Complex Manifolds

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Abstract

These notes accompany Math 241 (Complex Manifolds), taught by Professor Constantin Teleman, Fall 2025, UC Berkeley. They present a self-contained treatment of the elementary theory of Riemann surfaces functions and differentials, line bundles, Riemann-Roch, Abel-Jacobi maps, and theta functions. This is followed by an introduction to higher-dimensional complex manifolds: holomorphic vector bundles, sheaves, and Dolbeault cohomology. The exposition culminates in the central results for Kähler geometry, including Hodge decomposition, the Hard Lefschetz theorem, Serre duality for vector bundles, and Kodaira vanishing and embedding theorems.

Contents

1 Riemann Surfaces

Introduction here.

1.1 Structure theory

Definition 1.1 (Holomorphic function in two variables)**.** A complex function $F(z, w)$ defined in an open set in $\mathbb{C}^2$ is called **holomorphic** if, near each point (z_0, w_0) in its domain, F admits a convergent power series expansion

$$F(z, w) = \sum_{m,n \geq 0} F_{mn}(z - z_0)^m (w - w_0)^n.$$

Definition 1.2 (Analytic Riemann surface in $\mathbb{C}^2$)**.** A subset $S \subseteq \mathbb{C}^2$ is called a **(concrete, possibly singular) Riemann surface** if, for each point $s \in S$, there exists a neighbourhood U of s and a holomorphic function F on U such that $S \cap U$ is the zero-set of F in U. Moreover, we require that

$$\frac{\partial^n F}{\partial w^n}(s) \neq 0 \quad \text{for some } n.$$

In particular, the continuity of F implies that S is locally closed. The condition $\partial^n F/\partial w^n(s) \neq 0$ rules out vertical lines through s, which cannot reasonably be viewed as graphs. (Indeed, from the power series expansion we see that $S \cap U$ contains a vertical line precisely when $F_{0n} = 0$ for all n.)

Definition 1.3 (Non-singular point)**.** The Riemann surface S is called **non-singular** at $s \in S$ if a function F defining S near s can be found with gradient vector $(\partial F/\partial z,\, \partial F/\partial w)$ nonzero at s.

Definition 1.4. An **abstract Riemann surface** is a topological surface R equipped with a maximal atlas of charts to open subsets of $\mathbb{C}$ such that the transition functions are holomorphic.

Every non-singular concrete Riemann surface is an abstract Riemann surface in a natural way, by using the implicit function theorem to produce local charts. However, the converse fails for the obvious reason that compact Riemann surfaces cannot be embedded in $\mathbb{C}^2$ as concrete Riemann surfaces.

Definition 1.5. Let R and S be Riemann surfaces. A map $f : R \to S$ is called **holomorphic** if for every holomorphic function $g : U \to \mathbb{C}$ defined on an open subset $U \subseteq S$, the composition $g \circ f : f^{-1}(U) \to \mathbb{C}$ is a holomorphic function on R.

Theorem 1.6 (Local normal form and ramification index)**.** Let $f : R \to S$ be a nonconstant holomorphic map between Riemann surfaces, and let $p \in R$ with $q := f(p)$. Then there exist local holomorphic coordinates

$$z : (R, p) \xrightarrow{\sim} (\mathbb{C}, 0), \qquad w : (S, q) \xrightarrow{\sim} (\mathbb{C}, 0),$$

and an integer $k \geq 1$ such that

$$w \circ f \circ z^{-1}(\zeta) \;=\; \zeta^k$$

for all ζ in a neighbourhood of 0. The integer k is uniquely determined and is called the **ramification index** of f at p.

Proof. Choose local coordinates z near p and w near $q = f(p)$ with $z(p) = 0$, $w(q) = 0$, and consider the resulting holomorphic map

$$F := w \circ f \circ z^{-1} : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0).$$

Then F is holomorphic, $F(0) = 0$, and F is not constant.

Write the Taylor expansion of F at 0:

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Since $F(0) = 0$ and F is not constant, there is a smallest integer $k \geq 1$ such that $a_k \neq 0$. Thus we can factor

$$F(z) = z^k u(z),$$

where

$$u(z) := a_k + a_{k+1} z + a_{k+2} z^2 + \cdots$$

is holomorphic and satisfies $u(0) = a_k \neq 0$.

Shrink the coordinate disc so that $u(z)$ is never zero there. On a simply connected neighbourhood of 0 avoiding the zeros of u, a nowhere vanishing holomorphic function admits a holomorphic logarithm: there exists a holomorphic function h such that

$$u(z) = e^{h(z)}.$$

Define a holomorphic k-th root of u by

$$v(z) := u(z)^{1/k} := e^{h(z)/k}.$$

Then v is holomorphic, $v(0) \neq 0$, and $v(z)^k = u(z)$.

Now define a new local coordinate on the source by

$$\zeta := z\, v(z).$$

Since v is holomorphic and $v(0) \neq 0$, we have

$$\left.\frac{d\zeta}{dz}\right|_{z=0} = v(0) \neq 0,$$

so by the holomorphic inverse function theorem, ζ is a valid holomorphic coordinate near $z = 0$.

In terms of this new coordinate, we compute

$$F(z) = z^k u(z) = z^k v(z)^k = (zv(z))^k = \zeta^k.$$

Thus, in the coordinates ζ on R and w on S, the map f is given by

$$w = \zeta^k$$

near p. This proves the existence of the local normal form.

Uniqueness of k follows from the uniqueness of the order of vanishing: if $F(z) = z^k u(z)$ with $u(0) \neq 0$, then k is exactly the order of the zero of F at 0, which is independent of the choice of local coordinates. Hence k is well defined and is called the ramification index of f at p. □

Corollary 1.7. Let $f : R \to S$ be a nonconstant holomorphic map between Riemann surfaces. Then f is an open map: the image of every open set in R is open in S.

Proof. Let $U \subseteq R$ be an open set and let $q \in f(U)$. Choose $p \in U$ with $f(p) = q$. By the local normal form theorem, there exist local coordinates z near p and w near q such that $w \circ f \circ z^{-1}(\zeta) = \zeta^k$ for some $k \geq 1$. Since $z(U)$ is open in $\mathbb{C}$, it contains a small disc around 0. The image of this disc under $\zeta \mapsto \zeta^k$ is also a small disc around 0, which shows that $f(U)$ contains a neighbourhood of q. Thus $f(U)$ is open in S. □

Corollary 1.8. Every holomorphic function defined everywhere on a compact connected Riemann surface is constant.

Proof. Let $f : R \to \mathbb{C}$ be a holomorphic function on a compact connected Riemann surface R. Since R is compact, the image $f(R)$ is a compact subset of $\mathbb{C}$. The only compact open subsets of $\mathbb{C}$ are the empty set and $\mathbb{C}$ itself. Since $f(R)$ is nonempty, it must be all of $\mathbb{C}$. However, $\mathbb{C}$ is not compact, so this is a contradiction unless $f(R)$ is a single point. Thus f is constant. □

Theorem 1.9. Let $f : R \to S$ be a nonconstant holomorphic map between compact Riemann surfaces. Pick $s \in S$ and define the quantity

$$d := \sum_{p \in f^{-1}(s)} e_p$$

Then d is independent of the choice of $s \in S$ and is called the **degree** of the map f.

Theorem 1.10 (Riemann–Hurwitz)**.** Let $f : R \to S$ be a nonconstant holomorphic map of compact Riemann surfaces of degree d. Write

$$B \;:=\; \sum_{p \in R} (e_p - 1)\, p$$

for the ramification divisor of f. Then

$$2g_R - 2 \;=\; d(2g_S - 2) + \deg(B),$$

or equivalently,

$$\chi(R) \;=\; d\,\chi(S) - \sum_{p \in R} (e_p - 1).$$

Proof via Euler characteristic. If f were an unramified covering map, then R would be a d-fold cover of S and by the description of Euler characteristic as the alternating sum of cells in a cell decomposition, we would have

$$\chi(R) = d\,\chi(S).$$

However, ramification changes this count. We have to delete $e_p - 1$ points of R for each ramification point p, and each deletion drops the Euler characteristic by 1. Thus we obtain the desired formula. □

Proof via canonical bundles. Let K_R and K_S denote the canonical bundles of R and S. Choose local coordinates w near $p \in R$ and z near $f(p)$ so that the map is written $z = w^{e_p}$. Then

$$f^*(dz) = d(w^{e_p}) = e_p\, w^{e_p - 1}\, dw,$$

so $f^*(dz)$ has a zero of order $e_p - 1$ at p. Globalizing, one obtains an isomorphism of line bundles

$$K_R \;\cong\; f^*K_S \otimes \mathcal{O}_R(B),$$

where $B = \sum(e_p - 1)p$ is the ramification divisor.

Taking degrees on both sides, and using $\deg K_X = 2g_X - 2$, yields

$$2g_R - 2 = d(2g_S - 2) + \sum_p (e_p - 1),$$

the desired identity. $\square$

Remark 1.11. These two proofs are related by the fact a holomorphic 1–form ω on a compact Riemann surface satisfies $\deg(\omega) = 2g - 2 = -\chi(X)$. In particular, there is the isomorphism of line bundles $K_R = f^*K_S + B$. which refines the Riemann–Hurwitz formula.

Exercise 1.12. Let $f(z)$ be a polynomial of odd degree, with simple zeroes. Identify the topology of the Riemann surface of the double-valued function defined by $w^2 = f(z)$.

Solution 1.13. Consider $X_{\text{aff}} = \{(z, w) \in \mathbb{C}^2 : w^2 = f(z)\}$. Its projection $\pi_{\text{aff}} : (z, w) \mapsto z$ is a 2-sheeted branched covering of $\mathbb{C}$ away from the zeros of f. We compactify to a projective curve $X = \overline{X_{\text{aff}}} \subset \mathbb{P}^1_z \times \mathbb{P}^1_w$ and extend the projection to $\pi : X \longrightarrow \mathbb{P}^1_z$. The map π has degree 2. To study the topology of X_{aff}, we will use the Riemann-Hurwitz formula to compute the genus of X and delete the point(s) over $z = \infty$.

If a is a simple zero of f, write locally $f(z) = (z - a)u(z)$ with $u(a) \neq 0$. Then $w^2 = (z - a)u(z)$ has a single point of X lying over $z = a$ and the local model is $w^2 = z - a$, so the ramification index is $e = 2$. Thus each simple zero gives one branch point of ramification index 2. There are d of these in $\mathbb{C}$. Put $t = 1/z$ as a coordinate near $z = \infty$ and write

$$f(z) = z^d\, g(1/z) = t^{-d}\, g(t), \qquad g(0) \neq 0$$

The equation becomes $w^2 = t^{-d}g(t) \iff (w\, t^{\frac{d-1}{2}})^2 = t^{-1}g(t)$. Let $u = w\, t^{\frac{d-1}{2}}$. Then $u^2 = t^{-1}g(t)$, so near $t = 0$ we have the model $u^2 \sim t^{-1}$. Therefore, there is one point of X over $z = \infty$ and it is ramified of order 2. Hence the total number of simple branch points is $B = d + 1$.

Apply Riemann-Hurwitz to the degree-2 map $\pi : X \to \mathbb{P}^1$:

$$2g(X) - 2 = 2 \cdot (-2) + \sum_{p \in X} (e_p - 1).$$

Every simple ramification contributes $e_p - 1 = 1$, so

$$2g(X) - 2 = -4 + B = -4 + (d + 1) = d - 3.$$

Therefore

$$g(X) = \frac{d-1}{2}.$$

The compact Riemann surface X is a closed orientable surface of genus $g = \frac{d-1}{2}$. Recall that there is only one point of X over $z = \infty$. Therefore, X_{aff} is homeomorphic to X with one point removed. Hence X_{aff} is homeomorphic to a genus $\frac{d-1}{2}$ surface with one puncture.

1.2 Weierstrass theorems

The suprising fact about holomorphic functions in several variables is that they behave like polynomials in many respects. The following theorems establish that analytic sets have algebraic-like local structure and are important in estbalishing general factorization, dimension, flatness, and coherence theorems.

Theorem 1.14 (Weierstrass Division Theorem)**.** Given a polynomial

$$P(w, z_1, \ldots, z_n) = w^n + \sum_{k=0}^{n-1} p_k(z) w^k,$$

with the functions $p_k(z)$ holomorphic in an open set $V \subset \mathbb{C}^n$ and satisfying $p_k(0) = 0$, every germ of holomorphic function $G(w, z)$ near $(w, z) = (0, 0)$ can be uniquely expressed as

$$G(w, z) = P(w, z) \cdot Q(w, z) + R(w, z),$$

where $Q(w, z)$ is a holomorphic germ near 0 and $R(w, z)$ is a polynomial in w of degree $< n$ with coefficients germs of holomorphic functions in z near $z = 0$.

Proof. Using Cauchy's integral formula we write

$$G(w, z) = \frac{1}{2\pi i} \oint_\Gamma \frac{G(\zeta, z)}{\zeta - w} d\zeta$$

from which we can write

$$R(w, z) = \frac{1}{2\pi i} \oint_\Gamma \frac{G(\zeta, z) P(\zeta, z)}{P(\zeta, z)(\zeta - w)} - \frac{G(\zeta, z) P(w, z)}{P(\zeta, z)(\zeta - w)} d\zeta$$

Now write

$$P(\zeta, z) - P(w, z) = (\zeta^k - w^k) + \sum_{i=1}^{k-1} p_k(z)(\zeta^i - w^i)$$

which is divisible by $\zeta - w$, and the quotient

$$\frac{P(\zeta, z) - P(w, z)}{\zeta - w}$$

is a polynomial in w of degree $n - 1$. If for a fixed value of z, we pick a contour Γ in the w plane for which $P(\zeta, z)$ does not vanish on Γ, then the function R is holomorphic in w and z because it is the contour integral of an integrand, holomorphic in both w and z. Since R is holomorphic, we may differentiate n times with respect to w under the integral and we find that the integrand becomes zero, since the integrand is a polynomial in w of degree $n - 1$. Therefore,

$$\frac{d^n R}{dw^n} = 0$$

so R is indeed polynomial of degree $n - 1$. $\square$

Theorem 1.15 (Weierstrass Preparation Theorem)**.** Let F be holomorphic in the variables $z = (z_1, \ldots, z_k)$ and w near the origin in $\mathbb{C}^{k+1}$ and let n be the smallest integer such that

$$\frac{\partial^n F}{\partial w^n}(0, 0) \neq 0.$$

Then:

(weak form) There exists a function Φ of the form

$$\Phi(z, w) = w^n + f_{n-1}(z) w^{n-1} + \cdots + f_1(z) w + f_0(z),$$

with $f_0, \ldots, f_{n-1}$ analytic near $z = 0$, such that the zero–set of $\Phi(z, w)$ agrees with that of $F(z, w)$ near $(0, 0)$.

(strong form) There exists, in addition, a holomorphic function $u(z, w)$, nonzero near $(0, 0)$, such that

$$F(z, w) \;=\; \Phi(z, w)\, u(z, w).$$

Moreover, this factorization of F is unique.

Example 1.16. The ring of germs of holomorphic functions on a complex manifold X can be identified with the ring of convergent power series in $\dim X$ local coordinates. The Weierstrass theorems show that the *local* analytic rings

$$\mathcal{O}_{X,p} \;\cong\; \mathbb{C}\{z_1, \ldots, z_n\}$$

are Noetherian and factorial (UFD). By contrast, the ring of *global* holomorphic functions on a domain is typically very far from either property.

1. For example, let $\mathcal{O}(\mathbb{C})$ denote the ring of entire functions. For each $m \geq 1$ define the ideal

$$I_m := \{\, f \in \mathcal{O}(\mathbb{C}) : f(1/k) = 0 \text{ for all } 1 \leq k \leq m \,\}.$$

   Then I_m consists of entire functions vanishing at the finite set $\{1, \ldots, 1/m\}$, hence

$$I_m = \Big(\prod_{k=1}^{m} (z - 1/k) \Big),$$

   so each I_m is principal. Clearly

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

   is a strictly ascending chain of ideals (each new condition $f(1/(m+1)) = 0$ cuts out a strictly smaller ideal), and the chain never stabilizes. Thus $\mathcal{O}(\mathbb{C})$ is not Noetherian. Similar chains can be constructed in $\mathcal{O}(U)$ for many noncompact domains $U \subset \mathbb{C}^n$.

2. Even more dramatically, the ring $\mathcal{O}(\mathbb{C})$ is not a UFD as exhibited by the infinite product expansion of the sine function

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Theorem 1.17. The local ring of convergent power series

$$A_n := \mathbb{C}\{z_1, \ldots, z_n\}$$

is Noetherian for every $n \geq 1$.

Proof. For $n = 1$, the ring $\mathbb{C}\{z\}$ is a discrete valuation ring: every $f \neq 0$ can be written uniquely as $f(z) = z^k u(z)$ with $u(0) \neq 0$, so every ideal is generated by a single power of z, hence $\mathbb{C}\{z\}$ is Noetherian.

Assume inductively that A_{n-1} is Noetherian, and write

$$A_n = A_{n-1}\{w\}$$

with $w = z_n$. Let $I \subset A_n$ be an arbitrary ideal. Choose $f \in I$ such that its order in w is minimal among all elements of I. By the Weierstrass Preparation Theorem, after multiplying by a unit we may assume that f is a monic Weierstrass polynomial

$$P(w) = w^d + a_{d-1}(z')w^{d-1} + \cdots + a_0(z'), \qquad a_i(0) = 0,$$

lying in $A_{n-1}[w] \subset A_n$.

For any $g \in I$, the Weierstrass Division Theorem gives a unique expression

$$g = qP + r,$$

where $q \in A_n$ and the remainder has the form

$$r(z', w) = r_0(z') + r_1(z')w + \cdots + r_{d-1}(z')w^{d-1}, \qquad r_i \in A_{n-1}.$$

Thus

$$I = (P) + I',$$

where I' consists of those elements of I of w–degree $< d$. Define a map

$$\Phi : I' \longrightarrow A_{n-1}^d, \qquad r \longmapsto (r_0, \ldots, r_{d-1}).$$

The image $J := \Phi(I')$ is a submodule of the finitely generated A_{n-1}–module A_{n-1}^d. Since A_{n-1} is Noetherian, J is finitely generated, say by the vectors

$$(r_0^{(j)}, \ldots, r_{d-1}^{(j)}) \in A_{n-1}^d, \qquad j = 1, \ldots, N.$$

Lift each generator to an element

$$r^{(j)}(z', w) := r_0^{(j)}(z') + \cdots + r_{d-1}^{(j)}(z')w^{d-1} \in I'.$$

By construction, every element $r \in I'$ is an A_{n-1}–linear combination of the $r^{(j)}$, and hence also an A_n–linear combination. Thus

$$I = (P, r^{(1)}, \ldots, r^{(N)}),$$

so I is finitely generated. Since I was arbitrary, A_n is Noetherian. $\square$

Theorem 1.18. The ring $A_n = \mathbb{C}\{z_1, \ldots, z_n\}$ is a unique factorization domain.

Proof. We proceed by induction on n. For $n = 1$, the ring $\mathbb{C}\{z\}$ is a discrete valuation ring, hence a UFD.

Assume the result holds for A_{n-1}. Every nonunit $f \in A_n = A_{n-1}\{w\}$ can be made regular in w after a linear change of coordinates. By the Weierstrass Preparation Theorem, we may write

$$f = u \cdot P,$$

where u is a unit in A_n and P is a *monic Weierstrass polynomial*

$$P(w) = w^d + a_{d-1}(z')w^{d-1} + \cdots + a_0(z') \in A_{n-1}[w], \qquad a_i(0) = 0.$$

Thus factorization questions for A_n reduce to factorization questions for monic polynomials in $A_{n-1}[w]$.

Since A_{n-1} is a UFD by induction, the polynomial ring $A_{n-1}[w]$ is also a UFD. Hence every monic polynomial P admits a unique factorization

$$P = P_1 \cdots P_r$$

into monic irreducible polynomials $P_i \in A_{n-1}[w]$.

We claim that these irreducible factors P_i remain irreducible in A_n. Indeed, suppose $P_i = QR$ in A_n. By the Weierstrass Division Theorem, dividing Q by P_i in the variable w gives remainder zero (since P_i is monic), hence Q lies in $A_{n-1}[w]$. Likewise for R. Thus any factorization of P_i in A_n already occurs in $A_{n-1}[w]$, contradicting its irreducibility there. Hence P_i is irreducible in A_n, and being monic it is also prime.

Therefore every nonunit $f \in A_n$ factors as

$$f = u \cdot P_1 \cdots P_r,$$

with u a unit and the P_i pairwise nonassociate irreducible elements. Uniqueness of this factorization follows from the uniqueness of factorization in the UFD $A_{n-1}[w]$.

Thus A_n is a unique factorization domain. $\square$

Exercise 1.19. Let C_1 and C_2 be two circles in the w- and z-planes in $\mathbb{C}^2$, and $\Delta_{1,2}$ the disks that they bound. Show that a holomorphic function defined in an open set containing

$$C_1 \times \Delta_2 \;\cup\; \Delta_1 \times C_2$$

has a unique holomorphic extension over $\Delta_1 \times \Delta_2$.

Solution 1.20. Suppose f is holomorphic on a neighborhood of $X = (\partial\Delta \times \Delta) \,\cup\, (\Delta \times \partial\Delta) \subset \mathbb{C}^2$. For each fixed $z_2 \in \Delta$, the map $\zeta_1 \mapsto f(\zeta_1, z_2)$ is holomorphic on a neighborhood of $\partial\Delta$. Define

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_1|=1} \frac{f(\zeta_1, z_2)}{\zeta_1 - z_1}\, d\zeta_1, \qquad (z_1, z_2) \in \Delta \times \Delta.$$

F is holomorphic on $\Delta \times \Delta$. Moreover, on a neighborhood of $\partial\Delta \times \Delta$ (where f is defined in a full annulus in ζ_1), Cauchy's formula gives $F = f$. Similarly, for each fixed $z_1 \in \Delta$ the map $\zeta_2 \mapsto f(z_1, \zeta_2)$ is holomorphic near $\partial\Delta$. Define

$$G(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_2|=1} \frac{f(z_1, \zeta_2)}{\zeta_2 - z_2}\, d\zeta_2.$$

Then G is holomorphic on $\Delta \times \Delta$ and $G = f$ on a neighborhood of $\Delta \times \partial\Delta$. On a neighborhood of the torus $\partial\Delta \times \partial\Delta \subset X$, both representations are valid and equal f, hence $F = G$ there. By the identity theorem for holomorphic functions on $\Delta \times \Delta$, $F \equiv G$ on all of $\Delta \times \Delta$. Thus this common function extends f holomorphically to the full interior.

The extension is unique because if H were another holomorphic extension, then $H = F$ on $\partial\Delta \times \Delta$ by the identity theorem applied to the first variable, hence $H = F$ on all of $\Delta \times \Delta$ by the identity theorem applied to the second variable.

Exercise 1.21. Let F, G be two irreducible holomorphic functions in $n > 1$ variables defined on an open set U, and call their common zero-set Z. Using the Weierstrass Preparation Theorem (twice) and Q6, show that any holomorphic function defined on $U \setminus Z$ extends holomorphically over Z.

Remark 1.22. This is a version of *Hartogs' theorem* for holomorphic functions of several variables; somewhat loosely, the singular set of a holomorphic function defined on "most of" an open $U \subset \mathbb{C}^n$ cannot lie in an analytic subset of co-dimension 2, unless it's empty. Contrast that with the real function $1/(x^2 + y^2)$ on $\mathbb{R}^2$.

Solution 1.23. Let F, G be irreducible holomorphic functions on $U \subset \mathbb{C}^n$, $n > 1$, and set

$$Z = \{F = 0\} \cap \{G = 0\}$$

Fix $p \in Z$ and change coordinates so that $p = 0$, with F regular in w and G regular in z. By Weierstrass Preparation we may write $F = U_F \cdot P(w; z, t)$ and $G = U_G \cdot Q(z; w, t)$ where P is a Weierstrass polynomial in w, Q one in z, and t denotes the other coordinates. Thus for small polydisks, the zeros of F in w and of G in z form finite sets of roots varying holomorphically with the parameters. Choosing circles $C_1 = \{|w| = r_1\}$ and $C_2 = \{|z| = r_2\}$ that avoid these roots (uniformly in t), we see that $(C_1 \times \Delta_2) \cup (\Delta_1 \times C_2)$ is contained in $U \setminus Z$, so f is holomorphic there. By Exercise 1.19, f extends holomorphically to $\Delta_1 \times \Delta_2$ for each fixed t, and the extension depends holomorphically on t. Hence f extends to a neighborhood of p, and by uniqueness these local extensions glue to give a holomorphic extension of f to all of U.

1.3 Elliptic functions

The classical story begins with the Weierstrass $\wp$-function, defined by

$$\wp(z; L) = \frac{1}{z^2} + \sum_{\omega \in L \setminus \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)$$

which has the properties that it is an L-periodic meromorphic function on $\mathbb{C}$ with double poles at the lattice points, and that it satisfies the differential equation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 = 4(z - e_1)(z - e_2)(z - e_3)$$

where g_2, g_3 are constants depending on L, given explicitly by

$$g_2 = 60 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}$$

$$g_3 = 140 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}$$

and e_i are the values of $\wp$ at the half-lattice points $\omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2$. The e_i are distinct as we will show in Prop 1.25. The convergence is uniform on any compact subset $K \subset \mathbb{C}$, once the terms with poles in K are set aside.

Uniform convergence implies that the series can be differentiated term-by-term, so we get a formula for $\wp'(z)$ given by

$$\wp'(z) = -2 \sum_{\omega \in L} \frac{1}{(z-\omega)^3}$$

is an doubly periodic meromorphic function with triple poles at the lattice points. Moreover, one can see directly from the series expansion that $\wp$ is even and $\wp'$ is odd.

The oddness implies that $\wp'(z)$ vanishes at the half-lattice points. Moreover, one can check that these are simple zeros of $\wp'$, and moreover the only zeros of $\wp'$ modulo L. Thus $\wp'$ has only poles at lattice points, each of order 3. In a fundamental parallelogram there is exactly one pole (mod L), of total multiplicity 3. This implies the following proposition.

Proposition 1.24. $\wp(z)$ and $\wp'(z)$ define holomorphic maps $\mathbb{C}/L \to \mathbb{P}^1$ of degree 2 and 3 respectively.

We conclude that each of the half-lattice points must be a simple zero of $\wp'$ and moreover that these are all of the zeros, because any meromorphic function has divisor of degree 0.

Proposition 1.25 (Properties of the $\wp$-map)**.**

(i) The numbers e_1, e_2, e_3 are all distinct.

(ii) For any $a \in \mathbb{C}$ with $a \neq e_1, e_2, e_3$, the equation $\wp(u) = a$ has two simple roots in a fundamental period parallelogram. For the three exceptional values $a = e_i$, it has a single double root.

Proof.

(ii) General theory of meromorphic functions on a torus shows that we either have two simple roots or one double root. Since a double root corresponds to a zero of the derivative $\wp'$, the claim follows. Note that the two simple roots always differ by a sign modulo L, by the parity of $\wp$.

(i) Suppose, for contradiction, that $e_1 = e_2$. Then $\wp(u) = e_1$ would have a double root at $\frac{\omega_1}{2}$ and another double root at $\frac{\omega_2}{2}$. This would give too many roots (multiplicity 4 in a fundamental parallelogram), contradicting the fact that $\wp$ is a double covering of $\mathbb{P}^1$. Hence the e_i are distinct. $\square$

Need passage on theta functions here.

Theorem 1.26. Let $\theta_1, \ldots, \theta_4$ be the four Jacobi theta functions. Then there is a map

$$E/L \to \mathbb{CP}^3, \quad z \mapsto [\theta_1(z,\tau) : \theta_2(z,\tau) : \theta_3(z,\tau) : \theta_4(z,\tau)]$$

which is a smooth embedding of the complex torus $E = \mathbb{C}/L$ into projective space. It is a degree 4 map and its image is the intersection of two quadrics.

Proposition 1.27. The function $\wp : \mathbb{C}/L \to \mathbb{P}^1$ is a degree 2 holomorphic map with branch points over e_1, e_2, e_3, ∞.

We have seen the same picture of branching for the Riemann surface of the cubic equation

$$w^2 = (z - e_1)(z - e_2)(z - e_3);$$

We will use the $\wp$-function to prove the Unique Presentation by principal parts. Uniqueness being clear on general grounds (cf. Lecture 4), we merely need to prove the existence statement; and this will emerge from the proof of the first theorem below. Remarkably, this will also allow us to describe the field of meromorphic functions over $\mathbb{C}/L$.

Theorem 1.28. Every elliptic function is a rational function of $\wp$ and $\wp'$. Specifically, every **even** elliptic function is a rational function of $\wp$, every **odd** elliptic function is $\wp'$ times a rational function of $\wp$; and every elliptic function can be expressed uniquely as

$$f(u) = R_0(\wp(u)) + \wp'(u)\, R_1(\wp(u)),$$

with R_0, R_1 rational functions, where the two terms are the even and odd parts of f.

Proof. It suffices to prove the statement for even elliptic functions; division by $\wp'$ reduces odd ones to even ones. Recall that

$$\wp : \mathbb{C}/L \longrightarrow \mathbb{P}^1$$

is a degree 2 holomorphic map. This map realizes $\mathbb{P}^1$ as the quotient space of the torus $\mathbb{C}/L$ under the identification of u with $-u$. Certainly the map is surjective because general theory of holomorphic maps between compact Riemann surfaces shows that any nonconstant holomorphic map is surjective. The map is injective because $\wp(u) = \wp(v)$ if and only if $u \equiv \pm v \mod L$.

A bijective holomorphic map between compact Riemann surfaces is automatically biholomorphic. Let $f : R \to S$ be such a map. The inverse function theorem guarantees that the inverse function f^{-1} is smooth. Moreover, it guarantees that

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

Since f is bijective, it has nonzero derivative everywhere because if it did not, it would look like $z \mapsto z^k$ for some $k \geq 2$ and thus it would fail to be locally bijective. Since it has nonzero derivative everywhere, $(f^{-1})'$ is defined everywhere and is in fact a complex number. Hence f^{-1} is holomorphic. Δ and $\mathbb{C}$ are homeomorphic but they are not biholomorphic.

So indeed $\mathbb{P}^1$ is the quotient of $\mathbb{C}/L$ by the involution $u \mapsto -u$. Hence, any even **continuous** map

$$f : \mathbb{C}/L \to \mathbb{P}^1$$

has the form $f = R \circ \wp$, for some continuous map $R : \mathbb{P}^1 \to \mathbb{P}^1$. Moreover, $\wp$ is a local analytic isomorphism away from the four branch points, which implies that R is holomorphic there, if $R \circ \wp$ was so. So we know that R is continuous everywhere and holomorphic away from the four branch points.

The following result shows that R is holomorphic everywhere, hence a rational function.

$$R(z) = P(z)/Q(z) \implies f(u) = P(\wp(u))/Q(\wp(u))$$

Writing every elliptic function as a sum of an even and an odd one, and the odd ones as $\wp'$ times an even one, we get the desired result. $\square$

Proposition 1.29. Let $f : S \to R$ be a continuous map between Riemann surfaces, known to be holomorphic except at isolated points. Then f is holomorphic everywhere.

Proof. Choosing coordinate neighbourhoods near the questionable points and their images, we are reduced to the statement that a continuous function on Δ which is holomorphic on $\Delta^\times$ is, in fact, holomorphic at 0 as well. This follows from Riemann's theorem on removable singularities. $\square$

A remarkable consequence is that the function $\wp'(u)^2$, being elliptic and even, is expressible in terms of $\wp$. Explicitly, we have the following.

Theorem 1.30 (Differential equation for $\wp$)**.**

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3,$$

where $g_2 = 60G_4$, $g_3 = 140G_6$, and

$$G_r = G_r(L) = \sum_{\omega \in L^*} \omega^{-r}.$$

Proof. Recall the Laurent expansion of the Weierstrass function

$$\wp(u) = u^{-2} + 3G_4(L)u^2 + 5G_6(L)u^4 + \cdots , \qquad \wp'(u) = -2u^{-3} + 6G_4(L)u + 20G_6(L)u^3 + \cdots .$$

For $|u| < |\omega|$ and any integer $k \geq 1$,

$$(u-\omega)^{-k} = \frac{(-1)^k}{\omega^k}\left[1 + k\frac{u}{\omega} + \frac{k(k+1)}{2!}\frac{u^2}{\omega^2} + \frac{k(k+1)(k+2)}{3!}\frac{u^3}{\omega^3} + \cdots\right].$$

Expanding each term in the defining series for $\wp$ with the above, and (for small u) interchanging sums, the odd powers in u cancel, giving

$$\wp(u) = u^{-2} + \sum_{m=1}^{\infty}\binom{-2}{2m} G_{2m+2}(L)\, u^{2m} = u^{-2} + \sum_{m=1}^{\infty}(2m+1)\, G_{2m+2}(L)\, u^{2m}.$$

Similarly,

$$\wp'(u) = -2u^{-3} + \sum_{m=0}^{\infty}(-2)\binom{-3}{2m+1} G_{2m+4}(L)\, u^{2m+1}$$

Using these expansions, the first few terms of $(\wp'(u))^2$ and $4\wp(u)^3 - g_2\wp(u) - g_3$ agree at $u = 0$; hence their difference is an elliptic function with no poles that vanishes at $u = 0$, so it is identically zero. $\square$

The two theorems immediately lead to a description of the field of meromorphic functions on $\mathbb{C}/L$.

Corollary 1.31. The field of meromorphic functions on $\mathbb{C}/L$ is isomorphic to

$$\mathbb{C}(z)[w]/(w^2 - 4z^3 + g_2 z + g_3),$$

the degree 2 extension of the field of rational functions $\mathbb{C}(z)$ obtained by adjoining the solutions w to the equation

$$w^2 = 4z^3 - g_2 z - g_3.$$

Theorem 1.32. Let $z_1, \ldots, z_n$ and $p_1, \ldots, p_m$ denote the zeroes and poles of a non-constant elliptic function f in the period parallelogram, repeated according to multiplicity. Then:

(i) $m = n$,

(ii) $\displaystyle\sum_{k=1}^{m} \mathrm{Res}_{p_k}(f) = 0$,

(iii) $\displaystyle\sum_{k=1}^{n} z_k = \sum_{k=1}^{m} p_k \pmod{L}$.

Proof. (i) follows from the fact that

$$\frac{1}{2\pi i}\int_{\partial P} \frac{f'(z)}{f(z)}\,dz = n - m$$

where P is the period parallelogram. The integral is zero because opposite sides cancel out due to periodicity of f. (ii) follows from the fact that

$$\sum_k \mathrm{Res}_{p_k}(f) = \frac{1}{2\pi i}\int_{\partial P} f(z)\,dz$$

and the integral is zero by periodicity.

Let's integrate $z\frac{f'(z)}{f(z)}$ over the boundary ∂P:

$$I = \int_{\partial P} z\frac{f'(z)}{f(z)}\,dz$$

By the residue theorem:

$$I = 2\pi i \sum_k \mathrm{Res}_{z=z_k}(zf'/f) + 2\pi i \sum_k \mathrm{Res}_{z=p_k}(zf'/f)$$

At a zero z_k of order r, we have $\frac{f'(z)}{f(z)} \sim \frac{r}{z-z_k}$, so $\mathrm{Res}_{z_k}(zf'/f) = rz_k$. At a pole p_k of order s, we have $\frac{f'(z)}{f(z)} \sim -\frac{s}{z-p_k}$, so $\mathrm{Res}_{p_k}(zf'/f) = -sp_k$. Hence

$$\frac{I}{2\pi i} = \sum_{k=1}^{n} r_k z_k - \sum_{k=1}^{m} s_k p_k = \sum z_k - \sum p_k$$

using multiplicities.

Now consider how zf'/f behaves under translation by a period ω. When you shift $z \mapsto z + \omega_i$, f'/f stays the same (because it's periodic), but z changes by $+\omega_i$.

When traversing the parallelogram boundary, the two vertical sides differ by the period ω_1, and the two horizontal sides differ by ω_2. Careful calculation gives:

$$I = 2\pi i\,(\omega_1 k_1 + \omega_2 k_2)$$

for some integers k_1, k_2, because the integral around the boundary shifts by integer multiples of the lattice periods (this is the quasi-periodicity of the logarithm of f).

Thus

$$\sum z_k - \sum p_k \equiv 0 \pmod L$$

as desired. $\square$

Remark 1.33. Let ω be a meromorphic differential on a compact Riemann surface S. Then the sum of its residues at all poles is zero. Let p_i be the poles of ω and take disks D_i around each p_i. Then the boundary of $S \setminus \cup D_i$ is $\sum \partial D_i$. By Stokes' theorem,

$$\int_{S \setminus \cup D_i} d\omega = \int_{\partial(S \setminus \cup D_i)} \omega = \sum_i \int_{\partial D_i} \omega$$

The left side is zero because $d\omega = 0$ (since ω is a holomorphic 1-form). The right hand side is $2\pi i \sum \operatorname{Res}_{p_i}(\omega)$. Hence the sum of residues is zero.

