Stacks and Moduli Spaces

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Abstract

A detailed exploration of the theory of stacks and moduli spaces. We aim to bridge the abstract formalism of stacks with concrete examples, such as moduli spaces of vector bundles and principal bundles. The interplay between algebraic geometry, category theory, and topology is emphasized, showcasing the power of stacks as a unifying framework. Key results, including the Verlinde formula and the classification of principal bundles via loop groups, are discussed in depth.

Contents

1 Goals

We should try to understand the following results from various historical papers. Each of them really should merit its own discussion.

1.1 Statements from the general theory of stacks

Proposition 1.1. The stack $*/G$, defined as the sheafification of $(*/G)^{naive}$, represents the following moduli problem:

$$(*/G)(X) = \text{Groupoid of principal } G\text{-torsors over } X.$$

Proposition 1.2. There is an equivalence of categories:

$$\mathrm{QCoh}(BG) \leftrightarrow \mathrm{QCoh}^G(pt) \leftrightarrow \mathrm{Rep}(G).$$

Theorem 1.3. For any morphism of schemes $X \to Y$, the functor h_X is a sheaf in the fppf topology (and therefore also in the etale topology) on the category of Y-schemes.

1.2 Statements from moduli theory of Riemann surfaces

There is a canonical isomorphism between two vector spaces associated to a Riemann surface X. The first of these spaces is the space of **conformal blocks** $\mathcal{B}_c(r)$ (also called the space of vacua), which plays an important role in conformal field theory.

Definition 1.4. Choose a point $p \in X$, and let A_X be the ring of algebraic functions on $X - p$. To each integer $c \geq 0$ is associated a representation V_c of the Lie algebra $\mathfrak{sl}_r(\mathbb{C}((z)))$, the **basic representation** of level c (more correctly it is a representation of the universal extension of $\mathfrak{sl}_r(\mathbb{C}((z)))$. The ring A_X embeds into $\mathbb{C}((z))$ by associating to a function its Laurent development at p; then $\mathcal{B}_c(r)$ is the space of linear forms on V_c which vanish on the elements $A(z)v$ for $A(z) \in \mathfrak{sl}_r(A_X)$, $v \in V_c$.

The second space comes from algebraic geometry, and is defined as follows.

Definition 1.5. Let $\mathcal{SU}_X(r)$ be the moduli space of semi-stable rank r vector bundles on X with trivial determinant. One can define a theta divisor on $\mathcal{SU}_X(r)$ in the same way one does in the rank 1 case: one chooses a line bundle L on X of degree $g-1$, and considers the locus of vector bundles $E \in \mathcal{SU}_X(r)$ such that $E \otimes L$ has a nonzero section. The associated line bundle $\mathcal{L}$ is called the **determinant bundle**; the space we are interested in is $H^0(\mathcal{SU}_X(r), \mathcal{L}^c)$.

This space can be considered as a non-Abelian version of the space of c^{th}-order theta functions on the Jacobian of X, and is sometimes called the space of **generalized theta functions**.

We will prove that it is canonically isomorphic to $\mathcal{B}_c(r)$. This implies that $H^0(\mathcal{SU}_X(r), \mathcal{L}^c)$ satisfies the **fusion rules**, which allow to compute its dimension in a purely combinatorial way. A closed expression for this dimension is known as the **Verlinde formula**.

Theorem 1.6 (Verlinde formula)**.** We have

$$\dim H^0(\mathcal{SU}_X(r), \mathcal{L}^c) = \left(\frac{r}{r+c}\right)^g \sum_{\substack{S\subset[1,r+c]\\ |S|=r}} \prod_{\substack{s\in S\\ t\notin S}} \left|2\sin\pi\frac{s-t}{r+c}\right|^{g-1}$$

2 Motivating example $B\mathbb{Z}_2$

We begin by recalling vector bundles and Cech cohomology. A rank n vector bundle E over a topological space X is a topological space E together with a continuous map $\pi : E \to X$ such that:

1. For each $x \in X$, the fiber $\pi^{-1}(x)$ has the structure of an n-dimensional vector space.

2. For each $x \in X$, there is open U of x and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that:

   - $\pi = \mathrm{pr}_1 \circ \varphi$ where $\mathrm{pr}_1 : U \times \mathbb{R}^n \to U$ is the projection.
   - For each $y \in U$, the restriction $\varphi|_{\pi^{-1}(y)} : \pi^{-1}(y) \to \{y\} \times \mathbb{R}^n$ is a linear isomorphism.

In particular, vector bundles can be glued together from local data using transition functions. Given an open cover $\{U_i\}_{i\in I}$ of X, a vector bundle can be specified by transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{R})$ satisfying the cocycle condition:

$$g_{ij} \cdot g_{jk} = g_{ik} \quad \text{on} \quad U_i \cap U_j \cap U_k$$

This naturally leads us to Čech cohomology, which provides a framework for understanding when local data can be glued to form global structures. For a sheaf $\mathcal{F}$ on X and open cover $\mathcal{U} = \{U_i\}_{i\in I}$, we define the Čech complex:

$$\check{C}^0(\mathcal{U}, \mathcal{F}) \to \check{C}^1(\mathcal{U}, \mathcal{F}) \to \check{C}^2(\mathcal{U}, \mathcal{F}) \to \cdots$$

where

$$\check{C}^k(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1 < \cdots < i_k} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_k})$$

and the coboundary map is given by:

$$(\delta s)_{i_0,\ldots,i_{k+1}} = \sum_{j=0}^{k+1} (-1)^j s_{i_0,\ldots,\hat{i_j},\ldots,i_{k+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{k+1}}}$$

In particular, a 1-cochain is an arbitrary collection of sections $s_{ij} \in \mathcal{F}(U_i \cap U_j)$, a 1-cocycle is a collection of sections s_{ij} exactly satisfying the cocycle condition, and a 1-coboundary is a collection of sections of the form $s_i - s_j$ for some $s_i \in \mathcal{F}(U_i)$ and $s_j \in \mathcal{F}(U_j)$.

The cohomology of this complex gives the Čech cohomology groups $\check{H}^i(\mathcal{U}, \mathcal{F})$. We obtain the Cech cohomology groups $\check{H}^i(X, \mathcal{F})$ by taking the direct limit over all open covers of X. In particular, $\check{H}^1(X, \mathrm{GL}_n)$ classifies rank n vector bundles on X, where GL_n is the sheaf of functions $X \to \mathrm{GL}_n(\mathbb{R})$.

The key takeaway is that isomorphism classes of vector bundles on X are classified by elements of $\check{H}^1(X, \mathrm{GL}_n)$, and that this cohomology group captures symmetries of the fiber $\mathrm{Aut}(V) = \mathrm{GL}_n(\mathbb{R})$ and how these symmetries can be glued together to form a global object. These concepts will be generalized as we develop the theory of stacks and algebraic spaces.

2.1 Category theory

Definition 2.1. Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\eta : F \to G$ is a collection of morphisms $\eta_X : F(X) \to G(X)$ for each object $X \in \mathcal{C}$ such that for any morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

A natural transformation is an **isomorphism** if each η_X is an isomorphism in $\mathcal{D}$.

Definition 2.2. Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence** if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\mu : G \circ F \to \mathrm{id}_{\mathcal{C}}$ and $\nu : F \circ G \to \mathrm{id}_{\mathcal{D}}$.

Proposition 2.3. Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is full, faithful, and essentially surjective. This means that

- Full: For any $X, Y \in \mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective.
- Faithful: For any $X, Y \in \mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective.
- Essentially surjective: For any $Z \in \mathcal{D}$, there exists $X \in \mathcal{C}$ such that $F(X) \simeq Z$ where $\simeq$ denotes isomorphism in $\mathcal{D}$.

Definition 2.4 (Fiber Product of Groupoids)**.** If A, B, and C are groupoids, and $F : A \to C$ and $G : B \to C$ are functors, then the fiber product $A \times_C B$ is the groupoid defined as follows:

- **Objects:** Triples (a, b, ϕ) where $a \in \mathrm{ob}(A)$, $b \in \mathrm{ob}(B)$, and $\phi : F(a) \xrightarrow{\sim} G(b)$ is an isomorphism in C
- **Morphisms:** A morphism $(a, b, \phi) \to (a', b', \phi')$ consists of a pair (φ, ψ) of isomorphisms $\varphi : a \xrightarrow{\sim} a'$, $\psi : b \xrightarrow{\sim} b'$ so that

$$\begin{array}{ccc} F(a) & \xrightarrow{\phi} & F(b) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle G(\psi)} \\ F(a') & \xrightarrow[\phi']{} & F(b'). \end{array}$$

Example 2.5. The fiber product of $* \times_{(*/G)} *$, where $*$ is the trivial groupoid, is given by:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & */G. \end{tikzcd}$$

where G represents the set of elements of the group G thought of as a set in **Gpd**. This follows directly from the construction of fiber products described above.

2.2 The stack $B\mathbb{Z}_2$

Consider the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ with the discrete topology. We will answer the question: for a topological space T, what are the T-points of $B\mathbb{Z}_2 = */\mathbb{Z}_2$, i.e. a map $T \to B\mathbb{Z}_2$? It turns out that these correspond to principal $\mathbb{Z}_2$-torsors on T, i.e. a space P with a free transitive action of $\mathbb{Z}_2$ with a map $P \to T$ that is $\mathbb{Z}_2$-equivariant.

Our first guess to define the functor of points of $B\mathbb{Z}_2$, which we shall call $(*/\mathbb{Z}_2)^{\text{naive}} : \text{Top} \to \text{Gpd}$, is given by $(*/\mathbb{Z}_2)^{\text{naive}}(T) = \text{Maps}(T, *)/\text{Maps}(T, \mathbb{Z}_2) = */\text{Maps}(\mathbb{Z}_2^{\pi_0(T)})$. This is almost right; however, this functor into groupoids does not actually define a sheaf.

To see this, let $T = S^1$. We see that $(*/\mathbb{Z}_2)^{\text{naive}}(S^1) = */\text{Maps}(S^1, \mathbb{Z}_2) = */\mathbb{Z}_2$, where the latter is thought of as just the groupoid. Now, a sheaf is characterized by the property that compatible local sections patch uniquely into global sections. Cover S^1 by two open arcs U_0 and U_1 such that $U_0 \cup U_1 = S^1$, and such that $U_0 \cap U_1$ consists of two disconnected arcs $A \sqcup B$.

We see that

$$(*/\mathbb{Z}_2)^{\text{naive}}(U_0) = */\mathbb{Z}_2, \tag{1}$$
$$(*/\mathbb{Z}_2)^{\text{naive}}(U_1) = */\mathbb{Z}_2 \tag{2}$$

and

$$(*/\mathbb{Z}_2)^{\text{naive}}(U_0 \cap U_1) = */(\mathbb{Z}_2 \times \mathbb{Z}_2), \tag{3}$$

since $U_0 \cap U_1 = A \sqcup B$ has two disconnected components. The restrictions $\text{res}^{U_0}_{U_0 \cap U_1}$ and $\text{res}^{U_1}_{U_0 \cap U_1}$ both correspond to the diagonal embedding $\Delta : */\mathbb{Z}_2 \to */(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Therefore, if $(*/\mathbb{Z}_2)^{\text{naive}}$ were to be a sheaf, we need that

$$*/\mathbb{Z}_2^{\text{naive}}(S^1) = */\mathbb{Z}_2 \times_{*/(\mathbb{Z}_2 \times \mathbb{Z}_2)} */\mathbb{Z}_2$$

where this fiber product is taken in the category of groupoids.

We can check that the objects of the category $*/\mathbb{Z}_2 \times_{*/(\mathbb{Z}_2 \times \mathbb{Z}_2)} */\mathbb{Z}_2$ are given by triples $(*, *, g)$ where $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Morphisms from $(*, *, g)$ to $(*, *, g')$ are pairs of isomorphisms $\phi : *_A \to *_A$ and $\psi : *_B \to *_B$ so that

$$\begin{tikzcd}
*_C \arrow[r, "g_C"] \arrow[d, "F(\varphi)"'] & *_C \arrow[d, "G(\psi)"] \\
*_C \arrow[r, "g'_C"'] & *_C
\end{tikzcd}$$

in particular pairs of elements $\phi \in \Delta_{\mathbb{Z}/2}$ and $\psi \in \Delta_{\mathbb{Z}/2}$ so that $\phi g = g'\psi$.

1. Consider the morphisms from (e, e) to (e, e). If $\psi = \phi = \Delta(e)$ then

   $$(e, e) * \Delta(e) = (e, e) = (e, e) * \Delta(e)$$

   If $\psi = \phi = \Delta(g)$ then similarly

   $$(e, e) * \Delta(g) = (g, g) = (e, e) * \Delta(g)$$

   However if $\psi = \Delta(e)$ and $\phi = \Delta(g)$ then

   $$(e, e) * \Delta(g) = (g, g) \neq (e, e) = (e, e) * \Delta(e)$$

   and similarly if $\psi = \Delta(g)$ and $\phi = \Delta(e)$.

2. The morphisms from (g, g) to (g, g) are similar.

3. The morphisms from (e, e) to (g, g) are $\phi = \Delta(g), \psi = \Delta(e)$ and $\phi = \Delta(e), \psi = \Delta(g)$. Likewise for the morphisms from (g, g) to (e, e).

4. There are no morphisms from (e, g) to (e, e).

So one connected component of the fiber product is the following category $\mathcal{C}$. It has two objects (e, e) and (g, g) and

$$\begin{aligned}
\mathrm{Hom}((e, e), (e, e)) = \mathrm{Hom}((g, g), (g, g)) = \{(e, e), (g, g)\} \\
\mathrm{Hom}((e, e), (g, g)) = \mathrm{Hom}((g, g), (e, e)) = \{(e, g), (g, e)\}
\end{aligned}$$

This category is equivalent to the groupoid $*/\mathbb{Z}_2$. There is a functor $F : \mathcal{C} \to */\mathbb{Z}_2$ which we will define as follows. We send all objects to $*$ and

$$\begin{aligned}
(e, e) \in \mathrm{Hom}((e, e), (e, e)) \mapsto e \in \mathrm{Hom}(*, *) \\
(g, g) \in \mathrm{Hom}((e, e), (e, e)) \mapsto g \in \mathrm{Hom}(*, *) \\
(e, e) \in \mathrm{Hom}((g, g), (g, g)) \mapsto e \in \mathrm{Hom}(*, *) \\
(g, g) \in \mathrm{Hom}((g, g), (g, g)) \mapsto g \in \mathrm{Hom}(*, *) \\
(e, g) \in \mathrm{Hom}((e, e), (g, g)) \mapsto e \in \mathrm{Hom}(*, *) \\
(g, e) \in \mathrm{Hom}((e, e), (g, g)) \mapsto g \in \mathrm{Hom}(*, *) \\
(e, g) \in \mathrm{Hom}((g, g), (e, e)) \mapsto e \in \mathrm{Hom}(*, *) \\
(g, e) \in \mathrm{Hom}((g, g), (e, e)) \mapsto g \in \mathrm{Hom}(*, *)
\end{aligned}$$

In terms of compatibilty, since we have that $(g,g) = (e,g) \circ (g,e)$ as a morphism from (e,e) to (g,g) to (e,e) which maps to g, we must insist that $(e,g) \mapsto e$ and $(g,e) \mapsto g$ or vice versa. The functoriality constraints manifest themselves in this form.

The functor $G : */\mathbb{Z}_2 \to \mathcal{C}$ is defined in the following way.

$$\begin{aligned} * &\mapsto (e,e) \\ e &\mapsto (e,e) \\ g &\mapsto (g,g) \end{aligned}$$

Finally it remains to check that F and G define an equivalence of categories, in particular that $F \circ G \simeq \mathrm{id}_{*/\mathbb{Z}_2}$ and $G \circ F \simeq \mathrm{id}_{\mathcal{C}}$. Certainly the first equivalence is clear. As for the second equivalence, I will give a natural transformation $\mu : H = GF \to \mathrm{id}_{\mathcal{C}}$ which will in fact be a natural isomorphism. In particular, for the objects (e,e) and (g,g) we define the following morphisms

$$\begin{aligned} \mu_{(e,e)} &: H(e,e) \to \mathrm{id}(e,e) \\ \mu_{(g,g)} &: H(g,g) \to \mathrm{id}(g,g) \end{aligned}$$

by

$$\begin{aligned} \mu_{(e,e)} &: (e,e) \to (e,e) \quad \mu_{(e,e)} = (e,e) \\ \mu_{(g,g)} &: (e,e) \to (g,g) \quad \mu_{(g,g)} = (e,g) \end{aligned}$$

These are clearly isomorphisms. The naturality of μ is tedious to check but straightforward. Alternatively, we can verify that $F : \mathcal{C} \to */\mathbb{Z}_{\in}$ is an equivalence by checking that it is full, faithful, and essentially surjective. In fact, after writing this, I realize that this is the easier way to check that F is an equivalence.

This establishes that the fiber product $*/\mathbb{Z}_2 \times_{*/(\mathbb{Z}_2 \times \mathbb{Z}_2)} */\mathbb{Z}_2$ is a disjoint union of two copies of $*/\mathbb{Z}_2$. On the other hand, we have already seen that $(*/\mathbb{Z}_2)^{\text{naive}}(S^1) = */\mathbb{Z}_2$. Therefore, the naive functor $(*/\mathbb{Z}_2)^{\text{naive}}$ is not a sheaf. Thus we need to sheafify/stackify this functor to obtain the correct functor of points of $B\mathbb{Z}_2$. This regards the naive functor as true "only locally", and builds the general functor by gluing these local functors. This is the true definition of the functor of points of $B\mathbb{Z}_2$.

2.3 Topological interpretation

In the topological setting, we can define the stack $*/\mathbb{Z}_2$ directly. Recall that for any space T, we have the corresponding fundamental groupoid $\pi_{\leq 1}(T)$. Then we can define:

$$(*/\mathbb{Z}_2)(T) = \mathrm{Fun}(\pi_{\leq 1}(T), */\mathbb{Z}_2), \tag{4}$$

where the isomorphisms are given by natural isomorphisms of functors. This is automatically a sheaf, and it tells us why the naive $*/\mathbb{Z}_2$ did not work for S^1: the fundamental group of S^1 is nontrivial. On the other hand, for simply connected test spaces T, the naive functor does indeed give the correct groupoid.

Given a general space X, we use a good cover of X; i.e., one for which all the open sets and finite intersections of the open sets in the cover are contractible. (In fact, we can relax this constraint: we need only have all single, double, and triple intersections in our open cover be **simply-connected**.)

