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This is my plan for research this week.

1 Tools for solving the problem

1.1 Toric degeneration

Definition 1.1. *A toric degeneration of a projective variety V is a flat morphism $\phi : X \to \mathbb{A}^1$ which trivializes away from the fiber over 0 in $\mathbb{A}^1$*

$$X \setminus \phi^{-1}(0) \cong V \times \mathbb{A}^1 \setminus 0$$

We should really consider toric degenerations which are T-equivariant meaning that the action on the central fiber is an extension of the torus action on X.

One place toric degenrations come from is valuations.

Why should I care?

1.2 Intersection theory

The matter at hand is to expand the class of a subvariety in the geometric basis for cohomology. We know that the cohomology ring of the flag variety has a geometric basis given by the Schubert classes. There is this classical result which sort of characterizes the geometry of this basis.

Theorem 1.2. *Let $X = G/B$ be the flag variety. Then*

1. *The bases $\{[X_w]\}$ and $\{[X^v = X_{w_0 v}]\}$ are dual to each other for the Poincare pairing.*
2. *For any subvariety $Y \subset X$ there is an expansion*

$$[Y] = \sum_{w \in W} a^w(Y)[X_w]$$

where $a^w(Y) = \langle [Y], [X^w] \rangle = \#(Y \cap gX^w)$ for general $g \in G$.

3. *There are nonnegative structure constants for the expansion*

$$[X_v^{w_0 w}] = [X_v][X^w] = \sum_{u \in W} c^u_{vw}[X^u]$$

Moreover the map $f : G/B \to G/P$ induces a map on cohomology which sends any Schubert class $[X_{wP}]$ to the Schubert class $[X_{ww_{0,P}}]$ where $w \in W^P$.

Recall that W^P parametrizes the coset space W/W_P where $W_P = S_n/S_{d_1} \times \cdots \times S_{d_k}$ where $G/P = \mathrm{Fl}(d_1, \ldots, d_k)$ where the indexing of the flag variety denotes the step. In particular $\mathrm{Fl}(1, \ldots, 1)$ is the full flag variety. The elements of W^P are the minimal length coset representatives, the unique permutation so that $w(1) < w(2) < \cdots < w(d_1), w(d_1{+}1) < w(d_1{+}2) < \cdots < w(d_1{+}d_2)$, and so on.

It looks like that this theorem should tell us how to expand the class of any subvariety. This is not the case because we do not even know how to expand $[X_v^{w_0 w}]$ in the Schubert basis. This is the object of Schubert calculus and it is one of the oldest problems in algebraic combinatorics.

For the wonderful compactification, Brion tells us about his geometric basis.

Theorem 1.3. *$A^*_{T\times T}(X)$ has a basis given by the classes*

$$X(\omega, \tau) = [\overline{B \times B^-(\omega, \tau) z_\Phi}]$$

where $\Phi = \{\alpha \in \Sigma \mid \tau(\alpha) \in R^+\}$ and z_Φ is the basepoint of the $G \times G$ orbit corresponding to Φ. The restriction to $G \times B^- \times G/B$ is equal to

$$(D_\omega \otimes D_\tau) \prod_{\alpha \in \Sigma \setminus \Phi} c^{T\times T}(\alpha, -\alpha) \sum_{w \in W_P(\Phi)} [\overline{B^- w B}/B \times \overline{B w_{0,P} B}/B]$$

where $c^{T\times T} : X^*(T \times T) \to A^*_{T\times T}(G/B \times G/B)$ is the characteristic map.
So the $T\times T$-equivariant cohomology ring of the wonderful compactification has a geometric basis given by certain $B \times B^-$ orbit closures, in particular the ones which contain $T \times T$ fixed point.

Our hope is to expand the class of a given $B \times B^-$ orbit closure intersect $B^- \times B$ orbit closure in terms of the geometric basis. There is a subproblem of expanding the class of a general $B \times B^-$ orbit closure in terms of the geometric basis. Of course this should be very hard but we hope to reduce the problem to the Schubert calculus for G/B.

1.3 GKM

A thing that one might try is to write down Brion's basis explicitly in terms of GKM data. Then one can try to write down what an arbitrary $B \times B^-$ orbit closure intersect $B^- \times B$ orbit closure looks like in terms of GKM data. If we can do this, then we can try to expand these classes on the GKM side.

I don't even know how to do this for $\overline{\mathrm{PGL}(2)}$. We should talk to Tara about the computation.

1.4 Basis of anticanonical divisors

The canonical bundle of a nonsingular algebraic variety V is the line bundle $\Omega^n = \omega$, the nth exterior power of the cotangent bundle Ω on V. In particular over the complex numbers it is the determinant bundle of the holomorphic cotangent bundle Ω on V. Bundles have classes

Therefore the canonical bundle defines a divisor class, denoted the canonical class.

Definition 1.4. *An anticanonical divisor is any divisor class $-K$ with K canonical.*

One thing that we worried about is that the boundary divisors of the Richardson stratification are anticanonical in the Richardson variety. Why does this matter?