Teleman Woodward

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Abstract

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1 Overview of the finiteness theorem from Teleman Woodward

Let G be a connected reductive group and $\mathfrak{M} = \mathrm{Bun}_G(C)$ the moduli stack of G-bundles on a smooth projective curve C. The goal of Teleman-Woodward is to compute the index of certain K-theory classes on $\mathfrak{M}$, generalizing the Verlinde formula for line bundles.

However, our goal is to merely establish the finiteness of the index in the case of nodal curves. Abstracting from the paper, the finiteness theorem has the following structure:

1. Stratify the full stack $\mathfrak{M}$ by Harder–Narasimhan type $\mathfrak{M} = \bigsqcup_{\xi} \mathfrak{M}_{\xi}$. Include some explanation of HN type here.
2. Filter $R\Gamma(\mathfrak{M}, E)$ by local cohomology along the closed unions $\bigcup_{\xi' \geq \xi} \mathfrak{M}_{\xi'}$.
3. Show that admissibility forces the local terms for ξ sufficiently large to vanish.
4. Deduce that only finitely many strata contribute, hence the index is finite.

Admissible classes (Teleman–Woodward)

Let Σ be a smooth projective curve, G a connected reductive group, and $\mathfrak{M} = \mathrm{Bun}_G(\Sigma)$ the moduli stack of G–bundles. Let $\mathcal{E}$ denote the universal G–bundle on $\Sigma \times \mathfrak{M}$, and for a finite–dimensional representation V of G write $\mathcal{E}^*V$ for the associated vector bundle.

Teleman–Woodward single out the following natural K–theory classes on $\mathfrak{M}$:

(i) $E^*_x V \in K^0(\mathfrak{M})$, the restriction of $\mathcal{E}^*V$ to $\{x\} \times \mathfrak{M}$, for a point $x \in \Sigma$;

(ii) $E^*_C V := \mathcal{E}^*V/[C] \in K^{-1}(\mathfrak{M})$, the slant product of $\mathcal{E}^*V$ with a 1–cycle C on Σ;

(iii) $E^*_\Sigma V := R\pi_*(\mathcal{E}^*V \otimes \sqrt{K}) \in K^0(\mathfrak{M})$, the Dirac index bundle along Σ, where $\pi : \Sigma \times \mathfrak{M} \to \mathfrak{M}$ and $\sqrt{K}$ is a square root of the canonical bundle of Σ;

(iv) $D_\Sigma V := \det^{-1} E^*_\Sigma V$, the inverse determinant of cohomology.

The classes in (i)–(iii) are called the *Atiyah–Bott generators*. The classes in (iv) are determinant line bundles on $\mathfrak{M}$.

The first Chern class of any determinant line bundle $\mathcal{L}$ defines an invariant quadratic form

$$h_{\mathcal{L}} \in H^4(BG; \mathbb{R}) \cong \mathrm{Sym}^2(\mathfrak{g}^*)^G \cong \{\text{invariant symmetric bilinear forms on } \mathfrak{g}\},$$

called the *level* of $\mathcal{L}$. Let c denote the distinguished quadratic form corresponding to the canonical bundle $\mathcal{K} = \det E^*_\Sigma \mathfrak{g}$.

Definition 1.1 (Teleman–Woodward)**.** A line bundle $\mathcal{L}$ on $\mathfrak{M}$ is called *admissible* if the shifted quadratic form

$$h_{\mathcal{L}} + c$$

is positive definite on $\mathfrak{g}$. An *admissible class* in $K^*(\mathfrak{M})$ is any finite product of an admissible line bundle with Atiyah–Bott generators.

Shatz stratification

Recall that any G–bundle over Σ admits a canonical reduction of structure group to a standard parabolic subgroup $P \subset G$, for which the associated bundle with Levi structure group is semistable.

Remark 1.2. For a principal G-bundle P on a smooth curve Σ, there is a Harder–Narasimhan (HN) theory generalizing the usual HN filtration of vector bundles. The outcome is a canonical reduction of P to a parabolic subgroup $P \subset G$. "Canonical" means: determined functorially by P (up to unique isomorphism), not a choice. The defining property is that if you pass from P (the parabolic) to its Levi quotient $L = P/R_u(P)$, the induced L-bundle is semistable. "Standard parabolic" means: a parabolic containing a fixed Borel B (chosen once), so parabolics are indexed by subsets of simple roots.

Intuition: this parabolic reduction packages "the most destabilizing" subobject(s) of the bundle.

Topologically, this reduction is classified by a coweight of $P/[P, P]$; we identify this with a (possibly fractional) dominant coweight ξ of $\mathfrak{g}$, called the *instability type* of the original bundle. Then P is the standard parabolic subgroup defined by ξ; we denote it by P_ξ and its Levi subgroup by G_ξ. If $\mathfrak{M}_\xi$ denotes the stack of G–bundles of type ξ, we have an algebraic stratification

$$\mathfrak{M} = \bigsqcup_\xi \mathfrak{M}_\xi.$$

Sending a P_ξ–bundle to its associated Levi bundle defines a morphism from $\mathfrak{M}_\xi$ to the stack $\mathfrak{M}^{ss}_{G_\xi,\xi}$ of semistable principal G_ξ–bundles of type ξ; the fibres are quotient stacks of affine spaces by unipotent groups (equivalently the corresponding Lie algebra is nilpotent). The virtual normal bundle for the morphism $\mathfrak{M}^{ss}_{G_\xi,\xi} \to \mathfrak{M}$ is the complex

$$\nu_\xi = R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1].$$

Its K–theory Euler class should be the alternating sum of exterior powers

$$\lambda_{-1}(\nu_\xi^\vee) := \sum_p (-1)^p \wedge^p (\nu_\xi^\vee),$$

but for now this infinite sum is only a formal expression, whose meaning is to be spelled out.

Local cohomology

Finite open unions of Shatz strata

$$\mathfrak{M}_{\le\xi} = \bigcup_{\mu\le\xi} \mathfrak{M}_\mu$$

can be presented as quotient stacks of smooth quasi–projective varieties by reductive groups. The cohomology with supports over $\mathfrak{M}_\xi$ of a vector bundle $\mathcal{E}$ is

$$H^\bullet_{\mathfrak{M}_\xi}(\mathfrak{M}_{\le\xi}, \mathcal{E}_{\le\xi}) = H^{\bullet+d_\xi}(\mathfrak{M}_\xi, \mathcal{R}_\xi\mathcal{E}), \tag{1.9}$$

where d_ξ is the codimension of $\mathfrak{M}_\xi$ and $\mathcal{R}_\xi\mathcal{E}$ denotes the sheaf of "$\mathcal{E}$–valued residues along $\mathfrak{M}_\xi$." In particular

$$\mathcal{R}_\xi\mathcal{E} \ := \ i_\xi^!(\mathcal{E}_{\leq\xi})[-d_\xi]$$

where $i_\xi : \mathfrak{M}_\xi \hookrightarrow \mathfrak{M}_{\leq\xi}$ is the inclusion and $i^!$ is the extraordinary pullback (local duality functor).

This is a stacky derived version of the fact that for a smooth closed subvariety $Z \subset X$, local cohomology along Z equals cohomology on Z twisted by the normal bundle and shifted by codimension.

Basically I think we need to find the right stratification and Dan HL has a machine that produces such stratifications, known as θ-stratifications.

Role of the Shatz stratification in Teleman–Woodward

The proof of the finiteness theorem in [1] is organized around the Harder–Narasimhan (Shatz) stratification of the moduli stack

$$\mathfrak{M} = \mathrm{Bun}_G(\Sigma) = \bigsqcup_\xi \mathfrak{M}_\xi,$$

indexed by dominant rational coweights ξ. This stratification plays the role of a Morse stratification for the Yang–Mills functional, and replaces compactness in the non–finite–type stack $\mathfrak{M}$.

(1) Filtration by supports. The partial order on instability types defines an increasing filtration by open substacks

$$\mathfrak{M}_{\leq\xi} := \bigcup_{\mu\leq\xi} \mathfrak{M}_\mu.$$

For any sheaf or complex $\mathcal{E}$ on $\mathfrak{M}$, this filtration produces a filtration on derived global sections $R\Gamma(\mathfrak{M},\mathcal{E})$ by the subcomplexes $R\Gamma_{\mathfrak{M}_{\leq\xi_k}}(\mathfrak{M},\mathcal{E})$.

Remark 1.3 (General mechanism of local cohomology filtration)**.** Suppose you have a space/stack X and an increasing sequence of open substacks $\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset X$ with closed complements $Z_k := X \setminus U_k$. In our situation $X = \mathfrak{M}$, $U_k = \mathfrak{M}_{\leq\xi_k}$, $Z_k = \bigcup_{\mu>\xi_k} \mathfrak{M}_\mu$. For any sheaf or complex $\mathcal{E}$ on X, there is a canonical exact triangle

$$R\Gamma_{Z_k}(X,\mathcal{E}) \ \to \ R\Gamma(X,\mathcal{E}) \ \to \ R\Gamma(U_k,\mathcal{E}|_{U_k}) \ \to$$

This triangle is the definition of local cohomology with supports in Z_k.

The decreasing family of closed substacks $Z_k = X\setminus U_k$ induces a decreasing filtration $F^k := R\Gamma_{Z_k}(X,\mathcal{E})$ of $R\Gamma(X,\mathcal{E})$. Its successive graded pieces are

$$\mathrm{gr}^k F \simeq R\Gamma_{Z_k\setminus Z_{k+1}}(U_{k+1},\mathcal{E}|_{U_{k+1}}).$$

In the Shatz situation (refining the indexing so that $U_k = U_{k-1} \sqcup \mathfrak{M}_{\xi_k}$), this becomes

$$\mathrm{gr}^{k-1} F \simeq R\Gamma_{\mathfrak{M}_{\xi_k}}(\mathfrak{M}_{\leq \xi_k}, \mathcal{E}_{\leq \xi_k}).$$

Equivalently there is a spectral sequence with

$$E_1^{\xi,*} = R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi}) \quad \Longrightarrow \quad R\Gamma(\mathfrak{M}, \mathcal{E}).$$

(2) Reduction to semistable Levi moduli. Each stratum $\mathfrak{M}_\xi$ carries a canonical morphism

$$\mathfrak{M}_\xi \longrightarrow \mathfrak{M}^{ss}_{G_\xi, \xi}$$

to the moduli stack of semistable principal G_ξ–bundles of fixed topological type, whose fibres are quotient stacks of affine spaces by unipotent groups. This identifies $\mathfrak{M}_\xi$ as a bundle of unstable directions over a semistable core.

(3) Virtual normal complex. The stratification provides a uniform description of the virtual normal complex of $\mathfrak{M}^{ss}_{G_\xi,\xi}$ in $\mathfrak{M}$:

$$\nu_\xi = R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1].$$

Consequently, local cohomology along $\mathfrak{M}_\xi$ may be expressed formally as

$$R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}) \simeq R\Gamma\big(\mathfrak{M}_\xi, \mathcal{E} \otimes \lambda_{-1}(\nu_\xi^\vee)^{-1}\big),$$

where $\lambda_{-1}(\nu_\xi^\vee)$ is the K–theoretic Euler class of the normal complex.

(4) Weight decomposition and admissibility. The representation $\mathfrak{g}/\mathfrak{g}_\xi$ decomposes into positive ξ–weight spaces. This induces a natural grading on ν_ξ and hence on $\lambda_{-1}(\nu_\xi^\vee)^{-1}$. Admissibility of the twisting line bundle forces all sufficiently unstable types ξ to contribute only strictly negative weights, so that the formal inverse Euler class becomes summable and the local contributions vanish for ξ sufficiently large.

(5) Finiteness. Since only finitely many instability types can contribute nontrivially, the local–cohomology filtration terminates after finitely many steps. This yields the finiteness of the index.

In summary, the Shatz stratification supplies a canonical filtration, a reduction to semistable Levi moduli, and a uniform normal complex whose weight decomposition is controlled by admissibility. All finiteness statements in [1] are ultimately consequences of this structure.

2 Toy model: $G = \mathrm{GL}_2$ on a smooth curve

Let Σ be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$, and let

$$\mathfrak{M} = \mathrm{Bun}_{\mathrm{GL}_2}(\Sigma)$$

be the moduli stack of rank 2 vector bundles on Σ. We explain explicitly the Shatz stratification, the Levi description of strata, the virtual normal complex, and the weight bookkeeping behind the Teleman–Woodward finiteness mechanism in this case.

2.1 Harder–Narasimhan type and Shatz strata

Every $E \in \mathfrak{M}$ admits a unique Harder–Narasimhan filtration

$$0 \subset L \subset E, \qquad M := E/L,$$

where L is a line subbundle of maximal slope. Write

$$\deg(L) = d_1, \qquad \deg(M) = d_2, \qquad m := d_1 - d_2 \geq 0.$$

Then E is semistable iff $m = 0$ (equivalently $d_1 = d_2$).

The Shatz (HN) stratum of type (d_1, d_2) is the locally closed substack

$$\mathfrak{M}_{d_1,d_2} \subset \mathfrak{M}$$

parametrizing bundles whose HN filtration has graded pieces (L, M) of degrees (d_1, d_2) (so $m > 0$ on unstable strata). One has the stratification

$$\mathfrak{M} = \bigsqcup_{d_1 \geq d_2} \mathfrak{M}_{d_1,d_2}.$$

2.2 Parabolic and Levi

Fix the standard Borel $B \subset \mathrm{GL}_2$ of upper triangular matrices. For $m > 0$, the destabilizing reduction is to the standard parabolic

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\},$$

with Levi subgroup

$$G_\xi \cong \mathrm{GL}_1 \times \mathrm{GL}_1 \quad \text{(diagonal matrices)}.$$

Equivalently, the associated dominant coweight (instability type) may be taken as

$$\xi = \left(\frac{m}{2}, -\frac{m}{2} \right) \in \mathfrak{t}_{\mathbb{Q}},$$

so that the positive root α satisfies $\alpha(\xi) = m$.