Gluing two sections $\pi_{\leq 1}(U_i \cap U_j) \to */\mathbb{Z}_2$ and $\pi_{\leq 1}(U_j \cap U_j) \to */\mathbb{Z}_2$ (which we imagine to be coming from $\pi_{\leq 1}(U_i) \to */\mathbb{Z}_2$ and $\pi_{\leq 1}(U_j) \to */\mathbb{Z}_2$, respectively) is the same as providing a natural transformation between these two functors $\pi_{\leq 1}(U_i \cap U_j) \to */\mathbb{Z}_2$. Since $\pi_{\leq 1}(U_i \cap U_j)$ is equivalent to trivial category $*$, we see that this is the same as an isomorphism $* \to *$ in $*/\mathbb{Z}_2$; i.e., and element $\mathbb{Z}_2$, which we call g_{ji}. We see that the g_{ji} must satisfy a cocycle condition, and that two cocycles correspond to the same family if the usual coboundary equivalence holds. Thus we have that

$$(*/Z_2)(X) = \text{Groupoid of 2-point families over } X$$

2.4 BG in generality (note we have not defined stacks yet)

Proposition 2.6. The stack $*/G$, defined as the sheafification of $(*/G)^{\text{naive}}$, represents the following moduli problem:

$$(*/G)(X) = \text{Groupoid of principal } G\text{-torsors over } X.$$

Definition 2.7. Let G be an algebraic group. The classifying stack $BG = */G$ is the stack whose S-points are $BG(S) =$ groupoid of principal G-bundles on S.

For exactly the formal reasons outlined above (in the topological setting), this is the sheafification of

$$(*/G)^{\text{naive}} : S \mapsto */G(S).$$

We note that there is a canonical map of stacks $* \to */G$. For an arbitrary test-scheme S, the composition of the map $S \to *$ with the vertical quotient map must provide us with a particular isomorphism class of G-torsor over S: this is simply the trivial G-torsor. And given a torsor P over S and the bottom map is the corresponding map $S \to */G$, we have a Cartesian diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & * \arrow[d] \\ S \arrow[r, "P"] & */G. \end{tikzcd}$$

Because any isomorphism class of torsor can thus be "pulled back" from the torsor $* \to */G$ along a map $S \to */G$, we say that $* \to */G$ is the "universal G-torsor."

3 Formalities

We introduce formal framework underlying the theory of stacks, beginning with Grothendieck topologies and their role in defining sheaves on sites. We then explore the foundational concepts of morphisms of schemes, which are essential for understanding the various topologies used in algebraic geometry, such as the Zariski, étale, fppf, and fpqc topologies. These topologies provide the scaffolding for the definition of stacks, which generalize sheaves to categories fibered in groupoids.

3.1 Grothendieck Topologies

Definition 3.1 (Grothendieck Topology)**.** A Grothendieck topology J on a category $\mathcal{C}$ assigns to each object U in $\mathcal{C}$ a collection $J(U)$ of families of morphisms $\{f_i : U_i \to U\}_{i \in I}$ (called covering families or sieves) satisfying:

1. **Stability under isomorphism**: If $\{f_i : U_i \to U\}_{i \in I} \in J(U)$ and $g : V \to U$ is an isomorphism, then $\{f_i \circ g^{-1} : U_i \to V\}_{i \in I} \in J(V)$.
2. **Stability under base change**: If $\{f_i : U_i \to U\}_{i \in I} \in J(U)$ and $g : V \to U$ is any morphism, then the family of pullbacks $\{V \times_U U_i \to V\}_{i \in I} \in J(V)$.
3. **Transitivity**: If $\{f_i : U_i \to U\}_{i \in I} \in J(U)$ and for each $i \in I$, we have
$$\{g_{ij} : V_{ij} \to U_i\}_{j \in J_i} \in J(U_i)$$
then the composite family $\{f_i \circ g_{ij} : V_{ij} \to U\}_{i \in I, j \in J_i} \in J(U)$.

Definition 3.2 (Site)**.** A site is a category $\mathcal{C}$ equipped with a Grothendieck topology.

Example 3.3. Let $\mathcal{C}$ be the category of open sets in a topological space X, with inclusions as morphisms. We can define a Grothendieck topology by declaring a family $\{U_i \hookrightarrow U\}_{i \in I} \in J(U)$ if and only if $\bigcup_{i \in I} U_i = U$. This is called the **small classical site**.

If X is a scheme, we can do the same thing with the category of Zariski open sets. This is called the **small Zariski site**.

Example 3.4. Let X be a scheme, and let $\mathcal{C}$ be the category of X-schemes. For $(U \to X) \in \mathcal{C}$ define $\mathrm{Cov}(U)$ to be the set of collections of X-morphisms $\{U_i \to U\}_{i \in I}$ for which each $U_i \to U$ is an open embedding and $U = \bigcup_{i \in I} U_i$. Then Cov defines a Grothendieck topology on $\mathcal{C}$, called the **big Zariski topology** on the category of X-schemes.

Remark 3.5. Recall the small classical site for a topological space X. The key observation is that the notion of a sheaf on X depends only on the underlying category of opens and the distinguished collections of maps $\{U_i \to U\}$ which are coverings. This abstraction allows us to generalize the concept of sheaves beyond topological spaces to any category equipped with a notion of "covering," which is precisely what a Grothendieck topology provides.

In other words, once we have defined a Grothendieck topology on a category, we can define and work with sheaves in exactly the same way as we do for topological spaces, without requiring any underlying topological structure. This allows us to work with sheaves on schemes, algebraic spaces, and stacks using various topologies (Zariski, étale, fppf, etc.).

Definition 3.6 (Sheaf on a Site)**.** Let $(\mathcal{C}, J)$ be a site. A presheaf $F : \mathcal{C}^{op} \to \mathsf{Sets}$ is a **sheaf** if for every covering family $\{f_i : U_i \to U\}_{i\in I} \in J(U)$, the following sequence is exact:

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

where the two parallel arrows represent the two natural projections.

3.2 On morphisms of schemes

We include a few definitions and results about morphisms of schemes which are relevant to the Grothendieck topologies we will introduce. For more general details, see the Appendix.

Definition 3.7. If A is a ring and M is an A-module, then M is called of **finite presentation** if there exists an exact sequence

$$A^r \to A^s \to M \to 0$$

for some integers r and s. Note that in the case when A is noetherian, this is equivalent to M being finitely generated (as the kernel of any surjection $A^s \to M$ is automatically finitely generated), but in general M being of finite presentation is a stronger condition than being finitely generated.

If $A \to B$ is a ring homomorphism, then we say that B is of **finite presentation over** A (or that B is a **finitely presented** A**-algebra**) if there exists a surjection

$$\pi : A[X_1, \ldots, X_s] \to B$$

with kernel $\mathrm{Ker}(\pi)$ a finitely generated ideal in $A[X_1, \ldots, X_s]$. If A is noetherian this is equivalent to B being a finitely generated A-algebra, but in general B being of finite presentation is a stronger condition than being finitely generated.

Let X be a scheme. A quasi-coherent sheaf $\mathcal{F}$ on X is called **locally finitely presented** if for every affine open subset $\mathrm{Spec}(B) \subset X$ the module $\Gamma(\mathrm{Spec}(B), \mathcal{F})$ is a finitely presented B-module.

Note that if X is locally noetherian then a quasi-coherent sheaf is locally finitely presented if and only if it is coherent.

In the case when Y is noetherian, the morphism f is locally of finite presentation if and only if f is locally of finite type, and finitely presented if and only if of finite type.

Definition 3.8 (Flat Module)**.** Let R be a ring and let M be an R-module. We say that M is **flat** over R if the functor $M \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is exact.

Equivalently, M is flat if for every injective homomorphism of R-modules $N_1 \to N_2$, the induced map $M \otimes_R N_1 \to M \otimes_R N_2$ is also injective. This is because the functor $M \otimes_R -$ is right exact, so we only need to check that it preserves injections.

Proposition 3.9 (Characterizations of Flatness)**.** For an R-module M, the following are equivalent:

1. M is flat over R.
2. For every ideal $I \subseteq R$, the natural map $I \otimes_R M \to IM$ is an isomorphism.
3. For every finitely generated ideal $I \subseteq R$, the natural map $I \otimes_R M \to IM$ is an isomorphism.
4. $\mathrm{Tor}_1^R(M, R/I) = 0$ for every ideal $I \subseteq R$.
5. $\mathrm{Tor}_1^R(M, N) = 0$ for every R-module N.

Proof. We'll prove the equivalence through a cycle of implications.

(5) $\Rightarrow$ (4): This is immediate, as we're restricting to the special case where $N = R/I$.

(4) $\Rightarrow$ (3): Let $I \subseteq R$ be a finitely generated ideal. Consider the exact sequence

$$0 \to I \to R \to R/I \to 0$$

Applying $- \otimes_R M$, we get the long exact sequence for Tor:

$$\cdots \to \mathrm{Tor}_1^R(R, M) \to \mathrm{Tor}_1^R(R/I, M) \to I \otimes_R M \to R \otimes_R M \to (R/I) \otimes_R M \to 0$$

Since R is free (hence flat), $\mathrm{Tor}_1^R(R, M) = 0$. By assumption (4), $\mathrm{Tor}_1^R(R/I, M) = 0$. Thus, the sequence becomes

$$0 \to I \otimes_R M \to M \to M/IM \to 0$$

which shows that $I \otimes_R M \cong IM$, as required.

(3) $\Rightarrow$ (2): Let $I \subseteq R$ be any ideal. We can write I as the direct limit of its finitely generated subideals: $I = \varinjlim I_\alpha$.

Since tensor products commute with direct limits, we have:

$$\begin{aligned}
I \otimes_R M &= (\varinjlim I_\alpha) \otimes_R M \\
&\cong \varinjlim (I_\alpha \otimes_R M) \\
&\cong \varinjlim I_\alpha M \quad \text{(by assumption (3))} \\
&= IM
\end{aligned}$$

(2) $\Rightarrow$ (5): The proof of this proposition uses the fact that any module can be built from modules of the form R/I through direct limits and extensions, and Tor preserves these constructions.

(5) $\Rightarrow$ (1): This is the definition of flatness. If $\mathrm{Tor}_1^R(M, N) = 0$ for all R-modules N, then $M \otimes_R -$ is exact, which means M is flat.

(1) $\Rightarrow$ (5): If M is flat, then $M \otimes_R -$ is an exact functor, which implies $\mathrm{Tor}_1^R(M, N) = 0$ for all R-modules N. $\square$

Let **Sch** denote the category of schemes. Before defining the following Grothendieck topologies (Zariski, étale, fppf, and fpqc), we recall several types of morphisms in algebraic geometry.

Definition 3.10. Let $f : X \to Y$ be a morphism of schemes.

1. f is **flat** if for every point $x \in X$, the induced map on local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ into a flat $\mathcal{O}_{Y,f(x)}$-module.

2. f is **locally of finite presentation** if Y can be covered by affine open subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i)$ can be covered by affine open subsets $U_{ij} = \operatorname{Spec} A_{ij}$ where each A_{ij} is a finitely presented B_i-algebra.

3. f is **locally of finite type** if Y can be covered by affine open subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i)$ can be covered by affine open subsets $U_{ij} = \operatorname{Spec} A_{ij}$ where each A_{ij} is a finitely generated B_i-algebra.

4. f is **quasi-compact** if for every quasi-compact open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is quasi-compact.

5. f is **faithfully flat** if f is flat and surjective.

6. f is **unramified** if it is locally of finite presentation and the relative cotangent sheaf $\Omega_{X/Y}$ vanishes.

7. f is **étale** if it is flat and unramified, or equivalently, if it is flat, locally of finite presentation, and has relative dimension 0.

8. f is **of finite presentation** (or a **finitely presented morphism**) if f is locally of finite presentation and quasi-compact and quasi-separated (recall that by definition a morphism of schemes $f : X \to Y$ is quasi-separated if the diagonal morphism is quasi-compact).

3.3 Zariski Topology

Definition 3.11. The **Zariski topology** J_{Zar} on $\mathbf{Sch}$ is defined as follows: A family of morphisms $\{f_i : U_i \to U\}_{i \in I}$ is a Zariski covering if:

1. Each f_i is an open immersion.

2. The images of the f_i collectively cover U, i.e., $\cup_{i \in I} f_i(U_i) = U$.

Remark 3.12. The Zariski topology corresponds most closely to the classical notion of a topological covering and is the coarsest of the four topologies discussed here. For an affine scheme $\operatorname{Spec}(R)$, a standard Zariski covering arises from a set of elements $\{f_i\}$ generating the unit ideal in R, giving the covering $\{\operatorname{Spec}(R_{f_i}) \to \operatorname{Spec}(R)\}$.

3.4 Étale Topology

Definition 3.13. The **étale topology** $J_{\text{ét}}$ on $\mathbf{Sch}$ is defined as follows: A family of morphisms $\{f_i : U_i \to U\}_{i \in I}$ is an étale covering if:

1. Each f_i is étale.

2. The family is jointly surjective, i.e., $\cup_{i \in I} f_i(U_i) = U$.

Proposition 3.14. Let X be a scheme. The following are equivalent for a morphism $f : Y \to X$:

1. f is étale.

2. f is flat, locally of finite presentation, and for every $y \in Y$, the fiber $Y_{\kappa(f(y))}$ is a disjoint union of spectra of finite separable field extensions of $\kappa(f(y))$.

3. f is locally of finite presentation and formally étale, meaning that for every affine X-scheme Z and every nilpotent closed subscheme $Z_0 \subset Z$, the induced map

$$\operatorname{Hom}_X(Z, Y) \to \operatorname{Hom}_X(Z_0, Y)$$

is bijective.

Example 3.15. If L/K is a finite separable field extension, then $\operatorname{Spec}(L) \to \operatorname{Spec}(K)$ is an étale morphism. More generally, if R is a ring and S is a finite étale R-algebra, then $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is an étale covering.

3.5 fppf Topology

Definition 3.16. The **fppf topology** (fidèlement plat et de présentation finie) J_{fppf} on **Sch** is defined as follows: A family of morphisms $\{f_i : U_i \to U\}_{i\in I}$ is an fppf covering if:

1. Each f_i is flat and locally of finite presentation.

2. The family is jointly surjective, i.e., $\cup_{i\in I} f_i(U_i) = U$.

Example 3.17. If R is a ring and p prime, the morphism $\operatorname{Spec}(R[x]/(x^p - a)) \to \operatorname{Spec}(R)$ for $a \in R$ is flat and of finite presentation, but generally not étale in characteristic p.

Theorem 3.18 (Grothendieck)**.** Let G be an affine group scheme of finite type over a scheme S. Then any G-torsor over S is trivial in the fppf topology.

3.6 fpqc Topology

Provides the most general framework for descent theory.

Definition 3.19. The **fpqc topology** (fidèlement plat et quasi-compact) J_{fpqc} on **Sch** is defined as follows: A family of morphisms $\{f_i : U_i \to U\}_{i\in I}$ is an fpqc covering if:

1. Each f_i is flat and quasi-compact.

2. The family is jointly surjective, i.e., $\cup_{i\in I} f_i(U_i) = U$.

Proposition 3.20. Let G be a quasi-compact and quasi-separated group scheme over a scheme S. If P is a G-torsor over S in the fpqc topology, then P is already a G-torsor in the fppf topology.

The four topologies form a hierarchy of refinements:

Theorem 3.21. For the category **Sch** of schemes, the following inclusions hold:

$$J_{\text{Zar}} \subset J_{\text{ét}} \subset J_{\text{fppf}} \subset J_{\text{fpqc}}$$

That is, every Zariski covering is an étale covering, every étale covering is an fppf covering, and every fppf covering is an fpqc covering.

Definition 3.22. Let G be a group scheme over a scheme S, and let X be an S-scheme. A **principal G-bundle** over X is an X-scheme P with a right G-action $P \times_S G \to P$ such that the morphism $P \to X$ is locally trivial with respect to a given Grothendieck topology J on **Sch**. That is, there exists a covering $\{U_i \to X\}_{i\in I}$ in J such that $P \times_X U_i \cong U_i \times_S G$ as G-schemes over U_i for each $i \in I$.

Proposition 3.23. Let G be a group scheme over a scheme S.

1. If G is smooth over S, then every principal G-bundle that is fppf-locally trivial is also étale-locally trivial.

2. If G is finite and étale over S, then every principal G-bundle that is étale-locally trivial is also Zariski-locally trivial.

3. In general, a principal G-bundle that is fpqc-locally trivial is also fppf-locally trivial.

4 Stacks

Definition 4.1 (Category fibered in groupoids)**.** A category fibered in groupoids over a category $\mathcal{C}$ is a functor $p : \mathcal{F} \to \mathcal{C}$ such that:

1. For every morphism $f : U \to V$ in $\mathcal{C}$ and every object $y \in \mathcal{F}$ with $p(y) = V$, there exists an object $x \in \mathcal{F}$ and a morphism $\phi : x \to y$ in $\mathcal{F}$ such that $p(\phi) = f$.

2. For every pair of morphisms $\phi : x \to z$ and $\psi : y \to z$ in $\mathcal{F}$ and every morphism $f : p(x) \to p(y)$ in $\mathcal{C}$ such that $p(\psi) \circ f = p(\phi)$, there exists a unique morphism $\chi : x \to y$ in $\mathcal{F}$ such that $\psi \circ \chi = \phi$ and $p(\chi) = f$.

4.1 k-spaces and k-stacks.

Definition 4.2 (k-space, k-group)**.** A **k-space** (resp. **k-group**) is a sheaf of sets (resp. groups) over the big site $(\mathrm{Aff}\,/k)_{\mathrm{fppf}}$.

Definition 4.3 (Lax functor)**.** A **lax functor** $\mathcal{X} : \mathrm{Aff}\,/k^{\mathrm{op}} \to \mathbf{Gpd}$ associates to any $U \in \mathrm{ob}(\mathrm{Aff}\,/k)$ a groupoid $\mathcal{X}(U)$ and to every arrow $f : U' \to U$ in $\mathrm{Aff}\,/k$ a functor $f^* : \mathcal{X}(U) \to \mathcal{X}(U')$ together with isomorphisms of functors $g^* \circ f^* \simeq (f \circ g)^*$ for every arrow $g : U'' \to U'$ in $\mathrm{Aff}\,/k$. These isomorphisms should satisfy the following compatibility relation: for $h : U''' \to U''$ the following diagram commutes:

$$\begin{tikzcd}
h^* \circ g^* \circ f^* \arrow[r, "\sim"] \arrow[d, "\sim"] & h^*(f \circ g)^* \arrow[d, "\sim"] \\
(g \circ h)^* f^* \arrow[r, "\sim"] & (f \circ g \circ h)^*
\end{tikzcd}$$

If $x \in \mathrm{ob}(\mathcal{X}(U))$ and $f : U' \to U$ it is convenient to denote $f^*x \in \mathrm{ob}(\mathcal{X}(U'))$ by $x|_{U'}$.

