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These are remarks about intersection theory.
For a long time, algebraic geometers had ad hoc procedures of making sense of intersection multiplicities. It is intuitive that one should count some points with multiplicity in order to get the right answer, but Fulton and MacPherson really laid out groundwork for making sense of these things in the 1970s.

1 Introduction

If $A, B \subset X$ are subvarieties their intersection product should reflect geometry. Let's consider two extreme cases.

If A and B intersect properly, meaning that $\operatorname{codim} A + \operatorname{codim} B = \operatorname{codim}(A \cap B)$, then we have

$$A \cdot B = \sum_{\text{irreducible components}} \text{intersection multiplicity} \cdot C$$

This is why one should care about stratifications. What are the irreducible components of $A \cap B$? Well, if you have a stratification, you can just look at the poset

In the other case if $A = B$ then one has the self intersection formula, which tells us that $A \cdot A$ is the top Chern class of the normal bundle of A in X.

In general we want formulas for the intersection product $A \cdot B$ as rational equivalence classes of cycles on $A \cap B$. This is the subject of Fulton's *Schubert Varieties and Degeneracy Loci*.

2 The hard work

Definition 2.1. *If $f, g \in K[x, y]$ are plane curves F, G the intersection scheme Z is the subscheme of $\mathbb{A}^2$ corresponding to the ideal $\langle f, g \rangle$. The intersection multiplicity of Z at a point p is the dimension of the local ring ${}_{Z,p}$ as a K-vector space.*

Let $V \subset X$ be a subvariety of codimension 1. Then $A = O_{V,X}$ is a local ring of dimension 1.

Recall that $O_{V,X}$ is the set of equivalence classes of pairs (U, f) where U is an open set in X with $V \cap U = \emptyset$ and $f : U \to K$ is a regular function.

Recall that a regular function on an affine variety is a map $f : X \to K$ which is locally a quotient of polynomials. A regular function on an abstract variety is the result of gluing regular functions on affine open sets.

Recall that the field of rational functions on X is the set of equivalence classes of pairs (U, f) where U is an open set in X and $f : U \to K$ is a regular function.

Consider A. It is a local ring with maximal ideal $m = \{[f] \mid f|_V = 0\}$. It is a discrete valuation ring if V is a smooth subvariety of X. The prime ideals of A correspond to the irreducible subvarieties of X containing V.

For a given $r \in A$ we define the order of vanishinging of r along V as

$$\mathrm{ord}_V(r) = \ell_A(A/r)$$

the length of A/r as an A-module, this is defined as $1+$ the length of a maximal chain of submodules.
When A is regular (i.e. V is smooth) then A is a discrete valuation ring and ord_V agrees with the discrete valuation.

In particular recall the following, see Hartshorne.

- When A is a dimension 1 local ring, it is a DVR if and only if it is regular
- Regular is defined as $\dim_k m/m^2 = \dim A$.
- Affine Y is nonsingular at p if and only if $\mathcal{O}_{Y,p}$ is a regular local ring.

For any rational function $r \in K(X)^*$ we define its order by writing it as a fraction f/g for $f, g \in A$ (one can always do this) defining

$$\mathrm{ord}_V(r) = \mathrm{ord}_V(f) - \mathrm{ord}_V(g)$$

In particular we are interested in ord to be a homomorphism

$$\mathrm{ord}(rs) = \mathrm{ord}(r) + \mathrm{ord}(s)$$

for $r, s \in K(X)^*$. For a fixed $r \in K(X)^*$ there are only finitely many codimension 1 subvarieties V so that $\mathrm{ord}_V(r) \neq 0$ (See Fulton appendix).

3 Cycles

Let X be a variety. A K-cycle on X is a formal $\mathbb{Z}$-sum

$$Z = \sum n_i Z_i$$

where $Z_i \subset X$ is a subvariety of dimension k. For any $k+1$-dimensional variety W of X, any $r \in K(W)^*$ we have a divisor (homotopy)

$$[\mathrm{div}(r)] = \sum_{\text{codim } 1} \mathrm{ord}_V(r)[V]$$

Then we say that a K-cycle α is **rationally equivalent** to 0 if there is a there exist finitely many $k+1$-dimensional subvarieties W_i and rational functions $r_i \in K(W_i)^*$ so that

$$\alpha = \sum_i [\mathrm{div}(r_i)]$$

The kth Chow group of X is the group of K-cycles modulo rational equivalence and is denoted $A_k(X)$. We can put the Chow groups together to form the Chow ring $A_*(X)$.

Here come some deep results which I don't understand yet.

If $f : X \to Y$ is proper then subvarieties $V \subset X$ are sent to subvarieties $W = f(V) \subset Y$. There is an induced embedding $K(W) \hookrightarrow K(V)$ which is a finite field extension if W has the same dimension as V (in this case W, V have the same transcendence degree). Put

$$\deg(V/W) = [K(V) : K(W)] \text{ if } \dim V = \dim W$$

and 0 otherwise and define the pushfoward of a proper map

$$f_* : Z_k(X) \to Z_k(Y)$$
$$f_*[V] = \deg(V/W)[W]$$

Then the fact is that f_* preserves rationally equivalent to 0.

4 Another persective

The Chow group of X is the group of cycles of X modulo rational equivalence. Another way of thinking about rational equivalence is the following.