2.3 Semistable Levi core and structure of the stratum

A principal G_ξ–bundle is the same as a pair of line bundles (L, M), hence the moduli stack of semistable G_ξ–bundles of type (d_1, d_2) is

$$\mathfrak{M}^{ss}_{G_\xi,\xi} \;\cong\; \mathrm{Pic}^{d_1}(\Sigma) \times \mathrm{Pic}^{d_2}(\Sigma).$$

There is a canonical morphism

$$q : \mathfrak{M}_{d_1,d_2} \longrightarrow \mathrm{Pic}^{d_1}(\Sigma) \times \mathrm{Pic}^{d_2}(\Sigma), \qquad E \mapsto (L, E/L).$$

Fixing (L, M), the fibre of q over (L, M) classifies extensions

$$0 \to L \to E \to M \to 0,$$

hence is governed by

$$\mathrm{Ext}^1(M, L) \cong H^1(\Sigma, L \otimes M^{-1}).$$

Automorphisms of a given extension (fixing (L, M)) come from

$$\mathrm{Hom}(M, L) \cong H^0(\Sigma, L \otimes M^{-1}),$$

which is a unipotent group (additively) acting on the affine space $H^1(\Sigma, L \otimes M^{-1})$ by the usual change-of-splitting. Thus the fibre is a quotient stack $\left[H^1(\Sigma, L \otimes M^{-1})\middle/\; H^0(\Sigma, L \otimes M^{-1})\right]$ making $\mathfrak{M}_{d_1,d_2}$ a (stacky) affine fibration over the semistable Levi core.

2.4 Virtual normal complex

The tangent complex of Bun_G at a G–bundle E is

$$T_{\mathfrak{M},E} \simeq R\Gamma(\Sigma, \mathrm{End}(E))[1].$$

Over the Levi core (L, M) the adjoint representation decomposes as

$$\mathrm{End}(L \oplus M) \;=\; \underbrace{\mathrm{End}(L) \oplus \mathrm{End}(M)}_{\mathfrak{g}_\xi} \;\oplus\; \underbrace{\mathrm{Hom}(M, L) \oplus \mathrm{Hom}(L, M)}_{\mathfrak{g}/\mathfrak{g}_\xi}.$$

Along the stratum $\mathfrak{M}_{d_1,d_2}$, the relevant (unstable) normal directions are governed by the positive–weight root space $\mathrm{Hom}(M, L)$, and the virtual normal complex for the inclusion of the Levi moduli into $\mathfrak{M}$ is

$$\nu_\xi \;\simeq\; R\Gamma(\Sigma, L \otimes M^{-1})[1].$$

Equivalently, ν_ξ has cohomology

$$H^{-1}(\nu_\xi) \cong H^0(\Sigma, L \otimes M^{-1}), \qquad H^0(\nu_\xi) \cong H^1(\Sigma, L \otimes M^{-1}).$$

2.5 K–theoretic Euler class and its formal inverse

Formally, the K–theory Euler class of the dual normal complex is

$$\lambda_{-1}(\nu_\xi^\vee) = \sum_{p\geq 0}(-1)^p \wedge^p (\nu_\xi^\vee).$$

Because ν_ξ is a shifted cohomology complex, its inverse Euler class expands into exterior powers of the H^0–piece and symmetric powers of the H^1–piece. Schematically one may think of

$$\lambda_{-1}(\nu_\xi^\vee)^{-1} \sim \frac{\mathrm{Sym}^\bullet\big(H^1(\Sigma, L\otimes M^{-1})^\vee\big)}{\Lambda^\bullet\big(H^0(\Sigma, L\otimes M^{-1})^\vee\big)}$$

an infinite sum in ordinary K–theory which is made meaningful in Teleman–Woodward by working in a suitable completion determined by ξ–weights.

2.6 Weight bookkeeping: linear vs. quadratic growth

The one–parameter subgroup ξ acts on $\mathrm{Hom}(M, L)$ with weight $\alpha(\xi) = m$. Consequently, ξ acts on $H^i(\Sigma, L\otimes M^{-1})$ with weight m, and hence on the graded summand

$$\mathrm{Sym}^p\big(H^1(\Sigma, L\otimes M^{-1})^\vee\big)$$

with weight $p\,m$. This is the *linear* growth in the instability parameter m.

On the other hand, a determinant line bundle $\mathcal{L}$ on $\mathfrak{M}$ has a level $h_{\mathcal{L}} \in \mathrm{Sym}^2(\mathfrak{g}^*)^G$, and Teleman–Woodward introduce the canonical correction c coming from $\mathcal{K} = \det E_\Sigma^*\mathfrak{g}$. For an *admissible* $\mathcal{L}$, the form $h_{\mathcal{L}} + c$ is positive definite, so

$$(h_{\mathcal{L}} + c)(\xi, \xi) \to +\infty \quad \text{as } \|\xi\| \to \infty.$$

In the GL_2 normalization $\xi = (m/2, -m/2)$ and the standard invariant form $(X, Y) = \mathrm{tr}(XY)$ on diagonal matrices gives

$$(\xi, \xi) = \frac{m^2}{2},$$

so $(h_{\mathcal{L}} + c)(\xi, \xi)$ grows like a positive constant times m^2. In Teleman–Woodward's local cohomology calculation, twisting by $\mathcal{L}$ shifts the ξ–weight spectrum by a *negative* amount with leading term

$$-(h_{\mathcal{L}} + c)(\xi, \xi) \sim -\kappa m^2 \qquad (\kappa > 0).$$

Thus, on the ξ–stratum, the inverse Euler class contributes graded pieces with weights growing at most *linearly* in m (e.g. $p\,m$), while an admissible twist shifts weights by a *quadratic* negative term $\sim -\kappa m^2$. This is the mechanism behind the eventual vanishing of sufficiently unstable strata in the Teleman–Woodward finiteness theorem.

Why ξ–invariants control finiteness of the index

Let Σ be a smooth projective curve and $\mathfrak{M} = \mathrm{Bun}_G(\Sigma)$. For a class $\mathcal{E} \in K^*(\mathfrak{M})$ one defines its index by the Euler characteristic

$$\mathrm{Ind}(\mathfrak{M}, \mathcal{E}) \ := \ \chi(\mathfrak{M}, \mathcal{E}) \ = \ \sum_i (-1)^i \dim H^i(\mathfrak{M}, \mathcal{E}),$$

whenever the right-hand side is finite.

Since $\mathfrak{M}$ is not of finite type, finiteness is proved by filtering $\mathfrak{M}$ by finite-type open substacks using the Shatz stratification

$$\mathfrak{M} = \bigsqcup_{\xi} \mathfrak{M}_\xi, \qquad \mathfrak{M}_{\leq \xi} := \bigcup_{\mu \leq \xi} \mathfrak{M}_\mu.$$

The open substacks $\mathfrak{M}_{\leq \xi}$ form an increasing filtration of $\mathfrak{M}$, and for any complex $\mathcal{E}$ on $\mathfrak{M}$ this induces a filtration of $R\Gamma(\mathfrak{M}, \mathcal{E})$ by local cohomology with supports in the complements. Equivalently, there is a spectral sequence whose E_1–page is built from the local cohomology complexes

$$E_1^{\xi,*} \ = \ R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi}) \quad \Longrightarrow \quad R\Gamma(\mathfrak{M}, \mathcal{E}).$$

In particular, finiteness of $\mathrm{Ind}(\mathfrak{M}, \mathcal{E})$ follows once one knows:

(a) for each ξ, the contribution of $R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi})$ to Euler characteristic is finite-dimensional; and

(b) all but finitely many ξ contribute trivially.

Teleman–Woodward identify each local term by a purity/local-duality statement:

$$R\Gamma_{\mathfrak{M}_\xi}(\mathfrak{M}_{\leq \xi}, \mathcal{E}_{\leq \xi}) \simeq R\Gamma\big(\mathfrak{M}_\xi, \mathcal{R}_\xi \mathcal{E}\big)[d_\xi], \qquad \mathcal{R}_\xi \mathcal{E} := i_\xi^!(\mathcal{E}_{\leq \xi})[-d_\xi],$$

where $d_\xi = \mathrm{codim}(\mathfrak{M}_\xi, \mathfrak{M}_{\leq \xi})$ and $i_\xi : \mathfrak{M}_\xi \hookrightarrow \mathfrak{M}_{\leq \xi}$ is the inclusion.

Moreover, the residue object $\mathcal{R}_\xi \mathcal{E}$ may be expressed formally in terms of the virtual normal complex

$$\nu_\xi = R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1]$$

as

$$\mathcal{R}_\xi \mathcal{E} \ \sim \ \mathcal{E}|_{\mathfrak{M}_\xi} \otimes \mathrm{Eul}(\nu_\xi)^{-1}.$$

Remark 2.1. Let X be an algebraic stack, let $i : Z \hookrightarrow X$ be a closed immersion, and let $\mathcal{E}$ be a (bounded) complex of coherent sheaves on X. Recall that the *local cohomology* of $\mathcal{E}$ with supports in Z is defined by the exact triangle

$$R\Gamma_Z(X, \mathcal{E}) \ \longrightarrow \ R\Gamma(X, \mathcal{E}) \ \longrightarrow \ R\Gamma(X \setminus Z, \mathcal{E}|_{X \setminus Z}) \ \longrightarrow,$$

or, equivalently, by the functor of sections with supports $R\Gamma_Z(X, -) = R\Gamma(X, R\Gamma_Z(-))$. Local duality packages these groups as ordinary cohomology on Z: one defines the *residue object* along Z by

$$\mathcal{R}_Z(\mathcal{E}) \;:=\; i^!(\mathcal{E})[-\operatorname{codim}(Z)],$$

so that (under standard hypotheses, e.g. i a local complete intersection)

$$R\Gamma_Z(X, \mathcal{E}) \;\simeq\; R\Gamma\big(Z, \mathcal{R}_Z(\mathcal{E})\big)[\operatorname{codim}(Z)].$$

In Teleman–Woodward's setting one takes $X = \mathfrak{M}_{\leq \xi}$ and $Z = \mathfrak{M}_\xi$, so

$$\mathcal{R}_\xi \mathcal{E} := i_\xi^!(\mathcal{E}_{\leq \xi})[-d_\xi], \qquad d_\xi = \operatorname{codim}(\mathfrak{M}_\xi, \mathfrak{M}_{\leq \xi}).$$

Why an Euler factor appears. If $i : Z \hookrightarrow X$ is a *regular embedding* between smooth schemes with normal bundle $N_{Z/X}$, then $i^!$ is controlled by the normal directions. At the level of K–theory one has the standard identity

$$\big[i^!(\mathcal{E})\big] = \big[i^*(\mathcal{E})\big] \cdot \lambda_{-1}(N_{Z/X}^\vee)^{-1} \qquad \text{in } K^0(Z), \tag{1}$$

where

$$\lambda_{-1}(W) := \sum_{p \geq 0} (-1)^p [\wedge^p W]$$

is the K–theoretic Euler class. Heuristically, local cohomology measures "principal parts along Z", and principal parts are obtained by expanding in the normal directions; the inverse Euler class $\lambda_{-1}(N_{Z/X}^\vee)^{-1}$ is the K–theoretic avatar of this expansion.

Virtual normal complex. In the Shatz situation, $\mathfrak{M}_\xi \hookrightarrow \mathfrak{M}_{\leq \xi}$ is not presented globally as a single regular embedding into a smooth ambient space. Instead, Teleman–Woodward use the fact that $\mathfrak{M}_\xi$ maps to a finite–type semistable Levi stack $\mathfrak{M}^{ss}_{G_\xi, \xi}$ with fibres quotient stacks of affine spaces by unipotent groups, and there is a canonical *virtual normal complex* (perfect complex playing the role of $N_{Z/X}$)

$$\nu_\xi \;=\; R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1] \qquad \text{on } \mathfrak{M}^{ss}_{G_\xi, \xi},$$

whose pullback to $\mathfrak{M}_\xi$ controls the unstable directions transverse to the Levi moduli. Consequently, the K–theory Euler class is defined by

$$\operatorname{Eul}(\nu_\xi^\vee) \;:=\; \lambda_{-1}(\nu_\xi^\vee),$$

and the same formal identity as (1) holds with $N_{Z/X}$ replaced by ν_ξ:

$$\big[\mathcal{R}_\xi \mathcal{E}\big] = \big[\mathcal{E}|_{\mathfrak{M}_\xi}\big] \cdot \lambda_{-1}(\nu_\xi^\vee)^{-1}. \tag{2}$$

2.6.1 The polarized inverse Euler class and the formula for $\operatorname{Eul}(\nu_\xi)_+^{-1}$

We explain the origin of Teleman–Woodward's formula

$$\operatorname{Eul}(\nu_\xi)_+^{-1} := \operatorname{Sym}\Big(R\pi_* \mathcal{E}^*(\mathfrak{p}_\xi/\mathfrak{g}_\xi)[1]^\vee \,\oplus\, R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big) \,\otimes\, \det\Big(R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big)[d_\xi], \quad (*)$$

It is a formal inverse to the Euler class with a weight decomposition satisfing the key properties that only weights ≤ 0 occur and each weight space is finite.

(1) ξ–weights and the parabolic splitting. Fix a maximal torus $T \subset G$ and a Borel $B \supset T$. For a dominant rational coweight $\xi \in X_*(T) \otimes \mathbb{Q}$, let P_ξ be the associated *standard* parabolic subgroup (so $B \subset P_\xi$), and let G_ξ be its Levi subgroup. At the Lie algebra level one has a canonical ξ–weight decomposition

$$\mathfrak{g} = \mathfrak{g}_\xi \oplus \mathfrak{n}_\xi \oplus \mathfrak{n}_\xi^-, \qquad \mathfrak{n}_\xi = \bigoplus_{\langle \alpha, \xi \rangle > 0} \mathfrak{g}_\alpha, \quad \mathfrak{n}_\xi^- = \bigoplus_{\langle \alpha, \xi \rangle < 0} \mathfrak{g}_\alpha.$$

Equivalently,

$$\mathfrak{p}_\xi = \mathfrak{g}_\xi \oplus \mathfrak{n}_\xi, \qquad \mathfrak{g}/\mathfrak{g}_\xi \cong (\mathfrak{p}_\xi/\mathfrak{g}_\xi) \oplus (\mathfrak{g}/\mathfrak{p}_\xi),$$

where $\mathfrak{p}_\xi/\mathfrak{g}_\xi \cong \mathfrak{n}_\xi$ carries strictly *positive* ξ–weights and $\mathfrak{g}/\mathfrak{p}_\xi \cong \mathfrak{n}_\xi^-$ carries strictly *negative* ξ–weights.