Definition 4.4. A lax functor is a k-**stack** if it satisfies the following two topological properties:

(i) For every $U \in \mathrm{ob}(\mathrm{Aff}/k)$ and all $x, y \in \mathrm{ob}(\mathcal{X}(U))$ the presheaf

$$\mathrm{Isom}(x,y) : \mathrm{Aff}/U \to \mathbf{Set}, \quad (U' \to U) \mapsto \mathrm{Hom}_{\mathcal{X}(U')}(x|_{U'}, y|_{U'})$$

is a sheaf (with respect to the fppf topology on Aff/U).

(ii) Every descent datum is effective.

Remark 4.5. Any **k-space** X may be seen as a **k-stack**, by considering a set as a groupoid (with the identity as the only morphism). Conversely, any **k-stack** $\mathcal{X}$ such that $\mathcal{X}(R)$ is a discrete groupoid (**i.e.**, has only the identity as automorphisms) for all affine k-schemes U, is a **k-space**.

4.2 Descent datum

Definition 4.6 (Descent datum, effective descent datum)**.** A **descent datum** for a lax functor $\mathcal{X}$ for a covering family $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ is a system of the form $(x_i, \varphi_{ji})_{i,j\in I}$ with the following properties: each x_i is an object of $\mathcal{X}(U_i)$, and each $\varphi_{ji} : x_i|_{U_{ji}} \to x_j|_{U_{ji}}$ is an arrow in $\mathcal{X}(U_{ji})$. Moreover, we have the co-cycle condition

$$\varphi_{ki}|_{U_{kji}} = \varphi_{kj}|_{U_{kji}} \circ \varphi_{ji}|_{U_{kji}}$$

where $U_{ji} = U_j \times_U U_i$ and $U_{kji} = U_k \times_U U_j \times_U U_i$, for all i, j, k.

A descent datum is **effective** if there exists an object $x \in \mathcal{X}(U)$ and invertible arrows $\varphi_i : x|_{U_i} \xrightarrow{\sim} x_i$ in $\mathcal{X}(U_i)$ for each i such that

$$\varphi_j|_{U_{ji}} = \varphi_{ji} \circ \varphi_i|_{U_{ji}}$$

for all $i, j \in I$.

The most classical form of descent data is faithfully flat descent for quasicoherent sheaves.

Theorem 4.7 (Faithfully flat descent)**.** Let k be a field. Then the following hold:

(i) (Faithfully flat descent for morphisms) For any k-scheme Z the functor of points

$$\mathrm{Hom}_{(\mathbf{Aff}/k)}(-, Z) : (\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Set}$$

is a k-space.

(ii) (Faithfully flat descent for flat families of quasi-coherent sheaves) For any scheme Z, the lax functor $(\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Gpd}$ defined by

$$S \mapsto \{\text{quasi-coherent } \mathcal{O}_{Z\times_k S}\text{-modules flat over } S\} + \{\text{isomorphisms}\}$$

is a k-stack.

More verbosely, we have the following theorem from [4].

Theorem 4.8. Let $f : T' \to T$ be a flat morphism of schemes. Assume, further, that f is surjective and either quasi-compact or locally of finite presentation.

(a) Let $\mathcal{E}'$ be a quasi-coherent sheaf on T' and $\varphi : p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ an isomorphism on T'' such that

$$p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$$

on T'''. Then there exists a quasi-coherent sheaf $\mathcal{E}$ on T and an isomorphism $\lambda : f^*\mathcal{E} \to \mathcal{E}'$ on T' satisfying

$$p_2^*\lambda = \varphi \circ p_1^*\lambda$$

on T''. Moreover the pair consisting of the sheaf $\mathcal{E}$ and the isomorphism λ is unique up to canonical isomorphism.

(b) With notation as in (a), suppose $(\mathcal{F}', \psi)$ is another descent datum with solution given by $\mathcal{F}$ and μ. Then, for every morphism $h' : \mathcal{E}' \to \mathcal{F}'$ on T' satisfying

$$p_2^*h' \circ \varphi = \psi \circ p_1^*h'$$

on T'', there is a unique morphism $h : \mathcal{E} \to \mathcal{F}$ on T such that $\mu \circ f^*h = h' \circ \lambda$ on T'.

4.3 Effective descent for modules along faithfully flat ring maps

The general case of Theorem 4.8 will be reduced to the affine case, which amounts to some of the following commutative algebra. No Noetherian or finiteness conditions on either rings or modules are required.

Definition 4.9 (Flat and faithfully flat homomorphisms)**.** A homomorphism $A \to A'$ of commutative rings with unit is **flat** if, for any exact sequence $M_1 \to M_2 \to M_3$ of A-modules, the induced sequence $A' \otimes_A M_1 \to A' \otimes_A M_2 \to A' \otimes_A M_3$ (of A'-modules) is exact. The homomorphism is called **faithfully flat** if it is flat and the corresponding map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is surjective.

Exercise 4.10. 1. Show that a flat homomorphism $A \to A'$ is faithfully flat if and only if, for any nonzero A-module M, $A' \otimes_A M \neq 0$.

2. Show that a homomorphism $A \to A'$ is faithfully flat if and only if the following condition is satisfied: a sequence $M' \to M \to M''$ of A-modules is exact if and only if the sequence $A' \otimes_A M' \to A' \otimes_A M \to A' \otimes_A M''$ is exact.

1. Suppose that $A \to A'$ is faithfully flat. We can consider a cyclic submodule of M, and it is enough to show that this will be nonzero upon tensoring with A'. But every cyclic submodule looks like A/I for some ideal $I = \mathrm{Ann}_A(m)$. Pick a prime containing I. Such a prime exists

by Zorn's lemma, in particular the set of ideals $\{J \subseteq A \mid I \subseteq J, J \neq A\}$ has a maximal element, and such a maximal element must be a prime ideal. Let $\mathfrak{p}$ be such a prime ideal containing I, and let $\mathfrak{q}$ be the prime ideal of A' lying over $\mathfrak{p}$. Then

$$IA' \subseteq \mathfrak{p}A' \subseteq \mathfrak{q} \neq A',$$

so A'/IA' is nonzero. Conversely, we can let $M = k(\mathfrak{p})$ be the residue field at a prime $\mathfrak{p}$ in A. Then we see that $M \otimes_A A' \cong A'/\mathfrak{p}A' \neq 0$.

Recall that $\mathfrak{p}$ is in the image of $f : \operatorname{Spec} A' \to \operatorname{Spec} A$ if and only if the fiber $f^{-1}(\mathfrak{p})$ is nonempty if and only if $\mathrm{Spec(A' \otimes_A} \kappa(\mathfrak{p})) \neq \varnothing$ if and only if $A' \otimes_A \kappa(\mathfrak{p})$ is not the zero ring. This is because

$$\operatorname{Spec}(A' \otimes_A \kappa(\mathfrak{p})) \;\cong\; \operatorname{Spec}(A'_{\mathfrak{p}}/\mathfrak{p}A'_{\mathfrak{p}}) \;\cong\; \{\text{primes of } A'_{\mathfrak{p}} \text{ lying over } \mathfrak{p}A_{\mathfrak{p}}\} \cong f^{-1}(\mathfrak{p})$$

Recall that the points of $\operatorname{Spec} A_{\mathfrak{p}}$ are in bijection with the primes of A contained in $\mathfrak{p}$. Therefore, the primes of $A'_{\mathfrak{p}}$ correspond to primes of A' lying inside $\mathfrak{p}$ after contraction.

Therefore, the map $\operatorname{Spec} A' \to \operatorname{Spec} A$ is surjective.

2. Suppose $A \to A'$ is faithfully flat. Then we need that $A' \otimes_A M' \to A' \otimes_A M \to A' \otimes_A M''$ is exact implies that $M' \to M \to M''$ is exact. Let $S : N' \longrightarrow N \longrightarrow N''$ be a sequence of A-modules, and suppose that

$$S \otimes A' : M' \otimes A' \xrightarrow{f_M} M \otimes A' \xrightarrow{g_M} M'' \otimes A'$$

is exact. As A' is flat, the exact functor $\otimes A'$ transforms kernel into kernel and image into image. Thus $\operatorname{Im}(g \circ f) \otimes A' = \operatorname{Im}(g_M \circ f_M) = 0$, and by the assumption we get $\operatorname{Im}(g \circ f) = 0$, i.e. $g \circ f = 0$. Hence S is a complex, and if $H(S)$ denotes its homology (at N), we have $H(S) \otimes A' = H(S \otimes A') = 0$. Using again the assumption we obtain $H(S) = 0$, which implies that S is exact.

Conversely, suppose that $M' \to M \to M''$ is exact iff $M' \otimes A' \to M \otimes A' \to M'' \otimes A'$ is exact. By definition, A' is flat. We need to see that $\operatorname{Spec} A' \to \operatorname{Spec} A$ is surjective. By the above reasoning, it is enough to show that the localization $A' \otimes_A \kappa(\mathfrak{p})$ is nonzero for every prime $\mathfrak{p}$ of A. This follows from tensoring the three term sequence $0 \to \kappa(\mathfrak{p}) \to 0$. Since $A' \otimes_A \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p}) \otimes_A A' \neq 0$, we conclude that the localization is indeed nonzero.

Exercise 4.11. Suppose $A \to A'$ is faithfully flat.

1. Show that a homomorphism $M \to N$ of A-modules is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if the homomorphism $A' \otimes_A M \to A' \otimes_A N$ is a monomorphism (resp. epimorphism, resp. isomorphism).

2. Show that an A-module M is finitely generated (resp. finitely presented, resp. flat, resp. locally free of finite rank n) if and only if the A'-module $A' \otimes_A M$ is finitely generated (resp. finitely presented, resp. flat, resp. locally free of finite rank n).

Recall the two following facts about faithfully flat morphisms:

L1 If $0 \to K \to M \xrightarrow{f} N$ is exact, then $0 \to B \otimes_A K \to B \otimes_A M \xrightarrow{B \otimes f} B \otimes_A N$ is exact (flatness). Dually for cokernels.

L2 If $B \otimes_A X = 0$ then $X = 0$ (faithfulness). More generally, for submodules $N_1, N_2 \subseteq M$, $B \otimes_A N_1 = B \otimes_A N_2 \subseteq B \otimes_A M \Rightarrow N_1 = N_2$. In particular, one has submodule detection under faithful flatness. This is because if $A' \otimes N_1 = A' \otimes N_2$, then $A' \otimes (N_1/N_2) = 0$. Faithfulness will force $N_1/N_2 = 0$, i.e. $N_1 = N_2$.

1. Let $f : M \to N$ be an A-linear map. If f is mono, then $0 \to M \xrightarrow{f} N$ is exact, hence by (L1) $0 \to B \otimes M \xrightarrow{B \otimes f} B \otimes N$ is exact, so $B \otimes f$ is mono. Conversely, if $B \otimes f$ is mono, then $\ker(B \otimes f) = B \otimes \ker f = 0$ by (L1), hence $\ker f = 0$ by (L2), so f is mono. Let $C = \operatorname{coker}(f)$. If f is epi then $C = 0$, hence $B \otimes C = 0$, so $B \otimes f$ is epi. Conversely, if $B \otimes f$ is epi then $B \otimes C = 0$, hence $C = 0$ by (L2), so f is epi. f is an iso iff it is both mono and epi; equivalently iff $B \otimes f$ is both mono and epi (by the two bullets), hence an iso.

2. • **Finitely generated**

   Ascent. If M is generated by $m_1, \ldots, m_r$, then $1 \otimes m_i$ generate $B \otimes M$ as a B-module.

   Descent. Suppose $B \otimes M$ is generated by $y_1, \ldots, y_s$. Write each $y_j = \sum_k b_{jk} \otimes n_{jk}$ with $b_{jk} \in B, n_{jk} \in M$. Let $S = \{n_{jk}\}$ (finite). The map $\phi : A^{|S|} \to M$ sending basis vectors to the elements of S becomes surjective after tensoring with B (its image contains the y_j, hence all of $B \otimes M$). Thus $B \otimes \operatorname{coker} \phi = 0$, so$\operatorname{coker} \phi = 0$ by (L2). Hence ϕ is surjective and M is finitely generated.

   • **Finitely presented.**

   Ascent. If $A^r \xrightarrow{\alpha} A^s \to M \to 0$ is exact with $r, s < \infty$, then tensoring gives $B^r \xrightarrow{B \otimes \alpha} B^s \to B \otimes M \to 0$: finite presentation ascends by flatness.

   Descent. Assume $B \otimes M$ is finitely presented. By (a), pick a surjection $\pi : A^s \to M$ whose base change $B^s \to B \otimes M$ is surjective. Let $K = \ker(\pi)$. Then

$$K' = \ker(B^s \to B \otimes M) = B \otimes K$$

   by flatness. Since $B \otimes M$ is finitely presented, K' is finitely generated over B. By (a, descent) applied to the A-module K, from $B \otimes K$ f.g. we conclude K is f.g. Hence $A^s \to M$ has finitely generated kernel, i.e. M is finitely presented.

- **Flat.**

  Ascent. If M is flat over A, then $B \otimes M$ is flat over B (tensoring with a flat B preserves exactness).

  Descent. Assume $B \otimes M$ is flat over B. For any A-module N and $i \geq 1$,

  $$\mathrm{Tor}_i^A(N, M) \otimes_A B \cong \mathrm{Tor}_i^B(N \otimes_A B, M \otimes_A B)$$

  (because B is flat). The RHS is 0 when $i = 1$ by flatness of $B \otimes M$. Hence $\mathrm{Tor}_1^A(N, M) \otimes_A B = 0$, and by (L2) we get $\mathrm{Tor}_1^A(N, M) = 0$ for all N. Thus M is flat.

- **Locally free of finite rank n.**

  Ascent. If M is locally free of rank n, then $B \otimes M$ is locally free of rank n (localization commutes with tensor).

  Descent. Assume $B \otimes M$ is locally free of rank n over B. From (b) and (c), M is finitely presented and flat. Let $\mathfrak{p} \in \operatorname{Spec} A$ and pick $\mathfrak{q} \in \operatorname{Spec} B$ over $\mathfrak{p}$. Then

  $$(B \otimes M) \otimes_B \kappa(\mathfrak{q}) \cong M \otimes_A \kappa(\mathfrak{q}) \cong (M \otimes_A \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}).$$

  The LHS is an n-dimensional $\kappa(\mathfrak{q})$-vector space, hence $\dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p})) = n$. Thus the fiber-rank function $p \mapsto \dim_{\kappa(p)}(M \otimes_A \kappa(p))$ is constantly n.

  Over the local ring $A_{\mathfrak{p}}$, a finitely presented flat module is free; hence $M_{\mathfrak{p}} \cong A_{\mathfrak{p}}^n$ for all $\mathfrak{p}$. Therefore M is locally free of rank n.

Lemma 4.12. Let M be an A-module. If $A \to A'$ is faithfully flat, then

$$M \xrightarrow{\gamma} A' \otimes_A M \rightrightarrows A' \otimes_A A' \otimes_A M$$

is exact, that is, the canonical homomorphism γ maps M isomorphically to the set of elements in $A' \otimes_A M$ that have the same image in $A' \otimes_A A' \otimes_A M$ by the two projection homomorphisms. Equivalently, if one defines a homomorphism δ from $A' \otimes_A M$ to $A' \otimes_A A' \otimes_A M$ by the formula $\delta(x \otimes u) = 1 \otimes x \otimes u - x \otimes 1 \otimes u$, then the sequence

$$0 \to M \xrightarrow{\gamma} A' \otimes_A M \xrightarrow{\delta} A' \otimes_A A' \otimes_A M$$

of A-modules is exact.

Proof. By the previous exercise, it suffices to show that the sequence becomes exact after tensoring it (on the left) over A by A', i.e., that the sequence

$$0 \longrightarrow A' \otimes_A M \xrightarrow{A' \otimes \gamma} A' \otimes_A A' \otimes_A M \xrightarrow{A' \otimes \delta} A' \otimes_A A' \otimes_A A' \otimes_A M$$

is exact. Let $\mu : A' \otimes_A A' \to A'$ be the multiplication map, $\mu(x \otimes y) = xy$. The injectivity of the first map $A' \otimes \gamma$ is now immediate, since the mapping $\mu \otimes M : A' \otimes_A A' \otimes_A M \to A' \otimes_A M$ gives a left inverse to it. Suppose an element $\sum x_i \otimes y_i \otimes u_i$ is in the kernel of $A' \otimes \delta$, i.e.

$$\sum x_i \otimes 1 \otimes y_i \otimes u_i = \sum x_i \otimes y_i \otimes 1 \otimes u_i.$$

Applying μ to the first two factors yields

$$\sum x_i \otimes y_i \otimes u_i = \sum x_i y_i \otimes 1 \otimes u_i,$$

and $\sum x_i y_i \otimes 1 \otimes u_i$ is the image of $\sum x_i y_i \otimes u_i$ in $A' \otimes_A A' \otimes_A M$, as required. □

Remark 4.13. The proof of this lemma is a common one in descent theory: one makes a faithfully flat base extension to achieve the situation where the covering map $T' \to T$ has a section, in which case the assertion proves itself.

Lemma 4.14. If $A \to A'$ is faithfully flat, and M and N are A-modules, then the sequence

$$\mathrm{Hom}_A(M, N) \;\to\; \mathrm{Hom}_{A'}(A' \otimes_A M,\; A' \otimes_A N) \;\rightrightarrows\; \mathrm{Hom}_{A' \otimes_A A'}(A' \otimes_A A' \otimes_A M,\; A' \otimes_A A' \otimes_A N)$$

is exact.

Proof. The exactness of Lemma A.4, applied to N, together with the left exactness of Hom, gives the exactness of

$$\mathrm{Hom}_A(M, N) \;\to\; \mathrm{Hom}_A(M,\; A' \otimes_A N) \;\rightrightarrows\; \mathrm{Hom}_A(M,\; A' \otimes_A A' \otimes_A N).$$

Using the identifications

$$\mathrm{Hom}_A(M, P) \;=\; \mathrm{Hom}_B(B \otimes_A M,\; P)$$

for any homomorphism $A \to B$ and any B-module P, first for $B = A'$ and then for $B = A' \otimes_A A'$, translates this exact sequence into the exact sequence of the lemma. □

Now let M' be an A'-module. We have, as we recall, projection maps p_1 and p_2 from $\mathrm{Spec}(A'')$ to $\mathrm{Spec}(A')$, where $A'' = A' \otimes_A A'$. Hence we have pullbacks

$$p_1^*(M') = A'' \otimes_{p_1, A'} M', \qquad p_2^*(M') = A'' \otimes_{p_2, A'} M'.$$

The two pullbacks $p_1^*(M')$ and $p_2^*(M')$ can be identified with $M' \otimes_A A'$ and $A' \otimes_A M'$ respectively, where the actions of A'' on these modules are given by

$$(x \otimes y) \cdot (u \otimes z) = xu \otimes yz, \qquad (x \otimes y) \cdot (z \otimes u) = xz \otimes yu,$$

with $x, y, z \in A'$ and $u \in M'$.

Similarly, the three pullbacks of M' by q_1, q_2, and q_3 to A''' can be identified with

$$M' \otimes_A A' \otimes_A A', \qquad A' \otimes_A M' \otimes_A A', \qquad A' \otimes_A A' \otimes_A M',$$

respectively, again with the diagonal actions of $A''' = A' \otimes_A A' \otimes_A A'$.

Suppose

$$\varphi : M' \otimes_A A' = p_1^*(M') \longrightarrow p_2^*(M') = A' \otimes_A M'$$

is an isomorphism of A''-modules. This determines, by the three pullbacks p_{ij}, isomorphisms

$$\varphi_{ij} = p_{ij}^*(\varphi) : q_i^*(M') = p_{ij}^*(p_1^*(M')) \longrightarrow p_{ij}^*(p_2^*(M')) = q_j^*(M').$$

For example, φ_{12} is the map

$$M' \otimes_A A' \otimes_A A' \longrightarrow A' \otimes_A M' \otimes_A A'$$

that takes $u \otimes x \otimes y$ to $\varphi(u \otimes x) \otimes y$. That is, if

$$\varphi(u \otimes x) = \sum x_i \otimes u_i,$$

then

$$\varphi_{12}(u \otimes x \otimes y) = \sum x_i \otimes u_i \otimes y.$$

Similarly,

$$\varphi_{13}(u \otimes y \otimes x) = \sum x_i \otimes y \otimes u_i, \qquad \varphi_{23}(y \otimes u \otimes x) = \sum y \otimes x_i \otimes u_i.$$

Descent for modules amounts to the following assertion:

Lemma 4.15 (Descent for modules)**.** Suppose $A \to A'$ is faithfully flat, M' is an A'-module, and $\varphi : M' \otimes_A A' \to A' \otimes_A M'$ is an isomorphism of A''-modules such that $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$ from $q_1^*(M')$ to $q_3^*(M')$. Define the A-module M by

$$M \;=\; \{\, u \in M' \mid \varphi(u \otimes 1) = 1 \otimes u \,\}.$$

Then the canonical homomorphism

$$\lambda : A' \otimes_A M \longrightarrow M', \qquad x \otimes u \longmapsto x \cdot u,$$

is an isomorphism.

Proof. Let $\tau : M' \to A' \otimes_A M'$ be defined by

$$\tau(u) = 1 \otimes u - \varphi(u \otimes 1).$$

We have an exact sequence

$$0 \longrightarrow M \longrightarrow M' \xrightarrow{\tau} A' \otimes_A M'.$$

Tensoring this on the right with A' over A gives the top row of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_A A' \arrow[r] \arrow[d, "\psi"] & M' \otimes_A A' \arrow[r, "\tau\otimes 1"] \arrow[d, "\varphi"] & A' \otimes_A M' \otimes_A A' \arrow[d, "A'\otimes\varphi"] \\
0 \arrow[r] & M' \arrow[r] & A' \otimes_A M' \arrow[r] & A' \otimes_A A' \otimes_A M'
\end{tikzcd}$$

The bottom row is the exact sequence from Lemma 4.12, applied to the A-module M'. The map ψ is defined by $\psi(u \otimes x) = x \cdot u$, and we want to show ψ is an isomorphism. Since the rows are exact, and the right two vertical maps are isomorphisms, this conclusion will follow if we verify that the diagram is commutative.

The left square commutes since, for $u \in M$ and $x \in A'$,

$$\varphi(u \otimes x) = (1 \otimes x)\varphi(u \otimes 1) = (1 \otimes x)(1 \otimes u) = 1 \otimes xu.$$

To prove that the right diagram commutes, we must show that, for any $u \in M'$ and $x \in A'$, the element $u \otimes x \in M' \otimes_A A'$ has the same image by either route around the square.

Let $\varphi(u \otimes 1) = \sum y_i \otimes v_i$, with $y_i \in A'$ and $v_i \in M'$. Then

$$\varphi(u \otimes x) = (1 \otimes x)\varphi(u \otimes 1) = \sum y_i \otimes xv_i,$$

so the image of $u \otimes x$ by the lower route is

$$\sum 1 \otimes y_i \otimes xv_i \; - \; \sum y_i \otimes 1 \otimes xv_i.$$

On the upper route, $u \otimes x$ maps to the right to

$$1 \otimes u \otimes x \; - \; \varphi(u \otimes 1) \otimes x = 1 \otimes u \otimes x - \sum y_i \otimes v_i \otimes x,$$

which maps down to

$$1 \otimes \varphi(u \otimes x) - \sum y_i \otimes \varphi(v_i \otimes x) = \sum 1 \otimes y_i \otimes xv_i - \sum y_i \otimes \varphi(v_i \otimes x).$$

We are therefore reduced to verifying that

$$\sum y_i \otimes \varphi(v_i \otimes x) = \sum y_i \otimes 1 \otimes xv_i.$$

But this is exactly the assertion that $\varphi_{23}(\varphi_{12}(u \otimes 1 \otimes x)) = \varphi_{13}(u \otimes 1 \otimes x)$. $\square$

This shows that the module M that we constructed satisfies an isomorphism $\lambda : A' \otimes_A M \to M'$. In order to complete the proof of faithfully flat descent for modules, we must show that the isomorphism λ is compatible with the given isomorphism $\varphi : p_1^*(M') \to p_2^*(M')$. That is, we need to know that the descent data for M induced by λ is the same as the given descent data φ. Formally:

$$\varphi \circ p_1^* \lambda \; = \; p_2^* \lambda.$$

This amounts to verifying that the diagram

$$\begin{array}{ccc} A' \otimes_A M \otimes_A A' & \xrightarrow{p_1^*\lambda} & M' \otimes_A A' \\ \kappa \big\downarrow & & \big\downarrow \varphi \\ A' \otimes_A A' \otimes_A M & \xrightarrow[p_2^*\lambda]{} & A' \otimes_A M' \end{array}$$

commutes, where

$$\kappa(x \otimes u \otimes y) = x \otimes y \otimes u.$$

This is because κ is the canonical isomorphism from $p_1^*(A' \otimes_A M)$ to $p_2^*(A' \otimes_A M)$, which we have when we know that $M' \cong A' \otimes_A M$.

This amounts to the identity

$$x \otimes \lambda(y \otimes u) \; = \; \varphi\big(\lambda(x \otimes u) \otimes y\big),$$

i.e.

$$x \otimes yu = \varphi(xu \otimes y),$$

or equivalently

$$(x \otimes y)(1 \otimes u) = (x \otimes y)\varphi(u \otimes 1),$$

which follows from the fact that $u \in M$.

Finally, we need to check that a morphism of descent data $h' : M' \to N'$ "descends" uniquely to a morphism $h : M \to N$ of A-modules. This means that we have A'-modules M' and N', with isomorphisms

$$\varphi : M' \otimes_A A' \; \to \; A' \otimes_A M' \qquad \text{and} \qquad \psi : N' \otimes_A A' \; \to \; A' \otimes_A N',$$

and we have A-modules M and N, with isomorphisms

$$\lambda : A' \otimes_A M \; \to \; M', \qquad \mu : A' \otimes_A N \; \to \; N',$$

satisfying $\varphi \circ p_1^*(\lambda) = p_2^*(\lambda)$ and $\psi \circ p_1^*(\mu) = p_2^*(\mu)$.

We are given a homomorphism

$$h' : M' \to N'$$

of A'-modules, satisfying the identity

$$p_2^*(h') \circ \varphi = \psi \circ p_1^*(h').$$

We must show that there is a unique homomorphism $h : M \to N$ of A-modules such that

$$\mu \circ (A' \otimes h) = h' \circ \lambda.$$

The trick is to package h' into $g' = \mu^{-1} h' \lambda$, show $p_1^*(g') = p_2^*(g')$, and then invoke Lemma 4.14 to conclude that g' comes from a unique A-linear map h.

Set

$$g' \;=\; \mu^{-1} \circ h' \circ \lambda : A' \otimes_A M \;\to\; A' \otimes_A N.$$

If we show that $p_1^*(g') = p_2^*(g')$, then Lemma A.5 will produce a unique homomorphism $h : M \to N$ such that $g' = A' \otimes h$. This says that

$$h' \circ \lambda = \mu \circ (A' \otimes h),$$

as required.

To conclude the proof, we calculate:

$$\begin{aligned} p_1^*(g') &= p_1^*(\mu^{-1} \circ h' \circ \lambda) \\ &= p_1^*(\mu)^{-1} \circ p_1^*(h') \circ p_1^*(\lambda) \\ &= p_2^*(\mu)^{-1} \circ \psi \circ p_1^*(h') \circ p_1^*(\lambda) \\ &= p_2^*(\mu)^{-1} \circ p_2^*(h') \circ \varphi \circ p_1^*(\lambda) \\ &= p_2^*(\mu)^{-1} \circ p_2^*(h') \circ p_2^*(\lambda) \\ &= p_2^*(g'), \end{aligned}$$

as required.

Remark 4.16. The overall structure of the proofs in this section is worth noting. First, we proved descent for morphisms in the case of objects pulled back from the base (4.14). Then we showed that every descent datum is effective (4.15). We saw as a formal consequence that descent for morphisms holds in the case of an arbitrary pair of descent data, each admitting a solution, and from this that the solution to any descent problem is unique up to canonical isomorphism.

4.4 Extending to quasi-coherent sheaves on schemes

In this section, we complete the proof of Theorem 4.8. Recall that a morphism $f : T' \to T$ of schemes is *faithfully flat* if it is flat and surjective. It is not enough to assume f is faithfully flat for the conclusions of the theorem to hold, as we'll see below. To pass from the affine case to the case of general schemes we'll need some additional hypothesis on the morphism f. In fact, there are two additional hypotheses that we may impose, and either one will suffice to establish descent for objects and morphisms, in the context of quasi-coherent sheaves:

(i) f is **fpqc**, that is, faithfully flat and *quasi-compact*. We recall this means that the pre-image, under f, of any affine open subset of the base is covered by finitely many affine open subsets.

(ii) f is **fppf**, that is, faithfully flat and *locally of finite presentation*. The important fact needed here is that every morphism that is flat and locally of finite presentation is open [5] IV.2.4.6.

The notations **fpqc** and **fppf** come from the French terminology for the conditions on f (*fidèlement plat, quasi-compact* and *fidèlement plat, de présentation finie*).

As described at the end of the previous section, to prove Theorem 4.8, it suffices to prove descent for morphisms of objects pulled back from the base and to show that every descent datum is effective. In other words, Theorem 4.8 follows from the following pair of assertions.

Lemma 4.17 (Descent for morphisms)**.** Assume $f : T' \to T$ is (i) fpqc or (ii) fppf. Let $\mathcal{E}$ and $\mathcal{F}$ be quasi-coherent sheaves on T. Then, for every morphism

$$h' : f^*\mathcal{E} \to f^*\mathcal{F}$$

on T' such that $p_1^*h' = p_2^*h'$ on T'', there is a unique morphism

$$h : \mathcal{E} \to \mathcal{F}$$

on T such that $f^*h = h'$.

Lemma 4.18 (Effective descent)**.** Assume $f : T' \to T$ is (i) fpqc or (ii) fppf. Let $\mathcal{E}'$ be a quasi-coherent sheaf on T' and

$$\varphi : p_1^*\mathcal{E}' \longrightarrow p_2^*\mathcal{E}'$$

an isomorphism on T'' such that

$$p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$$

on T'''. Then there exists a quasi-coherent sheaf $\mathcal{E}$ on T and an isomorphism

$$\lambda : f^*\mathcal{E} \longrightarrow \mathcal{E}'$$

on T' such that

$$p_2^*\lambda = \varphi \circ p_1^*\lambda$$

on T''.

We will say that a morphism $f : T' \to T$ of schemes has **descent for morphisms** if it satisfies the conclusion of Lemma 4.17, and that it has **effective descent** if it satisfies the conclusions of both Lemma 4.17 and Lemma 4.18.

Recall the sheaf gluing theorem for honest open covers of a topological space. If you have sheaves $\mathcal{F}_\lambda$ on each piece of an open cover, and isomorphisms $\theta_{\lambda\mu}$ on overlaps satisfying the cocycle condition, then there exists a unique sheaf $\mathcal{F}$ on X which restricts to the given sheaves, compatible with the θ. In our setting, this theorem reads that quasi-coherent sheaves satisfy effective descent for Zariski open covers.

We have proved in the previous section that every faithfully flat morphism of affine schemes satisfies effective descent. Moreover, every Zariski open covering satisfies effective descent.

4.5 Morphisms of k-stacks, fiber products, representable morphisms

Definition 4.19 (Morphism of k-stacks)**.** A **1-morphism** $F : \mathcal{X} \to \mathcal{Y}$ will associate, for every $U \in \mathrm{ob}(\mathrm{Aff}\,/k)$, a functor

$$F(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$$

and for every arrow $U' \xrightarrow{f} U$ an isomorphism of functors

$$\alpha(f) : f^*_{\mathcal{X}} \circ F(U') \xrightarrow{\sim} F(U) \circ f^*_{\mathcal{Y}}$$

satisfying the obvious compatibility conditions:

(i) If $f = 1_U$ is an identity, then $\alpha(1_U) = 1_{F(U)}$ is an identity.

(ii) If f and g are composable, then $F(gf)$ is the composite of the squares $\alpha(f)$ and $\alpha(g)$, further composed with the composition of pullback isomorphisms $g^* \circ f^* \simeq (f \circ g)^*$ for $\mathcal{X}$ and $\mathcal{Y}$ (we will not draw the diagram here).

The structure of this morphism can be visualized in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{X}(U) \arrow[r, "F(U)"] \arrow[d, "f^*_{\mathcal{X}}"'] \arrow[dr, Rightarrow, "\alpha(f)"] & \mathcal{Y}(U) \arrow[d, "f^*_{\mathcal{Y}}"] \\
\mathcal{X}(U') \arrow[r, "F(U')"'] & \mathcal{Y}(U')
\end{tikzcd}$$

A **2-morphism** between 1-morphisms $\phi : F \to G$ associates for every $U \in \mathrm{ob}(\mathrm{Aff}\,/k)$, an isomorphism of functors

$$\phi(U) : F(U) \to G(U)$$

represented by the following diagram:

$$\begin{tikzcd}
\mathfrak{X}(U) \arrow[rr, bend left, "F(U)"] \arrow[rr, bend right, "G(U)"'] \arrow[rr, phantom, "\phi(U)\,\Vert"] & & \mathfrak{Y}(U)
\end{tikzcd}$$

subject to some compatibility conditions.

Definition 4.20 (Fiber product, representable morphisms)**.** Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-stacks. The **fiber product** $\mathfrak{X}_\eta$ of $\mathfrak{X}$ over $\mathfrak{Y}$ at an object $\eta \in \mathfrak{Y}(U)$ is the k-stack defined by the rule

$$\mathfrak{X}_\eta(V) = \mathfrak{X}(V) \times_{\mathfrak{Y}(V)} \{\eta|_V\}$$

In particular, such an object η can be thought of as a morphism $\eta : U \to \mathfrak{Y}$. Then the fiber of the morphism F over η is precisely the stack $\mathfrak{X} \times_{\mathfrak{Y}} U$.

The morphism $F : \mathfrak{X} \to \mathfrak{Y}$ is **representable** if $\mathfrak{X}_\eta$ is representable as a scheme for all $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and all $\eta \in \mathrm{ob}\,\mathfrak{Y}(U)$.

We say F has propety P if for every $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and every $\eta \in \mathrm{ob}(Y(\eta))$ the canonical morphism (coming from forming the fiber stack as a pullback) of schemes $\mathfrak{X}_\eta \to U$ has P. Examples of such properties are flat, smooth, surjective, étale, etc.

Definition 4.21. A k-stack $\mathfrak{X}$ is **algebraic** if

(i) the diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable, separated and quasi-compact

(ii) there is a k-scheme P and a smooth, surjective morphism $P \xrightarrow{p} \mathfrak{X}$.

Remark 4.22. Given any scheme T with two maps $f, g : T \to \mathfrak{X}$ (representing two families of objects parameterized by T), the fiber product:

$$T \times_{(\mathfrak{X} \times \mathfrak{X})} \mathfrak{X} \cong \mathrm{Isom}_T(f, g)$$

represents the "scheme of isomorphisms" between the objects corresponding to f and g. The condition that $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable means that for any scheme T with a morphism $T \to \mathfrak{X} \times \mathfrak{X}$, the fiber product $T \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$ is (represented by) an algebraic space. This controls the "size" of automorphism groups.

4.6 Examples of algebraic stacks

Definition 4.23. Let G be an algebraic group acting on a scheme X. The action groupoid X/G is the category whose objects are the points of X and morphisms from x to y are the elements of G such that

$gx = y$.

Note that the isomorphism classes of the action groupoid are in bijection with the orbits of G on X. There is a canonical map $X/G \to */G$ which is obvious on the level of groupoids.

Definition 4.24. We define the quotient stack $X/G : \mathrm{Sch} \to \mathrm{Gpd}$ by

$$(X/G)(S) = \text{sheafification of the presheaf } S \mapsto X(S)/G(S)$$

From the moduli perspective, we have to ask: what family over S is parameterized by $(X/G)(S)$ for a test scheme S? We can answer this question as follows.

The first thing we notice is that the map $X \to *$ should induce a canonical map $X/G \to */G$. Thus an S-point of $S \to X/G$ induces by composition an S-point $S \to */G$; i.e., a G-torsor P over S.

Now, say we have a G-torsor P over S. We can form the fiber product:

$$\begin{tikzcd}
X \times^G P \arrow[r] \arrow[d] & X/G \arrow[d] \\
S \arrow[r, "P"] & */G
\end{tikzcd}$$

We call the stack $X \times^G P$ the X-bundle associated to P, or the associated bundle of P with fiber X. In particular, there is the following correspondence:

Proposition 4.25. There is a canonical bijection between:

1. Maps from a scheme S to the quotient stack X/G
2. Sections of the associated bundle $S \to X \times^G P$
3. G-equivariant maps from P to X

where P is the principal G-bundle on S corresponding to $S \to X/G \to */G$.

