Perverse sheaves and the decomposition theorem

Songyu Ye

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Abstract

These are notes on the decomposition theorem for perverse sheaves.

Contents

1 Classical

We want to understand the decomposition theorem for perverse sheaves. The classical precursors to this theorem include the theorems of Lefschetz, Hodge, Deligne, and the invariant cycle theorems. We discuss them here below. Let X be a smooth projective variety over $\mathbb{C}$ and let $D = X \cap H$ be the intersection of X with a generic hyperplane.

1.1 Gysin maps

The discussion of the Lefschetz hyperplane theorem is based on Totaro's PCMI notes ([**?**] should do just as well) and [**?**]. In general, suppose we have $i : X \to Y$ the inclusion of a closed submanifold into a smooth manifold and we have an orientation of the normal bundle $N_{X/Y}$.

Then we have the Gysin map $i_* : H^*(X) \to H^{*+d}(Y)$ where d is the codimension of X in Y. The Gysin map is formally defined as follows. The map of pairs $(Y, \emptyset) \to (Y, Y \backslash X) \cong \mathrm{Th}_X N_{X/Y}$ induces a map

$$i_* : H^*(X) \to H^{*+d}(\mathrm{Th}_X N_{X/Y}) \to H^{*+d}(Y)$$

where the first map is the Thom isomorphism. Recall that the Thom space of a vector bundle E is disk bundle $D(E)$ of E with the boundary sphere bundle $S(E)$ collapsed to a point. The Thom isomorphism is the map $H^*(E) \to H^{*+d}(\mathrm{Th}\, E)$ given by cupping with the Thom class. The Thom class $u \in H^d(E, E \backslash 0)$ makes $H^*(E, E \backslash 0)$ into a free $H^*(E)$-module.

There is also the push-pull formula which says that the map

$$i^* i_* : H^*(X) \to H^{*+d}(X)$$

is equal to

$$i^* i_* = c_d(N_{X/Y}) \cup = [X]|_X \cup$$

In particular, an ample line bundle L on X gives a divisor D in X and the Gysin map for the divisor $i : D \to X$ is given by $i_* = c_1(L) \cup = [D] \cup$. The pull-push formula for $i_* i^* : H^*(Y) \to H^{*+d}(Y)$ is similarly given by $i_* i^* = c_d(N_{X/Y}) \cup = [D] \cup$.

Remark 1.1. *The intuition behind this map is, given a class $\alpha \in H^i(X)$ represented by a submanifold Z, we are embedding Z in the normal bundle $N_{X/Y}$ via the zero section, and capping with Chern classes is about looking at the homology class of the intersection of the embedded Z with general sections of $N_{X/Y}$.*

The Gysin maps satisfy very special properties when we have smooth projective varieties.

Theorem 1.2 (Lefschetz hyperplane theorem). *The restriction map $H^i(X) \to H^i(D)$ is an isomorphism for $i < n - 1$ and injective for $i = n - 1$.*

Theorem 1.3 (Hard Lefschetz). *The i-fold product $\cup c_1(L)^i : H^{j-i}(X) \to H^{j+i}(X)$ is an isomorphism for $j < n - 2i$ and injective for $j = n - 2i$.*

Remark 1.4. *Hyperplane line bunldes are positive are ample and satisfy Hard Lefschetz.*

1.2 Hodge theory

The cohomology of a smooth projective variety X is very special compared to that of general manifolds. In particular $H^*(X)$ has a Hodge structure which we will state some of the consequences of here.

We consider complex valued i-forms on X with p holomorphic coordinates and q anti-holomorphic coordinates. There are differential operators

$$\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$$
$$\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$$

which satisfy $\partial^2 = 0$ and $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. There is the Hodge decomposition

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X)$$

where

$$H^{p,q}(X) = \ker \bar{\partial} / \operatorname{im} \bar{\partial}$$

is the Dolbeault cohomology. There is also the Dolbeault isomorphism

$$H^{p,q}(X) \cong H^q(X, \Omega^p)$$

where Ω^p is the sheaf of holomorphic p-forms on X.

1.3 Proof of the Lefschetz hyperplane theorem

We follow the proof of the Lefschetz hyperplane theorem in [**?**]. I have to rename some things. Let M be a smooth projective variety of dimension n and let V be a hyperplane section in M, i.e. a positive line bundle. There are short exact sequences of sheaves

$$0 \to \Omega_M^p(-V) \to \Omega_M^p \xrightarrow{r} \Omega_M^p|_V \to 0$$
$$0 \to \Omega_V^{p-1}(-V) \to \Omega_M^p|_V \xrightarrow{i} \Omega_V^p \to 0$$

It is enough to show that

$$H^q(\Omega_M^p) \cong H^q(\Omega_V^p)$$

when $p + q < n - 1$ and that the map

$$H^q(M, \Omega_M^p) \xrightarrow{r^*} H^q(M, \Omega_M^p|_V) \to H^q(V, \Omega_M^p|_V) \xrightarrow{i^*} H^q(V, \Omega_V^p)$$

is injective when $p + q = n - 1$.

Proof. Look at the long exact sequence in cohomology associated to the short exact sequence of sheaves. The desired properties are equivalent to the vanishing of the cohomology groups $H^q(\Omega_M^p(-V))$ and $H^q(\Omega_V^{p-1}(-V))$ for $p + q \leq n$. This follows from the Kodaira vanishing theorem. $\square$

Theorem 1.5 (Kodaira vanishing theorem)**.** *Let L be a positive line bundle on a smooth projective variety M. Then $H^q(M, \Omega_M^p(L)) = 0$ for $p + q \geq n$. The dual statement is*

$$H^q(M, \Omega_M^p(-L)) = 0$$

for $p + q \leq n$.

Remark 1.6. *The definition of positive line bundle and the Kodaira vanishing theorem will be black-boxed.*

2 Sheaves and derived categories

This section is also based on [**?**].

2.1 Derived category

On a space X, we consider the category where the objects are complexes of sheaves on X and the morphisms are morphisms of complexes. There are quasi-isomorphisms of complexes which are those which induce isomorphisms on cohomology sheaves.

The derived category $D(X)$ is obtained by inverting quasi-isomorphisms (this is not precise!). Different complexes of sheaves which give the same cohomology theories are identified in $D(X)$. Given a complex of sheaves K, one can produce an injective resolution $K \to I$ where I is a complex of sheaves with injective components and $K \to I$ is a quasi-isomorphism.

Definition 2.1. *The **cohomology groups** $H^i(K)$ of K are defined as the cohomology of the complex $\Gamma(X, I)$. Beware that these are different than the cohomology sheaves, which are the "cohomology objects" in $D(X)$.*

More generally, the derived category is important because it admits derived functors. A map $f : X \to Y$ of spaces induces a map $f_* : D(X) \to D(Y)$ of derived categories. Given a bounded below complex of sheaves K on X, choose an injective resolution $K \to I$. The pushforward complex f_*I is a complex of sheaves on Y so that

$$H^i(U, f_*I) \cong H^i(f^{-1}(U), I)$$

and is well defined up to canonical isomorphism in $D^+(Y)$, denoted Rf_*K.

Remark 2.2. *When $f : X \to *$ is a map to a point, the derived pushforward Rf_*K sends K to the cohomology of K.*

2.2 Constructible sheaves

Definition 2.3. *A subset $V \subset Z$ of a complex variety is **constructible** if it is a finite sequence of unions, intersections, or complements of algebraic subvarieties of Z.*

*A **local system** on Z is a locally constant sheaf with finite-dimensional stalks.*

*A complex of sheaves K on Z is **constructible** if Z has a decomposition into constructible subsets $Z = \bigsqcup Z_i$ such that each of the cohomology sheaves $\mathcal{H}^i(K|_{Z_i})$ is a local system.*

*The **constructible bounded derived category** $\mathcal{D}_Z$ is defined to be the full subcategory of the bounded derived category $D_b(Z)$ whose objects are the constructible complexes.*

*A **perverse sheaf** is a constructible complex with certain restrictions on the dimension of the support of its stalk cohomology and of its stalk cohomology with compact supports. These restrictions are called the support and co-support conditions, respectively.*

Remark 2.4. *These definitions are right because we get a category in which we have very many nice properties, in particular Verdier duality, six-functor formalism, etc.*

In the derived category, we can embed sheaves as complexes concentrated in degree zero, and the image of this embedding is characterized by the property that $\mathcal{H}^i(K) = 0$ for $i \neq 0$. Every constructible complex K comes with a canonical collection of perverse sheaves called the **perverse cohomology sheaves** ${}^{\mathfrak{p}}\mathcal{H}^i(K)$ of K which are characterized among the constructible complexes by ${}^{\mathfrak{p}}\mathcal{H}^i(K) = 0$ for every $i \neq 0$.

2.3 Intersection complex

Perverse sheaves are interesting objects besides their role in developing intersection cohomology. The intersection complex is a fundamental example of a perverse sheaf, in the sense that every perverse sheaf is a finite iterated extension of intersection complexes.

Definition 2.5. *Given a complex algebraic variety Y and a local system L defined on a smooth open subset $U \subset Y$, the **intersection complex** $\mathrm{IC}_Y(L)$ is a constructible complex of sheaves, unique up to canonical isomorphism in D_Y so that $\mathrm{IC}(L)|_U = L$ and the support and cosupport conditions hold:*

$$\dim\{y \in Y \mid \mathcal{H}^i_y(\mathrm{IC}(L)) \neq 0\} < -i \text{ if } i > -n$$
$$\dim\{y \in Y \mid \mathcal{H}^i_{c,y}(\mathrm{IC}(L)) \neq 0\} < i \text{ if } i < n$$

where the c denotes compactly-supported cohomology stalks. The intersection cohomology of Y with coefficients in L is defined to be the cohomology of the complex $\mathrm{IC}_Y(L)$ up to a shift:

$$IH^{i+n}(Y, L) := \mathcal{H}^i(\mathrm{IC}_Y(L))$$

2.4 Intersection complex of the link of an affine cone

Recall that a complex projective manifold has hard Lefschetz.

