Homework 7

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Let K be a non-Archimedean complete valued field (CVF). By $\mathcal{A}$ we mean its valuation ring, $\mathfrak{m}$ the unique maximal ideal of $\mathcal{A}$, and $k := \mathcal{A}/\mathfrak{m}$ the residue field.

Definition 1. *We say K is a **perfectoid field** if the following three conditions hold:*

(P1) $\mathrm{char}(k) = p > 0$ *(but* $\mathrm{char}(K)$ *can be either* 0 *or* p*);*

*(P2) the valuation is non-*discrete*, i.e. the value group*

$$\Gamma := |K^\times| \subset \mathbb{R}^\times_{>0}$$

is not a discrete subgroup;

(P3) the Frobenius map

$$\Phi : \mathcal{A}/p\mathcal{A} \to \mathcal{A}/p\mathcal{A}, \quad x \mapsto x^p$$

is surjective.

In typical examples, one does not have an isomorphism in (P3).

Non-Example 1. *If K is a finite extension of $\mathbb{Q}_p$, then K is not perfectoid as the valuation is still discrete. Similarly, the completion of $\mathbb{Q}_p^{\mathrm{unr}}$ is not perfectoid. A finite field is perfect of characteristic > 0 but* not *perfectoid because it has trivial valuation (thus not a CVF).*

Example 1. *If $\mathrm{char}(K) = p$, then it is a simple exercise to show that K is a perfectoid field if and only if K is a perfect field (in addition to being a non-Archimedean CVF). A concrete example is*

$$K = \mathbb{F}_p\big((t^{1/p^\infty})\big) := \bigcup_{n \geq 1} \mathbb{F}_p\big((t^{1/p^n})\big),$$

where $|t| = a$, $|t^{1/p^n}| = a^{1/p^n}$ for a fixed constant $0 < a < 1$.

Example 2. *The completion $C = \widehat{\overline{\mathbb{Q}}_p}$, which is a CVF, appeared in the previous problem set. It is perfectoid — this will be verified in the first problem below. In this case, we usually write $\mathcal{O}_C$ for the valuation ring $\mathcal{A}$. Write $\mathcal{O}_{\overline{\mathbb{Q}}_p}$ for the valuation ring of $\overline{\mathbb{Q}}_p$, and $\mathfrak{m}_C, \mathfrak{m}_{\overline{\mathbb{Q}}_p}$ for the maximal ideals in the corresponding valuation rings. We verify this example in the following problem.*

Example 3. *The completion of $\mathbb{Q}_p(\mu_{p^\infty})$ turns out to be a perfectoid field as well. A general intuition is that a perfectoid field is "infinitely ramified" to allow the valuation to be non-discrete.*

Problem 1

(1) Show that the inclusion $\mathcal{O}_{\overline{\mathbb{Q}}_p} \subset \mathcal{O}_C$ induces isomorphisms

$$\mathcal{O}_{\overline{\mathbb{Q}}_p}/p\mathcal{O}_{\overline{\mathbb{Q}}_p} \cong \mathcal{O}_C/p\mathcal{O}_C \quad \text{and} \quad \mathcal{O}_{\overline{\mathbb{Q}}_p}/\mathfrak{m}_{\overline{\mathbb{Q}}_p} \cong \mathcal{O}_C/\mathfrak{m}_C.$$

(It is not hard to see that the residue field $\mathcal{O}_{\overline{\mathbb{Q}}_p}/\mathfrak{m}_{\overline{\mathbb{Q}}_p} \cong \overline{\mathbb{F}}_p$, since it is an algebraic extension of $\mathbb{F}_p$ containing an arbitrary finite extension of $\mathbb{F}_p$.)

(2) Show that $\Gamma = p^{\mathbb{Q}} = \{p^a : a \in \mathbb{Q}\}$ in this case, if we normalize the valuation such that $|p| = 1/p$.

(3) Check that the Frobenius map

$$\mathcal{O}_C/p\mathcal{O}_C \to \mathcal{O}_C/p\mathcal{O}_C$$

is surjective.

Solution: We first show that the natural inclusion $\mathcal{O}_{\overline{\mathbb{Q}}_p} \subset \mathcal{O}_C$ induces isomorphisms

$$\mathcal{O}_{\overline{\mathbb{Q}}_p}/p \;\cong\; \mathcal{O}_C/p \qquad \text{and} \qquad \mathcal{O}_{\overline{\mathbb{Q}}_p}/\mathfrak{m}_{\overline{\mathbb{Q}}_p} \;\cong\; \mathcal{O}_C/\mathfrak{m}_C.$$