Proof. A map $f : S \to X/G$ of stacks corresponds to a principal G-bundle P on S together with a G-equivariant map $\phi : P \to X$. Given a G-equivariant map $\phi : P \to X$, we can construct a section $\sigma : S \to X \times^G P$ of the associated bundle as follows:

For each point $s \in S$, define $\sigma(s) = [\phi(p), p]$ where p is any point in the fiber P_s and $[\phi(p), p]$ denotes the equivalence class in $X \times^G P$. The G-equivariance of ϕ ensures this is well-defined regardless of which $p \in P_s$ we choose.

Conversely, given a section $\sigma : S \to X \times^G P$ where $\sigma(s) = [x_s, p_s]$ for each $s \in S$, we can define a G-equivariant map $\phi : P \to X$ as follows:

For any $p \in P$ with $p \in P_s$ for some $s \in S$, we have $p = p_s \cdot g$ for some $g \in G$. We define $\phi(p) = g^{-1} \cdot x_s$. The properties of the associated bundle ensure this is well-defined and G-equivariant. $\square$

This motivates the following definition:

Definition 4.26. Let an algebraic group G act on a scheme X. Then the **quotient stack** X/G is the functor Sch $\to$ Gpd given by

$$(X/G)(S) = \text{Groupoid of principal } G\text{-torsors } P \text{ over } S \text{ with a } G\text{-equivariant map } P \to X.$$

Example 4.27. If Z is a k-scheme and H is a linear algebraic group over k acting on Z, then the quotient stack $[Z/H]$ is an algebraic stack. A presentation is given by the trivial H-bundle $p : Z \to [Z/H]$.

Example 4.28. Let X be a projective connected smooth curve over k. The moduli problem $\mathcal{M}_{G,X}$ which associated to a scheme S the groupoid of principal G-bundles over $X \times S$ is an algebraic stack.

$$\begin{aligned} \mathcal{M}_{G,X}(S) : \mathrm{Sch}\,/k^{op} &\to \mathrm{Gpd} \\ S &\mapsto \{\text{principal } G\text{-bundles over } X \times S\} + \text{isomorphisms} \end{aligned}$$

Proposition 4.29 (Algebraic stack of principal G-bundles)**.** If G is reductive, the algebraic stack $\mathcal{M}_{G,X}$ is smooth, dimension $\dim G(g-1)$.

Proof. Recall that given any principal G-bundle $E \to X$ and a representation V of G, you can form the associated vector bundle $E(V) := E \times^G V$. This is the quotient of $E \times V$ by the right G-action $(e, v) \sim (eg, g^{-1}v)$.

In particular, for the adjoint representation $V = \mathfrak{g}$, we have

$$E(\mathfrak{g}) := E \times^G \mathfrak{g}$$

is a vector bundle on X whose fibers are copies of the Lie algebra $\mathfrak{g}$, twisted by the bundle E.

The geometry of this stack is controlled by deformation theory and in particular the cohomology groups of these adjoint bundles. Given a principal G-bundle E over X:

- The infinitesimal automorphisms of E are given by $H^1(X, E(\mathfrak{g}))$ where $E(\mathfrak{g})$ is the adjoint bundle associated to E.

- Automorphisms of E are given by $H^0(X, E(\mathfrak{g}))$.
- The obstructions to deformations of E lie in $H^2(X, E(\mathfrak{g}))$. Since X is a curve, this group is zero.

In particular, this shows that the moduli stack is smooth. For algebraic stacks, we have the tangent complex formalism which reduces to the following:

$$T_{\mathcal{M}_{G,X}}|_{[E]} = R\Gamma(X, E(\mathfrak{g}))[1]$$
$$\mathbb{L}_{\mathcal{M}_{G,X}}|_{[E]} = R\Gamma(X, E(\mathfrak{g}))^*[-1].$$

At the point corresponding to E, we have:

$$\dim T_{[E]}\mathcal{M}_{G,X} = \dim H^1(X, E(\mathfrak{g})) - \dim H^0(X, E(\mathfrak{g})) = -\chi(X, E(\mathfrak{g}))$$

Since $\deg(E(\mathfrak{g})) = 0$ and $\mathrm{rk}(E(\mathfrak{g})) = \dim(G)$, we have by Riemann-Roch:

$$\chi(X, E(\mathfrak{g})) = \mathrm{rk}(E(\mathfrak{g}))(1-g) = \dim(G)(1-g)$$

Therefore:

$$\dim \mathcal{M}_{G,X} = \dim(G)(g-1)$$

See Sam Raskin notes $\square$

5 Uniformization

The theory of uniformization relates these moduli spaces to loop groups and associated Grassmannians. We introduce the uniformization of moduli stacks of principal G-bundles, beginning with the topological perspective and transitioning to the algebraic setting.

5.1 Topological loop groups

Let X be a smooth projective curve over k. Let G connected reductive. Then isomorphism classes of topological principal G-bundles over X are in bijection with elements of $\pi_1(G)$.

To see this, consider a basepoint $x \in X$ and a disk D around x. Then the restriction of a principal G-bundle P to D is trivial because D is contractible. Let $X^* = X\backslash\{x\}$. Then U is homotopy equivalent to a wedge of circles and therefore any topological principal G-bundle over U is also trivial. This is because of the general theory of obstruction theory for CW complexes.

In general, given a CW complex X and a map from the i-skeleton $X^i \to Y$, the obstruction to extending this map to the $(i+1)$-skeleton lies in the cellular cohomology group $H^{i+1}(X \cdot \pi_i(Y))$. We also make use of the fact that a topological principal G-bundle P over a space X is trivial if and only if it admits a global section. In particular, to trivialize a principal G-bundle over X is precisely to specify a map $X \to G$. Therefore, the obstruction to lifting a map $X^{*0} \to G$ to $X^{*1} \to G$ lies in the group $H^1(U, \pi_0(G))$ but $\pi_0(G) = e$ so this group is trivial. Therefore, all topological principal G-bundle over U are trivial.

Another way to see this is using the theory of classifying spaces. Prinicipal G-bundles over X^* are classified by homotopy classes of maps $X^* \to BG$. By the cellular approximation theorem, any map $X^* \to BG$ can be homotoped to a map $X^* \to BG^1$ where BG^1 is the 1-skeleton of BG. But BG carries a cell structure with cells in only even dimensions, so such homotopy classes of maps amount to picking a connected component of BG. But BG is connected because G is connected. Therefore, all principal G-bundles over U are trivial.

Return to X. The only data that is important, since the bundle is trivial over X^* and D, is the transition function $g_{X^*D} \in G$. This amounts to a map $D^* = D\backslash\{x\} \to G$. But D^* is homotopy equivalent to a circle, so this map is classified by an element of $\pi_1(G)$. Therefore, the isomorphism classes of principal G-bundles over X are in bijection with $\pi_1(G)$.

We recast the argument given above:

Definition 5.1. We have the following groups:

$$\begin{aligned} L^{top}G &= \{\text{continuous maps } D^* \to G\} \\ L_+^{top}G &= \{\text{continuous maps } D \to G\} \\ L_X^{top}G &= \{\text{continuous maps } X^* \to G\} \end{aligned}$$

and natural inclusions $L_+^{top}G \to L^{top}G$ and $L_X^{top}G \to L^{top}G$.

Proposition 5.2. There is a canonical bijection

$$L_X^{top}G\backslash L^{top}G/L_+^{top}G \cong \mathcal{M}_{G,X}^{top} \cong \pi_1(G)$$

Proof. One thinks of the space

$$L^{top}G = \{E, \sigma, \tau\}$$

of triples where $E \to X$ is a principal G bundle and $\sigma : E|_D \cong D \times G$ and $\tau : E|_{X^*} \cong X^* \times G$ are choices of trivializations. Then one divides out by the choice of trivializations. □

5.2 Algebraic loop groups

The algebraic analagoue of a neighborhood homeomorphic to x is given by looking at the completion of the local ring $\mathcal{O}_{X,x}$.

$$D_x = \operatorname{Spec} \hat{\mathcal{O}}_{X,x}$$

Choosing a local coordinate z near x gives the idenitfication

$$D_x = \operatorname{Spec} k[[z]]$$

The punctured disk is the field of fractions K_x of the completion of the local ring

$$D_x^* = \operatorname{Spec} K_x \cong \operatorname{Spec} k((z))$$

Introduce the notation $U = \operatorname{Spec} R$, $D_U^* = \operatorname{Spec} R((z))$ and $D_U = \operatorname{Spec} R[[z]]$ and $X_U^* = X^* \times U$.

The algebraic analogue of the topological loop group $L^{top}G$ is the group scheme

$$LG = \underline{\operatorname{Hom}}_{alg}(D^*, G)$$

the points of G with values in D^*, i.e. $G(k((z)))$.

Definition 5.3. We have the functor of points for **algebraic loop groups**

$$\begin{aligned} LG : \operatorname{Aff}/k &\to \operatorname{Grp} \\ LG(U) &= \operatorname{Hom}_{alg}(D_U^*, G) = G(R((z))) \end{aligned}$$

and the analagous k-groups

$$\begin{aligned} L_+G(U) &= \operatorname{Hom}_{alg}(D_U, G) = G(R[[z]]) \\ L_XG(U) &= \operatorname{Hom}_{alg}(X_U^*, G) = G(\mathcal{O}_{X_U^*}) \end{aligned}$$

The quotient space $\mathcal{Q}_G = LG/L_+G$ is the sheafification of the presheaf

$$U \mapsto LG(U)/L_+G(U)$$

carries an action of L_XG.

Consider the quotient stack $[L_XG \backslash \mathcal{Q}_G]$.

Theorem 5.4 (Uniformization)**.** Let G semisimple. Then there is a canonical isomorphism of stacks

$$[L_XG \backslash \mathcal{Q}_G] \cong \mathcal{M}_{G,X}$$

Moreover the L_XG-bundle $\mathcal{Q}_G \to \mathcal{M}_{G,X}$ is even locally trivial for the etale topology if the characteristic of k does not divide the order of $\pi_1(G(\mathbb{C}))$.

We consider triples (E, ρ, σ) where E is a vector bundle on X_R, $\rho : \mathcal{O}^r_{X_R^*} \longrightarrow E|_{X_R^*}$ a trivialization of E over X_R^*, $\sigma : \mathcal{O}^r_{D_R} \longrightarrow E|_{D_R}$ a trivialization of E over D_R. We let $\mathrm{T}(R)$ be the set of isomorphism classes of triples (E, ρ, σ) (with the obvious notion of isomorphism).

Proposition 5.5. The ind-group $\mathbf{GL}_r(K)$ represents the functor T.

Proposition 5.6. The ind-group $\mathbf{SL}_r(K)$ represents the subfunctor T_0 of T which associates to a k-algebra R the set of isomorphism classes of triples (E, ρ, σ) where E is a vector bundle on $\mathrm{X_R}$, $\rho : \mathcal{O}^r_{\mathrm{X}^*_{\mathrm{R}}} \longrightarrow E|_{\mathrm{X}^*_{\mathrm{R}}}$ and $\sigma : \mathcal{O}^r_{\mathrm{D_R}} \longrightarrow E|_{\mathrm{D_R}}$ are isomorphisms such that $\wedge^r \rho$ and $\wedge^r \sigma$ coincide over $\mathrm{D}^*_{\mathrm{R}}$.

Remark 5.7. The condition that the trivializations $\wedge^r \rho$ and $\wedge^r \sigma$ coincide over $\mathrm{D}^*_{\mathrm{R}}$ means that they come from a global trivialization of $\wedge^r E$. So we can rephrase by saying that $\mathrm{T}_0(R)$ is the set of isomorphism classes of data $(E, \rho, \sigma, \delta)$ where δ is a trivialization of $\wedge^r E$, ρ and σ are trivializations of $E|_{\mathrm{X}^*_{\mathrm{R}}}$ and $E|_{\mathrm{D_R}}$, respectively, such that $\wedge^r \rho$ coincide with $\delta|_{\mathrm{X}^*_{\mathrm{R}}}$ and $\wedge^r \sigma$ with $\delta|_{\mathrm{D_R}}$.

Corollary 5.8. Let us denote by $\mathrm{A_X}$ the affine algebra $\Gamma(X - p, \mathcal{O}_X)$. There is a canonical bijective correspondence between the set of isomorphism classes of rank r vector bundles on X with trivial determinant (resp. with determinant of the form $\mathcal{O}_X(np)$ for some integer n) and the double coset space $\mathrm{SL}_r(\mathrm{A_X})\backslash \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O})$ (resp. $\mathrm{GL}_r(\mathrm{A_X})\backslash \mathrm{GL}_r(K)/\mathrm{GL}_r(\mathcal{O})$).

Proof. Since two trivializations of $E|_{\mathrm{D}}$ differ by an element of $\mathrm{GL}_r(\mathcal{O})$, and two trivializations of $E|_{\mathrm{X}^*}$ by an element of $\mathrm{GL}_r(\mathrm{A_X})$, bijection between $\mathrm{GL}_r(\mathrm{A_X})\backslash \mathrm{GL}_r(K)/\mathrm{GL}_r(\mathcal{O})$ and the set of isomorphism classes of rank r vector bundles on X which are trivial on X^*. But a projective module over a Dedekind ring is free if and only if its determinant is free, hence our assertion for GL_r. The same proof applies for SL_r. □

Remark 5.9. Note that saying that a line bundle is trivial on the open complement $X^* = X - p$ is equivalent to saying that it is of the form $\mathcal{O}_X(np)$ for some integer n. This follows from the exact sequence

$$\mathbb{Z} \to \mathrm{Pic}(X) \to \mathrm{Pic}(X^*) \to 0$$

where the first map sends $1 \mapsto \mathcal{O}_X(p)$.

Lemma 5.10. Let G be any semisimple group. Given a principal G-bundle $\mathcal{E}$, and any representation $\rho : G \to \mathrm{GL}(V)$, the contracted product $E = \mathcal{E} \times_G V$ has trivial determinant.

Proof. To see that $\det(E)$ is trivial, we note that since G is semisimple, $[G, G] = G$, and so the image $\rho(G)$ is contained in the kernel of the determinant map which is $\mathrm{SL}(V)$. This is because ρ preserves commutator subgroups and $[\mathrm{GL}_n, \mathrm{GL}_n] \subset \mathrm{SL}_n$.

In particular, E has transition functions given by matrices with trivial determinant. These are the transition functions of the line bundle $\det(E)$, and so $\det(E)$ is necessarily trivial. □

5.3 As a moduli stack

In the last section, we described a bijection between the set of isomorphism classes of rnak r vector bundles on X with trivial determinant and the double coset space $\mathrm{SL}_r(A_X)\backslash \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O})$ by considering triples (E,σ,τ) corresponding to vector bundles E along with choices of trivializations over the open complement of a point, and the unit disk respectively. This in fact gives a description of the moduli stack. This section is about understanding the algebraic structure of the stack $\mathrm{SL}_r(A_X)\backslash \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O})$.

We begin with result about the infinite Grassmannian.

Proposition 5.11. The k-space $\mathcal{Q} := \mathbf{SL}_r(K)/\mathbf{SL}_r(\mathcal{O})$ represents the functor which associates to a k-algebra R the set of isomorphism classes of pairs (E,ρ), where E is a vector bundle over $\mathrm{X_R}$ and ρ a trivialization of E over X^*_R such that $\wedge^r\rho$ extends to a trivialization of $\wedge^r E$.

Proof. A standard proof using descent. Let R be a k-algebra and q an element of $Q(R)$. By definition there exists a faithfully flat homomorphism $R\to R'$ and an element γ of $\mathrm{SL}_r(R'((z)))$ such that the image of q in $\mathcal{Q}(R')$ is the class of γ. Effective, we are checking that the proposition holds for an fppf covering of R, and then it will necessarily hold for R by descent.

To γ corresponds a triple (E',ρ',σ') over $X_{R'}$. Let $R'' = R'\otimes_R R'$, and let $(E''_1,\rho''_1),(E''_2,\rho''_2)$ denote the pull-backs of (E',ρ') by the two projections of $X_{R''}$ onto $X_{R'}$. Since the two images of γ in $\mathrm{SL}_r(R''((z)))$ differ by an element of $\mathrm{SL}_r(R''[[z]])$, these pairs are isomorphic; this means that the isomorphism $\rho''_2\rho''^{-1}_1$ over $X^*_{R''}$ extends to an isomorphism $u : E''_1 \to E''_2$ over $X_{R''}$. This isomorphism satisfies the usual cocycle condition, because it is enough to check it over X^*, where it is obvious. Therefore (E',ρ') descends to a pair (E,ρ) on X_R as in the statement of the proposition.

Conversely, given a pair (E,ρ) as above over X_R, we can find a faithfully flat homomorphism $R\to R'$ and a trivialization σ' of the pull back of E over $D_{R'}$ such that $\wedge^r\sigma'$ coincides with $\wedge^r\rho$ over $D^*_{R'}$ (in fact $\mathrm{Spec}(R)$ is covered by open subsets $\mathrm{Spec}(R_\alpha)$ such that E is trivial over D_{R_α}, and we can take $R' = \prod R_\alpha$). By prop. 1.5 we get an element γ' of $\mathbf{SL}_r(R'((z)))$ such that the two images of γ' in $\mathbf{SL}_r(R''((z)))$ (with $R'' = R'\otimes_R R'$) differ by an element of $\mathbf{SL}_r(R''[[z]])$; this gives an element of $Q(R)$. The two constructions are clearly inverse one of each other. $\square$

5.4 As a Grassmannian of lattices

For any k-algebra R define **lattice** in $R((z))^r$ as a sub-$R[[z]]$ module W of $R((z))$ which is projective of rank R and so that $\cup z^{-n}W = R((z))^r$. In particular this implies that

$$z^{-N}R[[z]]\subset W\subset z^N R[[z]]$$

for some integer N, and so that the R-module $W/z^N R[[z]]^r$ is projective. Moreover we say that the lattice W is **special** if the projective R-module $W/z^N R[[z]]$ is of rank Nr. This is equivalent to saying that the determinant $\Lambda^r W$ is trivial $= R[[z]]$.

