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Problem 1 Show that the n-sheeted Riemann surface of the multi-valued function

$$w = z^{1/n}, \quad z \in \mathbb{C},$$

is topologically a sphere with 1 puncture.

Solution: Let $\mathcal{R} = \{(z,w) \in \mathbb{C}^2 : \; w^n = z\}$. $\mathcal{R}$ carries the structure of a Riemann surface so that the projection $\pi : \mathcal{R} \to \mathbb{C}, \; \pi(z,w) = z$ is holomorphic. Now consider the map

$$\Phi : \mathbb{C} \longrightarrow \mathcal{R}, \qquad \Phi(w) = (w^n, w)$$

Φ is bijective: given $(z,w) \in \mathcal{R}$ we must have $z = w^n$, so the inverse is simply $(z,w) \mapsto w$. Φ and its inverse are holomorphic because one is given by a polynomial, the other is a projection. Hence Φ is a biholomorphism. Therefore $\mathcal{R}$ is (as a Riemann surface, hence also topologically) just $\mathbb{C}$. Topologically, $\mathbb{C}$ is a sphere with one point removed (a "punctured sphere"): $\mathbb{C} \simeq \widehat{\mathbb{C}} \setminus \{\infty\}$. Thus the n-sheeted Riemann surface of $w = z^{1/n}$ is topologically a sphere with one puncture.

Problem Problem2 Let $f(z)$ be a polynomial of odd degree, with simple zeroes. Identify the topology of the Riemann surface of the double-valued function defined by $w^2 = f(z)$.

Solution: Consider the affine curve $X_{\text{aff}} = \{(z,w) \in \mathbb{C}^2 : w^2 = f(z)\}$. Its projection $\pi_{\text{aff}} : (z,w) \mapsto z$ is a 2-sheeted branched covering of $\mathbb{C}$ away from the zeros of f. We compactify to a projective curve $X = \overline{X_{\text{aff}}} \subset \mathbb{P}^1_z \times \mathbb{P}^1_w$ and extend the projection to $\pi : X \longrightarrow \mathbb{P}^1_z$. The map π has degree 2. To study the topology of X_{aff}, we will use the Riemann-Hurwitz formula to compute the genus of X and delete the point(s) over $z = \infty$.

If a is a simple zero of f, write locally $f(z) = (z-a)u(z)$ with $u(a) \neq 0$. Then $w^2 = (z-a)u(z)$ has a single point of X lying over $z = a$ and the local model is $w^2 = z - a$, so the ramification index is $e = 2$. Thus each simple zero gives one branch point of ramification index 2. There are d of these in $\mathbb{C}$. Put $t = 1/z$ as a coordinate near $z = \infty$ and write

$$f(z) = z^d \, g(1/z) = t^{-d} \, g(t), \qquad g(0) \neq 0$$

The equation becomes $w^2 = t^{-d} g(t) \iff (w \, t^{\frac{d-1}{2}})^2 = t^{-1} g(t)$. Let $u = w \, t^{\frac{d-1}{2}}$. Then $u^2 = t^{-1} g(t)$, so near $t = 0$ we have the model $u^2 \sim t^{-1}$. Therefore, there is one point of X over $z = \infty$ and it is ramified of order 2. Hence the total number of simple branch points is $B = d + 1$.

Apply Riemann-Hurwitz to the degree-2 map $\pi : X \to \mathbb{P}^1$:

$$2g(X) - 2 = 2 \cdot (-2) + \sum_{p \in X} (e_p - 1).$$

Every simple ramification contributes $e_p - 1 = 1$, so

$$2g(X) - 2 = -4 + B = -4 + (d+1) = d - 3.$$

Therefore

$$g(X) = \frac{d-1}{2}.$$

The compact Riemann surface X is a closed orientable surface of genus $g = \frac{d-1}{2}$. Recall that there is only one point of X over $z = \infty$. Therefore, X_{aff} is homeomorphic to X with one point removed. Hence X_{aff} is homeomorphic to a genus $\frac{d-1}{2}$ surface with one puncture.

