Derived Categories and Bondal–Orlov Reconstruction

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December 9, 2025

Abstract

These notes discuss the concept of derived categories in the context of algebraic geometry, particularly focusing on the Bondal–Orlov reconstruction theorem. We explore how derived categories encapsulate information about coherent sheaves on smooth projective varieties and how this framework leads to a powerful reconstruction result under certain positivity conditions on the canonical bundle.

Contents

1 Algebraic vs analytic

My talk will be about algebraic geometry, but the course is about complex manifolds, so let me start with a brief comparison of the two subjects.

We had the notion of a Hodge manifold: a compact Kahler manifold X which admits an ample line bundle $\mathcal{L}$. By Kodaira's embedding theorem, such an X is biholomorphic to a smooth projective variety. Conversely, any projective variety over $\mathbb{C}$ (by definition a closed subvariety of projective space $\mathbb{P}^n$) which is smooth is a Hodge manifold by the pull back of $\mathcal{O}_{\mathbb{P}^n}(1)$. We also defined coherent analytic sheaves on complex manifolds, and coherent sheaf cohomology $H^i(X, \mathcal{F})$ for coherent analytic sheaves $\mathcal{F}$.

This talk will focus on coherent algebraic sheaves on smooth projective varieties, and their derived categories. The two theories are closely related: by Serre's GAGA theorem, for a smooth projective variety X over $\mathbb{C}$, the category of coherent algebraic sheaves on X is equivalent to the category of coherent analytic sheaves on the associated complex manifold X^{an}. Moreover, the cohomology groups $H^i(X, \mathcal{F})$ computed in either category are isomorphic.

2 Derived Categories

We've spent a lot of time in this course on sheaf cohomology: $H^i(X, \mathcal{F})$, defined via resolutions: injective resolutions, Čech resolutions, Dolbeault resolutions on complex manifolds, etc.

A basic pattern is: whenever we have a left exact functor like

$$\Gamma(X, -), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes \mathcal{G}, \quad f_*, \ldots$$

we get derived functors R^iF measuring the failure of exactness.

Key observation: To compute $R^iF(\mathcal{F})$ you choose a resolution $\mathcal{F} \to I^\bullet$ and apply F termwise. All choices of resolution give the same cohomology groups, but not canonically the same map between complexes.

This motivates keeping the whole resolution instead of just its cohomology.

- A (cochain) complex in an abelian category $\mathcal{A}$ is

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots, \qquad d^i \circ d^{i-1} = 0.$$

- A morphism of complexes is a collection of maps commuting with the differentials.
- Two morphisms $f, g : A^\bullet \to B^\bullet$ are *homotopic* if

$$f^i - g^i = d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i$$

  for some $h^i : A^i \to B^{i-1}$.

The **homotopy category** $K(\mathcal{A})$ has:

- objects = complexes in $\mathcal{A}$,
- morphisms = maps of complexes modulo homotopy.

This is natural because in a first course on algebraic topology, one of the key results is that homotopic maps induce the same map on cohomology.

A morphism of complexes $f : A^\bullet \to B^\bullet$ is a **quasi-isomorphism** if it induces isomorphisms on all cohomology sheaves:

$$\mathcal{H}^i(A^\bullet) \xrightarrow{\cong} \mathcal{H}^i(B^\bullet) \quad \forall i.$$

Definition (very informally). The derived category $D(\mathcal{A})$ is obtained from $K(\mathcal{A})$ by *localizing* at all quasi-isomorphisms:

$$D(\mathcal{A}) := K(\mathcal{A})[\text{quasi-isos}^{-1}].$$

This means:

- we freely adjoin formal inverses to all quasi-isomorphisms,
- two complexes that are quasi-isomorphic become isomorphic objects in $D(\mathcal{A})$,
- morphisms can be represented by "roofs"

  $$A^\bullet \xleftarrow{\simeq} C^\bullet \to B^\bullet,$$

  with the left arrow a quasi-isomorphism.

For a smooth projective variety or complex manifold X, we write:

$$D^b(X) := D^b_{\text{coh}}(X)$$

for the bounded derived category of coherent sheaves on X. Objects are bounded complexes of coherent sheaves on X, up to quasi-isomorphism.

Major payoff: A fundamental fact is:

$$\operatorname{Ext}^i_X(\mathcal{F}, \mathcal{G}) \simeq \operatorname{Hom}_{D^b(X)}(\mathcal{F}, \mathcal{G}[i]).$$

In other words, all the Ext^i-groups between two sheaves are packaged as morphisms between shifts in $D^b(X)$.

Let X be a smooth projective variety of dimension n with canonical bundle ω_X. For us, Serre duality for vector bundles on a curve says that

$$H^1(X, V) \;\cong\; H^0(X, V^* \otimes \omega_X)^*$$

functorially in V. This extends to all coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X:

$$H^1(X, \mathcal{F}) \cong H^0(X, \mathcal{F} \otimes \omega_X)^*,$$

because every coherent sheaf admits a finite resolution by vector bundles of length ≤ 1.

