

Homework 5

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For Questions 1 and 2, you may use the correspondence indicated in class between the representation of H^1 classes by classes by principal parts versus Dolbeault distributions.

Problem 1 For a compact Riemann surface R , verify that the Serre duality pairing

$$H^1(R; \mathcal{O}) \otimes H^0(R; \Omega^1) \longrightarrow \mathbb{C}$$

defined by principal parts and residues agrees with the one given by integration of Dolbeault representatives.

Using the relation to harmonic forms, explain how this relates to Poincaré duality on R .

Solution: Choose a meromorphic function f on R whose principal part at each p_i with prescribed principal parts. Let U_i be pairwise disjoint coordinate discs around p_i , and choose $\chi \in C^\infty(R)$ such that $\chi \equiv 1$ on smaller discs $U'_i \subset U_i$ and $\chi \equiv 0$ outside $\bigcup_i U_i$. Define a $(0, 1)$ -current

$$T_f := \bar{\partial}(\chi f).$$

Since $\bar{\partial}^2 = 0$, T_f is $\bar{\partial}$ -closed. If we replace f by $f + g$ for a global meromorphic function g (with poles in D) or change χ within the same constraints, T_f changes by a current of the form $\bar{\partial}u$, so the class $[T_f]$ in

$$H_{\bar{\partial}}^{0,1}(R) \cong H^1(R, \mathcal{O})$$

depends only on the underlying principal parts.

Let $\omega \in H^0(R, \Omega^1)$ be a holomorphic 1-form. The *Dolbeault* definition of the pairing is

$$\langle \alpha, \omega \rangle_{Dol} := \frac{1}{2\pi i} \int_R T_f \wedge \omega = \frac{1}{2\pi i} \int_R \bar{\partial}(\chi f) \wedge \omega.$$

Since ω is of type $(1, 0)$ and holomorphic, $\bar{\partial}\omega = 0$, hence

$$\bar{\partial}(\chi f) \wedge \omega = \bar{\partial}(\chi f \omega).$$

Let $D_i \subset U'_i$ be small closed discs around p_i and set

$$R_\varepsilon := R \setminus \bigcup_i D_i(\varepsilon),$$

where $D_i(\varepsilon)$ are concentric discs of radius ε . On R_ε the form $\chi f \omega$ is smooth with compact support, so Stokes' theorem gives

$$\int_{R_\varepsilon} \bar{\partial}(\chi f \omega) = \int_{\partial R_\varepsilon} \chi f \omega = - \sum_i \int_{\partial D_i(\varepsilon)} f \omega,$$

the sign coming from the induced orientation on the boundary.

Letting $\varepsilon \rightarrow 0$ and using the residue theorem,

$$\int_{\partial D_i(\varepsilon)} f\omega \longrightarrow 2\pi i \operatorname{Res}_{p_i}(f\omega),$$

we obtain

$$\frac{1}{2\pi i} \int_R \bar{\partial}(\chi f) \wedge \omega = \sum_i \operatorname{Res}_{p_i}(f\omega).$$

This is precisely the *principal parts* definition of the Serre pairing.

Now equip R with any Hermitian (necessarily Kähler) metric. Hodge theory yields the decompositions

$$H_{\text{dR}}^1(R, \mathbb{C}) \cong \mathcal{H}^1(R) \cong H_{\bar{\partial}}^{1,0}(R) \oplus H_{\bar{\partial}}^{0,1}(R),$$

and every class has a unique harmonic representative. Moreover,

$$H^0(R, \Omega^1) \cong H_{\bar{\partial}}^{1,0}(R)$$

consists of harmonic $(1, 0)$ -forms, and

$$H^1(R, \mathcal{O}) \cong H_{\bar{\partial}}^{0,1}(R)$$

is represented by harmonic $(0, 1)$ -forms. Complex conjugation gives an isomorphism

$$\overline{H_{\bar{\partial}}^{1,0}(R)} \cong H_{\bar{\partial}}^{0,1}(R)$$

Poincaré duality on R is given by the nondegenerate pairing

$$H_{\text{dR}}^1(R, \mathbb{C}) \times H_{\text{dR}}^1(R, \mathbb{C}) \longrightarrow \mathbb{C}, \quad ([\alpha], [\beta]) \mapsto \int_R \alpha \wedge \beta.$$

It is clear that $\alpha \wedge \beta$ is nonzero only if α and β are of complementary types, i.e. their wedge is of type $(1, 1)$, since $(1, 0) \wedge (1, 0)$ and $(0, 1) \wedge (0, 1)$ necessarily vanish. Thus the Poincaré pairing restricts to a nondegenerate pairing

$$H_{\bar{\partial}}^{0,1}(R) \otimes H_{\bar{\partial}}^{1,0}(R) \longrightarrow \mathbb{C}, \quad (\eta, \omega) \mapsto \int_R \eta \wedge \omega,$$

with η, ω harmonic representatives.

Under the identifications

$$H^1(R, \mathcal{O}) \cong H_{\bar{\partial}}^{0,1}(R), \quad H^0(R, \Omega^1) \cong H_{\bar{\partial}}^{1,0}(R),$$

the Serre pairing of α and ω is

$$\langle \alpha, \omega \rangle = \frac{1}{2\pi i} \int_R \eta \wedge \omega,$$

where η is the harmonic $(0, 1)$ -representative of α . In particular, on a compact Riemann surface the Serre duality

$$H^1(R, \mathcal{O}) \cong H^0(R, \Omega^1)^\vee$$

is nothing but Poincaré duality in degree 1 up to the constant factor $2\pi i$, expressed via the Hodge decomposition of $H_{\text{dR}}^1(R, \mathbb{C})$.

Problem 2 For a compact Riemann surface R , verify that the map

$$H^1(R; \mathbb{Z}) \longrightarrow H^1(R; \mathcal{O})$$

corresponds to the period map

$$H_1(R; \mathbb{Z}) \otimes H^0(R; \Omega^1) \longrightarrow \mathbb{C}$$

under integral Poincaré duality and Serre duality on R .

Solution: Let $i : H^1(R; \mathbb{Z}) \rightarrow H^1(R; \mathcal{O})$ be the given homomorphism. We need to show for every $c \in H^1(R; \mathbb{Z})$ and $\omega \in H^0(R, \Omega^1)$, the Serre pairing $\langle i(c), \omega \rangle_{\text{Serre}}$ equals the period of ω along the 1-cycle Poincaré dual to c .

