\mathcal{Y}$ is itself a stack (i.e. satisfies descent for objects and morphisms).

Definition 3.11 (Open and closed substacks)**.** A substack $\mathcal{T} \subseteq \mathcal{X}$ of a stack over $\mathrm{Sch}_{\text{ét}}$ is called an **open substack** (resp. **closed substack**) if the inclusion $\mathcal{T} \to \mathcal{X}$ is representable by schemes and an open immersion (resp. closed immersion).

4 Derived categories

In this appendix we collect some definitions and facts about derived categories. We prove the classical reconstruction theorem of Bondal-Orlov for varieties with ample or anti-ample canonical bundle.

4.1 Basic definitions

Let $\mathcal{A}$ be an abelian category. The derived category $D(\mathcal{A})$ is constructed in several steps. Consider the category $C(\mathcal{A})$ of complexes in $\mathcal{A}$, whose objects are cochain complexes and morphisms are chain maps that commute with the differentials.

Form the homotopy category $K(\mathcal{A})$ whose objects are the same as $C(\mathcal{A})$.

The morphisms are chain maps modulo homotopy equivalence. Two chain maps

$$f, g : A^\bullet \to B^\bullet$$

are homotopic if there exist morphisms $h^i : A^i \to B^{i-1}$ such that

$$f^i - g^i = d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i$$

It is a routine check that two maps which are homotopic induce the same map on cohomology.

Finally form $D(\mathcal{A})$ by formally inverting all quasi-isomorphisms in $K(\mathcal{A})$. The morphisms in $D(\mathcal{A})$ are a little subtle. For example, one cannot just introduce formal inverses to quasi-isomorphisms. If X is not an injective object in $\mathcal{A}$, then the inclusion map $X[0] \to I^\bullet$ into an injective resolution is a quasi-isomorphism. If we formally invert by introducing $p : I^\bullet \to X[0]$ with

$$\begin{aligned} [p] \circ [i] &= [\mathrm{id}_{X[0]}] \quad \text{in } K(\mathcal{A}) \\ [i] \circ [p] &= [\mathrm{id}_{I^\bullet}] \quad \text{in } K(\mathcal{A}) \end{aligned}$$

then by definition, we impose that i, p are homotopy equivalences. This is too strong, since not every quasi-isomorphism is a homotopy equivalence.

Abstractly, let S be the set of quasi-isomorphisms in $K(\mathcal{A})$. The derived category

$$D(\mathcal{A}) = K(\mathcal{A})[S^{-1}]$$

is characterized by a universal property: there is a functor

$$Q : K(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

sending every $s \in S$ to an isomorphism, and universal with that property (any other functor inverting all quasi-isomorphisms factors uniquely through Q). One can also describe morphisms in $D(\mathcal{A})$ concretely as "roofs" via Verdier localization. The bounded derived category $D^b(\mathcal{A})$ is the full subcategory of complexes with bounded cohomology.

Definition 4.1 (Mapping cone)**.** For a chain map $s : X^\bullet \to I^\bullet$ (cohomological grading), the **mapping cone** $\mathrm{Cone}(s)$ is the complex

$$\mathrm{Cone}(s)^n = I^n \oplus X^{n+1}, \qquad d(b,a) = \big(d_I b + s(a),\ -d_X a\big).$$

There's a short exact sequence of complexes

$$0 \to I^\bullet \xrightarrow{\iota} \mathrm{Cone}(s) \xrightarrow{\pi} X^\bullet[1] \to 0,$$

giving rise to a long exact sequence in cohomology

$$\cdots \to H^n(I^\bullet) \xrightarrow{H^n(\iota)} H^n(\mathrm{Cone}(s)) \xrightarrow{H^n(\pi)} H^{n+1}(X^\bullet) \xrightarrow{H^{n+1}(s)} H^{n+1}(I^\bullet) \to \cdots$$

Proposition 4.2. Let $s : X^\bullet \to I^\bullet$ be a chain map in $C(\mathcal{A})$. Then:

1. s is a quasi-isomorphism if and only if $\mathrm{Cone}(s)$ is acyclic (all cohomology groups vanish).
2. s is an isomorphism in $K(\mathcal{A})$ (i.e., a homotopy equivalence) if and only if $\mathrm{Cone}(s)$ is contractible (chain-homotopic to 0).

Proof.

1. ($\Rightarrow$) If s is a quasi-isomorphism, then each $H^{n+1}(s)$ is an isomorphism. In the exact segment

   $$H^n(I) \to H^n(\mathrm{Cone}(s)) \to H^{n+1}(X) \xrightarrow{H^{n+1}(s)} H^{n+1}(I),$$

   the image of $H^n(\mathrm{Cone}(s)) \to H^{n+1}(X)$ is $\ker H^{n+1}(s) = 0$, so $H^n(I) \to H^n(\mathrm{Cone}(s))$ is surjective. Looking one step earlier,

   $$H^n(X) \xrightarrow{H^n(s)} H^n(I) \to H^n(\mathrm{Cone}(s)),$$

   the image of $H^n(s)$ is all of $H^n(I)$, so the map $H^n(I) \to H^n(\mathrm{Cone}(s))$ has zero kernel. Combining "surjective" and "zero kernel" forces $H^n(\mathrm{Cone}(s)) = 0$ for all n. So $\mathrm{Cone}(s)$ is acyclic.

   ($\Leftarrow$) If $\mathrm{Cone}(s)$ is acyclic, then $H^n(\mathrm{Cone}(s)) = 0$ for all n. The exact segment becomes

   $$0 \to H^{n+1}(X) \xrightarrow{H^{n+1}(s)} H^{n+1}(I) \to 0,$$

   so each $H^{n+1}(s)$ is an isomorphism. Hence s is a quasi-isomorphism.