Hartshorne actually delays the proof of the residue theorem until he has developed sheaf cohomology, because the general proof uses Serre duality. In the analytic setting, the proof is more elementary, as above.

Definition 1.34. Fix a local coordinate z at a point p. The **principal part** of a meromorphic function f at p is the part of its Laurent expansion in negative powers of $(z - p)$:

$$\sum_{n=1}^{N} a_{-n}(z-p)^{-n}$$

Theorem 1.35 (Unique Presentation by principal parts)**.** An elliptic function is specified uniquely, up to an additive constant, by prescribing its principal parts at all poles in the period parallelogram. The prescription is subject only to condition (ii).

Proof. This is more computational, but also more concrete. We first show that we can realize any even assignment of principal parts on $\mathbb{C}/L$ using a suitable rational function $R(\wp(u))$. Such an assignment involves finitely many points $\lambda \in \mathbb{C}/L$ and principal parts

$$\sum_{k=1}^{n_\lambda} a_k^{(\lambda)}(u-\lambda)^{-k},$$

with the properties that:

- if $2\lambda \notin L$, then $(-\lambda)$ also appears, with assignment

  $$\sum_{k=1}^{n_\lambda} (-1)^k a_k^{(\lambda)}(u+\lambda)^{-k},$$

  i.e. $a_k^{(-\lambda)} = (-1)^k a_k^{(\lambda)}$;

- if $2\lambda \in L$, then only even powers of $(u-\lambda)^{-1}$ are present.

This is because the local coordinates at λ and $-\lambda$ are opposite signs. Write the principal part at λ (using $v = u - \lambda$): $f(u) = \sum_{k=1}^{n_\lambda} a_k^{(\lambda)} v^{-k} + \cdots$. Near $-\lambda$ use $w = u + \lambda$. Evenness gives

$$f(-\lambda + w) = f\big(-(-\lambda + w)\big) = f(\lambda - w) = \sum_{k\geq 1} a_k^{(\lambda)}(-w)^{-k} = \sum_{k\geq 1}(-1)^k a_k^{(\lambda)}\, w^{-k}$$

If $2\lambda \in L$ (so $-\lambda \equiv \lambda$ on $\mathbb{C}/L$), the same calculation forces $\sum_{k\geq 1} a_k^{(\lambda)} v^{-k} = \sum_{k\geq 1} a_k^{(\lambda)}(-v)^{-k}$, hence $a_k^{(\lambda)} = 0$ for all odd k: only even powers $(u-\lambda)^{-2j}$ can appear.

Now if $2\lambda \notin L$, $(\wp(u) - \wp(\lambda))^{-1}$ has a simple pole at $u = \lambda$ and we can create any principal part there as a sum of $(\wp(u) - \wp(\lambda))^{-k}$. Evenness of $\wp$ takes care of the symmetry. If $2\lambda \in L$, then we can use either powers of $\wp$, if $\lambda \in L$, or powers of $(\wp(u) - e_{1,2,3})^{-1}$, which have double poles with no residue.

Now, onto the odd functions. Odd assignments of principal parts are of the form

$$\sum_{k=1}^{n_\lambda} a_k^{(\lambda)}(u-\lambda)^{-k},$$

with a matching term

$$-\sum_{k=1}^{n_\lambda}(-1)^k a_k^{(\lambda)}(u+\lambda)^{-k}$$

at $-\lambda$ (i.e. $a_k^{(-\lambda)} = (-1)^{k+1} a_k^{(\lambda)}$), or else with vanishing $a_k^{(\lambda)}$ (for even k) if $2\lambda \in L$.

The principal parts

$$\left(\frac{P_\lambda}{\wp'(u)} - \frac{P_{-\lambda}}{\wp'(u)}\right)$$

can be realized by a sum of powers of $(\wp(u) - \wp(\lambda))^{-1}$. If $2\lambda \in L$ but $\lambda \notin L$ (not 0), then $P_\lambda^{(u)}/\wp'(u)$ is also a well-defined even principal part, expressible via $(\wp(u) - \wp(\lambda))^{-1}$. The same goes for $P_0^{(u)}/\wp'(u)$. So there exists a function of the form $R_1(\wp(u))$ whose principal parts agree with the $P_\lambda(u)/\wp'(u)$ everywhere.

The principal parts of $R_1(\wp(u))\,\wp'(u)$ agree with the P_λ, except possibly at $\lambda = 0$, where the cubic pole of $\wp'$ could introduce unwanted or incorrect u^{-3} and u^{-1} terms. We can adjust the u^{-3} term by shifting R_1 by a constant. We have no control over the u^{-1} term, but that is determined from the condition $\sum \mathrm{Res} = 0$, which indeed must be met if a function with the prescribed principal parts is to exist. $\square$

Theorem 1.36 (Unique Presentation by zeroes and poles)**.** An elliptic function is specified uniquely, up to a multiplicative constant, by prescribing the location of its zeroes and poles in the period parallelogram, with multiplicities. The prescription is subject to conditions (i) and (iii).

Lemma 1.37. $g_2^3 \neq 27g_3^2$ and e_1, e_2, e_3 are the roots of the equation

$$4z^3 - g_2 z - g_3 = 0.$$

Proof. $\wp'$ vanishes at the half-lattice points, while $\wp$ takes the values e_1, e_2, e_3 there. The roots are distinct so the discriminant of the cubic is nonzero, i.e. $g_2^3 \neq 27g_3^2$. $\square$

Theorem 1.38 (Geometric interpretation)**.** The map $\mathbb{C}/L \setminus \{0\} \to \mathbb{C}^2$ given by

$$u \longmapsto (z(u), w(u)) = (\wp(u), \wp'(u))$$

gives an analytic isomorphism between the Riemann surface $\mathbb{C}/L \setminus \{0\}$ and the (concrete) Riemann surface R of the function

$$w^2 = 4z^3 - g_2 z - g_3$$

in $\mathbb{C}^2$.

Proof. We have the commutative diagram:

$$\begin{tikzcd} \mathbb{C}/L \setminus \{0\} \arrow[r, "(\wp,\wp')"] \arrow[dr, "\wp"'] & R \arrow[d, "\pi"] \\ & \mathbb{C} \end{tikzcd}$$

and we know that:

- π is proper and 2-to-1 except at the branch points e_1, e_2, e_3, which are the roots of $4z^3 - g_2 z - g_3$.
- $\wp$ is proper and 2-to-1 except at the half-period points $\omega_1/2, \omega_2/2, \omega_1/2 + \omega_2/2$, which map to the roots e_1, e_2, e_3.
- $\wp(u) = \wp(-u)$ and $\wp'(u) = -\wp'(-u)$: this means that, unless u is a half-period, $\wp'$ takes both values $\pm w = \pm\wp'(u)$ at the two points $\pm u$ mapping to the same $z = \wp(u)$ of $\mathbb{C}$.

Together, these three properties show that the map we just constructed is bijective. Note further that, at no point $u \in \mathbb{C}/L \setminus \{0\}$, is $\wp'(u) = \wp''(u) = 0$, because $\wp'$ has simple zeros only (there are three of them); this means that for every $u \in \mathbb{C}/L \setminus \{0\}$, either the map $\wp$ or the map $\wp'$ gives an analytic isomorphism of a neighbourhood of u with a small disc in the z-plane or in the w-plane.

Since the Riemann surface structure on the (concrete, non-singular) Riemann surface R is defined by the projections to the z- and w-planes, appropriately, we conclude that $(\wp, \wp')$ gives an analytic isomorphism

$$\mathbb{C}/L \longrightarrow R.$$

$\square$

1.4 Riemann surfaces and field extensions

This section is devoted to explaining the equivalence of three categories.

Theorem 1.39. The following categories are equivalent:

- Compact Riemann surfaces with nonconstant holomorphic maps
- Smooth proper (and hence projective) algebraic curves over $\mathbb{C}$ with nonconstant morphisms
- Field extensions of $\mathbb{C}$ of transcendence degree 1, of finite degree over $\mathbb{C}(t)$ where t is transcendental over $\mathbb{C}$, with field homomorphisms over $\mathbb{C}$

The correspondence in one direction is:

$$\begin{aligned} \text{Riemann surface } S &\mapsto \text{ function field } \mathbb{C}(S) \\ \text{Holomorphic map } f : S \to S' &\mapsto \text{ field homomorphism } f^* : \mathbb{C}(S') \to \mathbb{C}(S) \end{aligned}$$

Remark 1.40. For curves, smooth and proper implies projective. This is false in higher dimensions.

Common to both is the construction of nonconstant meromorphic functions. It suffices to find

- A map $f : R \to \mathbb{P}^1$ which realizes R as a branched cover of $\mathbb{P}^1$ (the transcendental part of the function field)

$$\begin{aligned} f^* : \mathbb{C}(z) &\hookrightarrow \mathbb{C}(R) \\ z &\mapsto f \end{aligned}$$

- A nonconstant meromorphic function g on S which separates the sheets (the finite part of the function field)

Once you have these functions, consider the set of pairs $\{(f(p), g(p)) : p \in S\} \subset \mathbb{P}^1 \times \mathbb{P}^1$. This is an analytic curve. By a theorem of Riemann (or later by Chow's theorem), an analytic curve in projective space is algebraic. So there exists a nonzero polynomial $P(x, y)$ such that

$$P(f, g) = 0 \quad \text{on } S.$$

Thus, the image of S under (f, g) is contained in the algebraic curve $P(x, y) = 0$. Moreover, because g separates the sheets, (f, g) is generically injective, so the map is birational. Hence S and the curve $P(x, y) = 0$ have the same function field. So you've now explicitly realized $\mathbb{C}(S) = \mathbb{C}(f, g)$.

We state Riemann's theorem which allows us to pass from the analytic setting to the algebraic setting.

Theorem 1.41. Let R be a compact Riemann surface and $p \in R$. There exists a meromorphic function f with poles of arbitrary order n at p and holomorphic elsewhere, provided that n is sufficiently large.

The method of proof involves constructing holomorphic differentials with poles at p, and in fact one can get them to any order of pole ≥ 2. Then if these differentials are exact, their integrals give a single valued function with pole only at p.

Theorem 1.42. Every compact Riemann surface is algebraic.

We have an idea what this means, because we have considered Riemann surfaces defined by polynomial equations

$$P(z, w) = w^n + a_{n-1}(z)w^{n-1} + \cdots + a_1(z)w + a_0(z) = 0,$$

and we have seen how to compactify these; and indeed, the result does imply that every compact Riemann surface arises in such manner. But we would like now to do more than just explain the meaning of the theorem, and survey the basic algebraic tools available for the study of compact Riemann surfaces.

The truly hard part of the theorem is to get started. Nothing in the definition of an abstract Riemann surface implies in any obvious way the existence of the basic algebraic objects of study, the meromorphic functions.

Theorem 1.43. Every compact Riemann surface carries a non-constant meromorphic function.

Remark 1.44. This is the difficult part of the theorem; once we have a branched cover of $\mathbb{P}^1$, we can start studying it by algebraic methods. The proof involves serious analysis, specifically finding solutions of the Laplace equation in various surface domains, with prescribed singularities ("Green's functions").

Contained in Riemann's theorem, there is a second result which we shall use without proof.

Proposition 1.45. Let $\pi : R \to \mathbb{P}^1$ be a holomorphic map of degree $n > 0$. There exists, then, an additional meromorphic function f on R which separates the sheets of R over $\mathbb{P}^1$, in the following sense: there exists a point $z_0 \in \mathbb{P}^1$ such that f takes n distinct values at the points of R over z_0.

Assuming now that the Riemann surface R is connected, let $\mathbb{C}(R)$ be its field of meromorphic functions. A non-constant meromorphic function z defines an inclusion of fields

$$\mathbb{C}(z) \subset \mathbb{C}(R).$$

Theorem 1.46. Let $\pi : R \to \mathbb{P}^1$ be the holomorphic map associated to the meromorphic function z.

1. $[\mathbb{C}(R) : \mathbb{C}(z)] = \deg \pi$.

2. Any $f \in \mathbb{C}(R)$ satisfies a polynomial equation of degree $\leq n$ with coefficients in $\mathbb{C}(z)$:

$$f^n + a_{n-1}(z) f^{n-1} + \cdots + a_0(z) = 0.$$

3. Let f be a meromorphic function on R which separates the sheets of R over $\mathbb{P}^1$. Then $\mathbb{C}(R)$ is generated by f over $\mathbb{C}(z)$:

$$\mathbb{C}(R) = \mathbb{C}(z)[f].$$

4. Let now $f^n + a_{n-1}(z) f^{n-1} + \cdots + a_0(z) = 0$ be the equation satisfied by the f in (iii). Then R is isomorphic to the non-singular, compactified Riemann surface of the equation

$$w^n + a_{n-1}(z) w^{n-1} + \cdots + a_1(z) w + a_0(z) = 0.$$

Theorem 1.47. There is a bijection between isomorphism classes of field extensions of $\mathbb{C}(z)$ on one hand, and isomorphism classes of compact Riemann surfaces, together with a degree n map to $\mathbb{P}^1$.

Forgetting the map to $\mathbb{P}^1$, we have:

Theorem 1.48. There is a bijection between isomorphism classes of fields which can be realized as finite extensions of $\mathbb{C}(z)$, on one hand, and isomorphism classes of compact Riemann surfaces, on the other.

The theorem follows essentially from part (4) of the previous result; the only missing ingredient, which rounds up the correspondence between Riemann surfaces and their fields of functions, is:

Theorem 1.49. Homomorphisms from $\mathbb{C}(S)$ to $\mathbb{C}(R)$ are in bijection with holomorphic maps from R to S.

Recall that a finite field extension $k \subset K$ is called **Galois**, with group Γ, if Γ acts by automorphisms of K and k is precisely the set of elements fixed by Γ.

Proposition 1.50. The automorphisms of a Riemann surface R are in bijection with those of its field of meromorphic functions $\mathbb{C}(R)$.

Let now $\pi : R \to S$ be holomorphic; it gives a field extension $\mathbb{C}(S) \subset \mathbb{C}(R)$.

Proposition 1.51. The automorphisms of R that commute with π are precisely the automorphisms of $\mathbb{C}(R)$ which fix $\mathbb{C}(S)$.

Corollary 1.52. A map $\pi : R \to S$ defines a Galois extension on the fields of meromorphic functions if and only if there exists a group Γ of automorphisms of R, commuting with π, and acting simply transitively on the fibres $\pi^{-1}(s)$, for a general $s \in S$. Such a map is called a **Galois cover with group** Γ.

Proof. Note first that any automorphism of R, commuting with π, which fixes a point of valency 1 must be the identity. Indeed, by continuity, it will fix an open neighbourhood of the point in question, and the unique continuation property of analytic maps shows it to be the identity. Now, if $\mathbb{C}(R)$ is Galois over $\mathbb{C}(S)$, the order of the group of automorphisms is $[\mathbb{C}(R) : \mathbb{C}(S)]$. So the automorphism group must act simply transitively on the fibres which do not contain branch points. Conversely, an automorphism group acting simply transitively on even one fibre with no branch points must have order $\deg \pi$. But since that is $[\mathbb{C}(R) : \mathbb{C}(S)]$, it follows that the extension is Galois. $\square$

Remark 1.53. Note that $R/\Gamma = S$, set theoretically. Topology tells us that the Γ-invariant continuous functions on R are precisely the continuous functions on S. We have just shown the same for the meromorphic functions.

Example 1.54 (Galois covers)**.**

(i) $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$, with $w \longmapsto z = w^3$. The automorphisms are $z \mapsto \zeta z$, where ζ is any cube root of 1.

(ii) $\mathbb{C}/L \longrightarrow \mathbb{P}^1$, with $u \longmapsto \wp(u)$. The non-trivial automorphism is $u \longmapsto -u$. The surface $w^2 = 4z^3 - g_2 z - g_3$ is a Galois cover of the z-plane, with Galois group $\mathbb{Z}/2$ and automorphism $w \mapsto -w$.

2 Coherent sheaves

Let X be a complex manifold and $\mathcal{O}_X$ the sheaf of holomorphic functions on X.

Definition 2.1. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is **coherent** if

1. $\mathcal{F}$ is locally of finite type: for every $x \in X$ there is a neighborhood U and a surjection

$$\mathcal{O}_X^{\oplus n}\big|_U \twoheadrightarrow \mathcal{F}\big|_U,$$

2. for every open set $U \subseteq X$ and every morphism $\phi : \mathcal{O}_X^{\oplus p}\big|_U \to \mathcal{F}\big|_U$, the kernel $\ker \phi$ is locally of finite type.

Remark 2.2. For quasi-coherent sheaves on a locally Noetherian scheme, being locally of finite type is a local/stalkwise condition, and kernels of maps of finite free modules are again finite type automatically (because any submodule of a finitely generated module over a Noetherian ring is finitely generated). Thus, a locally Noetherian scheme, a quasi-coherent sheaf $\mathcal{F}$ is coherent iff it is of finite type.

Example 2.3. $\mathcal{O}_X$ is coherent. The condition that $\mathcal{O}_X$ is locally of finite type is trivial. We need to see that for every map

$$\phi : \mathcal{O}_X^{\oplus m} \longrightarrow \mathcal{O}_X$$

the kernel $\ker \phi$ should be locally generated by finitely many sections. Equivalently, we can show that

1. for each $x \in X$, the local ring $\mathcal{O}_{X,x}$ is Noetherian. We can do this by identifing $\mathcal{O}_{X,x}$ with convergent power series in $n = \dim X$ variables, which is known to be Noetherian by the Weierstrass division theorem (See Example **??**).

2. For every homomorphism of free modules over $\mathcal{O}_{X,x}$,

$$\varphi : \mathcal{O}_{X,x}^{\oplus m} \to \mathcal{O}_{X,x},$$

the kernel $\ker \varphi$ is a finitely generated $\mathcal{O}_{X,x}$-module. Over a Noetherian ring, submodules of finitely generated modules are finitely generated. These generators can be lifted holomorphically to a neighborhood of x to give local generators of $\ker \phi$.

Definition 2.4. An **invertible sheaf** (or line bundle) on a complex manifold X is a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ which is locally isomorphic to $\mathcal{O}_X$, i.e. for every $x \in X$ there exists an open neighborhood U of x such that $\mathcal{L}\big|_U \cong \mathcal{O}_X\big|_U$.

Example 2.5. The sheaf of holomorphic sections of a holomorphic line bundle is an invertible sheaf.

Example 2.6. Let K be the sheaf of holomorphic differentials on a Riemann surface R. Then K is an invertible sheaf.

Example 2.7 (Holomorphic differentials on $\mathbb{P}^1$)**.** Indeed, over the usual chart $\mathbb{C}$, the differential must take the form $f(z)\,dz$ with f holomorphic. Near ∞, with $w = 1/z$ as a coordinate, the differential becomes

$$f(1/w)\,d(1/w) = -f(1/w)\,dw/w^2.$$

So we need $f(1/w)/w^2$ to be holomorphic at $w = 0$, so f should extend holomorphically at ∞ and have a double zero there. But then f must be zero.

Example 2.8. Consider the Riemann surface R defined by

$$w^4 = 1 - z^4.$$

We will find holomorphic differentials on its compactification R^{cpt}.

The branch points of the projection to the z-plane are at $z = \pm 1, \pm i$; $w = 0$ at all of them. The map has degree 4 and branching index 3 at each of the points. At ∞, we have four separate sheets defined by $w = \sqrt[4]{1-z^4}$ which has four convergent expansions in $1/z$, as soon as $|z| > 1$. So Riemann-Hurwitz gives

$$g(R) - 1 = -4 + \tfrac{1}{2} \cdot 12 = 2, \qquad g(R) = 3.$$

Thus R is a genus 3 surface with 4 points at ∞.

Now dz defines a meromorphic differential on R^{cpt}, because z is a meromorphic function there. At ∞, on R^{cpt}, $u = z^{-1}$ is a local holomorphic coordinate, and $dz = -u^{-2}du$ has a double pole.

On the other hand, I claim that dz has a triple zero at each of the branch points. Indeed, by the theorem on the local form of an analytic map, there is a local coordinate v with $z - 1 = v^4$. So

$$dz = d(v^4) = 4v^3\, dv$$

has a triple zero over $z = 1$, and similarly over the other branch points.

So $dz/w^2, dz/w^3$ are still holomorphic at the branch points (and everywhere else when $z \neq \infty$, because $w \neq 0$). At $z = \infty$, w has a simple pole on R^{cpt} and we see that $w^{-2}dz$ and $w^{-3}dz$ (and higher powers) are non-singular there. Moreover, we can even afford to add $z\, dz/w^3$ to our list, and we have produced three holomorphic differentials on R^{cpt}.

The ratios of holomorphic differentials on R^{cpt} generate the field of meromorphic functions.

$$\frac{dz/w^2}{dz/w^3} = w, \qquad \frac{z\, dz/w^3}{dz/w^3} = z,$$

and z, w generate the field of meromorphic functions, by our theorem from last time.

2.1 Line bundles and divisors

If f is a nonconstant meromorphic function on a compact Riemann surface R, then we defined the divisor of f to be

$$(f) = \sum_{p \in R} \mathrm{ord}_p(f) p$$

where $\mathrm{ord}_p(f)$ is the order of vanishing of f at p (negative if f has a pole at p).

We defined the following sets:

$$\begin{aligned} \mathrm{Div}(R) &= \{\text{formal finite sums } \textstyle\sum n_p p, n_p \in \mathbb{Z}\} \\ \mathrm{PDiv}(R) &= \{\text{divisors of meromorphic functions}\} \\ \mathrm{Cl}(R) &= \mathrm{Div}(R)/\,\mathrm{PDiv}(R) \end{aligned}$$

and there is a map

$$\begin{aligned} \mathrm{Div}(R) &\to \mathrm{Pic}(R) \\ D &\mapsto \mathcal{O}(D) \end{aligned}$$

where

$$\mathcal{O}(D)(U) = \{f \text{ meromorphic on } U : (f)|_U + D|_U \geq 0\}$$

is an invertible sheaf. More precisely, from D one gets an invertible sheaf $\mathcal{O}(D)$ along with a meromorphic section s_D such that $(s_D) = D$.

One can think of s_D as the constant function 1. In particular, recall that $\mathcal{O}(D)$ is locally isomorphic to $\mathcal{O}_R$ by picking local defining equations η_α for D on an open cover U_α. Recall that on a smooth variety there is an equivalence between Cartier divisors and Weil divisors. Then the isomorphism $\mathcal{O}(D)|_{U_\alpha} \to \mathcal{O}_R|_{U_\alpha}$ is given by multiplication by η_α. Then the canonical meromorphic section s_D, when restricted to U_α, is given by η_α which has divisor $D|_{U_\alpha}$.

Therefore, there is an isomorphism of abelian groups

$$\begin{aligned} \mathrm{Cl}(R) &\to \text{subgroup of } \mathrm{Pic}(R) \text{ consisting of invertible sheaves admitting meromorphic sections} \\ D &\mapsto (\mathcal{O}(D), 1) \end{aligned}$$

and this is in fact an isomorphism of groups because of the following theorem.

Theorem 2.9. Every line bundle $\mathcal{L}$ on a Riemann surface has a nonzero meromorphic section. More generally, every vector bundle admits a global meromorphic frame.

Remark 2.10. The compact case follows from the Kodaira vanishing theorem. In the noncompact case, all holomorphic vector bundles on noncompact R are trivializable and therefore admit a global holomorphic frame.

Remark 2.11 (Confusion about degrees of meromorphic functions/sections)**.** By definition of the invertible sheaf $\mathcal{O}(D)$, its holomorphic sections are those rational functions $f \in \mathbb{C}(R)$ such that $(f)+D \geq 0$. The degree of a meromorphic function is zero. However, the same function interpreted as a section of $\mathcal{O}(D)$ has degree $\deg D$. This is because the transition functions of $\mathcal{O}(D)$ introduce additional zeroes/poles to the section.

For example, consider the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$, and let z be the coordinate near zero, w the coordinate near infinity. Then z is a meromorphic function on $\mathbb{P}^1$, it has a simple zero at 0 and a simple pole at ∞, so it has degree zero as a meromorphic function.

However we can also consider it as a section of $\mathcal{O}_{\mathbb{P}^1}(1)$. Recall that $\mathcal{O}_{\mathbb{P}^1}(1)$ is constructed by taking two copies of $\mathbb{C}$ with coordinates e_0, e_∞ and gluing them on the overlap by $e_\infty = we_0$.

A section of $\mathcal{O}_{\mathbb{P}^1}(1)$ is given by a pair of functions $(s_0(z)e_0, s_\infty(w)e_\infty)$ so that on the overlap, both expressions agree. In particular

$$s_0(z)e_0 = s_\infty(w)e_\infty = s_\infty(w)(we_0)$$

so the rule becomes

$$s_\infty(w) = ws_0(1/w)$$

Therefore the section corresponding to the meromorphic function z is given by

$$\begin{aligned} s_0(z) &= z \\ s_\infty(w) &= ws_0(1/w) = w \cdot (1/w) = 1 \end{aligned}$$

so as a section of $\mathcal{O}_{\mathbb{P}^1}(1)$, z has a simple zero at 0 and is nonvanishing at ∞. Therefore as a section of $\mathcal{O}_{\mathbb{P}^1}(1)$, z has degree 1. In particular it is a global holomorphic section of $\mathcal{O}_{\mathbb{P}^1}(1 \cdot \infty)$.

A meromorphic section of $\mathcal{O}(D)$ also comes from a meromorphic function and it will have the same degree as D. In particular, if $\mathcal{O}(D)$ has a nonzero holomorphic section, then the degree of a meromorphic section will equal the degree of a holomorphic section, which we aptly call the degree of the line bundle.

The point is that sometimes line bundles do not admit holomorphic sections, for example $\mathcal{O}_{\mathbb{P}^1}(-1)$ has no nonzero holomorphic sections. In that case, the degree of the line bundle is still defined, and it is precisely the degree of any meromorphic section (which always exists by the above theorem).

Those line bundles with a nonzero holomorphic section are precisely those which come from effective divisors, i.e. $\mathcal{L} \cong \mathcal{O}(D)$ with $D \geq 0$. This is because if $\mathcal{L}$ has a nonzero holomorphic section s, then $(s) \geq 0$ is an effective divisor and $\mathcal{L} \cong \mathcal{O}((s))$. Conversely if $\mathcal{L} \cong \mathcal{O}(D)$ with $D \geq 0$, then the constant function 1 is a nonzero holomorphic section of $\mathcal{O}(D)$.

However it is not true that every line bundle of nonnegative degree is linearly equivalent to an effective divisor, unless $g = 0$. Consider the line bundle $\mathcal{O}_X(p - q)$ of degree zero for distinct points $p, q \in X$. If this line bundle were effective, then there would exist a nontrivial holomorphic section s of $\mathcal{O}_X(p - q)$. This would immediately give an isomorphism to $\mathbb{P}^1$. So every curve of genus $g \geq 1$ has non-effective degree-zero line bundles. See Remark 3.12 for another viewpoint.

> Recall that the multiplicative Cousin problem is the problem of finding a global meromorphic function with prescribed zeroes and poles. The additive Cousin problem is the problem of finding a global meromorphic function with prescribed principal parts. The above theorem shows that both problems are always solvable on a noncompact Riemann surface.

Theorem 2.12. On a noncompact Riemann surface, the multiplicative and additive Cousin problems are always solvable. Moreover every holomorphic vector bundle on a noncompact Riemann surface is trivializable.

Definition 2.13 (Degree of a line/vector bundle)**.** The degree of a line bundle $\mathcal{L}$ on a compact Riemann surface R is defined to be the degree of any meromorphic section of $\mathcal{L}$. This is well defined because if s, s' are two meromorphic sections of $\mathcal{L}$, then s/s' is a meromorphic function on R and has degree 0.

The degree of a vector bundle $\mathcal{E}$ is defined to be the degree of its determinant line bundle $\det \mathcal{E} = \wedge^{\operatorname{rank}\mathcal{E}}\mathcal{E}$.

Fact 2.14. On a compact Riemann surface, the degree and dimension of a vector bundle completely determine the topology of the bundle.

Proposition 2.15. Every holomorphic line bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(n)$ for some integer n.

Proof. We can solve the multiplicative Cousin problem on $\mathbb{P}^1$ for degree zero divisors. $\square$

Proposition 2.16. Let $E = \mathbb{C}/L$ be an elliptic curve. Then

$$0 \to E \to \operatorname{Pic}(E) \to \mathbb{Z} \to 0$$

is a short exact sequence of abelian groups. It splits, so $\operatorname{Pic}(E) \cong E \times \mathbb{Z}$.

Example 2.17 (Doubled lattice)**.** Recall that every ellptic curve $E = \mathbb{C}/L$ has a degree four cover by $\tilde{E} = \mathbb{C}/2L$. We defined four θ functions on E, let $\mathcal{L}_i$ be the corresponding line bundles. Then $\pi^*\mathcal{L}_i$ are all isomorphic on $\tilde{E}$ because the beriodicity conditions all become the same after doubling the lattice. Moreover recall that there is a map

$$E \to \mathbb{P}^3, \quad z \mapsto [\theta_1(z,\tau) : \theta_2(z,\tau) : \theta_3(z,\tau) : \theta_4(z,\tau)]$$

which is in fact a projective embedding by a line bundle.

Recall that in general if one has $\mathcal{L}$ a line bundle on X, then we can consider the evaluation map $X \to \mathbb{P}(H^0(X,\mathcal{L})^*)$ given by $x \mapsto \{s \in H^0(X,\mathcal{L}) : s(x) = 0\}$ when $\mathcal{L}$ has enough sections. For example, if $\mathcal{L}$ has negative degree than it has no sections. If $\mathcal{L}$ has degree 0 then it has a section if and only if it is trivial.

2.2 Vector bundles and elementary transformations

The analog of $\otimes\mathcal{O}(D)$ for vector bundles is called an elementary transformation. Let V be a vector bundle on R and choose a subspace $S \subset V_p$.

Define $\operatorname{elm}(V,p,S)$ to be the sheaf of sections of V whose value at p lies in S. This is a vector bundle whose degree is $\deg V - \operatorname{codim} S$.

$$0 \to \operatorname{elm}(V,p,S) \to V \to (V_p/S) \otimes \mathcal{O}_p \to 0$$

Therefore there is a short exact sequence of vector spaces

$$0 \to K \to \mathrm{elm}(V, p, S)_p \to S \to 0$$

with the property that

$$\mathrm{elm}(\mathrm{elm}(V, p, S), p, K) \cong V(-x)$$

and so elementary transformations are invertible up to twisting by a line bundle (which is also invertible). This resolves the obvious obstruction that elementary transformations reduce the degree of a vector bundle.

Proposition 2.18. Every vector bundle is obtained from a trivial vector bundle by a finite sequence of elementary transformations and tensoring by line bundles.

Exercise 2.19. Let V be a rank 2 (for simplicity) vector bundle over a Riemann surface R. Assume that V has two meromorphic sections s_1, s_2 which, at some point, are holomorphic and span the fiber.

(a) Show that this will be the case everywhere except at a set of isolated points.

(b) At an exceptional point, show that we can modify V by a finite sequence of elementary transformations so that s_1 and s_2 form a holomorphic frame of the new bundle.

Solution 2.20. Let s_1, s_2 be two meromorphic sections of a rank 2 vector bundle V over a Riemann surface R. Since V is a holomorphic vector bundle, there exists a local trivialization of V around p.

$$V|_U \cong \mathcal{O}_U e_1 \oplus \mathcal{O}_U e_2$$

and we can write

$$s_1 = f_1 e_1 + f_2 e_2, \quad s_2 = g_1 e_1 + g_2 e_2$$

where f_i, g_i are meromorphic functions on U. The failure of s_1, s_2 to span the fiber at a point $q \in U$ is given by the vanishing of the determinant

$$D(q) = f_1(q) g_2(q) - f_2(q) g_1(q).$$

which is a meromorphic function on U. The zeroes of a meromorphic function are isolated unless the function is identically zero. Since s_1, s_2 span the fiber at p, D is not identically zero. Therefore, the set of points where s_1, s_2 fail to be holomorphic or fail to span the fiber is a discrete set of isolated points in R, because meromorphic functions can only have isolated singularities and the determinant D is meromorphic.

Let D be the effective divisor of the poles of s_1, s_2. We can make s_1, s_2 holomorphic by twisting V with the line bundle $\mathcal{O}(D)$, i.e. consider the new vector bundle

$$V(D) = V \otimes \mathcal{O}(D)$$

Then s_1, s_2 are holomorphic sections of $V(D)$. Now consider a point p where s_1, s_2 fail to span the fiber of $V(D)$. If $s_1(p)$ and $s_2(p)$ both vanish, then twist by an appropriate power of $\mathcal{O}(-p)$ to make at least one of them non-vanishing at p, say $s_1(p) \neq 0$. In a chart near $V(D)$ we have a local trivialization $V(D)|_U \cong \mathcal{O}_U e_1 \oplus \mathcal{O}_U e_2$ so that $s_1 = e_1$ and $s_2 = f(z)e_1 + g(z)e_2$ for some holomorphic functions $f(z), g(z)$. Let $L = \mathbb{C}e_1 \subset V_p$. We can perform an elementary transformation of $V(D)$ at p with respect to L to obtain a new vector bundle V' which fits into the short exact sequence of coherent sheaves

$$0 \to V' \to V(D) \to (V(D)_p/L) \otimes \mathcal{O}_p \to 0. \tag{1}$$

The wedge product of the sections is given by

$$s_1 \wedge s_2 = g(z)e_1 \wedge e_2.$$

Since s_1, s_2 fail to span the fiber at p, we have $g(0) = 0$, so we can write $g(z) = z^n h(z)$ for some $n \geq 1$ and unit $h(0) \neq 0$. After absorbing the unit $h(z)$ into e_2, we can assume $g(z) = z^n$. Then we have in local coordinates sections $s_1 = e_1$ and $s_2 = f(z)e_1 + z^n e_2$.

The elementary transformation V' is locally generated by the sections $s_1' = e_1$ and $s_2' = ze_2$. This is because $V'(U)$ consists of sections of $V(D)(U)$ whose value at p lies in $L = \mathbb{C}e_1$. Any section of $V(D)(U)$ can be written as $a(z)e_1 + b(z)e_2$ for some holomorphic functions $a(z), b(z)$. The condition that the value at p lies in L means that $b(0) = 0$, so we can write $b(z) = zc(z)$ for some holomorphic function $c(z)$. Therefore, sections of $V'(U)$ are of the form

$$a(z)e_1 + zc(z)e_2, \quad a(z), c(z) \in \mathcal{O}_U$$

which means $V'(U)$ is a $\mathcal{O}_U$-module freely generated by e_1 and ze_2. In particular, the bundle V' is locally trivialized by the sections e_1 and $e_2' = ze_2$. In the new bundle V', the sections s_1 and s_2 have wedge product

$$s_1' \wedge s_2' = z^{n-1} e_1 \wedge e_2'.$$

Thus, the order of vanishing of the wedge product at p has decreased by 1. By repeating this process a finite number of times, we can obtain a vector bundle where s_1, s_2 span the fiber at p. By performing this procedure at each point where s_1, s_2 fail to span the fiber, we can obtain a vector bundle where s_1, s_2 form a holomorphic frame everywhere.

Remark 2.21. The argument generalizes to any dimension. If R is compact, it follows that we can trivialize V by a finite number of elementary transformations. If R is non-compact, one can show that every vector bundle is in fact trivial.

Theorem 2.22 (Grothendieck's theorem)**.** Every vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles.

$$V \cong \bigoplus_{i=1}^{\operatorname{rank} V} \mathcal{O}_{\mathbb{P}^1}(n_i)$$

where $n_i \geq n_{i+1}$. Moreover, the n_i are uniquely determined by V.

The degree of V is $\sum n_i$.

Example 2.23. On $\mathbb{P}^1$, we have homeomorphic but not biholomorphic vector bundles $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O}$. They both have degreee zero and the same number of sections, but the sections sit inside the bundles differently.

2.3 Riemann-Roch theorem for vector bundles

To a divisor $D = \sum n_p p$ on a compact Riemann surface R, we can associate the invertible sheaf $\mathcal{O}(D)$, defined by

$$\mathcal{O}(D)(U) = \{f \text{ meromorphic on } U : (f)|_U + D|_U \geq 0\}$$

The fact that $\mathcal{O}(D)$ is a locally free sheaf of rank one follows from the following fact:

Lemma 2.24. The space of holomorphic functions in the unit disk having a zero of order $\geq n$ at 0 is a free module of rank one over the ring of all holomorphic functions, generated by z^n.

In fact every invertible sheaf on a Riemann surface is of the form $\mathcal{O}(D)$ for some divisor D.