Definition 2.6. *The **primitive cohomology** is*

$$P^{n-k}(X) = \ker \cup c_1^{k+1} : H^{n-k}(X) \to H^{n+k+2}(X)$$

It is precisely which classes die after you cup too one many times with the hyperplane class. There is the Lefschetz decomposition

$$H^m X = \sum_k L^k P^{n-2k}$$

Hard Lefschetz says that there is a map $\mathfrak{sl}_2 \to \operatorname{End} H^*(X)$ where L acts as a lowering operator for the $\mathfrak{sl}_2$ action, H acts on $H^{n-i}(X)$ by weight i, and there is a raising operator given by the restriction to the harmonic forms of the the formal adjoint of $\omega \wedge \cdot$ In this setup, the primitive vectors are precisely the highest weight vectors.

Another way to think about it is as follows: the primitive cohomology in dimension $n - k$ is the classes corresponding to submanifolds of dimension $n+k$ which have empty intersection with $k+1$ generic hyperplanes. These are precisely classes which do not meet some $n - k - 1$-dimensional linear subspace of $\mathbb{CP}^n$. When $k = 0$, this is precisely those n-dimensional submanifolds which live in the $\mathbb{C}^n$ part of $\mathbb{CP}^n$.

See this math overflow post.

Given a smooth projective manifold $E^{n-1} \subset \mathbb{CP}^N$ take its affine cone $Y^n \subset \mathbb{C}^{N+1}$ and consider $\mathcal{L}$ the link of the cone, defined as the intersection of Y with a small sphere centered at the cone point. The link $\mathcal{L}$ is a smooth oriented compact manifold of real dimension $2n - 1$ and S^1-fibers over E. Its cohomology is equal to

$$\begin{aligned} H^j(\mathcal{L}) &= P^j(E) \quad \text{for } j \leq n - 1 \\ H^{n-1+j}(\mathcal{L}) &= P^{n-j}(E) \quad \text{for } 0 \leq j \leq n \end{aligned}$$

The intersection cohomology groups equal

$$\begin{aligned} IH^j(\mathcal{L}) &= P^j(E) \quad \text{for } j \leq n - 1 \\ IH^j(\mathcal{L}) &= 0 \quad \text{for } j > n - 1 \end{aligned}$$

With compact supports:

$$\begin{aligned} IH_c^{2n-j}(\mathcal{L}) &= H_j(\mathcal{L}) \quad \text{for } 0 \leq j \leq n - 1 \\ IH_c^{n-j}(\mathcal{L}) &= 0 \quad \text{for } j > n - 1 \end{aligned}$$

Poincare duality is the isomorphism

$$IH^j(\mathcal{L}) \cong IH_c^{2n-j}(\mathcal{L})$$

Remark 2.7. *A major motivation behind the introduction of intersection cohomology is the failure of Poincare duality for singular spaces. Note that in this example the link is smooth, so we have Poincare duality.*

3   Decomposition Theorem

The decomposition theorem for perverse sheaves is a very important result and computational tool. First we state it and then we do some examples.

Theorem 3.1 (Decomposition Theorem). *Let $f : X \to Y$ be a proper map of complex algebraic varieties. Then there is an isomorphism in the constructible bounded derived category D_Y*

$$Rf_* \operatorname{IC}_X \cong \bigoplus_{i \in \mathbb{Z}}^{\mathfrak{p}} \mathcal{H}^i(Rf_* \operatorname{IC}_X)[i]$$

where the sum is finite and the summands are the perverse cohomology sheaves. The perverse sheaves are semisimple:

$$^{\mathfrak{p}}\mathcal{H}^i(Rf_* \operatorname{IC}_X) = \bigoplus_{\beta} \operatorname{IC}_{\bar{S}_\beta}(L_\beta)$$

where L_β are local systems on the smooth open sets $S_\beta \subset Y$. In other words, there is a "essentially unique" decomposition into triples (Y_a, L_a, d_a) so that

$$Rf_* \operatorname{IC}_X \cong \bigoplus_{a} \operatorname{IC}_{\bar{Y}_a}(L_a)[\dim X - \dim Y_a - d_a]$$

where

1. *Y_a are locally closed, smooth, irreducible subvarieties*

2. *L_a are local systems on Y_a*

3. *d_a are integers*

Example 3.2. *Let Y be the projective cone over a Riemann surface Σ of genus g. The Betti numbers of Y are $1, 0, 1, 2g, 1$ and the IH Betti numbers are $1, 2g, 1, 2g, 1$.*

Blowing up Y at the vertex $X \to Y$ and we can apply the decomposition theorem to the proper map $X \to Y$. The decomposition theorem says that

$$0 \to \operatorname{IC}_Y \to f_* \mathbb{Q}_X[2] \to H^2(\Sigma)[0] \to 0$$

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4 First look at perverse sheaves

Let $K \in \mathcal{D}_Y$ be a constructible complex on the variety Y. Recall that this means K has cohomology sheaves which are all constructible. Recall that the **support** of a sheaf is the closure of the set of points with nontrivial stalks.

Definition 4.1. *We say K satisfies the **support condition** if the support of the cohomology sheaves*

$$\dim \operatorname{Supp} \mathcal{H}^{-i}(K) \leq i \text{ for all } i \in \mathbb{Z}$$

*has the right dimension. We say K satisfies the **cosupport condition** if the support of the compactly supported cohomology sheaves*

$$\dim \operatorname{Supp} \mathcal{H}_c^{i}(K) \leq i \text{ for all } i \in \mathbb{Z}$$

has the right dimension. This is equivalent to the support condition for the Verdier dual complex $K^{\vee}$. Quite reminiscent of Poincare duality.

*We say K is **perverse** if it satisfies both the support and cosupport conditions. The cateogry of perverse sheaves is denoted $\mathcal{P}_Y$ and is a full subcategory of the constructible bounded derived category $\mathcal{D}_Y$.*

Remark 4.2. *What do perverse sheaves in the context of spaces which are not algebraic varieties look like? This is largely an aside toward what perverse sheaves and knot contact homology.*

Example 4.3. *Let C be a Riemann surface of genus g and consider the map $\mathbb{P}^1 \times C \to X$ which collapses $0 \times C$ to a point. Then this map is semismall but nonalgebraic, in particular the decomposition theorem does not hold, but the pushforward $f_* \mathbb{Q}_{\mathbb{P}^1 \times C}$ is a perverse sheaf.*

4.1 Perverse sheaves and decomposition theorem for toric varieties

A polytope P is simplicial if all its faces are simplices. Given a simplicial polytope, the toric variety X_P who has moment polytope equal to the dual of P is smooth. Under this correpsondence, subdivisions of P (defined as a collection K of polytopes whose union is the boundary of P and the intersection of any two elements in K is in K) correspond to subdivisions of the fan of X_P correspond to corner chops of the dual of P. In particular, any subdivision $\tilde{P}$ of P gives rise to a proper equivariant birational map $X_{\tilde{P}} \to X_P$. One can then apply the decomposition theorem to this map.

Given a fan one constructs a convex polytope P by intersecting the fan with a ball centered at the origin and then flattening the faces. If the moment polytope is simple, then the fan polytope is simplicial.

Definition 4.4. *Given a simplicial polytope P, there is a face vector $(f_0, f_1, \cdots, f_d)$ where f_i is the number of i-dimensional faces of P. Consider the **h-polynomial***

$$h_P(t) = (t-1)^d + f_0(t-1)^{d-1} + f_1(t-1)^{d-2} + \cdots + f_{d-1}(t-1) + f_d$$
$$- \sum h_i(P)t^i$$

*and the **g-polynomial***

$$g_P(t) = h_0 + (h_1 - h_0)t + (h_2 - h_1)t^2 + \cdots + (h_d - h_{d-1})t^d$$
$$= \sum g_i(P)t^i$$

where $h_i(P)$ and $g_i(P)$ are the coefficients of the h-polynomial and g-polynomial.

Proposition 4.5. *The coefficient h_i computes the Betti number b_{2i} of the toric variety X_P.*

Proposition 4.6. *The coefficient g_l equal to the dimension of the primitive cohomology in degree $2l$.*

The proof of these statements and their generalizations to intersection cohomology can be found in [?]. Poincare duality and Hard Lefschetz therefore become statements about the face vector of a simplicial polytope P.

When the polytope is not simplicial, the toric variety is singular and therefore we expect interesting statements about the intersection cohomology of the toric variety in terms of the combinatorics of the polytope. Indeed the following is true.

Definition 4.7. *If P is a polytope of dimension d and h, g have been defined for all polytopes of dimension $< d$, we define*

$$h_P(t) = \sum_{F < P} g(F, t)(t-1)^{d-1-\dim F}$$

where F is a proper face, including the empty face $\emptyset$ for which $g(\emptyset, t) = h(\emptyset, t) = 1$ and $\dim \emptyset = -1$. g is defined from h as before.

Proposition 4.8. *The coefficients of these polynomials compute the intersection cohomology and the primitive intersection cohomology of the toric variety X_P.*

Remark 4.9. *There are a bunch of interesting calculations in the text that I don't have the details for. In particular they claim that if you take the cube and subdivide it along barycenters of all the faces, let f be the corresponding map, then we have*

$$^{\mathfrak{p}}\mathcal{H}^{\pm 1}(f_* \mathbb{Q}_{X_{cube}})[3] = \sum \mathbb{Q}_{p_i}$$

4.2 Semismall maps

In the case of semismall maps the decomposition theorem takes a particularly simple form. An important example of a semismall map is the resolution of the nilpotent cone (Springer resolution stuff)

Definition 4.10. *A **stratification** for $f : X \to Y$ is a decomposition of Y into locally closed smooth subsets so that $f^{-1}(Y_i) \to Y$ is topologically locally trivial fibration.*

The relationship to perverse sheaves is immediate.

