Nearby Vanishing Cycles and Gluing Perverse Sheaves

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Abstract

These are a second set of notes on perverse sheaves, specifically taking a look at the nearby and vanishing cycles functors, gluing perverse sheaves, and perverse sheaves on $(\mathbb{C}, 0)$ and square matrices with the rank stratification. We are following [1], [4], and [3]. There is a connection with representation theory which I am interested in exploring about Soergel bimodules and quiver represntations.

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1 Goals

We want to understand the following result of Beilinson:

Proposition 1.1. *Let X be a small disk around 0 in $\mathbb{A}^1$. Then the category $\mathrm{Perv}(X)$ of perverse sheaves on X with singularities at 0 only is equivalent to the category of representations of the quiver*

$$V_1 \underset{u}{\overset{v}{\rightleftarrows}} V_2$$

with the relations that $I - uv$ and $I - vu$ are invertible.

Let $j : Z \hookrightarrow X$ be a closed inclusion and $i : U \hookrightarrow X$ the open complement and $f : X \to \mathbb{A}^1$. Beilinson shows that the category $\mathrm{Perv}(X)$ is equivalent to the category $\mathrm{Perv}_f(Z, U)$ of "gluing data of perverse sheaves on Z and U".

Let $X \subset \mathbb{A}^1$ be a complex disk around 0. Then $\mathrm{Perv}(Z) = \mathrm{Perv}(\{0\}) = \mathrm{Vect}$ and $\mathrm{Perv}\, U$ is vector spaces with an automorphism (monodromy), and the **unipotent nearby cycles functor** ψ_f for the map $f : X \subset A^1$ takes a perverse sheaf on U to a perverse sheaf on Z. Beilinson computes this functor to be the one which takes (V, T) to (W, T) where W is the maximal subspace of V on which T acts nilpotently.

Therefore, the category $\mathrm{Perv}_f(Z, U)$ is equivalent to the category of data V_0', V_1', ϕ, u, v where V_0', V_1' are vector spaces, ϕ an automorphism of V_1 and u, v are maps

$$v : V_0' \leftrightarrow \psi(V_1', I - \phi) : u$$

so that $vu = I - \phi$. Then the equivalence is given by

$$\begin{aligned} V_0' &= \psi_f(V_0, uv) \\ V_1' &= V_1 \\ \phi &= I - vu \\ u &= u \\ v &= v \end{aligned}$$

The invertibility of $I - vu$, $I - uv$ is about the image of the functor being those vector spaces for which the prescribed maps are acting **maximally** unipotently.

2 Nearby Cycles

Let $f : X \to \mathbb{A}^1$ so that $Z = f^{-1}(0)$. Given i, j the open and closed inclusions of Z, U into X, we have the **nearby cycles** functor $R\psi_f : \mathrm{Perv}(U) \to \mathrm{Perv}(Z)$ defined as follows. Let $u : \tilde{\mathfrak{G}_m} \to \mathfrak{G}_m$ be the universal cover and let $\tilde{U} = U \times_{\mathfrak{G}_m} \tilde{\mathfrak{G}_m}$. Then $R\psi_f$ is the derived functor

$$R\psi_f = R(i^* j^* v_* v^*)$$

Note that v is not an algebraic map, but nonetheless Deligne proved that the nearby cycles functor preserves constructibility.

Lemma 2.1. *The nearby cycles functor $R\psi_f$ decomposes $R\psi_f = R\psi_f^{un} \oplus R\psi_f^{\geq 1}$ where for any choice of generator t of $\pi_1 \mathfrak{G}_m$, we have $1 - t$ acts nilpotently on $R\psi_f^{un}(A_U^*)$ for any complex A_U^* and is an automorphism of $R\psi_f^{\geq 1}(A_U^*)$.*

The unipotent piece is called the **unipotent nearby cycles** functor. One can show that $R\psi_f^{un}[-1]$ acts on perverse sheaves and we denote this functor Ψ_f^{un}.

3 Vanishing Cycles

4 Cellular perverse sheaves and the representation theory of category $\mathcal{O}$

4.1 Category $\mathcal{O}$ and quivers

The story starts with the BGG Category $\mathcal{O}$ and Soergel's results about the endomorphism ring of projective modules in $\mathcal{O}$. In particular, we have the following phenomenom which we will try to relate to what comes next.