Theorem 2.25. There is a group isomorphism

$$\begin{aligned} \mathrm{Cl}(R) &\to \mathrm{Pic}(R) \\ D &\mapsto \mathcal{O}(D) \end{aligned}$$

where $\mathrm{Cl}(R)$ is the divisor class group of R defined by $\mathrm{Cl}(R) = \mathrm{Div}(R)/\,\mathrm{PDiv}(R)$ and $\mathrm{Pic}(R)$ is the Picard group of R. The inverse map is given by

$$\mathcal{L} \mapsto (s)$$

where s is any nonzero meromorphic section of $\mathcal{L}$.

Remark 2.26. The above theorem is special to Riemann surfaces.

1. The local ring $\mathcal{O}_{X,p}$ is a discrete valuation ring (a DVR) and thus every rank-1 torsion-free module is free.

2. Every line bundle admits a meromorphic section.

In higher dimension these fail:

1. In higher dimension, the local ring at a point is not a DVR, so there exist rank-1 torsion-free sheaves that are not locally free (i.e. not line bundles).

2. Not every line bundle admits a meromorphic section in higher dimensions. For example, on a complex torus of dimension ≥ 2, there exist line bundles with no nontrivial meromorphic sections.

The comparison between line bundles and divisors can be formalized to the statement that on a Riemann surface, Cartier and Weil divisors coincide. This is because Cartier divisors correspond to the transition functions of line bundles. In general, Cartier divisors are Weil divisors when X is regular in codimension one.

The identification $\operatorname{Div}(X)/\operatorname{PDiv}(X) = Pic(X)$ continues to hold for projective algebraic manifolds X, by a similar argument, but can fail for non-algebraic manifolds. It also fails for singular varieties.

The Riemann–Roch theorem is the best possible answer to the question of finding the dimension of the space $\Gamma(R; L)$ of holomorphic sections of a line bundle by topological methods. This question does not have a purely topological answer, as seen from the following.

Example 2.27. Let $g(R) > 0$, fix a point $y \in R$ and consider the family of divisors $D_x = x - y$ on R, parametrized by $x \in R$. For $x \neq y$, D_x is not principal, because a meromorphic function with a simple pole at x and no other poles would describe a degree one map $R \to \mathbb{P}^1$, which would have to be an isomorphism. However, $D_y = 0$ has a nontrivial holomorphic section: the function 1. So the dimension of the space of holomorphic sections can jump in a family. Note that all line bundles $\mathcal{O}(D_x)$ have degree zero, so they are topologically trivial. This means that no topological information will detect $\dim \Gamma(R; L)$.

The topologically invariant quantity turns out to be

$$\chi(R; L) := \dim H^0(R; L) - \dim H^1(R; L).$$

Here, H^0 is the desired space of global sections, while the first cohomology group H^1 will be defined in a moment. The result is especially important in the case of Riemann surfaces because $\dim H^1(R; L)$ has another interpretation, as the dimension of the space

$$\dim H^0(R; K \otimes L^\vee)$$

of holomorphic differentials with values in the dual line bundle $L^\vee$. The relation to differentials will be our second major result, the **Serre duality theorem**. I separate the two theorems (which are often combined in the Riemann surface literature) because their higher-dimensional generalisations are quite different.

Definition 2.28. The space $\mathcal{P}_x(V)$ of **principal parts** at $x \in R$ with coefficients in a holomorphic vector bundle V is the quotient

$$\mathcal{M}_x(V)/\mathcal{O}_x(V)$$

of the space of germs at x of meromorphic sections of V by the subspace of germs of holomorphic sections. The space $\mathcal{P}(R; V)$ of principal parts over R is the direct sum, over all points $x \in R$, of the $\mathcal{P}_x(V)$.

In a local coordinate centred at x, $\mathcal{P}_x(V)$ is the space of negative Laurent polynomials with values in V. For later use, we note that $\mathcal{P}(R; V)$ is the space of sections of a sheaf $\mathcal{P}(V)$ over R; however, in order to satisfy the sheaf condition for (possibly infinite) coverings, one must allow on a general open set U a distribution of principal parts over any discrete, but possibly infinite set. It is the compactness of R that forces the restriction to finitely many points, and limits us to the direct sum. There is a linear map induced by the morphism of sheaves $\mathcal{M}(V) \to \mathcal{P}(V)$

$$p : \Gamma(R; \mathcal{M}(V)) \longrightarrow \Gamma(R; \mathcal{P}(V))$$

which assigns to each meromorphic section its principal parts. The kernel of p consists of (the sheaf of) holomorphic sections of V.

Definition 2.29. The **first cohomology group** $H^1(R; V)$ with coefficients in the (sheaf of holomorphic sections of the) vector bundle V is the ratio

$$H^1(R; V) := \mathcal{P}(R; V)/\operatorname{Im}(p),$$

that is, principal parts modulo global meromorphic sections.

In other words, we have a four-term exact sequence

$$0 \longrightarrow H^0(R; V) \longrightarrow \Gamma(R; \mathcal{M}(V)) \xrightarrow{p} \mathcal{P}(R; V) \longrightarrow H^1(R; V) \longrightarrow 0 \tag{2}$$

Remark 2.30. We will rediscover this construction of H^1 in the context of sheaf cohomology, and the sequence (2) will be the long exact sequence of sheaf cohomology associated to the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(V) \longrightarrow \mathcal{M}(V) \longrightarrow \mathcal{P}(V) \longrightarrow 0.$$

The key point will be the vanishing of higher cohomologies for the sheaves $\mathcal{M}$ and $\mathcal{P}$.

Example 2.31. Any distribution of principal parts of functions on $\mathbb{P}^1$ can be realised by a meromorphic function, unique up to an additive constant, showing that $H^1(\mathbb{P}^1; \mathcal{O}) = 0$. The additive constant even gives us one dimension to spare, so that we also have $H^1(\mathbb{P}^1; \mathcal{O}(-1)) = 0$: indeed, we can realise $\mathcal{O}(-1)$ by imposing simple vanishing at ∞ on $\mathcal{O}$; principal parts in $\mathcal{O}(-1)$ then have their usual meaning at every finite point, whereas at ∞ the constant term is now included in the principal part.

However, for $\mathcal{O}(-n)$, $n > 1$, we start meeting obstructions: indeed, we can now give the first n non-negative terms in the Laurent series at ∞, as part of our principal part specification there, and we can only match the constant term with a meromorphic function; every additional term contributes one dimension to H^1.

Leaving the remaining easy cases to the reader, we get the following answers:

$$\dim H^0(\mathbb{P}^1; \mathcal{O}(n)) = \begin{cases} n+1, & n \geq -1, \\ 0, & n < -1, \end{cases} \qquad \dim H^1((\mathbb{P}^1; \mathcal{O}(n)) = \begin{cases} 0, & n \geq -1, \\ -n-1, & n < -1. \end{cases}$$

Theorem 2.32 (Riemann–Roch theorem for vector bundles)**.** Let R be a compact Riemann surface of genus g, and let V be a holomorphic vector bundle of rank r and degree d on R. Then the Euler characteristic of V is given by

$$\chi(R, V) = \dim H^0(R, V) - \dim H^1(R, V) = d + r(1-g)$$

Lemma 2.33. Let $f : X \to Y$ be a branched cover of Riemann surfaces. If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module. The rank of $f_*\mathcal{F}$ is $n \cdot \operatorname{rank} \mathcal{F}$, where n is the degree of the cover.

Proof. Suppose y is not a branch point. Then the fibre $f^{-1}(y)$ consists of n distinct points $x_1, \ldots, x_n$ and there are neighbourhoods U of y and V_i of x_i such that

$$f^{-1}(U) = \bigsqcup_{i=1}^{n} V_i \quad \text{and} \quad f|_{V_i} : V_i \to U$$

is a biholomorphism.

Then

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)) = \bigoplus_{i=1}^{n} \mathcal{F}(V_i)$$

Shrinking U if necessary, we may assume that $\mathcal{F}$ is trivial on each V_i, so that

$$\mathcal{F}(V_i) \cong \mathcal{O}_X(V_i)^{\oplus r}$$

where r is the rank of $\mathcal{F}$. Since $f|_{V_i}$ is a biholomorphism, we have

$$\mathcal{O}_X(V_i) \cong \mathcal{O}_Y(U)$$

Therefore,

$$(f_*\mathcal{F})(U) \cong \bigoplus_{i=1}^{n} \mathcal{O}_Y(U)^{\oplus r} \cong \mathcal{O}_Y(U)^{\oplus nr}$$

showing that $f_*\mathcal{F}$ is locally free of rank nr near y.

Now suppose that y is a branch point. To show $f_*\mathcal{F}$ is a vector bundle, we must show that its stalks are free $\mathcal{O}_{Y,y}$-modules. By definition, for any sheaf $\mathcal{F}$ on X we have

$$(f_*\mathcal{F})_y \;=\; \varinjlim_{y\in U} \mathcal{F}\big(f^{-1}(U)\big).$$

Given a finite map, the topology of the fiber decomposes the stalk into a direct sum over preimages:

$$(f_*\mathcal{F})_y \;\cong\; \bigoplus_{x_i\in f^{-1}(y)} \mathcal{F}_{x_i}$$

Note that $\mathcal{F}_x$ is naturally an $\mathcal{O}_{X,x}$-module, and $\mathcal{O}_{Y,y}$ acts on it via the pullback map

$$f^\# : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}.$$

Thus $(f_*\mathcal{F})_y$ is an $\mathcal{O}_{Y,y}$-module through $f^\#$, and thus we want to show that $\mathcal{O}_X$ near x is a free module of rank n over $\mathcal{O}_Y$ near y. We identify

$$\mathcal{O}_{Y,y} \;\cong\; \mathbb{C}\{z\}, \qquad \mathcal{O}_{X,x} \;\cong\; \mathbb{C}\{w\}$$

Take any holomorphic function in w.

$$g(w) = \sum_{k=0}^{\infty} a_k w^k.$$

Grouping the terms according to their residue modulo n, we have

$$g(w) = h_0(z) + h_1(z)\,w + \cdots + h_{n-1}(z)\,w^{n-1},$$

Thus $\mathbb{C}\{w\}$ is a free $\mathbb{C}\{z\}$-module of rank n with basis $1, w, \ldots, w^{n-1}$. In other words, $\mathcal{O}_{X,x}$ is a free $\mathcal{O}_{Y,y}$-module of rank n. This completes the proof. $\square$

Example 2.34 (Riemann-Roch on $\mathbb{P}^1$)**.** We will directly verify the Riemann-Roch theorem for vector bundles on $\mathbb{P}^1$.

Proof of Riemann-Roch for vector bundles. We realize our surface R as an n-sheeted branched cover $\pi : R \to \mathbb{P}^1$ for some n. We will prove the theorem on R by pushing V down to $\mathbb{P}^1$ and using Riemann-Roch there. For $\mathbb{P}^1$, the theorem follows from Grothendieck's classification (Theorem 2.22) of vector bundles and the explicit calculation for line bundles.

Let $B \in \mathrm{Div}(R)$ be the branch divisor of the cover, the sum of branch points repeated according to the branching index. The degree of B is the total branching index b.

Consider the direct image bundle π_*V on $\mathbb{P}^1$, associated to the sheaf $\pi_*\mathcal{O}(V)$. I now make the following three claims:

(i) $H^i(R;V) \cong H^i(\mathbb{P}^1;\pi_*V)$ for $i=1,2$;

(ii) $\operatorname{rank}\,\pi_*V = n\cdot\operatorname{rank}V$;

(iii) $\deg\pi_*V = \deg V - \frac{1}{2}b\cdot\operatorname{rank}V$,

where b is the total branching index.

Claim (ii) follows from the previous lemma. (i) is clear for H^0 but less so for H^1; (iii) seems obscure. Granting the claims for now and writing $r = \operatorname{rank}V$, we find

$$\chi(R;V) = \chi(\mathbb{P}^1;\pi_*V) = (\deg V - \tfrac{1}{2}b\,r) + n\cdot r = \deg V + (1-g)\cdot r,$$

using Riemann-Hurwitz, thus proving the Riemann-Roch theorem.

It remains to prove the claims. Actually, claim (i) follows from the two observations that

$$\mathcal{M}(\mathbb{P}^1;\pi_*V) = \mathcal{M}(R;V) \qquad \text{and} \qquad \mathcal{P}(\mathbb{P}^1;\pi_*V) = \mathcal{P}(R;V).$$

Both statements actually hold at the level of sheaves, $\pi_*\mathcal{M}(V) = \mathcal{M}(\pi_*V)$ and similarly for $\mathcal{P}$, but we do not need that fact.

Where π is unramified, a holomorphic section is a tuple of r-vectors of holomorphic functions on each sheet, and a meromorphic section is a tuple of r-vectors of meromorphic functions on each sheet, so the equality of sheaves $\pi_*\mathcal{M}(V) = \mathcal{M}(\pi_*V)$ and $\pi_*\mathcal{O}(V) = \mathcal{O}(\pi_*V)$ is clear.

Likewise, at a branch point, in the local form $w \mapsto z = w^n$, we again have the decomposition of every holomorphic/meromorphic section, which after trivializing V, can be written as an r-vector of holomorphic/meromorphic functions in w. Grouping terms according to their residue modulo n again gives the desired equalities of sheaves. Thus again we have $\pi_*\mathcal{M}(V) = \mathcal{M}(\pi_*V)$ and $\pi_*\mathcal{O}(V) = \mathcal{O}(\pi_*V)$. Quotienting gives the equality for principal parts.

Now claim (i) follows from the two exact sequences

$$0 \to H^0(R,V) \to \Gamma(R,\mathcal{M}(V)) \to \Gamma(R,\mathcal{P}(V)) \to H^1(R,V) \to 0$$
$$0 \to H^0(\mathbb{P}^1,\pi_*V) \to \Gamma(\mathbb{P}^1,\mathcal{M}(\pi_*V)) \to \Gamma(\mathbb{P}^1,\mathcal{P}(\pi_*V)) \to H^1(\mathbb{P}^1,\pi_*V) \to 0.$$

since we have identified the middle two terms in each sequence.

The curious claim is (iii), which we now prove. For starters, observe that the fibre of π_*V at some $z \in \mathbb{P}^1$ where R is not branched is the sum of the fibres of V at the points in $\pi^{-1}(z)$.

An elementary transformation on V at some $y \in \pi^{-1}(z)$ effects an elementary transformation on π_*V at z, with a subspace of the same codimension (the original S, plus the sum of the other fibres). So the degrees of the two bundles change by the same amount. Transforming all the way to the trivial bundle $\mathbb{C}^r$ gives the relation

$$\deg \pi_* V - \deg V = \deg \pi_* \mathbb{C}^r - \deg \mathbb{C}^r = r \cdot \deg \pi_* \mathbb{C}.$$

So we need to show that $\deg \pi_* \mathcal{O} = -b/2$. We now consider the dual vector bundle $(\pi_* \mathcal{O})^\vee$, which has degree opposite that of $\pi_* \mathcal{O}$; the desired equality is equivalent to

$$\deg \big((\pi_* \mathcal{O})^\vee \big) = \deg \pi_* \mathcal{O} + b.$$

This equality, and with it the theorem, follows by comparing degrees in the following lemma. $\square$

Lemma 2.35. With notation as in the proof above,

$$(\pi_* \mathcal{O})^\vee \cong \pi_* \big(\mathcal{O}(B) \big).$$

More generally,

$$(\pi_* (V^\vee))^\vee \cong \pi_* \big(V^\vee(V)(B) \big).$$

Proof. To exhibit the duality, we must construct a perfect pairing between the two direct image bundles. For a meromorphic function on R, define its trace $\mathrm{Tr}_\pi a$ along π to be the sum of the values along the fibres of π (with multiplicities, at branch points): this is a meromorphic function on $\mathbb{P}^1$.

A key point is that the trace of a function in $\mathcal{O}(B)$ is in fact holomorphic: the order of the pole on R is too small to create a pole on $\mathbb{P}^1$. (The trace will have growth less than $|z|^{-1}$ in a local coordinate, as can be checked in the local form of π.)

Define now a bilinear pairing

$$\pi_* \mathcal{O} \ \times \ \pi_* \mathcal{O}\big((B)\big) \ \longrightarrow \ \mathcal{O}, \qquad \varphi \times \psi \ \longmapsto \ \mathrm{Tr}_\pi (\varphi \cdot \psi), \tag{3}$$

where the sections φ, ψ of the sheaves on $U \in \mathbb{P}^1$ are being viewed as functions on $\pi^{-1}(U)$, multiplied pointwise and fed into the trace. This is bilinear over $\mathcal{O}_{\mathbb{P}^1}$.

To check that this is a perfect duality between direct image bundles, we can pass to the local form $w \mapsto z = w^n$ of the map; then $\mathcal{O}_w$ has basis $\{1, w, \ldots, w^{n-1}\}$ over $\mathcal{O}_z$, while $\mathcal{O}_w(B)$ has basis $\{1, w^{-1}, \ldots, w^{-(n-1)}\}$. The trace sums the values at $w, \zeta w, \ldots, \zeta^{n-1} w$, where $\zeta = e^{2\pi i/n}$, and we obtain from the above equation the natural pairing between the two bases.

For general V, the same argument applies, but now the trace pairing must include the trace pairing on $V \times V^\vee$ as well as Tr_π. $\square$

2.4 Coherent sheaf cohomology

Recall if V is a vector bundle on a compact Riemann surface R, then we proved that

$$\chi(R,V) = \dim H^0(R,V) - \dim H^1(R,V) = \deg V + \operatorname{rank} V(1-g)$$

Moreover, we saw that any coherent sheaf $\mathcal{S}$ on R locally splits as a direct sum of a vector bundle and a torsion sheaf supported at finitely many points. More precisely, there is a short exact sequence

$$0 \to \mathcal{T} \to \mathcal{S} \to V \to 0$$

where $\mathcal{T}$ is the subsheaf of torsion sections of $\mathcal{S}$ and V is a vector bundle. This short exact sequence is canonical but the splitting is not.

We have $H^0(R,\mathcal{T})$ has dimension the global length of $\mathcal{T}$. We define

$$H^1(R,\mathcal{T}) = 0$$

and this is justified because

$$\begin{aligned} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{T} &= 0 \\ \mathcal{P} \otimes_{\mathcal{O}} \mathcal{T} &= 0 \end{aligned}$$

where $\mathcal{M}$ is the sheaf of meromorphic functions and $\mathcal{P} = \mathcal{M}/\mathcal{O}$ is the sheaf of principal parts. Define

$$\deg \mathcal{T} = \dim H^0(R,\mathcal{T})$$

In particular, this definition of $H^*(R,\mathcal{T})$ is justified because it is compatible with the Riemann-Roch theorem for coherent sheaves and it makes degree additive in short exact sequences. For example, one can check that

$$0 \to \mathcal{O}(-np) \to \mathcal{O} \to \mathbb{C}\{z\}/(z^n) \to 0$$

where p is a point with local coordinate z and $\mathbb{C}z$ denotes the ring of germs of holomorphic functions at p.

Theorem 2.36 (Long exact sequence in cohomology)**.** Let

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$$

be a short exact sequence of coherent sheaves on a compact Riemann surface R. Then there is a long exact sequence in cohomology

$$0 \to H^0(R,\mathcal{S}') \to H^0(R,\mathcal{S}) \to H^0(R,\mathcal{S}'') \to H^1(R,\mathcal{S}') \to H^1(R,\mathcal{S}) \to H^1(R,\mathcal{S}'') \to 0$$

and the Euler characteristic is additive

$$\chi(R,\mathcal{S}) = \chi(R,\mathcal{S}') + \chi(R,\mathcal{S}'')$$

To prove this theorem, we will introduce spectral sequences in a limited context. Let $(\mathcal{S}^\bullet, d^\bullet)$ be a complex of coherent sheaves of finite length on a compact Riemann surface R. We can form a double complex

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathcal{P}(\mathcal{S}^i) & \xrightarrow{d^i} & \mathcal{P}(\mathcal{S}^{i+1}) & \longrightarrow & \cdots \\
 & & {\scriptstyle p^i}\Big\uparrow & & {\scriptstyle p^{i+1}}\Big\uparrow & & \\
\cdots & \longrightarrow & \mathcal{M}(\mathcal{S}^i) & \xrightarrow{d^i} & \mathcal{M}(\mathcal{S}^{i+1}) & \longrightarrow & \cdots
\end{array}$$

where $\mathcal{M}(\mathcal{S}^i)$ is the sheaf of meromorphic sections of $\mathcal{S}^i$ and $\mathcal{P}(\mathcal{S}^i) = \mathcal{M}(\mathcal{S}^i)/\mathcal{S}^i$ is the sheaf of principal parts. The vertical maps p^i are the natural projections. We can form the total complex

$$\begin{gathered}
\mathrm{Tot}^n(\mathcal{S}^\bullet) = \mathcal{P}(\mathcal{S}^{n-1}) \oplus \mathcal{M}(\mathcal{S}^n) \\
d^n_{\mathrm{Tot}} = d^{n-1}_{\mathcal{P}} + (-1)^n d^n_{\mathcal{M}}
\end{gathered}$$

We define the hypercohomology of the complex $\mathcal{S}^\bullet$ to be

$$\mathbb{H}^k(R, \mathcal{S}^\bullet) = H^k(\Gamma(R, \mathrm{Tot}^\bullet(\mathcal{S}^\bullet)))$$

We want to relate $\mathbb{H}^k(R, \mathcal{S}^\bullet)$ to the cohomology of the individual sheaves $\mathcal{S}^i$. Note the vertical differential computes exactly $H^*(R, \mathcal{S}^i)$. So if all the horizontal differentials were zero, we would have

$$\mathbb{H}^k(R, \mathcal{S}^\bullet) = \bigoplus_{i+j=k} H^j(R, \mathcal{S}^i) = H^0(R, \mathcal{S}^k) \oplus H^1(R, \mathcal{S}^{k-1})$$

In general, the horizontal differentials are not zero and we only have the following approximation.

Theorem 2.37 (Hypercohomology long exact sequence)**.** There is a long exact sequence

$$\cdots \to \ker d^k|_{H^0(R,\mathcal{S}^k)} \Big/ \operatorname{im} d^{k-1}|_{H^0(R,\mathcal{S}^{k-1})} \to \mathbb{H}^k(R, \mathcal{S}^\bullet) \to \ker d^{k-1}|_{H^1(R,\mathcal{S}^{k-1})} \Big/ \operatorname{im} d^{k-2}|_{H^1(R,\mathcal{S}^{k-2})} \to \cdots$$

We will get this theorem as a special case of the spectral sequence associated to a double complex. This presents the hypercohomology $\mathbb{H}^k(R, \mathcal{S}^\bullet)$ as an extension of a subspace of the right term by a quotient of the left term. These terms are computable from $H^*(R, \mathcal{S}^i)$ and the differentials d^i.

This is an example of the spectral sequence associated to a double complex.

$$T^k = \bigoplus_{p+q=k} C^{p,q}$$

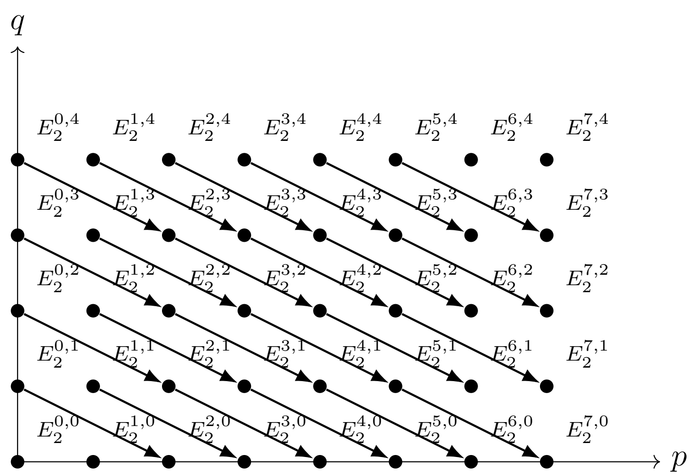
Then we can compute the hypercohomology of the total complex knowing something about the vertical differential. We can do a page by page computation, taking the total complex as the E^0

page. Then the E^1 page is given by taking the cohomology with respect to the vertical differential. The only maps which survive are the horizontal differentials induced on the cohomology groups.

$$\cdots \longrightarrow H^{q+1}(C^{p,\bullet}) \xrightarrow{d_1} H^{q+1}(C^{p+1,\bullet}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{q}(C^{p,\bullet}) \xrightarrow{d_1} H^{q}(C^{p+1,\bullet}) \longrightarrow \cdots$$

The E^2 page looks like:



Theorem 2.38. There exists an induced second differential of bidegree $(2, -1)$ on the E^2 page

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$$

Our example had only two rows, so the d_2 differential was zero. In general, one can continue this process to get higher differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

Theorem 2.39. If or when the procedure stops, the $E_\infty^{p,q}$ page contains the associated graded of the cohomology of the total complex with respect to the filtration by the horizontal differential. More precisely, there is a filtration

$$0 = F^{n+1}H^k \subset F^n H^k \subset \cdots \subset F^0 H^k = H^k(\mathrm{Tot}^\bullet(C^{\bullet,\bullet}))$$

such that

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$$

Proof of Theorem 2.37. The existence of the long exact sequence follows from applying the general theory. Suppose we have a spectral sequence $E_2^{p,q}$ converging to L^{p+q} and we are in the nice case $E_2 = E_\infty$. So for each total degree n there is a finite filtration

$$0 = F^{n+1}L^n \subset F^n L^n \subset \cdots \subset F^1 L^n \subset F^0 L^n = L^n$$

Then we can write down the beginning of a long exact sequence by looking at total degree 1:

$$0 \;\to\; E_2^{1,0} \;\to\; L^1 \;\to\; E_2^{0,1} \;\xrightarrow{d_2}\; E_2^{2,0}$$

where the first map is the inclusion of F^1L^1 into L^1 and the second map is the projection onto F^0L^1/F^1L^1. Now we can proceed by looking at total degree 2. We have a filtration

$$0 = F^3L^2 \subset F^2L^2 \subset F^1L^2 \subset F^0L^2 = L^2$$

with associated graded pieces

$$E_\infty^{2,0} \cong F^2L^2, \quad E_\infty^{1,1} \cong F^1L^2/F^2L^2, \quad E_\infty^{0,2} \cong F^0L^2/F^1L^2$$

Thus we have a short exact sequence

$$E_2^{2,0} \;\to\; \ker(L^2 \to E_2^{0,2}) \;\to\; E_2^{1,1} \;\xrightarrow{d_2}\; E_2^{3,0}$$

where the first map comes from the identification $\ker(L^2 \to E_2^{0,2}) = F^1L^2$ and the second map is the projection onto F^1L^2/F^2L^2. Splicing this with the previous exact sequence gives the long exact sequence

$$0 \to E_2^{1,0} \to L^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to \ker\big(L^2 \to E_2^{0,2}\big) \to E_2^{1,1} \xrightarrow{d_2} E_2^{3,0}$$

This is exactly the long exact sequence in the statement of Theorem 2.36 since $E^{3,0}$ vanishes. More generally, we can proceed by looking at total degree n and splicing the resulting short exact sequence with the previous long exact sequence to get the full long exact sequence.

$$\cdots \longrightarrow E_2^{p,q} \longrightarrow \ker(L^{p+q} \to E_2^{p-2,q+2}) \longrightarrow E_2^{p-1,q+1} \xrightarrow{d_2} E_2^{p+1,q} \longrightarrow \cdots$$

□

2.5 Extensions of vector bundles

Let

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence of vector bundles on a compact Riemann surface R. On a small open set U we get an exact sequence of sheaves of sections

$$0 \to V'(U) \to V(U) \to V''(U) \to 0$$

which are all free $\mathcal{O}(U)$-modules of finite rank. A splitting of this sequence is given by

$$s : V''(U) \to V(U)$$

a choice of free generators of $V''(U)$ lifted to $V(U)$, and can be thought of as a local choice of frame of V'' (i.e. a choice of inclusion $U \times \mathbb{C}^{\operatorname{rank} V''} \to U \times \mathbb{C}^{\operatorname{rank}} V$). Such a splitting always exists locally, however there may not exist a global splitting.

Example 2.40. Let $R = \mathbb{P}^1$ and consider the sequence

$$0 \to \mathcal{O} \to \mathcal{O}(0) \oplus \mathcal{O}(\infty) \to \mathcal{O}(0 + \infty) \to 0$$

where the maps are the diagonal inclusion and the subtraction map $(a, b) \mapsto a - b$. This sequence is exact.

Exactness is clear away from 0 and ∞. Near 0, injectivity and surjectivity are clear. To see exactness in the middle, suppose (a, b) maps to zero. Then $a = b$ as meromorphic functions on $\mathbb{P}^1$ and a has no pole at 0, b has no pole at ∞, so $a = b$ is a constant function. Therefore, (a, b) is in the image of the inclusion.

However this sequence is not split because

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \ncong \mathcal{O} \oplus \mathcal{O}(2)$$

because we can tensor by $\mathcal{O}(-2)$ and take global sections to see that the dimensions of global sections differ.

Theorem 2.41. An exact sequence of vector bundles on a compact Riemann surface

$$0 \to V' \to V \to V'' \to 0$$

determines a cohomology class in

$$H^1(R, \mathcal{H}om(V'', V')) \cong H^1(R, V' \otimes (V'')^*)$$

and the sequence splits if and only if this class is zero.

Example 2.42. In the previous example, we calcuated the unique (up to scale) nonsplit extension of $\mathcal{O}(2)$ by $\mathcal{O}$, which determines a cohomology class in

$$H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong \mathbb{C}$$

Addition of extensions corresponds to the operation

$$(A \to E \to B) + (A \to E' \to B) = A \to E \oplus_B E' \to B$$

which makes sense in any category with kernels and cokernels.

Proof. First we construct the class. Recall that we defined

$$H^1(R,V) = \Gamma(R,\mathcal{P}(V))/\Gamma(R,\mathcal{M}(V))$$

There is a diagram of vector spaces

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}(V')(R) \arrow[r] \arrow[d, "p'"] & \mathcal{M}(V)(R) \arrow[r] \arrow[d, "p"] & \mathcal{M}(V'')(R) \arrow[r] \arrow[d, "p''"] & 0 \\
0 \arrow[r] & \mathcal{P}(V')(R) \arrow[r] & \mathcal{P}(V)(R) \arrow[r, "Q"] & \mathcal{P}(V'')(R) \arrow[r] & 0
\end{tikzcd}$$

and since the top row is a sequence of finite dimensional vector spaces over $\mathcal{M}(R)$, we can product a splutting

$$s : \mathcal{M}(V'')(R) \to \mathcal{M}(V)(R)$$

This splitting in fact defines a splitting of sheaves

$$s : \mathcal{M}(V'') \to \mathcal{M}(V)$$

by restricting to smaller open sets.

Now observe that $p \circ s : \mathcal{O}(V'') \to \mathcal{P}(V)$ in fact lands in $\mathcal{P}(V')$ because

$$Q \circ p \circ s = p''$$

and p'' vanishes on $\mathcal{O}(V'')$. This gives a map $\mathcal{O}(V'') \to \mathcal{P}(V')$ of $\mathcal{O}$-modules. Dualizing, we get

$$1 \in \mathcal{O} \to \mathcal{P}(V' \otimes V''^{\vee})$$

Changing s changes the result by a meromorphic section of V'. So its class in cohomlogy is well defined, independent of s. $\square$

Remark 2.43. On $\mathbb{P}^1$, every vector bundle is a sum of line bundles and we still have nontrivial extensions. On higher genus Riemann surfaces, vector bundles are characterized by the Harder-Narasimhan filtration.

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

with successive quotients semistable and decreasing slopes. The extensions between the successive quotients are nontrivial in general.

In general, $0 \to V' \to V \to V'' \to 0$ we should expect that

$$\frac{\deg V''}{\operatorname{rank} V''} \geq \frac{\deg V'}{\operatorname{rank} V'}$$

and $H^1(R, \mathcal{H}om(V'', V'))$ is bigger if degree of V'' is large or degree of V' is small.

2.6 Serre duality

Let R be a compact Riemann surface, V vector bundle on R. We have the canonical bundle $K_R = \Omega^1_R$ of holomorphic one-forms on R.

Theorem 2.44 (Serre duality)**.** There is an isomorphism of vector spaces, functorial in R:

$$H^0(R, V) \cong H^1(R, K_R \otimes V^*)^*$$
$$H^1(R, V) \cong H^0(R, K_R \otimes V^*)^*$$

Functorial means that for $f : R \to S$ a nonconstant map of compact Riemann surfaces, we have the following commutative diagram:

$$\begin{tikzcd} H^0(R,V) \arrow[r, "\cong"] \arrow[d, "f^*"] & H^1(R, K_R \otimes V^*)^* \arrow[d, "f^*"] \\ H^0(S,V) \arrow[r, "\cong"] & H^1(S, K_S \otimes V^*)^* \end{tikzcd}$$

We will say a little bit more about the map on H^1.

Example 2.45. We check the isomorphism for $\mathcal{O}(n)$ on $\mathbb{P}^1$ for $n \geq -1$. Move to $\mathcal{O}(n\infty)$. Check the residue pairing at $z = 0$. May assume the principal parts are at 0 and that meromorphic functions have only poles at 0. In this case

$$H^0(\mathbb{P}^1, \mathcal{O}(n\infty)) = \operatorname{span}\{1, z, z^2, \ldots, z^n\}$$
$$H^1(\mathbb{P}^1, K_{\mathbb{P}^1} \otimes \mathcal{O}(-n\infty)) = \{\frac{dz}{z}, \frac{dz}{z^2}, \ldots, \frac{dz}{z^{n+1}}\}$$

because by the principal parts argument $dz/z^{n+2} = z^{-n}dz/z^n$ is holomorphic at ∞. The residue pairing is clear.

Proof. The global residue pairing

$$H^0(R, V) \times H^1(R, K_R \otimes V^*) \to \mathbb{C}$$

comes from a local residue pairing at a point $x \in R$.

$$\mathcal{O}(V)_x \otimes \mathcal{P}(K_R \otimes V^*)_x \to \mathbb{C}$$
$$(\sigma, \tau) \mapsto \operatorname{Res}_x(\langle \sigma, \tau \rangle)$$

In a local coordinate z near x, the pairing takes the form

$$z^n u \otimes \frac{v}{z^m} dz \mapsto \operatorname{Res}_x\left(\frac{uv}{z^{m-n}} dz\right) = \begin{cases} u(x)v(x) & m = n+1 \\ 0 & \text{otherwise} \end{cases}$$

Note that the holomorphic part of a differential has no residue so the map is defined on $\mathcal{P}(K_R \otimes V^*)_x$. The pairing is nondegenerate. The global residue pairing then is simply the sum of the local pairings over all points $x \in R$.

We will reduce the proof to the case $R = \mathbb{P}^1$. In this case, we have already explicitly verified the isomorphism. We wish to construct an isomorphism of sheaves on $\mathbb{P}^1$

$$f_*\big(K_R \otimes V^\vee\big) \;\cong\; K_{\mathbb{P}^1} \otimes (f_*V)^\vee. \tag{4}$$

Tensoring both sides by $K_{\mathbb{P}^1}^{-1}$, this is equivalent to

$$K_{\mathbb{P}^1}^{-1} \otimes f_*(K_R \otimes V^\vee) \;\cong\; (f_*V)^\vee. \tag{5}$$

By the projection formula,

$$K_{\mathbb{P}^1}^{-1} \otimes f_*(K_R \otimes V^\vee) \;\cong\; f_*\big(f^*K_{\mathbb{P}^1}^{-1} \otimes K_R \otimes V^\vee\big),$$

so (5) is equivalent to the existence of an isomorphism

$$f_*\big(f^*K_{\mathbb{P}^1}^{-1} \otimes K_R \otimes V^\vee\big) \;\cong\; (f_*V)^\vee. \tag{6}$$

By Lemma 2.35, we have

$$(f_*V)^\vee \;\cong\; f_*\big(V^\vee(B)\big),$$

where B is the branch divisor of f. Hence (6) will follow provided we can identify the bundles upstairs:

$$f^*K_{\mathbb{P}^1}^{-1} \otimes K_R \otimes V^\vee \;\cong\; V^\vee(B). \tag{7}$$

Cancelling $V^\vee$, condition (7) is equivalent to the line-bundle identity

$$K_R \otimes f^*K_{\mathbb{P}^1}^{-1} \;\cong\; \mathcal{O}_R(B), \tag{8}$$

which is precisely the Riemann–Hurwitz formula in line-bundle form.

To check the compatibility of residue pairings on R and $\mathbb{P}^1$. We can use the isomorphisms

$$f_*\mathcal{P}_x(V) \;\cong\; \mathcal{P}_{f(x)}(f_*V),$$

and the discussion of differentials at branch points. Choose local coordinates in standard form and write the residue pairings in the Laurent basis. $\square$

Now we state some consequences of Riemann-Roch and Serre duality:

- The degree of the canonical bundle is

$$\deg K_R = 2g - 2$$

In particular every meromorphic differential has exactly $2g - 2$ zeros (counted with multiplicity). This agrees with the statement of the Poincare-Hopf theorem for vector fields, which says that if M is a compact oriented surface, then every vector field with isolated zeros has total index equal to the Euler characteristic $\chi(M) = 2 - 2g$. For a holomorphic differential, the index of a zero is its multiplicity as a zero of the differential.