Proposition 4.11. *If X smooth connected variety of dimension n and $f : X \to Y$ proper surjective map of varieties. Say Y has a stratification $Y = \bigsqcup Y_i$ and let $y_i \in Y_i$, $d_i = \dim f^{-1}(Y_i) - \dim Y_i$. Then the following are equivalent*

- $f_* \mathbb{Q}_X$ *is a perverse sheaf*

- $\dim X \times_Y X \leq n$

- $\dim Y_i + 2d_i \leq n$ *for all i*

*A map which satisfies these conditions is called **semismall**. A stratum is **relevant** if equality holds in the third condiiton.*

In other words, if the map has been stratified, then for each stratum $S \subset Y$ the dimension of the fiber over S is at most $\frac{1}{2} \operatorname{codim} S$, and it is small if the inequality is strict.

Proposition 4.12. *If $\pi : X \to Y$ is small, then the intersection cohomology of Y is canonically isomorphic to the cohomology of X.*

5 Perverse sheaves in representation theory

The language of perverse sheaves and D-modules are critical in modern methods of geometric representation theory. For example, applications of perverse sheaf methods played an essential role in the proof of the Kazhdan-Lusztig conjecture by Beilinson-Bernstein and Brylinski-Kashiwara , Lusztig's construction of canonical bases in quantum groups, and the work of Beilinson-Drinfeld on the Geometric Langlands conjecture. Historically, these tools have also proved very effective in the study of a central object in geometric representation theory called the affine Grassmannian.

5.1 Springer correspondence

5.1.1 Springer resolution

The Springer correspondence is a way of realizing the irreducible representations of the Weyl group in a geometric way. In particular Springer realizes the group algebra

$$\mathbb{Q}[W] \cong H^{BM}_{2 \dim \tilde{N}} \tilde{N} \times_N \tilde{N}$$

where the right hand side has a canonical basis given by the irreducible components.

Let G/B be the flag variety for a connected reductive algebraic group. The Lie algebras are $\mathfrak{g}$ and $\mathfrak{b}$. If $x \in G$ and $xBx^{-1} = B$ then $x \in B$ and so we can identify G/B with the set of subgroups of G which are conjugate to B, or equivalently the set of all subalgebras of $\mathfrak{g}$ that are conjugate to $\mathfrak{b}$, that is **the variety of all Borel subalgebras** of $\mathfrak{g}$.

Let $\mathcal{N} \subset \mathfrak{g}$ be the cone of nilpotent elements and let

$$\tilde{\mathcal{N}} = \{(x, \mathfrak{b}) \in \mathcal{N} \times G/B \mid x \in \mathfrak{b}\}$$

Lemma 5.1. *The projection $\tilde{\mathcal{N}} \to G/B$ gives an isomorphism of bundles $\tilde{\mathcal{N}} \cong T^*G/B$*

Proof. The tangent space to G/B at the identity is $\mathfrak{g}/\mathfrak{b}$ so its dual space is

$$T^*_I(G/B) = \{\phi : \mathfrak{g} \to \mathbb{C} \mid \phi(\mathfrak{b}) = 0\}$$

The Killing form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $x, y \mapsto \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$ is nondegenerate and we pair $\mathfrak{g}$ with $\mathfrak{g}^*$ to get

$$T^*_I(G/B) = \{x \in \mathfrak{g} \mid \langle x, \mathfrak{b}\rangle = 0\} = \mathfrak{n}$$

is the nilradical of $\mathfrak{b}$. So for each Borel subgroup $A \subset G$, the cotangent space $T^*_A(G/B)$ is the nilradical of $\mathrm{Lie}(A)$, is exactly the fiber of $\tilde{\mathcal{N}}$ over A. $\square$

The **Springer resolution** is the map $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ given by projection. Therefore $\tilde{\mathcal{N}}$ carries a natural holomorphic symplectic form (i.e. a 2,0 form) and the Springer resolution map is semismall. One constructs an action of W on $\pi_*\mathbb{Q}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}]$ and then extends it to an algebra homomorphism $\mathbb{Q}[W] \to \mathrm{End}_{\mathcal{D}_\mathcal{N}}(\pi_*\mathbb{Q}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}])$ which is isomorphic to the desired BM homology group. To construct the initial action, one observes that there is a special W-fibration and then one pushes this action of W on the fiber around.

There is the Chevalley map $q : \mathfrak{g} \to \mathfrak{t}/W$ which sends a matrix to the roots of its characterstic polynomial. Consider $\mathfrak{t}_{rs}$ the regular semisimple elements of $\mathfrak{t}$, obtained by ripping out the root hyperplanes, and consider a dominant chamber $\mathfrak{t}_{rs}/W$. Consider $\mathfrak{g}_{rs} = q^{-1}(\mathfrak{t}_{rs}/W)$ and

$$\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times G/B \mid x \in \mathfrak{b}\}$$
$$\tilde{\mathfrak{g}_{rs}} = \pi^{-1}(\mathfrak{g}_{rs})$$

Then the map $\tilde{\mathfrak{g}_{rs}} \to \mathfrak{g}_{rs}$ is a W-fibration and the map $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is small. Associated to the W-covering is a local system

$$L = \pi'_*\mathbb{Q}_{\tilde{\mathfrak{g}}_{rs}}$$

where π' is the restriction of π to $\tilde{\mathfrak{g}}_{rs}$. Then they extend to an action of W on intersection cohomology and pushforward perverse sheaves along the small map.

Remark 5.2. *The image of the root hyperplanes is the discriminant variety of all polynomials with multiple roots. The complement of the union of root hyperplanes is the **configuration space of** n **ordered points** in $\mathbb{C}$ with $\pi_1 = $ the **colored braid group** The complement of the image is the configuration space on n unordered points and has $\pi_1 = $ **braid group**.*

5.1.2 Algebra of correspondences

There is a general construction described in [**?**] which he advertises as a method of geometrically constructing representations of finite dimensional algebras. The idea is to introduce the convolution product on Borel Moore homology, which is supposed to generalize the convolution of functions.

Definition 5.3. *Let M_1, M_2, M_3 connected oriented smooth manifolds and let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ be closed subsets. The **composition** of Z_{12} and Z_{23} is the set*

$$Z_{12} \circ Z_{23} = \{(x_1, x_3) \in M_1 \times M_3 \mid \exists x_2 \in M_2 \text{ s.t. } (x_1, x_2) \in Z_{12} \text{ and } (x_2, x_3) \in Z_{23}\}$$

*The **convolution in Borel-Moore homology** generalizes this to cycles and is defined as follows:*

$$H_i^{BM}(Z_{12}) \times H_j^{BM}(Z_{23}) \to H_{i+j}^{BM}(Z_{12} \circ Z_{23}) c_1, c_2 \mapsto c_1 * c_2$$

where

$$c_1 * c_2 := \pi_{13*}(\pi_{12}^* c_1 \cap \pi_{23}^* c_2)$$

The convolution product is associative. Now let $\mu : M \to N$ a proper map of complex varieties and consider $M_1 = M_2 = M_3 = M$ and $Z = Z_{12} = Z_{23} = M \times_N M$. Then we get convolution maps

$$H_*^{BM}(Z) \times H_*^{BM}(Z) \to H_*^{BM}(Z)$$

The convolution product is not graded, but it does preserve the middle dimension. If $\dim_C M = n$ then the convolution product

$$H_{2n}^{BM}(Z) \times H_{2n}^{BM}(Z) \to H_{2n}^{BM}(Z)$$

and we call this **middle dimensional subalgebra** $H(Z)$.

5.1.3 Sheaf theory applied to the convolution algebra

The convolution product makes $H_*^{BM}(Z)$ into an algebra. It turns out that we can express the algebra structure as the Ext algebra of a particular generator. Let $\mathcal{C}_M$ be the constant perverse sheaf on M i.e. $\mathcal{C}_M = \mathbb{C}_M[\dim M]$ extended along irreducible components.

Proposition 5.4. *There is a (not necessarily grading preserving) natural algebra isomorphism*

$$H_*^{BM}(Z) \to \mathrm{Ext}_{D^b N}^*(\mu_* \mathcal{C}_M, \mu_* \mathcal{C}_M)$$

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Assume that $\mu : M \to N$ is productive and that N is stratified so that the restriction maps are all locally trivial topological fibrations. We can study the convolution algebra by analyzing the pushforward $\mu_* \mathcal{C}_M$ of the constant sheaf on M. Applying the decomposition theorem we find that

$$
\begin{aligned}
H_*^{BM}(Z) &\cong \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}^k_{D^b N}(\mu_* \mathcal{C}_M, \mu_* \mathcal{C}_M) \\
&= \bigoplus_{i,j,k \in \mathbb{Z}, \phi, \psi} \mathrm{Hom}_{\mathbb{C}}(L_\phi(i), L_\psi(j)) \otimes \mathrm{Ext}^k_{D^b N}(IC_\phi[i], IC_\psi[j]) \\
&= \bigoplus_{i,j,k \in \mathbb{Z}, \phi, \psi} \mathrm{Hom}_{\mathbb{C}}(L_\phi(i), L_\psi(j)) \otimes \mathrm{Ext}^k_{D^b N}(IC_\phi, IC_\psi) \ \text{reindexing}
\end{aligned}
$$

where L_ψ is the multiplicity space of the decomposition of $\mu_* \mathcal{C}_M$ into irreducible IC sheaves. Now it is a fact that $\mathrm{Ext}^k_{D^b N}(IC_\phi, IC_\psi)$ vanishes if $k < 0$ Also $\mathrm{Hom}(IC_\phi, IC_\psi)$ is nonzero only if $\phi = \psi$. Therefore we find that

$$
H_*^{BM}(Z) \cong \bigoplus_\phi \mathrm{End}_C(L_\phi) \oplus \Big(\bigoplus_{k>0, \phi, \psi} \mathrm{Hom}_{\mathbb{C}} L_\phi, L_\psi \otimes \mathrm{Ext}^k_{D^b N}(IC_\phi, IC_\psi) \Big)
$$

The first summand is semisimple and the second $H_*^{BM}(Z)^+$ is nilpotent because it is concentrated in degrees $k > 0$. Moreover, this nilpotent ideal is the radical of our algebra because

$$
H_*^{BM}(Z)/H_*^{BM}(Z)^+ = \bigoplus_\phi \mathrm{End}_C(L_\phi)
$$

is semisimple. The composition

$$
H_*^{BM}(Z) \to H_*^{BM}(Z)/H_*^{BM}(Z)^+ \cong \bigoplus_\phi \mathrm{End}_C(L_\phi) \twoheadrightarrow \mathrm{End}_C(L_\phi)
$$

yields an irreducible representation of the algebra $H_*^{BM}(Z)$ on the vector space L_ψ.