(2) The virtual normal complex and its ξ–grading. Let $\mathfrak{M}_\xi$ be the Shatz stratum of instability type ξ and let $\mathfrak{M}^{ss}_{G_\xi,\xi}$ be the semistable Levi moduli.

Sending a P_ξ–bundle to its associated Levi bundle defines a morphism $q_\xi : \mathfrak{M}_\xi \to \mathfrak{M}^{ss}_{G_\xi,\xi}$. The fibres are quotient stacks of affine spaces by unipotent groups. Whenever we define our stratification, we need to make sure this is true. The deformation theory transverse to the Levi directions is governed by the perfect complex on $\mathfrak{M}^{ss}_{G_\xi,\xi}$

$$\nu_\xi := R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1],$$

How did Teleman-Woodward identify this? Is there some general theory? whose pullback along q_ξ controls the virtual normal directions along the stratum and where

$$\pi : \Sigma \times \mathfrak{M}^{ss}_{G_\xi,\xi} \to \mathfrak{M}^{ss}_{G_\xi,\xi}$$

and $\mathcal{E}$ is the universal bundle. Using the splitting above,

$$\nu_\xi \simeq \nu_\xi^+ \oplus \nu_\xi^-, \qquad \nu_\xi^+ := R\pi_* \mathcal{E}^*(\mathfrak{p}_\xi/\mathfrak{g}_\xi)[1], \quad \nu_\xi^- := R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1].$$

Thus ν_ξ carries a canonical $\mathbb{Z}$–grading by ξ–weights: ν_ξ^+ has strictly positive weights and ν_ξ^- has strictly negative weights.

(3) Why an "inverse Euler class" is not a genuine K–class. For a vector bundle W, the K–theoretic Euler class is

$$\lambda_{-1}(W^\vee) = \sum_{p \ge 0} (-1)^p [\wedge^p W^\vee].$$

Even for a line bundle L, the inverse of $1 - L^\vee$ is *not* a finite K–class:

$$(1 - L^\vee)^{-1} = \sum_{n \ge 0} (L^\vee)^n \qquad \text{(a formal geometric series).}$$

Teleman–Woodward therefore work in a *completion* of equivariant K–theory determined by the ξ–weights. In such a completion one is allowed to expand $1 - L^\vee$ as a geometric series *in whichever direction is convergent in the chosen completion*. This is the meaning of the phrase "prefers ξ–negative eigenvalues."

(4) The basic one–dimensional identity and the determinant correction. Let $\mathfrak{G}_m$ act on a one–dimensional representation of weight $w \neq 0$, so the character is t^w. Then

$$(1 - t^w)^{-1} = \sum_{n \geq 0} t^{nw} \qquad \text{as a formal series in the direction of weights } w, w, 2w, \dots.$$

If we instead want an expansion which involves only *nonpositive* weights (i.e. which "prefers negative eigenvalues"), we rewrite

$$(1 - t^w)^{-1} = -t^{-w}\,(1 - t^{-w})^{-1} = -t^{-w} \sum_{n \geq 0} t^{-nw}.$$

Compared to the naive geometric series, this introduces a prefactor $-t^{-w}$. In higher rank, multiplying these prefactors over all negative–weight lines produces a *determinant factor*.

(5) From weights to a polarized inverse for a split complex. Suppose a perfect complex K carries a ξ–grading and splits as

$$K \simeq K^+ \oplus K^-,$$

where all ξ–weights in K^+ are > 0 and all ξ–weights in K^- are < 0. Then

$$\lambda_{-1}(K^\vee) = \lambda_{-1}((K^+)^\vee) \cdot \lambda_{-1}((K^-)^\vee)$$

and one defines a *polarized inverse* $\lambda_{-1}(K^\vee)_+^{-1}$ by inverting each factor in the completion which expands in the direction of *ξ–negative weights*. The outcome is the standard schematic identity

$$\lambda_{-1}(K^\vee)_+^{-1} \;=\; \mathrm{Sym}\big((K^+)^\vee \oplus K^-\big) \;\otimes\; \det(K^-)\,[\text{shift}], \tag{3}$$

where:

- $\mathrm{Sym}(-)$ denotes the total symmetric algebra $\mathrm{Sym}^\bullet(-) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(-)$, interpreted in K–theory (or in the corresponding completed K–group) as a formal power series;
- $\det(K^-)$ is the determinant line of the perfect complex K^-, which precisely packages the product of the one–dimensional prefactors in **(4)**;
- $[\text{shift}]$ is the cohomological degree shift dictated by local duality/purity (and in the Shatz situation becomes $[d_\xi]$).

Remark 2.2 (Origin of the determinant factor in the polarized inverse)**.** Let $\mathfrak{G}_m$ act with a $\mathbb{Z}$–grading, and let W be a finite–rank $\mathfrak{G}_m$–equivariant vector bundle with *strictly negative* weights. Write the K–theoretic Euler class as

$$\lambda_{-1}(W^\vee) = \prod_i (1 - L_i^\vee),$$

after splitting $W = \bigoplus_i L_i$ into $\mathfrak{G}_m$–eigenlines (locally on the base). Formally,

$$(1 - L_i^\vee)^{-1} = \sum_{n \geq 0} (L_i^\vee)^n$$

is the geometric expansion in nonnegative powers of $L_i^\vee$. However, if L_i has *negative* ξ–weight, then $L_i^\vee$ has *positive* weight, so this expansion lives in the completion which prefers *positive* weights. To invert in the opposite completion (the one preferring negative weights), we rewrite

$$(1 - L_i^\vee)^{-1} = -\, L_i \cdot (1 - L_i)^{-1} = -\, L_i \sum_{n \geq 0} L_i^n,$$

which is now a series in nonnegative powers of L_i (hence in nonpositive weights). The price paid for using this expansion is the prefactor $(-L_i)$.

Multiplying over i gives

$$\lambda_{-1}(W^\vee)^{-1}\Big|_{\text{prefer negative}} = \Big(\prod_i (-L_i)\Big) \cdot \prod_i (1 - L_i)^{-1} = (-1)^{\operatorname{rank} W} \, \det(W) \cdot \operatorname{Sym}(W),$$

where $\operatorname{Sym}(W) := \bigoplus_{n \geq 0} \operatorname{Sym}^n(W)$. Up to the harmless sign $(-1)^{\operatorname{rank} W}$ (often suppressed in K–theory conventions), this explains the appearance of the factor $\det(W)$ in the polarized inverse.

For a perfect complex K^- of strictly negative weights, the same argument applied to any local splitting into graded line bundles (together with the multiplicativity of λ_{-1} in K–theory) produces the factor $\det(K^-)$ in the polarized inverse $\lambda_{-1}(K^\vee)_+^{-1}$.

(6) Specialization to ν_ξ. Apply (3) to $K = \nu_\xi$ and the splitting $\nu_\xi \simeq \nu_\xi^+ \oplus \nu_\xi^-$ from **(2)**. Then $(K^+)^\vee = (\nu_\xi^+)^\vee$ and $K^- = \nu_\xi^-$, and we obtain

$$\operatorname{Eul}(\nu_\xi)_+^{-1} := \lambda_{-1}(\nu_\xi^\vee)_+^{-1} = \operatorname{Sym}\big((\nu_\xi^+)^\vee \oplus \nu_\xi^-\big) \otimes \det(\nu_\xi^-)\,[d_\xi].$$

Unwinding the definitions of $\nu_\xi^\pm$ gives exactly the formula $(*)$ above.

Weight bookkeeping and the finiteness mechanism

Fix an instability type ξ and consider the local contribution supported on the Shatz stratum $\mathfrak{M}_\xi$. Teleman–Woodward control this contribution by analysing the ξ–weight decomposition of the *polarized inverse Euler factor* $\operatorname{Eul}(\nu_\xi)_+^{-1}$ and its tensor product with an admissible class $\mathcal{E}$.

(A) The determinant weight and the quadratic form $c(\xi,\xi)$. Recall the polarized inverse Euler factor (cf. [1, §1.10–1.11])

$$\mathrm{Eul}(\nu_\xi)_+^{-1} := \mathrm{Sym}\Big(R\pi_*\mathcal{E}^*(\mathfrak{p}_\xi/\mathfrak{g}_\xi)[1]^\vee \ \oplus\ R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big) \ \otimes\ \det\Big(R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big)[d_\xi]. \quad (*)$$

The second tensor factor is a determinant line bundle

$$\det\Big(R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big).$$

Because $\mathfrak{g}/\mathfrak{p}_\xi$ is a direct sum of root spaces $\mathfrak{g}_\alpha$ with $\langle\alpha,\xi\rangle < 0$, the one–parameter subgroup determined by ξ acts on $\mathfrak{g}/\mathfrak{p}_\xi$ with strictly *negative* weights. Consequently, the induced $\mathfrak{G}_m$–action on the determinant line above has a well-defined ξ–weight which may be computed as a signed sum of these negative integers, counted with the appropriate cohomological multiplicities coming from $R\pi_*$.

Teleman–Woodward package this total determinant weight by a distinguished invariant quadratic form

$$c \in \mathrm{Sym}^2(\mathfrak{g}^*)^G,$$

namely the quadratic form attached (via Grothendieck–Riemann–Roch) to the canonical bundle

$$\mathcal{K} := \det(E_\Sigma^*\mathfrak{g}) \qquad \text{on } \mathfrak{M}.$$

With this notation, the determinant factor in $(*)$ has ξ–weight

$$\mathrm{wt}_\xi\Big(\det(R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1])\Big) = c(\xi,\xi). \tag{4}$$

In Teleman–Woodward's conventions, $c(\xi,\xi)$ is *negative* when ξ is viewed in the compact real form $i\mathfrak{t}_k$; equivalently, c is negative definite on $i\mathfrak{t}_k$.

Justification of (4). Fix ξ and let $\lambda_\xi : \mathfrak{G}_m \to G$ be the canonical one–parameter subgroup. Consider the determinant line

$$\mathcal{D}_\xi = \det\Big(R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big)$$

which appears in the polarized inverse Euler class $\mathrm{Eul}(\nu_\xi)_+^{-1}$.

As a T–module (and hence as a λ_ξ–module),

$$\mathfrak{g}/\mathfrak{p}_\xi \ \cong\ \bigoplus_{\langle\alpha,\xi\rangle<0} \mathfrak{g}_\alpha,$$

a direct sum of root spaces on which λ_ξ acts with weights $\langle\alpha,\xi\rangle < 0$. The corresponding trace form is

$$\mathrm{Tr}_{\mathfrak{g}/\mathfrak{p}_\xi}(\eta,\eta) \ =\ \sum_{\langle\alpha,\xi\rangle<0} \langle\alpha,\eta\rangle^2 \qquad (\eta\in\mathfrak{t}),$$

because η acts on $\mathfrak{g}_\alpha$ by the scalar $\langle \alpha, \eta \rangle$.

Now compare with the adjoint trace form:

$$\mathrm{Tr}_{\mathfrak{g}}(\eta, \eta) \;=\; \sum_{\alpha \in \Phi} \langle \alpha, \eta \rangle^2,$$

since η acts trivially on $\mathfrak{t}$ and by $\langle \alpha, \eta \rangle$ on $\mathfrak{g}_\alpha$. The root system is symmetric $\alpha \leftrightarrow -\alpha$, so the sum over $\{\alpha : \langle \alpha, \xi \rangle < 0\}$ is exactly half the sum over all roots:

$$\mathrm{Tr}_{\mathfrak{g}/\mathfrak{p}_\xi}(\eta, \eta) \;=\; \frac{1}{2}\, \mathrm{Tr}_{\mathfrak{g}}(\eta, \eta).$$

Therefore the level of $\mathcal{D}_\xi$ is

$$\mathrm{lev}(\mathcal{D}_\xi) \;=\; \mathrm{Tr}_{\mathfrak{g}/\mathfrak{p}_\xi} \;=\; \frac{1}{2}\, \mathrm{Tr}_{\mathfrak{g}}\,.$$

There is the classical indentification of the level of the Pfaffian square root $\mathcal{K}^{-1/2}$

$$c \;:=\; -\frac{1}{2}\, \mathrm{Tr}_{\mathfrak{g}},$$

where $\mathcal{K} = \det(E_\Sigma^* \mathfrak{g})$ is the canonical bundle on $\mathfrak{M}$. Combining with the computation above gives

$$\mathrm{lev}(\mathcal{D}_\xi) = -c.$$

However the shift $[1]$ in the determinant line $\mathcal{D}_\xi$:

$$\det\Big(R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\Big) = \det\Big(R\pi_* \mathcal{E}^*(\mathfrak{g}/\mathfrak{p}_\xi)\Big)^{-1}$$

leaves us with c as desired. $\square$

(B) Tensoring by an admissible class $\mathcal{E}$. Lemma [1, §1.11] concerns the ξ–invariant part of

$$\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}, \qquad \big(\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}\big)^{\xi\text{-inv}},$$

i.e. the weight–0 subobject for the $\mathfrak{G}_m$–action defined by ξ.

Write $\mathcal{E}$ as a product

$$\mathcal{E} \;=\; \mathcal{L} \,\otimes\, (\text{Atiyah–Bott generators}),$$

where $\mathcal{L}$ is a determinant line bundle and the remaining factors are built from the Atiyah–Bott generators $E_x^* V$, $E_C^* V$, $E_\Sigma^* V$.

(B1) Quadratic shift from $\mathcal{L}$. By Grothendieck–Riemann–Roch, the first Chern class of $\mathcal{L}$ determines an invariant quadratic form

$$h = h_{\mathcal{L}} \in \mathrm{Sym}^2(\mathfrak{g}^*)^G,$$

called the *level* of $\mathcal{L}$. Teleman–Woodward's GRR calculation shows that, on the ξ–stratum, the ξ–weight contributed by $\mathcal{L}$ has leading behaviour controlled by this level:

$$\mathrm{wt}_\xi(\mathcal{L}) \sim h(\xi, \xi), \qquad \text{quadratic in } \xi. \tag{5}$$

Combining (5) with the determinant contribution (4) coming from $\mathrm{Eul}(\nu_\xi)_+^{-1}$, the *net* quadratic behaviour is governed by

$$(h + c)(\xi, \xi). \tag{6}$$

Recall that $\mathcal{L}$ is *admissible* precisely when $h + c$ is positive definite on $\mathfrak{g}$, equivalently when

$$(h + c)(\xi, \xi) \to +\infty \qquad \text{as } \|\xi\| \to \infty.$$

(B2) Linear perturbation from Atiyah–Bott factors. The remaining factors in $\mathcal{E}$ are Atiyah–Bott generators attached to representations V of G. Their ξ–weights are governed by ordinary representation theory: if λ is a weight of V, then ξ acts with weight $\langle \lambda, \xi \rangle$. In particular, these contributions are at most *linear* in ξ:

$$\mathrm{wt}_\xi(\text{Atiyah–Bott factors}) = O(\|\xi\|). \tag{7}$$

(C) Finite-dimensionality of the ξ–invariant part. The polarized inverse Euler factor $\mathrm{Eul}(\nu_\xi)_+^{-1}$ has a ξ–weight decomposition with two crucial properties:

(i) only weights ≤ 0 occur; and

(ii) each weight space has finite multiplicity.

These follow from the fact that in $(*)$ the symmetric algebra is generated by strictly negative ξ–weight summands.

Fix ξ. The weight–0 piece of $\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}$ is obtained by summing those weight spaces of $\mathrm{Eul}(\nu_\xi)_+^{-1}$ whose weights cancel the (finite) set of weights appearing in $\mathcal{E}$. Since each weight space of $\mathrm{Eul}(\nu_\xi)_+^{-1}$ has finite multiplicity, it follows that

$$\left(\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}\right)^{\xi\text{-inv}} \text{ is finite-dimensional.} \tag{8}$$

This is the first conclusion of [1, §1.11].

(D) Eventual vanishing for $\|\xi\| \gg 0$. Now let ξ vary in the dominant cone. The symmetric algebra part of $\mathrm{Eul}(\nu_\xi)_+^{-1}$ produces weights by taking symmetric powers of negative–weight generators. The possible weights contributed in this way move away from 0 in steps controlled by the individual ξ–weights of the generators; these steps scale *linearly* in ξ (because root weights $\langle \alpha, \xi \rangle$ are linear in ξ).

On the other hand, twisting by an admissible line bundle $\mathcal{L}$ produces the quadratic shift (6). Combining with the linear perturbation (7) from Atiyah–Bott generators, the net effect is that the set of ξ–weights appearing in $\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}$ is translated by a term which grows like $(h+c)(\xi, \xi)$, up to linear error. Since $(h+c)(\xi,\xi)$ grows quadratically while all available "corrections" coming from symmetric powers grow at most linearly, it follows that for $\|\xi\|$ sufficiently large the total ξ–weight 0 cannot occur. Equivalently, there exists $B > 0$ such that

$$\|\xi\| > B \quad \Longrightarrow \quad \left(\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}\right)^{\xi\text{-inv}} = 0. \tag{9}$$

This is the second conclusion of [1, §1.11].

(E) Consequence for finiteness of the index. The local cohomology filtration of $R\Gamma(\mathfrak{M}, \mathcal{E})$ by Shatz supports has graded pieces controlled by the strata $\mathfrak{M}_\xi$. Identifying the residue contribution along $\mathfrak{M}_\xi$ with the ξ–invariant part of $\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}$, the finiteness statement (8) gives finite-dimensionality of each stratum contribution, while the vanishing (9) shows that only finitely many ξ contribute. Therefore the local-to-global spectral sequence has only finitely many nonzero columns, and the index $\mathrm{Ind}(\mathfrak{M}, \mathcal{E})$ is finite.

3 General idea

Let $S = \mathbb{C}[[s]], S^* = \mathbb{C}((s))$ and B be an S-scheme. Let $C_S \to S$ be a projective flat family of curves with generic fiber $\mathbb{C}_{S^*}$ smooth and special fiber C_0 nodal with unique node p. Let $C_B = C_S \times_S B$.

Solis [**?**] defines the S-stack $\mathcal{X}_G(C_S)$ whose points evaluated at a test scheme B/S are given by elements (C'_B, P_B) where C'_B is a twisted modification of C_B and P_B is an admissible G-bundle on C'_B. This stack is over a fixed curve C_S and Solis shows that it is algebraic, locally of finite type, and complete over S. It contains $M_G(C_S)$ and $M_G(C_{S^*})$ as dense open substacks, and the complement of $M_G(C_{S^*})$ is a divisor with normal crossings.

In this section, we discuss how to generalize Solis' construction to families of curves by working over the universal curve over the moduli stack of stable curves $\overline{\mathfrak{M}}_{g,I}$. Let $\pi : \overline{\mathcal{C}_{g,I}} \to \overline{\mathfrak{M}}_{g,I}$ be the universal curve over the moduli stack of stable curves of genus g with I marked points.

Let $\pi : C \to B$ be a prestable family of nodal curves. Let

$$\Sigma := \mathrm{Sing}(C/B) \subset C$$

be the relative singular locus. It is finite étale over B after restricting to the locus where the number of nodes is constant; globally it is at least finite unramified in good situations.

Definition 3.1. A **modification of** C/B is a proper morphism $m : C' \to C$ over B such that:

1. $C' \to B$ is flat prestable curve, and m is finitely presented and projective.
2. m is an isomorphism away from the nodes:

$$m: \ C' \setminus m^{-1}(\Sigma) \ \xrightarrow{\sim} \ C \setminus \Sigma.$$

3. For every geometric point $b \to B$ and every node $p \in \Sigma_b \subset C_b$, the fiber of $m_b : C'_b \to C_b$ over p is either a point (no modification at that node) or a chain of $\mathbb{P}^1$'s meeting the two branches in the standard way, and m_b contracts that chain to p and is an isomorphism elsewhere.

A **length** $\leq n$ **condition** can be stated as:

- for every b and every node $p \in \Sigma_b$, the chain over p has at most n components.

Definition 3.2 (Twisted nodal curves over a base)**.** Let B be a scheme over $\mathbb{C}$. A **twisted nodal curve over** B is a proper Deligne–Mumford stack

$$\pi : \mathcal{C} \longrightarrow B$$

such that:

1. The geometric fibers of π are connected, one–dimensional, and the coarse moduli space $\overline{\mathcal{C}}$ is a nodal curve.
2. Let $\mathcal{U} \subset \mathcal{C}$ be the complement of the relative singular locus $\mathrm{Sing}(\mathcal{C}/B)$. Then the restriction

$$\mathcal{U} \hookrightarrow \mathcal{C}$$

is an open immersion.

3. For any geometric point $p : \operatorname{Spec} k \to \mathcal{C}$ mapping to a node of the fiber over $b \in B$, there exists an integer $k \geq 1$ and an element $t \in \mathfrak{m}_{B,b}$ such that, étale-locally on B at b and strictly henselian locally on $\mathcal{C}$ at p, there is an isomorphism

$$\operatorname{Spec} \mathcal{O}^{sh}_{\mathcal{C},p} \ \cong \ \Big[\operatorname{Spec}\big(\mathcal{O}^{sh}_{B,b}[u,v]/(uv-t)\big) \ / \ \mu_k\Big],$$

where $\zeta \in \mu_k$ acts by

$$(u,v) \longmapsto (\zeta u, \zeta^{-1} v).$$

Definition 3.3. A **twisted modification of** C/B is a twisted nodal curve $\mathcal{C} \to B$ whose coarse moduli space $\overline{\mathcal{C}}$ is a modification of C/B.

Let $r = \mathrm{rk}(G)$. The ordered simple roots $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$ determine ordered vertices $\{\eta_0, \dots, \eta_r\}$ determined by the conditions

$$\langle \eta_i, \alpha_j \rangle = 0 \text{ for } i \neq j \quad \text{and} \quad \langle \eta_0, \alpha_0 \rangle = 1.$$

If we write $\theta = \sum_{i=1}^{r} n_i \alpha_i$ and set $n_0 = 1$ then one can check these conditions can be expressed as

$$\langle \alpha_i, \eta_j \rangle = \frac{1}{n_i} \delta_{i,j}. \tag{10}$$

Following [**?**], if C'_B is a twisted modification of length $\leq r$, then a G–bundle on C'_B is called **admissible** if the co–characters determining the equivariant structure at all nodes are linearly independent over $\mathbb{Q}$ and are given by a subset of $\{\eta_0, \dots, \eta_r\}$.

Definition 3.4. We define a stack $\mathcal{X}_{G,g,I}$ over $\overline{\mathfrak{M}}_{g,I}$ whose points over a test scheme $B \to \overline{\mathfrak{M}}_{g,I}$ are given by pairs (C'_B, P_B) where C'_B is a twisted modification of the pullback C_B of the universal curve $\overline{\mathfrak{C}_{g,I}}$ to B, and P_B is an admissible G-bundle on C'_B.

Proposition 3.5. The projection

$$F : \mathcal{X}_{G,g,I} \to \overline{\mathcal{M}}_{g,I}$$

is algebraic and locally of finite type.

3.1 The PGL_2 toy model: gluing at a node and the wonderful compactification

Let $\widetilde{C}$ be a smooth connected curve and fix two distinct points $p, q \in \widetilde{C}$. Let C be the nodal curve obtained by identifying $p \sim q$, and write $\nu : \widetilde{C} \to C$ for the normalization; the node $x \in C$ satisfies $\nu^{-1}(x) = \{p, q\}$.

1. A point of G produces a G–bundle on the nodal curve

Let $G = \mathrm{PGL}_2(\mathbb{C})$ (or any algebraic group). Fix a principal G–bundle E on $\widetilde{C}$ together with framings (trivializations of the fibres as G–torsors)

$$f_p : E|_p \xrightarrow{\sim} G, \qquad f_q : E|_q \xrightarrow{\sim} G.$$

Given $g \in G$, define a G–equivariant isomorphism of G–torsors

$$\phi_g : E|_p \longrightarrow E|_q$$

by the composite

$$E|_p \xrightarrow{f_p} G \xrightarrow{\cdot g} G \xrightarrow{f_q^{-1}} E|_q,$$

where $x \mapsto xg$ denotes right multiplication (any consistent left/right convention works).

Using ϕ_g, one descends E from $\widetilde{C}$ to a principal G–bundle on the nodal curve C: informally, one glues the two fibres $E|_p$ and $E|_q$ over the branches of the node using the identification ϕ_g. Denote the resulting descended bundle by $E(\phi_g)$.

Equivalently, a principal G–bundle on C is the same as a principal G–bundle on $\widetilde{C}$ together with an identification of the fibres over p and q; the isomorphism ϕ_g is exactly such an identification. Thus, once the data (E, f_p, f_q) is fixed, the element $g \in G$ determines a principal G–bundle on C.

2. Rephrasing the gluing as a $\Delta(G)$–reduction

Consider the $G \times G$–torsor $E|_p \times E|_q$ over $\operatorname{Spec} \mathbb{C}$. An isomorphism $\phi : E|_p \to E|_q$ is equivalent to the choice of a point in the contracted product

$$(E|_p \times E|_q)/\Delta(G),$$

since $\Delta(G)$ acts by simultaneous change of trivializations, and the graph of ϕ is a $\Delta(G)$–orbit.

Equivalently, giving ϕ is the same as giving a reduction of structure group

$$E|_p \times E|_q \quad \text{from } G \times G \text{ to } \Delta(G) \subset G \times G.$$

This is the sense in which one identifies $G \simeq (G \times G)/\Delta(G)$ and interprets the gluing map ϕ_g as a $\Delta(G)$–reduction at the pair (p, q).

3. One–parameter families and the need for compactification

A morphism

$$\gamma : \operatorname{Spec} \mathbb{C}[t^{\pm 1}] \longrightarrow G$$

gives a family of gluing maps $\phi_{\gamma(t)}$, hence a family of principal G–bundles on the fixed nodal curve C parametrized by $t \in \mathbb{C}^\times$. However, γ may fail to extend to $t = 0$ as a morphism $\operatorname{Spec} \mathbb{C}[t] \to G$. A basic example is

$$t \longmapsto \operatorname{diag}(t, t^{-1}) \in \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{C}),$$

which "goes to infinity" in G. In that case the family of glued bundles on C has no *a priori* limit inside the original moduli problem. One remedy is to enlarge the parameter space G to a compactification $\overline{G}$, so that γ extends and the boundary value can be given a modular interpretation.

4. For $G = \mathrm{PGL}_2$, the wonderful compactification is $\mathbb{P}^3$

An element of $G = \mathrm{PGL}_2(\mathbb{C})$ may be represented by a 2×2 matrix up to overall scaling,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad (a, b, c, d) \neq 0,$$

giving an open embedding

$$\mathrm{PGL}_2 \hookrightarrow \mathbb{P}^3, \qquad [a : b : c : d],$$

whose image is the open subset $\{ad - bc \neq 0\}$. The boundary is the quadric

$$\{ad - bc = 0\} \subset \mathbb{P}^3,$$

which is the locus of rank ≤ 1 matrices. Concretely,

$$\{ad - bc = 0\} = \{\text{rank} \leq 1\} = \big\{[uv^T] : u \in \mathbb{C}^2 \setminus \{0\},\, v \in (\mathbb{C}^2)^\vee \setminus \{0\}\big\}/\mathbb{C}^\times \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

sending a rank–one matrix uv^T to the pair $([u], [v])$.