- For all R with genus $g \geq 1$, for all $p \in R$ there exists a nontrivial holomorphic differential not vanishing at p. If not, then all holomorphic differentials zeroes at p and so $h^0(R, K_R(-p)) = g$. This implies that $h^0(R, \mathcal{O}(p)) = 2$ so there exists a nonconstant meromorphic function with a single simple pole at p. This function gives an isomorphism to $\mathbb{P}^1$ and so $g = 0$, a contradiction.

- There exists a map of degree $g + 1$ from R to $\mathbb{P}^1$. This is because

$$h^0(R, \mathcal{O}((g+1)p)) = h^1(R, K_R(-(g+1)p)) + (g+1) - g + 1 \geq 2$$

- Every genus 2 curve has a degree 2 map to $\mathbb{P}^1$ and is therefore hyperelliptic. This is because for $g = 2$, we have $h^0(\Omega^1) = 2$ so there exist two linearly independent holomorphic differentials ω_1, ω_2. They each have degree $2g - 2 = 2$ zeros. If they have a common zero at p, then ω_1/ω_2 is a meromorphic function with a single simple pole at p, which is impossible on a genus 2 curve.

For $\deg L \geq 2g - 1$, one always has $h^0(L(p)) = h^0(L) + 1$ for all $p \in R$. However, for smaller degree line bundles, this can fail. The failure is measured by the following theorem.

Theorem 2.46 (Weierstrass gap theorem)**.** At each point p of a compact Riemann surface of genus g, there are exactly g integers

$$1 = n_1(p) < n_2(p) < \cdots < n_g(p) \leq 2g - 1$$

such that

$$h^0(R, \mathcal{O}(n_i(p)p)) = h^0(R, \mathcal{O}((n_i(p) - 1)p))$$

These integers are called the Weierstrass gaps at p.

Proof. $g = 0$ case is clear since we have explicit meromorphic functions on $\mathbb{P}^1$ with a single pole of any order at any point. So we assume $g \geq 1$. A positive integer n is a non-gap at p if there exists a meromorphic function whose only pole is at p and has order exactly n.

For any divisor D and point p, $0 \leq l(D+p) - l(D) \leq 1$, so in particular every $n \geq 1$ is either a gap or a non-gap. Also, for $g \geq 1$, 1 is always a gap: a function with a single simple pole at p would give a degree-1 map $X \to \mathbb{P}^1$, so $X \cong \mathbb{P}^1$ (genus 0), contradiction.

Riemann-Roch and Serre duality immediately imply that every $m \geq 2g-1$ is a non-gap. Hence all gaps lie in $\{1, \ldots, 2g-1\}$.

Let $G(m)$ be the number of gaps in $\{1, \ldots, m\}$. The number of non-gaps in that interval are is $m - G(m)$. Starting from $l(0) = 1$ and using that each non-gap increases l by 1 while each gap leaves it unchanged, for any $m \geq 1$, which implies that $l(mp) = 1 + (m - G(m))$. Now take $m \geq 2g$. Then $l(mp) = m + 1 - g$ and equating, we get $G(m) = g$ for all $m \geq 2g - 1$. $\square$

The generic sequence of gaps is

$$1, 2, \ldots, g$$

A point p is called a **Weierstrass point** if its gap sequence differs from the generic one. For example, on a hyperelliptic curve, the Weierstrass points are the branch points of the degree 2 map to $\mathbb{P}^1$ and have gap sequence

$$1, 3, 5, \ldots, 2g-1$$

There are always finitely many Weierstrass points on a compact Riemann surface of genus $g \geq 2$.

Exercise 2.47. A nodal Riemann surface S is one which is smooth except for finitely many singularities that look locally like

$$\{(x, y) \mid xy = 0\} \subset \mathbb{C}^2.$$

It is obtained from a smooth Riemann surface $\widetilde{S}$ by identifying pairs of points; $\widetilde{S} \to S$ is called the normalization. Define the canonical bundle K_S of S to be the sheaf of differentials holomorphic on $\widetilde{S}$, except for simple poles at the nodes, with opposite residues on the two branches of S.

Show that K_S is a line bundle. When S is compact, prove the residue theorem for a meromorphic section of K_S that is holomorphic at the nodes. Prove Serre duality for vector bundles on S.

Solution 2.48. Let $\nu : \widetilde{S} \to S$ be the normalization. Each node of S has preimage a pair $\{p_i^+, p_i^-\} \subset \widetilde{S}$. Set $N := \sum_i (p_i^+ + p_i^-)$.

Let $K_{\widetilde{S}}$ be the canonical bundle on the smooth Riemann surface $\widetilde{S}$. Consider $K_{\widetilde{S}}(N)$, whose sections are meromorphic 1–forms with at worst simple poles at the points of N.

Define K_S as the subsheaf of $\nu_* K_{\widetilde{S}}(N)$ consisting of sections ω such that for each node with preimages $p_i^\pm$,

$$\operatorname{Res}_{p_i^+}(\omega) + \operatorname{Res}_{p_i^-}(\omega) = 0.$$

Equivalently, we have an exact sequence

$$0 \longrightarrow K_S \longrightarrow \nu_* K_{\widetilde{S}}(N) \xrightarrow{\text{Res}} \bigoplus_i \underline{\mathbb{C}}_{\text{node } i} \longrightarrow 0, \tag{nodal}$$

where $\underline{\mathbb{C}}_{\text{node } i}$ is the skyscraper sheaf at the i-th node and the residue map takes ω to the collection of sums of residues at $p_i^{\pm}$, explicitly at the node i:

$$\text{Res}_i(\omega) = \text{Res}_{p_i^+}(\omega) + \text{Res}_{p_i^-}(\omega)$$

We claim K_S is a line bundle. Over smooth points of S, ν is an isomorphism, and the residue condition is vacuous, so K_S agrees with $K_{\widetilde{S}}$; in particular it has rank 1 there.

Near a node, choose local coordinates so $\widetilde{S}$ is the disjoint union of two discs with coordinates z and w, and the node identifies $z = 0$ with $w = 0$. A section of $K_{\widetilde{S}}(N)$ has local form

$$\left(\frac{a}{z} + \text{hol}\right) dz \quad \text{on the } z\text{–branch}, \qquad \left(\frac{b}{w} + \text{hol}\right) dw \quad \text{on the } w\text{–branch}.$$

The condition $a + b = 0$ is one linear relation on the two residues, so the allowed principal parts form a 1-dimensional space. Thus the stalk of K_S at the node is also 1-dimensional.

Assume S compact. Let ω be a meromorphic section of K_S which is holomorphic at the nodes. Viewed on $\widetilde{S}$, this is a meromorphic 1–form $\widetilde{\omega}$ which has the same poles away from the preimages of nodes and is holomorphic at each $p_i^{\pm}$ (since ω has no poles at nodes).

Apply the usual residue theorem on $\widetilde{S}$:

$$\sum_{q \in \widetilde{S}} \text{Res}_q(\widetilde{\omega}) = 0.$$

There is no contribution from $p_i^{\pm}$, so this sum is exactly $\sum_{p \in S} \text{Res}_p(\omega)$. Hence

$$\sum_{p \in S} \text{Res}_p(\omega) = 0$$

Let E be a holomorphic vector bundle on S. We claim there is a natural perfect pairing

$$H^1(S, E) \times H^0(S, K_S \otimes E^\vee) \longrightarrow \mathbb{C},$$

inducing an isomorphism

$$H^1(S, E)^\vee \cong H^0(S, K_S \otimes E^\vee).$$

Hence $H^1(S, E)$ identifies with a subspace of $H^1(\widetilde{S}, \nu^* E)$ cut out by linear relations at the nodes. In particular, using the principal parts characterization of H^1, a class in $H^1(S, E)$ corresponds to a collection of principal parts of E at points of $\widetilde{S}$ (the poles of the sections) such that at each node the principal parts at $p_i^{\pm}$ have equal values residues in a common trivialization of E near the node.

Similarly, a section of $K_S \otimes E^\vee$ corresponds to a section

$$\phi \in H^0\big(\widetilde{S}, K_{\widetilde{S}}(N) \otimes \nu^* E^\vee\big)$$

such that at each node the residues at $p_i^{\pm}$ are opposite, i.e. to a subspace of $H^0\big(\widetilde{S}, K_{\widetilde{S}}(N) \otimes \nu^* E^\vee\big)$ defined by linear relations.

On the smooth curve $\widetilde{S}$ we have the usual Serre duality pairing

$$H^1(\widetilde{S}, \nu^* E) \times H^0\big(\widetilde{S}, K_{\widetilde{S}} \otimes (\nu^* E)^\vee\big) \to \mathbb{C},$$

which extends to allow simple poles at N, giving a perfect pairing

$$H^1(\widetilde{S}, \nu^* E) \times H^0\big(\widetilde{S}, K_{\widetilde{S}}(N) \otimes (\nu^* E)^\vee\big) \to \mathbb{C}.$$

The node-compatibility conditions defining $H^1(S, E)$ and $H^0(S, K_S \otimes E^\vee)$ are dual to each other with respect to this pairing. In particular, locally the principal part (v^+, v^-) pairs with (ℓ^+, ℓ^-) by $\ell^+(v^+) - \ell^-(v^-)$.

Thus its restriction yields a perfect pairing

$$H^1(S, E) \times H^0(S, K_S \otimes E^\vee) \to \mathbb{C}.$$

3 Abel Jacobi theory

Recall the additive and multiplicative Cousin problems:

- Additive Cousin: Given a collection of principal parts on R, when does there exist a global meromorphic function with those principal parts?

- Multiplicative Cousin: Given a divisor D on R, when does there exist a global meromorphic function f with divisor $(f) = D$?

We have seen that for $g = 0$, the additive Cousin problem always has a solution and the multiplicative Cousin problem has a solution if and only if the degree of the divisor is zero. For $g = 1$, the additive Cousin problem has a solution if and only if the sum of the residues is zero and the multiplicative Cousin problem has a solution if and only if the degree of the divisor is zero and the sum of the points (with multiplicity) modulo the period lattice is zero.

In general, the obstruction in the additive Cousin problem lies in $H^1(R, \mathcal{O})$ because we defined it to be so:

$$H^1(R, \mathcal{O}) = \Gamma(R, \mathcal{P}) / \Gamma(R, \mathcal{M})$$

where $\mathcal{P}$ is the sheaf of principal parts and $\mathcal{M}$ is the sheaf of meromorphic functions. We proved the Serre duality isomorphism

$$H^1(R, \mathcal{O}) \cong H^0(R, K_R)^*$$

so the obstruction to solving the additive Cousin problem can be computed by pairing with holomorphic one-forms. In particular, a collection of principal parts $\{p_i\}$ has a solution if and only if

$$\sum_i \mathrm{Res}_{x_i}(p_i \omega) = 0$$

for all $\omega \in H^0(R, K_R)$. The obstruction in the multiplicative Cousin problem is encoded by the Abel-Jacobi map, which this section is devoted to describing.

3.1 Differentials on a Riemann surface

Recall that on a Riemann surface R, we have the Hodge star operator defined on all smooth differential 1-forms

$$* : \Omega^1_{sm}(R) \to \Omega^1_{sm}(R)$$

defined in local holomorphic coordinates $z = x + iy$ by

$$*(f\,dx + g\,dy) = -g\,dx + f\,dy$$

so that

$$\begin{aligned} *(dz) &= -i\,dz \\ *(d\overline{z}) &= i\,d\overline{z} \end{aligned}$$

The Hodge star operator satisfies

$$** = -1$$

on 1-forms. A differential form ω is called **harmonic** if

$$\begin{aligned} d\omega &= 0 \\ d * \omega &= 0 \end{aligned}$$

Recall that we had a decomposition of the square integrable smooth 1 forms on a compact Riemann surface R

$$L^2\Omega^1_{sm}(R) = (dC^\infty(R)) \oplus H \oplus (*dC^\infty(R))$$

where H is the space of harmonic 1-forms. Moreover, H consists of smooth forms. Also recall that we defined an inner product on $L^2\Omega^1_{sm}(R)$ by

$$\langle \alpha, \beta \rangle = \int_R \alpha \wedge *\beta$$

The decomposition is orthogonal with respect to this inner product in the sense that

$$\langle df, *dg \rangle = 0$$

for all smooth functions f, g on R.

Any closed differential form has $2g$ periods given by integrating over the edges of the polygonal decomposition of R. The form is exact if and only if all its periods vanish.

1. In particular, the space of harmonic forms is at most $2g$-dimensional because it injects into the space of period vectors $\mathbb{R}^{2g}$.

2. There are g linearly independent holomorphic differentials on R and g linearly independent antiholomorphic differentials on R, where holomorphic and antiholomorphic are defined as the $\pm i$ eigenspaces of the Hodge star operator $*$. Since eigenspaces don't intersect, we have

   $$\dim H \geq 2g$$

   Combining with the previous point, we have

   $$\dim H = 2g$$

   and the holomorphic and antiholomorphic differentials give bases for the eigenspaces.

Let $a_1, \dots, a_g, b_1, \dots, b_g$ be the edges of the polygonal decomposition of R. Let ϕ, ϕ' be holomorphic differentials, and $A_i = \int_{a_i} \phi$, $B_i = \int_{b_i} \phi$ and similarly for ϕ'. Using Stokes' theorem, we can compute

$$0 = \int_R \phi \wedge \phi' = \sum_{i=1}^{g} \left(\int_{a_i} \phi \int_{b_i} \phi' - \int_{b_i} \phi \int_{a_i} \phi' \right) = \sum_{i=1}^{g} (A_i B_i' - B_i A_i')$$

where the second equality comes from writing $\phi = dF$ (locally! there is monodromy) so that

$$\begin{aligned}
\int_X \phi \wedge \phi' &= \int_P dF \wedge \phi' \\
&= \int_P d(F\phi') - F\, d\phi' \\
&= \oint_{\partial P} F\, \phi' \\
&= \sum_{i=1}^{g} \left(\int_{a_i} F\phi' + \int_{b_i} F\phi' - \int_{b_i^{-1}} F\phi' - \int_{a_i^{-1}} F\phi' \right) \\
&= \sum_{i=1}^{g} \left(\int_{a_i} F\phi' - \int_{a_i^{-1}} (F\phi') \right) + \sum_{i=1}^{g} \left(\int_{b_i} F\phi' - \int_{b_i^{-1}} (F\phi') \right)
\end{aligned}$$

and the difference between integrating over a_i and a_i^{-1} is given by the period of ϕ over the cycle, that is precisely A_i. Similarly for b_i. We also have the relation

$$\sum_i (A_i \overline{B_i} - B_i \overline{A_i}) = i\langle \phi, \phi' \rangle$$

We have the following consequences:

- For any nonzero holomorphic differential ϕ

$$\Im(\sum_i A_i \overline{B_i}) > 0$$

- A holomorphic differential with all periods real is zero. This is because

$$\sum_i (A_i \overline{B_i} - B_i \overline{A_i}) = 0 = i < \phi, \phi >$$

  so $\phi = 0$.

- We can choose a basis of holomorphic differentials $\phi_1, \ldots, \phi_g$ such that

$$\int_{a_i} \phi_j = \delta_{ij}$$

  Then the period matrix

$$\Pi_{ij} = \int_{b_i} \phi_j$$

is symmetric with positive definite imaginary part. In particular

$$[A|B] \to [I|\Pi]$$

where $[A|B]$ is the old period matrix, Π is symmetric and $\Im\Pi$ is positive definite.

- The periods form a lattice in $\mathbb{C}^g$ of rank $2g$.

We are finally able to give a criterion for when a divisor is principal, completing the multiplicative Cousin problem.

Theorem 3.1 (Abel's theorem)**.** A divisor $D = \sum n_i p_i$ on a compact Riemann surface R is principal if and only if

$$\deg D = 0$$
$$\sum n_i \int_{p_0}^{p_i} \vec{\Phi} \in \Lambda$$

for all $j = 1, \dots, g$, where $\vec{\Phi} = (\phi_1, \dots, \phi_g)$ is a vector whose entries are a basis of holomorphic differentials normalized so that $\int_{a_i} \phi_j = \delta_{ij}$ and Λ is the period lattice generated by the integrals of the ϕ_j over the a_i and b_i cycles.

Remark 3.2. The Schottky problem asks for a characterization of which complex tori arise as the Jacobian of a Riemann surface. In other words, which period matrices Π arise from Riemann surfaces. For $g = 2, 3$ it is an open subset of the space of symmetric $g \times g$ matrices with positive definite imaginary part. For $g = 4$ it is a hypersurface. The problem is still open.

3.2 Jacobian of a curve

Definition 3.3. The **Jacobian variety** of a compact Riemann surface R of genus g is the complex torus

$$J(R) = \mathbb{C}^g / \Lambda$$

where Λ is the period lattice generated by the integrals of a basis of holomorphic differentials over the a_i and b_i cycles.

The Jacobian is independent of the choice of basis of holomorphic differentials and choice of symplectic basis of $H_1(R, \mathbb{Z})$ up to isomorphism of complex tori. It is a compact complex manifold of dimension g, and an abelian group. One can show that every connected abelian group which is a complex manifold (and where the group structure is holomorphic) is a quotient of $\mathbb{C}^g$ by a co-compact lattice. The specific properties of $[A|B]$ ($A^{-1}B$ is symmetric with positive definite imaginary part) impose a further restriction. They give rise to principally polarized abelian varieties.

We can give another interpretation of Abel's theorem.

Theorem 3.4 (Abel's theorem)**.** The kernel of the Abel-Jacobi map

$$\mathrm{Div}^0(R) \to J(R)$$
$$D = \sum n_i p_i \mapsto \sum n_i \int_{p_0}^{p_i} \vec{\Phi}$$

is precisely the group of principal divisors on R. In particular, the Abel-Jacobi map descends to an isomorphism

$$\mathrm{Pic}^0(R) \to J(R)$$

J is isomorphic to the group $\mathrm{Div}^0(R)/\,\mathrm{PDiv}(R)$ of isomorphism classes of degree-zero line bundles on R.

On line bundles, the group operation is the tensor product. Note that both sides have natural holomorphic structures: the divisor side can be built from symmetric powers of R, and the period map to J was given by integrating holomorphic forms, and is thus holomorphic: so we should suspect a stronger statement behind this.

Choose a point $r \in R$. On the product $R \times J$, there is the family of holomorphic line bundles over R parametrized by the points of J, with trivialized fibers over the point r. Without trivializing at r, each line bundle has a group $\mathbb{C}^\times$ of automorphisms, and the family of line bundles is not uniquely determined: for instance, you can tensor it with your favorite line bundle over the base J, without changing the isomorphism type on any factor of R.

Remark 3.5 (Rigidity and the Poincaré line bundle)**.** Set $p : R \times J \to R$ and $q : R \times J \to J$. Given a family L on $R \times J$ and a line bundle M on J, define

$$L' = L \otimes q^* M$$

For any fixed $j \in J$, when we restrict to the fiber $R \times \{j\}$ we have:

$$L'\big|_{R\times\{j\}} \cong L\big|_{R\times\{j\}} \otimes \big(q^* M\big)\big|_{R\times\{j\}} \cong L_j \otimes (M_j \otimes \mathcal{O}_R)$$

where M_j is the one-dimensional vector space (the fiber of M at j). As a line bundle on R, $M_j \otimes \mathcal{O}_R$ is just the trivial line bundle on R since there's no R-variation. Hence

$$L_j \otimes (M_j \otimes \mathcal{O}_R) \cong L_j$$

non-canonically (requiring a choice of basis of M_j). So for each j, the isomorphism class in $\mathrm{Pic}(R)$ is unchanged, meaning twisting by $q^* M$ does not change the isomorphism type on any R-fiber.

Globally over $R \times J$ though, L and $L \otimes q^* M$ are typically not isomorphic unless M is (canonically) trivial—the noncanonical choices of bases at each j don't glue holomorphically in j.

The point is that locally near each $j \in J$, you can choose a trivialization of M_j, i.e. a nonvanishing holomorphic section of M over a small neighborhood of j in J. This choice gives an isomorphism between L and $L \otimes q^*M$ over that neighborhood because we can use the local section to identify M_j with $\mathbb{C}$. However, if one tries to do this globally over all of J, this is the same as choosing a global nowhere-vanishing holomorphic section of M, which exists if and only if M is (canonically) trivial. So we see that twisting by q^*M changes L to a different family of line bundles on R parametrized by J, even though the fiberwise isomorphism classes remain the same.

For a family $L \to R \times B$, any automorphism over $R \times B$ is multiplication by a nowhere-vanishing holomorphic function $u \in \Gamma(R \times B, \mathcal{O}^\times)$. Because R is a compact Riemann surface, every holomorphic function on R is constant, so

$$\Gamma(R \times B, \mathcal{O}^\times) \cong \Gamma(B, \mathcal{O}_B^\times).$$

For each $b \in B$, $u(\cdot, b)$ is a nonvanishing holomorphic function on R, hence constant in the R-direction; these constants vary holomorphically in b.

Both issues are resolved by choosing a trivialization $\alpha : L|_{\{r\}\times J} \xrightarrow{\cong} \mathcal{O}_J$. This is the data of a global holomorphic section of the line bundle $L|_{\{r\}\times J}$ which is nowhere vanishing.

Any automorphism of L must now restrict to the identity on $\{r\} \times J$, forcing the multiplier in $\mathcal{O}_J^\times$ to be 1. So $\mathrm{Aut}(L, \alpha) = 1$. Additionally, twisting by q^*M sends the rigidified fiber to

$$(L \otimes q^*M)|_{\{r\}\times J} \cong \mathcal{O}_J \otimes M \cong M$$

which violates the chosen trivialization unless $M \cong \mathcal{O}_J$ with its chosen trivialization. Thus the $\mathrm{Pic}(J)$-torsor ambiguity also disappears.

> It turns out that these individual line bundles on R assemble to a holomorphic line bundle
>
> $$\mathcal{P} \;\to\; R \times J,$$
>
> the **universal** or **Poincaré** line bundle (of degree zero). These objects $\mathcal{P}$ and J can be characterized by a universal property:

Proposition 3.6. For any complex space B and holomorphic line bundle $L \to R \times B$, equipped with a trivialization on $\{r\} \times B$, there is a unique holomorphic map $f_L : B \to J$ and isomorphism $f_L^*\mathcal{P} \cong L$.

Remark 3.7. The trivialization condition may seem curious but is necessary to make the statement work. The actual moduli that universally parametrizes degree-zero line bundles on R is an **algebraic stack** with underlying space J and automorphisms forming a $\mathbb{C}^\times$-gerbe, a principal $B\mathbb{C}^\times$-bundle over J. This is split as $B\mathbb{C}^\times \times J$ by a choice of point $r \in R$.

Remark 3.8. Let C be a smooth projective curve over $k = \mathbb{C}$, and $J = \mathrm{Pic}^0_{C/k}$ be the Jacobian (the Picard scheme). For the unrigidified Picard stack $\mathbf{Pic}^0_{C/k}$, we work in the site $(\mathrm{Sch}/k)_{\mathrm{fppf}}$. For a k-scheme

B, the fibered category $\mathbf{Pic}^0_{C/k}(B)$ is the groupoid of line bundles L on $C \times B$ with $\deg(L|_{C\times\{b\}}) = 0$ for all $b \in B$. Morphisms are isomorphisms of line bundles over $C \times B$. For any object L, we have

$$\underline{\mathrm{Aut}}(L) \cong \mathbb{G}_{m,B} \quad (\text{i.e. } \mathrm{Aut}(L) = \Gamma(B, \mathcal{O}_B^\times))$$

because $\underline{\mathrm{End}}(L) \cong \mathcal{O}_{C\times B}$.

There is a natural 1-morphism of stacks $p : \mathbf{Pic}^0_{C/k} \to \mathrm{Pic}^0_{C/k} = J$ sending a family L to its class in the relative Picard functor $\mathrm{Pic}(C \times B/B)$. Crucially, twisting L by q^*M (with $q : C \times B \to B$, $M \in \mathrm{Pic}(B)$) does not change its image in $J(B)$.

For the rigidified stack $\mathbf{Pic}^{0,r}_{C/k}$ (choosing a basepoint $r \in C(k)$), we add a trivialization along the section $i_r : B \hookrightarrow C \times B$, $i_r(b) = (r, b)$. Objects over B are pairs (L, α) with L as above and

$$\alpha : i_r^* L \xrightarrow{\sim} \mathcal{O}_B$$

a nowhere-vanishing holomorphic frame along $\{r\} \times B$. Morphisms are isomorphisms $\phi : L \to L'$ satisfying $i_r^*\phi \circ \alpha = \alpha'$.

The key fact is that $\mathbf{Pic}^{0,r}_{C/k}$ is representable by the Jacobian J. More precisely, there is a canonical equivalence of stacks $\mathbf{Pic}^{0,r}_{C/k} \simeq J$, where the right side is viewed as the discrete stack $B \mapsto \mathrm{Hom}(B, J)$. This gives the universal property of the Poincaré line bundle $\mathcal{P}$ on $C \times J$.

In the language of gerbes, the map $p : \mathbf{Pic}^0_{C/k} \to J$ has fibers locally equivalent to $B\mathbb{G}_m$, band $\mathbb{G}_m$, and a class in $H^2_{\mathrm{fppf}}(J, \mathbb{G}_m) = \mathrm{Br}(J)$ obstructing a global universal line bundle. Choosing r splits the gerbe:

$$\mathbf{Pic}^0_{C/k} \simeq (B\mathbb{G}_m) \times J$$

This splitting is noncanonical without a basepoint, but canonical once r is fixed.

The degree-zero restriction is easy to remove. We denote by J_d the set of isomorphism classes of line bundles of degree d (so $J_0 = J$). We can change the degree of any given line bundle on R to zero by twisting by a multiple of your favorite point, and we get:

Proposition 3.9. The group $\mathrm{Pic}(R)$ of isomorphism classes of holomorphic line bundles on R fits into a short exact sequence

$$1 \longrightarrow J \longrightarrow \mathrm{Pic}(R) \xrightarrow{\deg} \mathbb{Z}$$

The sequence is split by a choice of point in R. However, the splitting does depend on the point.

For each degree $d \geq 0$, we have a holomorphic map

$$AJ_d : \mathrm{Sym}^d R \longrightarrow J_d,$$

defined by integrating the holomorphic differentials: choose an auxiliary base point $r \in R$, and then send a divisor $D \in \mathrm{Sym}^d R$ to the sum of integrals along a chosen collection of paths from r

to the points of D. A change of paths will change the integrals by an integral linear combination of periods, so the map to J_0 is well defined; now use the divisor $d\{r\}$ to identify J_0 with J_d, and the resulting map to J_d is now also independent of your choice of r. The image of the divisor D is the isomorphism class of the line bundle $\mathcal{O}(D)$. In particular, the image of the Abel-Jacobi map is precisely the set of isomorphism classes of line bundles which admit a nontrivial holomorphic section (i.e., those line bundles which can be represented as $\mathcal{O}(D)$ for some effective divisor D).

Remark 3.10 (Identifing the differential of AJ_d)**.** First we identify the tangent space of J_d at a point $[L]$ representing a line bundle L on R. Since $J_d \cong J_0 \cong H^0(R, K_R)^*/H_1(R, \mathbb{Z})$, we have

$$T_{J_d,[L]} \cong H^0(R, K_R)^*.$$

Use the Abel-Jacobi map in holomorphic 1-form coordinates. Pick a basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(R, K_R)$ and periods so that

$$AJ_1(p) = \left(\int_{r_0}^{p} \omega_1, \ldots, \int_{r_0}^{p} \omega_g \right) \in J$$

For reduced $D = p_1 + \cdots + p_d$, we have $AJ_d(D) = \sum_{i=1}^{d} AJ_1(p_i)$. Let $v = (v_1, \ldots, v_d) \in \bigoplus_i T_{p_i}R$ be a tangent to $\mathrm{Sym}^d R$ at D. Vary $p_i(t)$ with $p_i'(0) = v_i$. Then the derivative of an integral of a holomorphic form along a path with moving endpoint is just evaluation of the form at that endpoint against the velocity vector:

$$\left.\frac{d}{dt}\right|_{t=0} \int_{r_0}^{p_i(t)} \omega_k = \omega_k(p_i)(v_i)$$

Summing over i gives the differential

$$d(AJ_d)_D : \bigoplus_{i=1}^{d} T_{p_i}R \longrightarrow H^0(R, K_R)^*$$

$$(v_1, \ldots, v_d) \longmapsto \left(\omega \mapsto \sum_{i=1}^{d} \omega(p_i)(v_i) \right)$$

The situation is more delicate if D is not reduced. Choose a local coordinate z near each p_i. Write $\omega = f(z)\, dz$, so $f(z)$ is holomorphic.

A tangent vector to $\mathrm{Sym}^d R$ at D corresponds to a choice of infinitesimal deformation of each $m_i p_i$, i.e. an element of $H^0(\mathcal{O}_R(D)|_D) \cong \bigoplus_i \mathbb{C}[z]/(z^{m_i})$. That means you specify the coefficients of $(1, z, z^2, \ldots, z^{m_i - 1})$ at each p_i.

Then the differential of the Abel-Jacobi map acts by:

$$d(AJ_d)_D : \bigoplus_{i} \mathbb{C}[z]/(z^{m_i}) \longrightarrow H^0(K_R)^*$$

$$(v_{i,0}, v_{i,1}, \ldots, v_{i,m_i-1})_i \mapsto \left(\omega \mapsto \sum_i \sum_{k=0}^{m_i-1} \frac{1}{k!}\, \omega^{(k)}(p_i)\, v_{i,k}\right)$$

where $\omega^{(k)}(p_i)$ means the k-th derivative (the k-jet) of the local coefficient function $f(z)$ of $\omega = f(z)\,dz$.

In general, there is a uniform way to describe the differential valid for all D: The differential

$$d(AJ_d)_D : H^0(\mathcal{O}_R(D)|_D) \to H^1(\mathcal{O}_R)$$

is given by the connecting homomorphism in the long exact sequence

$$0 \to \mathcal{O}_R \to \mathcal{O}_R(D) \to \mathcal{O}_R(D)|_D \to 0$$

The Abel Jacobi map will hold the key to Riemann's solution of the multiplicative Cousin problem, but we need to settle some of its properties first. The word generic in the next theorem means away from an analytic subspace of strictly lower dimension Analytic subspaces are locally cut out by a collection of holomorphic functions; they are usually not submanifolds, but they are always stratified by locally closed submanifolds of decreasing dimensions.

Theorem 3.11 (Properties of the Abel-Jacobi map)**.** Let $AJ_d : \mathrm{Sym}^d(R) \to J_d$ be the Abel-Jacobi map which sends a degree d effective divisor D on R to the isomorphism class of the line bundle $\mathcal{O}(D)$.

(i) The fiber of AJ_d at a point $j \in J$ representing a line bundle L on R is isomorphic to the projective space $\mathbb{P}H^0(R; L)$. (This is empty if $h^0(R; L) = 0$.)

(ii) For $d \leq g$, the map AJ_d is generically injective, and a local embedding in J_d.

(iii) For $d = g$, it is also surjective, and therefore a generic isomorphism.

(iv) For $g \geq 2g - 1$, the map AJ_d realizes $\mathrm{Sym}^d(R)$ as a holomorphic projective bundle over J_d.

Proof.

1. A point in the fiber over j is an effective divisor D of degree d such that $\mathcal{O}(D) \cong L$. The set of such divisors is in bijection with the set of nonzero holomorphic sections of L, modulo scaling, which is precisely $\mathbb{P}H^0(R; L)$.

2. The fibers are connected. Local injectivity/isomorphism can be checked by computing the differential of AJ_d and checking that it is injective at generic points. The differential at $(p_1 + \cdots + p_d) \in \mathrm{Sym}^d R$ is given by the map

$$T_{p_1}R \oplus \cdots \oplus T_{p_d}R \to T_{AJ_d(p_1+\cdots+p_d)}J_d \cong H^0(R, K_R)^*$$
$$(v_1, \ldots, v_d) \mapsto \left(\omega \mapsto \sum_{i=1}^{d} \omega(v_i)\right)$$

The map fails to be injective if and only if there exist nonzero $v_i \in T_{p_i}R$ such that

$$\sum_{i=1}^{d} \omega(v_i) = 0$$

for all $\omega \in H^0(R, K_R)$. This is equivalent to saying that there exists a holomorphic differential ω vanishing at all p_i. By Riemann-Roch, for generic choices of p_i, there are no nontrivial holomorphic differentials vanishing at all p_i if $d \leq g$. Hence, the differential is injective at generic points, making AJ_d a local embedding.

3. The argument is the same as local injectivity.

4. By Riemann-Roch, for $d \geq 2g - 1$, we have

$$h^0(R, L) = d - g + 1$$

for all line bundles L of degree d because $H^1(R, L) = 0$ by Serre duality. Therefore, the fibers of AJ_d are all projective spaces of dimension $d - g$. This gives $\mathrm{Sym}^d(R)$ the structure of a holomorphic projective bundle over J_d. The map is holomorphic submersion because the differential is surjective at all points, and moreover the map is proper since the domain and target are compact. Hence, by Ehresmann's fibration theorem, it is a holomorphic fiber bundle with fibers $\mathbb{P}^{d-g}$. $\square$

Remark 3.12 (Non-effective line bundles of nonnegative degree)**.** The discussion of the Abel Jacobi map shows that for $g \geq 1$, there exist line bundles of degree $0 \leq d < g$ with no nontrivial holomorphic sections.

Consider the Abel-Jacobi map

$$i_r : X \longrightarrow \mathrm{Pic}^0(X), \qquad s \longmapsto \mathcal{O}_X(s - r).$$

Its image is a 1-dimensional subvariety of the g-dimensional torus $\mathrm{Pic}^0(X)$ and hence cannot be all of $\mathrm{Pic}^0(X)$.

Pick a degree-zero line bundle $T \in \mathrm{Pic}^0(X)$ not in $i_r(X)$. Set

$$L = \mathcal{O}_X(r) \otimes T \in \mathrm{Pic}^1(X).$$

Then $\deg L = 1$, but $h^0(X, L) = 0$. This is because if $h^0(X, L) > 0$ then $L \simeq \mathcal{O}_X(p)$ for some $p \in X$. Hence $T \simeq L \otimes \mathcal{O}_X(-r) \simeq \mathcal{O}_X(p - r)$, so $T \in i_r(X)$, contradicting the choice of T. Thus L is a non-effective class of nonnegative degree. More generally, for any $0 \leq d < g$, choose $T \in \mathrm{Pic}^0(X) \setminus (W_d - D_0)$ and set $L = \mathcal{O}_X(D_0) \otimes T$ with $\deg D_0 = d$; then $h^0(X, L) = 0$.

Let $W_d \subset J_d$ denote the image of AJ_d. They are only interesting for $d < g$, and the most interesting one is W_{g-1}, also known as the **theta-divisor**.

Proposition 3.13. The points of W_{g-1} correspond to isomorphism classes of line bundles L on R with $h^0(R; L) \geq 1$. It has codimension 1 and therefore generically smooth. The smooth points correspond to line bundles L with $h^0(R; L) = 1$. Locally W_{g-1} is given as the zero locus of a single holomorphic function (a theta function).

Proof. The map AJ_{g-1} has domain of dimension $g-1$ and target of dimension g and is generically injective, so its image W_{g-1} has dimension $g-1$, hence codimension 1 in J_{g-1}. Its image is those line bundles which came from effective divisors of degree $g-1$. Such line bundles L satisfy $h^0(R; L) \geq 1$ since they have a nontrivial holomorphic section given by the constant section 1 in $\mathcal{O}_R(D)$ for some effective divisor D with $\mathcal{O}_R(D) \cong L$. Explicitly (because I always get confused by this) there's a natural inclusion

$$\mathcal{O}_R \hookrightarrow \mathcal{O}_R(D), \qquad f \longmapsto f \cdot 1$$

where the image of the constant function 1 is a holomorphic section of $\mathcal{O}_R(D)$.

Locally near p_i, pick a coordinate z vanishing at p_i, and note that $\mathcal{O}_R(D)|_U \cong z^{-m_i}\mathcal{O}_U$. In this local trivialization, the "section 1" of $\mathcal{O}_R(D)$ corresponds to z^{m_i} as a function. That function does vanish to order m_i at p_i.

By the implicit function theorem, the smooth points of W_{g-1} correspond to points where the differential of AJ_{g-1} is injective.

The rank of the differential of AJ_{g-1} controls the regularity of the image. From the differential computation, we have

$$d(AJ_d)_D : \bigoplus_i T_{p_i}R \to H^0(K_R)^*, \quad (v_i) \mapsto (\omega \mapsto \sum_i \omega(p_i)(v_i))$$

Its kernel is nontrivial if and only if there exists a nonzero $\omega \in H^0(K_R)$ such that $\omega(p_i)(v_i) = 0$ for all i, i.e. ω vanishes at all p_i. Hence $\ker d(AJ_{g-1})_D \neq 0$ if and only if $H^0(K_R(-D)) \neq 0$.

