Homework 1

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Problem 1 Show that the n-sheeted Riemann surface of the multi-valued function

$$w = z^{1/n}, \quad z \in \mathbb{C},$$

is topologically a sphere with 1 puncture.

Solution: Let $\mathcal{R} = \{(z, w) \in \mathbb{C}^2 : \ w^n = z\}$. $\mathcal{R}$ carries the structure of a Riemann surface so that the projection $\pi : \mathcal{R} \to \mathbb{C}, \ \pi(z, w) = z$ is holomorphic. Now consider the map

$$\Phi : \mathbb{C} \longrightarrow \mathcal{R}, \qquad \Phi(w) = (w^n, w)$$

Φ is bijective: given $(z, w) \in \mathcal{R}$ we must have $z = w^n$, so the inverse is simply $(z, w) \mapsto w$. Φ and its inverse are holomorphic because one is given by a polynomial, the other is a projection. Hence Φ is a biholomorphism. Therefore $\mathcal{R}$ is (as a Riemann surface, hence also topologically) just $\mathbb{C}$. Topologically, $\mathbb{C}$ is a sphere with one point removed (a "punctured sphere"): $\mathbb{C} \simeq \widehat{\mathbb{C}} \setminus \{\infty\}$. Thus the n-sheeted Riemann surface of $w = z^{1/n}$ is topologically a sphere with one puncture.

Problem 2 Let $f(z)$ be a polynomial of odd degree, with simple zeroes. Identify the topology of the Riemann surface of the double-valued function defined by $w^2 = f(z)$.

Solution: Consider the affine curve $X_{\text{aff}} = \{(z, w) \in \mathbb{C}^2 : w^2 = f(z)\}$. Its projection $\pi_{\text{aff}} : (z, w) \mapsto z$ is a 2-sheeted branched covering of $\mathbb{C}$ away from the zeros of f. We compactify to a projective curve $X = \overline{X_{\text{aff}}} \subset \mathbb{P}^1_z \times \mathbb{P}^1_w$ and extend the projection to $\pi : X \longrightarrow \mathbb{P}^1_z$. The map π has degree 2. To study the topology of X_{aff}, we will use the Riemann-Hurwitz formula to compute the genus of X and delete the point(s) over $z = \infty$.

If a is a simple zero of f, write locally $f(z) = (z-a)u(z)$ with $u(a) \neq 0$. Then $w^2 = (z-a)u(z)$ has a single point of X lying over $z = a$ and the local model is $w^2 = z - a$, so the ramification index is $e = 2$. Thus each simple zero gives one branch point of ramification index 2. There are d of these in $\mathbb{C}$. Put $t = 1/z$ as a coordinate near $z = \infty$ and write

$$f(z) = z^d \, g(1/z) = t^{-d} \, g(t), \qquad g(0) \neq 0$$

The equation becomes $w^2 = t^{-d} g(t) \iff (w \, t^{\frac{d-1}{2}})^2 = t^{-1} g(t)$. Let $u = w \, t^{\frac{d-1}{2}}$. Then $u^2 = t^{-1} g(t)$, so near $t = 0$ we have the model $u^2 \sim t^{-1}$. Therefore, there is one point of X over $z = \infty$ and it is ramified of order 2. Hence the total number of simple branch points is $B = d + 1$.

Apply Riemann-Hurwitz to the degree-2 map $\pi : X \to \mathbb{P}^1$:

$$2g(X) - 2 = 2 \cdot (-2) + \sum_{p \in X} (e_p - 1).$$

Every simple ramification contributes $e_p - 1 = 1$, so

$$2g(X) - 2 = -4 + B = -4 + (d+1) = d - 3.$$

Therefore

$$g(X) = \frac{d-1}{2}.$$

The compact Riemann surface X is a closed orientable surface of genus $g = \frac{d-1}{2}$. Recall that there is only one point of X over $z = \infty$. Therefore, X_{aff} is homeomorphic to X with one point removed. Hence X_{aff} is homeomorphic to a genus $\frac{d-1}{2}$ surface with one puncture.

Problem 3 Show that a bijective holomorphic map

$$f : R \to S$$

between Riemann surfaces is in fact bi-holomorphic (meaning, the inverse is also holomorphic). Show that two homeomorphic Riemann surfaces need not be bi-holomorphic. *(Hint: Use the unit disk Δ and the complex plane.)* Show that no two of the following three annuli in $\mathbb{C}$ are bi-holomorphic:

$$\begin{aligned} &(a)\ \{z \mid 0 < |z| < 1\}, \\ &(b)\ \{z \mid 1 < |z| < 2\}, \\ &(c)\ \{z \mid 0 < |z| < \infty\}. \end{aligned}$$

Problem 4 Prove the *Weierstrass division theorem*: Given a polynomial

$$P(w, z_1, \ldots, z_n) = w^n + \sum_{k=0}^{n-1} p_k(z) w^k,$$

with the functions $p_k(z)$ holomorphic in an open set $V \subset \mathbb{C}^n$ and satisfying $p_k(0) = 0$, every germ of holomorphic function $G(w, z)$ near $(w, z) = (0, 0)$ can be uniquely expressed as

$$G(w, z) = P(w, z) \cdot Q(w, z) + R(w, z),$$

where $Q(w, z)$ is a holomorphic germ near 0 and $R(w, z)$ is a polynomial in w of degree $< n$ with coefficients germs of holomorphic functions in z near $z = 0$.

To do this, define

$$Q(z, w) = \frac{1}{2\pi i} \oint_\Gamma \frac{G(\zeta, z)}{P(\zeta, z)(\zeta - w)}\, d\zeta$$

for a suitable choice of the line integral over each fixed value of z, and show that the difference

$$R(w, z) := G(w, z) - P(w, z) \cdot Q(w, z)$$

is a holomorphic function of (w, z) which is polynomial in w with degree $< n$. *Hint:* You will want to express that difference as a Cauchy integral to get your conclusion.

Problem 5 A *Reinhardt domain* $R \subset \mathbb{C}^n$ is an open set such that

$$(z_1, \ldots, z_n) \in R \;\Rightarrow\; (qz_1, \ldots, qz_n) \in R, \quad \forall q \in \mathbb{C} \text{ with } |q| < 1.$$

(a) Show that the intersection of finitely many Reinhardt domains is Reinhardt.

(b) Show that if a multi-variable power series centered at 0 converges at some point $(z_1, \ldots, z_n) \in \mathbb{C}^n$, then it converges uniformly in some Reinhardt domain containing z.

(c) Prove that the *domain of convergence* of an n-variable Taylor series centered at 0 — defined as the interior of the set of points where the series converges — is a Reinhardt domain.

Problem 6 Let C_1 and C_2 be two circles in the w- and z-planes in $\mathbb{C}^2$, and $\Delta_{1,2}$ the disks that they bound. Show that a holomorphic function defined in an open set containing

$$C_1 \times \Delta_2 \;\cup\; \Delta_1 \times C_2$$

has a unique holomorphic extension over $\Delta_1 \times \Delta_2$. *Hint:* Use Cauchy's formula in a way very similar to the one exploited above.

Problem 7 Let F, G be two irreducible holomorphic functions in $n > 1$ variables defined on an open set U, and call their common zero-set Z. Using the Weierstrass Preparation Theorem (twice) and Q6, show that any holomorphic function defined on $U \setminus Z$ extends holomorphically over Z.

Remark 1. This is a version of *Hartogs' theorem* for holomorphic functions of several variables; somewhat loosely, the singular set of a holomorphic function defined on "most of" an open $U \subset \mathbb{C}^n$ cannot lie in an analytic subset of co-dimension 2, unless it's empty. Contrast that with the real function $1/(x^2 + y^2)$ on $\mathbb{R}^2$.