

Title

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Abstract

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1 Virtual normal bundles for stacks

For a smooth embedding of smooth algebraic stacks $f : X \longrightarrow Y$, the *virtual normal bundle* is the class in $K^0(X)$ given by

$$N_{X/Y} = [f^*T_Y] - [T_X].$$

To compute this class one compares the tangent complexes of X and Y . In the setting of moduli of principal bundles, the comparison arises from the short exact sequence of G -representations

$$0 \rightarrow \mathfrak{g}_\xi \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{g}_\xi \rightarrow 0.$$

2 Tangent complex of $\mathrm{Bun}_G(C)$

Let C be a smooth projective curve and

$$\pi : C \times \mathrm{Bun}_G \rightarrow \mathrm{Bun}_G$$

the projection. Denote by \mathcal{P} the universal G -bundle and set

$$\mathrm{ad}(\mathcal{P}) = \mathcal{P} \times^G \mathfrak{g}.$$

Standard deformation theory of principal bundles gives

- infinitesimal automorphisms: $H^0(C, \mathrm{ad} P)$,
- infinitesimal deformations: $H^1(C, \mathrm{ad} P)$,
- obstructions: $H^2(C, \mathrm{ad} P) = 0$.

Hence the tangent complex of Bun_G is

$$\mathbb{T}_{\mathrm{Bun}_G} \simeq R\pi_* \mathrm{ad}(\mathcal{P})[1],$$

where the shift places automorphisms in degree -1 and deformations in degree 0 .

3 Harder–Narasimhan strata

Fix a dominant cocharacter $\xi \in X_*(T)_+$. It determines group–theoretic data

- a parabolic subgroup $P(\xi) \subset G$,
- its unipotent radical $U(\xi)$,
- the Levi quotient $L(\xi) = P(\xi)/U(\xi)$.

3.1 Reductions of structure group

Let P be a principal G –bundle on C .

Definition. A *reduction of P to $P(\xi)$* is a principal $P(\xi)$ –bundle \mathcal{P}_ξ on C equipped with an isomorphism

$$P \simeq \mathcal{P}_\xi \times^{P(\xi)} G.$$

Equivalently, it is a section $s : C \rightarrow P/P(\xi)$ of the associated fibration with fiber $G/P(\xi)$.

From such a reduction one obtains the induced Levi bundle

$$\mathrm{gr}_\xi(P) = \mathcal{P}_\xi/U(\xi),$$

a principal $L(\xi)$ –bundle, playing the role of an associated graded object.

3.2 Canonical (Harder–Narasimhan) reduction

Among all reductions of P to $P(\xi)$ there is a unique *canonical* one, characterized by:

1. **Semistability of the Levi bundle:** The bundle $\mathrm{gr}_\xi(P)$ is semistable as a principal $L(\xi)$ –bundle.
2. **Prescribed numerical type:** For every character $\chi : P(\xi) \rightarrow \mathbb{G}_m$,

$$\deg(\chi_*(\mathcal{P}_\xi)) = \langle \chi, \xi \rangle.$$

3. **Maximal destabilizing property:** For any reduction \mathcal{P}_Q of P to a parabolic $Q \subset G$ with type $\xi(Q)$,

$$\xi(Q) \leq \xi$$

in the dominance order.

The Harder–Narasimhan stratum \mathfrak{M}_ξ consists of pairs (P, \mathcal{P}_ξ) where \mathcal{P}_ξ is the canonical reduction of P .

4 Tangent complex of the stratum

Let $\mathfrak{M}_{L(\xi),\xi}^{\text{ss}}$ be the moduli of semistable Levi bundles of type ξ . Denote by \mathcal{P}_ξ the universal $P(\xi)$ -bundle and set

$$\text{ad}_\xi(\mathcal{P}_\xi) = \mathcal{P}_\xi \times^{L(\xi)} \mathfrak{g}_\xi.$$

The tangent complex of the Levi core is

$$\mathbb{T}_{\mathfrak{M}_{L(\xi),\xi}^{\text{ss}}} \simeq R\pi_* \text{ad}_\xi(\mathcal{P}_\xi)[1].$$

5 The virtual normal complex

Over the stratum the universal G -bundle is induced from \mathcal{P}_ξ , and the exact sequence of Lie algebras gives a distinguished triangle

$$R\pi_*(\mathcal{P}_\xi \times^{L(\xi)} \mathfrak{g}_\xi)[1] \rightarrow R\pi_*(\mathcal{P}_\xi \times^{L(\xi)} \mathfrak{g})[1] \rightarrow R\pi_*(\mathcal{P}_\xi \times^{L(\xi)} (\mathfrak{g}/\mathfrak{g}_\xi))[1].$$

The first term represents the tangent complex of the stratum and the middle term the restriction of the tangent complex of Bun_G . Hence the cone is the *virtual normal complex*

$$\nu_\xi \simeq R\pi_*(\mathcal{P}_\xi \times^{L(\xi)} (\mathfrak{g}/\mathfrak{g}_\xi))[1].$$

In K -theory,

$$[\nu_\xi] = [f^* \mathbb{T}_{\text{Bun}_G}] - [\mathbb{T}_{\mathfrak{M}_{L(\xi),\xi}^{\text{ss}}}].$$

6 Lattice interpretation of parahoric subgroups

Let $LG = G((z))$ be the loop group of a connected reductive group G over \mathbb{C} . Let $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$ be the affine simple roots, and let $\{\eta_0, \eta_1, \dots, \eta_r\}$ be the dual vertices of the fundamental alcove, characterized by $\langle \alpha_i, \eta_j \rangle = \delta_{ij}/n_i$.

For any subset $I \subset \{0, \dots, r\}$, let F_I be the facet obtained by imposing $\alpha_i = 0$ for $i \in I$ and $\alpha_j > 0$ for $j \notin I$. The associated *parahoric subgroup* of LG is

$$\mathcal{P}_I = \text{Stab}_{LG}(F_I),$$

equivalently the $\mathbb{C}[[z]]$ -points of the Bruhat–Tits parahoric group scheme attached to F_I .

- The singleton subsets $I = \{i\}$ correspond to the *vertices* η_i and hence to the *maximal parahoric subgroups* \mathcal{P}_{η_i} .
- For $i > 0$, the reduction map $\mathcal{P}_{\eta_i} \twoheadrightarrow P_i \subset G$ identifies the special fiber with the standard maximal parabolic P_i determined by the finite simple root α_i . See Kumar's book.

- The vertex η_0 is the special (hyperspecial) vertex of the alcove, and its parahoric is

$$\mathcal{P}_{\eta_0} = G[[z]] = L^+G.$$

Proposition 6.1 (Vertex parahorics for SL_3). Let $K = \mathbb{C}((z))$ and $\mathcal{O} = \mathbb{C}[[z]]$. Fix the standard basis e_1, e_2, e_3 of K^3 and define lattices

$$L_0 = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{O}e_3,$$

$$L_1 = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus z\mathcal{O}e_3,$$

$$L_2 = \mathcal{O}e_1 \oplus z\mathcal{O}e_2 \oplus z\mathcal{O}e_3.$$

These represent the three vertices $\{\eta_0\}, \{\eta_1\}, \{\eta_2\}$ of a fundamental alcove in the Bruhat–Tits building of $SL_3(K)$.

The corresponding maximal parahoric subgroups are

$$\mathcal{P}_{\eta_i} = \text{Stab}_{SL_3(K)}(L_i).$$

Explicitly:

(i) **Hyperspecial vertex η_0 .**

$$\mathcal{P}_{\eta_0} = SL_3(\mathcal{O}) = \left\{ g = (g_{ij}) \in SL_3(K) \mid g_{ij} \in \mathcal{O} \text{ for all } i, j \right\}.$$

(ii) **Vertex η_1 (type 2|1).**

$$\mathcal{P}_{\eta_1} = \left\{ g \in SL_3(K) \mid gL_1 = L_1 \right\} = \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} & z^{-1}\mathcal{O} \\ \mathcal{O} & \mathcal{O} & z^{-1}\mathcal{O} \\ z\mathcal{O} & z\mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL_3(K) \right\}.$$

Equivalently, if $D_1 = \text{diag}(1, 1, z)$, then

$$\mathcal{P}_{\eta_1} = \{ g \in SL_3(K) \mid D_1^{-1}gD_1 \in SL_3(\mathcal{O}) \}.$$

(iii) **Vertex η_2 (type 1|2).**

$$\mathcal{P}_{\eta_2} = \left\{ g \in SL_3(K) \mid gL_2 = L_2 \right\} = \left\{ \begin{pmatrix} \mathcal{O} & z^{-1}\mathcal{O} & z^{-1}\mathcal{O} \\ z\mathcal{O} & \mathcal{O} & \mathcal{O} \\ z\mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL_3(K) \right\}.$$

Equivalently, with $D_2 = \text{diag}(1, z, z)$,

$$\mathcal{P}_{\eta_2} = \{ g \in SL_3(K) \mid D_2^{-1}gD_2 \in SL_3(\mathcal{O}) \}.$$

Remark 6.2 (Geometric interpretation). The reductions modulo z of these parahorics recover the classical parabolics of $SL_3(\mathbb{C})$:

$$\begin{aligned}\mathcal{P}_{\eta_0}/\mathcal{P}_{\eta_0}^+ &\cong SL_3(\mathbb{C}), \\ \mathcal{P}_{\eta_1}/\mathcal{P}_{\eta_1}^+ &\cong P_{(2|1)}(\mathbb{C}) \quad (\text{stabilizer of a plane}), \\ \mathcal{P}_{\eta_2}/\mathcal{P}_{\eta_2}^+ &\cong P_{(1|2)}(\mathbb{C}) \quad (\text{stabilizer of a line}).\end{aligned}$$

Thus the vertices of the affine building correspond to integral models whose special fibers are the three standard parabolics of SL_3 .

Therefore we see that specifying a parahoric structure of type \mathcal{P}_{η_i} at p gives us a reduction of the structure group of the bundle at p to the parabolic P_i , i.e. a choice of i -dimensional subspace of the fiber at p .

7 Spreading out stacks

Let \mathcal{C} be a site and let $U \in \text{Ob}(\mathcal{C})$ such that the representable presheaf h_U is a sheaf. Denote by

$$j : \mathcal{C}/U \longrightarrow \mathcal{C}$$

the localization functor.

Lemma 7.1 (Stacks Project, §8.13). There is a 2-equivalence between

1. stacks $\mathcal{S} \rightarrow \mathcal{C}/U$, and
2. pairs (\mathcal{T}, p) consisting of a stack $\mathcal{T} \rightarrow \mathcal{C}$ together with a morphism of stacks $p : \mathcal{T} \rightarrow \mathcal{C}/U$ over \mathcal{C} .

Definition 7.2 (Pushforward of a stack over U). Let $p : \mathcal{S} \rightarrow \mathcal{C}/U$ be a stack over the slice site. Define a stack $j_! \mathcal{S} \rightarrow \mathcal{C}$ as follows:

1. as a category fibered in groupoids, $j_! \mathcal{S} = \mathcal{S}$;
2. the structure morphism is the composition $j \circ p : j_! \mathcal{S} \rightarrow \mathcal{C}$.

The assumption that h_U is a sheaf ensures that descent for morphisms to U is effective, so $j_! \mathcal{S}$ is a stack over \mathcal{C} .

Conversely, if $\mathcal{T} \rightarrow \mathcal{C}$ is a stack endowed with a morphism $p : \mathcal{T} \rightarrow \mathcal{C}/U$, then \mathcal{T} is automatically a stack over the slice site \mathcal{C}/U .