By Hodge theory, every class in $H^1(R; \mathbb{R})$ has a unique harmonic representative. An element $c \in H^1(R; \mathbb{Z})$ maps to a real class $c_{\mathbb{R}} \in H^1(R; \mathbb{R})$ whose harmonic representative we denote by α so

$$[\alpha]_{\text{dR}} = c_{\mathbb{R}} \in H^1_{\text{dR}}(R; \mathbb{R}).$$

Decompose α

$$\alpha = \alpha^{1,0} + \alpha^{0,1}, \quad \alpha^{0,1} = \overline{\alpha^{1,0}},$$

since α is real. Under the Dolbeault isomorphism and Hodge decomposition, we have

$$H^1(R, \mathcal{O}) \cong H_{\bar{\partial}}^{0,1}(R)$$

and the image $i(c) \in H^1(R, \mathcal{O})$ is represented by the harmonic $(0, 1)$ -form $\alpha^{0,1}$.

We know that the Serre pairing can be described as

$$\langle \beta, \omega \rangle_{\text{Serre}} = \frac{1}{2\pi i} \int_R \eta^{0,1} \wedge \omega$$

whenever $\beta \in H^1(R, \mathcal{O})$ is represented by a harmonic $(0, 1)$ -form $\eta^{0,1}$ and $\omega \in H^0(R, \Omega^1)$ is a holomorphic 1-form.

Applying this to $\beta = i(c)$ and $\eta^{0,1} = \alpha^{0,1}$ gives

$$\langle i(c), \omega \rangle_{\text{Serre}} = \frac{1}{2\pi i} \int_R \alpha^{0,1} \wedge \omega.$$

Since R has complex dimension 1, a $(2, 0)$ -form vanishes, hence $\alpha^{1,0} \wedge \omega = 0$, and therefore

$$\alpha^{0,1} \wedge \omega = (\alpha^{1,0} + \alpha^{0,1}) \wedge \omega = \alpha \wedge \omega.$$

Thus

$$\langle i^*c, \omega \rangle_{\text{Serre}} = \frac{1}{2\pi i} \int_R \alpha \wedge \omega. \tag{1}$$

Integral Poincaré duality gives a perfect pairing

$$H^1(R; \mathbb{Z}) \times H_1(R; \mathbb{Z}) \longrightarrow \mathbb{Z},$$

and we denote by $\gamma_c \in H_1(R; \mathbb{Z})$ the Poincaré dual of c .

The de Rham realization of this pairing is as follows. The class $c_{\mathbb{R}} \in H^1(R; \mathbb{R})$ is represented by the closed 1-form α with integral periods, i.e.

$$\int_{\gamma} \alpha \in \mathbb{Z} \quad \text{for all } \gamma \in H_1(R; \mathbb{Z}).$$

The Poincaré dual cycle γ_c is then characterized by

$$\int_{\gamma_c} \beta = \int_R \alpha \wedge \beta \quad \text{for all closed 1-forms } \beta,$$

Thus, if we identify

$$H^1(R; \mathbb{Z}) \xrightarrow{\text{PD}} H_1(R; \mathbb{Z}) \quad \text{and} \quad H^1(R; \mathcal{O}) \xrightarrow{\text{Serre}} H^0(R, \Omega^1)^\vee,$$

the class $c \in H^1(R; \mathbb{Z})$ maps to the functional

$$H^0(R, \Omega^1) \longrightarrow \mathbb{C}, \quad \omega \mapsto \frac{1}{2\pi i} \int_{\gamma_c} \omega.$$

This is precisely the period map (up to the factor $1/(2\pi i)$)

$$H_1(R; \mathbb{Z}) \otimes H^0(R, \Omega^1) \longrightarrow \mathbb{C}, \quad (\gamma, \omega) \mapsto \int_{\gamma} \omega,$$

with $\gamma = \gamma_c$ the Poincaré dual of c .

Problem 3 Show that the period mapping gives an isomorphism

$$H_1(R; \mathbb{Z}) \xrightarrow{\sim} H_1(J; \mathbb{Z}),$$

which can be realized geometrically by the Abel-Jacobi map

$$R \longrightarrow J_1.$$

Show that under this correspondence, $c_1(\Theta) \in \Lambda^2 H_1(R)$ is the intersection pairing on R .

Hints for the second part: You can deduce it from the periodicity formulas of the Riemann Θ -function. Alternatively, you can find this by exploiting the facts that the Poincaré dual of $c_1(\Theta)$ in J_{g-1} is the Theta divisor, the image of $\text{Sym}^{g-1}(R)$. The maps

$$\text{Sym}^g(R) \longrightarrow J_g \quad \text{and} \quad \text{Sym}^{g-1}(R) \longrightarrow \text{div}(\Theta)$$

have degree 1.

Solution: The presentation of the Jacobian J as

$$J \cong H^1(R; \mathcal{O}) / H_1(R; \mathbb{Z})$$

makes it clear that $H_1(J; \mathbb{Z})$ is naturally identified with $H_1(R; \mathbb{Z})$, since the universal cover of J is the vector space $H^1(R; \mathcal{O})$. The period mapping

$$H_1(R; \mathbb{Z}) \rightarrow H_1(J; \mathbb{Z})$$

is injective because of the Riemann bilinear relations, and since both groups are free abelian of rank $2g$, it is an isomorphism. Pick a base point $p_0 \in R$ and define the Abel-Jacobi map

$$\varphi : R \rightarrow J, \quad p \mapsto \left[\omega \mapsto \int_{p_0}^p \omega \right].$$

precisely implements the lift of the period mapping to the universal cover and hence induces the same isomorphism on H_1 .

To identify $c_1(\Theta)$ with the intersection pairing on $H_1(R, \mathbb{Z})$, we first note that by the universal coefficient theorem and the fact that $H^k(J, \mathbb{Z}) = \text{Alt}^k(H_1(J, \mathbb{Z}), \mathbb{Z})$ (the group law on J induces a map $H_1(J, \mathbb{Z}) \otimes \cdots \otimes H_1(J, \mathbb{Z}) \rightarrow H_k(J, \mathbb{Z})$ as follows. For each $\alpha \in H_1(J, \mathbb{Z})$ choose a loop $\ell_\alpha : S^1 \rightarrow J$ representing α . For $\alpha_1, \dots, \alpha_k$, consider the map $(S^1)^k \rightarrow J$ given by $(t_1, \dots, t_k) \mapsto \ell_{\alpha_1}(t_1) + \cdots + \ell_{\alpha_k}(t_k)$. For orientation reasons, this map is alternating in the α_i). We have

$$\begin{aligned} H^2(J, \mathbb{Z}) &\cong \text{Hom}(H_2(J, \mathbb{Z}), \mathbb{Z}) \\ &\cong \text{Hom}(\Lambda^2 H_1(J, \mathbb{Z}), \mathbb{Z}) \\ &\cong \text{Alt}^2(H_1(J, \mathbb{Z}), \mathbb{Z}) \xrightarrow{\iota^*} \text{Alt}^2(H_1(R, \mathbb{Z}), \mathbb{Z}) \end{aligned}$$

so indeed $c_1(\Theta)$ corresponds to an alternating bilinear form on $H_1(R, \mathbb{Z})$. Pick a symplectic basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ of $H_1(R, \mathbb{Z})$, i.e.