Since $\mathcal{O}_{\overline{\mathbb{Q}}_p}$ is dense in $\mathcal{O}_C$, given any $\bar{x} \in \mathcal{O}_C/p$, we may choose a lift $x \in \mathcal{O}_C$ and find $y \in \mathcal{O}_{\overline{\mathbb{Q}}_p}$ such that $|x - y| \leq |p|$. This implies $x \equiv y \pmod{p}$, establishing surjectivity. Conversely, if $a \in \mathcal{O}_{\overline{\mathbb{Q}}_p}$ maps to 0 in $\mathcal{O}_C/p$, then $a \in p\mathcal{O}_C$. Writing $a = pb$ with $b \in \mathcal{O}_C$ and using that $\mathcal{O}_C$ and $\mathcal{O}_{\overline{\mathbb{Q}}_p}$ share the same valuation, we have $b \in \mathcal{O}_{\overline{\mathbb{Q}}_p}$. Hence $a \in p\mathcal{O}_{\overline{\mathbb{Q}}_p}$, proving injectivity. The same argument, applied to the maximal ideals $\mathfrak{m}_{\overline{\mathbb{Q}}_p}$ and $\mathfrak{m}_C$, shows that the reduction maps mod $\mathfrak{m}$ also induce an isomorphism of residue fields. In particular, both residue fields are algebraic closures of $\mathbb{F}_p$.

Next we compute the value group. Since completion does not change valuations, we have

$$|C^\times| = |\overline{\mathbb{Q}}_p^\times|.$$

For each finite extension $L/\mathbb{Q}_p$ with ramification index e, the value group satisfies $|L^\times| = p^{\frac{1}{e}\mathbb{Z}}$. Taking the union over all finite extensions inside $\overline{\mathbb{Q}}_p$ gives

$$|\overline{\mathbb{Q}}_p^\times| = \bigcup_{e \geq 1} p^{\frac{1}{e}\mathbb{Z}} = p^{\mathbb{Q}}.$$

Hence $\Gamma = |C^\times| = p^{\mathbb{Q}}$, which is non-discrete and p-divisible when the normalization $|p| = p^{-1}$ is used.

Finally, we verify that Frobenius on $\mathcal{O}_C/p$ is surjective. By the first part, it suffices to check surjectivity on $\mathcal{O}_{\overline{\mathbb{Q}}_p}/p$. Given $\bar{a} \in \mathcal{O}_{\overline{\mathbb{Q}}_p}/p$, choose a lift $a \in \mathcal{O}_{\overline{\mathbb{Q}}_p}$, which lies in the ring of integers $\mathcal{O}_L$ of some finite extension $L/\mathbb{Q}_p$. In $\mathcal{O}_L/p \cong k_L$, the residue field k_L is finite of characteristic p, so Frobenius $x \mapsto x^p$ is bijective. Thus there exists $b \in \mathcal{O}_L \subset \mathcal{O}_{\overline{\mathbb{Q}}_p}$ such that $b^p \equiv a \pmod{p}$. Therefore Frobenius is surjective on $\mathcal{O}_{\overline{\mathbb{Q}}_p}/p$, and hence also on $\mathcal{O}_C/p$.

For problems 2–4, we work in the following general setting. Let K be a perfectoid field with valuation ring $\mathcal{A}$. Fix a nonzero element $\varpi \in \mathcal{A}$ such that

$$|p| \leq |\varpi| < 1.$$

Such an element is called a *pseudo-uniformizer*, as it plays a similar role to a uniformizer for a DVR.

If $\mathrm{char}(K) = 0$, we can take $\varpi = p$. If $\mathrm{char}(K) = p$, we should make a different choice, e.g. for $K = \mathbb{F}_p((t^{1/p^\infty}))$, take $\varpi = t$. In either case, (P3) implies that the p-th power map induces a surjection

$$\mathcal{A}/\varpi\mathcal{A} \twoheadrightarrow \mathcal{A}/\varpi\mathcal{A} \quad (x \mapsto x^p).$$

Consider the inverse limit along this p-th power map:

$$\mathcal{A}^\flat := \varprojlim_{x \mapsto x^p} \mathcal{A}/\varpi\mathcal{A} = \{(x_0, x_1, \dots) : x_i \in \mathcal{A}/\varpi\mathcal{A},\ x_{i+1}^p = x_i \text{ for all } i \geq 0\}.$$

Problem 2

(1) Prove that $\mathcal{A}^\flat$ is a perfect ring of characteristic p. (Addition and multiplication are defined term-by-term.)

(2) Show that the canonical map

$$f : \varprojlim_{x \mapsto x^p} \mathcal{A} = \{(y_0, y_1, \dots) : y_i \in \mathcal{A},\ y_{i+1}^p = y_i\} \to \mathcal{A}^\flat$$

given by $f((y_i)) = (y_i \bmod \varpi)$ is a bijection.