Problem Problem3 Show that a bijective holomorphic map

$$f : R \to S$$

between Riemann surfaces is in fact bi-holomorphic (meaning, the inverse is also holomorphic). Show that two homeomorphic Riemann surfaces need not be bi-holomorphic. *(Hint: Use the unit disk Δ and the complex plane.)* Show that no two of the following three annuli in $\mathbb{C}$ are bi-holomorphic:

$$(a)\ \{z \mid 0 < |z| < 1\},$$
$$(b)\ \{z \mid 1 < |z| < 2\},$$
$$(c)\ \{z \mid 0 < |z| < \infty\}.$$

Solution: The inverse function theorem guarantees that the inverse function f^{-1} is smooth. Moreover, it guarantees that

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

Since f is bijective, it has nonzero derivative everywhere because if it did not, it would look like $z \mapsto z^k$ for some $k \geq 2$ and thus it would fail to be locally bijective. Since it has nonzero derivative everywhere, $(f^{-1})'$ is defined everywhere and is in fact a complex number. Hence f^{-1} is holomorphic. Δ and $\mathbb{C}$ are homeomorphic but they are not biholomorphic. An explicit homeomorphism is given by the radial map

$$\varphi\colon \Delta \to \mathbb{C}, \quad \varphi(re^{i\theta}) = \frac{r}{1-r}\, e^{i\theta}, \quad 0 \leq r < 1.$$

If there were a biholomorphism the map $\mathbb{C} \to \Delta$ is bounded and entire, hence constant which is a contradiction. Suppose (a) and (c) are biholomorphic. Then the map from

$\{z \mid 0 < |z| < \infty\} \to \{z \mid 0 < |z| < 1\}$ could be extended across the origin because it is bounded in a neighborhood of the origin (apply Riemann's removable singularity theorem). So it extends to a bounded entire function and hence must be constant. This is a contradiction. The same argument shows that (b) and (c) are not holomorphic. Finally suppose that (a) and (b) were biholomorphic. A map $F : \{z \mid 0 < |z| < 1\} \to \{z \mid 1 < |z| < 2\}$ again extends holomorphically to $\tilde{F}$ across zero by Riemann's theorem. Moreover, $\tilde{F}'(0) \neq 0$ because in a punctured neighborhood of 0, $\tilde{F}$ is a biholomorphism. Thus $\tilde{F} : U \to V$ admits a local holomorphic inverse $G : V \to U$ where U is a neighborhood of 0 and V is a neighborhood of $\tilde{F}(0) \in A(1,2)$. But F already has a global holomorphic inverse, call it F^{-1} and so F^{-1} and G must agree on V. But G maps $\tilde{F}(0)$ to 0 so so must F^{-1} but this is a contradiction.

Problem Problem4 Prove the *Weierstrass division theorem*: Given a polynomial

$$P(w, z_1, \ldots, z_n) = w^n + \sum_{k=0}^{n-1} p_k(z) w^k,$$

with the functions $p_k(z)$ holomorphic in an open set $V \subset \mathbb{C}^n$ and satisfying $p_k(0) = 0$, every germ of holomorphic function $G(w, z)$ near $(w, z) = (0, 0)$ can be uniquely expressed as

$$G(w, z) = P(w, z) \cdot Q(w, z) + R(w, z),$$

where $Q(w, z)$ is a holomorphic germ near 0 and $R(w, z)$ is a polynomial in w of degree $< n$ with coefficients germs of holomorphic functions in z near $z = 0$.

To do this, define

$$Q(z, w) = \frac{1}{2\pi i} \oint_\Gamma \frac{G(\zeta, z)}{P(\zeta, z)(\zeta - w)}\, d\zeta$$

for a suitable choice of the line integral over each fixed value of z, and show that the difference

$$R(w, z) := G(w, z) - P(w, z) \cdot Q(w, z)$$

is a holomorphic function of (w, z) which is polynomial in w with degree $< n$. *Hint:* You will want to express that difference as a Cauchy integral to get your conclusion.