Theorem 5.5. *The nonzero members of the collection $\{L_\phi\}$ are the irreducible representations of the algebra $H_*^{BM}(Z)$.*

5.1.4 Semi-small maps

When μ is semismall then the previous calculation becomes nicer since the shifts go away. In particular, we have the following:

Theorem 5.6. *1. Let $\mathcal{C}_M$ be the constant perverse sheaf on M. If μ is semismall then $\mu_* \mathcal{C}_M$ is a perverse sheaf and we have a decomposition without shifts*

$$
\mu_* \mathcal{C}_M = \bigoplus_{N_\phi, \chi_\phi} L_\phi \otimes IC_\phi
$$

where $\phi = (N_\phi, \chi_\phi)$ is a pair of a stratum a local system on the stratum. Furthermore, $H(Z)$ is a subalgebra of $H_*^{BM}(Z)$ and

$$H(Z) \cong \bigoplus_\phi \operatorname{End}_C(L_\phi)$$

2. *Let $H(M_x)$ denote the top Borel-Moore homology of the fiber $\mu^{-1}(x)$. For any stratum N_α, $x \mapsto H(M_x)$ is a local system on N_α. If $x \in N_\phi, \chi_\phi$ then the corresponding multiplicity space is given by the isotopy invariants of the top cohomology of the fiber.*

$$L_\phi = H(M_x)^{\pi_1(N_\alpha, x)}$$

5.1.5 Applying the machinery to the Springer resolution

In this section we explain the following setup and theorem. Set $Z = \tilde{\mathcal{N}} \times_\mathcal{N} \tilde{\mathcal{N}}$ the Steinberg variety. If $x \in \mathcal{N}$ then M_x is formed by pairs $(x, \mathfrak{b})$ where $\mathfrak{b}$ runs over the subset $\mathcal{B}_x$ of x invariant Borel subalgebras. Equivalently any element $x \in \mathfrak{g}$ induces a vector field on G/B and $\mathcal{B}_x$ is the zeros of this vector field.

Let $G(x)$ be the centralizer of x in G, $A(x) = G(x)/G(x)_0$ the isotopy group acting on the connected components. Let $A(x)^*$ denote the set of isomorphism classes of $A(x)$-representations occuring in the BM homology groups $H_{top}(M_x)$. The main techincal result known as "Springer construction of Weyl groups" is the following:

Theorem 5.7 (Geometric Construction of W). *1. $H(Z) \cong \mathbb{C}[W]$*

2. *The collection $\{H(M_x)_\phi\}$ as (x, ϕ) runs over G conjugacy classes of points in $\mathcal{N}$ and $\phi \in A(x)^*$ is a complete set of irreducible representations of W.*

5.1.6 Fourier transform

We introduce the main tool of the construction, the Fourier transform on perverse sheaves (or D-modules).

5.2 Schubert varieties and Kazhdan Lustzig polynomials

Let G be an algebraic group. The Kazhdan Lustzig polynomials are a family of polynomials defined for two Weyl group elements v, w in a Weyl group W with a system of generators S.

Definition 5.8. *The **Hecke algebra** $\mathcal{H}$ is the algebra of B-bi-invariant functions of $G(\mathbb{F}_q)$. The algebra structure is given by normalized convolution. It has a basis consisting of functions*

$$\phi_w = \operatorname{id}_{BwB} \text{ for } w \in W$$

Lemma 5.9. *If $s \in S$ is a simple reflection and $w \in W$ then we have the following relations*

$$\phi_w * \phi_{w'} = \phi_{ww'} \text{ if } l(ww') = l(w) + l(w')$$
$$\phi_s * \phi_s = (q-1)\phi_s + q\phi_1$$
$$\phi_s * \phi_w = (q-1)\phi_w + q\phi_{sw} \text{ if } l(ws) = l(w) + 1$$

We have the "standard" description of Hecke algebra

Proposition 5.10. *$\mathcal{H}$ is the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{\phi_w\}_{w \in W}$ and relations*

$$\phi_s \phi_w = \phi_{sw} \text{ if } l(sw) = l(w) + 1$$
$$(\phi_s - q)(\phi_s + 1) = 0$$

If $q = 1$, then this is the group algebra $\mathbb{Z}[W]$.

The convention is to use $q^{\frac{1}{2}}$ and $q^{-\frac{1}{2}}$ as the formal parameters. Each ϕ_w is invertible and the algebra admits an involution

$$i(q^{\frac{1}{2}}) = q^{-\frac{1}{2}} \text{ and } i(\phi_w) = \phi_{w^{-1}}^{-1}$$

Theorem 5.11 (Kazhdan-Lusztig). *For each w there is a unique $c_w \in \mathcal{H}$ and a uniquely determined polynomial P_{yw} with $y \leq w$ so that $i(c_w) = c_w$ and $P_{ww} = 1$ and $P_{yw}(q)$ has degree less than $\frac{1}{2}(l(w) - l(y) - 1)$ when $y < w$ and*

$$c_w = q^{-l(w)/2} \sum_{y \leq w} P_{yw}(q)\phi_y$$

Kazhdan and Lusztig conjectured that the coefficients of the polynomials $P_{yw}(q)$ are nonnegative integers and that in the Grothendieck group of Verma modules

$$[L_w] = \sum_{y \leq w} (-1)^{l(w) - l(y)} P_{yw}(1)[M_y]$$

The second conjecture became known as the **Kazhdan-Lustzig conjecture** and was proven by Beilinson-Bernstein and Brylinski-Kashiwara independently. The interpretation of c_w and P_{yw} in terms of intersection cohomology was critical to the proof.

Theorem 5.12. *Set for any $v \leq w$ the number $h^i(\bar{X}_w)_v = \dim \mathcal{H}^i(\mathrm{IC}_{\bar{X}_w})_v$. Then*

$$P_{v,w}(q) = \sum_i h^i(\bar{X}_w)_v q^i$$

is a polynomial in q with nonnegative integer coefficients.

5.3 Geometric Satake Isomorphism

Theorem 5.13. *The category $\mathcal{P}_\mathcal{O}$ is equivalent to the category of representations of the Langlands dual group $^L G$, as categories with tensor and fiber structures.*

There is a bilinear functor $\star : \mathcal{P}_\mathcal{O} \times \mathcal{P}_\mathcal{O} \to \mathcal{P}_\mathcal{O}$ with compatible commutativity, associativity restraints, and a fiber functor which respects the tensor product. In fact, one defines the desired functor geometrically so that H, the operation of taking cohomology, is the fiber functor. Therefore the equivalence is between

$$(\mathcal{P}_\mathcal{O}, \star, H) \cong (\mathrm{Rep}(^L G), \otimes, \mathrm{Forget})$$

The $G(\mathcal{O})$ orbits of the affine Grassmannian are indexed by dominant coweights λ. A shadow of the geometric Satake isomorphism is the statement that the intersection cohomology of the λ-orbit is the irreducible representation of $^L G$ with highest weight λ.

6 Closer look at perverse sheaves

6.1 Precise definitions

The category of perverse sheaves is defined by relaxing the support and cosuppoort conditions for the IC sheaf by one.

Definition 6.1. *Let W be a n-dimensional Whitney stratified space. A **middle perversity perverse sheaf** on W is a complex of sheaves $A^\bullet$ in the bounded constructible derived category $D^b_c(W)$ such that if S is a stratum of dimension d, A^* satisfies the support and cosupport conditions*

$$H^i(j_S^* A^*) = 0 \text{ for all } i > -d/2$$
$$H^i(j_S^! A^*) = 0 \text{ for all } i < -d/2$$

Definition 6.2. *A **perversity** is a function on dimension $p : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $p(0) = 0$*

$$p(d) \geq p(d+1) \geq p(d) - 1$$

Middle perversity is the perversity $p(d) = -d/2$. The category of perverse sheaves with perversity p is those objects in $D^b_c(W)$ for which

$$H^i(j_S^* A^*) = 0 \text{ for all } i > p(\dim S)$$
$$H^i(j_S^! A^*) = 0 \text{ for all } i < p(\dim S)$$

Each perversity involves its own shift: for a space W of dimension n the stalk cohomology of the IC sheaf in the top stratum is nonzero in degree $p(n)$.