$$a_i \cdot a_j = 0, \quad b_i \cdot b_j = 0, \quad a_i \cdot b_j = \delta_{ij}.$$

Under the identification $H_1(R, \mathbb{Z}) \xrightarrow{\sim} \Lambda \cong H_1(J, \mathbb{Z})$ coming from the period map and the Abel-Jacobi embedding, a homology class $\gamma \in H_1(R, \mathbb{Z})$ corresponds to an integral vector $(m, n) \in \mathbb{Z}^{2g}$. The intersection pairing on $H_1(R, \mathbb{Z})$ is given in these coordinates by

$$(m, n) \cdot (m', n') = m^T n' - m'^T n.$$

The Riemann theta function with period matrix τ is

$$\theta(z \mid \tau) := \sum_{k \in \mathbb{Z}^g} \exp(\pi i k^T \tau k + 2\pi i k^T z), \quad z \in \mathbb{C}^g.$$

The Riemann theta function satisfies the quasi-periodicity property.

$$\theta(z + m + \tau n \mid \tau) = \exp(-\pi i n^\top \tau n - 2\pi i n^\top z) \theta(z \mid \tau)$$

In particular, the Riemann theta function defines a holomorphic section of the line bundle $\mathcal{O}_J(\Theta)$.

Hence, identifying $H^2(U, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ by the above isomorphism, the Chern class of L is simply $\delta(\text{cl}\{e_u\})$. Write $e_u(z) = e^{2\pi i f_u(z)}$ with f_u holomorphic in V . Then by definition, $\delta(\text{cl}\{e_u\}) \in H^2(U, \mathbb{Z})$ is given by the 2-cocycle $F(u_1, u_2)$ on U with coefficients in \mathbb{Z} defined by

$$F(u_1, u_2) = f_{u_2}(z + u_1) - f_{u_1+u_2}(z) + f_{u_1}(z) \in \mathbb{Z}. \quad (*)$$

Lemma 1 (Mumford) Let $U \subset V$ be a lattice in a complex vector space V . The map which associates to any map $F : U \times U \rightarrow \mathbb{Z}$ the map $AF : U \times U \rightarrow \mathbb{Z}$ defined by

$$AF(u_1, u_2) = F(u_1, u_2) - F(u_2, u_1)$$

maps the group of 2-cocycles $Z^2(U, \mathbb{Z})$ into the space of alternating linear maps $U \times U \rightarrow \mathbb{Z}$, and induces an isomorphism

$$A : H^2(U, \mathbb{Z}) \xrightarrow{\sim} \text{Hom}(\Lambda^2 U, \mathbb{Z}) \cong \Lambda^2 \text{Hom}(U, \mathbb{Z}).$$

Furthermore for $\xi, \eta \in \text{Hom}(U, \mathbb{Z}) = H^1(U, \mathbb{Z})$, we have $A(\xi \smile \eta) = \xi \wedge \eta$.

Proposition 2 (Mumford) The Chern class of the line bundle corresponding to $\{e_u\} \in Z^1(U, H^*)$ is the alternating 2-form on U with values in \mathbb{Z} given by

$$E(u_1, u_2) = f_{u_2}(z + u_1) + f_{u_1}(z) - f_{u_1}(z + u_2) - f_{u_2}(z), \quad (z \text{ arbitrary in } V), \quad (**)$$

where

$$e_u(z) = e^{2\pi i f_u(z)}.$$

Moreover if we extend E \mathbb{R} -linearly to a map $V \times V \rightarrow \mathbb{R}$, then E satisfies the identity

$$E(ix, iy) = E(x, y) \quad \text{for } x, y \in V.$$

Problem 4 Prove the following generalized Cauchy formula for a smooth function f defined in the unit disk Δ :

$$f(z, \bar{z}) = \frac{1}{2\pi i} \oint_{|\zeta-z|=r} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \iint_{\Delta'} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z},$$

where $\Delta' \subset \Delta$ is the subdisk of radius $r < 1$.

Solution: We prove the two lemmas below at the end.

Lemma 1 Let $X \subset \mathbb{C}$ be a bounded domain with smooth boundary ∂X . Then

$$\frac{\partial \chi_X}{\partial \bar{z}} = \frac{i}{2} \oint_{\partial X} dz,$$

where the distribution on the right denotes contour integration along ∂X .

We can now deduce the generalized Cauchy integral formula. Let

$$u = \frac{\chi_X}{\pi(z - z_0)} \in L^1_{\text{loc}}(X), \quad z_0 \in X.$$

where $L^1_{\text{loc}}(X)$ is the space of locally integrable functions on X . We may apply the Leibniz rule to compute

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \left(\frac{1}{\pi(z - z_0)} \right) \chi_X + \frac{1}{\pi(z - z_0)} \frac{\partial \chi_X}{\partial \bar{z}}.$$

Lemma 2

$$\frac{\partial}{\partial \bar{z}} \left(\frac{1}{\pi(z - z_0)} \right) = \delta_{z_0}$$

It follows from the lemma and the identity above that

$$\frac{\partial u}{\partial \bar{z}} = \delta_{z_0} + \frac{1}{\pi(z - z_0)} \frac{\partial \chi_X}{\partial \bar{z}}.$$

Applying both sides to $\varphi \in C_c^\infty(X)$ gives

$$\left\langle \frac{\partial u}{\partial \bar{z}}, \varphi \right\rangle = \varphi(z_0) + \left\langle \frac{\partial \chi_X}{\partial \bar{z}}, \frac{\varphi}{\pi(z - z_0)} \right\rangle = \varphi(z_0) + \frac{i}{2} \oint_{\partial X} \frac{\varphi(z)}{\pi(z - z_0)} dz.$$

Rearranging yields the desired generalized Cauchy formula:

$$\varphi(z_0) = \frac{1}{2\pi i} \oint_{\partial X} \frac{\varphi(z)}{z - z_0} dz + \frac{1}{2\pi i} \iint_X \frac{\partial \varphi}{\partial \bar{z}}(z) \frac{dx \wedge dy}{z - z_0}.$$

Proof of Lemma 1. For $\varphi \in C_c^\infty(X)$, compute

$$\left\langle \frac{\partial}{\partial \bar{z}} \chi_X, \varphi \right\rangle = - \int_X \frac{\partial \varphi}{\partial \bar{z}} dx dy = - \frac{1}{2} \int_X (\partial_x \varphi + i \partial_y \varphi) dx dy.$$

