Title

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Abstract

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1 The setup

Recall that we have a line bundle $\mathcal{L} := \mathcal{L}_{\det}$ on $\mathcal{X} := \mathcal{X}_{G,g,I}$ and an invariant norm $\|\cdot\|$ on $X_*(T)_{\mathbb{R}}$.

For a point $x \in \mathcal{X}(k)$ and a nontrivial $f : \Theta \to \mathcal{X}$ with $f(1) \simeq x$, define

$$\mu(x, f) \ := \ \frac{\mathrm{wt}_{\mathcal{L}}(f)}{\|\lambda_f\|},$$

where $\mathrm{wt}_{\mathcal{L}}(f)$ is the $\mathbb{G}_m$-weight on the fiber of $\mathcal{L}$ at the special point $f(0)$, and λ_f is the associated cocharacter data of f coming from the action of $\mathbb{G}_m$ on the object $f(0)$.

We wanted to prove the following theorem.

Theorem 1.1 (Need to prove)**.** For any $x \in \mathcal{X}_{G,g,I}(k)$, there exists a (unique up to something) maximally destabilizing Θ-filtration $f_{\mathrm{HN}} : \Theta \to \mathcal{X}_{G,g,I}$ of x with respect to $\mathcal{L}_{\det}$ and any invariant norm on $X_*(T)_{\mathbb{R}}$.

Constantin remarked that the uniqueness part of the theorem is not important for our application, and that we can get away with existence of a maximally destabilizing Θ-filtration for each $x \in \mathcal{X}(k)$, provided that we can organize the points x into strata in such a way which mirrors the setup of the Kirwan Ness stratification.

2 Kirwan Ness stratification

This section follows [**?**]. Suppose we have linear action of a reductive group G on a projective variety X, singular or nonsingular, defined over an algebraically closed field. Let $T \subset G$ be a maximal torus, V the representation giving the linearization. Pick an invariant norm on the cocharacter lattice of G. Then Kirwan and Ness construct a stratification of X into locally closed subvarieties

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta,$$

The index set $\mathcal{B}$ is in correspondence with connected components $Z \subset X^\beta$ of the fixed locus of a dominant cocharacter β of T such that the semistable locus Z° of the divided action of the Levi subgroup L_β on Z is nonempty.

If the fixed loci is connected, then the index set $\mathcal{B}$ can be identified with the set of G conjugacy classes of rays in $X_*(G) \otimes \mathbb{R}$ together with the condition that the associated "center" Z_β^{ss} is nonempty. Note that every cocharacter of G is G-conjugate to a dominant cocharacter of T. It is harmless to choose a rational dominant cocharacter β in each conjugacy class as they determine the same parabolic subgroup P_β and the same stratum S_β. Note that one cannot simply group the components together under a single conjugacy class of rays, as each connected component Z_i could possibly a different weight of the linearized line bundle.

Then we have a weight decomposition

$$V = \bigoplus_\chi V_\chi,$$

and so for any point $x \in X$, we can write its homogeneous coordinates as $x = [v]$ for some $v \in V$ with $v = \sum_\chi v_\chi$. For a point x, let W_x be the set of weights appearing in its support, meaning the set of χ such that $v_\chi \neq 0$. Then Kirwan identifies β as the closest point to 0 in $\mathrm{Conv}(W_x)$.

Alternatively, we can identify β as the G-conjugacy class of ray which minimzes the following function on $X_*(G) \otimes \mathbb{R}$:

$$\lambda \mapsto \frac{\mu(x, \lambda)}{\|\lambda\|}.$$

where $\mu(x,\lambda)$ is defined as the minimum λ-weight of the nonzero coordinates of x.

$$\mu(x,\lambda) = \min_{\chi \in W_x} \langle \chi, \lambda \rangle$$

For each $\beta \in \mathcal{B}$, pick a dominant rational representative of the corresponding conjugacy class of rays which we also denote by β. The unstable strata are indexed by those Z with dominant β for which the semistable locus $Z^\circ \subset Z$ of the divided L-action on $\mathcal{L}$ is not empty.

In particular we look at the fixed locus $X^\beta = \{x \in X \mid \beta(t) \cdot x = x \ \forall t\}$ and take a connected component $Z \subset X^\beta$. Now define the attracting set:

$$Y = \{x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x \in Z\}$$

which gives us a natural map $\varphi : Y \to Z$ defined by the limit of the T-flow.

Proposition 2.1. $\varphi : Y \to Z$ is a locally trivial fibration in affine spaces. At a point $z \in Z$, the tangent space decomposes into weight spaces under β as:

$$T_z X = T_z Z \oplus (T_z X)_{>0}$$

The positive-weight directions integrate to affine fibers.

Now we describe what Z° is. Let $L = Z_G(\beta)$ be the Levi subgroup. β acts on the fiber of the linearized line bundle $\mathcal{L}$ over Z by the character β.

To remove the destabilizing contribution, one twists the linearization by subtracting this character. After twisting, the central $\mathbb{G}_m$ coming from β acts trivially, and so we are left with a genuine GIT problem for the Levi L acting on Z. Let Z° be the semistable locus for this GIT problem, and put $Y^\circ = \varphi^{-1}(Z^\circ)$.

To recover the stratum S_β, we take the G-orbit of Y°, i.e. $S = G \cdot Y^\circ$ and then one can show that we have the following isomorphism of G-varieties:

$$S \cong G \times^{P_\beta} Y^\circ$$

where we are dividing by the relation

$$(g, y) \sim (gp^{-1}, py) \quad \forall p \in P_\beta.$$

So geometrically, we see $S \sim (G/P) \times (\text{affine}) \times Z^\circ$.

Proposition 2.2. To recap, in this setup, we have the following properties:

(i) Y is a fiber bundle over Z, with affine spaces as fibers, under the morphism φ defined by the limiting value of the T-flow.

(ii) Y is stabilized by the parabolic subgroup $P \subset G$ whose nilpotent Lie algebra radical $\mathfrak{u}$ is spanned by the negative T-eigenspaces in $\mathfrak{g}$.

(iii) The G-orbit S of Y° is isomorphic to $G \times^P Y^\circ$. Under φ, it fibers in affine spaces over $G \times^P Z^\circ$, if we let P act on Z° via its reductive quotient L.

(iv) The various S, together with $X^\circ = X^{ss}$, smoothly stratify X.

(v) Z° has a projective, good quotient under L; X^{ss} has a good projective quotient under G.

Then there is a smooth locally closed subvariety $Y_\beta \subset X$ acted on by a parabolic subgroup P_β of G such that

$$S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}. \tag{1}$$

There is also a nonsingular closed subvariety $Z_\beta \subset X$ and a locally trivial fibration

$$P_\beta : Y_\beta^{ss} \longrightarrow Z_\beta^{ss}, \tag{2}$$

whose fibres are all affine spaces. Here Z_β^{ss} is the set of semistable points of Z_β under the action of the Levi subgroup L_β of P_β.

3 Refernences

Kirwan Cohomology of quotients in symplectic and algebraic geometry, 1984.