Remark 6.3. *One can check that perversity 0 corresponds precisely to the category of constructible sheaves (not complexes!, just sheaves).*

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Theorem 6.4. *The category of middle perverse sheaves forms an abelian subcategory of $D_c^b(X)$ that is preserved by Verdier duality.*

Theorem 6.5. *If W is an algebraic variety, the simple objects are the shifted IC sheaves with irreducible local coefficients of irreducible subvarieties.*

Remark 6.6. *To a $\mathcal{D}$-module there is a corresponding sheaf of solutions which is constructible. Beilinson, Bernstein, Brylinski, and Kashiwara showed that each Verma modlue can be associated to a certain holonomic $\mathcal{D}$-module with regular singularities whose sheaf of solutions turns out to be the IC sheaf. However, the category of $\mathcal{D}$-modules is abelian whereas the constructible derived category is not, so it was conjectured that there is some abelian subcategory of the category which "receives the solution sheaves". This is precisely the category of perverse sheaves with middle perversity!*

6.2 Examples

Example 6.7. *There are some special examples where we have conmbinatorial descriptions of the middle perverse sheaves on an algebraic variety with respect to a stratification.*

1. *$\mathbb{C}^n$ with respect to hyperplane arrangements*

2. *Square matrices with respect to the rank stratification*

3. *The flag variety with respect to the Schubert stratification*

In particular the most simple example is $\mathbb{C}, \{0\}$. The category of perverse sheaves is equivalent to the category of representations of the quiver

$$ \bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet $$
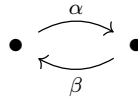
for which $I - \alpha\beta$ and $I - \beta\alpha$ are invertible.

Example 6.8. *Stratify $\mathbb{P}^1$ with a single zero dimensional stratum N the north pole. The support diagram for middle perversity sheaves is*

$i \setminus codim$	0	2
2	c	c
1		cx
0	x	x

The columns index codimension of the strata and the rows index the cohomological degree (the convention is 0 to n as opposed to $-n/2$ to $n/2$). x denotes that there may be nontrivial stalk cohomology sheaves supported along strata of that given codimension. c denotes that the same thing but with compactly supported cohomology stalks.

The category of perverse sheaves is equivalent to the category of representations of the quiver

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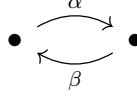
for $\alpha\beta = \beta\alpha = I$. *Somehow I want to relate this to the category of perverse sheaves on $\mathbb{C}, \{0\}$. Does a stratified map descend to a relationship between these categories?* *In particular there are 5 such indecomposable representations and consequently, 5 simple perverse sheaves. They are $\mathbb{Q}_N[-1], \mathbb{Q}_{\mathbb{P}^1}, j_!\mathbb{Q}_U, j_*\mathbb{Q}_U$ with the support diagrams*

$i \setminus codim$	0	2		$i \setminus codim$	0	2		$i \setminus codim$	0	2		$i \setminus codim$	0	2
2				2	c	c		2	c	c		2	c	
1		cx		1				1		c		1		x
0				0	x	x		0	x			0	x	x

and the last one which is not an IC sheaf. It is gotten by taking a closed disk and putting the constant sheaf on the interior and the 0 sheaf on the boundary, except for a point, and then pushing this sheaf forward along the map $D^2 \to S^2$ which collapses the boundary. It has stalk cohomology and compactly supported stalk cohomology in degree 1. Verdier duality interchanges these conditions and so this sheaf is self-dual. The first two sheaves are self-dual as well and the last two are dual to each other. Verdier duality can be seen by reflecting the support diagram across the horizontal axis and swapping x and c. *Does the support diagram of an IC sheaf determine the sheaf up to isomorphism?*

6.3 t-structures and perverse sheaves

6.3.1 Motivation: perversity 0 t-structure

Let W be a stratified space. The category of perversity 0 complexes of sheaves is equivalent to the category of constructible ordinary sheaves $\mathrm{Sh}_c(W)$.

Definition 6.9. *Let A^* be a complex of sheaves on W. We have **truncation functors** $\tau_{\leq r}$ and $\tau_{\geq r}$ defined by*

$$A^* = A^{r-1} \to A^r \to A^{r+1} \to \cdots$$
$$\tau_{\leq r}A^* := A^{r-1} \to \ker(d^r) \to 0 \to \cdots$$
$$\tau_{\geq r}A^* := 0 \to \mathrm{coker}(d^{r-1}) \to A^r \to A^{r+1} \to \cdots$$

Then there is a short exact sequence $0 \to \tau_{\leq 0}A^* \to A^* \to \tau_{\geq 1}A^* \to 0$ and the cohomology sheaf of A^* is given by

$$H^i(A^*) = \tau_{\leq i}\tau_{\geq i}A^*$$

In particular we have the following theorem which is the version which we can generalize.

Theorem 6.10. *The cohomology functor $H^r : D_c^b(W) \to \mathrm{Sh}_c(W)$ is given by the composition of the trunctation functors.The functor H^0 restricts to an equivalence of categories between $\mathrm{Sh}_c(W)$ and the full subcategory of $D_c^b(W)$ consisting of complexes A^* such that $H^i(A^*) = 0$ for $i \neq 0$. This category is Noetherian and Artinian and its simple objects are the sheaves $j_!(\mathcal{E})$ where $\mathcal{E}$ is a simple local system on a single stratum $j : X \to W$.*

6.3.2   General t-structures

Fix a perversity p and let $\mathcal{P}(W)$ be the category of perverse sheaves with perversity p on W. Then

Proposition 6.11. *There are truncation functors*

$$^p\tau_{\leq r} : D_c^b(W) \to D_c^b(W)$$
$$^p\tau_{\geq r} : D_c^b(W) \to D_c^b(W)$$

which take distinguished triangles to exact seuqences and satisfy

$$^p\tau_{\leq r}(A^*) = (^p\tau_{\leq 0}(A^*)[r])[-r]$$

Definition 6.12. *We have the **perverse cohomology***

$$^pH^r(A^*) =^p \tau_{\leq r}^p \tau_{\geq r} A^*$$

Theorem 6.13. *The perverse cohomology functor $^pH^r : D_c^b(W) \to \mathcal{P}(W)$ is given by the composition of the truncation functors. The functor $^pH^0$ restricts to an equivalence of categories between $\mathcal{P}(W)$ and the full subcategory of $D_c^b(W)$ consisting of complexes A^* such that $^pH^i(A^*) = 0$ for $i \neq 0$. This category is Noetherian and Artinian and its simple objects are the sheaves $Rj_*(\mathrm{IC}_p^*(\mathcal{E}))$ where $\mathcal{E}$ is a simple local system on a single stratum X and $j : \bar{X} \to W$.*

Remark 6.14. *More generally, a t-structure on a triangulated category is a pair of strictly full subcategories $\mathcal{D}^{\geq 0}$ and $\mathcal{D}_{\leq 0}$ satisfying technical conditions as above. In this case, the heart $P = \mathcal{D}^{\geq 0} \cap \mathcal{D}_{\leq 0}$ is an abelian full subcategory.*

6.4   Algebraic varieties and the decomposition theorem

6.4.1   Intersection cohomology

Intersection cohomology enjoys many of the same remarkable properties as ordinary cohomology for algebraic varieties. In particular, the Lefschetz hyperplane theorem, the hard Lefschetz theorem, the Lefschetz decomposition theorem, and the Hodge structure all hold for intersection cohomology.

Theorem 6.15. *Let $W \subset \mathbb{CP}^N$ projective algebraic variety of dimension n. Let L^j be a generic linear subspace of codimension j in $\mathbb{CP}^N$. If L^j is transverse to each stratum of a Whitney stratification of W, then there are natural restriction maps*

$$IH^r(W) \to IH^r(W \cap L^j)$$
$$IH^r(W \cap L^j) \to IH^{r+2j}(W)$$

If L^1 is transverse to W then the restriction map is an isomorphism for $r \leq n-2$ and an injection for $r = n-1$. If $j \geq 1$ and L^j transverse to W then the composition

$$L^j : IH^{n-j}(W) \to IH^{n-j}(W \cap L^j) \to IH^{n+j}(W)$$

is an isomorphism.

Theorem 6.16. *We have the Leftschetz decomposition which says that*

$$IH^r = \bigoplus_{j \geq 0} L^j IP^{r-2j}$$

where the primitive cohomology $IP^i \subset IH^i$ is the kernel of $\cdot L^{n-i+1}$. Poincare duality

$$IH^{n+r}(W, \mathbb{Q}) \cong \mathrm{Hom}(IH^{n-r}(W, \mathbb{Q}), \mathbb{Q})$$

and the Lefschetz isomorphism $L^r : IH^{n-r}(W) \to IH^{n+r}(W)$ induce a nondegenerate bilinear pairing on $IH^{n-r}(W)$. The Lefschetz decomposition is orthogonal with respect to this pairing and its signature is described by the Hodge-Riemann bilinear relations.

6.4.2 Decomposition theorem

The decomposition theorem was first formulated and proved by Beilinson, Bernstein, and Deligne. It is one of the deepest theorems we have about the topology of algebraic maps. Let $f : X \to Y$ be a proper map of algebraic varieties. The decomposition theorems says that $Rf_* IC_X$ breaks into a direct sum of intersection complexes of subvarieties of Y with coefficients in various local systems, with shifts.

Theorem 6.17 (Decomposition theorem). *Let $f : X \to Y$ be a proper map of algebraic varieties.*

1. *$Rf_* IC_X^* = \bigoplus_i {}^p \mathcal{H}^i(Rf_* IC_X^*)[-i]$ (this says that the push forward sheaf is a direct sum of perverse sheaves, in particular its own perverse cohomology sheaves)*

2. *Each summand is a semisimple perverse sheaf. In particular it is a direct sum of shifted IC sheaves with local coefficients on irreducible subvarieties of Y, and it enjoys all of the above properties of intersection cohomology.*

3. *Relative hard-Lefschetz theorem: if η is the class of a hyperplane of X then for all r the map $\eta^r : {}^p \mathcal{H}^{-r}(Rf_* IC_X^*) \to {}^p \mathcal{H}^r(Rf_* IC_X^*)$ is an isomorphism.*

6.4.3 Example: square pyramid

The square pyramid does not correspond to a smooth toric variety because it is singular at the tip where the four faces meet. However there is a resolution of singularities $\pi : \tilde{Y} \to Y$ given by corner chopping. We consider the decomposition theorem for this map.

