Title

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Abstract

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1 Vector bundles and connections

This section follows definitions from [**?**].

Lemma 1.1. *Let M be an $\mathcal{O}_X$-module. Giving a left $\mathcal{D}_X$-module structure on M extending the $\mathcal{O}_X$-module structure is equivalent to giving a $\mathbb{C}$-linear morphism*

$$\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(M) \quad (\theta \mapsto \nabla_\theta),$$

satisfying the following conditions:

1. $\nabla_{f\theta}(s) = f\nabla_\theta(s) \quad (f \in \mathcal{O}_X, \theta \in \Theta_X, s \in M)$

2. $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s) \quad (f \in \mathcal{O}_X, \theta \in \Theta_X, s \in M)$

3. $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s) \quad (\theta_1, \theta_2 \in \Theta_X, s \in M)$

In terms of ∇ the left $\mathcal{D}_X$-module structure on M is given by

$$\theta_s = \nabla_\theta(s) \quad (\theta \in \Theta, s \in M).$$

The condition (3) above is called the *integrability condition* on M.

For a locally free left $\mathcal{O}_X$-module M of finite rank, a $\mathbb{C}$-linear morphism $\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(M)$ satisfying the conditions (1), (2) is usually called a *connection* (of the corresponding vector bundle). If it also satisfies the condition (3), it is called an *integrable (or flat) connection*. Hence we may regard a (left) $\mathcal{D}_X$-module as an integrable connection of an $\mathcal{O}_X$-module which is not necessarily locally free of finite rank.

Definition 1.2. *We say that a $\mathcal{D}_X$-module M is an* integrable connection *if it is locally free of finite rank over $\mathcal{O}_X$.*

Definition 1.3. *A* local system *on X is a locally free $\mathbb{C}_X$-module of finite rank.*

2 Principal bundles and connections

Definition 2.1. *A principal G-bundle on X is a fiber bundle P on X with a right action of G on P such that the action is free and transitive on each fiber of P and the projection map $\pi : P \to X$ is G-equivariant.*

Definition 2.2. *Let $V = \ker \pi : TP \to TX$ be the vertical bundle of P. An* Ehresmann connection *on P is a smooth subbundle H of TP, called the horizontal bundle of the connection, which is complementary to V, in the sense that it defines a direct sum decomposition $TE = H \oplus V$.*

The *fundamental vector field $X^{\#}$* at a point $p \in P$ is defined as:

$$X_p^{\#} = \left.\frac{d}{dt}\right|_{t=0} (p \cdot \exp(tX)),$$

where:

- $\exp : \mathfrak{g} \to G$ is the exponential map,
- $\exp(tX)$ is the one-parameter subgroup of G generated by X,

- $p \cdot \exp(tX)$ is the action of $\exp(tX)$ on p.

In other words, $X^\#$ is the tangent vector to the curve $t \mapsto p \cdot \exp(tX)$ at $t = 0$. The fundamental vector field $X^\#$ is vertical, meaning it is tangent to the fibers of P. This is because the action of G preserves fibers, so $X^\#$ lies in the kernel of the differential $d\pi : TP \to TM$, where $\pi : P \to M$ is the projection. The map $X \mapsto X^\#$ is a Lie algebra homomorphism. Moreover, for $g \in G$, the pushforward of $X^\#$ by the right action R_g is:

$$(R_g)_* X^\# = (\mathrm{Ad}_{g^{-1}} X)^\#,$$

where Ad is the adjoint action of G on $\mathfrak{g}$.