5. Modular meaning of boundary points: Borel reductions at p and q

Let $B \subset \mathrm{PGL}_2$ be a Borel subgroup (e.g. the image of upper triangular matrices). Then

$$G/B \cong \mathbb{P}^1$$

is the flag variety. In this case, the boundary of the wonderful compactification admits an identification

$$\partial\overline{G} := \overline{G} \setminus G \;\cong\; (G/B) \times (G/B) \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Given a point $(s_p, s_q) \in (G/B) \times (G/B)$, the framings f_p, f_q identify the fibres $E|_p \simeq G$ and $E|_q \simeq G$, and hence:

- the point $s_p \in G/B$ is the same as a B–reduction of the framed fibre $E|_p$;
- the point $s_q \in G/B$ is the same as a B–reduction of the framed fibre $E|_q$.

Thus allowing the gluing parameter to land in the boundary replaces an honest identification $E|_p \simeq E|_q$ (equivalently a $\Delta(G)$–reduction of $E|_p \times E|_q$) by weaker boundary data: a $B \times B$–reduction of the $G \times G$–torsor $E|_p \times E|_q$.

In other words, the "completed" moduli problem includes:

- a principal G–bundle E on $\widetilde{C}$, and
- either a $\Delta(G)$–reduction at (p, q) (giving a genuine G–bundle on C),
- or a boundary datum consisting of B–reductions at p and q (i.e. a $B \times B$–reduction).

This is the simplest instance of completing a moduli problem by allowing controlled degenerations at the node.

6. Relation to the affine/loop–group story

In the affine story one replaces the finite–dimensional parameter space G by loop–group data (often together with loop rotation $\mathbb{C}^\times$). The boundary of an affine compactification encodes parahoric reductions (equivalently, coweight data) at the node; this is where affine Weyl combinatorics enters. For $G = \mathrm{PGL}_2$ in the finite–dimensional toy model, the boundary is merely $(G/B)^2$, whereas in the affine case the boundary stratifies by affine Weyl data and leads to admissibility constraints.

7. A concrete limit computation

Let $\gamma(t) = [\mathrm{diag}(t, t^{-1})] \in \mathrm{PGL}_2$. In homogeneous coordinates $[a : b : c : d]$ on $\mathbb{P}^3$, this is

$$[t : 0 : 0 : t^{-1}] = [t^2 : 0 : 0 : 1].$$

As $t \to 0$ this tends to $[0 : 0 : 0 : 1]$, a rank–one matrix, hence a boundary point. Under $\partial\overline{G} \cong \mathbb{P}^1 \times \mathbb{P}^1$, this point corresponds to a pair of flags, i.e. to a choice of Borels at p and q, which is exactly the $B \times B$–reduction boundary datum described above.

Let $S = \mathbb{C}[[t]]$ and take a family $C \to S$ smoothing a node, locally $xy = t$. Near the node you have two formal branches and the "middle" is a punctured disc in the generic fiber. To give a G-bundle on a curve, it is enough to give it on the complement of a point and on the formal disc, plus an identification on the punctured disc.

That identification is a transition function in $G(\!(z)\!)$, i.e. an element of LG. In particular, if we allow limits of such transition functions, we are forced into allowing other "integral models" on the disc, i.e. parahorics.

The classification of parahorics is affine-Weyl/alcove combinatorics. In partiuclar, parahorics are classified by facets of the fundamental alcove, and maximal parahorics correspond to vertices of the alcove, which are labeled by the η_i.

Remark 3.6 (Why the η_i should be viewed as *parahoric types*)**.** The slogan is that in Solis' compactification one is not parametrizing ordinary G–bundles alone, but rather torsors under a *sheaf of groups* $\mathcal{G}$ which agrees with G away from the nodes and is replaced by a *parahoric* subgroup of the loop group near each node. The labels η_i encode precisely which parahoric is allowed.

(1) G–bundles as torsors for a sheaf of groups. Let C be a smooth curve. Write

$$\mathcal{G}^{\mathrm{std}}(U) := \mathrm{Hom}_{\mathrm{Sch}}(U, G)$$

for the usual sheaf of groups on C. A principal G–bundle E determines a $\mathcal{G}^{\mathrm{std}}$–torsor by

$$\mathcal{F}_E(U) := \Gamma(U, E|_U),$$

and conversely $\mathcal{G}^{\mathrm{std}}$–torsors are the same thing as principal G–bundles (this is the standard equivalence between G–bundles and G–torsors).

(2) Modifying the local structure group: parabolics and parahorics. Fix a point $p \in C$ with a formal parameter z. Let $D = \operatorname{Spec} \mathbb{C}[[z]]$ and $D^\times = \operatorname{Spec} \mathbb{C}((z))$. A principal G–bundle may be described by giving it on $C \setminus \{p\}$ and on D, together with a gluing isomorphism on the overlap $D^\times$; after choosing trivialisations this gluing is an element of the loop group

$$G((z)) = LG.$$

Now let $P \subset G$ be a parabolic subgroup. Set

$$L_P^+ G := \{\gamma \in G[[z]] \mid \gamma(0) \in P\} \subset G[[z]].$$

One may package the condition "a G–bundle with reduction to P at p" by replacing the local gauge group $G[[z]]$ on the disc by L_P^+G. Equivalently, define a new sheaf of groups $\mathcal{G}^P$ on a neighborhood of p by

$$\mathcal{G}^P(D) = L_P^+ G, \qquad \mathcal{G}^P(D^\times) = G((z)),$$

and gluing this with $\mathcal{G}^{\mathrm{std}}|_{C\setminus\{p\}}$ along the overlap $D^\times$. Then $\mathcal{G}^P$–torsors are exactly quasi–parabolic G–bundles (i.e. G–bundles on C with a P–reduction at p).

More generally one may replace L_P^+G by any *parahoric* subgroup $K \subset G((z))$ (in the sense of Bruhat–Tits), obtaining a sheaf of groups $\mathcal{G}^K$ whose torsors are "G–bundles with parahoric structure at p". For *exotic* parahorics these torsors need not be identifiable with ordinary G–bundles plus extra structure; they are genuinely new objects.

(3) Where the η_i enter. Parahoric subgroups of $G((z))$ are classified by facets of the affine alcove, and the *vertices* of the fundamental alcove correspond to *maximal* parahorics. The affine simple roots $\{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ have dual "vertex" coweights $\{\eta_0, \ldots, \eta_r\}$, and one may view η_i as labels for these maximal parahorics:

$$\eta_i \quad \longleftrightarrow \quad \text{a maximal parahoric } \mathcal{P}_{\eta_i} \subset G((z)).$$

Thus, specifying "η_i at a node" should be interpreted as specifying that the local structure group near that node is the parahoric $\mathcal{P}_{\eta_i}$, i.e. that we are working with torsors for the corresponding modified sheaf of groups.

(4) Conceptual rephrasing of admissibility. In this language, Solis' admissibility condition is a finiteness restriction on the allowed local group schemes at the nodes: on a twisted modification one only allows those parahoric types indexed by the finite set $\{\eta_0, \ldots, \eta_r\}$ (and imposes a further independence condition when several nodes occur). This is why it is natural to think of the η_i as *parahoric types*.

Definition 3.7 ($\mathcal{P}$–parahoric G–bundles at a point)**.** Fix a smooth curve C over $\mathbb{C}$ and a point $p \in C$ with a choice of formal parameter z at p, so that the completed local ring is $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[z]]$ and the punctured disc is $D^\times = \operatorname{Spec} \mathbb{C}((z))$. Let

$$\mathcal{P} \subset G((z))$$

be a *parahoric* subgroup (for instance a *maximal* parahoric, i.e. one corresponding to a vertex of the fundamental alcove).

Define a sheaf of groups $\mathcal{G}^{\mathcal{P}}$ on C by gluing the standard sheaf $\mathcal{G}^{\mathrm{std}}$ away from p with the local sheaf determined by $\mathcal{P}$ at p as follows:

- if $U \subset C$ is an open subset with $p \notin U$, set $\mathcal{G}^{\mathcal{P}}(U) := \mathcal{G}^{\mathrm{std}}(U) = \mathrm{Hom}_{\mathrm{Sch}}(U, G)$;
- for the formal disc $D = \operatorname{Spec} \mathbb{C}[[z]]$ and punctured disc $D^\times = \operatorname{Spec} \mathbb{C}((z))$, set
$$\mathcal{G}^{\mathcal{P}}(D) := \mathcal{P}, \qquad \mathcal{G}^{\mathcal{P}}(D^\times) := G((z)),$$
with restriction map given by the inclusion $\mathcal{P} \hookrightarrow G((z))$;
- on the overlap $(C \setminus \{p\}) \cap D \simeq D^\times$, we identify both restrictions with $G((z))$ and glue.

A *$\mathcal{P}$–parahoric G–bundle on C (with parahoric structure of type $\mathcal{P}$ at p)* is a $\mathcal{G}^{\mathcal{P}}$–torsor on C. Equivalently, it is the data of:

(i) a principal G–bundle E on $C \setminus \{p\}$;

(ii) a $\mathcal{P}$–torsor E_D on the formal disc D (i.e. a principal homogeneous space under the group $\mathcal{P} = \mathcal{G}^{\mathcal{P}}(D)$);

(iii) an identification of the induced $G((z))$–torsors over $D^\times$.

When $\mathcal{P} = L_P^+G = \{\gamma \in G[[z]] \mid \gamma(0) \in P\}$ for a parabolic $P \subset G$, this recovers the usual notion of a quasi–parabolic G–bundle with a P–reduction at p. In particular $L_{P_J}^+G$ corresponds to the facet containing the hyperspecial vertex η_0 together with the vertices η_j for $j \in J$.

When $\mathcal{P}$ is a maximal parahoric corresponding to a vertex η_i of the fundamental alcove, we say the parahoric structure at p is *of type* η_i.

Proposition 3.8 (Normalization description of parahoric bundles)**.** Let C be a nodal curve with normalization $\nu : \widetilde{C} \to C$ and nodes x_i with preimages $\nu^{-1}(x_i) = \{p_i, q_i\}$. Fix parahoric subgroups $\mathcal{P}_i \subset G((z_i))$ at each node.

A $\mathcal{G}^{\mathcal{P}}$–torsor on C is equivalent to:

1. a principal G–bundle E on $\widetilde{C}$;
2. for each i, $\mathcal{P}_i$–torsors E_{p_i} and E_{q_i} on the formal discs at p_i, q_i whose restrictions to the punctured discs agree with $E|_{\widetilde{C}\setminus\{p_i,q_i\}}$;
3. for each i, a gluing class
$$[\phi_i] \in \mathcal{P}_i \backslash G((z_i))/\mathcal{P}_i,$$

giving an identification of the punctured restrictions $E_{p_i}|_{D^\times} \cong E_{q_i}|_{D^\times}$.

If the node is stacky with stabilizer μ_k, one must additionally specify homomorphisms $\rho_i : \mu_k \to \mathcal{P}_i$ and require the gluing to be μ_k–equivariant.

How the compactification includes ordinary bundles

How does this compactify G-bundles on a nodal curve? Ordinary bundles sit inside as the trivial parahoric structure.

(1) Ordinary bundles on nodal curves. Choose trivializations of E on formal discs at p, q. Then a G-bundle on C is the same as

- a principal G-bundle E on $\widetilde{C}$, and
- an identification of fibers $E_p \simeq E_q$.

After choosing framings, that identification is an element $g \in G$. So the gluing parameter space for bundles on the fixed nodal curve is G.

Equivalently (using discs), the gluing is a loop $\gamma(z) \in G((z))$ which actually lies in $G[[z]]$ and has the same value at $z = 0$ on both branches, so it descends across the node. Concretely, the descent condition forces you into the "diagonal" subgroup

$$\Delta(G[[z]]) \subset G[[z]] \times G[[z]].$$

(2) Parahoric bundles enlarge the allowed local data. Now pick a parahoric $\mathcal{P} \subset G((z))$. A $\mathcal{P}$-parahoric torsor on C is (after choosing local trivializations) given by a double coset

$$[\gamma] \in \mathcal{P}\backslash G((z))/\mathcal{P},$$

i.e. you allow gluing by any loop $\gamma(z)$, but you declare two loops equivalent if they differ by "integral" gauge transformations on each side lying in $\mathcal{P}$.

This enlarges the moduli because $\mathcal{P}\backslash G((z))/\mathcal{P}$ contains more than the constant loops G.

(3) Where ordinary bundles sit inside: the hyperspecial case. There is a distinguished maximal parahoric, the *hyperspecial*

$$\mathcal{P}_0 := G[[z]] \subset G((z)).$$

Then

$$\mathcal{P}_0\backslash G((z))/\mathcal{P}_0$$

contains the constant loops G as an open subset (the big cell) in the Bruhat decomposition

$$G((z)) = \bigsqcup_{\lambda \in X_*(T)_+} \mathcal{P}_0 z^\lambda \mathcal{P}_0$$

where $X_*(T)_+$ are the dominant coweights. The constant loops correspond to $\lambda = 0$.

In words:

- trivial parahoric structure means: at the node you are not allowing any degeneration of the local group scheme; you keep the standard integral model $G[[z]]$;
- ordinary G-bundles on C correspond to those parahoric torsors whose gluing class can be represented by a constant loop (equivalently, whose local modification at the node is trivial).

Fixed nodal curve

Let $C_{0,[k]}$ be a twisted nodal curve with a single twisted node p. Let C_0 be its coarse moduli space and, by abuse of notation, also write $p \in C_0$ for the coarse node. The stabilizer of $p \in C_{0,[k]}$ is μ_k, and in particular

$$C_{0,[k]} \times_{C_0} D_0 \;\cong\; \left[D_0^{1/k} / \mu_k \right],$$

For a parahoric $\mathcal{P}$, let $\mathcal{L}\mathcal{U}$ be its Levi decomposition and set

$$\mathcal{P}^\Delta \;:=\; \Delta(\mathcal{L}) \ltimes (\mathcal{U} \times \mathcal{U}).$$

One can construct a sheaf of groups $\mathcal{G}^\Delta$ over C_0 such that

$$\mathcal{G}^\Delta(\widehat{\mathcal{O}}_{C_0,p}) = \mathcal{P}^\Delta, \qquad \mathcal{G}^\Delta\big|_{C_0 - p} = \mathcal{G}^{\mathrm{std}}.$$

Let $\mathcal{M}_{\mathcal{G}^\Delta}(C_0)$ denote the moduli stack of $\mathcal{G}^\Delta$–torsors on C_0 and let $T_{\mathcal{G}^\Delta}(C_0)$ denote the moduli space of pairs $(\mathcal{F}, \tau)$ where $\mathcal{F} \in \mathcal{M}_{\mathcal{G}^\Delta}(C_0)$ and τ is a trivialization of $\mathcal{F}$ over $C_0 - p$. Define $T_{\mathcal{G}^\Delta}(D_0)$ similarly.

Let $\eta \in \mathrm{Hom}(\mathbb{C}^\times, T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and consider the moduli stack $\mathcal{M}_{G,\eta}(C_{0,[k]})$ of G–bundles on $C_{0,[k]}$ with equivariant structure at p determined by η. Let $T_{G,\eta}(C_{0,[k]})$ denote the moduli space of pairs (P, τ) with $P \in \mathcal{M}_{G,\eta}(C_{0,[k]})$ and τ a trivialization of P on $C_{0,[k]} - p$. Define $T_{G,\eta}\big([D_0^{1/k}/\mu_k]\big)$ similarly.

Proposition 3.9 ([?, Prop. 3.4.5])**.** Suppose $k\eta \in \mathrm{Hom}(\mathbb{C}^\times, T)$ and set $\mathcal{P} = \mathcal{P}(\eta)$. Let

$$D_0 = \operatorname{Spec} \mathbb{C}[[x,y]]/(xy), \qquad \left[D_0^{1/k} / \mu_k \right] \text{ as above.}$$

Choose kth roots u, v of x, y so that $D_0^{1/k} = \mathbb{C}[[u,v]]/(uv)$. Let

$$i_{0,[k]} : \left[D_0^{1/k}/\mu_k \right] \longrightarrow C_{0,[k]}, \qquad i_0 : D_0 \longrightarrow C_0$$

be the natural maps. Let

$$G^{\Delta}_{u,v} := \left\{ (g_1, g_2) \in L_u^+ G \times L_v^+ G \;\middle|\; g_1(0) = g_2(0) \right\}.$$

Then we have isomorphisms fitting into the diagram

$$\begin{tikzcd}
T_{\mathcal{G}^{\Delta}}(D_0) & T_{\mathcal{G}^{\Delta}}(C_0) \arrow[l, "i_0^*"'] \arrow[r, "\Xi_{C_0}"] \arrow[d, "\Psi_C^{\mathcal{P}^{\Delta}}"'] & T_{G,\eta}(C_{0,[k]}) \arrow[r, "i^*_{0,[k]}"] \arrow[d, "\Psi_C^{\eta}"] & T_{G,\eta}\Big(\big[D_0^{1/k}/\mu_k\big]\Big) \\
& LG \times LG \,/\, \mathcal{P}^{\Delta,\eta} \arrow[r, "\eta^{-1}(\,\cdot\,)\eta"'] & (L_u G \times L_v G)^{\mu_k} \big/ \big(G^{\Delta}_{u,v}\big)^{\mu_k} &
\end{tikzcd}$$

where Ξ_{C_0} is defined to be

$$\Xi_{C_0} := \left(\Psi_C^{\eta}\right)^{-1} \circ \eta(\,\cdot\,)\eta^{-1} \circ \Psi_C^{\mathcal{P}^{\Delta}},$$

$\Psi_C^{\mathcal{P}^{\Delta}}$ is the map in *(3.8)*, Ψ_C^{η} is the map in *(3.10)*, and the bottom horizontal map is the product map

$$g(z)\mathcal{P} \longmapsto \eta(w)\, g(w^k)\, \eta^{-1}(w) \cdot (L_w^+ G)^{\mu_k}.$$

The isomorphism Ξ_{C_0} descends to an isomorphism of stacks

$$\Xi : \; \mathcal{M}_{\mathcal{G}^{\mathcal{P}}}(C_0) \;\xrightarrow{\sim}\; \mathcal{M}_{G,\eta}(C_{0,[k]}).$$

Connection with $L^{\times}_{\text{poly}} G$

In Chapter 2 a stacky orbit closure $\partial X^{aff,poly}$ was constructed, analogous to the boundary of the wonderful compactification of a semisimple adjoint group. In particular, the boundary components are smooth and intersect transversely. Let $r = \operatorname{rank}(G)$. There are $2^{r+1} - 1$ boundary orbits labeled by the nonempty subsets of $\{0, \dots, r\}$.

Proposition 3.10 ([?, Prop. 3.4.6])**.** Let $L_I, \mathcal{P}_I^{\pm}, \mathcal{U}_I^{\pm}$ be as in *(3.2)* of Section 3.2 and let $Z_0(L_I)$ be the connected component of the center $Z(L_I)$. Define

$$\mathcal{P}_I^{\Delta,\pm} := \Delta(L_I) \ltimes (\mathcal{U}_I^+ \times \mathcal{U}_I^-).$$

Then the orbit $\mathcal{O}_I$ in the boundary of $X^{aff,poly}$ is

$$\mathcal{O}_I = \frac{L_{\text{poly}} G \times L_{\text{poly}} G}{Z_0(L_I) \times Z_0(L_I) \cdot \mathcal{P}_I^{\Delta,\pm}}.$$

In particular, the orbit $\mathcal{O}_I$ fibers over
$$LG/\mathcal{P}_I \times LG/\mathcal{P}_I^-$$
with fiber the adjoint Levi
$$L_{I,\mathrm{ad}} = L_I/Z_0(L_I).$$
Furthermore, when $I = \{i\}$ is a singleton the group $Z_0(L_I)$ is trivial, while for $|I| > 1$ one has $Z_0(L_I) = Z(L_I)$.

Remark 3.11. The isomorphisms of Proposition 3.9 allow one to identify the singleton orbit $\mathcal{O}_{\{i\}}$ with the moduli spaces
$$T_{G,\eta_i}\left(\left[D_0^{1/k}/\mu_k\right]\right) \qquad \text{and} \qquad T_{G,\eta_i}(C_{0,[k]}),$$
where η_i is the ith vertex of the affine alcove. The natural expectation is that the moduli problem $T_{G,\eta_i}\left(\left[D_0^{1/k}/\mu_k\right]\right)$ can further degenerate to moduli problems parametrized by the higher codimensional orbits in $\mathbb{C}^\times \times L_{\mathrm{poly}}G$, and similarly for $T_{G,\eta_i}(C_{0,[k]})$. This is established in the next subsection.

G–bundles on twisted chains

In the previous section we saw that associated to the singleton sets $\{i\} \subset \{0, \ldots, r+1\}$ there is a moduli space parametrizing G–bundles on a twisted nodal curve, and further the moduli space can be identified with an orbit of the wonderful embedding of the loop group. In this section we introduce a more general moduli problem which we show is isomorphic to the orbit O_I in the wonderful embedding for any $I \subset \{0, \ldots, r+1\}$.

Let R_n denote the rational chain of projective lines with n components; There is an action of $\mathbb{C}^\times$ on R_n which scales each component. Let $p_0, \ldots, p_n$ denote the fixed points of this action.

Recall that u, v are kth roots of x, y which are our coordinates near a node. Let p', p'' denote the closed points of $\operatorname{Spec}\mathbb{C}[[u]]$ and $\operatorname{Spec}\mathbb{C}[[v]]$. Finally, let $D_n^{1/k}$ be the curve obtained from
$$\operatorname{Spec}\mathbb{C}[[u]] \,\sqcup\, R_n \,\sqcup\, \operatorname{Spec}\mathbb{C}[[v]]$$
by identifying p' with p_0 and p'' with p_n.

The group μ_k acts on $D_n^{1/k}$ through its usual action on u, v and through the inclusion $\mu_k \subset \mathbb{C}^\times$ on the chain R_n. For an n–tuple $(\beta_0, \ldots, \beta_n) \in \operatorname{Hom}(\mathbb{C}^\times, T)^n$, we can speak about the equivariant G–bundles on $D_n^{1/k}$ with equivariant structure at p_i determined by β_i. We refer to this equivalently as G–bundles on
$$\left[D_n^{1/k}/\mu_k\right]$$
of type $(\beta_0, \ldots, \beta_n)$.

Further, we can also glue $\left[D_n^{1/k}/\mu_k\right]$ to $C_0 - p_0$ to obtain a curve $C_{n,[k]}$. Let C_n denote the coarse moduli space of $C_{n,[k]}$. We call C_n a *modification* of C_0 and $C_{n,[k]}$ a *twisted modification* of C_0.

Recall the specific cocharacters $\eta_0, \ldots, \eta_r$ defined in *(3.1)* in §3.2. For $I = \{i_1, \ldots, i_n\} \subset \{0, \ldots, r\}$, let $T_{G,I}\big([D_n^{1/k}/\mu_k]\big)$ denote the moduli space of pairs (P, τ) where P is a G–bundle on $\big[D_n^{1/k}/\mu_k\big]$ of type $(\eta_{i_1}, \ldots, \eta_{i_n})$ and τ is a trivialization on

$$\big[\operatorname{Spec}\mathbb{C}((u)) \times \mathbb{C}((v))/\mu_k\big].$$

Let $H = \operatorname{Aut}(P)$; then restriction to $\operatorname{Spec}\mathbb{C}[[u]]$ and $\operatorname{Spec}\mathbb{C}[[v]]$ realizes

$$H \;\subset\; (L_uG)^{\mu_k} \times (L_uG)^{\mu_k}.$$

Theorem 3.12 ([?, Thm. 3.4.7])**.** Let $I \subset \{0, \ldots, r\}$ and $T_{G,I}\big([D_n^{1/k}/\mu_k]\big)$ be as above. Then there is an isomorphism

$$T_{G,I}(C_{0,[k]}) \xrightarrow{\Psi^{\eta_I}} (L_uG)^{\mu_k} \times (L_uG)^{\mu_k}/H \xrightarrow{\eta_I^{-1}(\;)\eta_I} \frac{L_{\mathrm{poly}}G \times L_{\mathrm{poly}}G}{Z(L_I) \times Z(L_I) \cdot P_I^{\Delta,\pm}}.$$

Here Ψ^{η_I} is as in *(3.10)* and $\eta_I^{-1}(\;)\eta_I$ is described in Proposition 3.4.5. Let

$$i:\; [D_n^{1/k}/\mu_k] \;\longrightarrow\; C_{0,[k]}$$

be the natural map. Then

$$i^*:\; T_{G,I}(C_{0,[k]}) \;\longrightarrow\; [D_n^{1/k}/\mu_k]$$

is an isomorphism. In particular, $T_{G,I}(C_{0,[k]})$ and $T_{G,I}\big([D_n^{1/k}/\mu_k]\big)$ are isomorphic to an orbit in the wonderful embedding of $L^{\times}_{\mathrm{poly}}G$.

3.2 Refined stratification of $\mathcal{X}_{G,g,I}$

Let C/B be a prestable curve with dual graph Γ, and let (C'_B, P_B) be an object of $\mathcal{X}_{G,g,I}$; that is, C'_B is a twisted modification of C_B and P_B is an admissible G–bundle on C'_B.

For each vertex $v \in V(\Gamma)$ let ξ_v denote the Harder–Narasimhan type of the restriction of P_B to the normalization component indexed by v. For each node $e \in E(\Gamma)$ we have two additional pieces of boundary data:

1. a *parahoric type*
$$I_e \subset \{0, \ldots, r\},$$
specifying the parahoric subgroup $\mathcal{P}_{I_e} \subset G((z))$ which governs the local structure of the bundle at e;

2. a *relative position label*
$$\mathbf{w}_e \in W_{I_e}\backslash\widetilde{W}/W_{I_e},$$
equivalently an orbit $O_e \subset \mathcal{P}_{I_e}\backslash G((z))/\mathcal{P}_{I_e}$ describing the gluing of the two branches at e.

Let τ_e denote the length of the modification chain over the node e. Collect the data into

$$\alpha = (\Gamma, \tau, \mathbf{I}, \mathbf{w}, \boldsymbol{\xi}), \qquad \mathbf{I} = (I_e)_{e \in E(\Gamma)}, \ \mathbf{w} = (\mathbf{w}_e)_{e \in E(\Gamma)}, \ \boldsymbol{\xi} = (\xi_v)_{v \in V(\Gamma)}.$$

Definition 3.13. The *refined stratum* of type α is the locally closed substack

$$\mathcal{X}_\alpha \subset \mathcal{X}_{G,g,I}$$

consisting of objects (C'_B, P_B) such that:

(i) the coarse curve C_B has dual graph Γ and the modification lengths at the nodes are τ_e;

(ii) for every vertex v, the restriction of P_B to the corresponding normalization component has Harder–Narasimhan type ξ_v;

(iii) at each node e, the parahoric structure of P_B is of type I_e, and the gluing of the two branches lies in the orbit O_e corresponding to $\mathbf{w}_e$.

The collection $\{\mathcal{X}_\alpha\}_\alpha$ forms a stratification of $\mathcal{X}_{G,g,I}$, and its closure relations are governed by:

- the usual specialization of dual graphs and modification lengths;
- the dominance order on the HN types ξ_v;
- the Bruhat order on the double cosets $W_{I_e} \backslash \widetilde{W} / W_{I_e}$.

Remark 3.14 (Why Shatz data alone is insufficient)**.** For a smooth curve the Shatz stratification indexed by HN types ξ is adequate, because a G–bundle has no additional local structure. On a nodal curve, however, two bundles with identical Shatz types on the normalization can differ essentially at the node.

Consider two objects: (1) a genuine G-bundle on the nodal curve with gluing element $g \in G \subset G((z))$, and (2) a limit object where the gluing is $z^\lambda \in G((z))$ with $\lambda > 0$.

Both have the same normalization bundles, the same HN type $\xi_v = 0$, and the same parahoric type $I_e = \{0\}$. However, (1) lies in the open stratum corresponding to actual G-bundles, while (2) lies in a boundary stratum of the wonderful compactification.

(1) Parahoric choice. The admissible object is not an honest G–bundle but a torsor under a sheaf of groups that equals G away from the nodes and a parahoric $\mathcal{P}_{I_e}$ near each node. Different choices of I_e give non–isomorphic deformation theories and different normal complexes, so they must label distinct strata.