Let ∂X be oriented counterclockwise, and parametrize it by arc length $s \mapsto (x(s), y(s))$. Denote by $\tau = (x'(s), y'(s))$ the unit tangent vector and $\nu = (-y'(s), x'(s))$ the unit outward normal. Define $V = (\varphi, i\varphi) \in C_c^\infty(X)^2$, so that

$$\operatorname{div} V = \partial_x \varphi + i \partial_y \varphi.$$

By the divergence theorem,

$$-\frac{1}{2} \int_X (\partial_x \varphi + i \partial_y \varphi) dx dy = -\frac{1}{2} \int_{\partial X} V \cdot \nu ds = -\frac{1}{2} \int_0^\ell (\varphi \nu_1 + i \varphi \nu_2) ds.$$

Since $\nu = (-y', x')$, we get

$$V \cdot \nu = \varphi(-y' + ix') = i \varphi(x' + iy').$$

Thus

$$-\frac{1}{2} \int_X (\partial_x \varphi + i \partial_y \varphi) dx dy = \frac{i}{2} \int_0^\ell \varphi(x(s), y(s)) (x'(s) + iy'(s)) ds = \frac{i}{2} \oint_{\partial X} \varphi dz.$$

Hence

$$\frac{\partial \chi_X}{\partial \bar{z}} = \frac{i}{2} \oint_{\partial X} dz.$$

□

Proof of Lemma 2. Let $\varphi \in C_c^\infty(X)$ be a test function. By definition,

$$\left\langle \frac{\partial}{\partial \bar{z}} \left(\frac{1}{\pi(z - z_0)} \right), \varphi \right\rangle := -\frac{1}{\pi} \iint_{X \setminus \{z_0\}} \frac{1}{z - z_0} \frac{\partial \varphi}{\partial \bar{z}} dx dy.$$

Remove a small disk D_ε of radius ε centered at z_0 , and write $A_\varepsilon = X \setminus D_\varepsilon$. Then

$$\left\langle \frac{\partial}{\partial \bar{z}} \left(\frac{1}{\pi(z - z_0)} \right), \varphi \right\rangle = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \iint_{A_\varepsilon} \frac{1}{z - z_0} \frac{\partial \varphi}{\partial \bar{z}} dx dy.$$

We know that

$$d \left(\frac{\varphi}{z - z_0} dz \right) = d \left(\frac{\varphi}{z - z_0} \right) \wedge dz = \frac{1}{z - z_0} d\varphi \wedge dz + \varphi d \left(\frac{1}{z - z_0} \right) \wedge dz.$$

But

$$d \left(\frac{1}{z - z_0} \right) \wedge dz = g(z) dz \wedge dz = 0 \implies d \left(\frac{\varphi}{z - z_0} dz \right) = \frac{1}{z - z_0} d(\varphi dz).$$

This, together with the fact that

$$\frac{\partial \varphi}{\partial \bar{z}} dx dy = \frac{1}{2i} d(\varphi dz).$$

gives upon application of Stokes' theorem,

$$\iint_{A_\varepsilon} \frac{1}{z - z_0} \frac{\partial \varphi}{\partial \bar{z}} dx dy = \frac{1}{2i} \int_{\partial A_\varepsilon} \frac{\varphi(z)}{z - z_0} dz.$$

The boundary ∂A_ε consists of the small circle ∂D_ε , oriented negatively. Parametrizing ∂D_ε positively gives an extra minus sign, hence

$$\iint_{A_\varepsilon} \frac{1}{z - z_0} \frac{\partial \varphi}{\partial \bar{z}} dx dy = -\frac{1}{2i} \int_{|z-z_0|=\varepsilon} \frac{\varphi(z)}{z - z_0} dz.$$

Expand φ near z_0 :

$$\varphi(z) = \varphi(z_0) + O(\varepsilon).$$

Thus

$$\int_{|z-z_0|=\varepsilon} \frac{\varphi(z)}{z - z_0} dz = \varphi(z_0) \int_{|z-z_0|=\varepsilon} \frac{dz}{z - z_0} + O(\varepsilon).$$

But

$$\int_{|z-z_0|=\varepsilon} \frac{dz}{z - z_0} = 2\pi i.$$

Hence letting $\varepsilon \rightarrow 0$,

$$\lim_{\varepsilon \rightarrow 0} \int_{|z-z_0|=\varepsilon} \frac{\varphi(z)}{z - z_0} dz = 2\pi i \varphi(z_0).$$

Combining everything,

$$\left\langle \frac{\partial}{\partial \bar{z}} \left(\frac{1}{\pi(z - z_0)} \right), \varphi \right\rangle = -\frac{1}{\pi} \left(-\frac{1}{2i} \right) (2\pi i) \varphi(z_0) = \varphi(z_0).$$

This is precisely the defining property of the Dirac delta at z_0 . \square

Problem 5 Let $L \rightarrow X$ be a holomorphic line bundle on a complex manifold, and let $\alpha \in \mathcal{E}^{0,1}$ be a $\bar{\partial}$ -closed form. Show that the re-defined operator

$$\tilde{\bar{\partial}} = \bar{\partial} + \alpha$$

on sections of L defines a new holomorphic structure L' on the same underlying bundle, where local holomorphic sections are defined as those killed by $\tilde{\bar{\partial}}$. Show that $L \simeq L'$ if α is $\bar{\partial}$ -exact. Relate this to the exponential sequence.

Remark: For vector bundles, the same applies with an $\alpha \in \mathcal{E}^{0,1}(\text{End}(V))$ satisfying the non-linear equation

$$\bar{\partial}\alpha + \alpha \wedge \alpha = 0.$$

The new bundle is isomorphic to the old one if $\alpha = a^{-1}\bar{\partial}a$, for some smooth section a of $\text{Aut}(V)$.

Solution: From the given holomorphic structure, we have a \mathbb{C} -linear map

$$\bar{\partial}_L: \mathcal{E}^0(L) \longrightarrow \mathcal{E}^{0,1}(L)$$

satisfying the Leibniz rule and the condition $\bar{\partial}_L^2 = 0$. We have the new operator

$$\tilde{\bar{\partial}}s := \bar{\partial}s + \alpha \wedge s \in \mathcal{E}^{0,1}(L).$$

First we check that $\tilde{\bar{\partial}}$ is a $\bar{\partial}$ -operator. For $f \in C^\infty(X)$ and $s \in \mathcal{E}^0(L)$,

$$\tilde{\bar{\partial}}(fs) = \bar{\partial}(fs) + \alpha fs = (\bar{\partial}f)s + f\bar{\partial}s + f\alpha s = (\bar{\partial}f)s + f\tilde{\bar{\partial}}s,$$

so the Leibniz rule holds.

