Homework 5

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For Questions 1 and 2, you may use the correspondence indicated in class between the representation of H^1 classes by classes by principal parts versus Dolbeault distributions.

Problem 1 For a compact Riemann surface R, verify that the Serre duality pairing

$$H^1(R;\mathcal{O}) \otimes H^0(R;\Omega^1) \longrightarrow \mathbb{C}$$

defined by principal parts and residues agrees with the one given by integration of Dolbeault representatives.

Using the relation to harmonic forms, explain how this relates to Poincaré duality on R.

Solution: Choose a meromorphic function f on R whose principal part at each p_i with prescribed principal parts. Let U_i be pairwise disjoint coordinate discs around p_i, and choose $\chi \in C^\infty(R)$ such that $\chi \equiv 1$ on smaller discs $U_i' \subset U_i$ and $\chi \equiv 0$ outside $\bigcup_i U_i$. Define a $(0,1)$-*current*

$$T_f := \bar{\partial}(\chi f).$$

Since $\bar{\partial}^2 = 0$, T_f is $\bar{\partial}$-closed. If we replace f by $f+g$ for a global meromorphic function g (with poles in D) or change χ within the same constraints, T_f changes by a current of the form $\bar{\partial}u$, so the class $[T_f]$ in

$$H^{0,1}_{\bar{\partial}}(R) \cong H^1(R,\mathcal{O})$$

depends only on the underlying principal parts.

Let $\omega \in H^0(R,\Omega^1)$ be a holomorphic 1-form. The *Dolbeault* definition of the pairing is

$$\langle \alpha, \omega\rangle_{Dol} := \frac{1}{2\pi i}\int_R T_f \wedge \omega = \frac{1}{2\pi i}\int_R \bar{\partial}(\chi f)\wedge \omega.$$

Since ω is of type $(1,0)$ and holomorphic, $\bar{\partial}\omega = 0$, hence

$$\bar{\partial}(\chi f)\wedge \omega = \bar{\partial}(\chi f \omega).$$

Let $D_i \subset U_i'$ be small closed discs around p_i and set

$$R_\varepsilon := R \setminus \bigcup_i D_i(\varepsilon),$$

where $D_i(\varepsilon)$ are concentric discs of radius ε. On R_ε the form $\chi f \omega$ is smooth with compact support, so Stokes' theorem gives

$$\int_{R_\varepsilon} \bar{\partial}(\chi f\omega) = \int_{\partial R_\varepsilon} \chi f \omega = -\sum_i \int_{\partial D_i(\varepsilon)} f\omega,$$

the sign coming from the induced orientation on the boundary.

Letting $\varepsilon \to 0$ and using the residue theorem,

$$\int_{\partial D_i(\varepsilon)} f\omega \;\longrightarrow\; 2\pi i \,\mathrm{Res}_{p_i}(f\omega),$$

we obtain

$$\frac{1}{2\pi i}\int_R \bar\partial(\chi f)\wedge\omega = \sum_i \mathrm{Res}_{p_i}(f\omega).$$

This is precisely the *principal parts* definition of the Serre pairing.

Now equip R with any Hermitian (necessarily Kähler) metric. Hodge theory yields the decompositions

$$H^1_{\mathrm{dR}}(R,\mathbb{C}) \;\cong\; \mathcal{H}^1(R) \;\cong\; H^{1,0}_{\bar\partial}(R)\oplus H^{0,1}_{\bar\partial}(R),$$

and every class has a unique harmonic representative. Moreover,

$$H^0(R,\Omega^1)\cong H^{1,0}_{\bar\partial}(R)$$

consists of harmonic $(1,0)$-forms, and

$$H^1(R,\mathcal{O})\cong H^{0,1}_{\bar\partial}(R)$$

is represented by harmonic $(0,1)$-forms. Complex conjugation gives an isomorphism

$$\overline{H^{1,0}_{\bar\partial}(R)}\cong H^{0,1}_{\bar\partial}(R)$$

Poincaré duality on R is given by the nondegenerate pairing

$$H^1_{\mathrm{dR}}(R,\mathbb{C})\times H^1_{\mathrm{dR}}(R,\mathbb{C})\longrightarrow\mathbb{C},\qquad ([\alpha],[\beta])\mapsto\int_R\alpha\wedge\beta.$$