2. If s has a homotopy inverse t (so $ts \simeq \mathrm{id}_X$, $st \simeq \mathrm{id}_I$), then the triangle

$$X^\bullet \xrightarrow{s} I^\bullet \to \mathrm{Cone}(s) \to X^\bullet[1]$$

is isomorphic (in K) to

$$X^\bullet \xrightarrow{\mathrm{id}} X^\bullet \to \mathrm{Cone}(\mathrm{id}_X) \to X^\bullet[1].$$

For any complex $X^\bullet$, $\mathrm{Cone}(\mathrm{id}_X)$ is contractible with contracting homotopy

$$H^n : X^n \oplus X^{n+1} \longrightarrow X^{n-1} \oplus X^n, \qquad H^n(x, y) = (0, x).$$

One can check that $dH + Hd = \mathrm{id}$. Thus $\mathrm{Cone}(s)$ is contractible.

□

Example 4.3. Let $\mathcal{A} = \mathbf{Ab}$. Take the injective resolution of $\mathbb{Z}$:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{q} \mathbb{Q}/\mathbb{Z} \to 0$$

and regard $I^\bullet$ as $I^0 = \mathbb{Q}$, $I^1 = \mathbb{Q}/\mathbb{Z}$ with $d^0 = q$, and $X^\bullet = \mathbb{Z}[0]$. The resolution map $s : \mathbb{Z}[0] \to I^\bullet$ has $s^0 = i$.

Compute the cone. By the definition above,

$$\mathrm{Cone}(s)^{-1} = \mathbb{Z}, \quad \mathrm{Cone}(s)^0 = \mathbb{Q}, \quad \mathrm{Cone}(s)^1 = \mathbb{Q}/\mathbb{Z}$$

with differentials $d^{-1} = i : \mathbb{Z} \to \mathbb{Q}$ and $d^0 = q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. So $\mathrm{Cone}(s)$ is exactly the three-term complex sitting in degrees $-1, 0, 1$.

$$\mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{q} \mathbb{Q}/\mathbb{Z}$$

The cone is acyclic: the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is exact, so the cone's cohomology vanishes. However, it is not contractible: contractibility of this 3-term exact complex is equivalent to the short exact sequence splitting (a contracting homotopy gives splittings and vice versa). But $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ does not split: if it did, $\mathbb{Z}$ would be a direct summand of the divisible group $\mathbb{Q}$, hence divisible itself, which is false.

Therefore s is a quasi-isomorphism whose cone is acyclic but not contractible; hence s is not a homotopy equivalence and cannot be inverted in $K(\mathcal{A})$.

Definition 4.4 (Triangulated category)**.** A **triangulated category** is an additive category $\mathcal{T}$ equipped with an autoequivalence $[1] : \mathcal{T} \to \mathcal{T}$ (the shift functor) and a class of distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

satisfying the following axioms:

- (TR1) For every morphism $f : X \to Y$ in $\mathcal{T}$, there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1].$$

  Moreover, for every object $X \in \mathcal{T}$, the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$$

  is distinguished, and any triangle isomorphic to a distinguished triangle is distinguished.

- (TR2) A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

  is distinguished if and only if the rotated triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

  is distinguished.

- (TR3) Given two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

  and

$$U \xrightarrow{p} V \xrightarrow{q} W \xrightarrow{r} U[1],$$

  and morphisms $a : X \to U$, $b : Y \to V$ such that $b \circ f = p \circ a$, there exists a morphism $c : Z \to W$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"'] & Z \arrow[r, "h"] \arrow[d, "c"'] & X[1] \arrow[d, "{a[1]}"] \\
U \arrow[r, "p"'] & V \arrow[r, "q"'] & W \arrow[r, "r"'] & U[1]
\end{tikzcd}$$

- (TR4) (Octahedral axiom) Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{T}$, there exist distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} C(f) \xrightarrow{v} X[1],$$

$$Y \xrightarrow{g} Z \xrightarrow{u'} C(g) \xrightarrow{v'} Y[1],$$

  and

$$X \xrightarrow{g \circ f} Z \xrightarrow{u''} C(g \circ f) \xrightarrow{v''} X[1],$$

along with morphisms $C(f) \xrightarrow{w} C(g \circ f)$ and $C(g) \xrightarrow{w'} C(g \circ f)$ such that the following diagram commutes and the rows and columns are distinguished triangles:

$$\begin{tikzcd}
& Y \arrow[r, "u"] \arrow[dd, "g"'] & C(f) \arrow[dd, "w"] \\
X \arrow[ur, "f"] \arrow[dr, "g\circ f"'] & & \\
& Z \arrow[r, "u'"'] & C(g)
\end{tikzcd}$$

Proposition 4.5. This construction gives $D(\mathcal{A})$ the structure of a triangulated category, where:

- The shift functor $[1]$ moves complexes one place to the left:

$$X^{\bullet}[1]^n = X^{n+1}, \quad d^n_{X[1]} = -d^{n+1}_X$$

- Distinguished triangles come from mapping cones of chain maps.

- The cohomology functors are first defined on the homotopy category as functors

$$H^i_K : K(\mathcal{A}) \to \mathcal{A}$$

  Since these functors send quasi-isomorphisms to isomorphisms, they descend through the localization map $Q : K(\mathcal{A}) \to D(\mathcal{A})$.

  That is, there exists a unique functor $H^i_D : D(\mathcal{A}) \to \mathcal{A}$ such that $H^i_K = H^i_D \circ Q$.

For X a scheme, we write $D^b(X)$ for the bounded derived category of coherent sheaves on X.

5 References

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