Theorem 7.3. The constructions $\mathcal{S} \mapsto j_! \mathcal{S}$ and $(\mathcal{T}, p) \mapsto \mathcal{T}$ are mutually inverse, giving a 2-equivalence between stacks over \mathcal{C}/U and stacks over \mathcal{C} equipped with a morphism to \mathcal{C}/U .

7.1 Change of base schemes

Given a morphism $S \rightarrow S'$ of base schemes, any algebraic stack over S can be viewed as an algebraic stack over S' .

Lemma 7.4. Let Sch_{fppf} be the big fppf site. Let $S \rightarrow S'$ be a morphism in this site. The constructions described above give an isomorphism of 2-categories

$$\{ \text{2-category of algebraic stacks } \mathcal{X} \text{ over } S \} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{2-category of pairs } (\mathcal{X}', f) \text{ consisting of an} \\ \text{algebraic stack } \mathcal{X}' \text{ over } S' \text{ and a morphism} \\ f : \mathcal{X}' \rightarrow (\text{Sch}/S)_{\text{fppf}} \text{ of algebraic stacks over } S' \end{array} \right\}.$$

Definition 7.5 ([?, Definition 94.19.2]). Let Sch_{fppf} be the big fppf site. Let $S \rightarrow S'$ be a morphism in this site. If $p : \mathcal{X} \rightarrow (\text{Sch}/S)_{\text{fppf}}$ is an algebraic stack over S , then \mathcal{X} viewed as an algebraic stack over S' is the algebraic stack

$$\mathcal{X} \longrightarrow (\text{Sch}/S')_{\text{fppf}}$$

obtained by applying Construction A of Lemma 94.19.1 to \mathcal{X} .

Conversely, if we start with an algebraic stack \mathcal{X}' over S' and we want to get an algebraic stack over S , we consider the 2-fiber product

$$\mathcal{X}'_S = (\text{Sch}/S)_{\text{fppf}} \times_{(\text{Sch}/S')_{\text{fppf}}} \mathcal{X}',$$

which is an algebraic stack over S . Moreover, it comes equipped with a natural 1-morphism

$$p : \mathcal{X}'_S \longrightarrow (\text{Sch}/S)_{\text{fppf}},$$

and hence it corresponds in a canonical way to an algebraic stack over S .

8 Induced stacks over the moduli of curves

Let $S = \text{Spec } \mathbb{C}[[s]]$. Recall that Solis fixes a curve C over S whose generic fiber is smooth and whose special fiber is a nodal curve with a single node. In particular, C is classified by a morphism $f : S \rightarrow \overline{\mathcal{M}}_{g,I}$. Solis defines an S -stack whose B -points are given by twisted local modifications C'_B of our fixed family C/S and admissible G -bundles P'_B on C'_B . He christens this stack $\mathcal{X}_G(C)$ and shows that it is algebraic over S , locally of finite type over S , and complete over S .

In general, let

$$\pi : C \longrightarrow S = \text{Spec } \mathbb{C}[[s]]$$

be a stable I -marked curve and let

$$f : S \longrightarrow \overline{\mathcal{M}}_{g,I}$$

be the induced morphism to the moduli stack of stable curves.

Suppose we are given a stack of “widgets on C ” over S ,

$$\mathcal{W}_C \longrightarrow S, \quad \mathcal{W}_C(U) = \{\text{widgets on } C_U = C \times_S U\}.$$

Definition 8.1. The *induced widget stack over $\overline{\mathcal{M}}_{g,I}$* is the stack

$$\mathcal{W} := j_!(\mathcal{W}_C) \longrightarrow \overline{\mathcal{M}}_{g,I}$$

obtained from the localization construction with $U = S$ and $j : (\text{Sch}/S) \rightarrow (\text{Sch}/\overline{\mathcal{M}}_{g,I})$.

Note that the general theory described above ensures that \mathcal{W} is an algebraic stack over $\overline{\mathcal{M}}_{g,I}$.

Proposition 8.2 (Description on test schemes). Let $g : T \rightarrow \overline{\mathcal{M}}_{g,I}$ be a test scheme. Set

$$T_S := T \times_{\overline{\mathcal{M}}_{g,I}} S.$$

Then

$$\boxed{\mathcal{W}(T) = \mathcal{W}_C(T_S).}$$

Interpretation. The fiber product T_S parametrizes pairs

$$(t \in T, s \in S, \alpha : C_t \xrightarrow{\sim} C_s),$$

that is, points of T whose associated curve is identified with a base change of the fixed family C/S . Evaluating the original stack \mathcal{W}_C on this scheme produces widgets on the identified curve, which by definition gives $\mathcal{W}(T)$. \square

9 Cartier divisors on algebraic stacks

Let \mathcal{X} be an algebraic stack. An effective Cartier divisor on \mathcal{X} is one of the following equivalent pieces of data:

1. **Ideal sheaf description:** A quasi-coherent sheaf of ideals

$$\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$$

such that étale-locally on \mathcal{X} the ideal is generated by a single non-zero-divisor.

Equivalently, \mathcal{I} is an invertible $\mathcal{O}_{\mathcal{X}}$ -module embedded in $\mathcal{O}_{\mathcal{X}}$.

2. **Local equation description:** There exists a smooth surjective atlas $p : U \rightarrow \mathcal{X}$ with U a scheme such that the pullback ideal $p^{-1}\mathcal{I} \subset \mathcal{O}_U$ is an ordinary Cartier divisor on the scheme U , and the two pullbacks to $U \times_{\mathcal{X}} U$ agree.

3. **Line bundle and section description:** A line bundle \mathcal{L} on \mathcal{X} , together with a global section $s \in H^0(\mathcal{X}, \mathcal{L})$ such that étale-locally the pair (\mathcal{L}, s) is isomorphic to $(\mathcal{O}_{\mathcal{X}}, f)$ with f a non-zero-divisor. The divisor is the vanishing locus $V(s)$.

Definition 9.1 (Simple normal crossing divisors). Let \mathcal{X} be an algebraic stack locally of finite type over a field and $\mathcal{D} \subset \mathcal{X}$ a closed substack. We say \mathcal{D} is a *simple normal crossing divisor* on \mathcal{X} if:

1. \mathcal{D} is a Cartier divisor on \mathcal{X} , and
2. there exists a smooth surjective morphism (atlas) $p : U \rightarrow \mathcal{X}$ with U a scheme such that $D_U := p^{-1}(\mathcal{D}) \subset U$ is an SNC divisor in the usual scheme-theoretic sense.

10 The determinant line bundle on $\mathcal{X}_{G,g,I}$.

Let

$$\pi : \mathcal{C} \rightarrow \mathcal{X}_{G,g,I}$$

be the universal (twisted) curve/modification, and let \mathcal{P} be the universal torsor under the relevant Bruhat–Tits/parahoric group scheme \mathcal{G} on \mathcal{C} .

1. A representation of the group scheme \mathcal{G} .

Pick $\rho : G \rightarrow \mathrm{GL}(V)$ and at each parahoric type a compatible “integral model” (lattice) so that ρ extends to a morphism of group schemes

$$\rho : \mathcal{G} \rightarrow \mathrm{GL}(\mathcal{V})$$

on \mathcal{C} . Equivalently, you get an associated vector bundle

$$\mathcal{V}_\rho := \mathcal{P} \times^{\mathcal{G}} \mathcal{V}$$

on \mathcal{C} .

2. Perfectness of $R\pi_*\mathcal{V}_\rho$.

For a proper 1-dimensional DM morphism π and a vector bundle \mathcal{V}_ρ , the derived pushforward $R\pi_*\mathcal{V}_\rho$ is a perfect complex of amplitude $[0, 1]$ in the usual settings (the same hypothesis Teleman–Woodward implicitly uses on $\Sigma \times \mathfrak{M} \rightarrow \mathfrak{M}$).

3. Determinant functor.

Once $R\pi_*\mathcal{V}_\rho$ is perfect, the determinant construction gives an invertible sheaf

$$\mathcal{L}_{\det}(\rho) := \det R\pi_*(\mathcal{V}_\rho) \in \mathrm{Pic}(\mathcal{X}_{G,g,I}).$$

This is functorial in families, so it really is a line bundle on the stack.

Why this is compatible with the smooth-locus determinant.

On the open substack where your object is an honest G -bundle on a smooth curve, \mathcal{G} restricts to the constant group scheme G and \mathcal{V}_ρ restricts to the usual associated bundle \mathcal{E}_ρ . Then

$$\det R\pi_*(\mathcal{V}_\rho)|_{\text{smooth locus}} = \det R\pi_*(\mathcal{E}_\rho),$$

so it extends the familiar determinant-of-cohomology line bundle.

10.1 Bruhat–Tits (parahoric) group schemes à la Rapoport and comparison

We briefly recall the notion of a Bruhat–Tits (parahoric) group scheme used by Rapoport–Pappas and explain why Solís’ local group schemes coincide with it.

Rapoport–Pappas. Let S be a scheme and let $C \rightarrow S$ be a (possibly nodal) curve. Fix a connected reductive group G over the generic locus of C . A *Bruhat–Tits group scheme* over C (with generic fiber G) is a smooth affine group scheme $\mathcal{G} \rightarrow C$ such that

- (i) the restriction of \mathcal{G} to the open subset where $C \rightarrow S$ is smooth (equivalently, away from the marked points / nodes where we impose level structure) is the constant group scheme G ;
- (ii) for every geometric point $x \in C$ lying over a point of S , the base change of \mathcal{G} to the completed local ring $\widehat{\mathcal{O}}_{C,x}$ is isomorphic to the *Bruhat–Tits parahoric model* of G corresponding to some facet in the affine building of G over the local field $\text{Frac}(\widehat{\mathcal{O}}_{C,x})$.

Equivalently, after choosing a local parameter z at x (so $\widehat{\mathcal{O}}_{C,x} \simeq \mathbb{C}[[z]]$ in our equal-characteristic setting), condition (ii) says that $\mathcal{G}(\mathbb{C}[[z]]) \subset G(\mathbb{C}((z)))$ identifies with a *parahoric subgroup* (the stabilizer of a facet) and that \mathcal{G} is the corresponding smooth affine $\mathbb{C}[[z]]$ -model. (See e.g. [?] and, for the equal-characteristic global picture used in geometric representation theory, [?].)

Solís. In our situation we fix, at each marked point / (stacky) node, a parahoric subgroup $\mathcal{P} \subset G((z))$ (for instance a vertex parahoric \mathcal{P}_{η_i} , or a standard parabolic parahoric $L_P^+ G = \{\gamma \in G[[z]] \mid \gamma(0) \in P\}$). Solís’ construction produces a smooth affine group scheme $\mathcal{G}^\mathcal{P}$ over the formal disc $D = \text{Spec } \mathbb{C}[[z]]$ characterized by

$$\mathcal{G}^\mathcal{P}(D) = \mathcal{P}, \quad \mathcal{G}^\mathcal{P}(D^\times) = G((z)),$$

with restriction map induced by the inclusion $\mathcal{P} \hookrightarrow G((z))$. This is exactly the Bruhat–Tits parahoric model attached to the facet stabilized by \mathcal{P} : by Bruhat–Tits theory, parahoric subgroups of $G((z))$ are precisely the $\mathbb{C}[[z]]$ -points of (unique up to unique isomorphism) smooth affine parahoric group schemes over $\text{Spec } \mathbb{C}[[z]]$. Consequently, the global group scheme on the curve obtained by gluing the constant group scheme G away from the boundary with these local models at the chosen points satisfies Rapoport–Pappas’ conditions and is therefore a Bruhat–Tits (parahoric) group scheme in their sense.