By Riemann-Roch:

$$h^0(K_R(-D)) = h^0(\mathcal{O}_R(D)) + (g-1) - g + 1 = h^0(\mathcal{O}_R(D))$$

Thus $\ker d(AJ_{g-1})_D \neq 0$ if and only if $h^0(\mathcal{O}_R(D)) \geq 2$. In particular, if $h^0(\mathcal{O}_R(D)) = 1$, the differential is injective.

General theory of complex analytic spaces tells us that a codimension 1 analytic subspace is locally cut out by a single holomorphic function. In particular, every Weil divisor is locally principal, i.e. a Cartier divisor. However, we will see later that W_{g-1} is actually globally principal, cut out by a theta function. $\square$

Definition 3.14. A **theta characteristic** on a Riemann surface R is a line bundle ρ satisfying

$$\rho^{\otimes 2} \cong K_R$$

where K_R is the canonical bundle of R.

If we pick a basepoint $p_0 \in R$ to identify J with J_{g-1}, then the difference of two theta characteristics is a 2-torsion point in J, i.e. a line bundle τ satisfying

$$\tau^{\otimes 2} \cong \mathcal{O}_R$$

This is because if ρ_1, ρ_2 are two theta characteristics, then

$$(\rho_1 \otimes \rho_2^{-1})^{\otimes 2} \cong K_R \otimes K_R^{-1} \cong \mathcal{O}_R$$

The set of 2-torsion points in J forms a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$, so there are 2^{2g} theta characteristics on R.

Remark 3.15 (Cultural remark on theta characteristics)**.** A square root of K on a Riemann surface is the same as a spin structure on the tangent bundle. It is also given by a quadratic form on $H_1(R, \mathbb{Z}/2\mathbb{Z})$ compatible with the intersection form, i.e. a function $q : H_1(R, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ satisfying

$$q(a+b) = q(a) + q(b) + a \cup b$$

where $a \cup b$ is the intersection number modulo 2.

The following theorem describes the symmetries of W_{g-1}. Translating W_{g-1} by a theta characteristic corresponds to shifting the theta function by a half-period in the classical theory of theta functions.

Theorem 3.16.

1. W_{g-1} is invariant under the involution $L \mapsto K_R \otimes L^{-1}$ on J_{g-1}.
2. (equivalent to (1)) Let ρ be a theta characteristic, i.e. a line bundle satisfying $\rho^{\otimes 2} \cong K_R$. Then ρW_{g-1} is self-inverse under the group law on J_0
3. W_{g-1} is not invariant under any translation of the torus.
4. W_{g-1} is invariant under reflection by theta chraracteristics, but not under any reflections.

Proof of (3). The evenness of Θ implies symmetry under half-lattice points. We will see that:

- Topologically, $\Theta \cong \mathcal{O}(W_{g-1})$.

- Holomorphic line bundles of this type are related by translation on J. In particular, two line bundles of the same topological type on J differ only by a translation. This is a general theorem about line bundles on complex tori.

- Symmetry forces only half-lattice translations to be allowed. $\square$

Remark 3.17. There exists a universal vector bundle (the Poincaré bundle) $\mathcal{P}_d$ over $R \times J_d$ such that for any line bundle L on R, there is a natural isomorphism

$$L \cong \mathcal{P}_d|_{R \times \{[L]\}}$$

To make $\mathcal{P}_d$ unique, one must choose a base point $r \in R$ and require that $\mathcal{P}_d$ be trivial along $\{r\} \times J_d$:

$$\mathcal{P}_d|_{\{r\} \times J_d} \cong \mathcal{O}_{J_d}$$

Without this normalization, the universal line bundle is only defined up to tensoring by a line bundle pulled back from J_d.

Taking $d = g - 1$, we have finite dimensional holomorphic vector bundles on J_{g-1} denoted $\mathcal{H}_0$ and $\mathcal{H}_1$, a holomorphic map $\mathrm{pp} : \mathcal{H}_0 \to \mathcal{H}_1$, and an exact sequence of vector spaces for every $j \in J_{g-1}$

$$0 \to H^0(R, L_j) \to (\mathcal{H}_0)_j \to (\mathcal{H}_1)_j \to H^1(R, L_j) \to 0$$

This map $\mathrm{pp} : \mathcal{H}_0 \to \mathcal{H}_1$ is the principal parts map, which at a point $j \in J_{g-1}$ representing a line bundle L on R is given by taking a global meromorphic section of L and sending it to its principal parts at all points of R (i.e its obstruction to being holomorphic).

Now we introduce an identification of the theta line bundle Θ on J_{g-1} with the inverse determinant line bundle of the complex of vector bundles $\mathcal{H}_0 \to \mathcal{H}_1$.

Theorem 3.18. $\Theta \cong \Lambda^{\mathrm{top}}\mathcal{H}_1 \otimes (\Lambda^{\mathrm{top}}\mathcal{H}_0)^{-1}$ where Θ is the line bundle on J_{g-1} with divisor W_{g-1}. The map $\det \mathrm{pp} : \mathcal{H}_0 \to \mathcal{H}_1$ induces a holomorphic section of Θ whose zero locus is precisely W_{g-1}.

Remark 3.19 (What $\mathcal{P}_d$ is not)**.** Consider the map $R \times \mathrm{Sym}^d R \to J_{d-1}$ given by

$$(p, D) \mapsto AJ_d(D) - AJ_1(p)$$

The universal line bundle $\mathcal{P}_{d-1}$ on $R \times J_{d-1}$ can be pulled back to $R \times \mathrm{Sym}^d R$ along this map. This pullback gives a divisor called the incidence divisor

$$I = \{(p, D) \in R \times \mathrm{Sym}^d R : p \in \mathrm{supp}(D)\}$$

It turns out that this is not quite $(\mathrm{id}_R \times AJ_d)^*\mathcal{P}_d$. Instead, we have an isomorphism of line bundles on $R \times \mathrm{Sym}^d R$

$$\mathcal{O}(I) \cong (\mathrm{id}_R \times AJ_d)^*\mathcal{P}_d \otimes \mathcal{O}(1)$$

where $\mathcal{O}(1)$ is the hyperplane line bundle along the projective fibers of $\operatorname{Sym}^d R \to J_d$. More precisely, if we denote by $\pi : R \times \operatorname{Sym}^d R \to \operatorname{Sym}^d R$ the projection map, then

$$\mathcal{O}(1) = \pi^* \mathcal{O}_{\operatorname{Sym}^d R}(1)$$

This discrepancy arises because the incidence divisor I only keeps track of the points in the support of the divisor D, while the universal line bundle $\mathcal{P}_d$ also encodes the tautological line on the projectivized space of sections

3.3 Riemann Theta function

Recall that we had a basis of holomorphic differentials $\phi_1, \dots, \phi_g$ on R normalized so that the period matrix

$$\Omega = [I|B]$$

with B symmetric and $\Im B$ positive definite. Define the Riemann theta function

$$\theta(z|B) = \sum_{n \in \mathbb{Z}^g} \exp\left(2\pi i \left(\frac{1}{2} n^t B n + n^t z\right)\right)$$

where $z \in \mathbb{C}^g$.

Theorem 3.20 (Properties of the Riemann theta function)**.**

(1) Θ converges uniformly on compact subsets.

(2) Θ is periodic under translations

$$\vec{z} \longmapsto \vec{z} + \vec{e}_j,$$

and satisfies the quasi-periodicity relation

$$\Theta(\vec{z} + B\vec{m}; B) = \Theta(\vec{z}|B) \exp\bigl(-2\pi i(\vec{m}'\vec{z} + \tfrac{1}{2}\vec{m}^\top B\vec{m})\bigr).$$

(3) The zero locus of Θ is the translate

$$W_{g-1} - \rho,$$

where ρ is a theta characteristic determined by the choice of a and b cycles.

We are now prepared to solve the multiplicative Cousin problem on R. Consider the map

$$F : R \times \operatorname{Sym}^g R \longrightarrow J_{g-1}, \qquad (r, D) \longmapsto AJ(D) - AJ(r).$$

Claim 3.21. $F^{-1}(W_{g-1}^{\text{smooth}}) = \{\, (r, D) \mid r \in D \,\}$.

Remark 3.22. If $\dim H^0(R, \mathcal{O}(D)) > 1$, then $F^{-1}(AJ(D))$ projects surjectively onto all of R.

Therefore the function $\Theta(F(r, D) + \rho)$ as a function of $r \in R$ has divisor exactly D. This implies that we can solve almost any prescription of zeros and poles by a ratio of products of shifted theta functions restricted to $AJ_1(R) \subset J$.

Abel's condition ensures we can arrange the shifts so that the result is periodic with respect to the lattice, hence we obtain a single valued **meromorphic function on** R.

Remark 3.23. Torelli's theorem states that a compact Riemann surface R is determined up to isomorphism by its Jacobian $J(R)$ together with the theta divisor $W_{g-1} \subset J(R)$. In other words, if two compact Riemann surfaces have isomorphic Jacobians as principally polarized abelian varieties (i.e. the isomorphism identifies the theta divisors), then the Riemann surfaces themselves are isomorphic.

4 Sheaf theory

We introduce sheaves which offer the right language to study local-to-global problems on complex manifolds.

4.1 Presheaves and sheaves

Every presheaf $\mathcal{F}$ on a topological space X has an associated sheaf $\mathcal{F}^+$, called its sheafification, together with a natural morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{F}^+$, which is universal among morphisms from $\mathcal{F}$ to sheaves.

The construction of $\mathcal{F}^+$ passes through something called the **étalé space** of $\mathcal{F}$,

Definition 4.1. The **étalé space** $E(\mathcal{F})$ of a presheaf $\mathcal{F}$ on a topological space X is the disjoint union of the stalks of $\mathcal{F}$ at each point of X:

$$E(\mathcal{F}) = \bigsqcup_{x \in X} \mathcal{F}_x$$

equipped with the topology generated by the sets

$$U(s) = \{s_y \in \mathcal{F}_y : y \in U\}$$

for each open set $U \subseteq X$ and section $s \in \mathcal{F}(U)$, where s_y is the germ of s at the point y. The projection map $\pi : E(\mathcal{F}) \to X$ sends each germ $s_x \in \mathcal{F}_x$ to the point $x \in X$.

Proposition 4.2. $\pi : E(\mathcal{F}) \to X$ is a local homeomorphism.

The sheafification $\mathcal{F}^+$ is then defined as the sheaf of sections of the étalé space:

$$\mathcal{F}^+(U) = \{s : U \to E(\mathcal{F}) : \pi \circ s = \mathrm{id}_U \text{ and } s \text{ is continuous}\}$$

for each open set $U \subseteq X$. It turns out that any morphism of presheaves $\psi : \mathcal{F} \to \mathcal{G}$ into a sheaf $\mathcal{G}$ factors uniquely through the sheafification $\mathcal{F}^+$ via a morphism of sheaves $\psi^+ : \mathcal{F}^+ \to \mathcal{G}$.

Theorem 4.3. There is an adjunction between the category of presheaves and the category of sheaves on a topological space X:

$$\mathrm{Hom}_{\text{Sheaves}}(\mathcal{F}^+, \mathcal{G}) \cong \mathrm{Hom}_{\text{Presheaves}}(\mathcal{F}, \mathcal{G})$$

for any presheaf $\mathcal{F}$ and sheaf $\mathcal{G}$ on X. In particular, the sheafification functor $(-)^+$ is left adjoint to the inclusion functor from sheaves to presheaves.

Proposition 4.4. A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ induces a continuous map between their étalé spaces over X. Conversely, a continuous map between étalé spaces $E(\mathcal{F}) \to E(\mathcal{G})$ over X induces a morphism of presheaves by taking sections to their compositions with the continuous map.

In particular, $\mathcal{F} \to \mathrm{Forget}(\mathcal{G})$ induces a continuous map $E(\mathcal{F}) \to E(\mathcal{G})$ over X, and the induced map on sheaves of sections corresponds to the adjoint morphism $\mathcal{F}^+ \to \mathcal{G}$.

Example 4.5. Consider the embedding $i : \{0\} \hookrightarrow \mathbb{R}$ and the sheaf $\mathbb{C}$ on $\{0\}$. The direct image sheaf $i_*\mathbb{C}$ on $\mathbb{R}$ is given by

$$(i_*\mathbb{C})(U) = \begin{cases} \mathbb{C} & 0 \in U \\ 0 & 0 \notin U \end{cases}$$

The étalé space $E(i_*\mathbb{C})$ is a single point over every $x \neq 0$ and a copy of $\mathbb{C}$ over 0. The topology on $E(i_*\mathbb{C})$ is such that the only open set containing points over $x \neq 0$ is the entire space, while over 0 we have the usual topology of $\mathbb{C}$. It is topologized by starting with $\mathbb{C} \times \mathbb{R}$ and gluing all the sheets over $x \neq 0$ to a single point.

Definition 4.6. A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is **monic** (if for every open set $U \subseteq X$, the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective) if and only if the induced map between their étalé spaces $E(\mathcal{F}) \to E(\mathcal{G})$ is injective.

A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is **epic** (if it is surjective on stalks) if and only if the induced map between their étalé spaces $E(\mathcal{F}) \to E(\mathcal{G})$ is surjective.

Remark 4.7. These are functorial definitions which make sense in any abelian category. In particular we say that φ is monic if for any sheaf $\mathcal{H}$, the induced map

$$\mathrm{Hom}(\mathcal{H}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{H}, \mathcal{G})$$

is injective, and φ is epic if for any sheaf $\mathcal{H}$, the induced map

$$\mathrm{Hom}(\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{H})$$

is injective.

Remark 4.8. Cokernels are tricky in sheaf theory. The cokernel of a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is defined as the sheafification of the presheaf cokernel:

$$\operatorname{coker}\varphi = (\operatorname{Forget}(\mathcal{G})/\operatorname{Forget}(\mathcal{F}))^+$$

This is because the presheaf cokernel may fail to be a sheaf. In particular cokernels must be computed on stalks to get the correct answer, whereas kernels can be computed on sections or stalks interchangeably.

Example 4.9 (Surjective on stalks but not on global sections). There is a map of sheaves

$$\mathcal{O}_{\mathbb{P}^1}(-\infty) \oplus \mathcal{O}_{\mathbb{P}^1}(-0) \to \mathcal{O}_{\mathbb{P}^1}$$

given by $(s_1, s_2) \mapsto s_1 + s_2$. This map is surjective on stalks because at any point $p \in \mathbb{P}^1$, either s_1 or s_2 can generate the stalk of $\mathcal{O}_{\mathbb{P}^1}$ at p. However, it is not surjective on global sections because the only global sections of $\mathcal{O}_{\mathbb{P}^1}(-\infty)$ and $\mathcal{O}_{\mathbb{P}^1}(-0)$ are zero. Thus the image of the map on global sections is zero, which is a proper subset of the global sections of $\mathcal{O}_{\mathbb{P}^1}$. This gives an example of a morphism of sheaves which is surjective on stalks but not on global sections.

Homological algebra repairs this defect by considering derived functors of the global sections functor, leading to sheaf cohomology. Homological algebra works for any abelian category.

Definition 4.10. An abelian category $\mathcal{A}$ is a category where

- Hom sets are abelian groups and composition is bilinear,
- For any two objects $A, B \in \mathcal{A}$, there is a biproduct $A \oplus B$ which is both a product and a coproduct,
- there is a zero object 0 which is both initial and terminal,
- every morphism has a kernel and a cokernel, where by kernel we mean the cartesian square

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d] & 0 \arrow[d] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

  and by cokernel we mean the cocartesian square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & \operatorname{coker}\varphi \end{tikzcd}$$

- every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

The category of sheaves of abelian groups on a topological space is an abelian category.

Example 4.11 (Important observation)**.** If $\mathcal{F}$ is a sheaf of abelian groups on a topological space X, and $\mathbb{Z}_X$ is the constant sheaf with stalks $\mathbb{Z}$, then

$$\mathrm{Hom}(\mathbb{Z}_X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$$

so global sections can be interpreted as morphisms from the constant sheaf and in particular we can derive the global sections functor using homological algebra.

4.2 Some operations on sheaves

Let $f : X \to Y$ be a continuous map between topological spaces.

Definition 4.12. The **direct image** (or pushforward) sheaf $f_*\mathcal{F}$ of a sheaf $\mathcal{F}$ on X is defined on open sets $V \subseteq Y$ by

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for each open set $V \subseteq Y$. The restriction maps are induced from those of $\mathcal{F}$.

Definition 4.13. The **inverse image** (or pullback) sheaf $f^{-1}\mathcal{G}$ of a sheaf $\mathcal{G}$ on Y is defined as the sheafification of the presheaf given by

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

for each open set $U \subseteq X$, where the limit is taken over all open sets $V \subseteq Y$ containing $f(U)$.

There is a different notion of pullback for sheaves of modules over sheaves of rings.

Definition 4.14. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, and let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. The **pullback** (or inverse image) sheaf of $\mathcal{O}_X$-modules is defined as

$$f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

This definition replicates the usual extension of scalars for modules over rings. In particular on an affine open set $\mathrm{Spec}\, A \subseteq Y$ with preimage $\mathrm{Spec}\, B \subseteq X$, if $\mathcal{G}$ corresponds to an A-module M, then $f^*\mathcal{G}$ corresponds to the B-module $M \otimes_A B$.

Example 4.15 (The difference between f^{-1} and f^*)**.** Consider the projection $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the second factor. Let $\mathcal{O}_{\mathbb{P}^1}(1)$ be the hyperplane line bundle on $\mathbb{P}^1$. Then the inverse image sheaf $p_2^{-1}\mathcal{O}_{\mathbb{P}^1}(1)$ is the subsheaf of $\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ which is locally constant in fibers. Explicitly, over an open set $U \subseteq \mathbb{P}^1$, we have

$$p_2^{-1}\mathcal{O}_{\mathbb{P}^1}(1)(U \times \mathbb{P}^1) = \mathcal{O}_{\mathbb{P}^1}(1)(U)$$

It assigns to an open $U \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ the sections of $\mathcal{O}(1)$ on any open neighborhood of $p_2(U) \subseteq \mathbb{P}^1$, so sections vary only over the second coordinate.

However the pullback sheaf $p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ is by definition

$$p_2^*\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes_{p_2^{-1}\mathcal{O}_{\mathbb{P}^1}} p_2^{-1}\mathcal{O}(1)$$

is the full tensor product $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes p_2^{-1}\mathcal{O}_{\mathbb{P}^1}(1)$, which is the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ because extension of scalars doesn't change the transition functions.

Example 4.16 (Another difference)**.** Let $Y \to X$ be a closed imbedding of complex manifolds and $\mathcal{I}_Y \subset \mathcal{O}_X$ the subsheaf of holomorphic functions vanishing on Y. Then $i^{-1}\mathcal{O}_X$ is the sheaf of holomorphic functions on X restricted to Y (i.e. germs on Y of holomorphic functions on X) and in particular is much bigger than the sheaf of holomorphic functions on Y. In particular, ambient germs may vary transversely, meaning that they need not be constant in directions normal to Y. However $i^*\mathcal{O}_X = i^{-1}\mathcal{O}_X/\mathcal{I}_Y$ is the structure sheaf $\mathcal{O}_Y$ of holomorphic functions on Y.

4.3 Sheaf cohomology

We begin with the following important theorem.

Theorem 4.17. The category of sheaves of abelian groups on a topological space X has enough injectives, i.e. for any sheaf $\mathcal{F}$ on X, there exists a monomorphism $\mathcal{F} \to \mathcal{I}$ into an injective sheaf $\mathcal{I}$.

This allows us to define sheaf cohomology as the right derived functors of the global sections functor $\Gamma(X, -)$. We will give the definition now. Take an injective resolution of a sheaf $\mathcal{F}$ on X:

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

Applying the global sections functor $\Gamma(X, -)$ gives a complex of abelian groups

$$0 \to \Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \Gamma(X, \mathcal{I}^2) \to \cdots$$

Definition 4.18. The **sheaf cohomology** groups $H^i(X, \mathcal{F})$ are defined as the cohomology groups of the complex of global sections of an injective resolution of $\mathcal{F}$:

$$H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

However, in practice it is quite difficult to work with injective resolutions of sheaves. Instead, we use other types of resolutions which are easier to construct.

Definition 4.19. Let F be a left exact functor on $\mathrm{AbSh}(X)$. A sheaf $\mathcal{I}$ is F**-acyclic** if $R^i\mathcal{F}(\mathcal{I}) = 0$ for all $i > 0$.

Definition 4.20. A sheaf $\mathcal{F}$ on a topological space X is **flabby** if for every inclusion of open sets $V \subseteq U \subseteq X$, the restriction map

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

is surjective.

Definition 4.21. A sheaf $\mathcal{F}$ on a topological space X is **soft** if for every closed set $K \subseteq X$, the restriction map

$$\mathcal{F}(X) \to \mathcal{F}(K)$$

is surjective.

Proposition 4.22. Flabby sheaves are soft on any topological space.

Proof. Let $i : K \hookrightarrow X$ be the inclusion of a closed set. Then

$$\mathcal{F}(K) = \Gamma(K, i^{-1}\mathcal{F}) = \varinjlim_{U \supseteq K} \mathcal{F}(U),$$

the colimit taken over open neighborhoods U of K in X. Thus a section $s \in \mathcal{F}(K)$ is represented by some pair (U, s_U) with $K \subseteq U \subseteq X$ open and $s_U \in \mathcal{F}(U)$. Using flabbiness, the restriction map $\mathcal{F}(X) \twoheadrightarrow \mathcal{F}(U)$ is surjective, so choose $t \in \mathcal{F}(X)$ with $t|_U = s_U$.

In the colimit description of $\mathcal{F}(K)$ the class of (U, s_U) equals the class of (X, t) because they agree on $U \supseteq K$. Hence s is the restriction of t, so the restriction map $\mathcal{F}(X) \to \mathcal{F}(K)$ is surjective. Therefore $\mathcal{F}$ is soft. $\square$

Theorem 4.23. On any space X, one may compute $R^i\Gamma(X, \mathcal{F})$ using any resolution of $\mathcal{F}$ by $\Gamma(X, -)$-acyclic sheaves.

Theorem 4.24. On any space X, flabby sheaves are $\Gamma(X, -)$-acyclic.

Theorem 4.25. On a paracompact Hausdorff space X, soft sheaves are $\Gamma(X, -)$-acyclic.

Definition 4.26. A sheaf $\mathcal{F}$ on a topological space X is **fine** if for every locally finite open cover $\{U_i\}_{i \in I}$ of X, there exists a partition of unity subordinate to the cover, i.e. a collection of sheaf endomorphisms $\{\varphi_i : \mathcal{F} \to \mathcal{F}\}_{i \in I}$ such that

- For each $i \in I$, the support of φ_i is contained in U_i,
- The sum $\sum_{i \in I} \varphi_i$ is the identity endomorphism of $\mathcal{F}$.

Theorem 4.27. Fine sheaves are soft. Sheaves of modules over fine sheaves of rings are soft.

Example 4.28. Let $\mathcal{E}_X^k$ be the sheaf of smooth k-forms on a smooth manifold X. Then $\mathcal{E}_X^k$ is a fine sheaf because we can construct partitions of unity using smooth bump functions subordinate to any locally finite open cover of X.

Example 4.29. The sheaf of real analytic functions $\mathcal{O}$ on a real analytic manifold is not fine. This is because real analytic functions cannot be patched together using partitions of unity in the same way as smooth functions, due to the rigidity of real analyticity. The same holds for the sheaf of holomorphic functions on a complex manifold.

Example 4.30. The sheaf of principal parts on a Riemann surface is soft.

None of these examples are flabby. In fact, the failure of the sheaf of principal parts to be flabby is precise the obstruction to the Mittag-Leffler problem.

Example 4.31 (Complex of sheaves of $\mathbb{Z}$ singular cochains)**.** Consider the presheaf $\mathcal{C}^k$ on a topological space X defined by

$$\mathcal{C}^k(U) = C^k_{\text{sing}}(U, \mathbb{Z})$$

where $C^k_{\text{sing}}(U, \mathbb{Z})$ is the group of singular k-cochains on U with integer coefficients. The restriction maps are given by restricting cochains to smaller open sets. The sheafification $\mathcal{S}^k = \mathcal{C}^{k+}$ is called the sheaf of singular k-cochains. The complex of sheaves

$$\cdots \to \mathcal{S}^{k-1} \to \mathcal{S}^k \to \mathcal{S}^{k+1} \to \cdots$$

with the usual coboundary maps is a fine resolution of the constant sheaf $\mathbb{Z}_X$ on X. This is because partitions of unity can be constructed for singular cochains by using barycentric subdivisions of simplices.

The complex $(\mathcal{C}^\bullet(X), d)$ is exact on stalks except for when $k = 0$, where the cohomology is $\mathbb{Z}$. Thus this complex is a fine resolution of the constant sheaf $\mathbb{Z}_X$.

Corollary 4.32. The sheaf cohomology groups $H^i(X, \mathbb{Z})$ of the constant sheaf $\mathbb{Z}_X$ on a topological space X are isomorphic to the singular cohomology groups $H^i_{\text{sing}}(X, \mathbb{Z})$ of X with integer coefficients.

Example 4.33 (Sheaf cohomology depends on the ambient category)**.** Let $X = S^2$ and consider the category of constant sheaves of abelian groups on X. Since X is connected, this category is equivalent to the category of abelian groups.

In the category $\mathrm{CstAbSh}(X)$ of constant sheaves of abelian groups on X, $\mathrm{Hom}(\cdot, \cdot)$ derives just as in abelian group. In particular we get $R^0 \operatorname{Hom} = \operatorname{Hom}$ and $R^1 \operatorname{Hom} = \operatorname{Ext}$ and there are no higher derived functors.

In $\mathrm{AbSh}(X)$ the category of all sheaves of abelian groups on X, the sheaf cohomology groups $H^i(X,\mathbb{Z})$ are isomorphic to the singular cohomology groups $H^i_{\text{sing}}(X,\mathbb{Z})$. Since S^2 has nontrivial second singular cohomology group $H^2_{\text{sing}}(S^2,\mathbb{Z}) \cong \mathbb{Z}$, we have a nontrivial second sheaf cohomology group $H^2(S^2,\mathbb{Z}) \cong \mathbb{Z}$ in $\mathrm{AbSh}(X)$. Thus the sheaf cohomology groups depend on the ambient category of sheaves we are working in.

Remark 4.34. This issue is not remedied by considering locally constant sheaves. These are the sheaves for which the etale space $E(\mathcal{F}) \to X$ is a covering space. The issue is that the category of locally constant sheaves on X is equivalent to the category of representations of the fundamental group $\pi_1(X)$, and so it only sees topological information about X.

Remark 4.35. For projective algebraic varieties, one may derive Hom correctly using complexes of coherent sheaves.

This is not true for generic $K3$ surfaces X, which do not have enough coherent sheaves. In particular, one only has points and X as subvarieties of X, so the only coherent sheaves are skyscraper sheaves at points, TX, and related as coherent sheaves.

In this situation, one must derive Hom using larger categories of $\mathcal{O}$-modules to get the correct derived functors. In particular, one may consider complexes of sheaves with coherent cohomology sheaves.

Exercise 4.36. Show that, on a compact Hausdorff space X, sheaf cohomology commutes with filtered colimits: if $\mathcal{S} = \varinjlim \mathcal{S}_n$, then

$$\varinjlim H^q(X;\mathcal{S}_n) = H^q(X;\mathcal{S}).$$

Use this to prove that the sheaf of principal parts of meromorphic sections of a vector bundle on a Riemann surface has no H^1 or higher cohomology.

Solution 4.37. Let X compact Hausdorff and $\{\mathcal{S}_\alpha\}$ a filtered direct system of sheaves. Let $\mathcal{S} = \varinjlim_\alpha \mathcal{S}_\alpha$ be the colimit in sheaves.

Take a finite open cover $\mathfrak{U} = \{U_i\}_{i=1}^m$ of X (exists since X is compact). For any p,

$$C^p(\mathfrak{U},\mathcal{S}) = \prod_{i_0<\cdots<i_p} \mathcal{S}(U_{i_0} \cap \cdots \cap U_{i_p}).$$

We will use a couple of lemmas and justify them at the end of the solution.

Lemma 4.38. Filtered colimits are computed stalkwise and commute with finite products.

The lemma implies that the Čech complex for $\mathcal{S}$ is the filtered colimit of the Čech complexes for $\mathcal{S}_\alpha$ since we have the following isomorphisms which commute with the differentials:

$$C^p(\mathfrak{U},\mathcal{S}) = \prod_{i_0<\cdots<i_p} \mathcal{S}(U_{i_0} \cap \cdots \cap U_{i_p}) = \prod_{i_0<\cdots<i_p} \varinjlim_\alpha \mathcal{S}_\alpha(U_{i_0} \cap \cdots \cap U_{i_p})$$

$$\cong \varinjlim_{\alpha} \prod_{i_0<\cdots<i_p} \mathcal{S}_\alpha(U_{i_0} \cap \cdots \cap U_{i_p}) = \varinjlim_{\alpha} C^p(\mathfrak{U}, \mathcal{S}_\alpha).$$

Lemma 4.39. Filtered colimits of abelian groups are exact, and in particular taking cohomology commutes with $\varinjlim$:

$$\check{H}^q(\mathfrak{U}; \mathcal{S}) \cong \varinjlim_{\alpha} \check{H}^q(\mathfrak{U}; \mathcal{S}_\alpha).$$

On a compact Hausdorff space (in particular, on a Riemann surface) Čech cohomology with respect to a good cover computes sheaf cohomology. Hence

$$H^q(X; \mathcal{S}) \cong \varinjlim_{\alpha} H^q(X; \mathcal{S}_\alpha)$$

for all q.

For the second part, let R be a Riemann surface and V a holomorphic vector bundle. For each effective divisor D on R, let $\mathcal{P}_D(V)$ be the sheaf of principal parts of meromorphic sections of V with poles bounded by D.

Observe that $\mathcal{P}_D(V)$ is supported on the finite set $\mathrm{Supp}(D)$, so it is a finite direct sum of skyscraper sheaves.

Lemma 4.40. A sheaf on R with finite support has no higher cohomology. In particular, for $q \geq 1$,

$$H^q(R; \mathcal{P}_D(V)) = 0,$$

The full principal parts sheaf is the filtered colimit over all effective divisors:

$$\mathcal{P}(V) = \varinjlim_{D} \mathcal{P}_D(V),$$

directed by $D \leq D'$ if D divides D'. Applying the result from the first part:

$$H^q(R; \mathcal{P}(V)) \cong \varinjlim_{D} H^q(R; \mathcal{P}_D(V)).$$

But each $H^q(R; \mathcal{P}_D(V)) = 0$ for $q \geq 1$, so the colimit is 0. Therefore

$$H^q(R; \mathcal{P}(V)) = 0 \quad \text{for all } q \geq 1.$$

Now we justify the lemmas used above.

Proof of Lemma 4.38. Define a presheaf $\mathcal{F}(U) := \varinjlim_\alpha \mathcal{S}_\alpha(U)$. We claim $\mathcal{F}$ is already a sheaf. Then $\mathcal{F}$ is the colimit in the sheaf category, so for any open U, we have

$$\mathcal{S} := \varinjlim_{\alpha} \mathcal{S}_\alpha \quad \text{satisfies} \quad \mathcal{S}(U) = \varinjlim_{\alpha} \mathcal{S}_\alpha(U)$$

To show $\mathcal{F}$ is a sheaf, take an open cover $U = \bigcup_i U_i$. For separatedness, suppose $s \in \mathcal{F}(U)$ restricts to zero in every $\mathcal{F}(U_i)$. Represent s by some $s_\alpha \in \mathcal{S}_\alpha(U)$. Its restriction to $\mathcal{F}(U_i)$ is the image of $s_\alpha|_{U_i}$. If that image is zero in the colimit, then for each i there is some stage $\beta_i \geq \alpha$ with $s_{\beta_i}|_{U_i} = 0$. Since the system is filtered, find γ dominating all β_i. Then $(s_\gamma)|_{U_i} = 0$ for all i, so by the sheaf property of $\mathcal{S}_\gamma$, $s_\gamma = 0$, hence $s = 0$ in the colimit.

For gluing, suppose we have $s_i \in \mathcal{F}(U_i)$ that agree on overlaps. Represent each s_i by some $s_{i,\alpha_i} \in \mathcal{S}_{\alpha_i}(U_i)$. Filteredness gives a stage β dominating all α_i. Push all s_{i,α_i} to $\mathcal{S}_\beta$; they still agree on overlaps, so they glue to some $s_\beta \in \mathcal{S}_\beta(U)$. Its class in $\mathcal{F}(U)$ glues the original s_i.

Therefore the presheaf colimit is a sheaf, and $\mathcal{S}(U) = \varinjlim_\alpha \mathcal{S}_\alpha(U)$. Thus filtered colimits are computed stalkwise, so it is enough to show that in the category of abelian groups filtered colimits commute with finite products. $\square$

Proof of Lemma 4.39. Take a short exact sequence of direct systems

$$0 \to A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha \to 0,$$

with all maps compatible. We need to show that

$$0 \to \varinjlim A_\alpha \to \varinjlim B_\alpha \to \varinjlim C_\alpha \to 0$$

is exact.

Left-exactness (injectivity at $\varinjlim A_\alpha$) holds for any colimit: if an element becomes zero in $\varinjlim B_\alpha$, it is already zero after some stage, and then in A_α by exactness there.

Surjectivity at $\varinjlim C_\alpha$ is where filteredness is used. Let $[c_\alpha] \in \varinjlim C_\alpha$. Pick representative $c_\alpha \in C_\alpha$. Exactness at C_α gives $b_\alpha \in B_\alpha$ with $g_\alpha(b_\alpha) = c_\alpha$. Let $[b_\alpha]$ be its class in $\varinjlim B_\alpha$; its image is $[c_\alpha]$.

We need to check this does not depend on choices: if we change stage or lift, filteredness gives a common stage where the choices agree, because kernels and images are compatible. That is standard and uses only that the system is filtered.

Hence the right map is surjective and the colimit of a short exact sequence is exact. $\square$

Proof of Lemma 4.40. Let $\mathcal{F}$ be a sheaf on R with finite support $\{p_1, \ldots, p_n\}$. Then

$$\mathcal{F} \cong \bigoplus_{i=1}^{n} (i_{p_i})_* \mathcal{F}_{p_i},$$

where $i_{p_i} : \{p_i\} \hookrightarrow R$ is the inclusion. Each $(i_{p_i})_* \mathcal{F}_{p_i}$ is a skyscraper sheaf supported at p_i.

Skyscraper sheaves are flasque. For $\mathcal{G} = (i_x)_* A$, where A is an abelian group and $x \in R$, we have for $V \subset U$, the restriction map $\mathcal{G}(U) \to \mathcal{G}(V)$ is either:

$$\begin{cases} \mathrm{id}_A & \text{if both contain } x \\ A \to 0 & \text{if } x \in U, x \notin V \\ 0 \to 0 & \text{otherwise} \end{cases}$$

In all cases it is surjective. Therefore every restriction map is surjective, so $\mathcal{G}$ is flasque.

Finite direct sums of flasque sheaves are flasque since surjectivity is preserved under finite products. Flasque sheaves have vanishing higher cohomology. Since cohomology commutes with finite direct sums, we conclude that $H^q(R; \mathcal{F}) = 0$ for $q \geq 1$. $\square$

Exercise 4.41. Kodaira's theorem implies that for any vector bundle $V \to R$ on a compact Riemann surface, there exists a (bundle-dependent) degree $d \in \mathbb{Z}$ such that

$$H^1(R; \mathcal{O}(V \otimes L)) = 0$$

for all line bundles L of degree $> d$.

Use this fact, the long exact sequence for cohomology of

$$\mathcal{O} \to \mathcal{M} \to \mathcal{P},$$

and the result of Exercise 4.36 to show that the ad hoc definition of cohomology of vector bundles via principal parts computes the genuine sheaf cohomology of vector bundles.

Solution 4.42. For any V there exists d_0 such that for every line bundle L with $\deg L > d_0$,

$$H^1(R; \mathcal{O}(V \otimes L)) = 0$$

In particular, choose $L = \mathcal{O}(D)$ for some effective divisor D with $\deg D > d_0$.