It is clear that $\alpha\wedge\beta$ is nonzero only if α and β are of complementary types, i.e. their wedge is of type $(1,1)$, since $(1,0)\wedge(1,0)$ and $(0,1)\wedge(0,1)$ necessarily vanish. Thus the Poincaré pairing restricts to a nondegenerate pairing

$$H^{0,1}_{\bar\partial}(R)\;\otimes\;H^{1,0}_{\bar\partial}(R)\longrightarrow\mathbb{C},\qquad(\eta,\omega)\mapsto\int_R\eta\wedge\omega,$$

with η,ω harmonic representatives.

Under the identifications

$$H^1(R,\mathcal{O})\cong H^{0,1}_{\bar\partial}(R),\qquad H^0(R,\Omega^1)\cong H^{1,0}_{\bar\partial}(R),$$

the Serre pairing of α and ω is

$$\langle\alpha,\omega\rangle=\frac{1}{2\pi i}\int_R\eta\wedge\omega,$$

where η is the harmonic $(0,1)$-representative of α. In particular, on a compact Riemann surface the Serre duality

$$H^1(R,\mathcal{O})\;\cong\;H^0(R,\Omega^1)^\vee$$

is nothing but Poincaré duality in degree 1 up to the constant factor $2\pi i$, expressed via the Hodge decomposition of $H^1_{\mathrm{dR}}(R,\mathbb{C})$.

Problem 2 For a compact Riemann surface R, verify that the map

$$H^1(R;\mathbb{Z}) \longrightarrow H^1(R;\mathcal{O})$$

corresponds to the period map

$$H_1(R;\mathbb{Z}) \otimes H^0(R;\Omega^1) \longrightarrow \mathbb{C}$$

under integral Poincaré duality and Serre duality on R.

Solution: Let $i : H^1(R;\mathbb{Z}) \to H^1(R;\mathcal{O})$ be the given homomorphism. We need to show for every $c \in H^1(R;\mathbb{Z})$ and $\omega \in H^0(R,\Omega^1)$, the Serre pairing $\langle i(c),\omega\rangle_{\text{Serre}}$ equals the period of ω along the 1–cycle Poincaré dual to c.

By Hodge theory, every class in $H^1(R;\mathbb{R})$ has a unique harmonic representative. An element $c \in H^1(R;\mathbb{Z})$ maps to a real class $c_{\mathbb{R}} \in H^1(R;\mathbb{R})$ whose harmonic representative we denote by α so

$$[\alpha]_{\text{dR}} = c_{\mathbb{R}} \in H^1_{\text{dR}}(R;\mathbb{R}).$$

Decompose α

$$\alpha = \alpha^{1,0} + \alpha^{0,1}, \qquad \alpha^{0,1} = \overline{\alpha^{1,0}},$$

since α is real. Under the Dolbeault isomorphism and Hodge decomposition, we have

$$H^1(R,\mathcal{O}) \;\cong\; H^{0,1}_{\bar\partial}(R)$$

and the image $i(c) \in H^1(R,\mathcal{O})$ is represented by the harmonic $(0,1)$–form $\alpha^{0,1}$.

We know that the Serre pairing can be described as

$$\langle \beta,\omega\rangle_{\text{Serre}} = \frac{1}{2\pi i}\int_R \eta^{0,1}\wedge\omega$$

whenever $\beta \in H^1(R,\mathcal{O})$ is represented by a harmonic $(0,1)$–form $\eta^{0,1}$ and $\omega \in H^0(R,\Omega^1)$ is a holomorphic 1–form.

Applying this to $\beta = i(c)$ and $\eta^{0,1} = \alpha^{0,1}$ gives

$$\langle i(c),\omega\rangle_{\text{Serre}} = \frac{1}{2\pi i}\int_R \alpha^{0,1}\wedge\omega.$$

Since R has complex dimension 1, a $(2,0)$–form vanishes, hence $\alpha^{1,0}\wedge\omega = 0$, and therefore

$$\alpha^{0,1}\wedge\omega = (\alpha^{1,0}+\alpha^{0,1})\wedge\omega = \alpha\wedge\omega.$$

Thus

$$\langle \iota^* c,\omega\rangle_{\text{Serre}} = \frac{1}{2\pi i}\int_R \alpha\wedge\omega. \tag{1}$$

Integral Poincaré duality gives a perfect pairing

$$H^1(R;\mathbb{Z}) \times H_1(R;\mathbb{Z}) \longrightarrow \mathbb{Z},$$

and we denote by $\gamma_c \in H_1(R;\mathbb{Z})$ the Poincaré dual of c.