Representations (what data is needed). To define the determinant line bundle we need a representation of the group scheme \mathcal{G} on the universal curve. Concretely, the input consists of

- (1) an algebraic representation $\rho : G \rightarrow \mathrm{GL}(V)$ over \mathbb{C} ;
- (2) for each parahoric type that occurs on our curves (each facet / vertex label), a choice of an \mathcal{P} -stable lattice $\Lambda \subset V \otimes_{\mathbb{C}} \mathbb{C}((z))$, i.e. $\rho(\mathcal{P}) \cdot \Lambda \subset \Lambda$.

A choice of such lattices is equivalent to an extension of ρ to a morphism of group schemes on the disc $\rho : \mathcal{G}^{\mathcal{P}} \rightarrow \mathrm{GL}(\Lambda)$. Gluing these local extensions with the constant representation away from the boundary yields a global morphism $\rho : \mathcal{G} \rightarrow \mathrm{GL}(\mathcal{V})$ on the universal curve and hence an associated vector bundle $\mathcal{V}_{\rho} = \mathcal{P} \times^{\mathcal{G}} \mathcal{V}$.

10.2 Determinant of cohomology line bundles and level

We explain how to construct a natural “determinant of cohomology” line bundle on the compactified stack $\mathcal{X}_{G,g,I}$ of admissible G -bundles on twisted local modifications, and how its numerical behavior is governed by the *level* (i.e. an invariant bilinear form on \mathfrak{g}).

Universal curve and universal torsor

By definition, an object of $\mathcal{X}_{G,g,I}$ over a test scheme B consists of a family of twisted local modifications

$$\pi_B : C'_B \rightarrow B$$

of the pullback curve $C_B \rightarrow B$ together with an admissible G -bundle in the sense of Solís. Equivalently (using Solís’ comparison between parahoric torsors on the coarse space and honest G -bundles on the associated stacky curve), this is a torsor under a sheaf of groups \mathcal{G}_B on C'_B such that $\mathcal{G}_B|_{C'_B \setminus \{\text{nodes/stacky points}\}} \cong G$, while at each (node/stacky) point it is given by a fixed parahoric Bruhat–Tits local model. Gluing in families yields the universal twisted curve

$$\pi : \mathcal{C} \longrightarrow \mathcal{X}_{G,g,I}$$

together with a universal sheaf of groups \mathcal{G} on \mathcal{C} and a universal \mathcal{G} -torsor \mathcal{P} on \mathcal{C} .

Representations and associated vector bundles

To a usual algebraic representation $\rho : G \rightarrow \mathrm{GL}(V)$ one must add local integral data at each allowed parahoric type: for each parahoric subgroup $\mathcal{P}_{\eta} \subset G((z))$ occurring on our objects, choose a \mathcal{P}_{η} -stable lattice

$$\Lambda_{\eta} \subset V \otimes_{\mathbb{C}} \mathbb{C}((z)), \quad \rho(\mathcal{P}_{\eta}) \cdot \Lambda_{\eta} \subset \Lambda_{\eta}.$$

Such a choice is equivalent to an extension of ρ to a morphism of group schemes on the formal disc (the corresponding Bruhat–Tits model)

$$\rho : \mathcal{G}_{\eta} \longrightarrow \mathrm{GL}(\Lambda_{\eta}).$$

These local extensions glue with the constant representation away from the boundary to give a global morphism of group sheaves on \mathcal{C}

$$\rho : \mathcal{G} \longrightarrow \mathrm{GL}(\mathcal{V}),$$

hence an associated vector bundle on the universal curve

$$\mathcal{E}_\rho := \mathcal{P} \times^{\mathcal{G}} \mathcal{V} \in \mathrm{Vect}(\mathcal{C}).$$

Determinant of cohomology on $\mathcal{X}_{G,g,I}$

Since $\pi : \mathcal{C} \rightarrow \mathcal{X}_{G,g,I}$ is a proper flat family of (twisted) curves and \mathcal{E}_ρ is locally free on \mathcal{C} , the derived pushforward $R\pi_*\mathcal{E}_\rho$ is a perfect complex of Tor-amplitude $[0, 1]$ on $\mathcal{X}_{G,g,I}$. Therefore the Knudsen–Mumford determinant functor produces an *honest* line bundle

$$\mathcal{L}_{\mathrm{det}}(\rho) := \det R\pi_*(\mathcal{E}_\rho)^{-1} \in \mathrm{Pic}(\mathcal{X}_{G,g,I}).$$

By construction, on the open substack where the universal curve is smooth and the admissible object is an honest G -bundle, \mathcal{E}_ρ restricts to the usual associated vector bundle, hence $\mathcal{L}_{\mathrm{det}}(\rho)$ restricts to the classical determinant of cohomology line bundle on Bun_G .

Level and dependence on the representation

The isomorphism class of $\mathcal{L}_{\mathrm{det}}(\rho)$ depends on ρ . Recall that for G simple, ρ defines an invariant symmetric bilinear form on $\mathfrak{g} = \mathrm{Lie}(G)$

$$\kappa_\rho(x, y) := \mathrm{Tr}(d\rho(x) d\rho(y)), \quad x, y \in \mathfrak{g},$$

and since $\mathrm{Sym}^2(\mathfrak{g})^G$ is one-dimensional for G simple, there is a unique scalar m_ρ (an integer after fixing the basic normalization) such that

$$\kappa_\rho = m_\rho \kappa_{\mathrm{bas}}.$$

We call m_ρ the *level* (Dynkin index) of ρ . In particular, tensor products and direct sums behave additively:

$$\mathcal{L}_{\mathrm{det}}(\rho_1 \oplus \rho_2) \cong \mathcal{L}_{\mathrm{det}}(\rho_1) \otimes \mathcal{L}_{\mathrm{det}}(\rho_2), \quad m_{\rho_1 \oplus \rho_2} = m_{\rho_1} + m_{\rho_2}.$$

The level governs the leading (quadratic) term in the Hilbert–Mumford/ Θ numerical invariant associated to $\mathcal{L}_{\mathrm{det}}(\rho)$: along a degeneration of type ξ (dominant rational coweight) the weight of $\mathcal{L}_{\mathrm{det}}(\rho)$ grows as

$$\mathrm{wt}_\xi(\mathcal{L}_{\mathrm{det}}(\rho)) \sim -m_\rho \langle \xi, \xi \rangle_{\mathrm{bas}} + (\text{terms at most linear in } \xi),$$

For $\mathrm{Bun}_G(C)$ of a smooth curve and simple simply connected G , the level m_ρ is the only invariant of ρ that matters for the isomorphism class of $\mathcal{L}_{\mathrm{det}}(\rho)$, and $\mathcal{L}_{\mathrm{det}}(\rho)$ is ample if and only if $m_\rho > 0$.

The ample generator of $\text{Pic}(\text{Bun}_G(C)) \cong \mathbb{Z}$ is the determinant line bundle associated to the adjoint representation, which has level $m_{\text{adj}} = 2h^\vee$ where h^\vee is the dual Coxeter number of G .

For $\mathcal{X}_{G,g,I}$, the situation is more complicated and it is not clear at all. In order to get a stratification we need a condition like ℓ is positive on unstable directions (so μ goes to $-\infty$ appropriately), and b is positive definite on nontrivial filtrations.

10.3 The diagonal construction at a twisted node

Let $C_{0,[k]}$ be a twisted nodal curve with a single twisted node p . Let C_0 be its coarse moduli space and, by abuse of notation, also write $p \in C_0$ for the node. Assume the stabilizer of $p \in C_{0,[k]}$ is μ_k . Then

$$C_{0,[k]} \times_{C_0} D_0 \cong [D_0^{1/k} / \mu_k],$$

where

$$D_0 = \text{Spec } \mathbb{C}[[x, y]] / (xy), \quad D_0^{1/k} = \text{Spec } \mathbb{C}[[u, v]] / (uv),$$

with $u^k = x$ and $v^k = y$.

For a parahoric subgroup \mathcal{P} with Levi decomposition $\mathcal{P} = L \ltimes U$, set

$$\mathcal{P}^\Delta = \Delta(L) \ltimes (U \times U).$$

One constructs a sheaf of groups \mathcal{G}^Δ over C_0 such that

$$\mathcal{G}^\Delta(\widehat{\mathcal{O}}_{C_0,p}) = \mathcal{P}^\Delta, \quad \mathcal{G}^\Delta|_{C_0-p} = G^{\text{std}}$$

Let $\mathcal{M}_{\mathcal{G}^\Delta}(C_0)$ denote the moduli stack of \mathcal{G}^Δ -torsors on C_0 and let $T_{\mathcal{G}^\Delta}(C_0)$ denote the moduli space of pairs (\mathcal{F}, τ) where $\mathcal{F} \in \mathcal{M}_{\mathcal{G}^\Delta}(C_0)$ and τ is a trivialization of \mathcal{F} over $C_0 - p$. Define $T_{\mathcal{G}^\Delta}(D_0)$ similarly.

Let $\eta \in \text{Hom}(\mathbb{C}^\times, T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and consider the moduli stack $\mathcal{M}_{G,\eta}(C_{0,[k]})$ of G -bundles on $C_{0,[k]}$ with equivariant structure at p determined by η . In particular, a choice of η determines by restriction to $\mu_k \subset \mathbb{C}^\times$ a μ_k -equivariant structure on the fiber of the G -bundle at p .

Remark 10.1. Near a twisted node p , the curve looks like

$$[D^{1/k} / \mu_k]$$

where

$$D^{1/k} = \text{Spec } \mathbb{C}[[u, v]] / (uv), \quad \zeta \cdot (u, v) = (\zeta u, \zeta^{-1} v).$$

So the geometric point has stabilizer group μ_k and a G -bundle on this stack must carry a compatible μ_k -action.

Note that there are two ways that the cocharacter data is being used, on the coarse side to determine the parahoric type and on the stacky side to determine the equivariant structure at the node.

Let $T_{G,\eta}(C_{0,[k]})$ denote the moduli space of pairs (P, τ) with $P \in \mathcal{M}_{G,\eta}(C_{0,[k]})$ and τ a trivialization of P over $C_{0,[k]} - p$. Define $T_{G,\eta}([D_0^{1/k}/\mu_k])$ similarly.

Proposition 10.2. Suppose $k\eta \in \text{Hom}(\mathbb{C}^\times, T)$ and set $\mathcal{P} = \mathcal{P}(\eta)$. Choose k th roots u, v of x, y so that $D_0^{1/k} = \mathbb{C}[[u, v]]/(uv)$. Let

$$i_{0,[k]} : [D_0^{1/k}/\mu_k] \rightarrow C_{0,[k]}, \quad i_0 : D_0 \rightarrow C_0$$

be the natural maps. Set

$$G_{u,v}^\Delta = \{(g_1, g_2) \in L_u^+ G \times L_v^+ G \mid g_1(0) = g_2(0)\}.$$

Then there are isomorphisms

$$T_{\mathcal{G}^\Delta}(D_0) \xleftarrow{i_0^*} T_{\mathcal{G}^\Delta}(C_0) \xrightarrow{\Xi_{C_0}} T_{G,\eta}(C_{0,[k]}) \xrightarrow{i_{0,[k]}^*} T_{G,\eta}([D_0^{1/k}/\mu_k]),$$

and these are compatible with the loop group descriptions

$$\begin{aligned} T_{\mathcal{G}^\Delta}(C_0) &\longrightarrow LG \times LG/\mathcal{P}^{\Delta(\eta)}, \\ T_{G,\eta}(C_{0,[k]}) &\longrightarrow (L_u G \times L_v G)^{\mu_k}/(G_{u,v}^\Delta)^{\mu_k}. \end{aligned}$$

Moreover Ξ_{C_0} descends to an isomorphism of stacks

$$\Xi : \mathcal{M}_{\mathcal{G}^\Delta}(C_0) \xrightarrow{\sim} \mathcal{M}_{G,\eta}(C_{0,[k]}).$$

Remark 10.3. This theorem gives a comparison between stacky data and coarse data. In particular, to give a G -bundle of type η on $C_{0,[k]}$ is equivalent to giving a \mathcal{G}^Δ -torsor on C_0 (via Ξ). After choosing a representation ρ (and the requisite local integral data so that ρ extends across the parahoric model), the equivalence Ξ transports associated vector bundles.