4.2 Cellular perverse sheaves

Suppose K is a finite simiplicial complex, which we will identify with its geometric realization $|K|$. Given a perversity p, there are two types of integers, the $*$ for which $k \in *$ if $p(k) = p(k-1)$ and the ! for which $k \in !$ if $p(k) = p(k-1) - 1$. MacPherson defines the **perverse dimension** of a d-simplex σ to be

$$\delta(\sigma) = -p(d) \text{ if } \sigma \in * \text{ and } \delta(\sigma) = -p(d) - d \text{ if } \sigma \in !$$

Definition 4.1. *A* ***cellular perverse sheaf*** *S on the simplicial complex K is a rule which assigns to each simplex σ a vector space S_σ and "attaching homomorphisms" $s_{\sigma,\tau} : S_\sigma \to S_\tau$ whenever $\sigma \iff \tau$ and $\delta(\sigma) = \delta(\tau)$, so that the resulting sequence*

$$\underset{d}{\to} \bigoplus_{\delta(\sigma)=r} S_\sigma \underset{d}{\to} \bigoplus_{\delta(\sigma)=r-1} S_\sigma \underset{d}{\to} \cdots$$

Equivalently, this is saying that whenever $\delta(\sigma) = r+1$ and $\delta(\tau) = r-1$ then we have

$$\sum_{\delta(\theta)=r, \sigma \iff \theta \iff \tau} s_{\sigma,\theta} \circ s_{\theta,\tau} = 0$$

There is a cohomology functor $T : \mathrm{Perv}(K) \to \mathrm{Perv}^{\Delta}(K)$ which is an equivalence between the categories of perverse sheaves on K with repsect to the triangulation K and cellular perverse sheaves on K.

Given a simplicial complex, there is a quiver whose vertices are the simplices of K and whose arrows are the elementary relations $\sigma > \tau$ with $\delta(\sigma) = \delta(\tau) + 1$. Form the corresponding path algebra F of the quiver and consider the two-sided ideal J generated by the elements

$$\sum_{\delta(\theta)=r, \sigma \iff \theta \iff \tau} s_{\sigma,\theta} \circ s_{\theta,\tau}$$

Theorem 4.2. *The category* $\mathrm{Perv}(K)$ *with respect to the triangulation of* K *is equivalent to the category of modules over the ring* F/J.

4.3 Vybornov's theorem

Classically, the problem of computing an explicit quiver for $\mathcal{O}_0$ was posed by Gelfand and solved by Vybornov. Vybornov constructs a sequence of "IC modules" which are computing perverse sheaves, but then he proves that $d^2 = 0$ and this complex imposes a bunch of relations on the quiver, of the above shape.

5 DG categories

Recall that the bounded derived construcible category is a triangulated category and carries truncation functors, the heart of the t-structure being the perverse sheaves which form an abelian category.

Because of some abstract nonsense about triangulated categories, there is an equivalence of categories between the heart of a t-structure, i.e. the perverse sheaves, and the differential graded category of the endomorphisms of a particular "generator."

Because of this, you can always consider the category of perverse sheaves on a space wrt a stratification as the category of finite length "dg" modules over a "dg" quiver, and this "dg" quiver is an ordinary quiver precisely when the nonzero cohomology vanishes. Yuri referred to this as these sort of Bondal-Kapranov type results. See [2].

Many of the calculations for quivers, i.e. hyperplane arrangements and rank stratifications, can be thought of as doing hard work to understand this endomorphism algebra of a generator. There is also another endomorphism algebra I've been thinking about which is the endomorphism algebra of the antidominant projective in category $\mathcal{O}$.

Yuri also talked about the Artin representation, his attempt to generalize it, exotic solutions to the coCartesian Yang-Baxter equation maybe work on this and which ones extend to the Gelfand MacPherson Vilonen action, its connection to Ng's result, and why he thinks there should be a more general result about the homotopy theory of spaces. Finally we also thought about the perverse sheaves on the knot complement in $\mathbb{R}^3$ open question, what's happening in S^3, its different topologically because π_2 is nontrivial in the S^3 knot complement.

6 References

References

[1] Alexander Beilinson. How to glue perverse sheaves.

[2] Alexei Bondal and Mikhail Kapranov. Representable functors, serre functors, and mutations.

[3] Tom Braden. Characterstic cycles for toric varieties; perverse sheaves on the rank stratification.

[4] Ryan Reich. Notes on beilinson's how to glue perverse sheaves. *arXiv*, 2010.