Definition 4.1. *Z_1, Z_2 are rationally equivalent if there exist rationally parametrized family of cycles interpolating between them, i.e. a cycle on $\mathbb{P}^1 \times X$ so that the fibers over t_1, t_2 are Z_1, Z_2. The group of cycles rationally equivalent to 0 is then generated by the differences*

$$[\Phi \cap t_0 \times X] - [\Phi \cap t_1 \times X]$$

for any subvariety $\Phi \subset \mathbb{P}^1 \times X$ not contained in a fiber.

Remark 4.2. *What makes Chow groups usefeul is that under good conditions, the rational equivalence class of $A \cap B$ depends only on the rational equivalence classes of A and B.*

$A(X)$ is a ring under the intersection product.

Theorem 4.3. *Let X be a smooth quasi-projective variety. Then there exists a unique product structure on $A(X)$ so that if A and B are generically transverse, then*

$$[A] \cdot [B] = [A \cap B]$$

This makes $A(X) = \bigoplus A^c(X)$ into an associative ring graded by codimension.

4.1 Via divisors

If X affine, $f \in \mathcal{O}_X$ nonzero, then the irredducible components of the subscheme cut out by f are all of codim 1 (Krull's Principal Ideal Theorem) so to this subscheme we associate a cycle $[\text{div}(f)]$.

If X is not affine we can pick $U \subset X$ open affine $K(X) = K(U)$. This is because rational functions are just regular functions on an open subset, and $V \subset X$ open and $f = g$ on U open. Then we use the fact that if two functions agree on an open set then they are equal on X wherever they are defined.

In particular we get a divisor $\text{div}(f|_U)$ on each U and they agree on overlaps, so we get $\text{div}(\alpha)$ on X itself.

These two notions agree [Fulton 1984]

Example 4.4. *2 points on a curve are birationally equivalent if and only if the curve is birational to $\mathbb{P}^1$.*

5 Yet another perspective

Ravi Vakil has an intersection theory course website which is very good for intuition and examples. In particular he points out where the difficult parts lie.

5.1 Chow groups

- Two points on $\mathbb{P}^1$ are defined to be rationally equivalent
- If $\pi : X \to Y$ flat, then there is a pullbaack. If $\dim X - \dim Y = d$ then the map $\pi^* : H_n(Y) \to H_{n+d}(X)$.
- If $\pi : X \to Y$ is proper (which implies that the image of closed is closed) then we have a pushforward $\pi_* : H_n(X) \to H_n(Y)$.

What should multiplicity of scheme theoretic intersection be?

Example 5.1 (Two planes in $\mathbb{P}^4$). *Put coordinates a, b, c, d, e and consider two planes $X_1 = \langle a, b\rangle$ and $X_2 = \langle c, d\rangle$. Then $X_1 \cup X_2 = \langle a, b\rangle \cup \langle c, d\rangle = \langle a, b, c, d\rangle$ should have degree two ("number of points of intersection with arbitrary linear subspace")*

How should we count this? Let $P = \langle a - c, b - d\rangle$ be a third plane. Then P meets each of X_1, X_2 in a point as long as $\dim X \cap P = 0$. But if P passes through $X_1 \cap X_2$ then we have trouble. Let's see why.

Work on affine open chart $e = 1$. The ring in question is $k[a, b, c, d]$. The planes X_1 and X_2 correspond to ideals $\langle a, b\rangle$ and $\langle c, d\rangle$. Then we consider the ring

$$k[a, b, c, d]/\langle a, b, c, d\rangle \cup \langle a - c, b - d\rangle \cong k[a, b]/\langle ab, a^2, b^2\rangle$$

The problem is that this ring is 3-diml as a vector space over k. What's the fix?

The current formula says that we should look at

$$\dim_k R/I_1 \otimes R/I_2$$

But Serre says that $\otimes$ is not exact. It is only right exact and there is cohomology

$$\begin{aligned}
&\to \mathrm{Tor}^2(M, A) \to \mathrm{Tor}^2(M, B) \to \mathrm{Tor}^2(M, C)\\
&\to \mathrm{Tor}^1(M, A) \to \mathrm{Tor}^1(M, B) \to \mathrm{Tor}^1(M, C)\\
&\to M \otimes A \to M \otimes B \to M \otimes C \to 0
\end{aligned}$$

Instead Serre says you should look at the Tor groups.

$$\chi = \sum (-1)^i \dim \mathrm{Tor}_i(R/I_1, R/I_2)$$

and this is the right answer.

Cohen Macauley is about higher Tor groups vanishing.

A common thing to do is to localize at the generic point of a subvariety X of a scheme Y. What this means is that we consider rational functions on Y defined on a dense open subset of X, of dimension $= \dim Y - \dim X$.

Recall that the points of Y correspond to irreducible subvariety of Y, and the closed points correspond to the "old points" of Y.

Moreover points of $\operatorname{Spec} O_{X,Y}$ correspond to prime ideals correspond to irreducible subvarieties of Y containing X. The maximal ideal corresponds to X itself.

Example 5.2. *What is the order of vanishing of y/x at 0 in $y^2 = x^3$? What about $y^2 = x^3 + x^2$? Note that these guys are singular.*

6 References

- Fulton's *Intersection Theory*
- Eisenbud Harris *3264 and all that*
- Ravi Vakil's Course Website