Solis also proves an expanded version of this result to deal with chains of rational curves. Let R_n denote the rational chain of projective lines with n -components. There is an action of \mathbb{C}^\times on R_n which scales each component. Let p_0, \dots, p_n denote the fixed points of this action.

Recall u, v are k th roots of x, y which are our coordinates near a node. Let p', p'' be denote the closed points of $\text{Spec } \mathbb{C}[[u]]$, $\text{Spec } \mathbb{C}[[v]]$ and finally let $D_n^{\frac{1}{k}}$ be the curve obtained from $\text{Spec } \mathbb{C}[[u]] \sqcup R_n \sqcup \text{Spec } \mathbb{C}[[v]]$ by identifying p' with p_0 and p'' with p_n .

The group μ_k acts on $D_n^{\frac{1}{k}}$ through its usual action on u, v and through the inclusion $\mu_k \subset \mathbb{C}^\times$ on the chain R_n . For an n -tuple $(\beta_0, \dots, \beta_n) \in \text{hom}(\mathbb{C}^\times, T)^n$, we can speak about the equivariant

G -bundles on $D_n^{\frac{1}{k}}$ with equivariant structure at p_i determined by β_i . We refer to this equivalently as a G -bundles on $[D_n^{\frac{1}{k}}/\mu_k]$ of type $(\beta_1, \dots, \beta_n)$.

Further, we can also glue $[D_n^{\frac{1}{k}}/\mu_k]$ to $C_0 - p_0$ to obtain a curve $C_{n,[k]}$. Let C_n denote the coarse moduli space of $C_{n,[k]}$.

We call C_n a *modification* of C_0 and $C_{n,[k]}$ a *twisted modification* of C_0 .

Recall the specific co-characters η_0, \dots, η_r . For $I = \{i_1, \dots, i_n\} \subset \{0, \dots, r\}$, let $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ denote the moduli space of pairs (P, τ) where P is a G -bundles on $[D_n^{\frac{1}{k}}/\mu_k]$ of type $(\eta_{i_1}, \dots, \eta_{i_n})$ and τ is a trivialization on $[\text{Spec } \mathbb{C}((u)) \times \mathbb{C}((v))/\mu_k]$. Let $H = \text{Aut}(P)$ then restriction to $\text{Spec } \mathbb{C}[[u]]$ and $\text{Spec } \mathbb{C}[[v]]$ realizes $H \subset (L_u G)^{\mu_k} \times (L_u G)^{\mu_k}$.

Theorem 10.4. Let $I \subset \{0, \dots, r\}$ and $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ be as above. Then there is an isomorphism

$$T_{G,I}(C_{0,[k]}) \xrightarrow{\Psi^{\eta_I}} (L_u G)^{\mu_k} \times (L_u G)^{\mu_k} / H \xrightarrow{\eta_I^{-1}(\cdot)\eta_I} \frac{L_{\text{poly}} G \times L_{\text{poly}} G}{Z(L_I) \times Z(L_I) \cdot \mathcal{P}_I^{\Delta, \pm}}.$$

where Ψ^{η_I} is as in (3.10) and $\eta_I^{-1}(\cdot)\eta_I$ is described in proposition 3.4.5. Let $i: [D_n^{\frac{1}{k}}/\mu_k] \rightarrow C_{n,[k]}$ be the natural map. Then $i^*: T_{G,I}(C_{n,[k]}) \rightarrow [D_n^{\frac{1}{k}}/\mu_k]$ is an isomorphism. In particular, $T_{G,I}(C_{n,[k]})$, $T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ are isomorphic to an orbit in the wonderful embedding of $\overline{L_{\text{poly}}^\times G}$. Moreover, the isomorphism $T_{G,I}(C_{n,[k]}) \cong T_{G,I}([D_n^{\frac{1}{k}}/\mu_k])$ descends to an isomorphism of stacks

$$\mathcal{M}_{G,I}(C_{n,[k]}) \cong \mathcal{M}_{G,I}([D_n^{\frac{1}{k}}/\mu_k]).$$

I want to write this theorem to give a comparison between the stacky and coarse data once we have introduced modifications. In particular, we can construct the parahoric Bruhat-Tits group scheme \mathcal{G}^Δ on C_n by gluing the local constructions at each node, using the choice of parabolic data I .

We spell this out precisely. At the i th node, let the parahoric subgroup \mathcal{P}_i with Levi decomposition $\mathcal{P}_i = L_i \ltimes U_i$, set

$$\mathcal{P}_i^\Delta = \Delta(L_i) \ltimes (U_i \times U_i).$$

One constructs a sheaf of groups \mathcal{G}^Δ over C_n such that

$$\mathcal{G}^\Delta(\widehat{\mathcal{O}}_{C_n, p_i}) = \mathcal{P}_i^\Delta, \quad \mathcal{G}^\Delta|_{C_n - \{p_0, \dots, p_n\}} = G^{\text{std}}$$

Then similar arguments give an isomorphism

$$\mathcal{M}_{G,I}(C_{n,[k]}) \cong \mathcal{M}_{\mathcal{G}^\Delta}(C_n).$$

Therefore, once for each parabolic type I one fixes a representation

$$\rho_I : \mathcal{G}_I^\Delta \longrightarrow \mathrm{GL}(\mathcal{V}_I)$$

of the corresponding affine Bruhat–Tits group scheme, the universal \mathcal{G}_I^Δ –torsor on C_n produces, via the associated bundle construction, a vector bundle on the universal curve over the stratum of type I . Under the identification

$$\mathcal{M}_{G,I}(C_{n,[k]}) \cong \mathcal{M}_{\mathcal{G}_I^\Delta}(C_n),$$

this agrees with the associated bundle constructed from the corresponding G –bundle on the stacky curve.

Thus the question is, given a parabolic type I , how do we write down suitable representations

$$\rho_I : \mathcal{G}_I^\Delta \longrightarrow \mathrm{GL}(\mathcal{V}_I)$$

of the corresponding affine Bruhat–Tits group scheme? Fix $\rho : G \rightarrow \mathrm{GL}(V)$. For each parahoric type I (at each node p_i), choose \mathcal{P}_i –stable lattices $\Lambda_{i,x} \subset V \otimes \mathbb{C}((x))$ and $\Lambda_{i,y} \subset V \otimes \mathbb{C}((y))$ whose Levi reductions agree; equivalently choose an extension of ρ to the local Bruhat–Tits model and impose the diagonal Levi condition. These local extensions glue (because away from the nodes the group scheme is constant) to give $\rho_I : \mathcal{G}_I^\Delta \rightarrow \mathrm{GL}(\mathcal{V}_I)$.

Remark 10.5. A morphism of group schemes over C_n

$$\rho_I : \mathcal{G}_I^\Delta \longrightarrow \mathrm{GL}(\mathcal{V}_I)$$

is equivalent to the following local–global data: 1. On the smooth locus $U := C_n \setminus \{p_0, \dots, p_n\}$: choose an algebraic representation

$$\rho : G \rightarrow \mathrm{GL}(V),$$

and set $\mathcal{V}_I|_U := V \otimes_{\mathbb{C}} \mathcal{O}_U$. 2. At each node p_i : you must extend ρ from the generic fiber to the parahoric integral model

$$\rho : \mathcal{P}_i^\Delta \rightarrow \mathrm{GL}(\Lambda_i)$$

for some \mathcal{P}_i^Δ –stable lattice Λ_i over the completed local ring $\widehat{\mathcal{O}}_{C_n, p_i} \cong \mathbb{C}[[x, y]]/(xy)$.

So the whole problem is: construct Λ_i stable under \mathcal{P}_i^Δ .

2. Reduce to lattices on the two branches + a Levi compatibility

Let \tilde{C}_n be the normalization at the node p_i , with preimages p'_i, p''_i and local parameters x and y on the two branches. Then

$$\widehat{\mathcal{O}}_{C_n, p_i} \cong \mathbb{C}[[x, y]]/(xy) \quad \text{and} \quad \mathrm{Frac}(\mathbb{C}[[x, y]]/(xy)) \cong \mathbb{C}((x)) \times \mathbb{C}((y)).$$

Your diagonal parahoric sits naturally in the product:

$$\mathcal{P}_i^\Delta = \Delta(L_i) \ltimes (U_i \times U_i) \subset \mathcal{P}_i(\mathbb{C}((x))) \times \mathcal{P}_i(\mathbb{C}((y))).$$

A representation of \mathcal{P}_i^Δ on a “node lattice” is equivalent to: • a \mathcal{P}_i -stable lattice $\Lambda_{i,x} \subset V \otimes \mathbb{C}((x))$, • a \mathcal{P}_i -stable lattice $\Lambda_{i,y} \subset V \otimes \mathbb{C}((y))$, • plus a compatibility along the Levi: the induced L_i -lattices (or reductions) match so that the diagonal L_i acts.

Concretely: since $\mathcal{P}_i \rightarrow L_i$ is the Levi quotient, you want the two reductions $\Lambda_{i,x}/x\Lambda_{i,x}$ and $\Lambda_{i,y}/y\Lambda_{i,y}$ to carry the same L_i -module structure (so that $\Delta(L_i)$ acts via the same representation on both sides). This is the “diagonal” condition.

Canonical way to choose the branch lattices from η_I

Fix I and the corresponding coweight data η_I (or a vertex η in the simplest case). The standard construction is: 1. Restrict ρ to the torus T . Then V decomposes into weights: $V = \bigoplus_{\lambda \in X^*(T)} V_\lambda$. 2. Pair weights with the coweight η_I to get rational numbers $\langle \lambda, \eta_I \rangle$. 3. Define an η_I -lattice in $V \otimes \mathbb{C}((z))$ by

$$\Lambda(\eta_I) := \bigoplus_{\lambda} z^{-\lceil \langle \lambda, \eta_I \rangle \rceil} V_\lambda \subset V \otimes \mathbb{C}((z)).$$

Then a basic fact from Bruhat-Tits/Moy-Prasad theory is: • $\Lambda(\eta_I)$ is stable under the parahoric $\mathcal{P}(\eta_I) \subset G((z))$, • hence ρ extends to a map of group schemes on the disc

$$\mathcal{G}_{\eta_I} \rightarrow \mathrm{GL}(\Lambda(\eta_I)).$$

For a node p_i , you do this twice:

$$\Lambda_{i,x} := \Lambda_x(\eta_i) \subset V \otimes \mathbb{C}((x)), \quad \Lambda_{i,y} := \Lambda_y(\eta_i) \subset V \otimes \mathbb{C}((y)).$$

Because the same η_i is used on both branches, the resulting Levi actions match, and you get a canonical action of \mathcal{P}_i^Δ on the “node module” $\Lambda_{i,\Delta} := \{(s_x, s_y) \in \Lambda_{i,x} \oplus \Lambda_{i,y} \mid \text{their Levi reductions agree}\}$. This $\Lambda_{i,\Delta}$ is the correct object over $\mathbb{C}[[x, y]]/(xy)$ that \mathcal{P}_i^Δ preserves.