Proposition 5.12. The k-space Q (resp. $\mathbf{GL}_r(\mathrm{K})/\mathbf{GL}_r(\mathcal{O})$) represents the functor which associates to a k-algebra R the set of special lattices (resp. of lattices) $\mathrm{W} \subset \mathrm{R}((z))^r$. The group $\mathbf{SL}_r(\mathrm{K})$ acts on Q by $(\gamma, \mathrm{W}) \mapsto \gamma\mathrm{W}$ (for $\gamma \in \mathbf{GL}_r(\mathrm{R}((z)))$, $\mathrm{W} \subset \mathrm{R}((z))^r$).

$$\begin{tikzcd} \mathrm{D}^* \arrow[r, hook] \arrow[d] & \mathrm{D} \arrow[d] \\ \mathrm{X}^* \arrow[r, hook] & \mathrm{X} \end{tikzcd}$$

Proof. Consider the diagram where for simplicity we have dropped the suffix R. Let us start with a pair (E, ρ) over X. The trivialization ρ gives an isomorphism $R((z))^r \longrightarrow H^0(D^*, E|_{D^*})$; the inverse image W of $H^0(D, E|_D)$ is a lattice in $R((z))^r$, and it is a special lattice if $\wedge^r \rho$ extends to a trivialization of $\wedge^r E$ over X.

Conversely, given a lattice W in $R((z))^r$, we define a vector bundle E_W on X by gluing the trivial bundle over X^* with the bundle on D associated to the $R[[z]]$-module W; the gluing isomorphism is the map $W \otimes_{R[[z]]} R((z)) \longrightarrow R((z))^r$ induced by the embedding $W \hookrightarrow R((z))^r$. By definition E_W has a natural trivialization ρ_W over X^*, and if W is a special lattice $\wedge^r \rho$ extends to a trivialization of $\wedge^r E$ over X. It is easy to check that these two constructions are inverse one of each other.

Let γ be an element of $\mathbf{GL}_r(R((z)))$, corresponding to a triple (E, ρ, σ). By construction the corresponding lattice is $\rho^{-1}\sigma(R[[z]]^r) = \gamma(R[[z]]^r)$. $\square$

Recall that we have denoted by $S^{(N)}$ the subscheme of $\mathrm{SL}_r(K)$ parametrizing matrices $A(z)$ such that $A(z)$ and $A(z)^{-1}$ have a pole of order $\leq N$; it is stable under right multiplication by $S^{(0)} = \mathrm{SL}_r(\mathcal{O})$. We will denote by $\mathcal{Q}^{(N)}$ its image in $\mathcal{Q}$, i.e. the quotient k-space $S^{(N)}/S^{(0)}$.

Corollary 5.13. Let $\mathbb{F}_N$ be a free module of rank r over the ring $k[z]/(z^{2N})$ (so that $\mathbb{F}_N$ is a k-vector space of dimension $2rN$). The k-space $\mathcal{Q}^{(N)} = S^{(N)}/S^{(0)}$ is isomorphic to the (projective) variety of rN-dimensional subspaces G of $\mathbb{F}_N$ such that $zG \subset G$.

Recall that we have denoted by $\mathrm{SL}_r(\mathcal{O}_-)$ the subgroup of $\mathrm{SL}_r(k[z^{-1}])$ parametrizing matrices $\sum_{n \geq 0} A_n z^{-n}$ with $A_0 = \mathrm{I}$. It is an ind-variety.

Theorem 5.14. The k-space $\mathcal{Q} = \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O})$ is an ind-variety, direct limit of the system of projective varieties $(\mathcal{Q}^{(N)})_{N \geq 0}$. It is covered by open subsets which are isomorphic to $\mathrm{SL}_r(\mathcal{O}_-)$, and over which the fibration $p : \mathrm{SL}_r(K) \longrightarrow \mathcal{Q}$ is trivial.

Proposition 5.15. Let ω be the class of the identity I in $\mathcal{Q}$.

1. The orbits of $\mathrm{SL}_r(\mathcal{O})$ in $\mathcal{Q}$ are the orbits of the points $z^d\omega$ where d runs through the sequences $d_1 \leq d_2 \leq \cdots \leq d_r$ and $\sum d_i = 0$.

2. The orbit of $z^{d'}\omega$ is in the closure of $z^d\omega$ if and only if $d' \geq d$ in dominance order, i.e. the pth partial sum of d' is greater than or equal to the pth partial sum of d for all p.

Proof. We have the formula

$$\begin{pmatrix} t^{-1}z & t^{-1} \\ -t & 0 \end{pmatrix} \begin{pmatrix} z^{d_1} & 0 \\ 0 & z^{d_2} \end{pmatrix} \begin{pmatrix} t & -t^{-1}z^{d_2-d_1-1} \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} z^{d_1+1} & 0 \\ -t^2 z^{d_2} & z^{d_2-1} \end{pmatrix}$$

and take the limit as $t \to 0$. $\square$

5.5 The moduli stack $\mathrm{SL}_r(A_X)\backslash \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O})$

Recall that a stack over k associates to any k-algebra R a groupoid $F(R)$, and to any morphism of k-algebras $u : R \to R'$ a functor $F(u) : F(R) \to F(R')$. This data should satisfy some compatibility conditions and as well as some localization properties.

Example 5.16. The *moduli stack* $\mathcal{GL}_X(r)$ of rank r vector bundles on X is defined by associating to a k-algebra R the groupoid of rank r vector bundles over X_R. Similarly, one defines a stack $\mathcal{SL}_X(r)$ by associating to R the groupoid of pairs (E, δ), where E is a vector bundle over X_R and $\delta : \mathcal{O}_{X_R} \to \bigwedge^r E$ an isomorphism; this is the fibre over the trivial bundle of the morphism of stacks $\det : \mathcal{GL}_X(r) \to \mathcal{GL}_X(1)$.

Definition 5.17. A Γ-torsor over R (in the fppf site) is a k-space P over R with an action of Γ_R which after a faithfully flat extension $R \to R'$ becomes isomorphic to $\Gamma_{R'}$ acting on itself by multiplication.

Example 5.18. Let Q be a k-space, and Γ a k-group acting on Q. The quotient stack $\Gamma\backslash Q$ is defined in the following way: an object of $F(R)$ is a Γ-torsor P over $\operatorname{Spec} R$ together with a Γ-equivariant morphism $\alpha : P \to Q$; arrows in $F(R)$ are defined in the obvious way, and so are the functors $F(u)$. The stack $\Gamma\backslash Q$ is indeed the quotient of Q by Γ in the category of stacks, in the sense that any Γ-invariant morphism from Q into a stack F factors through $\Gamma\backslash Q$ in a unique way. If Γ acts freely on Q (i.e. $\Gamma(R)$ acts freely on $Q(R)$ for each k-algebra R), then the stack $\Gamma\backslash Q$ is a k-space.

When $Q = \operatorname{Spec}(k)$ (with the trivial action), $\Gamma\backslash Q$ is the *classifying stack* $B\Gamma$: for each k-algebra R, $B\Gamma(R)$ is the groupoid of Γ-torsors over $\operatorname{Spec}(R)$.

Proposition 5.19. The quotient stack $\mathrm{SL}_r(A_X)\backslash \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O})$ is canonically isomorphic to the algebraic stack $\mathcal{SL}_X(r)$ of vector bundles on X with trivial determinant. The projection map

$$\pi : \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O}) \longrightarrow \mathcal{SL}_r(X)$$

is locally trivial for the Zariski topology.

5.6 Determinant line bundle on the moduli stack

Let X projective curve smooth and connected over the algebraically closed field k. Let F be a vector bundle over the fiber product $X_S = X \times_{\operatorname{Spec} k} S$ where S is a locally noetherian k-scheme.

Definition 5.20. A complex $K^\bullet$ of coherent locally free $\mathcal{O}_S$-modules

$$0 \longrightarrow K^0 \xrightarrow{\gamma} K^1 \longrightarrow 0$$

a **representative of the cohomology** of F if for every base change $T \xrightarrow{f} S$

$$\begin{tikzcd} X_T \arrow[r, "g"] \arrow[d, "u"'] & X_S \arrow[d, "p"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

we have $H^i(f^*K^\bullet) = R^i u_* g^* F$.

In particular, if $s \in S$ is a closed point:

$$H^i(K_s^\bullet) = H^i(X, F_s).$$

Remark 5.21. Without this condition, one could take any locally free resolution $K^\bullet \simeq Rp_*F$. But in general, base change does not commute with arbitrary choices of resolutions. For example, you might have $H^i(K_s^\bullet) \neq H^i(X_s, F_s)$ if your chosen resolution does not remain exact after tensoring with $\mathcal{O}_s$. Tensoring with $\mathcal{O}_s$ is only exact if the complex is flat over S.

Representatives of the cohomology of F are easy to construct in our setup. Indeed, we may choose a resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow F \longrightarrow 0$$

of F by S-flat coherent $\mathcal{O}_{X_S}$-modules such that $p_*P_0 = 0$ (use Serre's theorem A in its relative version to see its existence). Then we have $p_*P_1 = 0$ and, by base change for coherent cohomology, the complex

$$0 \longrightarrow R^1p_*P_1 \longrightarrow R^1p_*P_0 \longrightarrow 0$$

is convenient. This result is generally quoted as choosing a perfect complex of length one representing Rp_*F in the derived category $D^c(S)$.

Remark 5.22. Let $f : T \to S$ be any morphism of locally noetherian schemes. If $p : X \to S$ is proper and F is coherent and flat over S, then for each i the natural map

$$f^*R^ip_*F \longrightarrow R^iu_*g^*F$$

is an isomorphism whenever $R^{i+1}p_*F$ is locally free.

Pick $n \gg 0$ such that

(RS1) $F(n) := F \otimes p^*\mathcal{O}_X(n)$ is generated by global sections relatively over S

(RS2) $R^i p_* F(n) = 0$ for all $i > 0$.

Being generated by relative global sections over S means that there is a surjection of coherent sheaves on X $\mathrm{ev} : p^*p_*F(n) \to F(n)$. Let $K_1(n)$ be its kernel and so we have a short exact sequence

$$0 \longrightarrow K_1(n) \longrightarrow p^*p_*F(n) \longrightarrow F(n) \longrightarrow 0.$$

We untwist and let $P_0 = p^*p_*F(n) \otimes \mathcal{O}_X(-n)$ and $P_1 = K_1(n) \otimes \mathcal{O}_X(-n)$.

This gives a short exact sequence

$$0 \to P_1 \xrightarrow{\phi} P_0 \twoheadrightarrow F \to 0$$

where P_0 is a pullback of a vector bundle on S (so it is S-flat and p-acyclic) and P_1 is the kernel of a map between S-flat sheaves (so it is also S-flat).

Apply Rp_* to the above sequence. Because of (RS2) and how we built P_i, we get:

- We have $p_*P_0 = p_*(p^*p_*F(n) \otimes \mathcal{O}_X(-n)) \twoheadrightarrow p_*F$ and crucially $R^j p_* P_0 = 0$ for $j \geq 1$ (choose n so this holds; standard with projection formula + vanishing).

- Similarly arrange $p_*P_1 = 0$ and $R^j p_* P_1 = 0$ for $j \geq 2$.

Because the P_i are S-flat and p-acyclic (their higher direct images vanish), the cohomology of F can be computed from the 2-term complex

$$0 \longrightarrow R^1 p_* P_1 \xrightarrow{R^1 p_\phi} R^1 p_{P_0} \longrightarrow 0.$$

From the long exact sequence, only the R^1 terms survive in a controlled way, and you extract a 2-term complex on S:

$$K^\bullet := [K^0 \xrightarrow{\gamma} K^1] = [R^1 p_* P_1 \xrightarrow{R^1 p_* \phi} R^1 p_* P_0], \tag{K}$$

with K^0, K^1 locally free (by cohomology-and-base-change plus vanishing of the next R^2).

Definition 5.23. The **determinant** of a complex $K^\bullet$ of locally free coherent $\mathcal{O}_S$-modules

$$0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow 0$$

is defined by

$$\det(K^\bullet) = \bigwedge^{\max} K^0 \otimes \left(\bigwedge^{\max} K^1\right)^{-1}.$$

The determinant of our family F of vector bundles parameterized by S is the line bundle on S defined by

$$\mathcal{D}_F = \det(Rp_*F)^{-1}.$$

In general, in order to calculate $\mathcal{D}_F$, we choose a representative $K^\bullet$ of the cohomology of F and then calculate $\det(K^\bullet)^{-1}$. This does not depend, up to canonical isomorphism on the choice of $K^\bullet$. This means that if $L^\bullet$ is another representative of the cohomology of F, there is a canonical isomorphism $\det(K^\bullet) \simeq \det(L^\bullet)$ not depending on the choice of quasi-isomorphism between $K^\bullet$ and $L^\bullet$, functorial in the sense that if $M^\bullet$ is a third representative of the cohomology of F, the following diagram commutes:

$$\begin{tikzcd}
\det(K^\bullet) \arrow[r] \arrow[dr] & \det(L^\bullet) \arrow[d] \\
& \det(M^\bullet).
\end{tikzcd}$$

Remark 5.24. Note that this statement does not say that any quasi-isomorphism between $K^\bullet$ and $L^\bullet$ induces the same isomorphism between their determinants. This is false in general. If you take $K^\bullet = L^\bullet = [V \to W]$ with V, W vector spaces over k and the zero map between them, then a quasi-isomorphism between them is any automorphism of the complex, i.e. a pair of automorphisms (u, v) of V and W. The induced map on determinants is $\det(u)\det(v)^{-1}$, which depends on the choice of (u, v).

The point is that there is a systematic way to choose an isomorphism between the determinants that is compatible with composition.

By construction, the fiber of $\mathcal{D}_F$ at $s \in S$ is given as follows:

$$\mathcal{D}_F(s) = \left(\bigwedge^{\max} H^0(X, F_s)\right)^{-1} \otimes \bigwedge^{\max} H^1(X, F_s).$$

We may also twist our family F by bundles coming from X, i.e. consider $F \otimes q^*E$ where E is a vector bundle on X and $q : X_S \to X$ is the projection. We obtain the line bundle $\mathcal{D}_{F\otimes q^*E}$, and this line bundle actually depends only on the class of E in the Grothendieck group $K(X)$ of X (check this!). It follows that we get a group morphism, Le Potier's determinant morphism

$$\lambda_F : K(X) \longrightarrow \mathrm{Pic}(S), \qquad u \longmapsto \mathcal{D}_{F\otimes q^*u}.$$

If our bundle F comes from an SL_r-bundle, *i.e.* has trivial determinant, twisting F by an element $u \in K(X)$ then taking determinants just means taking the $r(u)$-th power of $\mathcal{D}_F$:

Lemma 5.25. Suppose F is a vector bundle on X_S such that $\bigwedge^{\max} F$ is the pullback of some line bundle on X. Then

$$\mathcal{D}_{F\otimes q^*u} = \mathcal{D}_F^{\otimes r(u)} \quad \text{in } \mathrm{Pic}(S),$$

where $r(u)$ is the rank of u.

Proof. $\mathcal{D}_{F\otimes q^*u}$ depends additively on the class $u \in K(X)$:

$$\mathcal{D}_{F\otimes q^*(u+v)} = \mathcal{D}_{F\otimes q^*u} \otimes \mathcal{D}_{F\otimes q^*v}.$$

because determinant is multiplicative in short exact sequences. Moreover for a smooth projective variety, every coherent sheaf has a finite resolution by soms of line bundles $\mathcal{O}_X(n)$. This follows from Serre's theorem and the existence of very ample $\mathcal{O}(1)$.

Thus it is enough to check it for $L = \mathcal{O}_X(-p)$, for $p \in X$, where it follows from considering

$$0 \longrightarrow \mathcal{O}_X(-p) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0$$

Pulling back along $q : X_S \to X$ and tensoring with F, we get a short exact sequence

$$0 \longrightarrow F \otimes q^*\mathcal{O}_X(-p) \longrightarrow F \longrightarrow F \otimes q^*\mathcal{O}_p \longrightarrow 0.$$

Taking determinants, we get

$$\mathcal{D}_F \cong \mathcal{D}_{F\otimes q^{\mathcal{O}}_X(-p)} \otimes \mathcal{D}_{F\otimes q^{\mathcal{O}}_p}.$$

i.e

$$\mathcal{D}_{F\otimes q^{\mathcal{O}}_X(-p)} \cong \mathcal{D}_F \otimes (\mathcal{D}_{F\otimes q^*\mathcal{O}_p})^{-1}.$$

Thus everything hinges on showing that $\mathcal{D}_{F\otimes q^*\mathcal{O}_p} \cong \mathcal{O}_S$.

The sheaf $q^*\mathcal{O}_p$ is just the structure sheaf of the fiber $q^{-1}(p) = \{p\} \times S$: $q^*\mathcal{O}_p = \mathcal{O}_{q^{-1}(p)} = \mathcal{O}_S$. Hence $F \otimes q^*\mathcal{O}_p = F|_{q^{-1}(p)}$. This is a vector bundle on S of rank $r = \operatorname{rk} F$.

Since $p : X_S \to S$ is projection, the pushforward of a sheaf supported entirely on $q^{-1}(p)$ is just that sheaf: $Rp_*(F|_{q^{-1}(p)}) = F|_{q^{-1}(p)}$. So

$$\mathcal{D}_{F\otimes q^*\mathcal{O}_p} = \det Rp_*(F|_{q^{-1}(p)}) = \det(F|_{q^{-1}(p)}) = \bigwedge^r F|_{q^{-1}(p)}.$$

By assumption, $\det(F)$ is pulled back from X: $\det(F) = q^*(L_X)$ for some line bundle L_X on X.

Then

$$\det(F|_{q^{-1}(p)}) = (\det F)|_{q^{-1}(p)} = (q^*L_X)|_{\{p\}\times S} = L_X|_p \otimes \mathcal{O}_S.$$

But $L_X|_p$ is just a 1-dimensional k-vector space, a constant line. As a line bundle on S, that's the trivial line bundle $\mathcal{O}_S$ (canonically trivial once you fix the base field).

Hence $\mathcal{D}_{F\otimes q^*\mathcal{O}_p} = \mathcal{O}_S$. Plugging this back we get $\mathcal{D}_{F\otimes q^*\mathcal{O}_X(-p)} \cong \mathcal{D}_F$ as desired. $\square$

5.7 Theta-functions

Associating to each bundle F on $X \times S$ the line bundle $\mathcal{D}_F$ defines a line bundle L on the stack $\mathrm{SL}_X(r)$ (or $\mathrm{GL}_X(r)$), the determinant line bundle.

Twisting is particularly useful in order to produce sections of (powers of) the determinant bundle. Suppose S is integral and that F is a vector bundle on X_S with trivial determinant. Choose a vector bundle E on Xsuch that $F_s \otimes E$ has trivial Euler characteristic for some s. If

$$0 \longrightarrow K^0 \xrightarrow{\gamma} K^1 \longrightarrow 0$$

is a representative of the cohomology of $F \otimes q^*E$, then we know that the rank n of K^0 is equal to the rank of K^1, hence γ may be locally represented as an $n \times n$ matrix. We get a section

$$\theta_E = \det(\gamma)$$

of $\mathcal{D}_F^{r(E)}$. Changing the representative $K^\bullet$ changes γ by a change of basis (automorphism of K^0, K^1), hence $\det(\gamma)$ changes by an invertible function on S. In particular, its divisor Θ_E is well defined with support the points $s \in S$ such that $H^0(F_s \otimes E) \neq 0$.