(2) Relative position. Even after fixing I_e, the gluing of the two branches is classified by orbits in $\mathcal{P}_{I_e} \backslash G((z)) / \mathcal{P}_{I_e}$, indexed by double cosets $W_{I_e} \backslash \widetilde{W} / W_{I_e}$. These correspond to different boundary directions in the wonderful compactification and produce different weights in the local Euler factors.

Hence Shatz types $\boldsymbol{\xi}$ describe only the instability on the normalization components; they do not control the parahoric structure nor the affine–Weyl relative position at the nodes. A filtration of $R\Gamma(\mathcal{X}_{G,g,I}, \mathcal{E})$ analogous to the Teleman–Woodward argument therefore requires the refined indexing $(\Gamma, \tau, \mathbf{I}, \mathbf{w}, \boldsymbol{\xi})$.

Requires proof and I am not even sure it is true

Lemma 3.15. For any fixed combinatorial type (graph Γ + expansion length bound + allowed parahoric types I_e in a finite set) and any bound on $(\boldsymbol{\xi}_v, \mathbf{w}_e)$, the open substack $\mathcal{X}_{\leq\alpha}$ is algebraic and of finite type over an étale chart of $\overline{\mathfrak{M}}_{g,I}$.

3.3 Virtual normal complex and local cohomology for a refined stratum

Fix a refined type

$$\alpha = (\Gamma, \tau, \mathbf{I}, \mathbf{w}, \boldsymbol{\xi})$$

as in the refined stratification of $\mathcal{X}_{G,g,I}$, and let

$$i_\alpha : \mathcal{X}_\alpha \hookrightarrow \mathcal{X}_{\leq\alpha}$$

be the inclusion into a finite–type open substack $\mathcal{X}_{\leq\alpha}$ obtained by bounding the HN types on vertices, the modification lengths, and restricting the node data to the prescribed finite sets $(I_e, \mathbf{w}_e)$.

Let

$$\pi : \mathcal{C}_{\leq\alpha} \longrightarrow \mathcal{X}_{\leq\alpha}$$

be the universal twisted modification and let $\mathcal{P}$ be the universal (parahoric) G–torsor on $\mathcal{C}_{\leq\alpha}$. Write

$$\mathrm{ad}(\mathcal{P}) := \mathcal{P} \times^G \mathfrak{g}$$

for the adjoint vector bundle on $\mathcal{C}_{\leq\alpha}$.

1. Tangent complexes: global principle

A basic deformation–theoretic fact (for principal bundles on a curve, and likewise for torsors under a smooth affine group scheme on a twisted curve) is:

$$T_{\mathrm{Bun}_G,\mathcal{P}} \;\simeq\; R\pi_*\,\mathrm{ad}(\mathcal{P})[1]. \tag{11}$$

Intuitively: first–order deformations of $\mathcal{P}$ are controlled by $H^1(\mathrm{ad}(\mathcal{P}))$ and infinitesimal automorphisms by $H^0(\mathrm{ad}(\mathcal{P}))$, hence the shift $[1]$.

In our setting, $\mathcal{X}_{\leq\alpha}$ is not just Bun_G on a fixed curve: it also includes the expansion/twisting data. However, after fixing Γ and bounding τ, the expansion part is finite type and its tangent directions are independent of the bundle instability directions. Thus, for the purpose of the *normal directions to the refined stratum*, one isolates the contribution coming from the G–torsor deformation theory, which is governed by (11). (One can either include the expansion tangent complex everywhere and cancel it in cones below, or simply work relative to the expansion stack.)

2. The "Levi/parahoric core" controlling the stratum

By definition of the refined stratum $\mathcal{X}_\alpha$:

- for each vertex $v \in V(\Gamma)$, the restriction of $\mathcal{P}$ to the corresponding normalization component has instability type ξ_v, hence admits a canonical reduction to a parabolic $P_{\xi_v} \subset G$ with Levi G_{ξ_v};
- for each node $e \in E(\Gamma)$, the local structure group is the parahoric $\mathcal{P}_{I_e} \subset G((z))$, and the gluing lies in the orbit indexed by $\mathbf{w}_e \in W_{I_e} \backslash \widetilde{W} / W_{I_e}$.

Package these constraints into a subsheaf of Lie algebras

$$\mathrm{ad}_\alpha(\mathcal{P}) \;\subset\; \mathrm{ad}(\mathcal{P})$$

on $\mathcal{C}_{\leq \alpha}$ defined as follows:

(i) on the smooth locus away from the nodes, $\mathrm{ad}_\alpha(\mathcal{P}) = \mathrm{ad}(\mathcal{P})$;

(ii) on the normalization component corresponding to v, $\mathrm{ad}_\alpha(\mathcal{P})$ is the Lie algebra bundle associated to the *Levi* reduction, i.e.

$$\mathrm{ad}_\alpha(\mathcal{P})|_{C_v^{\mathrm{sm}}} \;=\; \mathcal{P}_{G_{\xi_v}} \times^{G_{\xi_v}} \mathfrak{g}_{\xi_v} \;\subset\; \mathcal{P}|_{C_v^{\mathrm{sm}}} \times^G \mathfrak{g};$$

(iii) at a node e, $\mathrm{ad}_\alpha(\mathcal{P})$ is the Lie algebra of the *parahoric Levi* dictated by I_e and the chosen orbit $\mathbf{w}_e$. Concretely, after choosing a formal parameter z and trivializing on the punctured disc, the allowed gauge transformations are $\mathcal{P}_{I_e}$, and the orbit $\mathbf{w}_e$ fixes a relative position; infinitesimally, this replaces $\mathfrak{g}((z))$ by the Lie algebra $\mathfrak{p}_{I_e} \subset \mathfrak{g}((z))$ and, on the stratum, by the Levi subalgebra $\mathfrak{l}_{I_e} \subset \mathfrak{p}_{I_e}$.

Define the *unstable quotient sheaf*

$$\mathcal{Q}_\alpha \;:=\; \mathrm{ad}(\mathcal{P}) \big/ \mathrm{ad}_\alpha(\mathcal{P}).$$

This $\mathcal{Q}_\alpha$ is the geometric object which simultaneously encodes:

- the usual unstable normal directions on components (root spaces outside $\mathfrak{g}_{\xi_v}$), and
- the boundary/gluing directions at nodes (tangent directions to the Schubert strata in $\mathcal{P}_{I_e} \backslash G((z)) / \mathcal{P}_{I_e}$ determined by $\mathbf{w}_e$).

3. Definition of the virtual normal complex ν_α

Definition 3.16 (Virtual normal complex for the refined stratum)**.** The *virtual normal complex* to $\mathcal{X}_\alpha$ inside $\mathcal{X}_{\le\alpha}$ (relative to the fixed expansion data) is the perfect complex on $\mathcal{X}_\alpha$

$$\nu_\alpha \ := \ R\pi_*(\mathcal{Q}_\alpha)[1]\Big|_{\mathcal{X}_\alpha}.$$

4. Justification: why this is the correct normal complex

The key point is that the refined stratum is cut out by imposing *linear conditions on infinitesimal gauge data*, encoded by the inclusion $\mathrm{ad}_\alpha(\mathcal{P}) \subset \mathrm{ad}(\mathcal{P})$.

Indeed, the standard deformation theory gives the tangent complex to the ambient moduli (again, relative to expansions) as $R\pi_* \,\mathrm{ad}(\mathcal{P})[1]$. On the stratum, allowed first–order deformations are precisely those preserving:

- the HN–Levi reductions on each component, and
- the parahoric type and relative position orbit at each node,

which infinitesimally means deformations governed by $R\pi_* \,\mathrm{ad}_\alpha(\mathcal{P})[1]$.

Since $\mathrm{ad}_\alpha(\mathcal{P}) \to \mathrm{ad}(\mathcal{P}) \to \mathcal{Q}_\alpha$ is exact, pushing forward and shifting yields a distinguished triangle

$$R\pi_* \,\mathrm{ad}_\alpha(\mathcal{P})[1] \ \longrightarrow \ R\pi_* \,\mathrm{ad}(\mathcal{P})[1] \ \longrightarrow \ R\pi_* \mathcal{Q}_\alpha[1] \ \longrightarrow,$$

which identifies $R\pi_*\mathcal{Q}_\alpha[1]$ as the cone of $T_{\mathcal{X}_\alpha} \to T_{\mathcal{X}_{\le\alpha}}$. This is exactly what "normal complex" means in derived deformation theory.

5. Local cohomology term attached to $\mathcal{X}_\alpha$

Let $\mathcal{E}$ be a coherent sheaf or perfect complex on $\mathcal{X}_{\le\alpha}$. Define the local cohomology of $\mathcal{E}$ with supports in $\mathcal{X}_\alpha$ by

$$R\Gamma_{\mathcal{X}_\alpha}(\mathcal{X}_{\le\alpha}, \mathcal{E}) \ := \ R\Gamma\big(\mathcal{X}_{\le\alpha},\, R\Gamma_{\mathcal{X}_\alpha}(\mathcal{E})\big),$$

characterized by the exact triangle

$$R\Gamma_{\mathcal{X}_\alpha}(\mathcal{X}_{\le\alpha}, \mathcal{E}) \ \to \ R\Gamma(\mathcal{X}_{\le\alpha}, \mathcal{E}) \ \to \ R\Gamma(\mathcal{X}_{\le\alpha} \setminus \mathcal{X}_\alpha, \mathcal{E}) \ \to .$$

Needs proof and I am not even sure it is true

Lemma 3.17. Each inclusion $i_\alpha : \mathcal{X}_\alpha \hookrightarrow \mathcal{X}_{\le\alpha}$ is (derived) lci/perfect so that

$$R\Gamma_{\mathcal{X}_\alpha}(\mathcal{X}_{\le\alpha}, \mathcal{E}) \ \simeq \ R\Gamma\big(\mathcal{X}_\alpha,\, i_\alpha^! \mathcal{E}\big)[d_\alpha]$$

for a codimension shift d_α. Then there is a purity/local-duality identification:

$$R\Gamma_{\mathcal{X}_\alpha}(\mathcal{X}_{\leq\alpha}, \mathcal{E}) \;\simeq\; R\Gamma\big(\mathcal{X}_\alpha,\; \mathcal{R}_\alpha(\mathcal{E})\big)[d_\alpha],$$

where d_α is the (virtual) codimension and

$$\mathcal{R}_\alpha(\mathcal{E}) \;:=\; i_\alpha^!(\mathcal{E})[-d_\alpha]$$

is the residue object along $\mathcal{X}_\alpha$.

6. Relation to the inverse Euler class (formal, but canonical once polarized)

Under the same lci/perfectness hypotheses, the extraordinary pullback $i_\alpha^!$ is controlled by the normal complex. In K–theory this yields the formal identity

$$[\mathcal{R}_\alpha(\mathcal{E})] \;\sim\; [\mathcal{E}|_{\mathcal{X}_\alpha}] \cdot \mathrm{Eul}(\nu_\alpha^\vee)^{-1}.$$

Here $\sim$ means equality in the appropriate *completed* (equivariant) K–group determined by a chosen polarization/weight convention, exactly as in Teleman–Woodward: one must choose a direction of expansion so that the inverse Euler factor is summable and weight spaces are finite.

In the refined situation, the relevant weight data comes from:

- the one–parameter subgroups determined by the vertex instability types ξ_v (componentwise Shatz data), and
- the parahoric/relative position label $(I_e, \mathbf{w}_e)$ at each node, which fixes the affine–Weyl combinatorics of the boundary directions and thus the weight decomposition of the node contribution to $\mathcal{Q}_\alpha$.

Consequently, the local cohomology term attached to the refined stratum is:

$$R\Gamma_{\mathcal{X}_\alpha}(\mathcal{X}_{\leq\alpha}, \mathcal{E}) \;\simeq\; R\Gamma\Big(\mathcal{X}_\alpha,\; \mathcal{E}|_{\mathcal{X}_\alpha} \otimes \mathrm{Eul}(\nu_\alpha^\vee)^{-1}\Big)[d_\alpha],$$

interpreted in the same completed sense as Teleman–Woodward (and with a polarized inverse Euler class if one wants uniform "weights ≤ 0" properties).

The analogue of the virtual normal complex ν_α

For each refined stratum $\mathcal{X}_\alpha$, you want:

- a "semistable Levi core" stack $\mathcal{Y}_\alpha$ (finite type), and
- a morphism $q_\alpha : \mathcal{X}_\alpha \to \mathcal{Y}_\alpha$ whose fibers are "unstable directions" (affine/unipotent), so that the normal theory is controlled by a perfect complex pulled back from $\mathcal{Y}_\alpha$.

3.1 What $\mathcal{Y}_\alpha$ should be

Let $\nu : \widetilde{C} \to C$ be the normalization of the underlying curve (over the relevant base). Over $\mathcal{X}_\alpha$, you have:

- components $\widetilde{C}_v$ indexed by vertices v,
- marked points corresponding to original markings plus the preimages of nodes (two per edge),
- at each preimage of a node: a specified parahoric type I_e and a relative-position orbit $\mathbf{w}_e$.

Define $\mathcal{Y}_\alpha$ as the product over vertices v of moduli of semistable G_{ξ_v}-bundles on $\widetilde{C}_v$ with the prescribed parahoric structures at the special points (the ones lying over nodes), together with whatever matching constraints encode $\mathbf{w}_e$ (often best phrased as belonging to a fixed Schubert cell in an affine flag space).

3.2 What ν_α should be

Let $\pi : \mathcal{C}_\alpha \to \mathcal{X}_\alpha$ be the universal (twisted) curve and $\mathcal{P}$ the universal parahoric G-torsor (or $\mathcal{G}$-torsor for the Bruhat–Tits group scheme $\mathcal{G}$ that equals G away from nodes).

Define a subsheaf of Lie algebras $\mathrm{ad}_\alpha(\mathcal{P}) \subset \mathrm{ad}(\mathcal{P})$ consisting of infinitesimal automorphisms that preserve:

- the vertexwise canonical P_{ξ_v}-reductions on each normalization component, and
- the edge constraints (parahoric type I_e and relative-position orbit $\mathbf{w}_e$) at each node.