Next, compute $\tilde{\bar{\partial}}^2$. View $\bar{\partial}$ as a derivation of degree $(0, 1)$ on $\mathcal{E}^{0,\bullet}(L)$; then for $\beta \in \mathcal{E}^{0,1}$ and $\eta \in \mathcal{E}^{0,q}(L)$,

$$\bar{\partial}(\beta \wedge \eta) = (\bar{\partial}\beta) \wedge \eta - \beta \wedge \bar{\partial}\eta.$$

Hence, for a section $s \in \mathcal{E}^0(L)$,

$$\begin{aligned} \tilde{\bar{\partial}}^2 s &= \tilde{\bar{\partial}}(\bar{\partial}s + \alpha \wedge s) \\ &= \bar{\partial}(\bar{\partial}s + \alpha \wedge s) + \alpha \wedge (\bar{\partial}s + \alpha \wedge s) \\ &= \bar{\partial}^2 s + \bar{\partial}(\alpha \wedge s) + \alpha \wedge \bar{\partial}s + \alpha \wedge \alpha \wedge s \\ &= 0 + (\bar{\partial}\alpha) \wedge s - \alpha \wedge \bar{\partial}s + \alpha \wedge \bar{\partial}s + \alpha \wedge \alpha \wedge s \\ &= (\bar{\partial}\alpha) \wedge s + \alpha \wedge \alpha \wedge s. \end{aligned}$$

By assumption $\bar{\partial}\alpha = 0$, and since α is a 1-form, $\alpha \wedge \alpha = 0$. Thus $\tilde{\bar{\partial}}^2 s = 0$ for all s , so $\tilde{\bar{\partial}}^2 = 0$ and $\tilde{\bar{\partial}}$ is a $\bar{\partial}$ -operator. It therefore defines a new holomorphic structure L' on the same underlying smooth bundle, whose local holomorphic sections are those killed by $\tilde{\bar{\partial}}$.

Now we check that if α is $\bar{\partial}$ -exact, then $L' \simeq L$. Suppose $\alpha = \bar{\partial}\phi$ for some smooth complex-valued function ϕ . Define an automorphism of the C^∞ line bundle L by multiplication with e^ϕ :

$$F: L \longrightarrow L, \quad s \longmapsto e^\phi s.$$

We claim that F is an isomorphism of holomorphic line bundles $L' \rightarrow L$, i.e.

$$\bar{\partial}(Fs) = F(\tilde{\bar{\partial}}s) \quad \text{for all } s.$$

Indeed,

$$\bar{\partial}(Fs) = \bar{\partial}(e^\phi s) = e^\phi(\bar{\partial}\phi \wedge s + \bar{\partial}s) = e^\phi(\alpha \wedge s + \bar{\partial}s) = F(\tilde{\bar{\partial}}s).$$

Thus F is holomorphic with respect to $\tilde{\bar{\partial}}$ on the domain and $\bar{\partial}$ on the target, so $L' \simeq L$.

The $(0, 1)$ -form α is $\bar{\partial}$ -closed, so it defines a Dolbeault cohomology class

$$[\alpha] \in H_{\bar{\partial}}^{0,1}(X) \cong H^1(X; \mathcal{O}).$$

Changing α by a $\bar{\partial}$ -exact form does not change this class, and by the computation above such a change yields an isomorphic holomorphic structure. Thus the isomorphism class of the new line bundle L' depends only on $[\alpha]$.

Recall the holomorphic exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}^\times \longrightarrow 1,$$

whose long exact cohomology sequence contains

$$H^1(X; \mathcal{O}) \xrightarrow{\exp} H^1(X; \mathcal{O}^\times),$$

and $H^1(X; \mathcal{O}^\times) \cong \text{Pic}(X)$ classifies holomorphic line bundles. The class $[\alpha] \in H^1(X; \mathcal{O})$ maps under the exponential to the class of the holomorphic line bundle $L' \otimes L^{-1}$.

Problem 6 Let V be a complex g -dimensional vector space and $L \simeq \mathbb{Z}^{2g} \subset V$ a lattice. Let $A = V/L$.

1. Using harmonic theory, compute the Dolbeault cohomology $H^*(A; \mathcal{O})$.
2. Show that the moduli space of holomorphic line bundles on A with zero Chern class is naturally identified with

$$A^\vee := \bar{V}^\vee / L^\vee.$$

3. Show that the moduli space of holomorphic line bundles on A^\vee with zero Chern class is naturally identified with A .
4. Define a line bundle

$$\mathcal{P} \longrightarrow A \times A^\vee$$

from the trivial line bundle over $V \times V^\vee$ with connection

$$\nabla = d + i(x d\xi + \xi dx),$$

by dividing the $L \times L^\vee$ -action as follows: identify the fiber \mathbb{C} over $(x, \xi) \in V \times V^\vee$ with that over $(x + \ell, \xi + \lambda)$ by multiplication by

$$\exp(2\pi i(\lambda(x) + \xi(\ell))).$$

Show that \mathcal{P} is holomorphic, that $\mathcal{P}|_{A \times \{a^\vee\}}$ is the line bundle over A classified by $a^\vee \in A^\vee$, and prove the corresponding statement for $\{a\} \times A^\vee$.

Solution: Write $g = \dim_{\mathbb{C}} V$. Choose a Hermitian inner product on V which is L -invariant. By translation invariance and the locality of Kahler geometry, this induces a flat Kahler metric on A .

1. The Dolbeault Laplacian $\square_{\bar{\partial}}$ on $(0, q)$ -forms is translation invariant. On a flat torus, a $(0, q)$ -form is harmonic iff it has constant coefficients. This is because in a global parallel frame, the Laplacian on forms acts coefficientwise as the usual scalar Laplacian

and the that any harmonic function on a compact manifold is constant by the maximum principle.

A translation-invariant $(0, 1)$ -form on A is determined by its value at a single point (say 0), and conversely any linear functional on $T_0^{0,1}A$ extends uniquely to a translation-invariant $(0, 1)$ -form. Thus

$$\mathcal{H}^{0,q}(A) \cong \Lambda^q(T_0^{0,1}A)^\vee \cong \Lambda^q \bar{V}^\vee,$$

where \bar{V} is V with the conjugate complex structure. By Hodge theory,

$$H^{0,q}(A) \cong \mathcal{H}^{0,q}(A),$$

and since $H^q(A; \mathcal{O}) \cong H^{0,q}(A)$ we obtain

$$H^q(A; \mathcal{O}) \cong \Lambda^q \bar{V}^\vee \cong \Lambda^q V^\vee, \quad 0 \leq q \leq g.$$