If we suppose moreover that $F_t \otimes q^*E$ has trivial cohomology for some $t \in S$, then $\Theta_E \neq S$, i.e. the section θ_E is nontrivial; if there is $t' \in S$ such that $H^0(X, F_{t'} \otimes E) \neq 0$, then $\Theta_E \neq \varnothing$.

5.8 Pfaffian bundles

Suppose $\mathrm{char}(k) \neq 2$ in this subsection. Let F be a vector bundle over $X_S = X \times S$, together with a quadratic nondegenerate form σ with values in the canonical bundle ω_X. We will view σ as an isomorphism

$$F \xrightarrow{\sim} F^\vee \quad \text{such that} \quad \sigma = \sigma^\vee,$$

where $F^\vee = \mathcal{H}om_{\mathcal{O}_{X_S}}(F, q^*\omega_X)$. By Grothendieck-Serre duality, this induces a pairing on the derived pushforward Rp_*F:

$$Rp_*F \xrightarrow{\sim} \mathcal{RH}om(Rp_*F, \mathcal{O}_S)[-1].$$

So the object Rp_*F (which represents the cohomology of F on each fiber) comes with a symmetric bilinear form in the derived category. The motivating question is: Can we represent Rp_*F by an actual complex of vector bundles with a symmetric pairing between the terms, not just in the derived category?

Lemma 5.26. If $K^\bullet$ is a representative of the cohomology of F, then

$$K^{\bullet *}[-1]$$

is a representative of the cohomology of $F^\vee$.

Here $K^{\bullet *}[-1]$ denotes the complex supported in degrees 0 and 1,

$$0 \longrightarrow K^{1*} \xrightarrow{-\gamma^*} K^{0*} \longrightarrow 0.$$

Proof. In the derived category $D_c(S)$, we have

$$Rp_*(F^\vee) \simeq Rp_*\big(\mathcal{RH}om_{\mathcal{O}_{X_S}}(F, q^*\omega_X)\big) \quad (F \text{ locally free})$$

$$\simeq \mathcal{RH}om\big(Rp_*F, \mathcal{O}_S\big)[-1] \quad \text{(Grothendieck–Serre duality)}.$$

Now if $K^\bullet$ represents the cohomology of F, we see that $\mathcal{RH}om(K^\bullet, \mathcal{O}_S)[-1]$ represents the cohomology of $F^\vee$. But this is nothing else than $K^{\bullet *}[-1]$, as the K^i are locally free. $\square$

Proposition 5.27 (6.2.2)**.** There exists, locally for the Zariski topology on S, a representative of the cohomology $K^\bullet$ of F and a symmetric isomorphism

$$\tau : K^\bullet \xrightarrow{\sim} K^{\bullet *}[-1]$$

such that τ and σ induce the same map in cohomology.

Proof. Choose a representative $\widetilde{K}^\bullet$ of the cohomology of F and remark that σ induces an isomorphism $\widetilde{\tau}$ in the derived category $D_c^b(S)$:

$$\widetilde{K}^\bullet \xrightarrow{\sim} Rp_*F \xrightarrow{\sigma} Rp_*(F^\vee) \xrightarrow{\sim} \widetilde{K}^{\bullet *}[-1],$$

which is still symmetric (follows from symmetry of σ and standard properties of Grothendieck–Serre duality).

The problem here is that this isomorphism is only defined in the derived category: the proposition actually claims that we can get a symmetric morphism of complexes, and this we only get Zariski locally.

First we may suppose that S is affine. Then the category of coherent sheaves on S has enough projectives, and as the $\widetilde{K}^i$ are locally free we see that $\widetilde{\tau}$ is an isomorphism in $K_c^b(S)$. Let φ be a lift of $\widetilde{\tau}$ to $C_c^b(S)$. We get a morphism of complexes

$$\begin{tikzcd} \widetilde{K}^0 \arrow[r, "\gamma"] \arrow[d, "\varphi_0"'] & \widetilde{K}^1 \arrow[d, "\varphi_1"] \\ \widetilde{K}^{1*} \arrow[r, "-\gamma^*"] & \widetilde{K}^{0*} \end{tikzcd}$$

which need neither be symmetric nor an isomorphism (it is only a quasi-isomorphism). We first symmetrize:

$$\phi_i = \frac{1}{2}\big(\varphi_i + \varphi_{1-i}^*\big), \qquad i = 0, 1.$$

Remark that ϕ is still a quasi-isomorphism, inducing σ in cohomology. Fix $s \in S$. A standard argument shows that there exists, in a neighborhood of s, another length-one complex $K^\bullet$ of free coherent $\mathcal{O}_S$-modules together with a quasi-isomorphism $u : K^\bullet \to \widetilde{K}^\bullet$, such that for the differential d we have $d|_s = 0$. Now set

$$\tau = u^*[-1]\phi\, u : K^\bullet \longrightarrow K^{\bullet *}[-1].$$

Then τ is a symmetric quasi-isomorphism inducing σ in cohomology, and τ_s is an isomorphism. Hence, in a neighborhood of s, τ_t remains an isomorphism, which proves the proposition. $\square$

Let $(K^\bullet, \tau)$ be as in the proposition and consider the following diagram:

$$\begin{tikzcd} K^0 \arrow[r, "\gamma"] \arrow[d, "\tau_0"'] & K^1 \arrow[d, "\tau_0^*"] \\ K^{1*} \arrow[r, "-\gamma^*"] & K^{0*} \end{tikzcd}$$

It follows that α is skew-symmetric. Therefore the cohomology of F may be represented, locally for the Zariski topology on S, by complexes of free coherent $\mathcal{O}_S$-modules

$$0 \longrightarrow K \xrightarrow{\alpha} K^* \longrightarrow 0,$$

with α skew-symmetric. Such complexes will be called **special** in the following.

6.3. The pfaffian bundle.

Let F be a vector bundle on X_S equipped with a nondegenerate quadratic form with values in ω_X, and cover S by Zariski open subsets U_i such that F admits a special representative $K_i^\bullet$ of the cohomology of F on U_i. Over U_i we have

$$\mathcal{D}_{F|U_i} = \bigwedge^{\max} K_i^* \otimes \bigwedge^{\max} K_i^*,$$

which is a square. This formula comes from the fact that $\mathcal{D}_F|_U = (\det K) \otimes (\det K^*)^{-1}$ and $\det K^* = (\det K)^{-1}$ via the differential $\alpha : K \to K^*$. The skew-symmetry of α induces a canonical orientation on the determinant lines.

It turns out, because the $K_i^\bullet$ are *special complexes*, that the $\bigwedge^{\max} K_i^*$ glue together over S and define a canonical square root of $\mathcal{D}_F$, called the *pfaffian bundle*.

Theorem 5.28 (6.3.1)**.** Let F be a vector bundle over X_S equipped with a nondegenerate quadratic form σ with values in ω_X. Then the determinant bundle $\mathcal{D}_F$ admits a canonical square root $\mathcal{P}_{(F,\sigma)}$. Moreover, if $f : S' \to S$ is a morphism of locally noetherian k-schemes, then

$$\mathcal{P}_{(f^*F, f^*\sigma)} = f^*\mathcal{P}_{(F,\sigma)}.$$

Example 5.29. Let $r \geq 3$ and (F, σ) be the universal SO_r-bundle over $\mathcal{M}_{\mathrm{SO}_r,X} \times X$. It comes with a non-degenerate form

$$\sigma : F \otimes F \to \mathcal{O}_{X \times \mathcal{M}_{\mathrm{SO}_r,X}}$$

because every SO_r-bundle has a non-degenerate quadratic form with values in $\mathcal{O}_X$ by definition.

If we twist by a theta-characteristic κ (i.e. a line bundle such that $\kappa \otimes \kappa = \omega_X$), then $F_\kappa = F \otimes q^*\kappa$ comes with a non-degenerate form with values in ω_X. Then we may apply the theorem in order to get the pfaffian bundle $\mathcal{P}_{(F_\kappa,\sigma)}$, which we denote simply by $\mathcal{P}_\kappa$.

Example 5.30 (The square-root of the dualizing sheaf)**.** Suppose G is semi-simple and consider its action on $\mathfrak{g}$ given by the adjoint representation. It follows from the proof of Proposition 4.29 that the dualizing sheaf $\omega_{\mathcal{M}_{G,X}}$ is $\mathcal{D}_{\mathcal{E}(\mathfrak{g})}$, where $\mathcal{E}$ is the universal G-bundle on $\mathcal{M}_{G,X}$. Remark that the bundle $\mathcal{E}(\mathfrak{g})$ comes with a natural quadratic form given by the Cartan–Killing form.

$$\begin{aligned} B : \mathcal{E}(\mathfrak{g}) \otimes \mathcal{E}(\mathfrak{g}) &\to \mathcal{O}_{\mathcal{M}_{G,X} \times X} \\ s \otimes t &\mapsto \operatorname{tr}(\operatorname{ad}(s) \operatorname{ad}(t)) \end{aligned}$$

Hence the choice of a theta-characteristic κ defines a skew symmetric form with values in ω_X on $\mathcal{E}(\mathfrak{g})_\kappa = \mathcal{E}(\mathfrak{g}) \otimes q^*\kappa$. By the previous theorem, we get a canonical square root of $\omega_{\mathcal{M}_{G,X}}$.

$$\omega^{1/2}_{\mathcal{M}_{G,X}}(\kappa)$$

6 The central extension

Recall that there is a canonical map of stacks $\pi : \mathcal{Q} = \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O}) \to \mathcal{SL}_X(r)$. Let $\mathcal{L}$ be the determinant line bundle on $\mathcal{SL}_X(r)$. It will turn out that, though it is invariant under the action of $\mathrm{SL}_r(K)$, this line bundle does not admit an action of $\mathrm{SL}_r(K)$. But it does admit an action of a canonical extension $\mathrm{SL}_{cr}(K)$ of $\mathrm{SL}_r(K)$.

6.1 Fredholm group

Let V be an infinite dimensional vector space over k. Let $\mathrm{End}^f(V)$ denote the two sided ideal generated by the finite rank endomorphisms of V, and let $\mathcal{F}(V) = (\mathrm{End}(V)/\mathrm{End}^f(V))^*$ be the equivalence classes of endomorphisms with finite dimensional kernel and cokernel. Let $\mathcal{F}(V)^0$ denote the subgroup of index 0 endomorphisms.

The map $\mathrm{Aut}(V) \hookrightarrow \mathrm{End}(V) \to \mathcal{F}(V)$ has image precisely $\mathcal{F}(V)^0$, and its kernel is those automorphisms $u \in \mathrm{Aut}(V)$ so that $u - I \in \mathrm{End}^f(V)$. The determinant of such u is well defined by the formula

$$\det(u) = \det(I + v) = \sum_{n \geq 0} \operatorname{tr} \Lambda^n v$$

so there is a short exact sequence

$$0 \to I + \operatorname{End}^f(V) \to \operatorname{Aut}(V) \to \mathcal{F}(V)^0 \to 0$$

Let $(I + \operatorname{End}^f(V))_1$ denote those automorphisms u such that $\det(u) = 1$. We get a short exact sequence

$$0 \to k^* \to \operatorname{Aut}(V)/(I + \operatorname{End}^f(V))_1 \to \mathcal{F}(V)^0 \to 0$$

For $v \in \operatorname{End}^f(V), u \in \operatorname{Aut}(V)$, we have

$$\det(I + uvu^{-1}) = \det(I + v)$$

so $\det(I+v)$ is invariant under conjugation and $I+v$ is in the center of $\operatorname{Aut}(V)/(I+\operatorname{End}^f(V))_1$. Thus we have defined a caonical central extension of the group $\mathcal{F}(V)^0$ by k^*.

6.2 Algebraic setting

Consider the k-space $\operatorname{End}(V)(R) = \operatorname{End}_R(V \otimes_k R)$, and has the k-group $\operatorname{Aut}(V)$ as its group of units. An endomorphism of $V \otimes R$ has finite rank if its image is contained in a finitely generated submodule, denote these endomorphisms $\operatorname{End}^f(V)$ and take $\mathcal{F}(V) = (\operatorname{End}(V)/\operatorname{End}^f(V))^*$. The group $\mathcal{F}(V)^0$ is defined as the image of $\operatorname{Aut}(V)$ in $\mathcal{F}(V)$. Then again there is a central extension

$$0 \to \mathfrak{G}_m \to \operatorname{Aut}(V)/(I + \operatorname{End}^f(V))_1 \to \mathcal{F}(V)^0 \to 0$$

Now consider the ind-group $\mathrm{GL}_r(K)$. Choose a supplement $K^r = V \oplus \mathcal{O}^r$ giving rise to a direct sum decomposition

$$R((z))^r = V \oplus R[[z]]^r$$

Let $\gamma \in \mathrm{GL}_r(R((z)))$ be a matrix with entries in $R((z))$ and decompose

$$\gamma = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$$

so $a(\gamma) : V_R \to V_R$ and consider the class $\bar{a}(\gamma) \in \operatorname{End}(V_R)/\operatorname{End}^f(V_R)$.

Proposition 6.1. The map

$$\gamma \mapsto \bar{a}(\gamma)$$

is a group homomorphism

$$\mathrm{GL}_r(R((z))) \to \mathcal{F}(V_R)$$

Another choice of supplement V' gives rise to $\bar{a}' : \mathrm{GL}_r(R((z))) \to \mathcal{F}(V'_R)$. Let $\phi : V \to V'$ be the map $V \to R((z))^r \to V'$. Then $\bar{a}'(\gamma) = \phi \circ \bar{a}(\gamma) \circ \phi^{-1}$.

Proposition 6.2. Let R be a k-algebra and γ an element of $\mathrm{SL}_r(R((z)))$; locally on $\mathrm{Spec}(R)$ (for the Zariski topology), the endomorphism $a(\gamma)$ of V_R is equivalent modulo $\mathrm{End}^f(V_R)$ to an automorphism.

It is enough to prove the result for one particular choice of V; we'll take $V = (z^{-1}k[z^{-1}])^r$. The assertion is clear when γ belongs to $\mathrm{SL}_r(R[[z]])$ or to $\mathrm{SL}_r(R[z^{-1}])$: in those cases the matrix (4.2) is triangular, so that $a(\gamma)$ itself is an isomorphism. The result then follows when R is a field, since any matrix $\gamma \in \mathrm{SL}_r(R((z)))$ can be written as a product of elementary matrices $I + \lambda E_{ij}$, where λ can be taken either in $R[[z]]$ or in $R[z^{-1}]$. The general case is a consequence of the following lemma:

Lemma 6.3. Locally over $\mathrm{Spec}(R)$, any element γ of $\mathrm{SL}_r(R((z)))$ can be written $\gamma_0\gamma^-\gamma^+$, with $\gamma_0 \in \mathrm{SL}_r(K)$, $\gamma^- \in \mathrm{SL}_r(R[z^{-1}])$, $\gamma^+ \in \mathrm{SL}_r(R[[z]])$.

Let us assume first that the k-algebra R is finitely generated. Let t be a closed point of $\mathrm{Spec}(R)$; put $\gamma_0 = \gamma(t)$. By (1.12) $\gamma_0^{-1}\gamma$ can be written in a neighborhood of t as $\gamma^-\gamma^+$, hence the result in this case.

In the general case, R is the union of its finitely generated subalgebras R_α. Let

$$p : \mathrm{SL}_r(K) \longrightarrow \mathcal{Q} = \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O})$$

be the quotient map. Since $\mathcal{Q}$ is an ind-variety, the morphism $p \circ \gamma : \mathrm{Spec}(R) \to \mathcal{Q}$ factors through $\mathrm{Spec}(R_\alpha)$ for some α. Locally over $\mathrm{Spec}(R_\alpha)$, this morphism can be written $p \circ \gamma_\alpha$ for some element γ_α of $\mathrm{SL}_r(R_\alpha((z)))$, which differs from γ by an element of $\mathrm{SL}_r(R[[z]])$ (thm. 2.5). Since R_α is of finite type, the lemma holds for γ_α, hence also for γ.

Corollary 6.4. The image of $\mathrm{SL}_r(K)$ by $\bar{a}$ is contained in the subgroup $\mathcal{F}(V)^0$.

Take the pullback of the central extension

$$0 \to \mathfrak{G}_m \to \mathrm{Aut}(V)/(I + \mathrm{End}^f(V))_1 \to \mathcal{F}(V)^0 \to 0$$

by $\bar{a}$ so that we get

$$0 \to \mathfrak{G}_m \to \hat{\mathrm{SL}}_r(K) \to \mathrm{SL}_r(K) \to 0$$

where $\hat{\mathrm{SL}}_r(K)$ is the central extension of $\mathrm{SL}_r(K)$ by $\mathfrak{G}_m$. It is also an ind-group.

Explicitly, an element of $\mathrm{SL}_r(K)(R)$ is given, locally on $\mathrm{Spec}\, R$, by a pair (γ, u) with γ in $\mathrm{SL}_r(R((z)))$, u in $\mathrm{Aut}(V_R)$, and $u \equiv a(\gamma) \bmod \mathrm{End}^f(V_R)$. Two pairs (γ, u) and (γ', u') give the same element of $\hat{\mathrm{SL}}_r(K)(R)$ if $u^{-1}u'$ (which is in $I + \mathrm{End}^f(V_R)$) has determinant 1. The map

$$\psi : \hat{\mathrm{SL}}_r(K)(R) \longrightarrow \mathrm{SL}_r(K)(R)$$

is given by $\psi(\gamma, u) = \gamma$. The kernel of ψ consists of the pairs (I, u) with $u \in \mathrm{Aut}(V_R)$, modulo the pairs (I, u) with $\det u = 1$; the map $u \mapsto \det u$ provides an isomorphism from $\ker \psi$ onto $\mathfrak{G}_m(R)$.

Remark 6.5. The interpretation of the central extension is as follows. Consider what happens when $\gamma \in \mathrm{SL}_r(R((z)))$ acts on V_R. The action $a(\gamma)$ is given by embedding V_R into $R((z))^r$, and then applying the matrix γ and projecting back to V_R. This map determined by γ fails to be an automorphism, but we showed that if we perturb it by a finite rank endomorphism, we can get an automorphism. However we can get different automorphisms, and the central extension carries around the data of the particular automorphism we choose to represent the element γ. Two automorphisms are equivalent precisely when $\det(u^{-1}u') = 1$ which makes a $\mathfrak{G}_m$-torsor.