Solution: Using Cauchy's integral formula we write

$$G(w, z) = \frac{1}{2\pi i} \oint_\Gamma \frac{G(\zeta, z)}{\zeta - w} d\zeta$$

from which we can write

$$R(w, z) = \frac{1}{2\pi i} \oint_\Gamma \frac{G(\zeta, z) P(\zeta, z)}{P(\zeta, z)(\zeta - w)} - \frac{G(\zeta, z) P(w, z)}{P(\zeta, z)(\zeta - w)} d\zeta$$

Now write

$$P(\zeta, z) - P(w, z) = (\zeta^k - w^k) + \sum_{i=1}^{k-1} p_k(z)(\zeta^i - w^i)$$

which is divisible by $\zeta - w$, and the quotient

$$\frac{P(\zeta, z) - P(w, z)}{\zeta - w}$$

is a polynomial in w of degree $n-1$. If for a fixed value of z, we pick a contour Γ in the w plane for which $P(\zeta, z)$ does not vanish on Γ, then the function R is holomorphic in w and z because it is the contour integral of an integrand, holomorphic in both w and z. Since R is holomorphic, we may differentiate n times with respect to w under the integral and we find that the integrand becomes zero, since the integrand is a polynomial in w of degree $n-1$. Therefore,

$$\frac{d^n R}{dw^n} = 0$$

so R is indeed polynomial of degree $n-1$.

Problem Problem5 A *Reinhardt domain* $R \subset \mathbb{C}^n$ is an open set such that

$$(z_1, \ldots, z_n) \in R \;\Rightarrow\; (qz_1, \ldots, qz_n) \in R, \quad \forall q \in \mathbb{C} \text{ with } |q| < 1.$$

(a) Show that the intersection of finitely many Reinhardt domains is Reinhardt.

(b) Show that if a multi-variable power series centered at 0 in some neighborhood of some point $(z_1, \ldots, z_n) \in \mathbb{C}^n$, then it converges uniformly in some Reinhardt domain containing z.

(c) Prove that the *domain of convergence* of an n-variable Taylor series centered at 0 — defined as the interior of the set of points where the series converges — is a Reinhardt domain.

Solution:

1. If $x \in R_i$ then $qx \in R_i$ and in particular $x \in \bigcap R_i$ implies $qx \in \bigcap R_i$ so $\bigcap R_i$ is Reinhardt. The intersection of finitely many open sets is open.

2. Let $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$ be a power series which converges in a neighborhood of some point $z = (z_1, \ldots, z_n)$. We want to show that it converges uniformly in some Reinhardt domain containing z. Define $b_m = \sum_{|\alpha|=m} a_\alpha z^\alpha$ and group the series by total degree:

$$f(z) = \sum_{m=0}^{\infty} b_m$$

For a scalar $q \in \mathbb{C}$, consider $F(q) = \sum_\alpha a_\alpha (qz)^\alpha$. Observe that

$$F(q) = \sum_{m=0}^{\infty} \Big(\sum_{|\alpha|=m} a_\alpha z^\alpha \Big) q^m = \sum_{m=0}^{\infty} b_m q^m$$

By assumption, the original series converges at z. That means $F(1) = \sum_{m=0}^{\infty} b_m$ converges. For any $r < 1$, the series $\sum b_m q^m$ converges absolutely and uniformly on $|q| \le r$ by the one-variable Weierstrass M-test, since $\sum |b_m| r^m < \infty$. Fix $r < 1$. Then for all $|q| \le r$, $F(q) = \sum_\alpha a_\alpha (qz)^\alpha$ converges uniformly. In other words, the original n-variable series converges uniformly on the set $\Omega_r(z) = \{qz : |q| \le r\}$. This set $\Omega_r(z)$ is a Reinhardt domain containing z.

3. Let $X \subset \mathbb{C}^n$ be the set of points where $\sum_\alpha a_\alpha z^\alpha$ converges. Let $Y := \operatorname{int} X$ be its domain of convergence. We claim that X is Reinhardt. Indeed, if $z \in X$, then as in part (b) the function $F(q) = \sum_{m=0}^{\infty} b_m q^m$ where $b_m = \sum_{|\alpha|=m} a_\alpha z^\alpha$, converges at $q = 1$, hence has one-variable radius of convergence $R \ge 1$. Therefore $F(q)$ converges for all $|q| < 1$, i.e. $|q| < 1 \;\Rightarrow\; qz \in X$. Thus X is stable under common scalings by $|q| < 1$. The interior Y is both open and stable under such scalings, hence is a Reinhardt domain as desired.