More generally, the same reasoning shows

$$H^{p,q}(A) \cong \Lambda^p V^\vee \otimes \Lambda^q \bar{V}^\vee$$

2. Isomorphism classes of holomorphic line bundles on A are classified by

$$\text{Pic}(A) \cong H^1(A; \mathcal{O}^\times).$$

Consider the holomorphic exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}^\times \longrightarrow 1,$$

with long exact cohomology sequence

$$H^1(A; \mathcal{O}) \longrightarrow H^1(A; \mathcal{O}^\times) \xrightarrow{c_1} H^2(A; \mathbb{Z}).$$

The subgroup

$$\text{Pic}^0(A) := \ker(c_1 : \text{Pic}(A) \rightarrow H^2(A; \mathbb{Z}))$$

is precisely those bundles with zero Chern class. Exactness shows

$$\text{Pic}^0(A) \cong H^1(A; \mathcal{O}) / \text{im}(H^1(A; \mathbb{Z}))$$

We have already computed

$$H^1(A; \mathcal{O}) \cong H^{0,1}(A) \cong \bar{V}^\vee$$

On the other hand, $H^1(A; \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong L^\vee$, and this sits inside $H^1(A; \mathcal{O})$ as a lattice (one way to see this is via $H^1(A; \mathbb{Z}) \subset H^1(A; \mathbb{R}) \subset H^1(A; \mathbb{C})$ and the Hodge decomposition). Thus

$$\text{Pic}^0(A) \cong H^1(A; \mathcal{O}) / H^1(A; \mathbb{Z}) \cong \bar{V}^\vee / L^\vee = A^\vee.$$

So the moduli space of holomorphic line bundles on A with $c_1 = 0$ is canonically identified with A^\vee .

3. Apply the same argument to $A^\vee = V^\vee/L^\vee$.

4. Consider the trivial line bundle

$$V \times V^\vee \times \mathbb{C} \longrightarrow V \times V^\vee$$

equipped with the connection

$$\nabla = d + i(x d\xi + \xi dx),$$

where $x \in V$, $\xi \in V^\vee$, and $x d\xi + \xi dx$ denotes the tautological pairing (so in coordinates it is linear in x and ξ).

The group $L \times L^\vee$ acts on $V \times V^\vee \times \mathbb{C}$ by

$$(\ell, \lambda) \cdot (x, \xi, z) = (x + \ell, \xi + \lambda, e^{2\pi i(\lambda(x) + \xi(\ell))} z).$$

The multiplier $e^{2\pi i(\lambda(x) + \xi(\ell))}$ is holomorphic in (x, ξ) (it is the exponential of a holomorphic linear function), so this action is by holomorphic bundle automorphisms of the trivial holomorphic line bundle $V \times V^\vee \times \mathbb{C}$.

The quotient

$$\mathcal{P} := (V \times V^\vee \times \mathbb{C}) / (L \times L^\vee) \longrightarrow (V/L) \times (V^\vee/L^\vee) = A \times A^\vee$$

is therefore a holomorphic line bundle, the *Poincaré bundle*. The connection ∇ is invariant under the $L \times L^\vee$ -action, hence descends to a connection on \mathcal{P} .

Now fix $a^\vee \in A^\vee$ and choose a lift $\xi_0 \in V^\vee$. The restriction $\mathcal{P}|_{A \times \{a^\vee\}}$ is obtained as the quotient of $V \times \mathbb{C}$ by the following L -action: an element $\ell \in L$ sends (x, z) to

$$(x + \ell, e^{2\pi i \xi_0(\ell)} z),$$

because along the slice $\xi = \xi_0$ the factor $\lambda(x)$ vanishes (we take $\lambda = 0$ to stay in the same fiber over a^\vee), while $\xi(\ell) = \xi_0(\ell)$ is constant in x . Thus the monodromy of the resulting line bundle around $\ell \in L$ is exactly

$$\chi_{\xi_0}(\ell) = e^{2\pi i \xi_0(\ell)}.$$

By part (2), a holomorphic line bundle on A with zero Chern class is classified precisely by such a character $L \rightarrow U(1)$, and changing ξ_0 by an element of L^\vee does not change the character χ_{ξ_0} . Hence the isomorphism class of $\mathcal{P}|_{A \times \{a^\vee\}}$ depends only on the class $a^\vee = [\xi_0] \in A^\vee$ and is exactly the line bundle over A classified by a^\vee .

The argument for the restriction to $\{a\} \times A^\vee$ is symmetric. Fix $a \in A$ with lift $x_0 \in V$. Then the effective L^\vee -action on $V^\vee \times \mathbb{C}$ along the slice $x = x_0$ is

$$\lambda \cdot (\xi, z) = (\xi + \lambda, e^{2\pi i \lambda(x_0)} z),$$

so the monodromy around $\lambda \in L^\vee$ is

$$\chi_{x_0}(\lambda) = e^{2\pi i \lambda(x_0)}.$$

This is precisely the character of L^\vee corresponding to the point $a = [x_0] \in A$, and hence $\mathcal{P}|_{\{a\} \times A^\vee}$ is the line bundle on A^\vee classified by a .

Remark 1 A unitary character is a homomorphism $\chi : L \longrightarrow U(1)$.

Any χ can be written as $\chi(\ell) = \exp(2\pi i \xi(\ell))$ for some real-valued group homomorphism $\xi : L \rightarrow \mathbb{R}$. Conversely, any such ξ defines a character this way. So

$$\text{Hom}(L, U(1)) \cong \text{Hom}(L, \mathbb{R}) / \text{Hom}(L, \mathbb{Z}).$$

But

$$\text{Hom}(L, \mathbb{R}) \cong L^\vee \otimes_{\mathbb{Z}} \mathbb{R} \cong \overline{V}^\vee$$

(using the inclusion $L \subset V$ and identifying $V \cong L \otimes_{\mathbb{Z}} \mathbb{R}$), and

$$\text{Hom}(L, \mathbb{Z}) \cong L^\vee.$$

Therefore

$$\text{Hom}(L, U(1)) \cong \text{Hom}(L, \mathbb{R}) / \text{Hom}(L, \mathbb{Z}) \cong \overline{V}^\vee / L^\vee = A^\vee.$$

Problem 7 Show that, in the case of the Jacobian J of a Riemann surface R , one has a natural isomorphism $J \simeq J^\vee$.

Hint: Remember the natural Hilbert space structure on holomorphic differentials.

Remark: This self-duality is a property of principally polarized Abelian varieties, those A equipped with a positive line bundle having a single holomorphic section (the Θ -function).

Solution: Let R be a compact Riemann surface of genus g and J its Jacobian. Recall that

$$J \simeq V/\Lambda, \quad V := H^0(R, \Omega^1)^\vee,$$

where the lattice Λ is the image of $H_1(R; \mathbb{Z})$ under the period map

$$H_1(R; \mathbb{Z}) \longrightarrow H^0(R, \Omega^1)^\vee, \quad \gamma \longmapsto (\omega \mapsto \int_\gamma \omega).$$

There is a natural Hermitian inner product on the space of holomorphic differentials

$$\langle \omega, \eta \rangle := \frac{i}{2} \int_R \omega \wedge \bar{\eta}, \quad \omega, \eta \in H^0(R, \Omega^1).$$

This is positive definite and defines a Hilbert space structure on $H^0(R, \Omega^1)$. This inner product gives a linear isomorphism

$$\bar{\rho} : \overline{H^0(R, \Omega^1)} \xrightarrow{\sim} V.$$

Dualizing, we obtain an antilinear isomorphism

$$\bar{V}^\vee \xrightarrow{\sim} H^0(R, \Omega^1).$$

The intersection pairing on $H_1(R; \mathbb{Z})$ is unimodular, so it induces an isomorphism

$$H_1(R; \mathbb{Z}) \xrightarrow{\sim} H^1(R; \mathbb{Z}) \cong \text{Hom}(H_1(R; \mathbb{Z}), \mathbb{Z}).$$

Translating this through the identification $\Lambda \cong H_1(R; \mathbb{Z})$, we obtain a canonical isomorphism of lattices

$$\Lambda \xrightarrow{\sim} \Lambda^\vee.$$

Recall that the first Chern class $c_1(\Theta) \in H^2(J; \mathbb{Z})$ of the theta line bundle corresponds, under the isomorphism $H^2(J; \mathbb{Z}) \cong \text{Alt}^2(H_1(J; \mathbb{Z}), \mathbb{Z})$, to the intersection form

$$H_1(R; \mathbb{Z}) \times H_1(R; \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

This implies that the inner product on holomorphic differentials actually refines the intersection pairing on $H_1(R; \mathbb{Z})$. Therefore the isomorphism $\bar{V}^\vee \xrightarrow{\sim} H^0(R, \Omega^1)$ actually descends to an isomorphism of complex tori

$$J^\vee = \bar{V}^\vee / \Lambda^\vee \xrightarrow{\sim} H^0(R, \Omega^1) / \Lambda \xrightarrow{\sim} J.$$

Problem 8

1. Given a holomorphic line bundle \mathcal{L} on a complex manifold and a smooth real closed 2-form ω in the cohomology class of $c_1(\mathcal{L})$, prove that there exists a Hermitian metric on \mathcal{L} whose holomorphic connection has curvature $-2\pi i \omega$.
2. Conclude (from Kodaira vanishing) that the holomorphic line bundles on a compact Riemann surface R which carry metrics of positive curvature are precisely those of positive degree.
3. Show also that for every holomorphic vector bundle V on R , there exists a d so that the twisted bundle $V(D)$ has no H^1 for any $D > d$.

Solution:

1. Let $L \rightarrow X$ be a holomorphic line bundle on a complex manifold and let ω be a smooth real closed 2-form representing $c_1(L) \in H^2(X; \mathbb{R})$. Choose any Hermitian metric h_0 on L and let F_0 denote the curvature of its Chern connection. Then

$$\frac{i}{2\pi} F_0 \in \Omega^{1,1}(X, \mathbb{R}) \quad \text{represents } c_1(L).$$

Since ω also represents $c_1(L)$, the difference

$$\omega - \frac{i}{2\pi} F_0$$

is an exact real $(1, 1)$ -form. By the $\partial\bar{\partial}$ -lemma (which holds on every complex curve and more generally on Kähler manifolds), there exists a real-valued smooth function φ such that

$$\omega - \frac{i}{2\pi} F_0 = \frac{i}{2\pi} \partial\bar{\partial}\varphi.$$

Define a new Hermitian metric h by $h = e^{-\varphi} h_0$. In a local holomorphic frame l , if $h(l, l) = e^{-\phi}$ the curvature is $F_h = \partial\bar{\partial}\phi$. Thus

$$F_h = F_0 + \partial\bar{\partial}\varphi = -2\pi i \omega.$$

Hence h is a Hermitian metric whose Chern connection has curvature $-2\pi i \omega$.

2. For a Hermitian metric h on a holomorphic line bundle $L \rightarrow R$, the degree is

$$\deg(L) = \int_R c_1(L) = \int_R \frac{i}{2\pi} F_h.$$

If h has positive curvature, then the form $\frac{i}{2\pi} F_h$ is positive on R , hence its integral is strictly positive and $\deg(L) > 0$.

The hyperplane bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ carries the Fubini–Study Hermitian metric h_{FS} whose curvature form F_{FS} is a positive $(1, 1)$ -form. Pulling back gives a metric $h_m = \Phi_m^* h_{\text{FS}}$ on $L^{\otimes m}$ with positive curvature

$$F_{h_m} = \Phi_m^* F_{\text{FS}} > 0.$$

Now define a Hermitian metric h on L by taking an m th root locally: in a local holomorphic frame e of L , write

$$h_m(e^{\otimes m}, e^{\otimes m}) = e^{-\phi_m}$$

and set

$$h(e, e) := e^{-\phi_m/m}.$$

Then the curvature satisfies

$$F_h = \frac{1}{m} F_{h_m},$$

which is still a positive $(1, 1)$ -form. Thus L admits a Hermitian metric of positive curvature. Therefore the holomorphic line bundles on R which carry metrics of positive curvature are precisely those of positive degree.

3. This follows immediately from Serre duality and the Riemann–Roch theorem. For a holomorphic vector bundle V on R and a divisor D , Serre duality gives

$$H^1(R, V(D)) \cong H^0(R, K_R \otimes V^\vee \otimes \mathcal{O}(-D))^\vee.$$

By Riemann–Roch, for $\deg(D)$ sufficiently large, the degree of the bundle $K_R \otimes V^\vee \otimes \mathcal{O}(-D)$ becomes negative, and hence

$$H^0(R, K_R \otimes V^\vee \otimes \mathcal{O}(-D)) = 0.$$

Thus, for such D , we have $H^1(R, V(D)) = 0$.

Problem 9 Show that isomorphism classes of *flat unitary* line bundles on a manifold X are classified by $H^1(X; U(1))$, with the constant sheaf $U(1)$ associated to the unit circle group in \mathbb{C}^\times .

When X is compact Kähler, compare the constant and holomorphic exponential sequences to conclude that the map

$$H^1(X; U(1)) \longrightarrow H^1(X; \mathcal{O}^\times)$$

induces a bijection from isomorphism classes of flat unitary line bundles to those of holomorphic line bundles with zero Chern class.

Remark: You probably need the Hodge decomposition theorem for the second part.