The de Rham realization of this pairing is as follows. The class $c_{\mathbb{R}} \in H^1(R;\mathbb{R})$ is represented by the closed 1–form α with integral periods, i.e.

$$\int_\gamma \alpha \in \mathbb{Z} \quad \text{for all } \gamma \in H_1(R;\mathbb{Z}).$$

The Poincaré dual cycle γ_c is then characterized by

$$\int_{\gamma_c} \beta = \int_R \alpha \wedge \beta \quad \text{for all closed 1–forms } \beta,$$

Thus, if we identify

$$H^1(R;\mathbb{Z}) \xrightarrow{\text{PD}} H_1(R;\mathbb{Z}) \quad \text{and} \quad H^1(R;\mathcal{O}) \xrightarrow{\text{Serre}} H^0(R,\Omega^1)^\vee,$$

the class $c \in H^1(R;\mathbb{Z})$ maps to the functional

$$H^0(R,\Omega^1) \longrightarrow \mathbb{C}, \qquad \omega \longmapsto \frac{1}{2\pi i}\int_{\gamma_c} \omega.$$

This is precisely the period map (up to the factor $1/(2\pi i)$)

$$H_1(R;\mathbb{Z}) \otimes H^0(R,\Omega^1) \longrightarrow \mathbb{C}, \qquad (\gamma,\omega) \longmapsto \int_\gamma \omega,$$

with $\gamma = \gamma_c$ the Poincaré dual of c.

Problem 3 Show that the period mapping gives an isomorphism

$$H_1(R;\mathbb{Z}) \xrightarrow{\sim} H_1(J;\mathbb{Z}),$$

which can be realized geometrically by the Abel–Jacobi map

$$R \longrightarrow J_1.$$

Show that under this correspondence, $c_1(\Theta) \in \Lambda^2 H_1(R)$ is the intersection pairing on R.

Hints for the second part: You can deduce it from the periodicity formulas of the Riemann Θ-function. Alternatively, you can find this by exploiting the facts that the Poincaré dual of $c_1(\Theta)$ in J_{g-1} is the Theta divisor, the image of $\mathrm{Sym}^{g-1}(R)$. The maps

$$\mathrm{Sym}^g(R) \longrightarrow J_g \qquad \text{and} \qquad \mathrm{Sym}^{g-1}(R) \longrightarrow \mathrm{div}(\Theta)$$

have degree 1.

Solution: The presentation of the Jacobian J as

$$J \cong H^1(R;\mathcal{O})/H_1(R;\mathbb{Z})$$

makes it clear that $H_1(J;\mathbb{Z})$ is naturally identified with $H_1(R;\mathbb{Z})$, since the universal cover of J is the vector space $H^1(R;\mathcal{O})$. The period mapping

$$H_1(R;\mathbb{Z}) \to H_1(J;\mathbb{Z})$$

is injective because of the Riemann bilinear relations, and since both groups are free abelian of rank $2g$, it is an isomorphism. Pick a base point $p_0 \in R$ and define the Abel–Jacobi map

$$\varphi : R \to J, \qquad p \mapsto \left[\omega \mapsto \int_{p_0}^{p} \omega\right].$$

precisely implements the lift of the period mapping to the universal cover and hence induces the same isomorphism on H_1.