For an effective divisor D, we have the short exact sequence of sheaves

$$0 \to \mathcal{O}(V) \to \mathcal{O}(V(D)) \to \mathcal{P}_D(V) \to 0,$$

where $\mathcal{P}_D(V)$ is the sheaf of principal parts of meromorphic sections of V with poles bounded by D. Taking the long exact sequence in cohomology, we have

$$0 \to H^0(R; \mathcal{O}(V)) \to H^0(R; \mathcal{O}(V(D))) \to H^0(R; \mathcal{P}_D(V)) \to H^1(R; \mathcal{O}(V)) \to H^1(R; \mathcal{O}(V(D))) \to \cdots$$

Since $\deg D > d_0$, we have $H^1(R; \mathcal{O}(V(D))) = 0$. Therefore,

$$H^1(R; \mathcal{O}(V)) \cong \frac{H^0(R; \mathcal{P}_D(V))}{\mathrm{Im}(H^0(R; \mathcal{O}(V(D))) \to H^0(R; \mathcal{P}_D(V)))}.$$

Taking the direct limit over all effective divisors D, we have

$$H^1(R; \mathcal{O}(V)) \cong \varinjlim_D \frac{H^0(R; \mathcal{P}_D(V))}{\mathrm{Im}(H^0(R; \mathcal{O}(V(D))) \to H^0(R; \mathcal{P}_D(V)))}.$$

By the result of Exercise 4.36, we have

$$\varinjlim_D H^0(R; \mathcal{P}_D(V)) \cong H^0(R; \mathcal{P}(V)),$$

where $\mathcal{P}(V)$ is the sheaf of principal parts of meromorphic sections of V. Therefore,

$$H^1(R; \mathcal{O}(V)) \cong \frac{H^0(R; \mathcal{P}(V))}{\mathrm{Im}(\varinjlim_D H^0(R; \mathcal{O}(V(D))) \to H^0(R; \mathcal{P}(V)))}.$$

But $\varinjlim_D H^0(R; \mathcal{O}(V(D)))$ is precisely the space of global meromorphic sections of V. Thus,

$$H^1(R; \mathcal{O}(V)) \cong \frac{H^0(R; \mathcal{P}(V))}{\mathrm{Im}(H^0(R; \mathcal{M}(V)) \to H^0(R; \mathcal{P}(V)))}$$

4.4 Currents

Let M be a $\mathcal{C}^\infty$ oriented differentiable manifold, $m = \dim_{\mathbb{R}} M$. We first introduce a topology on the space of differential forms $C^s(M, \Lambda^p T_M^*)$. Let $\Omega \subset M$ be a coordinate open set and u a p-form on M, written $u(x) = \sum u_I(x)\, dx_I$ on Ω. To every compact subset $L \subset \Omega$ and every integer $s \in \mathbb{N}$, we associate a seminorm

$$p_L^s(u) = \sup_{x \in L} \max_{\substack{|I|=p \\ |\alpha| \le s}} \big|D^\alpha u_I(x)\big|,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ runs over $\mathbb{N}^m$, and

$$D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m})$$

is a derivation of order $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Definition 4.43. We introduce the following spaces of p-forms on manifolds.

(a) We denote by $\mathcal{E}^p(M)$ the space $C^\infty(M, \Lambda^p T_M^*)$ equipped with the topology defined by all seminorms p_L^s as s, L, Ω vary.

(b) If $K \subset M$ is a compact subset, $\mathcal{D}^p(K)$ will denote the subspace of elements $u \in \mathcal{E}^p(M)$ with support contained in K, together with the induced topology; $\mathcal{D}^p(M)$ will stand for the set of all elements with compact support, i.e.

$$\mathcal{D}^p(M) = \bigcup_K \mathcal{D}^p(K).$$

Since our manifolds are assumed to be separable, the topology of $\mathcal{E}^p(M)$ can be defined by means of a countable set of seminorms p_L^s, hence $\mathcal{E}^p(M)$ is a Fréchet space. It should be observed however that $\mathcal{D}^p(M)$ is not a Fréchet space; in fact $\mathcal{D}^p(M)$ is dense in $\mathcal{E}^p(M)$ and thus non-complete for the induced topology. Spaces of **currents** are defined as the topological duals of the above spaces, in analogy with the usual definition of distributions.

Definition 4.44. The **space of currents of dimension** p (or degree $m-p$) on M is the space $\mathcal{D}'_p(M)$ of linear forms T on $\mathcal{D}^p(M)$ such that the restriction of T to all subspaces $\mathcal{D}^p(K)$, $K \Subset M$, is continuous. The degree is indicated by raising the index, hence we set

$$\mathcal{D}'^{\,m-p}(M) = \mathcal{D}'_p(M) := \text{topological dual } \big(\mathcal{D}^p(M)\big)'.$$

Currents can be given locally by integration against differential forms with distributional coefficients. In particular, there is an inclusion of differential forms into currents

$$\begin{aligned} \mathcal{E}^p(M) &\to \mathcal{D}'^{\,m-p}(M) \\ \omega &\mapsto \left(\eta \mapsto \int_M \omega \wedge \eta\right) \end{aligned}$$

Currents have a well-defined exterior derivative defined by duality. In particular, for a current $T \in \mathcal{D}'^{\,m-p}(M)$, the exterior derivative $dT \in \mathcal{D}'^{\,m-(p+1)}(M)$ is defined by

$$dT(\eta) = (-1)^{p+1} T(d\eta)$$

for any test form $\eta \in \mathcal{D}^{p+1}(M)$. Also, the construction admits a natural sheafification, where for each open set $U \subseteq M$, we define

$$\mathcal{D}'^{\,m-p}_M(U) = \mathcal{D}'^{\,m-p}(U)$$

and the restriction map for $j : V \hookrightarrow U$ is given by

$$\begin{aligned} \mathcal{D}'^{\,m-p}_M(U) &\to \mathcal{D}'^{\,m-p}_M(V) \\ T &\mapsto T|_V \end{aligned}$$

where for any test form $\varphi \in \mathcal{D}^p(V)$, we define

$$(T|_V)(\varphi) = T(\tilde{\varphi})$$

where $\tilde{\varphi} = j_*\varphi$ is the pushforward of φ by j, defined by

$$j_*\varphi(x) = \begin{cases} \varphi(x) & x \in V \\ 0 & x \notin V \end{cases}$$

This is an $\mathcal{E}^0_M$-module sheaf, where $\mathcal{E}^0_M$ is the sheaf of smooth functions on M. Moreover, it is a fine sheaf because one can construct partitions of unity using smooth bump functions.

4.5 De Rham/Dolbeault cohomology

Let X be a n-dimensional paracompact differential manifold. Then the de Rham complex resolves the constant sheaf $\mathbb{R}_X$ on X:

$$0 \to \mathbb{R}_X \to \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^n_X \to 0$$

where $\mathcal{E}^k_X$ is the sheaf of smooth k-forms on X. Local exactness holds by the Poincaré lemma.

Lemma 4.45. Let ω be a closed k-form defined on an open set $U \subseteq X$. Then for any $x \in U$, there exists an open neighborhood $V \subseteq U$ of x and a $(k-1)$-form η on V such that $d\eta = \omega|_V$.

Since each $\mathcal{E}^k_X$ is a fine sheaf, we can compute the sheaf cohomology groups $H^i(X, \mathbb{R})$ using the global sections of the de Rham complex:

$$H^i(X, \mathbb{R}) \cong H^i(\Gamma(X, \mathcal{E}^\bullet_X)) = H^i_{\text{dR}}(X)$$

Instead of using C^∞ differential forms, one can consider the resolution of $\mathbb{R}$ given by the exterior derivative d acting on currents. Being modules over the sheaf of smooth functions, the sheaves of currents are fine, so one can compute sheaf cohomology using currents as well.

$$0 \to \mathbb{R}_X \to \mathcal{D}'^m_X \xrightarrow{d} \mathcal{D}'^{m-1}_X \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}'^0_X \to 0$$

and the inclusion $\mathcal{E}^\bullet_X \to \mathcal{D}'^{m-\bullet}_X$ induces an isomorphism on cohomology. Thus we have

$$H^i(X, \mathbb{R}) \cong H^i(\Gamma(X, \mathcal{D}'^{m-\bullet}_X)) = H^i_{\text{dR}}(X)$$

Let X be a $\mathbb{C}$-analytic manifold of dimension n, and let $\mathcal{E}^{p,q}$ be the sheaf of germs of $\mathcal{C}^\infty$ differential forms of type (p, q) with complex values. For every $p = 0, 1, \ldots, n$, the Dolbeault-Grothendieck Lemma I-2.9 shows that $(\mathcal{E}^{p,\bullet}, d'')$ is a resolution of the sheaf Ω^p_X of germs of holomorphic forms of degree p on X. The complex of global sections

$$0 \to \mathcal{E}^{p,0}(X) \xrightarrow{d''} \mathcal{E}^{p,1}(X) \xrightarrow{d''} \cdots \xrightarrow{d''} \mathcal{E}^{p,n}(X) \to 0$$

then defines the $\overline{\partial}$-cohomology groups of X with coefficients in $\mathbb{C}$:

$$H^{p,q}(X,\mathbb{C}) = H^q\big(\mathcal{E}^{p,\bullet}(X)\big). \tag{9}$$

The sheaves $\mathcal{E}^{p,q}$ are acyclic, so we get the Dolbeault isomorphism theorem which relates $\overline{\partial}$-cohomology and sheaf cohomology:

$$H^{p,q}(X,\mathbb{C}) \;\simeq\; H^q\big(X,\Omega_X^p\big). \tag{6.15}$$

The case $p = 0$ is especially interesting:

$$H^{0,q}(X,\mathbb{C}) \;\simeq\; H^q\big(X,\mathcal{O}_X\big). \tag{6.16}$$

As in the case of De Rham cohomology, there is an inclusion $\mathcal{E}^{p,q} \subset \mathcal{D}'_{n-p,n-q}$, and the complex of currents $(\mathcal{D}'_{n-p,n-\bullet}, d'')$ defines also a resolution of Ω_X^p. Hence there is an isomorphism:

$$H^{p,q}(X,\mathbb{C}) = H^q\big(\mathcal{E}^{p,\bullet}(X)\big) \;\simeq\; H^q\big(\mathcal{D}'_{n-p,n-\bullet}(X)\big) \tag{10}$$

5 Hermitian connections, curvature, and Chern classes

Let E be a complex vector bundle over a complex manifold X. Let $\mathcal{E}^p(E)$ be the sheaf of smooth E-valued p-forms on X. In particular, $\mathcal{E}^0(E)$ is the sheaf of smooth sections of E.

For now, we let E be a real or complex vector bundle over a smooth manifold X.

Definition 5.1. A **connection** as a linear map

$$\nabla : \mathcal{E}^0(E) \to \mathcal{E}^1(E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

for any smooth function f and smooth section s of E. This map extends uniquely to a map

$$\nabla : \mathcal{E}^p(E) \to \mathcal{E}^{p+1}(E)$$

satisfying the graded Leibniz rule for E-valued forms:

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg\omega}\omega \wedge \nabla s$$

for any smooth p-form ω and smooth section s of E.

Remark 5.2 (Notation)**.** Juxtaposition always means wedge product of forms + composition of endomorphisms. In particular, for $\alpha = A \otimes \omega \in \mathcal{E}^p(\mathrm{End}(E))$ and $\beta = B \otimes \eta \in \mathcal{E}^q(\mathrm{End}(E))$, we have

$$\alpha\beta = (A \circ B) \otimes (\omega \wedge \eta)$$

where $A \circ B$ is composition of endomorphisms and $\omega \wedge \eta$ is wedge product of forms.

Definition 5.3. The **curvature** of a connection ∇ on a vector bundle E over a smooth manifold X is the map

$$F_\nabla = \nabla^2 : \mathcal{E}^0(E) \to \mathcal{E}^2(E)$$

is an endomorphism valued 2-form on E.

1. In a local trivialization, one can write $\nabla(s) = ds + As$ where s is a local section, A is a matrix of 1-forms, and As is matrix multiplication where entrywise wedge products are taken.

2. When extending ∇ to $\mathrm{End}(E)$-valued forms we require the graded Leibniz rule

   $$\nabla(\Phi s) = (\nabla\Phi) \wedge s \;+\; (-1)^{\deg\Phi}\, \Phi \wedge (\nabla s)$$

   for any section s of E and any $\Phi \in \Omega^p(X, \mathrm{End}(E))$. Note that Φs is the E-valued p-form obtained by applying the endomorphism $\Phi(x)(v_1, \ldots, v_p)$ to $s(x)$.

   To compute $\nabla\Phi$, write $\nabla = d + A$ in a local trivialization. Then

   $$\begin{aligned} \nabla(\Phi s) &= d(\Phi s) + A \wedge (\Phi s) \\ &= (d\Phi) \wedge s \;+\; (-1)^p\, \Phi \wedge ds \;+\; A \wedge \Phi \wedge s. \end{aligned}$$

   On the other hand,

   $$\Phi(\nabla s) = \Phi(ds + A \wedge s) = \Phi \wedge ds \;+\; \Phi \wedge A \wedge s.$$

   Subtracting the two expressions and using the graded Leibniz rule shows that the extra term multiplying s must be

   $$d\Phi + A \wedge \Phi - (-1)^p\, \Phi \wedge A.$$

   Thus

   $$\boxed{\nabla\Phi = d\Phi + A \wedge \Phi - (-1)^p\, \Phi \wedge A}$$

   which is equivalently

   $$\nabla\Phi = d\Phi + [A, \Phi], \qquad [A, \Phi] := A \wedge \Phi - (-1)^p\, \Phi \wedge A.$$

   Note that we are using the graded commutator here, which stems from the fact that differential forms supercommute.

3. The curvature is given by

$$F_\nabla = dA + A \wedge A = dA + \frac{1}{2}[A, A]$$

Proposition 5.4. The Bianchi identity states that

$$dF_\nabla + [A, F_\nabla] = \nabla F_\nabla = 0$$

Proof. We compute

$$\begin{aligned} dF_\nabla + [A, F_\nabla] &= d(dA + A \wedge A) + [A, dA + A \wedge A] \\ &= d^2 A + dA \wedge A - A \wedge dA + [A, dA] + [A, A \wedge A] \\ &= 0 \end{aligned}$$

since $d^2 = 0$, the middle two terms cancel, and the last term vanishes since $A \wedge A$ is of even degree and therefore commutes with A. $\square$

Definition 5.5 (Chern classes)**.** Let E be a complex vector bundle over X. The total Chern class of E is a cohomology class in $H^{\text{even}}(X, \mathbb{R})$ defined by

$$c(E) = \det\left(I + \frac{i}{2\pi} F_\nabla\right)$$

where F_∇ is the curvature of any connection ∇ on E.

Morally $c_k \in H^{2k}(X, \mathbb{R})$ is the kth symmetric polynomial in the eigenvalues of $\frac{i}{2\pi} F_\nabla$. Many things are not clear. It's not clear that this is a closed form, it's not clear that it's real. It's not clear that it depends only on R and not on the curvature.

1. We get a closed form

$$d(c(E)) = 0$$

In fact for any conjugation invariant polynomial $P : \mathrm{Mat}_r(\mathbb{C}) \to \mathbb{C}$, we have

$$dP(F_\nabla) = 0$$

This is checked by specializing to $P(M) = \mathrm{tr}(M^k)$ because these generate all conjugation invariant polynomials. Then

$$d\,\mathrm{tr}(F_\nabla^k) = k\,\mathrm{tr}(dF_\nabla \wedge F_\nabla^{k-1}) = -k\,\mathrm{tr}([A, F_\nabla] \wedge F_\nabla^{k-1}) = 0$$

by the Bianchi identity and cyclicity of the trace.

2. The Chern class $c(E)$ is independent of the choice of connection ∇ and depends only on E. We will check this by showing that if we change connections, the traces of powers $\operatorname{tr}(F^k)$ will change by exact forms. If you try to do this directly it gets very messy, so it is easiest to check infinitesimal changes. Then when we integrate up the infinitesimal changes, we integrate exact forms which remain exact.

   Let ∇_t be a family of connections on E with curvature F_t. We will show that $\frac{d}{dt}\operatorname{tr}(F_t^k)$ is exact for each k.

   First compute the derivative of the curvature. With A_t the connection 1-form and $\dot{A}_t = \frac{d}{dt}A_t$, we have

$$F_t = dA_t + A_t \wedge A_t, \qquad \frac{d}{dt}F_t = d\dot{A}_t + \dot{A}_t \wedge A_t + A_t \wedge \dot{A}_t.$$

   Recalling the covariant derivative $\nabla_t(\cdot) = d(\cdot) + [A_t, \cdot]$ with $[A_t, \dot{A}_t] = A_t \wedge \dot{A}_t + \dot{A}_t \wedge A_t$, we obtain

$$\frac{d}{dt}F_t = \nabla_t(\dot{A}_t).$$

   Thus we get that

$$\begin{aligned}
\frac{d}{dt}\operatorname{tr}(F_t^k) &= \sum_{j=0}^{k-1} \operatorname{tr}\left(F_t^j \,\frac{dF_t}{dt}\, F_t^{k-1-j}\right) \\
&= k\operatorname{tr}(F_t^{k-1} \wedge \frac{d}{dt}F_t) \quad \text{by cyclicity} \\
&= k\operatorname{tr}(F_t^{k-1} \wedge \nabla_t(\dot{A}_t)) \\
&= -k\sum_{j=0}^{k-2} \operatorname{tr}(F_t^j \wedge (\nabla_t F_t) \wedge F_t^{k-2-j} \wedge (\dot{A}_t)) + kd\operatorname{tr}(F_t^{k-1} \wedge \dot{A}_t)
\end{aligned}$$

   where the last line follows from the graded Leibniz rule for ∇_t applied to the product $F_t^{k-1} \wedge \dot{A}_t$. Explicitly,

$$\nabla_t\big(F_t^{k-1} \wedge \dot{A}_t\big) = \nabla_t(F_t^{k-1}) \wedge \dot{A}_t + F_t^{k-1} \wedge \nabla_t(\dot{A}_t).$$

   But

$$\nabla_t(F_t^{k-1}) = \sum_{j=0}^{k-2} F_t^j \,(\nabla_t F_t)\, F_t^{k-2-j},$$

   so

$$\nabla_t\big(F_t^{k-1} \wedge \dot{A}_t\big) = \sum_{j=0}^{k-2} F_t^j \wedge (\nabla_t F_t) \wedge F_t^{k-2-j} \wedge \dot{A}_t + F_t^{k-1} \wedge \nabla_t(\dot{A}_t).$$

   Rearrange to solve for the term we have:

$$F_t^{k-1} \wedge \nabla_t(\dot{A}_t) = \nabla_t\big(F_t^{k-1} \wedge \dot{A}_t\big) - \sum_{j=0}^{k-2} F_t^j \wedge (\nabla_t F_t) \wedge F_t^{k-2-j} \wedge \dot{A}_t.$$

Returning to our computation, the first term vanishes by the Bianchi identity, so we get that

$$\frac{d}{dt}\operatorname{tr}(F_t^k) = kd\operatorname{tr}(F_t^{k-1}\wedge \dot{A}_t)$$

which is exact. Thus the Chern classes are independent of the choice of connection.

3. We skip the check of reality. In fact it is only real up to exact forms. Need a hermetian connection to get a real form representative.

5.1 Key properties of Chern classes

Proposition 5.6. Chern classes are natural, i.e. for any smooth map $f : Y \to X$ between smooth manifolds and complex vector bundle E over X, we have

$$c(f^*E) = f^*c(E)$$

where f^*E is the pullback bundle of E over Y.

The reason is that when we pull back a vector bundle, one can also pull back a connection.

Proposition 5.7. c_0 is the identity, i.e.

$$c_0(E) = 1$$

If $E \cong \mathbb{C}^r \times X$ is the trivial bundle, then

$$c(E) = 1$$

i.e. all higher Chern classes vanish. If $k > \operatorname{rank}(E)$, then

$$c_k(E) = 0$$

Proposition 5.8. The top Chern class $c_r(E)$ of a complex vector bundle E of rank r over a smooth manifold X is the Euler class $e(E_{\mathbb{R}})$ of the underlying real vector bundle $E_{\mathbb{R}}$ of rank $2r$.

Proposition 5.9 (Whitney sum formula)**.** For any two complex vector bundles E and F over a smooth manifold X, we have

$$c(E \oplus F) = c(E) \wedge c(F)$$

where $E \oplus F$ is the direct sum bundle of E and F.

Proof. Choose ∇ on $E \oplus F$ which respects the direct sum decomposition, then ∇^E and ∇^F commute and the curvature decomposes as

$$F_\nabla = F_{\nabla^E} \oplus F_{\nabla^F}$$

and so the determinant factors. $\square$

Proposition 5.10 (Splitting principle)**.** Given $E \to X$ a complex vector bundle of rank r over a smooth manifold X, there exists a natural smooth manifold $\mathrm{Fl}(E)$ (holomorphic if E and X are), and a fiber bundle $\pi : \mathrm{Fl}(E) \to X$, so that π^*E splits (non-holomorphically) as a direct sum of complex line bundles

$$\pi^*E \cong L_1 \oplus L_2 \oplus \cdots \oplus L_r$$

At a point $x \in X$, the fiber $\mathrm{Fl}(E)_x$ is the variety of complete flags in the vector space E_x. If $\alpha_i = c_1(L_i)$ are the first Chern classes of the line bundles L_i, then

$$H^*(\mathrm{Fl}(E), \mathbb{R}) \cong H^*(X, \mathbb{R})[\alpha_1, \alpha_2, \ldots, \alpha_r] / \left(e_k(\alpha_1, \alpha_2, \ldots, \alpha_r) - \pi^* c_k(E) \mid k = 1, 2, \ldots, r\right)$$

In particular, the induced map on cohomology

$$\pi^* : H^*(X, \mathbb{R}) \to H^*(\mathrm{Fl}(E), \mathbb{R})$$

is injective, and the total Chern class of E splits in $\mathrm{Fl}(E)$ as

$$\pi^* c(E) = \prod_{i=1}^{r} (1 + \alpha_i)$$

where the α_i are the Chern roots of E.

Remark 5.11. Assume $E \to X$ is a holomorphic vector bundle of rank r. On the flag bundle $\pi : \mathrm{Fl}(E) \to X$ one has the tautological holomorphic flag

$$0 = S_0 \subset S_1 \subset \cdots \subset S_r = \pi^*E$$

of holomorphic subbundles, with $\operatorname{rk} S_i = i$. Define holomorphic line bundles

$$L_i := S_i / S_{i-1}, \qquad i = 1, \ldots, r.$$

From the short exact sequences

$$0 \longrightarrow S_{i-1} \longrightarrow S_i \longrightarrow L_i \longrightarrow 0$$

we see that each S_i is an extension of S_{i-1} by the line bundle L_i. In the holomorphic category these extensions need not split: the obstruction to splitting is an extension class in

$$\mathrm{Ext}^1(L_i, S_{i-1}) \cong H^1\big(\mathrm{Fl}(E), \mathcal{H}om(L_i, S_{i-1})\big),$$

which is in general nonzero.

In the C^∞ category, however, every such short exact sequence of complex vector bundles splits. Choosing a Hermitian metric on the bundles and taking the orthogonal complement of S_{i-1} inside S_i yields a smooth splitting

$$S_i \cong S_{i-1} \oplus L_i \qquad \text{as } C^\infty \text{ complex bundles.}$$

Inductively this gives a smooth isomorphism

$$\pi^* E = S_r \cong L_1 \oplus \cdots \oplus L_r$$

as complex C^∞-bundles. The complements obtained by orthogonal projection are not holomorphic subbundles in general, so the resulting splitting is not holomorphic.

Exercise 5.12. Let R be a compact Riemann surface of genus g. Show that the period mapping gives an isomorphism

$$H_1(R;\mathbb{Z}) \xrightarrow{\sim} H_1(J;\mathbb{Z}),$$

which can be realized geometrically by the Abel-Jacobi map

$$R \longrightarrow J_1.$$

Show that under this correspondence, $c_1(\Theta) \in \Lambda^2 H_1(R)$ is the intersection pairing on R.

Solution 5.13. The presentation of the Jacobian J as

$$J \cong H^1(R;\mathcal{O})/H_1(R;\mathbb{Z})$$

makes it clear that $H_1(J;\mathbb{Z})$ is naturally identified with $H_1(R;\mathbb{Z})$, since the universal cover of J is the vector space $H^1(R;\mathcal{O})$. The period mapping

$$H_1(R;\mathbb{Z}) \to H_1(J;\mathbb{Z})$$

is injective because of the Riemann bilinear relations, and since both groups are free abelian of rank $2g$, it is an isomorphism. Pick a base point $p_0 \in R$ and define the Abel-Jacobi map

$$\varphi : R \to J, \qquad p \mapsto \left[\omega \mapsto \int_{p_0}^{p} \omega\right].$$

precisely implements the lift of the period mapping to the universal cover and hence induces the same isomorphism on H_1.

To identify $c_1(\Theta)$ with the intersection pairing on $H_1(R,\mathbb{Z})$, we first note that by the universal coefficient theorem and the fact that $H^k(J,\mathbb{Z}) = \mathrm{Alt}^k(H_1(J,\mathbb{Z}),\mathbb{Z})$ (the group law on J induces a map $H_1(J,\mathbb{Z}) \otimes \cdots \otimes H_1(J,\mathbb{Z}) \to H_k(J,\mathbb{Z})$ as follows. For each $\alpha \in H_1(J,\mathbb{Z})$ choose a loop $\ell_\alpha : S^1 \to J$

representing α. For $\alpha_1, \dots, \alpha_k$, consider the map $(S^1)^k \to J$ given by $(t_1, \dots, t_k) \mapsto \ell_{\alpha_1}(t_1)+\cdots+\ell_{\alpha_k}(t_k)$. For orientation reasons, this map is alternating in the α_i). We have

$$\begin{aligned} H^2(J,\mathbb{Z}) &\cong \operatorname{Hom}(H_2(J,\mathbb{Z}),\mathbb{Z}) \\ &\cong \operatorname{Hom}(\Lambda^2 H_1(J,\mathbb{Z}),\mathbb{Z}) \\ &\cong \operatorname{Alt}^2(H_1(J,\mathbb{Z}),\mathbb{Z}) \xrightarrow{\iota^*} \operatorname{Alt}^2(H_1(R,\mathbb{Z}),\mathbb{Z}) \end{aligned}$$

so indeed $c_1(\Theta)$ corresponds to an alternating bilinear form on $H_1(R,\mathbb{Z})$. Pick a symplectic basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ of $H_1(R,\mathbb{Z})$, i.e.

$$a_i \cdot a_j = 0, \quad b_i \cdot b_j = 0, \quad a_i \cdot b_j = \delta_{ij}.$$

Under the identification $H_1(R,\mathbb{Z}) \xrightarrow{\sim} \Lambda \cong H_1(J,\mathbb{Z})$ coming from the period map and the Abel-Jacobi embedding, a homology class $\gamma \in H_1(R,\mathbb{Z})$ corresponds to an integral vector $(m,n) \in \mathbb{Z}^{2g}$. The intersection pairing on $H_1(R,\mathbb{Z})$ is given in these coordinates by

$$(m,n)\cdot(m',n') \;=\; m^{\mathrm{T}}n' - m'^{\mathrm{T}}n.$$

The Riemann theta function with period matrix τ is

$$\theta(z \mid \tau) := \sum_{k\in\mathbb{Z}^g} \exp\bigl(\pi i\, k^{\mathsf{T}}\tau k + 2\pi i\, k^{\mathsf{T}} z\bigr), \quad z \in \mathbb{C}^g.$$

The Riemann theta function satisfies the quasi-periodicity property:

$$\theta\bigl(z + m + \tau n \mid \tau\bigr) = \exp\bigl(-\pi i n^{\mathsf{T}}\tau n - 2\pi i n^{\mathsf{T}} z\bigr)\theta(z \mid \tau).$$

In particular, the Riemann theta function defines a holomorphic section of the line bundle $\mathcal{O}_J(\Theta)$. Thus the factor of automorphy is

$$e_u(z) = \exp\bigl(-\pi i\, n^{\mathsf{T}}\tau n - 2\pi i\, n^{\mathsf{T}} z\bigr) = \exp\bigl(2\pi i f_u(z)\bigr)$$

with

$$f_u(z) = -\tfrac{1}{2}\, n^{\mathsf{T}}\tau n - n^{\mathsf{T}} z, \qquad u = m + \tau n.$$

The following lemma is standard.

Lemma 5.14. Let $U \subset V$ be a lattice in a complex vector space V. The map which associates to any map $F : U \times U \to \mathbb{Z}$ the map $AF : U \times U \to \mathbb{Z}$ defined by

$$AF(u_1,u_2) = F(u_1,u_2) - F(u_2,u_1)$$

maps the group of 2-cocycles $Z^2(U,\mathbb{Z})$ into the space of alternating linear maps $U \times U \to \mathbb{Z}$, and induces an isomorphism

$$A :\; H^2(U,\mathbb{Z}) \;\xrightarrow{\sim}\; \operatorname{Hom}(\Lambda^2 U,\mathbb{Z}) \;\cong\; \Lambda^2 \operatorname{Hom}(U,\mathbb{Z}).$$

Furthermore for $\xi,\eta \in \operatorname{Hom}(U,\mathbb{Z}) = H^1(U,\mathbb{Z})$, we have $A(\xi \smile \eta) = \xi \wedge \eta$.

The proof of the following lemma is given at the end of the solution.

Lemma 5.15. If $e_u(z) = \exp(2\pi i f_u(z))$ are the factors of automorphy, then the Chern class

$$c_1(L) \in H^2(J, \mathbb{Z}) \cong \mathrm{Alt}^2(\Lambda, \mathbb{Z})$$

corresponds to the form

$$E(u_1, u_2) = f_{u_2}(z + u_1) + f_{u_1}(z) - f_{u_1}(z + u_2) - f_{u_2}(z)$$

which is bilinear, alternating, and independent of z.

Substituting into Mumford's formula and using the symmetry of τ, the quadratic terms cancel, leaving

$$E(u_1, u_2) = n_1^{\mathsf{T}} m_2 - n_2^{\mathsf{T}} m_1.$$

recovering the intersection pairing on $H_1(R, \mathbb{Z})$.

Proof of Lemma 5.15. A holomorphic line bundle on J is equivalent to the trivial bundle $V \times \mathbb{C} \to V$ endowed with the Λ–action

$$u \cdot (z, \xi) = (z + u, e_u(z)\xi).$$

Thus L corresponds to the multiplicative 1–cocycle $\{e_u\}$ satisfying the identity

$$e_{u_1+u_2}(z) = e_{u_2}(z + u_1) e_{u_1}(z)$$

Consider the exponential sequence on V,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_V \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}_V^\times \longrightarrow 1.$$

Passing to group cohomology of Λ gives an exact sequence

$$H^1(\Lambda, \mathcal{O}_V^\times) \xrightarrow{\delta} H^2(\Lambda, \mathbb{Z}),$$

and it is standard that the Chern class $c_1(L)$ is precisely the image under δ of the cocycle $\{e_u\}$.

Choose logarithms f_u with $e_u = \exp(2\pi i f_u)$. The formula for δ on group cochains gives the 2–cochain

$$F(u_1, u_2)(z) := f_{u_2}(z + u_1) - f_{u_1+u_2}(z) + f_{u_1}(z).$$

Because of the multiplicative cocycle identity taking logarithms yields

$$f_{u_1+u_2}(z) - f_{u_2}(z + u_1) - f_{u_1}(z) \in \mathbb{Z}.$$

Hence $F(u_1, u_2)(z)$ takes values in $\mathbb{Z}$. Being holomorphic and integer–valued, it is constant in z. Thus F defines a function

$$F : \Lambda \times \Lambda \longrightarrow \mathbb{Z},$$

and one checks directly that it satisfies the group cocycle condition $\delta F = 0$. Therefore F is a 2–cocycle representing the Chern class $c_1(L)$.

The previous lemma identifies

$$H^2(\Lambda, \mathbb{Z}) \;\cong\; \mathrm{Alt}^2(\Lambda, \mathbb{Z})$$

via the alternation map

$$AF(u_1, u_2) = F(u_1, u_2) - F(u_2, u_1).$$

Applying this to the cocycle F gives

$$\begin{aligned} E(u_1, u_2) &= F(u_1, u_2) - F(u_2, u_1) \\ &= f_{u_2}(z + u_1) + f_{u_1}(z) - f_{u_1}(z + u_2) - f_{u_2}(z), \end{aligned}$$

which is therefore a representative of $c_1(L)$ in the alternating form picture. In particular E is bilinear, alternating, and independent of z. $\square$

5.2 Hermitian structures on complex vector bundles

Let E be a complex vector bundle over a smooth manifold X.

Definition 5.16. A **Hermitian structure** on E is a smoothly varying family of Hermitian inner products $\langle \cdot, \cdot \rangle_x$ on the fibers E_x for each $x \in X$. In other words, for any two smooth sections s, t of E, the function

$$x \mapsto \langle s(x), t(x) \rangle_x$$

is a smooth complex-valued function on X.

In a local frame σ_i, a Hermitian structure is given by a positive definite Hermitian matrix of smooth functions, defining a positive definite inner product on each fiber. If $\sigma = \sum f_i \sigma_i$ and $\tau = \sum g_i \sigma_i$ are local sections, then

$$\langle \sigma, \tau \rangle = \sum_{i,j} h_{ij} \overline{f_i} g_j$$

where $h_{ij} = \langle \sigma_i, \sigma_j \rangle$. **Throughout the text, we use the convention that the inner product is conjugate linear in the first variable and linear in the second variable. Note that this convention results in some formulas looking different from other texts which use the opposite convention.**

Global Hermitian structures always exist on complex vector bundles over smooth manifolds by using partitions of unity to patch together local Hermitian structures.

Definition 5.17. A connection ∇ on a complex vector bundle E with Hermitian structure $\langle \cdot, \cdot \rangle$ is **Hermitian** if for any real tangent vector field X on X and any smooth sections s, t of E, we have

$$X\langle s, t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$$

Equivalently, the parallel transport defined by ∇ preserves the Hermitian structure on the fibers of E.

For any piecewise smooth path $\gamma : [0,1] \to X$ and an initial vector $\sigma(\gamma(0)) \in E_{\gamma(0)}$, the parallel transport of $\sigma(\gamma(0))$ along γ is the unique section σ of $\gamma^* E$ satisfying the following ordinary differential equation:

$$\begin{aligned} \nabla_{\dot\gamma(t)}\sigma &= 0 \\ \sigma(\gamma(0)) &= \sigma(\gamma(0)) \end{aligned}$$

We say σ is horizontal along γ for the connection ∇. Then ∇ is Hermitian if for any such γ and any two initial vectors $v, w \in E_{\gamma(0)}$, the parallel transports s_v and s_w of v and w along γ satisfy

$$\langle s_v(\gamma(t)), s_w(\gamma(t))\rangle = \langle v, w\rangle$$

for all $t \in [0,1]$.

> One way of thinking about the expression $X\langle s, t\rangle$ is that it is linear in s and t, but also $\langle , \rangle$ as well, and the Hermitian condition expresses the fact that the covariant derivative of the Hermitian metric is zero.

Example 5.18. On $X = S^1$, consider the trivial complex line bundle $E = S^1 \times \mathbb{C}$ with the standard Hermitian structure

$$\langle (x, v), (x, w)\rangle = \overline{v} w$$

for $x \in S^1$ and $v, w \in \mathbb{C}$. Define a connection ∇ on E by

$$\nabla_{\frac{d}{d\theta}} s = \frac{ds}{d\theta} + \cos(\theta) s$$

for any smooth section $s : S^1 \to E$. This choice of ∇ is not Hermitian because the factor of $\cos(\theta)$ causes the norm of a section to change under parallel transport. The connection

$$\nabla_{\frac{d}{d\theta}} s = \frac{ds}{d\theta} + i f(\theta) s$$

for any real-valued function $f : S^1 \to \mathbb{R}$ is Hermitian because the factor of $i f(\theta)$ acts only by rotation and preserves the norm of sections under parallel transport.

Theorem 5.19 (Chern connection)**.** Given E, h over X Hermitian and holomorphic, there exists a unique Hermitian connection ∇ on E whose $(0,1)$-component is the Dolbeault operator $\bar\partial_E$:

$$\nabla_{d/d\bar z_i} s = \frac{\partial}{\partial \bar z_i} s$$

Proof. First we check the uniqueness. Let s, t be local holomorphic sections of E. Then by the Hermitian condition, we have

$$\frac{\partial}{\partial z_i} h(s,t) = h(\nabla_{\frac{\partial}{\partial \bar{z}_i}} s, t) + h(s, \nabla_{\frac{\partial}{\partial z_i}} t) = h(s, \nabla_{\frac{\partial}{\partial z_i}} t)$$

since $\nabla_{\frac{\partial}{\partial \bar{z}_i}} s = \bar{\partial}_E s = 0$ by holomorphicity of s. In particular, any two holomorphic sections s, t can be written as column vectors $s = (s^1, s^2, \ldots, s^r)$ and $t = (t^1, t^2, \ldots, t^r)$ in a local holomorphic frame of E, and the Hermitian metric can be expressed as

$$h(s,t) = \sum_{i,j} h_{ij} \overline{s^i} t^j$$

where the s^i and t^j are holomorphic functions. Therefore, applying $\frac{\partial}{\partial z_i}$ to the s^i yields zero because $\overline{s^i}$ is antiholomorphic.