The map is an isomorphism except at the singular point whose fiber $\pi^{-1}(y) = \mathbb{P}^1 \times \mathbb{P}^1$. The stalk cohomology of the pushforward $R\pi_* \mathbb{Q}_{\tilde{Y}}$ is $\mathbb{Q}, 0, \mathbb{Q} \oplus \mathbb{Q}, 0, \mathbb{Q}$ is equal to the cohomology of the fiber. We consider the support diagram

i \ codim	0	2	4	6	$H^*(\pi^{-1}(y))$
6	c	c	c	c	
5			c	c	
4				c	$\mathbb{Q}$
3				0	0
2				x	$\mathbb{Q} \oplus \mathbb{Q}$
1				x	x
0	x	x	x	x	$\mathbb{Q}$

The support condition at the bottom of the table says $\mathbb{Q}[3]$ is part of the IC sheaf. The $\mathbb{Q}[-1]$ at the top of the table is not part of the IC sheaf. By duality, neither is one of the copies of $\mathbb{Q}$ in the middle. Finally, we can show that the other copy of $\mathbb{Q}$ in the middle is part of the IC sheaf. These two terms constitute the primitive cohomology of the fiber.

7 Appendix A: Spectral Sequences

What follows comes from [?] and [?].

7.1 First look

Recall that a short exact sequence of chain complexes

$$0 \to A^* \to B^* \to C^* \to 0$$

induces a long exact sequence in cohomology

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

We can view this as a piece of cohomological data coming from the two step filtration of the total complex B^* by $0 \subset A^* \subset B^*$. This is the beginning of a spectral sequence.

The filtration of the total complex descends to a filtration on cohomology. In particular each page E^r of the spectral sequence is filtered. There is a very special filtration on the $H^*(X, \mathbb{C})$ of a Kahler manifold coming from the Cech to de Rham spectral known as the **Hodge filtration**.

Definition 7.1. *A **spectral sequence** is a sequence (E^r, d^r) of bigraded abelian groups*

$$E^r = \bigoplus_{p,q} E_r^{p,q}$$

for $r \geq 0$ and differentials

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

so that $d_r^2 = 0$ and $H^*(E^r, d^r) = E_{r+1}$.

7.2 Spectral sequence of a filtered complex

Proposition 7.2. *Every filtered complex $K^* = F^0 K^* \supset F^1 K^* \supset \cdots$ gives rise to a spectral sequence with*

$$E_0^{p,q} = F^p K^{p+q} / F^{p+1} K^{p+q}$$
$$E_1^{p,q} = H^{p+q}(F^p K^* / F^{p+1} K^*) = H^{p+q}(\mathrm{Gr}^p K^*)$$
$$E_\infty^{p,q} = \mathrm{Gr}^p(H^{p+q}(K^*))$$

Proof. We have the differential inherited from K^*

$$d_0 : E_0^{p,q} \to E_0^{p,q+1}$$
$$d_0 : F^p K^{p+q} / F^{p+1} K^{p+q} \to F^p K^{p+q+1} / F^{p+1} K^{p+q+1}$$

Note that the differential only increases the cohomological degree of K^* by 1. We compute the first page

$$
\begin{aligned}
E_1^{p,q} &= \ker d_0 / \operatorname{im} d_0 \\
&= \{ a \in F^p K^{p+q} \mid da \in F^p K^{p+q+1} \} / d(F^p K^{p+q-1}) + F^{p+1} K^{p+q} \\
&= H^{p+q}(F^p K^* / F^{p+1} K^*) \\
&= H^{p+q}(\mathrm{Gr}^p K^*)
\end{aligned}
$$

The next differential

$$d_1 : E_1^{p,q} \to E_1^{p+1,q}$$
$$a \mapsto da$$

is well-defined because

$$da \in \{ b \in F^{p+1} K^{p+q+1} \mid db \in F^{p+2} K^{p+q+2} \}$$

One can compute the kernel and image of d_1 to get the second page, to find that

$$E_2^{p,q} = \{ a \in F^p K^{p+q} \mid da \in F^{p+2} K^{p+q+1} \} / d(F^{p-1} K^{p+q-1}) + F^{p+1} K^{p+q}$$

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Take denominator as written intersect the numerator. The general pattern is

$$E_r^{p,q} = \{a \in F^p K^{p+q} \mid da \in F^{p+r} K^{p+q+1}\}/d(F^{p-r+1} K^{p+q-1}) + F^{p+1} K^{p+q}$$

and then one can check that

$$H^*(E^r, d^r) = E^{r+1}$$

For r sufficiently large we get that

$$E_\infty^{p,q} = \{a \in F^p K^{p+q} \mid da = 0\}/d(K^{p+q-1}) + F^{p+1} K^{p+q}$$
$$= \mathrm{Gr}^p(H^{p+q}(K^*))$$

as desired.   □

7.3   Spectral sequence of a double complex

Let $K^{*,*}$ be a double complex with differentials

$$d : K^{p,q} \to K^{p+1,q} \text{ and } \delta : K^{p,q} \to K^{p,q+1}$$

so that

$$d^2 = \delta^2 = 0 \text{ and } d\delta + \delta d = 0$$

We form the associated single compex (K, D) with

$$K^* = \bigoplus_{p+q=n} K^{p,q} \text{ and } D = d + \delta$$

There are two filtrations on (K^*, D), by rows and columns

$$F'^p K^n = \bigoplus_{i \le p} K^{i, n-i}$$
$$F''^q K^n = \bigoplus_{i \le q} K^{n-i, i}$$

and there are two associated spectral sequences, both abuting to $H^*(K, D)$. The corresponding E_2 pages turn out to be

$$E_{p,q}'^2 = H_d^p(H_\delta^q(K^{*,*}))$$
$$E_{p,q}''^2 = H_\delta^q(H_d^p(K^{*,*}))$$

Example 7.3. *If you consider the complex of real-valued forms on a manifold*

$$0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^n \to 0$$

and filter it by the limit of all of the "good covers" (intersection of two opens is contractible) of the manifold, then the associated spectral sequence is the Cech to de Rham spectral sequence and realizes the de Rham isomorphism

$$H^*(X, \mathbb{R}) \cong H^*_{dR}(X)$$

Alternatively we can think about the de Rham isomorphism like this. Recall the following lemma:

Lemma 7.4. *A map of complexes of sheaves which is a quasi-isomorphism (i.e. induces an isomorphism on all cohomology sheaves) induces an isomorphism on hypercohomology.*

The point is is that this map will induce an isomorphism on the associated spectral sequences.

Example 7.5. *Again we have the real-valued de-Rham complex*

$$0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^n \to 0$$

and also the trivial complex $\mathbb{R}^*$

$$\mathbb{R} \to 0 \to 0$$

The d-Poincare lemma shows that these two complexes have the same cohomology sheaves and therefore inclusion is a quasiisomorphism. Therefore the lemma shows they have the same cohomology. Now consider the two spectral sequences at hand. We have

$$(E'^{p,q}_{\mathbb{R}^*})_2 = H^p(M, \mathbb{R}) \text{ if } q = 0 \text{ and } 0 \text{ otherwise}$$

so the spectral sequence is trivial and

$$H^p(M, \mathbb{R}) = \mathcal{H}^p(M, \mathbb{R}^*)$$

On the other hand we compute $H^q(M, \Omega^*) = 0$ if $q > 0$ by partition of unity and

$$E''^{p,q}_{\Omega^*} = H^p_{dR}(M) \text{ if } q = 0 \text{ and } 0 \text{ otherwise}$$

and so putting everything together we find that

$$H^p(M, \mathbb{R}) \cong H^p_{dR}(M)$$

7.4   Leray Spectral Sequence

Let $f : X \to Y$ be a continuous map of topological spaces. Recall that the q-th direct image sheaf $R_f^q \mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto H^q(f^{-1}(U), \mathcal{F})$$

The Leray spectral sequence is a spectral sequence with

$$E_2^{p,q} = H^p(Y, R_f^q \mathcal{F})$$
$$E_\infty \to H^*(X, \mathcal{F})$$

This spectral sequence is particularly special when $f : E \to B$ is a fiber bundle with fiber F. In this case, for the constant sheaf $\mathbb{Q}$ on E we have by the Kunneth formula

$$H^q(\pi^{-1}(U), \mathbb{Q}) \cong H^*(F, \mathbb{Q})$$

which suggests

$$R_f^q \mathbb{Q} \cong H^*(F, \mathbb{Q})$$

is a constant sheaf on B. However this is not so because we have to account for the monodromy action of $\pi_1(B)$ on the fibers.

Example 7.6. *We consider the derivation of the Leray spectral sequence for de Rham cohomology in the case of a fiber bundle. I don't understand this example but the relevant fact is that*

$$E_2^{p,q} = H_{DR}^p(B, H_{DR}^q(F, \mathbb{R}))$$

where we can interpret the right hand side as the the de Rham cohomology of B with coefficients in the local system $H_{DR}^q(F, \mathbb{R})$.

8   Appendix B: Homological algebra

What follows comes from [**?**].

8.1   The four functors $f_*, f^*, f_!, f^!$

There is a nice geometrical way to think about these functors which requires introducing the leaf space or espace etale of a presheaf.

Definition 8.1. *Given a presheaf $\mathcal{F}$ on a topological space X, the **espace etale** $L\mathcal{F}$ is the disjount union of the stalks of $\mathcal{F}$*

$$L\mathcal{F} = \bigsqcup_{x \in X} \mathcal{F}_x \to X$$

equipped with a topology that is discrete on each S_x and that makes π into a local homeomorphism.