Solution: Each $\mathcal{A}/\varpi$ has characteristic p, hence so does the inverse limit; addition and multiplication are defined termwise and respect the transition maps, so $\mathcal{A}^\flat$ is a ring. The Frobenius

$$\varphi : \mathcal{A}^\flat \to \mathcal{A}^\flat, \quad (x_0, x_1, \ldots) \mapsto (x_0^p, x_1^p, \ldots)$$

is bijective: its inverse is the right shift

$$\varphi^{-1}(x_0, x_1, \ldots) = (x_1, x_2, \ldots),$$

which is well-defined because $x_{i+1}^p = x_i$. Thus $\mathcal{A}^\flat$ is perfect.

Lemma 1 If $a \equiv b \pmod{\varpi^m}$ in $\mathcal{A}$, then $a^p \equiv b^p \pmod{\varpi^{m+1}}$. Consequently, for any $r \geq 1$,

$$a^{p^r} \equiv b^{p^r} \pmod{\varpi^{m+r}}.$$

Proof. Write $a = b + \varpi^m u$. Then

$$a^p - b^p = \sum_{j=1}^{p} \binom{p}{j} b^{p-j} (\varpi^m u)^j.$$

For $j \geq 2$ the term is divisible by ϖ^{2m}. For $j = 1$ it equals $p\, b^{p-1}\, \varpi^m u$, which is divisible by $p\varpi^m$; since $v(p) \geq v(\varpi)$ (i.e. $|p| \leq |\varpi|$), we have $p\varpi^m \in \varpi^{m+1}\mathcal{A}$. Hence $a^p \equiv b^p \pmod{\varpi^{m+1}}$. Iterate to get the p^r statement. $\square$

Injectivity of f. Suppose (y_i) and (y'_i) in $\varprojlim \mathcal{A}$ have the same reduction mod ϖ. Then $d_i := y_i - y'_i \in \varpi\mathcal{A}$ for all i, and

$$d_i \;=\; y_{i+1}^p - (y'_{i+1})^p.$$

By Lemma 1 with $m = 1$, $d_i \in \varpi^2\mathcal{A}$. Repeating, we get $d_i \in \varpi^n\mathcal{A}$ for all $n \geq 1$. Since $\mathcal{A}$ is ϖ-adically separated, $\bigcap_{n\geq 1} \varpi^n \mathcal{A} = \{0\}$, hence $d_i = 0$ for all i. Thus f is injective.

Surjectivity of f. Let $x = (x_0, x_1, \ldots) \in \mathcal{A}^\flat$. Choose arbitrary lifts $y_n^{(0)} \in \mathcal{A}$ of x_n for each $n \geq 0$. For fixed i, define a sequence (indexed by $n \geq i$)

$$z_i^{(n)} \;:=\; \left(y_n^{(0)}\right)^{p^{\,n-i}} \in \mathcal{A}.$$

If $n > m \geq i$ and $y_n^{(0)} \equiv y_m^{(0)} \pmod{\varpi}$ (true because both reduce to x_n transported along p-power to x_m), then by Lemma 1

$$\left(y_n^{(0)}\right)^{p^{\,n-i}} \equiv \left(y_m^{(0)}\right)^{p^{\,n-i}} \equiv \left(y_m^{(0)}\right)^{p^{\,m-i}} \pmod{\varpi^{\,(n-i)+1}}.$$

Hence $(z_i^{(n)})_{n\geq i}$ is Cauchy in the ϖ-adic topology. Since $\mathcal{A}$ is ϖ-adically complete, the limit

$$y_i \;:=\; \lim_{n\to\infty} z_i^{(n)} \in \mathcal{A}$$

exists. Define $y = (y_0, y_1, \ldots)$.

By construction, $y_i \equiv x_i \pmod{\varpi}$ (pass to the limit of the reductions), and

$$y_{i+1}^p \;=\; \left(\lim_{n\to\infty} \left(y_n^{(0)}\right)^{p^{\,n-(i+1)}}\right)^p \;=\; \lim_{n\to\infty} \left(y_n^{(0)}\right)^{p^{\,n-i}} \;=\; y_i,$$

using continuity of $t \mapsto t^p$. Thus $y \in \varprojlim \mathcal{A}$ and $f(y) = x$. So f is surjective and hence a bijection.

For each $x \in \mathcal{A}^\flat$, write $f^{-1}(x) = (y_0, y_1, \ldots)$. Define $x^\sharp \in \mathcal{A}$ to be y_0, the first coordinate in $f^{-1}(x)$. Thus we obtain a map

$$(\cdot)^\sharp : \mathcal{A}^\flat \to \mathcal{A}, \qquad x \mapsto x^\sharp.$$

Problem 3

(1) Given $x \in K^\times$, show there exists $y \in K^\times$ such that $|y^p| = |x|$. (This tells us that $\Gamma = |K^\times|$ is not only non-discrete but p-divisible.)

(2) Prove that there exists an element $\varpi^\flat \in \mathcal{A}^\flat$ such that $|(\varpi^\flat)^\sharp| = |\varpi|$.