That gives your local extension at p_i , and gluing over all nodes gives the global $\rho_I : \mathcal{G}_I^\Delta \rightarrow \mathrm{GL}(\mathcal{V}_I)$.

If $G = \mathrm{GL}_n$ and ρ is the standard representation, then “choosing a \mathcal{P} -stable lattice” is literally choosing a lattice chain. A parahoric of type I corresponds to a block decomposition $n = n_1 + \dots + n_m$, with Levi $L \simeq \mathrm{GL}_{n_1} \times \dots \times \mathrm{GL}_{n_m}$. On the x -branch pick a lattice $\Lambda_x \subset \mathbb{C}((x))^n$ whose reduction mod x carries the corresponding flag of type (n_1, \dots, n_m) . Similarly pick $\Lambda_y \subset \mathbb{C}((y))^n$. The diagonal condition $\Delta(L)$ is exactly: the induced graded pieces (or the Levi framings) are identified across the node. Then \mathcal{P}^Δ is precisely the subgroup of $\mathcal{P}_x \times \mathcal{P}_y$ preserving (Λ_x, Λ_y) with the same Levi action.

Thus in this way we can write down very many line bundles on our moduli stack $\mathcal{X}_{G,g,I}$ by choosing representations of the Bruhat–Tits group schemes on the universal curve, and then passing to the determinant of cohomology. These line bundles give a numerical criterion in the sense of Halpern–Leistner in the following way.

More precisely, to obtain a scale–invariant numerical invariant in the sense of Halpern–Leistner, one must normalize the weight by a quadratic norm on cocharacters. A map

$$f : \Theta = [\mathbb{A}^1/\mathbb{G}_m] \longrightarrow \mathcal{X}_{G,g,I}$$

determines, at the special fiber, a rational cocharacter $\xi \in X_*(T)_{\mathbb{Q}}$ encoding the associated parahoric reduction. For a line bundle $\mathcal{L} \in \text{Pic}(\mathcal{X}_{G,g,I})$, the pullback $f^*\mathcal{L}$ is a \mathbb{G}_m –linearized line bundle on \mathbb{A}^1 , hence determined by an integer weight $\text{wt}_{\mathcal{L}}(f)$, namely the weight of \mathbb{G}_m on the fiber $\mathcal{L}|_{f(0)}$.

Fix a positive definite invariant quadratic form

$$\langle -, - \rangle_{\text{bas}} \in \text{Sym}^2(\mathfrak{g}^*)^G,$$

normalized once and for all (for G simple this is unique up to scale). We define the norm

$$\|\xi\|^2 := \langle \xi, \xi \rangle_{\text{bas}}.$$

The normalized numerical invariant attached to \mathcal{L} is then

$$M_{\mathcal{L}}(f) := \frac{-\text{wt}_{\mathcal{L}}(f)}{\|\xi\|}.$$

This quantity is invariant under reparametrization $\xi \mapsto n\xi$ and hence depends only on the filtration. A point is \mathcal{L} –semistable if $M_{\mathcal{L}}(f) \geq 0$ for all nontrivial Θ –filtrations f .

In the case $\mathcal{L} = \mathcal{L}_{\det}(\rho)$, the leading term of $\text{wt}_{\mathcal{L}}(f)$ is quadratic in ξ :

$$\text{wt}_{\mathcal{L}_{\det}(\rho)}(f) = -m_{\rho} \langle \xi, \xi \rangle_{\text{bas}} + (\text{terms at most linear in } \xi),$$

where m_{ρ} is the level (Dynkin index) of the representation. Thus determinant line bundles provide precisely the type of quadratic numerical function required for the Halpern–Leistner Θ –stratification on $\mathcal{X}_{G,g,I}$.

The difficulty here is the following question. Given an unstable point of $\mathcal{X}_{G,g,I}$, is there a maximally destabilizing one parameter subgroup? In the classical GIT case, this recovers Kempf’s optimal 1-PS theorem.

For $\text{Bun}_G(C)$ on a smooth projective curve, the maximally destabilizing Θ –filtration is exactly the Harder–Narasimhan reduction of the bundle, and the “optimal” cocharacter is the HN type.

For the compactified/parahoric setting, the expected answer is the analogous one: an unstable admissible object should have a canonical parahoric HN reduction (compatible with your diagonal local models at the nodes), and the associated ξ should minimize $M_{\mathcal{L}_{\det}(\rho)}$.

I think we need to examine general theory of the Harder-Narasimhan filtration for G -bundles, which has been worked out for smooth curves. The hope is that we can generalize it to nodal curves, and then prove the analogous theorem that such a filtration minimizes the numerical invariant associated to the determinant line bundle $\mathcal{L}_{\det}(\rho)$. Note that any HN theory for nodal G -bundles should provide the data of a parabolic subgroup (i.e. a cocharacter up to conjugacy as well as a degree).

11 Saturday Feb 14, 2026

I was wondering what it means to say that the theta line bundle on $\text{Bun}_G(C)$ is ample on the moduli stack of G -bundles on a smooth projective curve C . A quick search of the literature suggests that there is no definition of ampleness for line bundles on stacks, but that the following is the correct interpretation.

Let $\text{Bun}_G^{ss}(C) \subset \text{Bun}_G(C)$ be the semistable open substack. It admits a (good/coarse) moduli space $M_G(C)$ (for G semisimple; over \mathbb{C} this is the usual projective moduli variety). On $M_G(C)$ the theta line bundle descends to witness its projectivity, in particular it embeds $M_G(C)$ into projective space.

Intimately related to this is the fact that the theta line bundle defines a stratification of $\text{Bun}_G(C)$. On $\text{Bun}_G(C)$ one has the determinant (theta) line bundle \mathcal{L}_{\det} whose numerical weight along a degeneration gives a numerical invariant $\text{wt}_{\mathcal{L}_{\det}}$.

A reduction of a G -bundle \mathcal{E} to a parabolic $P \subset G$ is by definition a P -bundle \mathcal{E}_P together with an isomorphism $\mathcal{E}_P \times^P G \cong \mathcal{E}$. This is equivalent to giving a section $C \rightarrow \mathcal{E}/P$ of the associated fiber bundle with fiber G/P .

Suppose we have a reduction $\mathcal{E}_P \rightarrow C$ with $\mathcal{E} \cong \mathcal{E}_P \times^P G$. Take any representation V of G . Then the G -bundle \mathcal{E} gives a vector bundle $\mathcal{E}(V) := \mathcal{E} \times^G V$, but since \mathcal{E} comes from \mathcal{E}_P , we also have $\mathcal{E}(V) \cong \mathcal{E}_P \times^P V$.

Now P preserves the flag $0 \subset F_1 \subset \dots \subset F_r = V$. Therefore each F_i is a P -subrepresentation, and we obtain subbundles $\mathcal{E}_P(F_i) \subset \mathcal{E}_P(V) = \mathcal{E}(V)$. The filtration

$$0 \subset \mathcal{E}_P(F_1) \subset \dots \subset \mathcal{E}_P(F_r) = \mathcal{E}(V)$$

is called the filtration of the associated vector bundle.

This gives a filtration type, but no numerical direction yet. We enrich this notion by introduction of 1-PS reduction is a parabolic reduction \mathcal{E}_P , together with a dominant cocharacter $\lambda : \mathbb{G}_m \rightarrow P$

(equivalently into a Levi of P). This cocharacter gives a decomposition of V into weight spaces $V = \bigoplus V_n$ where V_n is the subspace of V on which $\lambda(t)$ acts by t^n . Given the weight decomposition, we can recover the filtration by taking all weight spaces up to some cutoff:

$$F_{\leq n} V := \bigoplus_{m \leq n} V_m$$

For most n , $F_{\leq n} V$ will be the same as $F_{\leq n-1} V$, but at certain critical values of n we get a jump in the filtration. The following theorem of Harder-Narasimhan says that for any unstable G -bundle \mathcal{E} , there is a canonical 1-PS reduction (P, \mathcal{E}_P, μ) where P is a parabolic subgroup, \mathcal{E}_P is a reduction of \mathcal{E} to P , and μ is a dominant cocharacter of P such that the associated filtration of any representation V of G is the Harder-Narasimhan filtration of the associated vector bundle $\mathcal{E}(V)$.

11.1 The Harder-Narasimhan filtration

Let G be a connected reductive group over \mathbb{C} , let C be a smooth projective curve, and let \mathcal{E} be a principal G -bundle on C .

Degrees associated to reductions. Let $P \subset G$ be a parabolic subgroup and let \mathcal{E}_P be a reduction of \mathcal{E} to P . For any character

$$\chi : P \rightarrow \mathbb{G}_m,$$

one gets a line bundle $\mathcal{L}_\chi(\mathcal{E}_P) := \mathcal{E}_P \times^{P, \chi} \mathbb{A}^1$ on C , and hence an integer $\deg \mathcal{L}_\chi(\mathcal{E}_P) \in \mathbb{Z}$.

Definition 11.1. Let us say that the bundle \mathcal{E} is Ramanathan-semistable if for every parabolic reduction (P, \mathcal{E}_P) and every dominant character χ of P ,

$$\deg \mathcal{L}_\chi(\mathcal{E}_P) \leq 0.$$

It is enough to check this numerical criterion against the maximal parabolics.

Type of a reduction. Fix a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$. For a standard parabolic $P \supset B$ with Levi subgroup L , let $\{\chi_i\}$ denote the fundamental characters of P (trivial on $[L, L]$).

The degrees

$$d_i := \deg \mathcal{L}_{\chi_i}(\mathcal{E}_P)$$

determine a rational coweight

$$\mu(P, \mathcal{E}_P) \in X_*(T)_{\mathbb{Q}}^+$$

characterized by

$$\langle \chi_i, \mu(P, \mathcal{E}_P) \rangle = -d_i.$$

This coweight is called the *type* of the reduction. Using the Weyl group action, we can conjugate $\mu(P, \mathcal{E}_P)$ to a dominant coweight. Among all parabolic reductions of \mathcal{E} , the set of types $\mu(P, \mathcal{E}_P)$ has a unique maximal element for the dominance order. This element is denoted

$$\mu(\mathcal{E}) \in X_*(T)_{\mathbb{Q}}^+$$

and called the *Harder–Narasimhan (HN) type* of \mathcal{E} . The associated parabolic subgroup is

$$P_{\text{HN}} = P(\mu(\mathcal{E})) = \left\{ g \in G \mid \lim_{t \rightarrow 0} \mu(t)g\mu(t)^{-1} \text{ exists} \right\},$$

where μ is any integral multiple of $\mu(\mathcal{E})$.

Theorem 11.2. There exists a unique reduction $\mathcal{E}_{P_{\text{HN}}}$ of \mathcal{E} to P_{HN} such that:

(i) **Prescribed type:**

$$\mu(P_{\text{HN}}, \mathcal{E}_{P_{\text{HN}}}) = \mu(\mathcal{E}).$$

(ii) **Semistable Levi quotient:** if L_{HN} is a Levi subgroup of P_{HN} , then the induced principal L_{HN} -bundle

$$\mathcal{E}_{L_{\text{HN}}} = \mathcal{E}_{P_{\text{HN}}} / U_{\text{HN}}$$

is semistable.