Pick a symplectic basis $\{a_1,\dots,a_g,b_1,\dots,b_g\}$ of $H_1(R,\mathbb{Z})$, i.e.

$$a_i \cdot a_j = 0, \quad b_i \cdot b_j = 0, \quad a_i \cdot b_j = \delta_{ij}.$$

Under the identification $H_1(R,\mathbb{Z}) \xrightarrow{\sim} \Lambda \cong H_1(J,\mathbb{Z})$ coming from the period map and the Abel–Jacobi embedding, a homology class $\gamma \in H_1(R,\mathbb{Z})$ corresponds to an integral vector $(m,n) \in \mathbb{Z}^{2g}$. The intersection pairing on $H_1(R,\mathbb{Z})$ is given in these coordinates by

$$(m,n)\cdot(m',n') \;=\; m^{\mathrm{T}}n' - m'^{\mathrm{T}}n.$$

The Riemann theta function with period matrix τ is

$$\theta(z \mid \tau) := \sum_{k\in\mathbb{Z}^g} \exp\bigl(\pi i\, k^{\mathsf{T}}\tau k + 2\pi i\, k^{\mathsf{T}} z\bigr), \quad z \in \mathbb{C}^g.$$

The Riemann theta function satisfies the quasi-periodicity property.

$$\theta\bigl(z + m + \tau n \mid \tau\bigr) = \exp\bigl(-\pi i n^{\mathsf{T}}\tau n - 2\pi i n^{\mathsf{T}} z\bigr)\theta(z \mid \tau)$$

In particular, the Riemann theta function defines a holomorphic section of the line bundle $\mathcal{O}_J(\Theta)$.

Hence, identifying $H^2(U,\mathbb{Z})$ and $H^2(X,\mathbb{Z})$ by the above isomorphism, the Chern class of L is simply $\delta(\mathrm{cl}\{e_u\})$. Write $e_u(z) = e^{2\pi i f_u(z)}$ with f_u holomorphic in V. Then by definition, $\delta(\mathrm{cl}\{e_u\}) \in H^2(U,\mathbb{Z})$ is given by the 2–cocycle $F(u_1,u_2)$ on U with coefficients in $\mathbb{Z}$ defined by

$$F(u_1,u_2) = f_{u_2}(z+u_1) - f_{u_1+u_2}(z) + f_{u_1}(z) \in \mathbb{Z}. \tag{$*$}$$

Lemma 1 (Mumford) Let $U \subset V$ be a lattice in a complex vector space V. The map which associates to any map $F : U \times U \to \mathbb{Z}$ the map $AF : U \times U \to \mathbb{Z}$ defined by

$$AF(u_1, u_2) = F(u_1, u_2) - F(u_2, u_1)$$

maps the group of 2–cocycles $Z^2(U, \mathbb{Z})$ into the space of alternating linear maps $U \times U \to \mathbb{Z}$, and induces an isomorphism

$$A : \ H^2(U, \mathbb{Z}) \xrightarrow{\sim} \operatorname{Hom}(\Lambda^2 U, \mathbb{Z}) \cong \Lambda^2 \operatorname{Hom}(U, \mathbb{Z}).$$

Furthermore for $\xi, \eta \in \operatorname{Hom}(U, \mathbb{Z}) = H^1(U, \mathbb{Z})$, we have $A(\xi \smile \eta) = \xi \wedge \eta$.

Proposition 2 (Mumford) The Chern class of the line bundle corresponding to $\{e_u\} \in Z^1(U, H^*)$ is the alternating 2–form on U with values in $\mathbb{Z}$ given by

$$E(u_1, u_2) = f_{u_2}(z + u_1) + f_{u_1}(z) - f_{u_1}(z + u_2) - f_{u_2}(z), \qquad (z \text{ arbitrary in } V), \quad (**)$$

where

$$e_u(z) = e^{2\pi i\, f_u(z)}.$$

Moreover if we extend E $\mathbb{R}$–linearly to a map $V \times V \to \mathbb{R}$, then E satisfies the identity

$$E(ix, iy) = E(x, y) \qquad \text{for } x, y \in V.$$

Problem 4 Prove the following generalized Cauchy formula for a smooth function f defined in the unit disk Δ:

$$f(z, \bar{z}) = \frac{1}{2\pi i} \oint_{|\zeta - z| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \iint_{\Delta'} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z},$$

where $\Delta' \subset \Delta$ is the subdisk of radius $r < 1$.

Remark: When f is holomorphic, you recover Cauchy's formula.