Given the leaf space, we have the sheafification of $\mathcal{F}$ given by $U \mapsto \Gamma(U, L\mathcal{F})$ and the presheaf $\mathcal{F}$ is a sheaf if and only if it is isomorphic to its sheafification.

Definition 8.2. *Given a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X and $\mathcal{G}$ sheaf on Y, we define the **direct image** $f_*\mathcal{F}$ on Y by*

$$f_*\mathcal{F}(U) = \Gamma(f^{-1}(U), \mathcal{F})$$

*and the **inverse image** $f^*\mathcal{G}$ on X by*

$$f^*\mathcal{G}(U) = \lim_{V \supset f(U)} \mathcal{G}(V)$$

*We also have the **pushforward with proper support** $f_!$ with sections $\Gamma(U, f_!\mathcal{F})$ consisting of all sections $s \in \Gamma(f^{-1}(U), \mathcal{F}$ so that the mapping $\mathrm{closure}(x \in U \mid s(x) \neq 0) \to U$ is proper. In particular if $X \to Y$ is the inclusion of a subspace then $f_!\mathcal{F}(U)$ is those sections $s \in \Gamma(U \cap X, \mathcal{F})$ with compact support.*

Lemma 8.3. *When Y is locally compact and $X \to Y$ is the inclusion then $f_!\mathcal{F} = f_*\mathcal{F}$.*

8.2 Adjunction

Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y. Then there are natural maps

$$f^* f_*\mathcal{F} \to \mathcal{F} \text{ and } \mathcal{G} \to f_* f^*\mathcal{G}$$

To see this for the first one, consider an open set $U \subset X$. Then

$$\Gamma(U, f^* f_*\mathcal{F}) = \lim_{V \supset f(U)} \Gamma(V, f_*\mathcal{F}) = \lim_{V \supset f(U)} \Gamma(f^{-1}(V), \mathcal{F})$$

has a natural map to $\Gamma(U, \mathcal{F})$, compatible with restriction. For the second one, we have $V \subset Y$ open. If t is a section of $L\mathcal{G}$ over V, then the pullback by f is a section of the leaf space of $f^*\mathcal{G}$, over the set $f^{-1}(V)$. Therefore we have a map

$$\Gamma(V, \mathcal{G}) \to \Gamma(f^{-1}(V), f^*\mathcal{G}) = \Gamma(V, f_* f^*\mathcal{G})$$

compatible with restriction. Therefore we have the following adjunction

Theorem 8.4. *There are natural isomorphisms*

$$\mathrm{Hom}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$$

Remark 8.5. *One might ask if there is an adjoint to the functor $f_!$. The answer is a very subtle question and introduces Verdier duality.*

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8.3 Cohomology

Definition 8.6. *A sheaf $\mathcal{F}$ is **injective** if for every $0 \to A \to B$, any morphism $A \to \mathcal{F}$ can be extended to a morphism $B \to \mathcal{F}$. That is the diagram*

$$
\begin{array}{ccc}
0 \longrightarrow A \longrightarrow B \\
\downarrow \quad \swarrow \\
\mathcal{F}
\end{array}
$$

*commutes. An **injective resolution** of a sheaf $\mathcal{F}$ is an exact sequence*

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots$$

*where each I^i is injective sheaf. The **sheaf cohomology groups** $H^*(X, \mathcal{F})$ is the cohomology of the complex*

$$0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \cdots$$

*Any other injective resolution of $\mathcal{F}$ will give the same cohomology groups up to canonical isomorphism. Given a complex of sheaves $\mathcal{F}^*$ we can also take the cohomology sheaves $\mathcal{H}^*(\mathcal{F}^*) := \ker d / \operatorname{im} d$. The stalk of the cohomology sheaf coincides with the cohomology of the stalks. A morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves is a **quasi-isomorphism** if it induces an isomorphism on all cohomology sheaves. A quasi-isomorphism induces an isomorphism on cohomology groups.*

Definition 8.7. *The **mapping cone** $C(\phi)$ of a map $\phi : \mathcal{F}^* \to \mathcal{G}^*$ of complexes of sheaves is the total complex of the double complex corresponding to ϕ. We often write this as a "magic triangle"*

$$\mathcal{F}^* \xrightarrow{\phi} \mathcal{G}^* \longrightarrow \mathcal{C}(\phi) \longrightarrow \mathcal{F}^*[1]$$

Mapping cones are interesting because they give rise to long exact sequences in cohomology.

Lemma 8.8. *If ϕ is injective then there is a natural quasiisomorphism $\operatorname{coker} \phi \cong C(\phi)$. If ϕ is surjective then there is a natural quasiisomorphism $C(\phi) \cong \ker \phi[1]$. Moreover there is a long exact sequence on cohomology*

$$\cdots \to \mathcal{H}^{r-1}(\mathcal{G}^*) \to \mathcal{H}^{r-1}(C(\phi)) \to \mathcal{H}^r(\mathcal{F}^*) \to \mathcal{H}^r(\mathcal{G}^*) \to \mathcal{H}^r(C(\phi)) \to \cdots$$

Lemma 8.9. *Let C^** be a first quadrant double complex with exact rows and so that the zeroth horizontal arrows d_h^{0q} are injections (which is the same as saying we can add an extra zero to the left end of each row). Then the total complex T is exact.*

Proof. We will check that the total complex is exact at T^2. Let $x = x_{02} + x_{11} + x_{20} \in T^2$ and suppose that $dx = 0$. Then

$$
\begin{aligned}
d_v x_{02} &= 0 \\
d_v x_{11} + d_h x_{02} &= 0 \\
d_v x_{20} - d_h x_{11} &= 0 \\
d_h x_{20} &= 0
\end{aligned}
$$

and since the bottom row is exact, we have $x_{20} = d_h y_{10}$ for some y_{10}. Consider $x'_{11} = x_{11} - d_v y_{10}$. Then

$$d_h x'_{11} = d_h x_{11} - d_h d_v y_{10} = d_h x_{11} - d_v d_h y_{10} = d_h x_{11} - d_v x_{20} = 0$$

Since the row is exact, we have $x'_{11} = d_h y_{01}$ for some y_{01}, and the argument continues. Check that $d(y_{01} + y_{10}) = x$. The argument is the same in the general case. $\square$

Corollary 8.10. *Let C^{**} be a first quadrant double complex with exact rows. Let T^* be the total complex and $A^r = \ker(d^{0,r} : C^{0,r} \to C^{1,r})$ be the subcomplex of the zeroth column with vertical differential. Then the inclusion $A^* \to T^*$ is a quasiisomorphism.*

Proof. Augment C^{**} by putting the complex A^* in the -1th column. Then the total complex S^* of the augmented complex is precisely the mapping cone of the inclusion $A^* \to T^*$. Therefore there is a long exact sequence in cohomology

$$\cdots \to \mathcal{H}^{r-1}(T^*) \to \mathcal{H}^{r-1}(S^*) \to \mathcal{H}^r(A^*) \to \mathcal{H}^r(T^*) \to \cdots$$

and by the previous lemma, S^* is exact. Therefore the inclusion $A^* \to T^*$ is a quasiisomorphism. $\square$

Remark 8.11. *What have we done? Given a complex of sheaves A^*, take an injective resolution of each sheaf and stack them up to get a double complex. It's not so obvious you can do this but Goresky assures me we can. Then what we are saying is that we can take the total complex and the resulting map $A^* \to T^*$ is a quasiisomorphism.*

Thus we end up with the following definition which works in any abelian category.

Definition 8.12. *An **injective resolution** of a complex A^* is a quasi-isomorphism $A^* \to T^*$ where each T^r is an injective object. The cohomology $H^r(X, A^*)$ is the cohomology of the complex of global sections of any injective resolution of A^*.*

Example 8.13. *Consider the resolution of a single sheaf S. It is a quasi-isomorphism*

$$
\begin{array}{ccccccc}
0 & \longrightarrow & S & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & \downarrow & & & & & & \\
0 & \longrightarrow & I_0 & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow & \cdots
\end{array}
$$

Recall the Poincare lemma which says that closed forms on a smooth manifold are locally, when restricted to a Euclidean ball, exact. The statement about sheaves is that we have a resolution of the constant sheaf by the de Rham complex

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & \downarrow & & & & \\
0 & \longrightarrow & \Omega^0 & \longrightarrow & \Omega^1 & \longrightarrow & \cdots
\end{array}
$$

which one again proves the de Rham isomorphism $H^r(M, \mathbb{R}) \cong H_{dR}^r(M)$.

Definition 8.14. *The Cech cohomology of a sheaf $\mathcal{F}$ on a topological space X is defined for an open cover $\mathcal{U}$ of X by the complex*

$$0 \to \mathcal{F}(X) \to \prod_{U \in \mathcal{U}} \mathcal{F}(U) \to \prod_{U,V \in \mathcal{U}} \mathcal{F}(U \cap V) \to \cdots$$

*The cohomology of this complex is the **Cech cohomology** $\hat{H}^*(X, \mathcal{F})$.*

Theorem 8.15. *Suppose $\mathcal{F}$ is a sheaf on X and $\mathcal{U}$ is an open cover of X so that*

$$H^r(U_J, \mathcal{F}) = 0 \text{ for all } r > 0 \text{ and all finite intersections } U_J$$

Then there is a canonical isomorphism

$$H^*(X, \mathcal{F}) \cong \hat{H}^*(X, \mathcal{F})$$

The proof of this theorem is via first sheafifying the Cech complex which is functorial, picking an injective resolution of $\mathcal{F}$ and combining the two to get a double complex. The cohomology of this total complex is equal to both the LHS and RHS of the theorem.

8.4 Derived categories and derived functors

The motto is: derive by taking injective resolutions.