Then set the quotient

$$\mathcal{Q}_\alpha := \mathrm{ad}(\mathcal{P}) / \mathrm{ad}_\alpha(\mathcal{P}),$$

and define the virtual normal complex

$$\nu_\alpha := R\pi_*(\mathcal{Q}_\alpha)[1] \quad \in \mathrm{Perf}(\mathcal{X}_\alpha).$$

What you must check here (deformation theory input)

You need:

1. **Tangent complex for parahoric torsors:** $T_{\mathcal{X},(C',P)} \simeq R\Gamma(C', \mathrm{ad}(P))[1]$ still holds in the parahoric/twisted nodal setting.
2. **Compatibility of "refined stratum constraints" with deformation theory:** The tangent

complex of $\mathcal{X}_\alpha$ is the subcomplex cut out by $\mathrm{ad}_\alpha(\mathcal{P})$, so the quotient controls the normal directions.

3. **Perfectness:** $\mathrm{ad}(\mathcal{P})$, $\mathrm{ad}_\alpha(\mathcal{P})$, and $\mathcal{Q}_\alpha$ are perfect on the curve (ideally vector bundles), so ν_α is perfect of amplitude $[-1,0]$ after shifting.

This replaces TW's formula $R\pi_*\mathcal{E}^*(\mathfrak{g}/\mathfrak{g}_\xi)[1]$.

3.4 An explicit refined stratification in the one–node, two–component case

Setup

Let $C = C_1 \cup_x C_2$ be a connected nodal curve over $\mathbb{C}$ with a single node x, where C_1, C_2 are smooth and meet transversely at x. Let $\nu : \widetilde{C} = C_1 \sqcup C_2 \to C$ be the normalization and write $\nu^{-1}(x) = \{p \in C_1,\ q \in C_2\}$. Fix a connected reductive group G.

Fix a formal parameter z at p and q (after choosing étale neighborhoods if desired), so that the formal discs are $D_p = \operatorname{Spec}\mathbb{C}[[z]]$, $D_q = \operatorname{Spec}\mathbb{C}[[z]]$ and the punctured discs are $D_p^\times = \operatorname{Spec}\mathbb{C}((z))$, $D_q^\times = \operatorname{Spec}\mathbb{C}((z))$. Write $LG := G((z))$ and $L^+G := G[[z]]$.

Gluing description of $\mathrm{Bun}_G(C)$

Let $\mathrm{Bun}_G(C_i, p)$ denote the stack of G–bundles on C_i equipped with a framing (trivialization) along D_p (respectively D_q). Then a G–bundle on C is equivalent to a triple

$$(E_1, \tau_p) \in \mathrm{Bun}_G(C_1, p), \qquad (E_2, \tau_q) \in \mathrm{Bun}_G(C_2, q), \qquad \gamma \in LG,$$

modulo the change of framings by $L^+G \times L^+G$, acting by

$$(a,b)\cdot(E_1, \tau_p, E_2, \tau_q, \gamma) = (E_1, a\cdot\tau_p,\ E_2, b\cdot\tau_q,\ a\,\gamma\,b^{-1}).$$

Equivalently,

$$\mathrm{Bun}_G(C) \;\simeq\; \Big[\big(\mathrm{Bun}_G(C_1,p)\times \mathrm{Bun}_G(C_2,q)\times LG\big)\big/\big(L^+G\times L^+G\big)\Big]. \tag{12}$$

(One can formulate (12) invariantly without choosing z by working with torsors over punctured formal neighborhoods; the displayed model is the usual "loop group" presentation after choosing parameters.)

Parahoric variant and the refined labels

Fix a parahoric subgroup $\mathcal{P} \subset LG$. For definiteness, you may keep in mind either the hyperspecial parahoric $\mathcal{P} = L^+G$ or a maximal parahoric corresponding to a vertex η_i of the fundamental alcove in the affine–Weyl combinatorics.

Replacing the local gauge group L^+G by $\mathcal{P}$ gives the parahoric gluing stack

$$\mathrm{Bun}_{\mathcal{P}}(C) \;:=\; \Big[\big(\mathrm{Bun}_G(C_1,p)\times \mathrm{Bun}_G(C_2,q)\times LG\big)/\big(\mathcal{P}\times\mathcal{P}\big)\Big], \tag{13}$$

with $(a,b)\in\mathcal{P}\times\mathcal{P}$ acting as above.

The double coset space $\mathcal{P}\backslash LG/\mathcal{P}$ carries a stratification by $\mathcal{P}\times\mathcal{P}$–orbits. Fix an indexing set $\Omega_{\mathcal{P}}$ for these orbits; for example one may take

$$\Omega_{\mathcal{P}} \;=\; W_{\mathcal{P}}\backslash W_{\text{aff}}/W_{\mathcal{P}},$$

and denote the orbit corresponding to $w\in\Omega_{\mathcal{P}}$ by $\mathcal{O}_w\subset LG$.

Definition 3.18 (Orbit–type strata)**.** For $w\in\Omega_{\mathcal{P}}$ define the locally closed substack

$$\mathrm{Bun}_{\mathcal{P}}(C)_w \;:=\; \Big[\big(\mathrm{Bun}_G(C_1,p)\times \mathrm{Bun}_G(C_2,q)\times \mathcal{O}_w\big)/\big(\mathcal{P}\times\mathcal{P}\big)\Big] \;\subset\; \mathrm{Bun}_{\mathcal{P}}(C).$$

Remark 3.19 (Why this is genuinely new on nodal curves)**.** Even if one fixes a "Shatz type on each component", the gluing parameter $\gamma\in LG$ may move in distinct $\mathcal{P}$–double cosets, and the closure relations of these orbits (Bruhat order in $\Omega_{\mathcal{P}}$) contribute additional boundary directions at the node. Thus Shatz data alone does not control the boundary geometry once parahoric/loop–group gluing is allowed.

Refinement by Shatz types on the components

Let $\mathrm{Bun}_G(C_i)_{\xi_i}\subset\mathrm{Bun}_G(C_i)$ be the Shatz stratum of instability type ξ_i, and let $\mathrm{Bun}_G(C_i,p)_{\xi_i}$ be its pullback to framed bundles. (Equivalently: the HN type is a condition on the underlying bundle and does not depend on the framing.)

Definition 3.20 (Refined strata)**.** A *refined label* is a triple

$$\alpha=(\xi_1,\xi_2,w),\qquad \xi_i\ \text{dominant rational coweights},\qquad w\in\Omega_{\mathcal{P}}.$$

Define the refined stratum

$$\mathcal{X}_\alpha \;:=\; \Big[\big(\mathrm{Bun}_G(C_1,p)_{\xi_1}\times \mathrm{Bun}_G(C_2,q)_{\xi_2}\times \mathcal{O}_w\big)/\big(\mathcal{P}\times\mathcal{P}\big)\Big] \;\subset\; \mathrm{Bun}_{\mathcal{P}}(C). \tag{14}$$

Remark 3.21 (Closure order)**.** Let $\le$ denote the partial order on Shatz types (dominance order) and let $\le_{\mathrm{Br}}$ denote the Bruhat order on $\Omega_{\mathcal{P}}$. In all standard situations (e.g. hyperspecial $\mathcal{P}=L^+G$ or more generally any parahoric), one expects the closure relations to satisfy

$$\overline{\mathcal{X}_{\xi_1,\xi_2,w}}\;\supset\;\mathcal{X}_{\xi_1',\xi_2',w'}\qquad\Longleftrightarrow\qquad \xi_1'\le\xi_1,\ \xi_2'\le\xi_2,\ \text{and}\ w'\le_{\mathrm{Br}} w.$$

In particular, for a fixed bound Ξ_1, Ξ_2 and a fixed w_0, the union

$$\mathcal{X}_{\leq(\Xi_1,\Xi_2,w_0)} := \bigcup_{\xi_1\leq\Xi_1,\ \xi_2\leq\Xi_2,\ w\leq_{\mathrm{Br}}w_0} \mathcal{X}_{\xi_1,\xi_2,w}$$

is a *finite union* of strata (the Bruhat interval is finite, and Shatz truncations are finite). This is the basic finite–type truncation used for index finiteness arguments.

A workable virtual normal complex for $\mathcal{X}_\alpha \subset \mathcal{X}_{\leq\alpha}$

Write $\pi_i : C_i \times \mathrm{Bun}_G(C_i) \to \mathrm{Bun}_G(C_i)$ for the projections, and let $\mathcal{E}_i$ be the universal G–bundle on $C_i \times \mathrm{Bun}_G(C_i)$. Let $P_{\xi_i} \subset G$ be the canonical parabolic of type ξ_i and G_{ξ_i} its Levi.

On the smooth–curve side, the Teleman–Woodward normal complex for the Shatz stratum is (after pulling back to the semistable Levi core)

$$\nu^{(i)}_{\xi_i} \ := \ R(\pi_i)_*\big(\mathcal{E}_i^*(\mathfrak{g}/\mathfrak{g}_{\xi_i})\big)[1].$$

On the node side, the orbit closure $\overline{\mathcal{O}_w}$ carries a normal complex for the locally closed embedding $\mathcal{O}_w \hookrightarrow \overline{\mathcal{O}_w}$:

$$\nu^{\mathrm{node}}_w \ := \ \mathbb{L}_{\mathcal{O}_w/\overline{\mathcal{O}_w}}[-1],$$

a perfect complex concentrated in degrees $[-1, 0]$ whenever $\overline{\mathcal{O}_w}$ is normal with rational singularities (as happens for Schubert varieties in the standard cases). Pull this back to $\mathcal{X}_\alpha$ via the projection to the LG–factor.

Definition 3.22 (Candidate normal complex on the refined stratum)**.** Define the (virtual) normal complex on $\mathcal{X}_\alpha$ by

$$\nu_\alpha \ := \ \nu^{(1)}_{\xi_1} \ \oplus \ \nu^{(2)}_{\xi_2} \ \oplus \ \nu^{\mathrm{node}}_w. \tag{15}$$

Remark 3.23 (Justification for (15))**.** Heuristically, the refined stratum $\mathcal{X}_\alpha$ is cut out by three independent conditions: (i) the Shatz condition on C_1, (ii) the Shatz condition on C_2, and (iii) the orbit–type condition at the node (the $\mathcal{P}$–double coset of the gluing). Infinitesimally these contribute transverse deformation directions: the first two are measured by the smooth–curve deformation theory of G–bundles (TW's $\nu^{(i)}_{\xi_i}$), and the third is measured by the deformation theory of the orbit embedding in the local loop–group parameter space. The direct sum in (15) is the cleanest way to package this additivity; it becomes canonical after choosing a presentation (13) and identifying tangent complexes via the standard quotient formula $T_{[X/H]} \simeq [T_X \to \mathfrak{h}]$.

Local cohomology terms for the refined filtration

Fix an ordering of refined labels so that $\mathcal{X}_{\leq\alpha}$ is open and $\mathcal{X}_\alpha$ is locally closed with complement a union of "more unstable" labels. Let $i_\alpha : \mathcal{X}_\alpha \hookrightarrow \mathcal{X}_{\leq\alpha}$ be the inclusion.

For a sheaf/complex $\mathcal{E}$ on $\mathcal{X}_{\leq\alpha}$ define

$$R\Gamma_{\mathcal{X}_\alpha}(\mathcal{X}_{\leq\alpha}, \mathcal{E}) := R\Gamma\big(\mathcal{X}_{\leq\alpha},\ R\Gamma_{\mathcal{X}_\alpha}(\mathcal{E})\big), \qquad \mathcal{R}_\alpha(\mathcal{E}) := i_\alpha^!(\mathcal{E})[-\operatorname{codim}(\mathcal{X}_\alpha)].$$

Under standard purity hypotheses (e.g. $\mathcal{X}_\alpha$ regularly embedded in a smooth ambient presentation of $\mathcal{X}_{\leq\alpha}$, or more generally lci in the derived sense), one expects an identification parallel to the smooth–curve story:

$$R\Gamma_{\mathcal{X}_\alpha}(\mathcal{X}_{\leq\alpha}, \mathcal{E}) \ \simeq\ R\Gamma\Big(\mathcal{X}_\alpha,\ \mathcal{E}|_{\mathcal{X}_\alpha} \otimes \operatorname{Eul}(\nu_\alpha)^{-1}\Big)[d_\alpha], \tag{16}$$

where $d_\alpha = \operatorname{codim}(\mathcal{X}_\alpha, \mathcal{X}_{\leq\alpha})$ and $\operatorname{Eul}(\nu_\alpha)$ denotes the K–theoretic Euler class of the perfect complex ν_α.

Remark 3.24 (What one must check to use (16))**.** To make (16) a theorem (rather than a guiding formula), one needs:

(1) a filtration by finite–type opens $\mathcal{X}_{\leq\alpha}$ and a locally closed stratification $\mathcal{X}_{\leq\alpha} = \bigsqcup_{\beta\leq\alpha} \mathcal{X}_\beta$ with good closure relations;

(2) a purity/local–duality statement identifying $i_\alpha^!$ in terms of a normal complex;

(3) perfectness of ν_α (and compatibility with base change over the global base);

(4) a $\mathfrak{G}_m$–grading on ν_α (from Shatz 1–PS data on the components and the standard weight grading on the Schubert normal directions) so that a *polarized inverse Euler class* can be defined in a completed equivariant K–theory;

(5) an admissibility condition ensuring eventual vanishing of the $\mathfrak{G}_m$–weight–0 part of $\mathcal{E}\otimes\operatorname{Eul}(\nu_\alpha)_+^{-1}$ for α sufficiently unstable.

The one–node model above is useful because each item can be tested explicitly: (2)–(4) reduce to the known smooth–curve Shatz theory on C_i and to standard properties of Schubert strata in $\mathcal{P}\backslash LG/\mathcal{P}$.

4 References

1. Teleman, C., & Woodward, C. (2012). The index formula on the moduli of G-bundles. Annals of Mathematics, 176(2), 601-77.