Let $\mathcal{F}$ be a coherent sheaf on a smooth projective curve X, and let $\mathcal{O}_X(1)$ be a very ample line bundle. For $m \gg 0$, the twisted sheaf $\mathcal{F}(m) := \mathcal{F} \otimes \mathcal{O}_X(m)$ is generated by global sections. So there is a surjection

$$\mathcal{O}_X^{\oplus N} \twoheadrightarrow \mathcal{F}(m)$$

for some N. Twisting back by $\mathcal{O}_X(-m)$, we get a surjection from a vector bundle onto $\mathcal{F}$:

$$\mathcal{O}_X(-m)^{\oplus N} \twoheadrightarrow \mathcal{F}.$$

Call this vector bundle $E_0 := \mathcal{O}_X(-m)^{\oplus N}$.

Let $E_1 := \ker(E_0 \twoheadrightarrow \mathcal{F})$. Then we have an exact sequence

$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0.$$

Two key facts apply: a smooth projective curve is regular of dimension 1, so its local rings $\mathcal{O}_{X,x}$ are discrete valuation rings (DVRs), and over a DVR, any finitely generated torsion-free module is free. The kernel E_1 is a coherent subsheaf of the vector bundle E_0, so it is torsion-free. A torsion-free coherent sheaf on a smooth curve is therefore locally free (since its stalk at each point is a torsion-free $\mathcal{O}_{X,x}$-module, hence free). So E_1 is also a vector bundle, giving us a short exact sequence of vector bundles resolving $\mathcal{F}$:

$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0.$$

Because $\mathcal{F}$ has a finite locally free resolution, you can compute $\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{G})$ by applying $\mathrm{Hom}(-, \mathcal{G})$ to that resolution. Since Serre duality is already known for vector bundles (the terms E_0, E_1), you can pass to $\mathcal{F}$ by homological algebra and the same duality formula extends to arbitrary coherent $\mathcal{F}$.

The statement in fact generalizes to coherent sheaves on higher-dimensional varieties:

$$\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{G}) \;\cong\; \mathrm{Ext}^{n-i}_X(\mathcal{G}, \mathcal{F} \otimes \omega_X)^*$$

functorially in $\mathcal{F}, \mathcal{G}$. This recovers the previous Serre duality since

$$H^i(X, \mathcal{F}) \cong \mathrm{Ext}^i_X(\mathcal{O}_X, \mathcal{F}).$$

In the language of derived categories, this duality is encoded by a **Serre functor**:

$$S_X(-) := - \otimes \omega_X[n] : D^b(X) \to D^b(X)$$

such that

$$\mathrm{Hom}_{D^b(X)}(A, B) \;\cong\; \mathrm{Hom}_{D^b(X)}(B, S_X A)^*$$

naturally in A, B.

Definition 2.1. An object $P \in D^b(X)$ is a *point object* if:

1. $S_X(P) \simeq P[n]$ (it has the expected Serre duality of a codimension-n object),

2. $\operatorname{Hom}^{<0}(P, P) = 0$,

3. $\operatorname{Hom}^0(P, P)$ is a field (no nontrivial idempotents).

Theorem 2.2 (Bondal–Orlov, special case)**.** If ω_X is ample or anti-ample, then the point objects in $D^b(X)$ are exactly the shifts of skyscraper sheaves:

$$P \text{ point object} \quad \Longleftrightarrow \quad P \simeq \mathcal{O}_x[m] \quad \text{for some closed } x \in X,\ m \in \mathbb{Z}.$$

Definition 2.3. An object $L \in D^b(X)$ is *invertible* if for every point object P there is an integer s such that

$$\operatorname{Hom}^s(L, P) \cong k, \qquad \operatorname{Hom}^i(L, P) = 0 \text{ for } i \neq s.$$

Theorem 2.4. Under the same positivity assumption on ω_X, the invertible objects in $D^b(X)$ are exactly the shifts of line bundles:

$$L \text{ invertible} \quad \Longleftrightarrow \quad L \simeq \mathcal{L}[t], \quad \mathcal{L} \in \operatorname{Pic}(X).$$

Theorem 2.5 (Bondal–Orlov reconstruction)**.** Let X and Y be smooth projective varieties over k whose canonical bundles ω_X, ω_Y are ample or anti-ample. If there is a triangulated equivalence

$$F : D^b_{\mathrm{coh}}(X) \xrightarrow{\sim} D^b_{\mathrm{coh}}(Y),$$

then X and Y are isomorphic as varieties.

Proof. An equivalence F preserves Serre functors and Ext groups. Therefore F sends point objects to point objects and invertible objects to invertible objects.

This induces a bijection between closed points $\{\mathcal{O}_x[m]\} \leftrightarrow \{\mathcal{O}_y[m]\}$ and an isomorphism of Picard groups (line bundles).

Using compatibility with the Serre functor, one checks that $F(\omega_X^k) \simeq \omega_Y^k$ for all k, and hence

$$H^0(X, \omega_X^k) \cong H^0(Y, \omega_Y^k)$$

as rings in k. If ω_X is ample or anti-ample, the canonical (or anti-canonical) ring

$$R(X, \omega_X) := \bigoplus_{k \geq 0} H^0(X, \omega_X^k)$$

determines X as a projective variety:

$$X \cong \operatorname{Proj} R(X, \omega_X), \qquad Y \cong \operatorname{Proj} R(Y, \omega_Y).$$

Since the rings are isomorphic, $\operatorname{Proj} R(X, \omega_X) \cong \operatorname{Proj} R(Y, \omega_Y)$, so $X \cong Y$. $\square$