In this case, the left hand side is known and since the form $h(s, \cdot)$ is nondegenerate, this determines $\nabla_{\frac{\partial}{\partial z_i}} t$ uniquely. Now we check existence. Note that having established uniqueness, it suffices to establish existence via a formula locally since there is no ambiguity in gluing local connections.

Choose a local holomorphic frame

$$\{\sigma_1, \ldots, \sigma_r\}$$

for E. In this frame the metric is given by a positive definite Hermitian matrix of smooth functions

$$H = (h_{ij}), \qquad h_{ij} := h(\sigma_i, \sigma_j).$$

Any local section s of E can be written as a column vector of holomorphic functions

$$s = \sum_i s^i \sigma_i \quad \leftrightarrow \quad s = (s^1, \ldots, s^r)^\mathsf{T},$$

and similarly $t = (t^1, \ldots, t^r)^\mathsf{T}$. With our convention on h we have

$$h(s,t) = \sum_{i,j} h_{ij} \, \overline{s^i} \, t^j = \bar{s}^\mathsf{T} H t.$$

We look for a connection of the form

$$\nabla^{0,1} = \bar{\partial}_E, \qquad \nabla^{1,0} = \partial + A^{1,0},$$

where $A^{1,0}$ is an $r \times r$ matrix of $(1,0)$-forms. Thus, in this frame,

$$\nabla s = \bar{\partial} s + \partial s + A^{1,0} s.$$

For a vector field of type $(1,0)$, say $X = \partial/\partial z_k$, set $\nabla_k := \nabla_{\partial/\partial z_k}$ and $\partial_k := \partial/\partial z_k$. The Hermitian compatibility condition

$$\partial_k h(s,t) \;=\; h(\nabla_k s, t) + h(s, \nabla_k t)$$

now becomes, in matrix notation,

$$\begin{aligned}\partial_k(\overline{s}^{\mathsf{T}} H t) &= \overline{(\nabla_k s)}^{\mathsf{T}} H t \;+\; \overline{s}^{\mathsf{T}} H (\nabla_k t) \\ &= \overline{(\partial_k s + A_k s)}^{\mathsf{T}} H t \;+\; \overline{s}^{\mathsf{T}} H (\partial_k t + A_k t),\end{aligned}$$

where A_k is the matrix of functions such that $A^{1,0} = \sum_k A_k \, dz_k$.

On the other hand,

$$\partial_k(\overline{s}^{\mathsf{T}} H t) = \overline{s}^{\mathsf{T}} (\partial_k H) t + \overline{(\partial_k s)}^{\mathsf{T}} H t + \overline{s}^{\mathsf{T}} H (\partial_k t),$$

since $\partial_k \overline{s} = 0$ (the s^i are holomorphic).

Comparing the two expressions and cancelling the terms containing $\partial_k s$ and $\partial_k t$ gives

$$\overline{s}^{\mathsf{T}} (\partial_k H) t = \overline{(A_k s)}^{\mathsf{T}} H t + \overline{s}^{\mathsf{T}} H A_k t = \overline{s}^{\mathsf{T}} (A_k^* H + H A_k) t$$

for all s, t. Hence we obtain the matrix identity

$$\partial_k H = A_k^* H + H A_k.$$

Taking the $(1,0)$-part of both sides and using that A_k is of type $(1,0)$ while A_k^* is of type $(0,1)$, we get

$$\partial_k H = H A_k,$$

so

$$A_k = H^{-1} \partial_k H.$$

Equivalently, in differential form notation,

$$A^{1,0} = H^{-1} \partial H.$$

Thus in our holomorphic frame we have defined a connection by

$$\nabla_{\frac{\partial}{\partial \bar{z}_k}} s = \frac{\partial s}{\partial \bar{z}_k}, \qquad \nabla_{\frac{\partial}{\partial z_k}} s = \frac{\partial s}{\partial z_k} + H^{-1} \frac{\partial H}{\partial z_k}\, s,$$

which by construction satisfies $\nabla^{0,1} = \bar{\partial}_E$ and is compatible with the Hermitian metric h. One checks that these local formulas transform correctly under change of holomorphic frame, so they glue to a global connection. This is the desired Chern connection. $\square$

Proposition 5.20. The Chern connection has curvature given by the formula

$$F_\nabla = \bar{\partial}(H^{-1}\partial H)$$

provided H is defined in a local holomorphic frame. In particular, the curvature is of type $(1,1)$.

Theorem 5.21 (Weights and curvature of Hermitian metrics on holomorphic line bundles)**.** Let $L \to X$ be a holomorphic line bundle on a complex manifold, equipped with a Hermitian metric h. Choose a local holomorphic frame e of L over an open set $U \subset X$, and write

$$h(e,e) = e^{-\phi}, \qquad \phi \in C^\infty(U, \mathbb{R}).$$

Then ϕ is called the **local weight** of the metric h in the frame e. The following statements hold:

1. The Chern connection ∇ of (L,h) is given in the frame e by

$$\nabla e = (\partial\phi) \otimes e.$$

2. The curvature of the Chern connection is

$$F_h \;=\; \partial\bar{\partial}\phi.$$

   In particular, F_h is a real $(1,1)$–form representing $2\pi i\, c_1(L)$.

3. If h_0 and h are two Hermitian metrics related by

$$h = e^{-\varphi}\, h_0, \qquad \varphi \in C^\infty(X, \mathbb{R}),$$

   and if $h_0(e,e) = e^{-\phi_0}$, then $h(e,e) = e^{-(\phi_0+\varphi)}$. Thus the weights satisfy $\phi = \phi_0 + \varphi$, and the curvatures satisfy

$$F_h = F_0 + \partial\bar{\partial}\varphi.$$

4. Since $\log h(e,e) = -\phi$, one may also write

$$F_h = -\,\partial\bar{\partial}\log h(e,e).$$

Proof. Let e be a holomorphic frame. Since $h(e,e) = e^{-\phi}$, metric compatibility determines the connection 1-form θ by

$$\partial h(e,e) = h(e,e)\big(\theta + \overline{\theta}\big), \qquad \bar{\partial} h(e,e) = h(e,e)\,\overline{\theta}.$$

Because e is holomorphic, the $(0,1)$-part of the Chern connection equals $\bar{\partial}$, hence $\overline{\theta} = \bar{\partial}\phi$. It follows that $\theta = \partial\phi$, giving

$$\nabla e = \theta \otimes e = (\partial\phi) \otimes e,$$

which proves (1).

Since the curvature is $F_h = \nabla^2 = d\theta$ and $\theta = \partial\phi$ has type $(1,0)$, we obtain

$$F_h = \bar{\partial}(\partial\phi) = \partial\bar{\partial}\phi,$$

proving (2). If $h = e^{-\varphi}h_0$, then

$$h(e,e) = e^{-\varphi}h_0(e,e) = e^{-(\phi_0+\varphi)},$$

so $\phi = \phi_0 + \varphi$. Applying (2) yields

$$F_h = \partial\bar{\partial}\phi = \partial\bar{\partial}(\phi_0 + \varphi) = F_0 + \partial\bar{\partial}\varphi,$$

which proves (3). Finally, since $\log h(e,e) = -\phi$, statement (4) is the same as (2) written as

$$F_h = -\partial\bar{\partial}\log h(e,e).$$

□

Example 5.22. Consider the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^n$. It carries a hermitian structure invariant under the action of $U(n+1)$ induced from the standard hermitian structure on $\mathbb{C}^{n+1}$. In the standard affine chart $U_0 = \{[1 : z_1 : \cdots : z_n]\}$ with coordinates $z_i = x_i + iy_i$, a local holomorphic frame is given by the section

$$\sigma(z) = (1, z_1, z_2, \ldots, z_n)$$

whose norm is given by

$$h(\sigma(z), \sigma(z)) = 1 + |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$$

Therefore, in this local holomorphic frame, the hermitian metric is the bilinear form $h = 1 + \sum |z_i|^2$, i.e.

$$h(f\sigma, g\sigma) = (1 + \sum |z_i|^2)\overline{f}g$$

where $f\sigma, g\sigma$ are local sections. We can compute the curvature of the Chern connection on $\mathcal{O}(-1)$

$$\begin{aligned}
F_\nabla &= \bar{\partial}\left(h^{-1}\partial h\right) \\
&= \bar{\partial}\left(\frac{\sum \overline{z_i}dz_i}{1 + \sum |z_i|^2}\right) \\
&= \frac{\sum z_i dz_i \wedge \sum \overline{z_i}d\overline{z_i}}{(1 + \sum |z_i|^2)^2} - \frac{\sum dz_i \wedge d\overline{z_i}}{1 + \sum |z_i|^2} \\
&= \frac{-\sum(1 + \sum |z_j|^2)dz_i \wedge d\overline{z_i} + \sum z_i\overline{z_j}dz_i \wedge d\overline{z_j}}{(1 + \sum |z_i|^2)^2}
\end{aligned}$$

When $n = 1$, this reduces to

$$F_\nabla = \frac{-(1+|z|^2)dz \wedge d\overline{z} + z\overline{z}dz \wedge d\overline{z}}{(1+|z|^2)^2}$$
$$= \frac{-dz \wedge d\overline{z}}{(1+|z|^2)^2}$$
$$= \frac{2idx \wedge dy}{(1+r^2)^2}$$

This gives

$$c_1(\mathcal{O}(-1)) = \frac{i}{2\pi}F_\nabla = \frac{-1}{\pi}\frac{dx \wedge dy}{(1+r^2)^2}$$

Finally we can compute

$$\int_{\mathbb{CP}^1} c_1(\mathcal{O}(-1)) = \frac{-1}{\pi} \cdot 2\pi \cdot \int_0^\infty \frac{r}{(1+r^2)^2}dr$$
$$= -1$$

so we see why the normalization of Chern classes is chosen as it is.

The reason $\bar{\partial}$ operators are important is that they allow us to define holomorphic structures on complex line bundles. In fact, the following proposition holds for complex vector bundles of any rank, but we state it here for line bundles for simplicity.

Proposition 5.23. Let L be a smooth complex line bundle over a complex manifold X. A holomorphic structure on L is equivalent to the choice of a $\mathbb{C}$-linear operator

$$\bar{\partial}_L : \mathcal{E}^0(L) \longrightarrow \mathcal{E}^{0,1}(L)$$

satisfying:

(i) The Leibniz rule: $\bar{\partial}_L(fs) = (\bar{\partial}f) \otimes s \;+\; f\,\bar{\partial}_L s, \qquad f \in C^\infty(X),\; s \in \mathcal{E}^0(L)$

(ii) $\bar{\partial}_L^2 = 0$.

Proof. Suppose first that L is a holomorphic line bundle. Choose a holomorphic trivialization on an open set $U \subset X$ with holomorphic frame e. Every smooth section has the form $s = fe$, and we define

$$\bar{\partial}_L(s) := (\bar{\partial}f)\,e.$$

If $e' = ge$ is another holomorphic frame on U, then $f' = g^{-1}f$ and $\bar{\partial}g = 0$, so

$$\bar{\partial}f' = g^{-1}\bar{\partial}f.$$

Thus $(\bar\partial f')e' = (\bar\partial f)e$, so the definition glues on overlaps. The resulting operator clearly satisfies the Leibniz rule, and $\bar\partial_L^2 = 0$ because in a holomorphic frame it is just the usual $\bar\partial$ on functions.

Conversely, suppose we are given a $\mathbb{C}$-linear operator $\bar\partial_L$ satisfying (i)-(ii). On a trivializing open set U, pick a nowhere-vanishing smooth frame e, and write any section as $s = fe$. By the Leibniz rule there is a unique $(0,1)$-form α on U such that

$$\bar\partial_L e = \alpha \otimes e.$$

Then

$$\bar\partial_L(fe) = (\bar\partial f + \alpha f)\, e.$$

Applying $\bar\partial_L^2 = 0$ to e gives $\bar\partial\alpha = 0$. By the $\bar\partial$-Poincaré lemma, on a smaller U there exists a smooth function ϕ with $\alpha = \bar\partial\phi$. Define a new frame $e' := e\, e^{-\phi}$; then

$$\bar\partial_L e' = 0.$$

Thus e' is a $\bar\partial_L$-holomorphic frame, meaning that a section $s = fe'$ satisfies $\bar\partial_L s = 0$ iff $\bar\partial f = 0$. Hence the sheaf

$$\mathcal{O}_L(U) := \{\, s \in \mathcal{E}^0(L|_U) \mid \bar\partial_L s = 0 \,\}$$

is identified with the usual sheaf of holomorphic functions on U. On overlaps $U \cap V$, if e' and e'' are such holomorphic frames, the transition function g defined by $e'' = ge'$ satisfies

$$0 = \bar\partial_L e'' = (\bar\partial g)\, e',$$

so $\bar\partial g = 0$; hence g is holomorphic. Therefore these frames define a holomorphic line bundle structure on L. $\square$

Exercise 5.24. Let $L \to X$ be a holomorphic line bundle on a complex manifold, and let $\alpha \in \mathcal{E}^{0,1}$ be a $\bar\partial$-closed form. Show that the re-defined operator

$$\tilde{\bar\partial} = \bar\partial + \alpha$$

on sections of L defines a new holomorphic structure L' on the same underlying bundle, where local holomorphic sections are defined as those killed by $\tilde{\bar\partial}$. Show that $L \simeq L'$ if α is $\bar\partial$-exact. Relate this to the exponential sequence.

For vector bundles, the same applies with an $\alpha \in \mathcal{E}^{0,1}(\mathrm{End}(V))$ satisfying the non-linear equation

$$\bar\partial\alpha + \alpha \wedge \alpha = 0.$$

The new bundle is isomorphic to the old one if $\alpha = a^{-1}\bar\partial a$, for some smooth section a of $\mathrm{Aut}(V)$.

Solution 5.25. From the given holomorphic structure, we have a $\mathbb{C}$-linear map

$$\bar{\partial}_L \colon \mathcal{E}^0(L) \longrightarrow \mathcal{E}^{0,1}(L)$$

satisfying the Leibniz rule and the condition $\bar{\partial}_L^2 = 0$. We have the new operator

$$\tilde{\bar{\partial}} s := \bar{\partial} s + \alpha \wedge s \in \mathcal{E}^{0,1}(L).$$

First we check that $\tilde{\bar{\partial}}$ is a $\bar{\partial}$-operator. For $f \in C^\infty(X)$ and $s \in \mathcal{E}^0(L)$,

$$\tilde{\bar{\partial}}(fs) = \bar{\partial}(fs) + \alpha\, fs = (\bar{\partial} f)\, s + f\, \bar{\partial} s + f\, \alpha s = (\bar{\partial} f)\, s + f\, \tilde{\bar{\partial}} s,$$

so the Leibniz rule holds.

Next, compute $\tilde{\bar{\partial}}^2$. View $\bar{\partial}$ as a derivation of degree $(0,1)$ on $\mathcal{E}^{0,\bullet}(L)$; then for $\beta \in \mathcal{E}^{0,1}$ and $\eta \in \mathcal{E}^{0,q}(L)$,

$$\bar{\partial}(\beta \wedge \eta) = (\bar{\partial}\beta) \wedge \eta - \beta \wedge \bar{\partial}\eta.$$

Hence, for a section $s \in \mathcal{E}^0(L)$,

$$\begin{aligned}
\tilde{\bar{\partial}}^2 s &= \tilde{\bar{\partial}}\big(\bar{\partial} s + \alpha \wedge s\big) \\
&= \bar{\partial}(\bar{\partial} s + \alpha \wedge s) + \alpha \wedge (\bar{\partial} s + \alpha \wedge s) \\
&= \bar{\partial}^2 s + \bar{\partial}(\alpha \wedge s) + \alpha \wedge \bar{\partial} s + \alpha \wedge \alpha \wedge s \\
&= 0 + (\bar{\partial}\alpha) \wedge s - \alpha \wedge \bar{\partial} s + \alpha \wedge \bar{\partial} s + \alpha \wedge \alpha \wedge s \\
&= (\bar{\partial}\alpha) \wedge s + \alpha \wedge \alpha \wedge s.
\end{aligned}$$

By assumption $\bar{\partial}\alpha = 0$, and since α is a 1-form, $\alpha \wedge \alpha = 0$. Thus $\tilde{\bar{\partial}}^2 s = 0$ for all s, so $\tilde{\bar{\partial}}^2 = 0$ and $\tilde{\bar{\partial}}$ is a $\bar{\partial}$-operator. It therefore defines a new holomorphic structure L' on the same underlying smooth bundle, whose local holomorphic sections are those killed by $\tilde{\bar{\partial}}$.

Now we check that if α is $\bar{\partial}$-exact, then $L' \simeq L$. Suppose $\alpha = \bar{\partial}\phi$ for some smooth complex-valued function ϕ. Define an automorphism of the C^∞ line bundle L by multiplication with e^ϕ:

$$F \colon L \longrightarrow L, \qquad s \longmapsto e^\phi s.$$

We claim that F is an isomorphism of holomorphic line bundles $L' \to L$, i.e.

$$\bar{\partial}(Fs) = F(\tilde{\bar{\partial}} s) \quad \text{for all } s.$$

Indeed,

$$\bar{\partial}(Fs) = \bar{\partial}(e^\phi s) = e^\phi(\bar{\partial}\phi \wedge s + \bar{\partial} s) = e^\phi(\alpha \wedge s + \bar{\partial} s) = F(\tilde{\bar{\partial}} s).$$

Thus F is holomorphic with respect to $\tilde{\bar{\partial}}$ on the domain and $\bar{\partial}$ on the target, so $L' \simeq L$.

The $(0,1)$-form α is $\bar{\partial}$-closed, so it defines a Dolbeault cohomology class

$$[\alpha] \in H^{0,1}_{\bar{\partial}}(X) \cong H^1(X; \mathcal{O}).$$

Changing α by a $\bar{\partial}$-exact form does not change this class, and by the computation above such a change yields an isomorphic holomorphic structure. Thus the isomorphism class of the new line bundle L' depends only on $[\alpha]$.

Recall the holomorphic exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i\cdot)} \mathcal{O}^\times \longrightarrow 1,$$

whose long exact cohomology sequence contains

$$H^1(X; \mathcal{O}) \xrightarrow{\exp} H^1(X; \mathcal{O}^\times),$$

and $H^1(X; \mathcal{O}^\times) \cong \mathrm{Pic}(X)$ classifies holomorphic line bundles. The class $[\alpha] \in H^1(X; \mathcal{O})$ maps under the exponential to the class of the holomorphic line bundle $L' \otimes L^{-1}$.

6 Kahler manifolds

Recall that a Riemannian metric g on a smooth manifold X is a smoothly varying choice of inner product g_x on each tangent space T_xX for $x \in X$. In local coordinates x_i, the metric can be expressed as

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$$

where $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ are smooth functions on X.

Definition 6.1. The torsion T of a connection ∇ on the tangent bundle TX of a smooth manifold X is the map

$$T : \mathcal{E}^0(TX) \times \mathcal{E}^0(TX) \to \mathcal{E}^0(TX)$$
$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for any two smooth vector fields X, Y on X. A connection is torsion-free if $T(X,Y) = 0$ for all vector fields X, Y.

Theorem 6.2 (Levi-Civita)**.** There exists a unique torsion-free connection ∇ on the tangent bundle TX of a smooth manifold X which is compatible with a Riemannian metric g on X, i.e. for any vector fields X, Y, Z on X, we have

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

This connection is called the **Levi-Civita connection**.

One can obtain the Levi-Civita connection by symmetrizing any connection compatible with the metric. Let its torsion be $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. Define a new connection $\widetilde{\nabla}$ by $\widetilde{\nabla}_X Y := \nabla_X Y - \frac{1}{2}T(X,Y)$. Then one checks that $\widetilde{\nabla}$ is torsion-free and compatible with the metric.

Example 6.3. Let $V = \mathbb{C}^r$ and equip V with a hermitian inner product $\langle \cdot, \cdot \rangle$, i.e. one which is conjugate linear in the first argument and linear in the second argument. Then the Levi-Civita connection is the unique connection ∇ on the tangent bundle $TV \cong V \times V$ compatible with the Riemannian metric g defined by

$$g(u,v) = \mathrm{Re}\langle u, v\rangle$$

for $u, v \in V$. In this case, the Levi-Civita connection is simply the trivial connection

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0, \qquad \nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = 0$$

Equivalently, to covariantly differentiate the vector field Y in the direction of X at a point x, just take the directional derivative of the vector-valued function Y in the direction $X(x)$.

$$(\nabla_X Y)(x) = dY_x\big(X(x)\big)$$

In particular all constant vector fields are parallel and all coordinate functions have zero second derivatives. It has zero curvature via a long computation which can be done in coordinates.

In standard coordinates on V we can write

$$h \ = \ \sum_{i,j} h_{ij}\, d\bar{z}_i \otimes dz_j$$

with constants h_{ij} (because we chose a constant Hermitian inner product).

The associated Riemannian metric is $g = \Re(h)$, and the associated Kähler form is defined by

$$\omega(X,Y) \ := \ g(X, JY),$$

where J is the standard complex structure. In coordinates one has

$$\omega \ = \ \frac{i}{2}\sum_{i,j} h_{ij}\, dz_i \wedge d\bar{z}_j,$$

and again all coefficients h_{ij} are constant functions on V.

Since ω has constant coefficients in these linear coordinates, $d\omega = 0$ because d only differentiates the coefficients, which are constant. Hence (V, J, g) is Kähler.

Now if X is a complex manifold, then the Riemannian metric comes from a Hermitian metric h on the tangent bundle TX. We get a preferred hermitian connection on TX by taking the Chern connection. However, the induced connection on the underlying real tangent bundle $T_{\mathbb{R}}X$ need not be the Levi-Civita connection of the Riemannian metric associated to h. This is very special and is precisely the condition for X to be a Kahler manifold.

Theorem 6.4. The following are equivalent for a complex manifold X with Hermitian metric h:

1. The Levi-Civita connection of the Riemannian metric associated to h coincides with the Chern connection of h on TX.

2. Parallel transport with respect to the Levi-Civita connection is complex linear.

3. Let the Hermitian metric be given by
$$h = \sum_{i,j} h_{ij} dz_i \otimes d\overline{z_j}$$
in local holomorphic coordinates z_i. Then the associated $(1,1)$-form
$$\omega = \frac{i}{2} \sum_{i,j} h_{ij} dz_i \wedge d\overline{z_j}$$
is closed.

4. For all x there exist complex coordinates z_i centered at x so that at x,
$$h_{ij} = \delta_{ij} + O(|z|^2)$$
i.e. the first derivatives of h_{ij} vanish at x.

Remark 6.5. In flat space $\mathbb{C}^n$ with standard coordinates $z_i = x_i + iy_i$, the standard Hermitian metric is given by

$$h = \sum_{i=1}^{n} dz_i \otimes d\overline{z_i} = \sum_{i=1}^{n} (dx_i \otimes dx_i + dy_i \otimes dy_i)$$

The associated $(1,1)$-form is

$$\omega = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\overline{z_i} = \sum_{i=1}^{n} dx_i \wedge dy_i$$

In particular, ω is real, nondegenerate, and closed so it is a symplectic form.

Remark 6.6 (Sketch of proof of equivalences)**.** A strong consequence of (4) is that any coordinate independent identity involving the metric and its first derivatives which holds in flat space must hold on any Kahler manifold. One identity one sees immediately is that $d\omega = 0$ because it holds in flat space. Thus (4) implies (3). (3) implies (4) is an exercise in choosing coordinates. The equivalence of (1) and (2) is straightforward from the definitions. The equivalence of (1) and (3) is a computation in local coordinates.

Definition 6.7. A complex manifold X with Hermitian metric h satisfying the equivalent conditions above is called a **Kahler manifold**. The associated $(1,1)$-form ω is called the **Kahler form** of the Kahler manifold.

The formula relating the Kahler form to the metric is given by

$$h(X,Y) = -2i\omega(X,\overline{Y})$$

Theorem 6.8. Every compact Riemann surface is a Kahler manifold.

Proof. Let X be a compact Riemann surface and pick any Hermitian metric h on TX. The resulting $(1,1)$-form ω is a real 2-form on the 2-dimensional manifold X, so it is automatically closed because there are no 3-forms on a 2-manifold. Therefore, h is a Kahler metric on X. $\square$

6.1 Harmonic theory on Kahler manifolds

The key construction is the inner product on the space of complex-valued differential forms on a compact Kahler manifold, which allows us to define formal adjoints of the operators ∂ and $\bar{\partial}$. This leads to the definition of Laplacians and harmonic forms.

Recall that we have the decomposition of the de Rham differential

$$d : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q}(X) \oplus \mathcal{E}^{p,q+1}(X)$$

into its $(1,0)$ and $(0,1)$ components

$$\begin{aligned} \partial &: \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q}(X) \\ \bar{\partial} &: \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X) \end{aligned}$$

satisfying

$$\begin{aligned} \partial^2 &= 0 \\ \bar{\partial}^2 &= 0 \\ \partial\bar{\partial} + \bar{\partial}\partial &= 0 \end{aligned}$$

so that

$$d = \partial + \bar{\partial}$$

On a compact Kahler manifold X, we have the following harmonic theory. There is a Hodge star operator

$$* : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{n-q,n-p}(X)$$

characterized on real forms ϕ by

$$\overline{\phi} \wedge *\psi = \langle \phi, \psi \rangle dV$$

where $dV = \frac{\omega^n}{n!}$ is the volume form associated to the Kahler metric and $\langle \cdot, \cdot \rangle$ is the pointwise inner product on forms induced by the Kahler metric.

In coordinates x_i for which $\frac{\partial}{\partial x_i}$ is an orthonormal basis at a point, we have

$$*(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{n-k}} \varepsilon$$

where $\{j_1, j_2, \ldots, j_{n-k}\}$ is the complement of $\{i_1, i_2, \ldots, i_k\}$ in $\{1, 2, \ldots, n\}$ such that corresponding permutation has sign ε.

Then we extend $*$ complex linearly to complex-valued forms. This induces the complex inner product

$$\langle \phi, \psi \rangle_{L^2} = \int_X \langle \phi, \psi \rangle dV = \int_X \overline{\phi} \wedge *\psi$$

on the space of complex-valued forms. We define the formal adjoints of ∂ and $\bar{\partial}$ with respect to this inner product:

$$\partial^* : \mathcal{E}^{p+1,q}(X) \to \mathcal{E}^{p,q}(X)$$
$$\bar{\partial}^* : \mathcal{E}^{p,q+1}(X) \to \mathcal{E}^{p,q}(X)$$

characterized by the adjunction

$$\langle \partial\phi, \psi \rangle_{L^2} = \langle \phi, \partial^*\psi \rangle_{L^2}$$
$$\langle \bar{\partial}\phi, \psi \rangle_{L^2} = \langle \phi, \bar{\partial}^*\psi \rangle_{L^2}$$

for all compactly supported smooth forms ϕ, ψ of appropriate degrees.

Proposition 6.9. Let $\bar{\partial} : \mathcal{E}^k(X) \to \mathcal{E}^{k+1}(X)$ be the Dolbeault operator on k-forms. Its formal adjoint with respect to $\langle \cdot, \cdot \rangle_{L^2}$ is

$$\bar{\partial}^* = - *^{-1} \partial * .$$

Similarly,

$$\partial^* = - *^{-1} \bar{\partial} * .$$

Proof. Let $\phi \in \mathcal{E}^k(X)$ and $\psi \in \mathcal{E}^{k+1}(X)$ have compact support. Then

$$\langle \bar\partial\phi, \psi\rangle_{L^2} = \int_X \overline{\bar\partial\phi} \wedge *\psi = \int_X \partial\overline{\phi} \wedge *\psi,$$

since complex conjugation interchanges ∂ and $\bar\partial$.

By the Leibniz rule for ∂ on a k-form $\overline{\phi}$,

$$\partial(\overline{\phi} \wedge *\psi) = \partial\overline{\phi} \wedge *\psi + (-1)^k\, \overline{\phi} \wedge \partial(*\psi).$$

Integrating and using Stokes' theorem (the boundary term vanishes because the forms are compactly supported), we obtain

$$\int_X \partial\overline{\phi} \wedge *\psi = (-1)^{k+1} \int_X \overline{\phi} \wedge \partial(*\psi).$$

Hence

$$\langle \bar\partial\phi, \psi\rangle_{L^2} = (-1)^{k+1} \int_X \overline{\phi} \wedge \partial(*\psi).$$

Insert $*^{-1}$:

$$\partial(*\psi) = *\big(*^{-1}\partial(*\psi)\big).$$

Thus

$$\langle \bar\partial\phi, \psi\rangle_{L^2} = (-1)^{k+1} \int_X \overline{\phi} \wedge *\big(*^{-1}\partial(*\psi)\big).$$

By the definition of the L^2 inner product,

$$\langle \phi, \eta\rangle_{L^2} = \int_X \overline{\phi} \wedge *\eta \qquad \text{for any form } \eta,$$

so comparing with the previous line we see that

$$\bar\partial^*\psi = (-1)^{k+1}\ *^{-1}\partial(*\psi) \quad \text{on } k+1\text{-forms } \psi$$

$\square$

Remark 6.10. Note that these are only formal adjoints since $d, \partial, \bar\partial$ are unbounded operators on the L^2-completion of the space of smooth forms. The adjunction only holds on the dense subspace of compactly supported smooth forms.

6.2 Hodge decomposition

Definition 6.11. The **Laplacians** associated to the differentials $d, \partial, \bar{\partial}$ are defined as follows:

$$\Delta = dd^* + d^*d = (d + d^*)^2$$
$$\Box = \partial\partial^* + \partial^*\partial = (\partial + \partial^*)^2$$
$$\bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = (\bar{\partial} + \bar{\partial}^*)^2$$

On a general complex manifold with a Hermitian metric, these Laplacians are different. However, on a Kähler manifold, they essentially coincide.

Theorem 6.12. On a Kähler manifold X, the Laplacians satisfy the relations

$$\Delta = 2\Box = 2\bar{\Box}$$

Proof. The identity is true in flat space and involves only the metric and its first derivatives. □

Definition 6.13 (Harmonic forms). We say that a form $\varphi \in \mathcal{E}^{p,q}(X)$ is harmonic if it satisfies

$$\Delta\varphi = 0$$

The space of harmonic (p,q)-forms is denoted by $\mathcal{H}^{p,q}(X)$. Similarly, we define the space of harmonic r-forms by

$$\mathcal{H}^r(X) = \{\varphi \in \mathcal{E}^r(X) : \Delta\varphi = 0\}$$

On a compact complex manifold, we have the de Rham cohomology groups $H^r(X, \mathbb{C})$ whose classes are represented by closed r-forms with complex coefficients. We also have the Dolbeault cohomology groups $H^{p,q}(X)$ whose classes are represented by $\bar{\partial}$-closed (p,q)-forms. These groups are finite dimensional and are related by a spectral sequence.

However, if ϕ is a d-closed r-form, its (p,q)-components need not be $\bar{\partial}$-closed, and conversely if ψ is a $\bar{\partial}$-closed (p,q)-form, it need not be d-closed. However, on a compact Kahler manifold, it turns out that every d-closed form ϕ is d-cohomologous to a form whose (p,q)-components are all $\bar{\partial}$-closed, and conversely every $\bar{\partial}$-closed form ψ is $\bar{\partial}$-cohomologous to a form whose r-components are all d-closed. This leads to the Hodge decomposition theorem.

Theorem 6.14 (Hodge decomposition on Kähler manifolds). Let X be a compact Kähler manifold. Then there is a direct sum decomposition

$$H^r(X, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X), \tag{11}$$

and, moreover,

$$\overline{H^{p,q}(X)} = H^{q,p}(X). \tag{12}$$

Proof. We shall show that

$$\mathcal{H}^r(X) \;=\; \bigoplus_{p+q=r} \mathcal{H}^{p,q}(X),$$

and then (11) follows immediately from Hodge theory for de Rham cohomology

$$\begin{aligned} H^r(X,\mathbb{C}) &\cong \mathcal{H}^r(X) \\ [\varphi] &\longmapsto \text{the unique harmonic representative of } [\varphi] \end{aligned}$$

and Dolbeault cohomology

$$\begin{aligned} H^{p,q}(X) &\cong \mathcal{H}^{p,q}(X) \\ [\varphi] &\longmapsto \text{the unique harmonic representative of } [\varphi] \end{aligned}$$

Suppose that $\varphi \in \mathcal{H}^r(X)$, so $\Delta\varphi = 0$. On a Kähler manifold we have $2\bar{\Box} = \Delta$ and hence $\bar{\Box}\varphi = 0$. Writing φ as a sum of its bihomogeneous components,

$$\varphi = \varphi^{r,0} + \cdots + \varphi^{0,r},$$

we have

$$\bar{\Box}\varphi = \bar{\Box}\varphi^{r,0} + \cdots + \bar{\Box}\varphi^{0,r}.$$

Since $\bar{\Box}$ preserves bidegree, we see that $\bar{\Box}\varphi = 0$ implies that

$$\bar{\Box}\varphi^{r,0} = \cdots = \bar{\Box}\varphi^{0,r} = 0.$$

Therefore there is a mapping

$$\tau : \mathcal{H}^r(X) \longrightarrow \bigoplus_{p+q=r} \mathcal{H}^{p,q}(X)$$

given by

$$\varphi \longmapsto (\varphi^{r,0}, \ldots, \varphi^{0,r}).$$

The mapping is clearly injective, and moreover surjective, since if $\psi^{p,q} \in \mathcal{H}^{p,q}(X)$ for $p+q=r$, then

$$\varphi := \sum_{p+q=r} \psi^{p,q}$$

satisfies $\Delta\varphi = 2\bar{\Box}\varphi = 0$, so $\varphi \in \mathcal{H}^r(X)$ and $\tau(\varphi) = (\psi^{r,0}, \ldots, \psi^{0,r})$.

The isomorphism $\overline{H^{p,q}(X)} \cong H^{q,p}(X)$ follows immediately from the fact that $\bar{\Box}$ is real, i.e. commutes with complex conjugation, and that complex conjugation is an isomorphism from $\mathcal{E}^{p,q}(X)$ to $\mathcal{E}^{q,p}(X)$, hence also from $\mathcal{H}^{p,q}(X)$ to $\mathcal{H}^{q,p}(X)$. $\Box$

Remark 6.15. Every real cohomology class $[\alpha] \in H^1(X, \mathbb{R})$ on a compact Kähler manifold X has a unique harmonic representative α satisfying $\Delta_d \alpha = 0$. Since α is a real 1-form, we can write it as

$$\alpha = \alpha^{1,0} + \alpha^{0,1}$$

where $\alpha^{1,0}$ is a $(1,0)$-form and $\alpha^{0,1}$ is a $(0,1)$-form. Note that $\overline{\alpha^{1,0}} = \alpha^{0,1}$ since α is real. The Dolbeault cohomology class $[\alpha^{0,1}] \in H^{0,1}(X)$ corresponds to the image of $[\alpha]$ under the natural map

$$H^1(X, \mathbb{R}) \hookrightarrow H^1(X, \mathbb{C}) \to H^{0,1}(X)$$

induced by the inclusion of sheaves $\mathbb{R} \hookrightarrow \mathcal{O}_X$.

This is because on a Kahler manifold, the three operators

$$\begin{aligned} \Delta_d &= dd^* + d^*d \\ \Box &= \partial\partial^* + \partial^*\partial \\ \bar{\Box} &= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \end{aligned}$$

mapping $\mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q}(X)$ are related by

$$\Delta_d = 2\bar{\Box} = 2\Box$$

In particular, an $\alpha \in \mathcal{E}^{p,q}$ is Δ_d-harmonic if and only if it is $\bar{\Box}$-harmonic if and only if it is $\Box$-harmonic.

Therefore, the map induced by $\mathbb{R}_X \hookrightarrow \mathcal{O}_X$

$$\begin{aligned} H^1(X, \mathbb{R}_X) \to H^1(X, &\mathcal{O}_X) \cong H^{0,1}(X) \\ [\alpha] &\mapsto [\alpha^{0,1}] \end{aligned}$$

is an isomorphism of $\mathbb{R}$-vector spaces.

Remark 6.16. The Hodge decomposition theorem can also be obtained via direct spectral sequence arguments using the degeneration at the E_1 page of the Hodge-de Rham spectral sequence on a Kahler manifold. This argument is worth mentioning because it allows one to obtain the Hodge decomposition theorem for special compact complex manifolds which are not known to be Kahler but for which the Hodge-de Rham spectral sequence degenerates at E_1. This remark applies to K3 surfaces whose Hodge decomposition was obtained before the proof that all K3 surfaces are Kahler manifolds.

Corollary 6.17. Let X be a compact Kähler manifold. Then the Betti numbers $b_r = \dim H^r(X, \mathbb{C})$ and Hodge numbers $h^{p,q} = \dim H^{p,q}(X)$ satisfy the relations

1. $b_r = \sum_{p+q=r} h^{p,q}$
2. $h^{p,q} = h^{q,p}$

3. b_r is even when r is odd.

4. $h^{1,0}$ is a topological invariant since $h^{1,0} = \frac{1}{2}b_1$.

