Homework 1

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Problem 1 (from RS1) Δ is the unit disk, $\Delta^\times = \Delta \setminus \{0\}$.

1. Prove that a holomorphic map $f : \Delta^\times \to \mathbb{C}$ which has an *essential* (non–pole) singularity at 0 has dense image in $\mathbb{C}$.

2. Use this to show that any map $f : \Delta^\times \to \mathbb{P}$ which is never more than N-to-1, for a fixed number N, extends holomorphically to Δ.

3. Generalize (b) to the case when the target is an arbitrary compact Riemann surface R, by invoking Riemann's theorem which guarantees the existence of meromorphic functions on R.

Remark. A much stronger (and more difficult) version of (a) says that f assumes every value infinitely often, possibly with a single exception (such as 0, for $e^{1/z}$). This is the Great Picard Theorem.

Solution:

Problem 2 Identify successive pairs of edges of a $2n$-gon, labelled $a, a, b, b, c, c, \ldots$, by matching points on matching edge pairs in *parametric order.* (Equivalently, identify the points θ and $\theta + \pi/n$ on the boundary of the unit disk.)

Explain why the surface obtained is homeomorphic to the one obtained by sewing on n Möbius strips to an n-holed sphere, along matching boundaries.

Which of these gives a Klein bottle?

Remark. It's not hard to show that every closed non–orientable surface is obtained in this way, but please *do not* write a complete proof of that ...

Solution:

Problem 3 (from RS2) Show that any degree 2 holomorphic map $f : \mathbb{C}/L \to \mathbb{P}$ is a "Möbius transform of a shifted $\wp$–function":

$$f(u) = \frac{a\wp(u-w)+b}{c\wp(u-w)+d}, \qquad a, b, c, d, w \in \mathbb{C}.$$

Comment. You may assume standard facts about Möbius transformations.

Solution:

Problem 4 (from RS2) Prove that any two meromorphic functions f, g on a compact Riemann surface are *algebraically related*: $P(f, g) \equiv 0$ for some 2–variable polynomial P.

Hint. Recall that a meromorphic function without poles must be constant, and estimate, in terms of N, the dimension of the vector space spanned by the functions $f^m g^n$, for $0 \le m, n \le N$, to conclude that a linear dependence relation must hold for large N.

Solution:

Problem 5

1. Specializing the period lattice to the limiting case $\omega_1 = \pi$, $\omega_2 \to i \cdot \infty$, show that
$$\wp(u) \to \cot^2(u) + \tfrac{2}{3}, \qquad \zeta(u) \to \cot(u) + u, \qquad \sigma(u) \to \sin(u) \cdot \exp(u^2/2).$$
2. Do the series expansions apply?
3. Find and check the differential equation expressing $(\wp')^2$ in terms of $\wp$ in this limit.
4. Describe the (singular) analytic set in $\mathbb{C}^2$ parametrized as $z = \wp(u), w = \wp'(u)$.

Solution: