Conormal varieties

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Abstract

Spoke with Allen today about conormal varieties.

Contents

1 Introduction

We are interested in the $B_- \times B_+$ orbits in $M_{m,n}$. These are Matrix Schubert Varieties formed by looking at Northeast rank conditions, Fulton's essential set.

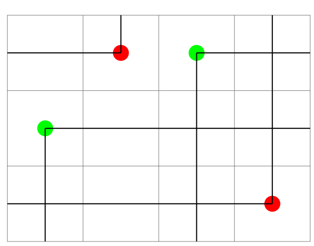
We are interested in the conormal variety to the $B_- \times B_+$ orbit through the partial permutation matrix π. This variety

$$C(B_- \pi B_+) \subset M_{m,n} \times M^*_{m,n}$$

satisfies the following equations (and more but they are not important for now)

$$p_1(C(B_-\pi B_+)) \text{ satisfies NE rank conditions for } \pi$$
$$p_2(C(B_-\pi B_+)) \text{ satisfies SW rank conditions for } \pi^*$$

where p_1, p_2 are the projections to the first and second factors, and π^* is defined combintorially as the partial permutation shown in red.



The resulting equations cut out the conormal variety and some other stuff of codimension 2 or more. Therefore, these equations are sufficient to know when components of the conormal variety to the stratification are smooth.

Let $\Lambda \subset T^*M_{m,n}$ be the conormal variety to the Matrix Schubert Variety stratification. This means that we are considering the union

$$\Lambda = \bigcup_{\pi} C(B_-\pi B_+)$$

There is the theorem that when G acts on M, then G acts on T^*M Hamiltonianly and there is a moment map $\mu : M \to \mathfrak{g}^*$ such that

$$\Phi^{-1}(0) = \Lambda = \bigcup C(\text{orbits in } M)$$

In our situation, we have the moment map for the action of $B_- \times B_+ \hookrightarrow G \times G$ acting on $M_{m,n}$. We have the composition

$$(X, Y) \mapsto (XY^T, -Y^TX) \mapsto (\text{Lower}(XY^T), \text{Upper}(-Y^TX))$$

where I want the lower triangular part of the first matrix and the upper triangular part of the second matrix (including the diagonal). Therefore when $j = k = n$ we have the following variety, the "lower-upper" variety.

$$\Lambda = \bigcup_{\pi} C(B_-\pi B_+) = \{(X, Y) \in M_n \times M_n \mid XY \text{ is lower triangular, } YX \text{ is upper triangular } \}$$

Remark 1.1 (Important)**.** *Braden points to tseveral pleasant features of Grassmannians that he studied which made the computation of* Perv_Λ *reasonable, all of which fail for the flag variety.*

1. *The action of the Borel on* Λ *has finitely many orbits.*
2. *The action of the Borel on* Λ *has connected stabilizers.*
3. *All of the singularities of the stratification are conical.*

This carries an action of $B_- \times B_+$ but not with finitely many orbits. However the equations from above provide us a way of detecting if two components of the conormal variety intersect codimension 1.

We can get perverse sheaves out of this by studying the microlocal geometry of Λ ala Gelfand MacPherson Vilonen, and Braden independently.