(3) Consider the localization

$$K^\flat := \mathcal{A}^\flat[1/\varpi^\flat].$$

Show that the map $(\cdot)^\sharp$ extends to a multiplicative map $K^\flat \to K$, still denoted $(\cdot)^\sharp$, making $K^\flat$ a field of characteristic p.

(4) Show that the function

$$|\cdot| : K^\flat \to \mathbb{R}_{\geq 0}, \qquad |y| := |y^\sharp|$$

is a valuation on $K^\flat$, and that $\mathcal{A}^\flat$ is its valuation ring.

Solution:

1. First treat the case $0 < v(x) < v(p)$. By surjectivity of Frobenius on $\mathcal{A}/p$, pick $b \in \mathcal{A}$ with $b^p \equiv x \pmod{p}$, so $x = b^p + pc$ for some $c \in \mathcal{A}$. Since $v(pc) \geq v(p) > v(x)$, the ultrametric inequality gives $v(x) = v(b^p)$ and hence $|b^p| = |x|$. Thus the claim holds with $y = b$ in this range.

   For general $x \neq 0$, choose $N \in \mathbb{Z}$ with $0 < v(xp^{-N}) < v(p)$; apply the previous paragraph to $x' := xp^{-N}$ to get $y_0 \in K^\times$ with $|y_0^p| = |x'|$.

2. By (1) choose $y \in \mathcal{A}$ with $|y^p| = |\varpi|$. Using surjectivity of Frobenius on $\mathcal{A}/\varpi$, choose inductively a sequence

$$x_0 := \overline{y} \in \mathcal{A}/\varpi, \qquad x_{i+1} \in \mathcal{A}/\varpi \ \text{ with } \ x_{i+1}^p = x_i \ \text{ for all } i \geq 0.$$

   Let $\varpi^\flat := (x_0, x_1, x_2, \ldots) \in \mathcal{A}^\flat$. Pick arbitrary lifts $\tilde{x}_i \in \mathcal{A}$ of x_i. Then by the defining formula,

$$(\varpi^\flat)^\sharp = \lim_{i\to\infty} \tilde{x}_i^{p^i},$$

   and (since $\tilde{x}_0$ may be taken to be y) we get $|(\varpi^\flat)^\sharp| = |y|^p = |\varpi|$.

3. Define $(\cdot)^\sharp$ on $K^\flat$ by

$$\left(\frac{a}{(\varpi^\flat)^n}\right)^\sharp := \frac{a^\sharp}{\left((\varpi^\flat)^\sharp\right)^n} \quad \in K \qquad (a \in \mathcal{A}^\flat,\ n \in \mathbb{Z}_{\geq 0}).$$

   This is well-defined because $(\cdot)^\sharp$ is multiplicative on $\mathcal{A}^\flat$ and $(\varpi^\flat)^\sharp \neq 0$. It is again multiplicative by construction.

Since $\mathcal{A}^\flat$ is perfect of characteristic p (Problem 2) and $\varpi^\flat$ is a nonzerodivisor in the valuation setting below, inverting $\varpi^\flat$ yields a nonzero characteristic-p domain.

4. Multiplicativity is immediate:

$$|xy|_\flat = |(xy)^\sharp| = |x^\sharp y^\sharp| = |x^\sharp| \cdot |y^\sharp| = |x|_\flat \, |y|_\flat.$$

The ultrametric inequality follows from that in K and continuity of $(\cdot)^\sharp$:

$$|x+y|_\flat = |(x+y)^\sharp| \leq \max\{|x^\sharp|, |y^\sharp|\} = \max\{|x|_\flat, |y|_\flat\}.$$

Nontriviality holds because $|(\varpi^\flat)|_\flat = |(\varpi^\flat)^\sharp| = |\varpi| < 1$.

We claim $\mathcal{A}^\flat = \{z \in K^\flat : |z|_\flat \leq 1\}$. If $a \in \mathcal{A}^\flat$, then $a^\sharp \in \mathcal{A}$ so $|a|_\flat = |a^\sharp| \leq 1$. Conversely, let $z = a/(\varpi^\flat)^n$ with $a \in \mathcal{A}^\flat$. If $|z|_\flat \leq 1$, then

$$|a^\sharp| \leq |(\varpi^\flat)^\sharp|^n = |\varpi|^n,$$

so $a^\sharp \in \varpi^n \mathcal{A}$. Using multiplicativity and that $(\cdot)^\sharp$ detects divisibility by $\varpi^\flat$ via absolute values, this implies $a \in (\varpi^\flat)^n \mathcal{A}^\flat$, hence $z \in \mathcal{A}^\flat$. Therefore $\mathcal{A}^\flat$ is exactly the valuation ring for $|\cdot|_\flat$, and its fraction field is $K^\flat$.