Problem 5 Let $L \to X$ be a holomorphic line bundle on a complex manifold, and let $\alpha \in \mathcal{E}^{0,1}$ be a $\bar{\partial}$-closed form. Show that the re-defined operator

$$\bar{\bar{\partial}} = \bar{\partial} + \alpha$$

on sections of L defines a new holomorphic structure L' on the same underlying bundle, where local holomorphic sections are defined as those killed by $\bar{\bar{\partial}}$. Show that $L \simeq L'$ if α is $\bar{\partial}$-exact. Relate this to the exponential sequence.

Remark: For vector bundles, the same applies with an $\alpha \in \mathcal{E}^{0,1}(\operatorname{End}(V))$ satisfying the non-linear equation

$$\bar{\partial}\alpha + \alpha \wedge \alpha = 0.$$

The new bundle is isomorphic to the old one if $\alpha = a^{-1}\bar{\partial}a$, for some smooth section a of $\mathrm{Aut}(V)$.

Problem 6 Let V be a complex g-dimensional vector space and $L \simeq \mathbb{Z}^{2g} \subset V$ a lattice. Let $A = V/L$.

1. Using harmonic theory, compute the Dolbeault cohomology $H^*(A; \mathcal{O})$.

2. Show that the moduli space of holomorphic line bundles on A with zero Chern class is naturally identified with

$$A^\vee := V^\vee/L^\vee.$$

3. Show that the moduli space of holomorphic line bundles on $A^\vee$ is naturally identified with A.

4. Define a line bundle

$$\mathcal{P} \longrightarrow A \times A^\vee$$

from the trivial line bundle over $V \times V^\vee$ with connection

$$\nabla = d + i(x\, d\xi + \xi\, dx),$$

by quotienting out the $L \times L^\vee$-action as follows: identify the fiber $\mathbb{C}$ over $(x, \xi) \in V \times V^\vee$ with that over $(x + \ell, \xi + \lambda)$ by multiplication by

$$\exp\bigl(2\pi i(\lambda(x) + \xi(\ell))\bigr).$$

Show that $\mathcal{P}$ is holomorphic, that $\mathcal{P}|_{A\times\{a^\vee\}}$ is the line bundle over A classified by $a^\vee \in A^\vee$, and prove the corresponding statement for $\{a\} \times A^\vee$.

Problem 7 Show that, in the case of the Jacobian J of a Riemann surface R, one has a natural isomorphism $J \simeq J^\vee$.

Hint: Remember the natural Hilbert space structure on holomorphic differentials.

Remark: This self-duality is a property of principally polarized Abelian varieties, those A equipped with a positive line bundle having a single holomorphic section (the Θ-function).

Problem 8 Given a holomorphic line bundle $\mathcal{L}$ on a complex manifold and a smooth real closed 2-form ω in the cohomology class of $c_1(\mathcal{L})$, prove that there exists a Hermitian metric on $\mathcal{L}$ whose holomorphic connection has curvature $-2\pi i\,\omega$.

Conclude (from Kodaira vanishing) that the holomorphic line bundles on a compact Riemann surface R which carry metrics of positive curvature are precisely those of positive degree.

Show also that for every holomorphic vector bundle V on R, there exists a d so that the twisted bundle $V(D)$ has no H^1 for any $D > d$.

Problem 9 Show that isomorphism classes of *flat unitary* line bundles on a manifold X are classified by $H^1(X; U(1))$, with the constant sheaf $U(1)$ associated to the unit circle group in $\mathbb{C}^\times$.

When X is compact Kähler, compare the constant and holomorphic exponential sequences to conclude that the map

$$H^1(X; U(1)) \longrightarrow H^1(X; \mathcal{O}^\times)$$

induces a bijection from isomorphism classes of flat unitary line bundles to those of holomorphic line bundles with zero Chern class.

Remark: You probably need the Hodge decomposition theorem for the second part.

Problem 10 Prove the global $\partial\bar{\partial}$-Lemma on a compact Kähler manifold X: for any d-exact form $\varphi \in \mathcal{E}^{p,q}$, there exists $\psi \in \mathcal{E}^{p-1,q-1}$ with

$$\partial\bar{\partial}\psi = \varphi.$$

Hint: Show that

$$\varphi = \partial\bar{\partial}^*\Box\varphi$$

and use this and similar identities to find ψ.