Remark 6.6. Let H be a sub-k-group of $\mathrm{SL}_r(K)$ such that $\mathcal{O}^r$ (resp. V) is stable under H. Then the extension $(\mathcal{E})$ is canonically split over H. For any element $\gamma \in H(R)$, we have $b(\gamma) = 0$ (resp. $c(\gamma) = 0$), so that the map $\gamma \mapsto a(\gamma)$ is a homomorphism from $H(R)$ into $\mathrm{Aut}(V_R)$. The map $\gamma \mapsto (\gamma, a(\gamma))$ defines a section of ψ over H. In particular, the pullback $\widehat{\mathrm{SL}}_r(\mathcal{O})$ of $\mathrm{SL}_r(\mathcal{O})$ is canonically isomorphic to $\mathrm{SL}_r(\mathcal{O}) \times \mathfrak{G}_m$.

We denote by $\chi_0 : \widehat{\mathrm{SL}}_r(\mathcal{O}) \to \mathfrak{G}_m$ the second projection. If an element $\tilde{\delta} \in \widehat{\mathrm{SL}}_r(\mathcal{O})(R)$ is represented by a pair (δ, v), then

$$\chi_0(\tilde{\delta}) = \det(a(\delta)^{-1}v).$$

More generally, suppose there exists an element $\lambda \in \mathrm{SL}_r(K)$ such that the subgroup H preserves the subspace $\lambda(\mathcal{O}^r)$ (resp. $\lambda(V)$). Choose an automorphism u of V such that $u \equiv a(\lambda) \pmod{\mathrm{End}^f(V)}$, and define a section of ψ over H by

$$\gamma \mapsto (\gamma, ua(\lambda^{-1}\gamma\lambda)u^{-1}).$$

This section is independent of the choice of u, so once again the group H embeds canonically into $\widehat{\mathrm{SL}}_r(K)$.

Let $\tilde{\gamma}$ an element of $\widehat{\mathrm{SL}}_r(K)(R)$. Locally on $\mathrm{Spec}(R)$ write $\tilde{\gamma} = (\gamma, u)$ with γ in $\mathrm{SL}_r(R((z)))$, $u \in \mathrm{Aut}(V_R)$, and $u \equiv a(\gamma) \pmod{\mathrm{End}^f(V_R)}$. We associate to this pair the element

$$\tau_V(\gamma, u) := \det(ua(\gamma^{-1}))$$

of R. This is clearly well-defined, so we get an algebraic function τ_V on $\widehat{\mathrm{SL}}_r(K)$.

Proposition 6.7. Let R be a k-algebra, $\tilde{\gamma}$ an element of $\widehat{\mathrm{SL}}_r(K)(R)$, γ its image in $\mathrm{SL}_r(R((z)))$. One has

$$\tau_V(\tilde{\gamma}\tilde{\delta}) = \chi_0(\tilde{\delta})\tau_V(\tilde{\gamma})$$

for all $\tilde{\delta}$ in $\widehat{\mathrm{SL}}_r(\mathcal{O})(R)$.

Proof. Let us choose representatives (γ, u) of $\tilde{\gamma}$ and (δ, v) of $\tilde{\delta}$. Since $b(\delta^{-1}) = 0$, one has

$$a(\delta^{-1}\gamma\gamma^{-1}) = a(\delta^{-1})a(\gamma^{-1}),$$

and

$$\tau_V(\tilde{\gamma}\tilde{\delta}) = \det(uva(\delta^{-1})a(\gamma^{-1})) = \det(va(\delta^{-1}))\det(ua(\gamma^{-1})) = \chi_0(\tilde{\delta})\tau_V(\tilde{\gamma}).$$

as claimed. □

Let us denote by χ the character χ_0^{-1} of $\widehat{\mathrm{SL}}_r(\mathcal{O})$. The function τ_V thus defines a section of the line bundle $\mathcal{L}_\chi$ on the ind-variety

$$\mathcal{Q} = \widehat{\mathrm{SL}}_r(K)/\widehat{\mathrm{SL}}_r(\mathcal{O})$$

More generally, let $\delta \in \mathrm{SL}_r(K)$, and let $\tilde{\delta}$ be a lift of δ in $\widehat{\mathrm{SL}}_r(K)$; the function

$$\tilde{\gamma} \mapsto \tau_V(\tilde{\delta}^{-1}\tilde{\gamma})$$

still defines an element of $H^0(\mathcal{Q}, \mathcal{L}_\chi)$, whose divisor is $\delta(\mathrm{div}(\tau_V))$.

6.3 The determinant bundle

We will now compare the pullback over $\mathcal{Q}$ of the determinant line bundle $\mathcal{L}$ on the moduli stack with the line bundle $\mathcal{L}_\chi$.

Proposition 6.8. Let R be a k-algebra, γ an element of $\mathrm{GL}_r(R((z)))$, and (E, ρ, σ) the corresponding triple over X_R. There is a canonical exact sequence

$$0 \to H^0(X_R, E) \to A_X^r \otimes_k R \to (R((z))/R[[z]])^r \to H^1(X_R, E) \to 0$$

where $\gamma : A_X^r \otimes_k R \to (R((z))/R[[z]])^r$ is the composition of the injection $A_X^r \otimes_k R \to R((z))^r$, the automorphism γ^{-1} of $R((z))^r$, and the projection $R((z))^r \to (R((z))/R[[z]])^r$.

Proof. Recall that there is a short exact sequence

$$0 \to \mathcal{O}_X \to j_* O_{X^*} \to f_*(\mathcal{K}_D/\mathcal{O}_D) \to 0$$

where $j : X^* \to X$ and $f : D \to X$ is the inclusion. The term $j_* O_{X^*}$ represents those sections of O_X which are regular on X^*. The term $K_D/\mathcal{O}_D$ represents the quotient of Laurent series by regular functions, and then we push it forward to X. The exact sequence says that a function on X^* comes from a function on X precisely when it has no poles along D. Tensoring with E and using our trivializations ρ, σ we get an exact sequence (tensoring with E is exact because E is locally free)

$$0 \to E \to j_* O_{X^*} \to f_*(\mathcal{K}_D/\mathcal{O}_D) \to 0$$

Now take the long exact sequence in cohomology. □

Choose an element $\gamma_0 \in \mathrm{GL}_r(K)$ so that $\bar{\gamma}_0 : A_X^r \to (K/\mathcal{O})^r$ is an isomorphism, i.e. so that E has no cohomology, i.e. so that $V = \gamma_0^{-1}(A_X^r)$ is a supplement of $\mathcal{O}^r$ in K^r. Identifying A_X^r with V and the quotient map $K^r \to K^r/\mathcal{O}^r$ with the projection of K^r onto V, we obtain that $\bar{\gamma}$ is the composition of the mappings

$$V \to K^r \xrightarrow{\gamma^{-1}\gamma_0} K^r \to V$$

so that $\bar{\gamma}$ is the coefficient $a(\gamma^{-1}\gamma_0)$ of the matrix $\gamma^{-1}\gamma_0$ in the decomposition $K^r = V \oplus \mathcal{O}^r$.

Therefore we have shown the following.

Proposition 6.9. Let $\gamma \in \mathrm{GL}_r(R((z)))$, and let E be the associated vector bundle over X_R. There is a canonical exact sequence:

$$0 \to H^0(X_R, E) \to V_R \xrightarrow{a(\gamma^{-1}\gamma_0)} V_R \to H^1(X_R, E) \to 0.$$

Corollary 6.10. Assume that there exists an automorphism u of V_R such that $u \equiv a(\gamma_0^{-1}\gamma) \pmod{\mathrm{End}^f(V_R)}$. Then there is an exact sequence:

$$0 \to H^0(X_R, E) \to V_0 \xrightarrow{v_0} V_0 \to H^1(X_R, E) \to 0,$$

where V_0 is a free finitely generated R-module, and $\det(v_0) = \tau_V(\gamma_0^{-1}\gamma, u)$.

Proof. Let $v = u \cdot a(\gamma^{-1}\gamma_0) \in I + \mathrm{End}^f(V_R)$, and let V_0 be a free finitely generated direct factor of V_R containing $\mathrm{Im}(v - I)$. Denote by v_0 the restriction of v to V_0. The matrix of v relative to a direct sum decomposition $V_R = V_0 \oplus V_1$ is of the form:

$$\begin{pmatrix} v_0 & * \\ 0 & I \end{pmatrix},$$

from which it follows that $\det v_0 = \det v = \tau_V(\gamma_0^{-1}\gamma, u)$. It also follows that $\ker v_0 = \ker v$, and the inclusion $V_0 \hookrightarrow V_R$ induces an isomorphism $\mathrm{Coker}\, v_0 \cong \mathrm{Coker}\, v$. $\square$

Remark 6.11. One can think of the map $\tau_V(\gamma_0^{-1}\gamma, u)$ as a determinant associated with the difference between two ways of representing the action of $\gamma_0^{-1}\gamma$ on V_R. Specfically, when we have $u \equiv a(\gamma_0^{-1}\gamma) \pmod{\mathrm{End}^f(V_R)}$, the difference $u - a(\gamma_0^{-1}\gamma)$ is a finite rank endomorphism of V_R, and the determinant $\det(u - a(\gamma_0^{-1}\gamma))$ is well defined. The map τ_V is a way of encoding this information in a more algebraic form, and it allows us to compare different representations of the same action.

The corollary says that we can "finitize" the cohomology computaiton to a finite-dimensional submodule V_0, and this localized computation preserves the key determinant information.

7 Appendix: Morphisms of Schemes

Other notions for morphisms of schemes that we will not need, but still worth mentioning and defining.

Definition 7.1. Let $f : X \to Y$ be a morphism of schemes.

1. f is **affine** if for every affine open subset $V = \operatorname{Spec}(B) \subset Y$, the preimage $f^{-1}(V)$ is affine. Equivalently, there exists an affine open cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i.

2. $f : X \to Y$ is **finite** if for every affine open subset $V = \operatorname{Spec}(B) \subset Y$, the preimage $f^{-1}(V) = \operatorname{Spec}(A)$ where A is a finite B-algebra (i.e., A is finitely generated as a B-module).

3. f is **of finite type** if it is locally of finite type and quasi-compact.

4. f is **quasicompact** if for every quasi-compact open subset $V \subset Y$, the preimage $f^{-1}(V)$ is quasi-compact.

5. $f : X \to Y$ is **separated** if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion.

6. $f : X \to Y$ is **quasi-separated** if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is quasi-compact.

7. $f : X \to Y$ is **proper** if it is separated, of finite type, and universally closed (the image of a closed subset remains closed after any base change).

8. $f : X \to Y$ is **unramified** at a point $x \in X$ if:

   (a) The extension of residue fields $\kappa(x)/\kappa(f(x))$ is finite and separable.

   (b) The cotangent space of the fiber at x, $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}/\mathfrak{m}^2_{f(x)}\mathcal{O}_{X,x}$, vanishes.

   It is **unramified** if it is unramified at every point of X.

9. A morphism $f : X \to Y$ is **formally smooth** (resp. **formally unramified**, **formally étale**) if for every affine Y-scheme $Y' \to Y$ and every closed immersion $Y'_0 \to Y'$ defined by a nilpotent ideal, the map

$$\operatorname{Hom}_Y(Y', X) \to \operatorname{Hom}_Y(Y'_0, X)$$

   is surjective (resp. injective, bijective).

10. A morphism $f : X \to Y$ is **smooth** (resp. **unramified**, **étale**) if it is formally smooth (resp.

formally unramified, formally étale) and locally of finite presentation.

11. A morphism $f : X \to Y$ is **smooth** of relative dimension n if it is flat, locally of finite presentation, and for each point $x \in X$, the fiber $X_{f(x)}$ is a smooth variety of dimension n over $\kappa(f(x))$.

12. A morphism $f : X \to Y$ is an **open immersion** if it induces a homeomorphism of X onto an open subset of Y and the induced map $f^\sharp : \mathcal{O}_Y|_{f(X)} \to f_*\mathcal{O}_X$ is an isomorphism.

13. A morphism $f : X \to Y$ is a **closed immersion** if it induces a homeomorphism of X onto a closed subset of Y and the induced map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

14. A morphism $f : X \to Y$ is **quasi-finite** at a point $x \in X$ if there exist open neighborhoods U of x and V of $f(x)$ such that $f|_U : U \to V$ has finite fibers. It is **quasi-finite** if it is quasi-finite at every point of X.

Theorem 7.2. For a morphism of schemes $f : X \to Y$, the following are equivalent:

1. f is formally smooth and locally of finite presentation.

2. f is flat, locally of finite presentation, and has geometrically regular fibers.

Proof. We will prove both implications to establish the equivalence.

(1) ⇒ (2): Assume f is formally smooth and locally of finite presentation.

We need to establish that f is flat and has geometrically regular fibers.

Step 1: Proving flatness.

Let $x \in X$ be a point and $y = f(x) \in Y$. We need to show that $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{Y,y}$-module. By standard criteria for flatness, it suffices to show that for every finitely generated ideal $I \subset \mathcal{O}_{Y,y}$, the natural map

$$\varphi : I \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to I\mathcal{O}_{X,x} \tag{5}$$

is an isomorphism, or equivalently, that $\mathrm{Tor}_1^{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}/I, \mathcal{O}_{X,x}) = 0$.

Since f is formally smooth, by definition, for every affine Y-scheme Y', every closed subscheme $Y'_0 \subset Y'$ defined by a nilpotent ideal J, and every Y-morphism $g_0 : Y'_0 \to X$, there exists a Y-morphism $g : Y' \to X$ extending g_0.

For our purposes, we consider the specific case where:

$$Y' = \mathrm{Spec}(\mathcal{O}_{Y,y}/I^2) \tag{6}$$
$$Y'_0 = \mathrm{Spec}(\mathcal{O}_{Y,y}/I) \tag{7}$$

The ideal $J = I/I^2$ is nilpotent in $\mathcal{O}_{Y,y}/I^2$ with $J^2 = 0$.

The obstruction to lifting $g_0 : Y'_0 \to X$ to $g : Y' \to X$ lies in

$$\mathrm{Ext}^1_{\mathcal{O}_{Y'_0}}(g_0^* L_{X/Y}, I/I^2) \tag{8}$$

where $L_{X/Y}$ is the cotangent complex of f.

Since f is formally smooth, this obstruction vanishes for all possible g_0. Moreover, as f is locally of finite presentation, the cotangent complex $L_{X/Y}$ is perfect and concentrated in degrees $[-1, 0]$.

By deformation theory, there is a connection between these Ext groups and the Tor groups relevant to flatness. Specifically, the vanishing of the obstruction for all g_0 implies that

$$\mathrm{Tor}_1^{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}/I, \mathcal{O}_{X,x}) = 0 \tag{9}$$

This connection is established through the local-to-global spectral sequence relating Ext groups of the cotangent complex to appropriate Tor groups. For a formally smooth morphism, the cotangent complex is quasi-isomorphic to the module of differentials placed in degree 0, which simplifies these relationships.

As this holds for all finitely generated ideals $I \subset \mathcal{O}_{Y,y}$, we conclude that $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$. Since this applies to all points $x \in X$, the morphism f is flat.

I am not confident in the truth of what is written here. It is also not complete. $\square$

Remark 7.3 (Properties of morphisms)**.** The following properties of morphisms of schemes are related in the following way:

1. finite $\Rightarrow$ proper $\Rightarrow$ separated
2. finite $\Rightarrow$ affine $\Rightarrow$ quasi-affine
3. finite $\Rightarrow$ quasi-finite
4. étale $\Rightarrow$ smooth $\Rightarrow$ flat

5. étale $\Rightarrow$ unramified

6. locally of finite presentation $\Rightarrow$ locally of finite type

7. proper + flat + finite type + locally of finite presentation $\Rightarrow$ cohomologically flat

Properties preserved under composition include: affine, finite, (locally) of finite type, (locally) of finite presentation, quasi-compact, separated, proper, closed immersion, and flat.

Properties preserved under base change include: affine, finite, (locally) of finite type, (locally) of finite presentation, flat, unramified, étale, smooth, open immersion, closed immersion, and proper.

8 Appendix: Associated Bundles

Let G be a group scheme and let $P \to X$ be a principal G-bundle over a scheme X. Suppose we have a scheme F equipped with a (left) G-action. We can construct the associated bundle with fiber F, denoted $P \times^G F$, as follows.

Consider the product $P \times F$ with the diagonal G-action given by $g \cdot (p, f) = (p \cdot g^{-1}, g \cdot f)$ for $g \in G$, $p \in P$, and $f \in F$. The associated bundle $P \times^G F$ is defined as the quotient of $P \times F$ by this G-action:

$$P \times^G F = (P \times F)/G$$

More precisely, $P \times^G F$ can be constructed as the sheafification of the presheaf quotient $(P \times F)/G$ in the appropriate topology (étale, fppf, etc.). This construction yields a bundle $\pi : P \times^G F \to X$ where the fiber over each point $x \in X$ is isomorphic to F.

We can also go in the other direction - starting from a bundle with fiber F and constructing a principal bundle.

Definition 8.1 (Frame Bundle)**.** Let $\pi : E \to X$ be a bundle whose fibers are isomorphic to a scheme F on which G acts. The **frame bundle** of E, denoted $\mathrm{Fr}_G(E)$, is the X-scheme representing the functor that assigns to each X-scheme T the set of G-equivariant isomorphisms:

$$\mathrm{Fr}_G(E)(T) = \{\phi : T \times F \xrightarrow{\sim} E \times_X T \text{ (as } T\text{-schemes)} \mid \phi \text{ is } G\text{-equivariant}\}$$

Proposition 8.2. Let $\pi : E \to X$ be a bundle with fiber F.

1. The frame bundle $\mathrm{Fr}_G(E)$ is a principal G-bundle over X.

2. If $E = P \times^G F$ is an associated bundle for some principal G-bundle P, then $\mathrm{Fr}_G(E) \cong P$.

3. For any bundle E with fiber F, we have $E \cong \mathrm{Fr}_G(E) \times^G F$.

This establishes a correspondence between principal G-bundles and bundles with fiber F (with G-action), showing that these two perspectives are equivalent.

Example 8.3. Let $E \to X$ be a vector bundle of rank n. Then the frame bundle $\mathrm{Fr}_{\mathrm{GL}_n}(E)$ is the principal GL_n-bundle whose fiber at $x \in X$ consists of all bases of the vector space E_x. Conversely, given a principal GL_n-bundle P, the associated bundle $P \times^{\mathrm{GL}_n} \mathbb{A}^n$ is a vector bundle of rank n.

9 References

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