Exercise 6.18. For a compact Riemann surface R, verify that the Serre duality pairing

$$H^1(R;\mathcal{O}) \otimes H^0(R;\Omega^1) \longrightarrow \mathbb{C}$$

defined by principal parts and residues agrees with the one given by integration of Dolbeault representatives. Using the relation to harmonic forms, explain how this relates to Poincaré duality on R.

Solution 6.19. Choose a meromorphic function f on R whose principal part at each p_i with prescribed principal parts. Let U_i be pairwise disjoint coordinate discs around p_i, and choose $\chi \in C^\infty(R)$ such that $\chi \equiv 1$ on smaller discs $U_i' \subset U_i$ and $\chi \equiv 0$ outside $\bigcup_i U_i$. Define a $(0,1)$-**current**

$$T_f := \bar\partial(\chi f).$$

Since $\bar\partial^2 = 0$, T_f is $\bar\partial$-closed. If we replace f by $f+g$ for a global meromorphic function g (with poles in D) or change χ within the same constraints, T_f changes by a current of the form $\bar\partial u$, so the class $[T_f]$ in

$$H^{0,1}_{\bar\partial}(R) \cong H^1(R,\mathcal{O})$$

depends only on the underlying principal parts.

Let $\omega \in H^0(R,\Omega^1)$ be a holomorphic 1-form. The **Dolbeault** definition of the pairing is

$$\langle \alpha, \omega\rangle_{Dol} := \frac{1}{2\pi i}\int_R T_f \wedge \omega = \frac{1}{2\pi i}\int_R \bar\partial(\chi f)\wedge\omega.$$

Since ω is of type $(1,0)$ and holomorphic, $\bar\partial\omega = 0$, hence

$$\bar\partial(\chi f)\wedge\omega = \bar\partial(\chi f\omega).$$

Let $D_i \subset U_i'$ be small closed discs around p_i and set

$$R_\varepsilon := R \setminus \bigcup_i D_i(\varepsilon),$$

where $D_i(\varepsilon)$ are concentric discs of radius ε. On R_ε the form $\chi f\omega$ is smooth with compact support, so Stokes' theorem gives

$$\int_{R_\varepsilon}\bar\partial(\chi f\omega) = \int_{\partial R_\varepsilon}\chi f\omega = -\sum_i\int_{\partial D_i(\varepsilon)} f\omega,$$

the sign coming from the induced orientation on the boundary.

Letting $\varepsilon \to 0$ and using the residue theorem,

$$\int_{\partial D_i(\varepsilon)} f\omega \;\longrightarrow\; 2\pi i\ \mathrm{Res}_{p_i}(f\omega),$$

we obtain

$$\frac{1}{2\pi i}\int_R \bar\partial(\chi f)\wedge\omega = \sum_i \mathrm{Res}_{p_i}(f\omega).$$

This is precisely the **principal parts** definition of the Serre pairing.

Now equip R with any Hermitian (necessarily Kähler) metric. Hodge theory yields the decompositions

$$H^1_{\mathrm{dR}}(R,\mathbb{C}) \;\cong\; \mathcal{H}^1(R) \;\cong\; H^{1,0}_{\bar\partial}(R)\oplus H^{0,1}_{\bar\partial}(R),$$

and every class has a unique harmonic representative. Moreover,

$$H^0(R,\Omega^1)\cong H^{1,0}_{\bar\partial}(R)$$

consists of harmonic $(1,0)$-forms, and

$$H^1(R,\mathcal{O})\cong H^{0,1}_{\bar\partial}(R)$$

is represented by harmonic $(0,1)$-forms. Complex conjugation gives an isomorphism

$$\overline{H^{1,0}_{\bar\partial}(R)}\cong H^{0,1}_{\bar\partial}(R)$$

Poincaré duality on R is given by the nondegenerate pairing

$$H^1_{\mathrm{dR}}(R,\mathbb{C})\times H^1_{\mathrm{dR}}(R,\mathbb{C})\longrightarrow\mathbb{C},\qquad ([\alpha],[\beta])\mapsto\int_R\alpha\wedge\beta.$$

It is clear that $\alpha\wedge\beta$ is nonzero only if α and β are of complementary types, i.e. their wedge is of type $(1,1)$, since $(1,0)\wedge(1,0)$ and $(0,1)\wedge(0,1)$ necessarily vanish. Thus the Poincaré pairing restricts to a nondegenerate pairing

$$H^{0,1}_{\bar\partial}(R)\;\otimes\;H^{1,0}_{\bar\partial}(R)\longrightarrow\mathbb{C},\qquad(\eta,\omega)\mapsto\int_R\eta\wedge\omega,$$

with η,ω harmonic representatives.

Under the identifications

$$H^1(R,\mathcal{O})\cong H^{0,1}_{\bar\partial}(R),\qquad H^0(R,\Omega^1)\cong H^{1,0}_{\bar\partial}(R),$$

the Serre pairing of α and ω is

$$\langle\alpha,\omega\rangle=\frac{1}{2\pi i}\int_R\eta\wedge\omega,$$

where η is the harmonic $(0,1)$-representative of α. In particular, on a compact Riemann surface the Serre duality

$$H^1(R, \mathcal{O}) \;\cong\; H^0(R, \Omega^1)^\vee$$

is nothing but Poincaré duality in degree 1 up to the constant factor $2\pi i$, expressed via the Hodge decomposition of $H^1_{\mathrm{dR}}(R, \mathbb{C})$.

Exercise 6.20. For a compact Riemann surface R, verify that the map

$$H^1(R; \mathbb{Z}) \longrightarrow H^1(R; \mathcal{O})$$

corresponds to the period map

$$H_1(R; \mathbb{Z}) \otimes H^0(R; \Omega^1) \longrightarrow \mathbb{C}$$

under integral Poincaré duality and Serre duality on R.

Solution 6.21. Let $i : H^1(R; \mathbb{Z}) \to H^1(R; \mathcal{O})$ be the given homomorphism. We need to show for every $c \in H^1(R; \mathbb{Z})$ and $\omega \in H^0(R, \Omega^1)$, the Serre pairing $\langle i(c), \omega \rangle_{\mathrm{Serre}}$ equals the period of ω along the 1-cycle Poincaré dual to c.

By Hodge theory, every class in $H^1(R; \mathbb{R})$ has a unique harmonic representative. An element $c \in H^1(R; \mathbb{Z})$ maps to a real class $c_{\mathbb{R}} \in H^1(R; \mathbb{R})$ whose harmonic representative we denote by α so

$$[\alpha]_{\mathrm{dR}} = c_{\mathbb{R}} \in H^1_{\mathrm{dR}}(R; \mathbb{R}).$$

Decompose α

$$\alpha = \alpha^{1,0} + \alpha^{0,1}, \qquad \alpha^{0,1} = \overline{\alpha^{1,0}},$$

since α is real. Under the Dolbeault isomorphism and Hodge decomposition, we have

$$H^1(R, \mathcal{O}) \;\cong\; H^{0,1}_{\bar\partial}(R)$$

and the image $i(c) \in H^1(R, \mathcal{O})$ is represented by the harmonic $(0,1)$-form $\alpha^{0,1}$.

We know that the Serre pairing can be described as

$$\langle \beta, \omega \rangle_{\mathrm{Serre}} = \frac{1}{2\pi i} \int_R \eta^{0,1} \wedge \omega$$

whenever $\beta \in H^1(R, \mathcal{O})$ is represented by a harmonic $(0,1)$-form $\eta^{0,1}$ and $\omega \in H^0(R, \Omega^1)$ is a holomorphic 1-form.

Applying this to $\beta = i(c)$ and $\eta^{0,1} = \alpha^{0,1}$ gives

$$\langle i(c), \omega \rangle_{\mathrm{Serre}} = \frac{1}{2\pi i} \int_R \alpha^{0,1} \wedge \omega.$$

Since R has complex dimension 1, a $(2,0)$-form vanishes, hence $\alpha^{1,0} \wedge \omega = 0$, and therefore

$$\alpha^{0,1} \wedge \omega = (\alpha^{1,0} + \alpha^{0,1}) \wedge \omega = \alpha \wedge \omega.$$

Thus

$$\langle \iota^* c, \omega \rangle_{\text{Serre}} = \frac{1}{2\pi i} \int_R \alpha \wedge \omega. \tag{13}$$

Integral Poincaré duality gives a perfect pairing

$$H^1(R;\mathbb{Z}) \times H_1(R;\mathbb{Z}) \longrightarrow \mathbb{Z},$$

and we denote by $\gamma_c \in H_1(R;\mathbb{Z})$ the Poincaré dual of c.

The de Rham realization of this pairing is as follows. The class $c_{\mathbb{R}} \in H^1(R;\mathbb{R})$ is represented by the closed 1-form α with integral periods, i.e.

$$\int_\gamma \alpha \in \mathbb{Z} \quad \text{for all } \gamma \in H_1(R;\mathbb{Z}).$$

The Poincaré dual cycle γ_c is then characterized by

$$\int_{\gamma_c} \beta = \int_R \alpha \wedge \beta \quad \text{for all closed 1-forms } \beta,$$

Thus, if we identify

$$H^1(R;\mathbb{Z}) \xrightarrow{\text{PD}} H_1(R;\mathbb{Z}) \quad \text{and} \quad H^1(R;\mathcal{O}) \xrightarrow{\text{Serre}} H^0(R,\Omega^1)^\vee,$$

the class $c \in H^1(R;\mathbb{Z})$ maps to the functional

$$H^0(R,\Omega^1) \longrightarrow \mathbb{C}, \qquad \omega \longmapsto \frac{1}{2\pi i} \int_{\gamma_c} \omega.$$

This is precisely the period map (up to the factor $1/(2\pi i)$)

$$H_1(R;\mathbb{Z}) \otimes H^0(R,\Omega^1) \longrightarrow \mathbb{C}, \qquad (\gamma,\omega) \longmapsto \int_\gamma \omega,$$

with $\gamma = \gamma_c$ the Poincaré dual of c.

Exercise 6.22. Let V be a complex g-dimensional vector space and $L \simeq \mathbb{Z}^{2g} \subset V$ a lattice. Let $A = V/L$.

1. Using harmonic theory, compute the Dolbeault cohomology $H^*(A;\mathcal{O})$.

2. Show that the moduli space of holomorphic line bundles on A with zero Chern class is naturally identified with
$$A^\vee := \bar{V}^\vee / L^\vee.$$

3. Define a line bundle
$$\mathcal{P} \longrightarrow A \times A^\vee$$
from the trivial line bundle over $V \times V^\vee$ with connection
$$\nabla = d + i(x\, d\xi + \xi\, dx),$$
by dividing the $L \times L^\vee$-action as follows: identify the fiber $\mathbb{C}$ over $(x, \xi) \in V \times V^\vee$ with that over $(x + \ell, \xi + \lambda)$ by multiplication by
$$\exp\bigl(2\pi i(\lambda(x) + \xi(\ell))\bigr).$$
Show that $\mathcal{P}$ is holomorphic, that $\mathcal{P}|_{A\times\{a^\vee\}}$ is the line bundle over A classified by $a^\vee \in A^\vee$, and prove the corresponding statement for $\{a\} \times A^\vee$.

Solution 6.23. Write $g = \dim_{\mathbb{C}} V$. Choose a Hermitian inner product on V which is L-invariant. By translation invariance and the locality of Kahler geometry, this induces a flat Kahler metric on A.

1. The Dolbeault Laplacian $\bar{\Box}$ on $(0, q)$-forms is translation invariant. On a flat torus, a $(0, q)$-form is harmonic iff it has constant coefficients. This is because in a global parallel frame, the Laplacian on forms acts coefficientwise as the usual scalar Laplacian and the that any harmonic function on a compact manifold is constant by the maximum principle.

   A translation-invariant $(0, 1)$-form on A is determined by its value at a single point (say 0), and conversely any linear functional on $T_0^{0,1}A$ extends uniquely to a translation-invariant $(0, 1)$-form. Thus
$$\mathcal{H}^{0,q}(A) \cong \Lambda^q(T_0^{0,1}A)^\vee \cong \Lambda^q\overline{V}^\vee,$$
   where $\overline{V}$ is V with the conjugate complex structure. By Hodge theory,
$$H^{0,q}(A) \cong \mathcal{H}^{0,q}(A),$$
   and since $H^q(A; \mathcal{O}) \cong H^{0,q}(A)$ we obtain
$$H^q(A; \mathcal{O}) \;\cong\; \Lambda^q\overline{V}^\vee \cong \Lambda^q V^\vee, \qquad 0 \le q \le g.$$
   More generally, the same reasoning shows
$$H^{p,q}(A) \;\cong\; \Lambda^p V^\vee \otimes \Lambda^q\overline{V}^\vee$$

2. Isomorphism classes of holomorphic line bundles on A are classified by

$$\mathrm{Pic}(A) \cong H^1(A; \mathcal{O}^\times).$$

Consider the holomorphic exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i\cdot)} \mathcal{O}^\times \longrightarrow 1,$$

with long exact cohomology sequence

$$H^1(A; \mathcal{O}) \longrightarrow H^1(A; \mathcal{O}^\times) \xrightarrow{c_1} H^2(A; \mathbb{Z}).$$

The subgroup

$$\mathrm{Pic}^0(A) \ := \ \ker\big(c_1 : \mathrm{Pic}(A) \to H^2(A; \mathbb{Z})\big)$$

is precisely those bundles with zero Chern class. Exactness shows

$$\mathrm{Pic}^0(A) \ \cong \ H^1(A; \mathcal{O}) / \operatorname{im}\big(H^1(A; \mathbb{Z})\big)$$

We have already computed

$$H^1(A; \mathcal{O}) \cong H^{0,1}(A) \cong \overline{V}^\vee$$

On the other hand, $H^1(A; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong L^\vee$, and this sits inside $H^1(A; \mathcal{O})$ as a lattice (one way to see this is via $H^1(A; \mathbb{Z}) \subset H^1(A; \mathbb{R}) \subset H^1(A; \mathbb{C})$ and the Hodge decomposition). Thus

$$\mathrm{Pic}^0(A) \cong H^1(A; \mathcal{O})/H^1(A; \mathbb{Z}) \cong \overline{V}^\vee / L^\vee = A^\vee.$$

So the moduli space of holomorphic line bundles on A with $c_1 = 0$ is canonically identified with $A^\vee$.

3. Consider the trivial line bundle

$$V \times V^\vee \times \mathbb{C} \longrightarrow V \times V^\vee$$

equipped with the connection

$$\nabla = d + i(x\, d\xi + \xi\, dx),$$

where $x \in V$, $\xi \in V^\vee$, and $x\, d\xi + \xi\, dx$ denotes the tautological pairing (so in coordinates it is linear in x and ξ).

The group $L \times L^\vee$ acts on $V \times V^\vee \times \mathbb{C}$ by

$$(\ell, \lambda) \cdot (x, \xi, z) = \big(x + \ell,\ \xi + \lambda,\ e^{2\pi i(\lambda(x) + \xi(\ell))}\, z\big).$$

The multiplier $e^{2\pi i(\lambda(x)+\xi(\ell))}$ is holomorphic in (x, ξ) (it is the exponential of a holomorphic linear function), so this action is by holomorphic bundle automorphisms of the trivial holomorphic line bundle $V \times V^\vee \times \mathbb{C}$.

The quotient

$$\mathcal{P} := (V \times V^\vee \times \mathbb{C})/(L \times L^\vee) \longrightarrow (V/L) \times (V^\vee/L^\vee) = A \times A^\vee$$

is therefore a holomorphic line bundle, the **Poincaré bundle**. The connection ∇ is invariant under the $L \times L^\vee$-action, hence descends to a connection on $\mathcal{P}$.

Now fix $a^\vee \in A^\vee$ and choose a lift $\xi_0 \in V^\vee$. The restriction $\mathcal{P}|_{A\times\{a^\vee\}}$ is obtained as the quotient of $V \times \mathbb{C}$ by the following L-action: an element $\ell \in L$ sends (x, z) to

$$(x + \ell,\ e^{2\pi i\,\xi_0(\ell)} z),$$

because along the slice $\xi = \xi_0$ the factor $\lambda(x)$ vanishes (we take $\lambda = 0$ to stay in the same fiber over $a^\vee$), while $\xi(\ell) = \xi_0(\ell)$ is constant in x. Thus the monodromy of the resulting line bundle around $\ell \in L$ is exactly

$$\chi_{\xi_0}(\ell) = e^{2\pi i\,\xi_0(\ell)}.$$

A holomorphic line bundle on A with zero Chern class is classified precisely by such a character $L \to U(1)$, and changing ξ_0 by an element of $L^\vee$ does not change the character χ_{ξ_0}. Hence the isomorphism class of $\mathcal{P}|_{A\times\{a^\vee\}}$ depends only on the class $a^\vee = [\xi_0] \in A^\vee$ and is exactly the line bundle over A classified by $a^\vee$.

The argument for the restriction to $\{a\} \times A^\vee$ is symmetric. Fix $a \in A$ with lift $x_0 \in V$. Then the effective $L^\vee$-action on $V^\vee \times \mathbb{C}$ along the slice $x = x_0$ is

$$\lambda \cdot (\xi, z) = \big(\xi + \lambda,\ e^{2\pi i\,\lambda(x_0)} z\big),$$

so the monodromy around $\lambda \in L^\vee$ is

$$\chi_{x_0}(\lambda) = e^{2\pi i\,\lambda(x_0)}.$$

This is precisely the character of $L^\vee$ corresponding to the point $a = [x_0] \in A$, and hence $\mathcal{P}|_{\{a\}\times A^\vee}$ is the line bundle on $A^\vee$ classified by a.

Remark 6.24. Let L be a lattice in a complex vector space V, and let $A = V/L$ be the corresponding complex torus. We will verify in detail the fact that a holomorphic line bundle on A with zero Chern class is classified precisely by such a character $L \to U(1)$, and therefore the moduli space of such line bundles is identified with

$$A^\vee = \overline{V}^\vee/L^\vee.$$

A unitary character is a homomorphism $\chi : L \longrightarrow U(1)$. Any χ can be written as $\chi(\ell) = \exp(2\pi i\,\xi(\ell))$ for some real-valued group homomorphism $\xi : L \to \mathbb{R}$. Conversely, any such ξ defines a character this way. So

$$\mathrm{Hom}(L, U(1)) \cong \mathrm{Hom}(L, \mathbb{R})/\,\mathrm{Hom}(L, \mathbb{Z}).$$

But

$$\mathrm{Hom}(L,\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \cong \overline{V}^{\vee}$$

where $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ has a canonical complex structure inherited from that of V by defining $(i\phi)(v) := \phi(iv)$ for $\phi \in \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ and $v \in V$. Finally,

$$\mathrm{Hom}(L,\mathbb{Z}) \cong L^{\vee}.$$

Therefore

$$\mathrm{Hom}(L,U(1)) \;\cong\; \mathrm{Hom}(L,\mathbb{R})/\,\mathrm{Hom}(L,\mathbb{Z}) \;\cong\; \overline{V}^{\vee}/L^{\vee} \;=\; A^{\vee}.$$

Exercise 6.25. Show that isomorphism classes of flat unitary line bundles on a manifold X are classified by $H^1(X;U(1))$, with the constant sheaf $U(1)$ associated to the unit circle group in $\mathbb{C}^{\times}$.

When X is compact Kähler, compare the constant and holomorphic exponential sequences to conclude that the map

$$H^1(X;U(1)) \longrightarrow H^1(X;\mathcal{O}^{\times})$$

induces a bijection from isomorphism classes of flat unitary line bundles to those of holomorphic line bundles with zero Chern class.

Solution 6.26. A flat unitary line bundle on X is a complex line bundle with structure group $U(1)$ and a flat unitary connection. Choosing a good open cover $\{U_i\}$, such a bundle is given by locally constant transition functions $g_{ij}\colon U_{ij} \to U(1)$ satisfying the cocycle condition $g_{ij}g_{jk}g_{ki} = 1$ on U_{ijk}, and two such bundles are isomorphic if their cocycles differ by a coboundary $g'_{ij} = h_i^{-1}g_{ij}h_j$ with $h_i\colon U_i \to U(1)$ locally constant. This is exactly the description of Čech 1–cocycles and coboundaries for the constant sheaf $U(1)$, so isomorphism classes of flat unitary line bundles are classified by

$$H^1(X;U(1)).$$

Now suppose X is compact Kähler. Consider the constant exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\exp(2\pi i\cdot)} U(1) \longrightarrow 1$$

and the holomorphic exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i\cdot)} \mathcal{O}^{\times} \longrightarrow 1.$$

The inclusions $\mathbb{R} \hookrightarrow \mathcal{O}$ and $U(1) \hookrightarrow \mathcal{O}^{\times}$ give a morphism of short exact sequences and so we get a commutative diagram of long exact cohomology sequences:

$$\begin{array}{ccccccc}
H^1(X;\mathbb{Z}) & \to & H^1(X;\mathbb{R}) & \to & H^1(X;U(1)) & \xrightarrow{c_1^{\mathrm{top}}} & H^2(X;\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(X;\mathbb{Z}) & \to & H^1(X;\mathcal{O}) & \to & H^1(X;\mathcal{O}^{\times}) & \xrightarrow{c_1^{\mathrm{hol}}} & H^2(X;\mathbb{Z})
\end{array}$$

Let

$$\mathrm{Pic}^0(X) := \ker\left(c_1^{\mathrm{hol}}\right)$$

be the subgroup of holomorphic line bundles with $c_1 = 0$. There is a commutative diagram

$$\begin{array}{ccc} H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}) & \cong & \ker c_1^{\mathrm{top}} \\ \downarrow & & \downarrow \\ H^1(X;\mathcal{O})/H^1(X;\mathbb{Z}) & = & \mathrm{Pic}^0(X) \end{array}$$

The left vertical map is an isomorphism by the Hodge decomposition theorem. Thus the right vertical map is also an isomorphism, and we conclude that every holomorphic line bundle with zero Chern class corresponds to a flat unitary line bundle with zero Chern class.

6.3 Kodaira Nakano vanishing and the embedding theorem

We introduce the operators

$$\begin{aligned} L = \omega \wedge - : \Omega^{p,q}(X) &\to \Omega^{p+1,q+1}(X) \\ \Lambda = L^* : \Omega^{p,q}(X) &\to \Omega^{p-1,q-1}(X) \\ H = [L,\Lambda] = p+q-n : \Omega^{p,q}(X) &\to \Omega^{p,q}(X) \end{aligned}$$

where ω is the Kähler form on a compact Kähler manifold X of complex dimension n. These operators satisfy the commutation relations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$:

$$\begin{aligned} [L,\partial^*] &= i\bar{\partial} \\ [L,\bar{\partial}^*] &= -i\partial \\ [\Lambda,\partial] &= -i\bar{\partial}^* \\ [\Lambda,\bar{\partial}] &= i\partial^* \end{aligned}$$

The Riemannian Hodge decomposition recovers Poincaré duality on X. The ordinary Poincaré duality pairing

$$H^k(X;\mathbb{C}) \otimes H^{2n-k}(X;\mathbb{C}) \to \mathbb{C}$$

becomes, via the isomorphism $* : \mathcal{H}^k \cong \mathcal{H}^{n-k}$, the L^2-pairing. Note that Poincaré duality is topological whereas the Hodge star operator depends on the metric.

6.4 Dolbeault cohomology for vector bundles

Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold X. The holomorphic structure on E gives rise to the Dolbeault operator

$$\bar{\partial}_E : \Omega^{p,q}(X,E) \to \Omega^{p,q+1}(X,E)$$

Put a Hermitian metric h on E. We also have the Chern connection ∇ on E compatible with the holomorphic and Hermitian structure, given by the formula

$$\nabla = \partial_E + \bar{\partial}$$

The Hodge decomposition says that $\bar\partial$ cohomology is computed by the kernel of the Dolbeault Laplacian

$$\bar\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

acting on $\Omega^{p,q}(X,E)$. In particular, on a compact Hermitian manifold,

$$\Omega^{p,q}(X,E) \;=\; \ker\Box_{\bar\partial} \;\oplus\; \operatorname{im}\bar\partial_E \;\oplus\; \operatorname{im}\bar\partial_E^*$$

In particular, $\ker\bar\partial_E = \ker\bar\Box \oplus \operatorname{im}\bar\partial_E$. Using the decomposition above, every cohomology class has a unique representative in $\ker\bar\Box$, and the map

$$\ker\bar\Box \longrightarrow H^{p,q}_{\bar\partial}(X,E), \qquad \alpha \longmapsto [\alpha]$$

is an isomorphism.

Theorem 6.27 (Kodaira-Nakano)**.** We have the Weitzenböck formula

$$\bar\Box = \Box - i[\Lambda, F\wedge]$$

In particular, $[\Lambda, F\wedge]$ is an order zero differential operator acting on $\Omega^{p,q}(X,E)$.

Remark 6.28. The zeroth order term $-i[\Lambda, F\wedge]$ is self adjoint. For a unitary connection, the curvature F is skew-Hermitian in the bundle indices, and the commutator $[\Lambda, F\wedge]$ is skew-adjoint. Multiplying a skew-adjoint operator T by $-i$ gives a self-adjoint operator: $(-iT)^* = iT^* = i(-T) = -iT$.

Remark 6.29. $\Box$ is semipositive because

$$\begin{aligned}\langle\Box\alpha,\alpha\rangle &= \langle(\partial\partial^* + \partial^*\partial)\alpha,\alpha\rangle\\ &= \langle\partial^*\alpha,\partial^*\alpha\rangle + \langle\partial\alpha,\partial\alpha\rangle\\ &= \|\partial^*\alpha\|^2 + \|\partial\alpha\|^2 \geq 0\end{aligned}$$

Remark 6.30. If $-i[\Lambda, F\wedge]$ is positive definite on $\Omega^{p,q}(X,E)$, then $\bar\Box$ is strictly positive on $\Omega^{p,q}(X,E)$, hence $\ker\bar\Box = 0$ and therefore $H^{p,q}_{\bar\partial}(X,E) = 0 = \ker\bar\Box = 0$.

Proof. By direct computation using the commutation relations above, we have

$$\begin{aligned}\bar\Box &= [\bar\partial, \bar\partial^*]\\ &= [\bar\partial, -i[\Lambda,\partial]]\\ &= -i[[\bar\partial,\Lambda],\partial] - i[\Lambda,[\bar\partial,\partial]]\\ &= -i[i\partial^*,\partial] - i[\Lambda, F\wedge]\\ &= \Box - i[\Lambda, F\wedge]\end{aligned}$$

□

Example 6.31. Let $E = \mathcal{L}$ and pick a Hermitian metric h on the holomorphic line bundle $\mathcal{L} \to X$ with curvature form F. We suppose that $F = -2\pi i\,\omega$ for a Kähler form ω. The line bundles which admit such metrics are by definition the positive line bundles. Then

$$\begin{aligned}[\Lambda, F\wedge] &= -2\pi i[\Lambda, \omega\wedge] \\ &= -2\pi i L^* L \\ &= -2\pi i H \\ &= -2\pi i(p+q-n)\end{aligned}$$

Therefore, if $p = n$ then $\bar{\Box} = \Box + 2\pi(p+q-n) = \Box + 2\pi q$ is strictly positive on $\Omega^{n,q}(X, \mathcal{L})$ for $q > 0$, hence we recover the Kodaira vanishing theorem

$$H^q(X, \mathcal{L} \otimes \Omega_X^n) = 0 \quad \text{for } q > 0$$

There is also the dual statement

$$H^q(X, \mathcal{L}^{-1}) = 0 \quad \text{for } q < n$$

We notice from this example that positivity seems like a very strong condition on a line bundle. We can relax this condition as follows. Let X be a compact Kähler manifold.

Definition 6.32. We say a class $\alpha \in H^{1,1}(X; \mathbb{R})$ is **positive** if it is in the real part of $H^{1,1}(X)$ and can be represented by a Kähler form.

Theorem 6.33. Suppose $\mathcal{L} \to X$ is holomorphic line bundle with positive $c_1(\mathcal{L}) \in H^{1,1}(X; \mathbb{R})$. Then there exists a Hermitian metric on $\mathcal{L}$ with curvature $2\pi/i\omega$, where ω is an integral Kähler form representing $c_1(\mathcal{L})$.

Remark 6.34. The constraints $[\omega] \in \Im H^2(X; \mathbb{Z}) \cap H^{1,1}(X; \mathbb{R})$ and $\omega > 0$ imply such Hermitian metric exists on $\mathcal{L}$. A compact complex X admitting such a form ω is called a **Hodge manifold**.

Theorem 6.35 (Kodaira embedding theorem)**.** Let X be a Hodge manifold. For a choice of $\mathcal{L}$, there exists $N > 0$ and a holomorphic embedding $X \to \mathbb{P}(H^0(X, \mathcal{L}^{\otimes N})^\vee)$.

An $\mathcal{L}$ whose power gives such an embedding is called an **ample line bundle**. If $\mathcal{L}$ gives an embedding itself, it is called **very ample**.

Sketch. We must arrange that there is no base locus, the map is injective, and the map is an immersion. All of these properties are witnessed by the surjectivity of certain evaluation maps. In particular

$$H^0(X, \mathcal{L}^{\otimes N}) \to H^0(X, I\mathcal{L}^{\otimes N})$$

where $I = \mathfrak{m}_x$ for no base locus, $I = \mathfrak{m}_x\mathfrak{m}_y$ for injectivity, and $I = \mathfrak{m}_x^2$ for immersion. It suffices to demonstrate the vanishing of the cohomology group $H^1(X, I\mathcal{L}^{\otimes N})$ for sufficiently large N. The issue is that when $\dim X > 1$, I is not a line bundle, so we cannot use degree arguments invoking Riemann Roch and Serre duality.

It turns out that you can pass to a blowup $\tilde{X} \to X$ at the finite set of points involved and compute there. In this context, we consider $H^1(\tilde{X}, \pi^* I\mathcal{L}^{\otimes N})$ where $I = I(E_x), I(E_x + E_y), I(2E_x)$ for the exceptional divisors E_x, E_y over $x, y \in X$. The exceptional divisor, being codimension 1, puts us back in the line bundle situation and so degree arguments can be used. $\square$

Example 6.36. The hyperplane bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ carries the Fubini–Study Hermitian metric h_{FS} whose curvature form F_{FS} is a positive $(1,1)$-form.

Work on the affine chart

$$U_0 = \{[1 : z_1 : \cdots : z_n]\} \subset \mathbb{P}^n, \qquad z = (z_1, \dots, z_n),$$

and let σ denote the standard holomorphic frame of $\mathcal{O}_{\mathbb{P}^n}(-1)$ given by the point $(1, z_1, \dots, z_n) \in \mathbb{C}^{n+1}$. With respect to this frame, the Hermitian metric is

$$h(\sigma, \sigma) \;=\; 1 + \sum_{i=1}^{n} |z_i|^2,$$

and we computed that the Chern curvature is

$$F_\nabla \;=\; \frac{-\Big(1 + \sum_{k=1}^n |z_k|^2\Big) \sum_{i=1}^n dz_i \wedge d\bar{z}_i \;+\; \sum_{i,j=1}^n z_i \overline{z_j}\, dz_i \wedge d\bar{z}_j}{(1 + \sum_{k=1}^n |z_k|^2)^2}.$$

Evaluate the associated real $(1,1)$-form on a tangent vector $v = (v_1, \dots, v_n) \in T_z^{1,0} U_0 \cong \mathbb{C}^n$:

$$\frac{i}{2\pi} F_\nabla(v, \bar{v}) \;=\; -\frac{1}{2\pi} \cdot \frac{\big(1 + \sum_{k=1}^n |z_k|^2\big)\, |v|^2 \;-\; |\langle z, v\rangle|^2}{(1 + \sum_{k=1}^n |z_k|^2)^2}.$$

By the Cauchy-Schwarz inequality, $|\langle z, v\rangle|^2 \le |z|^2\, |v|^2$, so the numerator satisfies

$$\big(1 + \sum |z_k|^2\big)\, |v|^2 - |\langle z, v\rangle|^2 \;\ge\; |v|^2 \;>\; 0 \qquad \text{for all } v \neq 0.$$

Thus the entire expression is strictly negative. Hence the real $(1,1)$-form $\frac{i}{2\pi} F_\nabla$ is negative definite, and we conclude that $\mathcal{O}_{\mathbb{P}^n}(-1)$ is negatively curved, while its dual $\mathcal{O}_{\mathbb{P}^n}(1)$ is positively curved.

Example 6.37. Positivity is preserved under pullback. If $f : Y \to X$ is a holomorphic map between compact Kähler manifolds and $\mathcal{L} \to X$ is a holomorphic line bundle with a Hermitian metric of positive curvature, then the pullback bundle $f^*\mathcal{L} \to Y$ carries the pullback metric whose curvature form is the pullback of the original curvature form, which is also positive because if one evaluates the pulled-back form on a tangent vector $v \in T_x^{1,0}X$:

$$(f^*\alpha)(v, \bar{v}) \;=\; \alpha(df_x v, \overline{df_x v})$$

Since α is positive, the right-hand side satisfies $\alpha(df_x v, \overline{df_x v}) \geq 0$.

Therefore, from Example 6.36, any ample line bundle L over a compact Kähler manifold X admits a Hermitian metric h whose curvature form F is a positive $(1,1)$-form by using Kodaira's embedding theorem to embed X into projective space and pulling back the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^N}(1)$ to L.

Exercise 6.38.

1. Given a holomorphic line bundle $\mathcal{L}$ on a complex manifold and a smooth real closed 2-form ω in the cohomology class of $c_1(\mathcal{L})$, prove that there exists a Hermitian metric on $\mathcal{L}$ whose holomorphic connection has curvature $-2\pi i\,\omega$.

2. Conclude (from Kodaira vanishing) that the holomorphic line bundles on a compact Riemann surface R which carry metrics of positive curvature are precisely those of positive degree.

3. Show also that for every holomorphic vector bundle V on R, there exists a d so that the twisted bundle $V(D)$ has no H^1 for any $D > d$.

Solution 6.39.

1. Let $L \to X$ be a holomorphic line bundle on a complex manifold and let ω be a smooth real closed 2–form representing $c_1(L) \in H^2(X;\mathbb{R})$. Choose any Hermitian metric h_0 on L and let F_0 denote the curvature of its Chern connection. Then

   $$\frac{i}{2\pi} F_0 \in \Omega^{1,1}(X, \mathbb{R})$$

   also represents $c_1(L)$, so the difference

   $$\omega - \frac{i}{2\pi} F_0$$

   is an exact real $(1,1)$–form. By the $\partial\bar{\partial}$–lemma (which holds on every complex curve and more generally on Kähler manifolds), there exists a real-valued smooth function φ such that

   $$\omega - \frac{i}{2\pi} F_0 = \frac{i}{2\pi}\,\partial\bar{\partial}\varphi.$$

   Define a new Hermitian metric h by $h = e^{-\varphi} h_0$. In a local holomorphic frame l, if $h(l,l) = e^{-\phi}$ the curvature is $F_h = \partial\bar{\partial}\phi$. Thus

   $$F_h = F_0 + \partial\bar{\partial}\varphi = -2\pi i\,\omega.$$

Hence h is a Hermitian metric whose Chern connection has curvature $-2\pi i\,\omega$.

2. For a Hermitian metric h on a holomorphic line bundle $L \to R$, the degree is

$$\deg(L) = \int_R c_1(L) = \int_R \frac{i}{2\pi} F_h.$$

If h has positive curvature, then the form $\frac{i}{2\pi} F_h$ is positive on R, hence its integral is strictly positive and $\deg(L) > 0$.

The hyperplane bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ carries the Fubini-Study Hermitian metric h_{FS} whose curvature form F_{FS} is a positive $(1,1)$-form. Pulling back gives a metric $h_m = \Phi_m^* h_{\mathrm{FS}}$ on $L^{\otimes m}$ with positive curvature

$$F_{h_m} = \Phi_m^* F_{\mathrm{FS}} > 0.$$

Now define a Hermitian metric h on L by taking an mth root locally: in a local holomorphic frame e of L, write

$$h_m(e^{\otimes m}, e^{\otimes m}) = e^{-\phi_m}$$

and set

$$h(e, e) := e^{-\phi_m/m}.$$

Then the curvature satisfies

$$F_h = \frac{1}{m} F_{h_m},$$

which is still a positive $(1,1)$-form. Thus L admits a Hermitian metric of positive curvature. Therefore the holomorphic line bundles on R which carry metrics of positivecurvature are precisely those of positive degree.

3. This follows immediately from Serre duality and the Riemann–Roch theorem. For a holomorphic vector bundle V on R and a divisor D, Serre duality gives

$$H^1(R, V(D)) \cong H^0(R, K_R \otimes V^\vee \otimes \mathcal{O}(-D))^\vee.$$

By Riemann–Roch, for $\deg(D)$ sufficiently large, the degree of the bundle $K_R \otimes V^\vee \otimes \mathcal{O}(-D)$ becomes negative, and hence

$$H^0(R, K_R \otimes V^\vee \otimes \mathcal{O}(-D)) = 0.$$

Thus, for such D, we have $H^1(R, V(D)) = 0$.

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