Solution: A flat unitary line bundle on X is a complex line bundle with structure group $U(1)$ and a flat unitary connection. Choosing a good open cover $\{U_i\}$, such a bundle is given by locally constant transition functions $g_{ij}: U_{ij} \rightarrow U(1)$ satisfying the cocycle condition $g_{ij}g_{jk}g_{ki} = 1$ on U_{ijk} , and two such bundles are isomorphic if their cocycles differ by a coboundary $g'_{ij} = h_i^{-1}g_{ij}h_j$ with $h_i: U_i \rightarrow U(1)$ locally constant. This is exactly the description of Čech 1-cocycles and coboundaries for the constant sheaf $U(1)$, so isomorphism classes of flat unitary line bundles are classified by

$$H^1(X; U(1)).$$

Now suppose X is compact Kähler. Consider the *constant exponential sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\exp(2\pi i \cdot)} U(1) \longrightarrow 1$$

and the *holomorphic exponential sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}^\times \longrightarrow 1.$$

The inclusions $\mathbb{R} \hookrightarrow \mathcal{O}$ and $U(1) \hookrightarrow \mathcal{O}^\times$ give a morphism of short exact sequences and therefore a natural map

$$H^1(X; U(1)) \longrightarrow H^1(X; \mathcal{O}^\times) \cong \text{Pic}(X).$$

The connecting homomorphism in the holomorphic sequence

$$c_1^{\text{hol}}: H^1(X; \mathcal{O}^\times) \longrightarrow H^2(X; \mathbb{Z})$$

is the first Chern class of the corresponding holomorphic line bundle; let

$$\text{Pic}^0(X) := \ker(c_1^{\text{hol}})$$

be the subgroup of holomorphic line bundles with $c_1 = 0$.

The long exact sequence of the constant exponential sequence yields

$$H^1(X; \mathbb{R}) \longrightarrow H^1(X; U(1)) \longrightarrow H^2(X; \mathbb{Z}) \longrightarrow H^2(X; \mathbb{R}),$$

so $H^1(X; U(1))$ is an extension of $H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$ by the torsion subgroup of $H^2(X; \mathbb{Z})$. Similarly, the long exact sequence of the holomorphic exponential sequence gives

$$H^1(X; \mathcal{O}) \longrightarrow H^1(X; \mathcal{O}^\times) \xrightarrow{c_1^{\text{hol}}} H^2(X; \mathbb{Z}) \longrightarrow H^2(X; \mathcal{O}),$$

and exactness shows

$$\text{Pic}^0(X) \cong H^1(X; \mathcal{O})/H^1(X; \mathbb{Z}).$$

By Hodge decomposition, on a compact Kähler manifold

$$H^1(X; \mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X),$$

and $H^1(X; \mathcal{O}) \cong H^{0,1}(X)$. The inclusion $\mathbb{R} \hookrightarrow \mathcal{O}$ induces a map

$$H^1(X; \mathbb{R}) \longrightarrow H^1(X; \mathcal{O}) \cong H^{0,1}(X)$$

which, under Hodge decomposition, is the projection $H^{1,0}(X) \oplus H^{0,1}(X) \rightarrow H^{0,1}(X)$.

Passing to the quotients by $H^1(X; \mathbb{Z})$, we obtain an isomorphism of real tori

$$\frac{H^1(X; \mathbb{R})}{H^1(X; \mathbb{Z})} \xrightarrow{\sim} \frac{H^1(X; \mathcal{O})}{H^1(X; \mathbb{Z})} \cong \text{Pic}^0(X).$$

Thus we see that the image of $H^1(X; U(1))$ is precisely $\text{Pic}^0(X)$ and that the induced map

$$H^1(X; U(1)) \xrightarrow{\sim} \text{Pic}^0(X)$$

is a bijection.

Problem 10 Prove the global $\partial\bar{\partial}$ -Lemma on a compact Kähler manifold X : for any d -exact form $\varphi \in \mathcal{E}^{p,q}$, there exists $\psi \in \mathcal{E}^{p-1,q-1}$ with

$$\partial\bar{\partial}\psi = \varphi.$$

Hint: Show that

$$\varphi = \partial\bar{\partial}^* \square \varphi$$

and use this and similar identities to find ψ .

Solution: Fix a Kähler metric on X and let Δ_d , Δ_∂ and $\Delta_{\bar{\partial}}$ denote the d -, ∂ - and $\bar{\partial}$ -Laplacians. On a Kähler manifold we have the identities

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$$

acting on each $\mathcal{E}^{p,q}$, and these Laplacians commute with ∂ and $\bar{\partial}$.

We define the Green operator $G_{\bar{\partial}} : \mathcal{E}^{p,q}(X) \rightarrow \mathcal{E}^{p,q}(X)$ as the inverse of $\Delta_{\bar{\partial}}$ on the orthogonal complement of the space of $\bar{\partial}$ -harmonic forms, and zero on the harmonic forms.

Regard φ as a $\bar{\partial}$ -exact form. Since its Dolbeault class in $H_{\bar{\partial}}^{p,q}(X)$ is zero, its $\bar{\partial}$ -harmonic part vanishes and we can write

$$\varphi = \Delta_{\bar{\partial}} G_{\bar{\partial}} \varphi = (\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) G_{\bar{\partial}} \varphi,$$

Using that ∂ commutes with $\Delta_{\bar{\partial}}$ on a Kähler manifold, we have

$$0 = \partial \varphi = \partial \Delta_{\bar{\partial}} G_{\bar{\partial}} \varphi = \Delta_{\bar{\partial}} \partial G_{\bar{\partial}} \varphi,$$

so $\partial G_{\bar{\partial}} \varphi$ is $\bar{\partial}$ -harmonic and $\bar{\partial}$ -exact, hence $\partial G_{\bar{\partial}} \varphi = 0$. It follows that $\bar{\partial}^* \bar{\partial} G_{\bar{\partial}} \varphi = 0$ and

$$\varphi = \bar{\partial} \bar{\partial}^* G_{\bar{\partial}} \varphi.$$

Now use the Kähler identity $\bar{\partial}^* = -i[\Lambda, \partial]$, where Λ is contraction with the Kähler form. Then

$$\varphi = -i \bar{\partial} [\Lambda, \partial] G_{\bar{\partial}} \varphi = -i (\bar{\partial} \Lambda \partial - \bar{\partial} \partial \Lambda) G_{\bar{\partial}} \varphi.$$

Since $\partial G_{\bar{\partial}} \varphi = 0$, the first term vanishes, and using $\bar{\partial} \partial = -\partial \bar{\partial}$ we obtain

$$\varphi = i \partial \bar{\partial} (\Lambda G_{\bar{\partial}} \varphi).$$

Thus, if we set

$$\psi := i \Lambda G_{\bar{\partial}} \varphi \in \mathcal{E}^{p-1,q-1}(X),$$

then

$$\partial \bar{\partial} \psi = \varphi.$$