(iii) **Maximal destabilizing property:** for every other reduction (Q, \mathcal{E}_Q) ,

$$\mu(Q, \mathcal{E}_Q) \leq \mu(\mathcal{E}).$$

Equality holds only when the reduction is isomorphic to $\mathcal{E}_{P_{\text{HN}}}$.

The pair

$$(P_{\text{HN}}, \mathcal{E}_{P_{\text{HN}}})$$

is called the *Harder–Narasimhan reduction* of \mathcal{E} . It is characterized entirely by the bundle \mathcal{E} itself and does not depend on a choice of representation of G . Note that choosing a representation $\rho : G \rightarrow \text{GL}(V)$ and applying the associated bundle construction to the HN reduction gives the HN filtration of the associated vector bundle $\mathcal{E}(V)$.

11.2 Very close degenerations

One can reformulate the numerical Hilbert–Mumford criterion in terms of stacks. The quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ has two geometric points 1 and 0 which are the images of the points of the same name in \mathbb{A}^1 . For any algebraic stack \mathcal{M} and $f : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \mathcal{M}$ we will write $f(0), f(1) \in \mathcal{M}(k)$ for the points given by the images of 0, 1 $\in \mathbb{A}^1(k)$.

Definition 11.3 (Very close degenerations). Let \mathcal{M} be an algebraic stack over k and $x \in \mathcal{M}(K)$ a geometric point for some algebraically closed field K/k . A *very close degeneration* of x is a morphism $f : [\mathbb{A}_K^1/\mathfrak{G}_{m,K}] \rightarrow \mathcal{M}$ with $f(1) \simeq x$ and $f(0) \not\simeq x$.

We emphasize that $f(0)$ is an object that lies in the closure of a K point of \mathcal{M}_K , which only happens for stacks and orbit spaces, but if $X = \mathcal{M}$ is a scheme, then there are no very close degenerations.

Definition 11.4 (\mathcal{L} -stability). Let \mathcal{M} be an algebraic stack over k , locally of finite type with affine diagonal and \mathcal{L} a line bundle on \mathcal{M} . A geometric point $x \in \mathcal{M}(K)$ is called *\mathcal{L} -stable* if

1. for all very close degenerations $f : [\mathbb{A}_K^1/\mathfrak{G}_{m,K}] \rightarrow \mathcal{M}$ of x we have

$$\text{wt}(f^*\mathcal{L}) < 0$$

and

2. $\dim_K(\text{Aut}_{\mathcal{M}}(x)) = 0$.

We can also introduce the notion of \mathcal{L} -semistable points, by requiring only \leq in (1) and dropping condition (2).

11.3 Very close degenerations of G -bundles

For a cocharacter

$$\lambda : \mathbb{G}_m \rightarrow G$$

we denote by P_λ , U_λ , L_λ the corresponding parabolic subgroup, its unipotent radical and the Levi subgroup.

The source of degenerations is the following analog of the Rees construction. Given $\lambda : \mathbb{G}_m \rightarrow G$ we obtain a homomorphism of group schemes over \mathbb{G}_m :

$$\text{conj}_\lambda : P_\lambda \times \mathbb{G}_m \longrightarrow P_\lambda \times \mathbb{G}_m, \quad (p, t) \longmapsto (\lambda(t)p\lambda(t)^{-1}, t).$$

This homomorphism extends to a morphism of group schemes over \mathbb{A}^1 :

$$\text{gr}_\lambda : P_\lambda \times \mathbb{A}^1 \longrightarrow P_\lambda \times \mathbb{A}^1$$

in such a way that

$$\text{gr}_\lambda(p, 0) = \lim_{t \rightarrow 0} \lambda(t)p\lambda(t)^{-1} \in L_\lambda \times 0.$$

These morphisms are \mathbb{G}_m -equivariant with respect to the action $(\text{conj}_\lambda, \text{act})$ on $P_\lambda \times \mathbb{A}^1$.

Given a P_λ -bundle \mathcal{E}_λ on a scheme X , this morphism defines a P_λ -bundle on $X \times [\mathbb{A}^1/\mathbb{G}_m]$ by

$$\text{Rees}(\mathcal{E}_\lambda, \lambda) := [((\mathcal{E}_\lambda \times \mathbb{A}^1) \times_{\mathbb{A}^1}^{\text{gr}_\lambda} (P_\lambda \times \mathbb{A}^1))/\mathbb{G}_m],$$

where $\times_{\mathbb{A}^1}^{\text{gr}_\lambda}$ denotes the bundle induced via the morphism gr_λ , i.e. we take the product over \mathbb{A}^1 and divide by the diagonal action of the group scheme $P_\lambda \times \mathbb{A}^1/\mathbb{A}^1$, which acts on the right factor via gr_λ .

By construction this bundle satisfies

$$\text{Rees}(\mathcal{E}_\lambda, \lambda)|_{X \times 1} \cong \mathcal{E}_\lambda$$

and

$$\text{Rees}(\mathcal{E}_\lambda, \lambda)|_{X \times 0} \cong \mathcal{E}_\lambda/U_\lambda \times_{L_\lambda} P_\lambda,$$

which is the analog of the associated graded bundle.

Remark 11.5. We unravel this construction in more detail, since it is the key to understanding the source of very close degenerations of G -bundles. Start with the trivial family $E_\lambda \times \mathbb{A}^1$ on $X \times \mathbb{A}^1$. Twist its P_λ -structure along \mathbb{A}^1 by gr_λ ; over $t \neq 0$ this is just conjugation by $\lambda(t)$, over $t = 0$ it collapses the U_λ -part. Because λ gives a grading, this twisting is \mathbb{G}_m -equivariant, so it descends to $X \times [\mathbb{A}^1/\mathbb{G}_m]$.

The output is a \mathbb{G}_m -equivariant family, i.e. a map $[\mathbb{A}^1/\mathbb{G}_m] \rightarrow \text{Bun}_{P_\lambda}$, and then extending structure group gives a map to Bun_G .

Over $t = 1$, gr_λ is just the identity automorphism, so nothing changes:

$$\text{Rees}(E_\lambda, \lambda)|_{t=1} \cong E_\lambda.$$

Over $t = 0$, the twisting morphism becomes project to the Levi $P_\lambda = L_\lambda \ltimes U_\lambda \rightarrow L_\lambda$. The bundle is given by the formula

$$\text{Rees}(E_\lambda, \lambda)|_{t=0} \cong (E_\lambda/U_\lambda) \times_{L_\lambda} P_\lambda.$$

where E_λ/U_λ is the quotient L_λ -bundle given by dividing the P_λ -bundle E_λ by the unipotent radical U_λ , and then we extend structure group back to P_λ via the inclusion $L_\lambda \subset P_\lambda$.

Heinloth shows that every very close degeneration of G -bundles arises from the Rees construction applied to a parabolic reduction. We have already seen that given $\lambda : \mathbb{G}_m \rightarrow G$ and a reduction E_λ of E to P_λ , the Rees construction gives a very close degeneration. We unpack the converse. A map

$$f : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \text{Bun}_G(X)$$

is a \mathbb{G}_m -equivariant G -bundle \mathcal{E} on $X \times \mathbb{A}^1$. Consider special fiber $X \times [0/\mathbb{G}_m] \subset X \times [\mathbb{A}^1/\mathbb{G}_m]$. Because 0 is fixed by scaling, the restriction $\mathcal{E}_0 := \mathcal{E}|_{X \times [0/\mathbb{G}_m]}$ is a G -bundle together with a \mathbb{G}_m -action. Choosing a trivialization of \mathcal{E}_0 gives a homomorphism $\mathbb{G}_m \rightarrow \text{Aut}(\mathcal{E}_0) \cong G$ and changing trivialization changes this homomorphism by conjugation, so we get a well-defined cocharacter $\lambda : \mathbb{G}_m \rightarrow G$ up to conjugation.

Once we have extracted the cocharacter λ , we recover the parabolic reduction as an *attractor subbundle*. Let \mathcal{E} be the \mathbb{G}_m -equivariant G -bundle on $X \times \mathbb{A}^1$ corresponding to $f : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \text{Bun}_G(X)$, and write $E := \mathcal{E}|_{X \times \{1\}}$ for the general fiber.

Choose a local trivialization of \mathcal{E} in the fpqc topology over $X \times \mathbb{A}^1$, so that over such a trivializing open the \mathbb{G}_m -action is given by λ up to G -conjugacy. In this local model, a point of the fiber E_x may be written as a frame $g \in G$, and the \mathbb{G}_m -action transports it by conjugation, so the condition that the orbit has a limit as $t \rightarrow 0$ is exactly

$$\lim_{t \rightarrow 0} \lambda(t) g \lambda(t)^{-1} \text{ exists.}$$

By definition this is equivalent to $g \in P_\lambda$. Therefore the subset of points of E whose \mathbb{G}_m -orbit admits a limit is stable under the right action of P_λ and defines a principal P_λ -subbundle

$$E_\lambda \subset E.$$

Equivalently, E_λ is the reduction of E corresponding to the canonical \mathbb{G}_m -fixed section of the associated bundle $E \times^G (G/P_\lambda)$ coming from the special fiber at $t = 0$.

Finally, applying the Rees construction to (E_λ, λ) yields a \mathbb{G}_m -equivariant family of G -bundles on $X \times \mathbb{A}^1$, and by construction it agrees with \mathcal{E} over $\mathbb{A}^1 \setminus \{0\}$; the \mathbb{G}_m -equivariant extension across 0 is unique, hence the Rees family recovers \mathcal{E} .

Heinloth shows that these two mechanisms are inverse to each other, so that the data of a very close degeneration is precisely the data of a parabolic reduction together with a cocharacter. This is the key to understanding the numerical criterion for stability in terms of parabolic reductions.

Lemma 11.6. Let G be a split reductive group over k . Given a very close degeneration

$$f : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \text{Bun}_G$$

corresponding to a family \mathcal{E} of G -bundles on $X \times [\mathbb{A}^1/\mathbb{G}_m]$, there exist:

1. a cocharacter $\lambda : \mathbb{G}_m \rightarrow G$, canonical up to conjugation,
2. a reduction \mathcal{E}_λ of the bundle \mathcal{E} to P_λ ,
3. an isomorphism

$$\mathcal{E} \cong \text{Rees}(\mathcal{E}_\lambda|_{X \times 1}, \lambda).$$

Remark 11.7 (The case of $\text{GL}(V)$). A cocharacter $\lambda : \mathbb{G}_m \rightarrow \text{GL}(V)$ is the same as a \mathbb{Z} -grading

$$V = \bigoplus V_n, \quad \lambda(t)|_{V_n} = t^n.$$

Then P_λ is the stabilizer of the induced filtration

$$F^{\geq m} V := \bigoplus_{n \geq m} V_n,$$

and $L_\lambda \cong \prod \mathrm{GL}(V_n)$. A reduction E_λ of a $\mathrm{GL}(V)$ -bundle is the same as a filtration of the associated vector bundle by subbundles (with weights). The Rees construction is literally the usual Rees module construction that deforms a filtered vector bundle to its graded.

Theorem 11.8. A G -bundle E is \mathcal{L}_{\det} -stable if and only if for all reductions E_P to maximal parabolic subgroups $P \subset G$ we have $\deg(E_P \times_P \mathrm{Lie}(P)) < 0$.