Definition 2.3. *A* principal connection *is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ (where $\mathfrak{g}$ is the Lie algebra of G) satisfying the following conditions:*

1. *For all $X \in \mathfrak{g}$,*
$$\omega(X^\#) = X,$$
*where $X^\#$ is the **fundamental vector field** on P generated by X.*

2. *For all $g \in G$,*
$$R_g^* \omega = Ad_{g^{-1}} \circ \omega,$$
where:
   - *$R_g : P \to P$ is the right action of g on P,*
   - *$R_g^* \omega$ is the pullback of ω by R_g,*
   - *$Ad_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of G on $\mathfrak{g}$.*

Definition 2.4. *The curvature of a $\mathfrak{g}$-valued 1-form ω on a principal G-bundle P is the $\mathfrak{g}$-valued 2-form $F = d\omega + \frac{1}{2}[\omega, \omega]$, where $d\omega$ is the exterior derivative of ω and $[\omega, \omega]$ is the Lie bracket of ω with itself. We say that ω is* flat, or integrable *if $F = 0$.*

Remark 2.5. *We can also write the curvature as the failure of commutativity of the covariant derivative:*
$$F(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

3 Associated bundle construction

Given a principal G-bundle P where G acts freely transitively on the fibers, and a representation $G \to GL(V)$, we can form an associated vector bundle $E = P \times_G V$ by taking the quotient of

$P \times V$ by the action of G given by

$$g \cdot (p, v) = (p \cdot g^{-1}, g \cdot v).$$

In other words, if P has transition functions $U_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, then E has transition functions $U_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$. Conversely, if E is a vector bundle with structure group G via a representation ρ, we can recover the principal G-bundle P as the frame bundle of E.

The notion of a linear connection on E coincides precisely with the notion of a principal connection on P. Likewise does the notion of integrability of a connection.

4 Local systems

Let G^{discrete} be G with the discrete topology. Then local systems are precisely G^{discrete}-bundles on X. Other interpretations of local systems include:

1. A local system is given by a covering of open sets, transition functions $\gamma_{ij} : U_i \cap U_j \to G$ which are locally constant and satisfy the 1-cocycle condition $g_{ij} g_{jk} g_{ki} = id$. Equivalence of local systems is given by a common refinement of two coverings and a family of maps to G which conjugate one system of transition functions to the other.

2. Suppose that X is connected. Then equivalence classes of local systems are in one-to-one correspondence with the equivalence classes of homomorphisms of the fundamental group $\pi_1(X)$ to G. Let M be an integrable connection of rank m. Then the sheaf of horizontal sections

   $$M^\nabla = \{s \in M : \nabla_X s = 0 \text{ for all } X \in \Theta_X\}$$

   is a locally free $\mathbb{C}_X$-module of rank m. This is how one thinks about local systems as locally constant sheaves of vector spaces.

   Moreover, by considering the parallel transport of sections of M along paths, we can define a representation of the fundamental group $\pi_1(X)$ on the germ of horizontal sections, which is a vector space of dimension m. This representation is called the monodromy representation of M.

5 Nonabelian Hodge theory

Nonabelian Hodge theory on a smooth projective complex curve X, as formulated by Simpson, studies three different moduli problems for bundles for a complex reductive group G:

deRham $\mathcal{C}onn_G(X)$: the moduli stack of flat G-connnections on X

Dolbeaut $\mathcal{H}iggs_G(X)$: the moduli stack of G-Higgs bundles on X

Betti $\mathcal{L}oc_G(X)$: the moduli stack of G-local systems on X

6 Geometric Langlands Program

The geometric Langlands program provides a nonabelian, global and categorical form of harmonic analysis. We fix a complex reductive group G and study the moduli stack $\mathrm{Bun}_G(X)$ of G-bundles on X. This stack comes equipped with a large commutative symmetry algebra: for any point $x \in X$we have a family of correspondences acting on $\mathrm{Bun}_G(X)$ by modifying G-bundles at x. The goal of the geometric Langlands program is to simultaneously diagonalize the action of Hecke correspondences on suitable categories of sheaves on $\mathrm{Bun}_G(X)$.

One can ask to label the common eigensheaves (Hecke eigensheaves) by their eigenvalues (Langlands parameters), or more ambitiously, to construct a Fourier transform identifying categories of sheaves with dual categories of sheaves on the space of Langlands parameters. The kernels for Hecke modifications are bi-equivariant sheaves on the loop group $G(K)$, $K = \mathbb{C}((t))$, with respect to the arc subgroup $G(O)$, $O = \mathbb{C}[[t]]$. The underlying double cosets are in bijection with irreducible representations of the Langlands dual group.