Problem Problem6 Let C_1 and C_2 be two circles in the w- and z-planes in $\mathbb{C}^2$, and $\Delta_{1,2}$ the disks that they bound. Show that a holomorphic function defined in an open set containing

$$C_1 \times \Delta_2 \,\cup\, \Delta_1 \times C_2$$

has a unique holomorphic extension over $\Delta_1 \times \Delta_2$. *Hint:* Use Cauchy's formula in a way very similar to the one exploited above.

Solution: Suppose f is holomorphic on a neighborhood of $X = (\partial\Delta \times \Delta) \cup (\Delta \times \partial\Delta) \subset \mathbb{C}^2$. For each fixed $z_2 \in \Delta$, the map $\zeta_1 \mapsto f(\zeta_1, z_2)$ is holomorphic on a neighborhood of $\partial\Delta$. Define

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_1|=1} \frac{f(\zeta_1, z_2)}{\zeta_1 - z_1}\, d\zeta_1, \qquad (z_1, z_2) \in \Delta \times \Delta.$$

F is holomorphic on $\Delta \times \Delta$. Moreover, on a neighborhood of $\partial\Delta \times \Delta$ (where f is defined in a full annulus in ζ_1), Cauchy's formula gives $F = f$. Similarly, for each fixed $z_1 \in \Delta$ the map $\zeta_2 \mapsto f(z_1, \zeta_2)$ is holomorphic near $\partial\Delta$. Define

$$G(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_2|=1} \frac{f(z_1, \zeta_2)}{\zeta_2 - z_2}\, d\zeta_2.$$

Then G is holomorphic on $\Delta \times \Delta$ and $G = f$ on a neighborhood of $\Delta \times \partial\Delta$. On a neighborhood of the torus $\partial\Delta \times \partial\Delta \subset X$, both representations are valid and equal f, hence $F = G$ there. By the identity theorem for holomorphic functions on $\Delta \times \Delta$, $F \equiv G$ on all of $\Delta \times \Delta$. Thus this common function extends f holomorphically to the full interior.

The extension is unique because if H were another holomorphic extension, then $H = F$ on $\partial\Delta \times \Delta$ by the identity theorem applied to the first variable, hence $H = F$ on all of $\Delta \times \Delta$ by the identity theorem applied to the second variable.

Problem Problem7 Let F, G be two irreducible holomorphic functions in $n > 1$ variables defined on an open set U, and call their common zero-set Z. Using the Weierstrass Preparation Theorem (twice) and Q6, show that any holomorphic function defined on $U \setminus Z$ extends holomorphically over Z.

Remark 1. This is a version of *Hartogs' theorem* for holomorphic functions of several variables; somewhat loosely, the singular set of a holomorphic function defined on "most of" an open $U \subset \mathbb{C}^n$ cannot lie in an analytic subset of co-dimension 2, unless it's empty. Contrast that with the real function $1/(x^2 + y^2)$ on $\mathbb{R}^2$.

Solution: Let F, G be irreducible holomorphic functions on $U \subset \mathbb{C}^n$, $n > 1$, and set $Z = \{F = 0\} \cap \{G = 0\}$. Fix $p \in Z$ and change coordinates so that $p = 0$, with F regular in w and G regular in z. By Weierstrass Preparation we may write $F = U_F \cdot P(w; z, t)$ and $G = U_G \cdot Q(z; w, t)$ where P is a Weierstrass polynomial in w, Q one in z, and t denotes the other coordinates. Thus for small polydisks, the zeros of F in w and of G in z form finite sets of roots varying holomorphically with the parameters. Choosing circles $C_1 = \{|w| = r_1\}$ and $C_2 = \{|z| = r_2\}$ that avoid these roots (uniformly in t), we see that $(C_1 \times \Delta_2) \cup (\Delta_1 \times C_2)$ is contained in $U \setminus Z$, so f is holomorphic there. By Question 6, f extends holomorphically to $\Delta_1 \times \Delta_2$ for each fixed t, and the extension depends holomorphically on t. Hence f extends to a neighborhood of p, and by uniqueness these local extensions glue to give a holomorphic extension of f to all of U.