Proof. We have ave to compute the weight of \mathcal{L}_{\det} on very close degenerations.

Let us choose $T \subset B \subset G$ a maximal torus and a Borel subgroup and

$$\lambda : \mathbb{G}_m \rightarrow G$$

a dominant cocharacter, i.e.

$$\langle \lambda, \alpha \rangle \geq 0$$

for all roots such that

$$\mathfrak{g}_\alpha \subset \mathrm{Lie}(B).$$

Let us denote by I the set of positive simple roots with respect to (T, B) and by

$$I_P := \{\alpha_i \in I \mid \lambda(\alpha_i) = 0\}$$

the simple roots α_i for which $-\alpha_i$ is also a root of P_λ . For $j \in I$ let us denote by

$$\tilde{\omega}_j \in X_*(T)_{\mathbb{R}}$$

the cocharacter defined by

$$\tilde{\omega}_j(\alpha_i) = \delta_{ij},$$

and by P_j the corresponding maximal parabolic subgroup.

Then

$$\lambda : \mathbb{G}_m \rightarrow Z(L_\lambda) \subset L_\lambda \subset P_\lambda.$$

Thus for any very close degeneration $f : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \mathrm{Bun}_G$ given by $\mathrm{Rees}(\mathcal{E}_\lambda, \lambda)$ the bundle \mathcal{L}_{\det} defines a morphism

$$\mathrm{wt}_{\mathcal{L}} : X_*(Z_\lambda) \subset \mathrm{Aut}_{\mathrm{Bun}_G}(f(0)) \rightarrow \mathbb{Z}.$$

Then the weight function is additive in the cocharacter so it is enough to compute for one fundamental direction at a time. Write $\lambda = \sum_{j \in I - I_P} a_j \tilde{\omega}_j$ for some $a_j > 0$. Then

$$\mathrm{wt}(\mathcal{L}_{\det}|_{f(0)}) = \mathrm{wt}_{\mathcal{L}}(\lambda) = \sum_{j \in I - I_P} a_j \mathrm{wt}_{\mathcal{L}}(\tilde{\omega}_j).$$

For each j we get a decomposition

$$\mathrm{Lie}(G) = \bigoplus_i \mathrm{Lie}(G)_i,$$

where $\mathrm{Lie}(G)_i$ is the subspace of the Lie algebra on which $\tilde{\omega}_j$ acts with weight i . Each of these spaces is a representation of L_λ and also of the Levi subgroups L_j of P_j . Using this decomposition we find as in the case of vector bundles:

$$\begin{aligned} \mathrm{wt}_{\mathcal{L}}(\tilde{\omega}_j) &= -\mathrm{wt}_{\mathbb{G}_m}(\det H^*(C, \mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i)) \\ &= \sum_i i \cdot \chi(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i) \\ &= \sum_i i \left(\deg(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i) + \dim(\mathrm{Lie}(G)_i)(1 - g) \right) \\ &= 2 \sum_{i>0} i \deg(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i). \end{aligned}$$

because the decomposition is symmetric with respect to $i \mapsto -i$ and so the terms with $1 - g$ cancel out. Now

$$\deg(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i) = \deg(\det(\mathcal{E}_{0,\lambda} \times^{L_\lambda} \mathrm{Lie}(G)_i)).$$

Since the Levi subgroups of maximal parabolics have only a one-dimensional space of characters, all of these degrees are positive multiples of

$$\det(\mathrm{Lie}(P_j)).$$

□

Remark 11.9. This agrees with the seemingly different numerical criterion of Ramanathan stability, basically for the reason that $\det \mathrm{Ad}_{\mathrm{Lie}(P)}$ is a positive multiple of the unique dominant character of P .

Let us examine some of the Lie theory here to understand. Note that for a maximal parabolic, there is a unique dominant character (up to scaling). For semisimple G , any parabolic P has

$$X^*(P) \cong X^*(L) \cong X^*(Z(L))$$

where L is a Levi and $Z(L)$ its center. If P is maximal, then $Z(L)$ is 1-dimensional, hence $X^*(P) \cong \mathbb{Z}$. Choosing a Borel $B \subset P$ picks out a notion of dominant characters, i.e. the submonoid $\mathbb{Z}_{\geq 0} \cdot \lambda_P$ generated by a single primitive dominant character λ_P . Therefore Ramanathan stability for maximal parabolics:

$$\deg(E_P(\chi)) < 0 \text{ for all dominant } \chi$$

is equivalent to checking it just for $\chi = \lambda_P$. The point is

$$\deg(E_P \times^P \mathrm{Lie}(P)) = \deg(E_P(\det \mathrm{Ad}_{\mathrm{Lie}(P)})).$$

Write $\mathrm{Lie}(P) = \mathrm{Lie}(L) \oplus \mathfrak{u}$. The determinant of the adjoint action on $\mathrm{Lie}(L)$ is trivial for semisimple reasons (the adjoint weights are roots summing to 0), so

$$\det(\mathrm{Ad}_{\mathrm{Lie}(P)}) = \det(\mathrm{Ad}_{\mathfrak{u}}).$$

But $\det(\mathrm{Ad}_{\mathfrak{u}})$ is the character corresponding to the weight $2\rho_P := \sum_{\alpha \in \Phi^+ \setminus \Phi_L^+} \alpha$, i.e. the sum of the positive roots appearing in \mathfrak{u} . This is a dominant character of P , and for a maximal parabolic it is a positive multiple of the generator λ_P :

$$\det(\mathrm{Ad}_{\mathfrak{u}}) = m_P \cdot \lambda_P \quad (m_P > 0).$$

Therefore the numerical criterion for stability is

$$\deg(E_P \times^{P, \lambda_P} \mathbb{A}^1) < 0 \iff \deg(E_P \times^P \mathrm{Lie}(P)) < 0.$$

In fact these numbers are positive multiples of each other, so the two criteria are exactly equivalent.

Remark 11.10 (Conceptual viewpoint). The weight computation for the determinant line bundle admits a conceptual interpretation independent of coordinates or root combinatorics. The key point is that a very close degeneration determines two pieces of data:

1. a cocharacter

$$\lambda : \mathbb{G}_m \rightarrow G,$$

which specifies the direction of the degeneration, and

2. a reduction E_λ of the general fiber to the corresponding parabolic P_λ

These two objects naturally pair. Given a P_λ -bundle E_λ on the curve C , there is a map

$$\deg(E_\lambda \times^P \cdot) : X^*(P_\lambda) \longrightarrow \mathbb{Z}.$$

Since characters factor through the Levi quotient,

$$X^*(P_\lambda) = X^*(L_\lambda) = X^*(Z(L_\lambda)),$$

and note that $Z(L_\lambda)$ is a torus by structure theory of reductive groups. The cocharacter λ factors through the center of the Levi $\lambda \in X_*(Z(L_\lambda))$ so there is a canonical pairing

$$\langle \lambda, \deg(E_\lambda \times^P \cdot) \rangle \in \mathbb{Z}.$$

This relates to the determinant line bundle as follows. The determinant line bundle is defined using the adjoint representation:

$$\mathcal{L}_{\det} = \det R\Gamma(C, \mathrm{Ad}(E)).$$

The cocharacter λ induces a grading

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i,$$

and hence a decomposition of the associated graded adjoint bundle

$$\mathrm{Ad}(E_0) = \bigoplus_i E_{0,\lambda} \times^{L_\lambda} \mathfrak{g}_i.$$

The \mathbb{G}_m -weight on the determinant line is therefore

$$\mathrm{wt}_{\mathcal{L}_{\det}}(f) = \sum_i i \chi(E_{0,\lambda} \times^{L_\lambda} \mathfrak{g}_i).$$

The determinant line is a moment-map functional, and this pairing is the algebraic analogue of the Kempf–Ness pairing between a 1-PS and a moment map value.

After applying Riemann–Roch and using the symmetry of the adjoint representation, only degree terms remain, yielding

$$\mathrm{wt}_{\mathcal{L}_{\det}}(f) = \langle \lambda, \deg(E_\lambda) \rangle.$$

In the case $G = \mathrm{GL}_n$, this recovers the familiar expression

$$\sum_i (\mathrm{weight}_i) \cdot (\deg \mathrm{gr}^i),$$

namely the pairing between a weighted filtration and the degrees of its graded pieces.

This theorem of Heinloth [2] is precisely the way we need to generalize the notion of stability to our compactified moduli stack $\mathcal{X}_{G,g,I}$. We need to understand the very close degenerations of points of $\mathcal{X}_{G,g,I}$, and then apply the numerical criterion to determine which points are semistable with respect to the determinant line bundles we have constructed.

Note that we need to know that the stack $\mathcal{X}_{G,g,I} \rightarrow \overline{\mathcal{M}}_{g,I}$ has affine diagonal in order to apply Heinloth’s theory of \mathcal{L} -stability. Honestly maybe not worth worrying about this.

Lemma 11.11 (Affine diagonal). The morphism $\mathcal{X}_{G,g,I} \rightarrow \overline{\mathcal{M}}_{g,I}$ has affine diagonal.

Sketch. Fix a test scheme B and two objects of $\mathcal{X}_{G,g,I}(B)$ lying over the same family of curves $C_B \rightarrow B$, i.e. two twisted local modifications $\pi : C'_B \rightarrow B$ together with admissible bundles on C'_B in the sense of Solís. Equivalently, we may view each admissible bundle as a torsor under an affine group scheme \mathcal{G}_B on C'_B (the Bruhat–Tits/parahoric group scheme determined by the local types at the stacky points/nodes).

Let $\mathcal{P}_1, \mathcal{P}_2$ be the corresponding \mathcal{G}_B -torsors on C'_B . The sheaf $\mathrm{Isom}_{C'_B}(\mathcal{P}_1, \mathcal{P}_2)$ on B is the push-forward along π of the sheaf of isomorphisms on C'_B :

$$\mathrm{Isom}_B((C'_B, \mathcal{P}_1), (C'_B, \mathcal{P}_2)) \cong \pi_* \mathrm{Isom}_{C'_B}(\mathcal{P}_1, \mathcal{P}_2).$$

Now $\text{Isom}_{C'_B}(\mathcal{P}_1, \mathcal{P}_2)$ is a torsor under the group scheme $\text{Aut}(\mathcal{P}_1) := \mathcal{P}_1 \times^{\mathcal{G}_B} \mathcal{G}_B$, which is affine over C'_B since \mathcal{G}_B is affine. Since $\pi : C'_B \rightarrow B$ is proper and of finite presentation, the relative Weil restriction $\text{Res}_{C'_B/B}$ sends affine C'_B -schemes to affine B -schemes; hence $\pi_* \text{Isom}_{C'_B}(\mathcal{P}_1, \mathcal{P}_2)$ is representable by an affine B -scheme. This identifies the relative diagonal of $\mathcal{X}_{G,g,I} \rightarrow \overline{\mathcal{M}}_{g,I}$ with an affine morphism. \square

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We need to think about very close degenerations of points of $\mathcal{X}_{G,g,I}$. In the smooth case, these are given by 1-PS reductions to parabolic subgroups. In our compactified setting, we expect that they are given by parahoric reductions to the local Bruhat–Tits group schemes at the nodes, together with a cocharacter of the Levi that gives the numerical direction. We need to understand how to construct these degenerations, and then apply Heinloth’s criterion to determine which points are semistable with respect to the determinant line bundles we have constructed.

We must first classify all very close degenerations of $\mathcal{X}_{G,g,I}$. We first survey some general theory introduced by Heinloth.