8.4.1 First construction

Definition 8.16. *Two morphisms of complexes are **homotopic** if there is a chain of maps $h^r : A^r \to B^{r-1}$ so that $d_B h + h d_A = d_A$. Let $[A^*, B^*]$ be the set of homotopy classes of morphisms of complexes. Define the complex of abelian groups*

$$\text{Hom}^n(A^*, B^*) = \prod_r \text{Hom}(A^r, B^{r+n}) \tag{1}$$

with differential $d(f) = d_B f - (-1)^n f d_A$.

We have the following

$$H^n(\text{Hom}^*(A^*, B^*)) = [A^*, B^*[n]]$$

Definition 8.17. *The **bounded homotopy category** $K^b(X)$ of complexes of sheaves on a topological space X is the category whose objects are bounded complexes of sheaves and whose morphisms are homotopy classes of maps.*

*The **bounded derived category** $D^b(X)$ is homotopy category of injective sheaves. Its objects are bounded complexes of injective sheaves and its morphisms are homotopy classes of maps.*

Remark 8.18. *In topology, homotopic maps induce the same map on cohomology. In the derived category, the same is true. However, the following lemmas will show that once you restrict your objects to injective complexes, every quasi-isomorphism between injective complexes is in fact a homotopy equivalence. Therefore, this interprets the statement "inverting quasi-isomorphisms" as being the same as restricting to the injective objects.*

This lemma is where the work happens and then everything else follows.

Lemma 8.19. *Let C^* be a bounded complex of sheaves and suppose that all cohomology sheaves are zero. Let J^* be injective. Then any map $C^* \to J^*$ is homotopic to zero.*

The proof is quite standard, but we get a lot out of it:

Corollary 8.20. *The following hold:*

1. *Suppose J^* is injective and has no cohomology. Then J is homotopy equivalent to the zero complex.*

2. *Let $\phi : X^* \to Y^*$ be a quasi-isomorphism of bounded complexes of injective sheaves. Then ϕ is a homotopy equivalence.*

3. *Let $A^* \to I^*$ and $B^* \to J^*$ be injective resolutions of complexes A^* and B^* and $f : A^* \to B^*$ a map of complexes. There is a lift $\tilde{f} : I^* \to J^*$ unique up to homotopy.*

4. *$f : A^* \to B^*$ a quasiisomorphism induces an isomorphism on hypercohomology.*

Remark 8.21. *Injective objects are like CW complexes, and these facts are basically about the homotopy theory of CW complexes. The first statement is Whitehead's theorem.*

There is a functor $K^b(X) \to D^b(X)$ which sends a complex of sheaves to its "canonical" Godemont injective resolution. If $A^* \to B^*$ is a quasi-isomorphism of complexes of sheaves, then it becomes an isomorphism in the derived category.

Definition 8.22. *The **right derived functor** $RT : D^b(X) \to D^b(Y)$ of a functor $T : \mathrm{Sh}(X) \to \mathcal{C}$ is defined by replacing a complex A^* by its injective resolution I^* and applying T to get TI^*.*

Remark 8.23. *We emphasize that the hypercohomology of $R\,\mathrm{Hom}$ is exactly the group of homomorphisms in the derived category:*

$$H^0(R\,\mathrm{Hom}(A^*, B^*)) = \mathrm{Hom}_{D^b(X)}(A^*, B^*)$$

8.4.2 Second construction

The derived category can also be interpreted as a quotient category of the category of complexes of sheaves, by inverting quasi-isomorphisms. Let $E^b(X)$ be the category of bounded complexes of sheaves on X with a morphism $A^* \to B^*$ being the data of a diagram

$$
\begin{array}{ccc}
& C^* & \\
{}^{qi}\swarrow & & \searrow \\
A^* & & B^*
\end{array}
$$
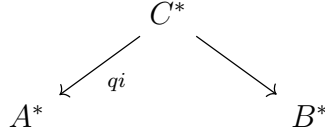
up to the equivalence relation between two morphisms C_1^* and C_2^* if there is a diagram like
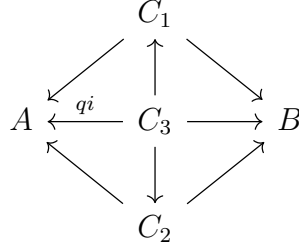
and then there is a natural functor $D^b(X) \to E^b(X)$ which is an equivalence of categories.

Definition 8.24. *A functor $T : C \to D$ between abelian categories is **exact** if it preserves exact sequences and **left exact** if it preserves kernels. An object X is T-**acyclic** if $H^r(TX) = 0$ for all $r > 0$.*

The advantage of T-acyclic objects is that one can use them in place of injective objects when computing derived functors.

Lemma 8.25. *Let T be left exact and $A^* \to X^*$ a quasi-isomorphism of complexes. If X^r is T-acyclic for all $r > 0$ then $RT(A^*)$ can be computed by the complex TX^*.*

Proof. The proof follows from the fact that if the rows of a double complex are exact, then the total complex is exact. The condition that X^r is T-acyclic for all $r > 0$ is about the rows of the double complex being exact once you apply T to an injective resolution. □

Remark 8.26. *Fine, flabby, soft sheaves are all Γ-acyclic. Therefore, sheaf cohomology may be computed with any of those resolutions.*

8.5 Verdier duality

8.5.1 Borel-Moore homology

Borel and Moore defined a sheaf C_{BM}^* whose presheaf of sections is the "locally finite r dimensional chians in U". The cohomology of this sheaf is called the **Borel-Moore homology** $H_{BM}^*(X)$. Moreover, the compactly supported cohomology of this sheaf is the same as the singular homology of X.

$$
H_c^{-i}(X, C_{BM}^*) = H_i(X)
$$

and the stalk cohomology is the local homology of X at x.

$$
H_x^{-r}(X, C_{BM}^*) = H_r(X, X - x)
$$

8.5.2 The dual of a complex

Borel and Moore also gave a way to define the dual $\mathbb{D}(S^*)$ of a complex of sheaves S^*. Unfortunately, the double dual of S^* is not S^*. Later Verdier discovered another way to interpret the BM dual sheaf theoretically:

$$\mathbb{D}(S^*) = R\operatorname{Hom}^*(S^*, \mathbb{D}^*)$$

where $\mathbb{D}^*$ is a universal sheaf called the dualizing complex. Then Verdier showed that there is a canonical quasiisomorphism in the derived category

$$S^* \to \mathbb{D}(\mathbb{D}(S^*))$$

thereby restoring double duality. When we are talking about $R = \mathbb{Z}$-modules, the dualizing complex is precisely $\mathbb{D}^* = R\operatorname{Hom}^*(\mathbb{Z}, \mathbb{D}^*) = \mathbb{D}(\mathbb{Z})$ is the Borel-Moore dual of the constant sheaf, so it is the sheaf of chains.

Definition 8.27. *If $f : X \to Y$ is a continuous map and S^* is a complex of sheaves on Y define $f^!(S^*) = \mathbb{D}_X(f^*(\mathbb{D}_Y(S^*)))$.*

Definition 8.28. *Let $i : Z \to X$ closed subspace and $j : U \to X$ open complement. If S is a sheaf on X define $i^!(S)$ to be the restriction to Z of the presheaf with sectinos supported in Z, that is*

$$i^!(S) = i^*(S^Z) \text{ and } S^Z(V) = \{s \in S(V) \mid \operatorname{supp} s \subset Z\}$$

Thus if $W \subset Z$ is open then

$$i^!(S)(W) = \varinjlim_{V \supset W} S(V)$$

Definition 8.29. *A triangle of morphisms*

$$A^* \to B^* \to C^* \to A^*[1]$$

*in $D^b(X)$ is **distinguished** if it is homotopy equivalent to a mapping cone $C(\phi)$ of a morphism $\phi : A^* \to B^*$. This means that there should be maps between the objects in the two triangles so that the squares commute up to homotopy.*

Theorem 8.30 (Verdier duality). *Let $f : X \to Y$ be a stratified mapping between Whitney stratified spaces. Let A^*, B^*, C^* be constructible sheaves of abelian groups of X, Y, Y respectively. Then $f^*, f^!, Rf_*, Rf_!$ take distinguished triangles to distinguished triangles. There are canonical isomorphisms in $D^b_c(X)$ as follows.*

1. $\mathbb{D}\mathbb{D}(A^*) \cong A^*$

2. $\mathbb{D}^*_X \cong f^!(\mathbb{D}^*_Y)$

3. $f^!(A^*) \cong \mathbb{D}_X(f^*(\mathbb{D}_Y(A^*)))$

4. $Rf_!(A^*) \cong \mathbb{D}_Y(Rf_*(\mathbb{D}_X(A^*)))$ *so $f^!$ is the dual of f^* and $Rf_!$ is the dual of Rf_*.*

5. $f^! R\operatorname{Hom}^*(B^*, C^*) \cong R\operatorname{Hom}^*(f^*B^*, f^!C^*)$

6. $Rf_*(R\operatorname{Hom}^*(A^*, f^!B^*)) \cong R\operatorname{Hom}^*(Rf_!A^*, B^*)$ *is the statement of Verdier duality. This says that $Rf_!$ and $f^!$ are adjoint just as Rf_* and f^* are adjoint.*

7. $Rf_*R\operatorname{Hom}^*(f^*B^*, A^*) \cong R\operatorname{Hom}^*(B^*, Rf_!A^*)$

8. $Rf_!(R\operatorname{Hom}^*(A^*, f^!B^*)) \cong R\operatorname{Hom}^*(Rf_!A^*, B^*)$

9. *If $f : X \to Y$ is an open inclusion then $f^!B^* \cong f^*B^*$*

10. *If $f : X \to Y$ closed inclusion then $Rf_!A^* \cong Rf_*A^*$.*

11. *If $f : X \to Y$ inclusion of oriented submanifold and B^* is cohomologically locally constant on Y then $f^!B^* \cong f^*B^*[\dim Y - \dim X]$.*