Lemma 12.1. Let \mathcal{M} be an algebraic stack locally of finite type over $k = \bar{k}$ with quasi-affine diagonal.

1. For any very close degeneration

$$f : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow \mathcal{M}$$

the induced morphism

$$\lambda_f : \mathbb{G}_m = \text{Aut}_{[\mathbb{A}^1/\mathbb{G}_m]}(0) \longrightarrow \text{Aut}_{\mathcal{M}}(f(0))$$

is nontrivial.

2. The restriction functor

$$\mathcal{M}([\mathbb{A}^1/\mathbb{G}_m]) \rightarrow \varprojlim \mathcal{M}([\text{Spec}(k[x]/x^n)/\mathbb{G}_m])$$

is an equivalence of categories.

Proof. As the statement produces the point $f(1)$ out of a formal datum let us explain briefly why this is possible: The composition

$$\phi : \mathbb{D} = \text{Spec } k[[x]] \rightarrow \mathbb{A}^1 \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$$

is faithfully flat, because both morphisms are flat and the map is surjective, because both points of $[\mathbb{A}^1/\mathbb{G}_m]$ are in the image.

By our assumptions \mathcal{M} is a stack for the fppf topology, we therefore see that $\mathcal{M}([\mathbb{A}^1/\mathbb{G}_m])$ can be described as objects in $\mathcal{M}(k[[x]])$ together with a descent datum with respect to ϕ .

Moreover, the canonical map

$$\mathcal{M}(\mathbb{D}) \xrightarrow{\sim} \varprojlim \mathcal{M}(k[x]/(x^n))$$

is an equivalence of categories: this follows for example, because the statement holds for schemes and choosing a smooth presentation $X \rightarrow \mathcal{M}$ one can reduce to this statement.

In particular this explains already that an element of

$$\varprojlim \mathcal{M}(\mathrm{Spec}(k[x]/x^n)/\mathbb{G}_m)$$

will produce a $k[[x]]$ -point of \mathcal{M} . The problem now lies in constructing a descent datum for this morphism, as

$$\mathbb{D} \times_{[\mathbb{A}^1/\mathbb{G}_m]} \mathbb{D} = \mathrm{Spec}(k[[x]] \otimes_{k[x]} k[x, t, t^{-1}] \otimes_{k[y]} k[[y]]),$$

where the last tensor product is taken via $y = xt$. The ring on the right hand side is not complete and the formal descent data coming from an element in

$$\varprojlim \mathcal{M}(\mathrm{Spec}(k[x]/x^n)/\mathbb{G}_m)$$

only seems to induce a descent datum on the completion of the above ring.

Let us deduce (1). First note that this holds automatically if \mathcal{M} is a scheme, because then $f(1)$ is a closed point and $f(0)$ lies in the closure of $f(1)$, and therefore must equal $f(1)$. This implies that there are no very close degenerations of schemes and so (1) is vacuously true for schemes.

In general choose a smooth presentation $p : X \rightarrow \mathcal{M}$. If λ_f is trivial, we can lift the morphism

$$f|_0 : [0/\mathbb{G}_m] \rightarrow \mathrm{Spec} k \rightarrow \mathcal{M}$$

to

$$\tilde{f}_0 : [0/\mathbb{G}_m] \rightarrow X.$$

Since p is smooth, we can inductively lift this morphism to obtain an element in

$$\varprojlim X([\mathrm{Spec}(k[x]/x^n)/\mathbb{G}_m).$$

Thus we reduced (1) to the case $\mathcal{M} = X$. \square

Lemma 12.2. For any cocharacter $\lambda : \mathbb{G}_m \rightarrow G$ and any geometric point $x \in X(K)$ that is not a fixed point of λ , the equivariant map

$$f_{\lambda, x} : \mathbb{A}_K^1 \rightarrow X$$

defines a very close degeneration

$$\bar{f}_{\lambda, x} : [\mathbb{A}_K^1/\mathbb{G}_{m, K}] \rightarrow [X/G].$$

Moreover, any very close degeneration in the stack $[X/G]$ is of the form $\bar{f}_{\lambda, x}$ for some x, λ .

Proof. Since $f_{\lambda,x}(0)$ is a fixed point of λ and x is not, we have $\bar{f}(0) \neq \bar{f}(1)$, thus \bar{f} is a very close degeneration. Conversely let

$$f : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow [X/G]$$

be any very close degeneration. We need to find a \mathbb{G}_m -equivariant morphism

$$\begin{array}{ccc} \mathbb{A}^1 & \rightarrow & X \\ \downarrow & & \downarrow \pi \\ [\mathbb{A}^1/\mathbb{G}_m] & \rightarrow & [X/G] \end{array}$$

Since $\pi : X \rightarrow [X/G]$ is a G -bundle, the pull-back

$$p : X \times_{[X/G]} [\mathbb{A}^1/\mathbb{G}_m] \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$$

is a G -bundle on $[\mathbb{A}^1/\mathbb{G}_m]$. A section of this bundle p is precisely a \mathbb{G}_m -equivariant morphism $\mathbb{A}^1 \rightarrow X$ lifts f because p is the fiber product of f and π . Thus we reduce to the classification of \mathbb{G}_m -equivariant G -bundles on \mathbb{A}^1 . \square

Lemma 12.3. Let G be a reductive group and \mathcal{P} a G -bundle on $[\mathbb{A}^1/\mathbb{G}_m]$. Denote by \mathcal{P}_0 the fiber of \mathcal{P} over $0 \in \mathbb{A}^1$.

1. If there exists $x_0 \in \mathcal{P}_0(k)$ (e.g. this holds if $k = \bar{k}$), then there exists a cocharacter

$$\lambda : \mathbb{G}_m \rightarrow G,$$

unique up to conjugation, and an isomorphism of G -bundles

$$\mathcal{P} \cong [(\mathbb{A}^1 \times G)/(\mathbb{G}_m, (\text{act}, \lambda))].$$

Moreover, \mathcal{P} has a canonical reduction to P_λ , the parabolic subgroup defined by λ .

2. Let $G_0 := \text{Aut}_G(\mathcal{P}_0)$ and $\lambda : \mathbb{G}_m \rightarrow G_0$ be the cocharacter defined by $\mathcal{P}|_{[0/\mathbb{G}_m]}$. Note that since 0 is a fixed point of the \mathbb{G}_m -action, \mathbb{G}_m acts on \mathcal{P}_0 whose automorphism group is G and therefore this gives a cocharacter $\lambda : \mathbb{G}_m \rightarrow G_0$.

Consider the G_0 -bundle $\mathcal{P}^{G_0} := \text{Isom}_G(\mathcal{P}, \mathcal{P}_0)$ on $[\mathbb{A}^1/\mathbb{G}_m]$. Then

$$\mathcal{P}^{G_0} \cong [(\mathbb{A}^1 \times G_0)/(\mathbb{G}_m, (\text{act}, \lambda))],$$

i.e.

$$\mathcal{P} \cong \text{Isom}_{G_0}([(\mathbb{A}^1 \times G_0)/(\mathbb{G}_m, (\text{act}, \lambda))], \mathcal{P}_0).$$

Moreover, \mathcal{P}^{G_0} has a canonical reduction to $P_{0,\lambda} \subset G_0$.

Proof of Lemma 1.7. The second part follows from the first, as the G_0 -bundle

$$\mathcal{P}^{G_0} = \text{Isom}_G(\mathcal{P}_0, \mathcal{P}_0)$$

has a canonical point id . We added (2), because it gives an intrinsic statement, independent of choices.

To prove (1) note that x_0 defines an isomorphism

$$G \xrightarrow{\sim} \text{Aut}_G(\mathcal{P}_0)$$

and a section

$$[\text{Spec } k/\mathbb{G}_m] \rightarrow \mathcal{P}|_{[0/\mathbb{G}_m]}.$$

This induces a section

$$[\text{Spec } k/\mathbb{G}_m] \rightarrow \mathcal{P}/P_\lambda.$$

As the map

$$\pi : \mathcal{P}/P_\lambda \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$$

is smooth, any section can be lifted infinitesimally to $\text{Spec } k[x]/x^n$ for all $n \geq 0$. Inductively the obstruction to the existence of a \mathbb{G}_m -equivariant section is an element in

$$H^1(B\mathbb{G}_m, T_{(\mathcal{P}/P_\lambda)/\mathbb{A}^1, 0} \otimes (x^{n-1})/(x^n)) = 0,$$

and the choices of such liftings form a torsor under

$$H^0(B\mathbb{G}_m, T_{(\mathcal{P}/P_\lambda)/\mathbb{A}^1, 0} \otimes (x^{n-1})/(x^n)).$$

Now by construction \mathbb{G}_m acts with negative weight on

$$T_{\mathcal{P}/P_\lambda, 0} = \text{Lie}(G)/\text{Lie}(P_\lambda),$$

and it also acts with negative weight on the cotangent space

$$(x)/(x^2),$$

so there exists a canonical \mathbb{G}_m -equivariant reduction \mathcal{P}_λ of \mathcal{P} to P_λ .

Similarly, the vanishing of H^1 implies that we can also find a compatible family of λ -equivariant sections

$$[(\text{Spec } k[t]/t^n)/\mathbb{G}_m] \rightarrow \mathcal{P}$$

and by Lemma 1.5 this defines a section over

$$[\mathbb{A}^1/\mathbb{G}_m],$$

i.e. a morphism of G -bundles

$$[(\mathbb{A}^1 \times G)/(\mathbb{G}_m, \lambda)] \rightarrow \mathcal{P}.$$

□

Remark 12.4. We unpack this statement in more elementary terms. The key point is that \mathbb{G}_m -equivariant G -bundles on \mathbb{A}^1 are classified by cocharacters

$$\lambda : \mathbb{G}_m \rightarrow G$$

up to conjugation.

In the forward direction, let \mathcal{P} be a \mathbb{G}_m -equivariant G -bundle on \mathbb{A}^1 . Since $0 \in \mathbb{A}^1$ is fixed by the \mathbb{G}_m -action, the induced action on the fiber \mathcal{P}_0 defines a homomorphism

$$\lambda : \mathbb{G}_m \rightarrow \text{Aut}_G(\mathcal{P}_0) \cong G,$$

well-defined up to conjugation after choosing an identification $\mathcal{P}_0 \cong G$. Because $\mathbb{A}^1 \setminus \{0\} \cong \mathbb{G}_m$ is a single \mathbb{G}_m -orbit, equivariance forces the action on every other fiber to be determined by this same cocharacter; thus the entire equivariant structure is encoded by λ .

Conversely, given a cocharacter

$$\lambda : \mathbb{G}_m \rightarrow G,$$

one constructs a \mathbb{G}_m -equivariant G -bundle on \mathbb{A}^1 by starting with the trivial bundle

$$\mathbb{A}^1 \times G \rightarrow \mathbb{A}^1$$

and defining the \mathbb{G}_m -action by

$$t \cdot (x, g) := (tx, \lambda(t)g).$$

The quotient

$$[(\mathbb{A}^1 \times G)/(\mathbb{G}_m, (\text{act}, \lambda))]$$

is then a \mathbb{G}_m -equivariant G -bundle on \mathbb{A}^1 . Changing the trivialization of the fiber over 0 conjugates λ , so the resulting bundle depends only on the conjugacy class of λ .

13 References

1. TORSORS ON SEMISTABLE CURVES AND DEGENERATIONS
2. Hilbert-Mumford stability on algebraic stacks and applications to G -bundles on curves