Jacobian varieties

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Abstract

Divisors on algebraic curves, their associated linear systems, and the construction of Jacobian varieties.

Contents

1 Divisors and Line Bundles

Let X be a smooth projective curve over an algebraically closed field k.

1.1 Divisors on Curves

Definition 1.1 (Divisor)**.** *Let X be a smooth projective curve over an algebraically closed field k. A **divisor** D on X is a formal finite sum*

$$D = \sum_{P \in X} n_P \cdot P$$

where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many points $P \in X$.

Definition 1.2 (Degree of a Divisor)**.** *The* ***degree*** *of a divisor* $D = \sum_{P \in X} n_P \cdot P$ *is defined as the sum of its coefficients:*

$$\deg(D) = \sum_{P \in X} n_P$$

Definition 1.3 (Effective Divisor)**.** *A divisor* $D = \sum_{P \in X} n_P \cdot P$ *is* ***effective****, denoted* $D \geq 0$*, if all coefficients are non-negative, i.e.,* $n_P \geq 0$ *for all* $P \in X$*.*

The set of all divisors on X forms an abelian group $\mathrm{Div}(X)$ under addition.

1.2 Principal Divisors

Definition 1.4 (Principal Divisor)**.** *Let* $f \in k(X)^*$ *be a non-zero rational function on* X*. The* ***principal divisor*** *of* f*, denoted* $div(f)$*, is defined as*

$$div(f) = \sum_{P \in X} \mathrm{ord}_P(f) \cdot P$$

where $\mathrm{ord}_P(f)$ *is the order of vanishing of* f *at* P *(with zeros counted positively and poles negatively).*

Proposition 1.5. *For any rational function* $f \in k(X)^*$*, the degree of its principal divisor is zero:*

$$\deg(div(f)) = 0$$

Remark 1.6. *Note that principal divisors of rational functions always have degree zero. However principal divisors of general line bundles have degree equal to the degree of the line bundle. This is a point that I often get confused about, and is an important distinction to keep in mind.*

This fundamental property follows from the fact that a rational function on a complete curve has an equal number of zeros and poles, counting multiplicity.

Definition 1.7 (Linear Equivalence)**.** *Two divisors* $D, E \in \mathrm{Div}(X)$ *are* ***linearly equivalent****, denoted* $D \sim E$*, if their difference is a principal divisor, i.e.,* $D - E = div(f)$ *for some* $f \in k(X)^*$*.*

The set of principal divisors $\mathrm{PDiv}(X)$ forms a subgroup of $\mathrm{Div}(X)$. The quotient group

$$\mathrm{Cl}(X) = \mathrm{Div}(X) / \mathrm{PDiv}(X)$$

is called the **divisor class group** or **Picard group** of X, denoted $\mathrm{Pic}(X)$.

1.3 Divisors and Line Bundles

For each divisor D on X, we can associate a line bundle $\mathcal{L}(D)$ defined as follows:

Definition 1.8 (Line Bundle Associated to a Divisor)**.** *For a divisor $D = \sum_P n_P \cdot P$ on X, the line bundle $\mathcal{L}(D)$ is the sheaf of sections s such that div$(s) + D \geq 0$.*

Theorem 1.9. *The map $D \mapsto \mathcal{L}(D)$ induces an isomorphism between $\mathrm{Pic}(X)$ and the group of isomorphism classes of line bundles on X.*

Under this correspondence:

- $\mathcal{L}(D_1 + D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)$
- $\mathcal{L}(-D) \cong \mathcal{L}(D)^{-1}$
- $\mathcal{L}(0) \cong \mathcal{O}_X$ (the structure sheaf)

Definition 1.10 (Complete Linear System)**.** *The **complete linear system** associated to a divisor D, denoted $|D|$, is the set of effective divisors linearly equivalent to D:*

$$|D| = \{E \in \mathrm{Div}(X) : E \geq 0, E \sim D\}$$

Proposition 1.11. *The complete linear system $|D|$ is isomorphic to $\mathbb{P}(H^0(X, \mathcal{L}(D)))$.*

Explicitly, this bijection works as follows:

- For a non-zero section $s \in H^0(X, \mathcal{L}(D))$, the divisor of zeros $(s)_0$ satisfies $(s)_0 \sim D$.
- For any effective divisor $E \sim D$, there exists a section $s \in H^0(X, \mathcal{L}(D))$, unique up to scalar multiplication, such that $(s)_0 = E$.

Theorem 1.12 (Riemann-Roch)**.** *Let X be a smooth projective curve of genus g, and let D be a divisor on X. Then*

$$l(D) - l(K_X - D) = \deg(D) + 1 - g$$

where K_X is the canonical divisor on X.

2 The Abel-Jacobi Map

Let X be a smooth projective curve of genus g, and let $P_0 \in X$ be a fixed base point.

Definition 2.1 (Abel-Jacobi Map). *The* ***Abel-Jacobi map*** $\Phi : X \to \mathrm{Pic}^0(X)$ *is defined by*

$$\Phi(P) = [P - P_0]$$

where $[P - P_0]$ *denotes the linear equivalence class of the divisor* $P - P_0$*.*

This map can be extended to the n-fold symmetric product of X:

Definition 2.2 (Extended Abel-Jacobi Map). *The* n*-th Abel-Jacobi map* $\Phi_n : \mathrm{Sym}^n(X) \to \mathrm{Pic}^0(X)$ *is defined by*

$$\Phi_n(P_1, P_2, \dots, P_n) = [P_1 + P_2 + \cdots + P_n - n \cdot P_0]$$

Theorem 2.3. *When* $n \geq g$*, the Abel-Jacobi map* $\Phi_n : \mathrm{Sym}^n(X) \to \mathrm{Pic}^0(X)$ *is surjective.*

Proof. Given a divisor class $[D] \in \mathrm{Pic}^0(X)$, we can represent it as $[D' - g \cdot P_0]$ where $\deg(D') = g$. By Riemann-Roch, $l(D') \geq 1$, which implies that the linear system $|D'|$ is non-empty. Therefore, there exists an effective divisor $E = P_1 + P_2 + \cdots + P_g$ linearly equivalent to D'. Thus, $[D] = [E - g \cdot P_0]$, which shows that Φ_g is surjective. $\square$

2.1 Analytical Construction over Complex Numbers

Over the complex numbers, the Abel-Jacobi map has a beautiful analytical interpretation:

Proposition 2.4. *Let* X *be a complex curve of genus* g*. Choose a basis* $\{\omega_1, \dots, \omega_g\}$ *of holomorphic differentials on* X*. The Abel-Jacobi map can be expressed as*

$$\Phi(P) = \left(\int_{P_0}^{P} \omega_1, \int_{P_0}^{P} \omega_2, \dots, \int_{P_0}^{P} \omega_g \right) \mod \Lambda$$

where Λ *is a lattice in* $\mathbb{C}^g$ *generated by the periods of these differentials along a basis of* $H_1(X, \mathbb{Z})$*.*

Theorem 2.5 (Abel's Theorem). *Points* $P_1, \dots, P_n, Q_1, \dots, Q_n$ *on* X *satisfy*

$$[P_1 + \cdots + P_n] = [Q_1 + \cdots + Q_n]$$

in $\mathrm{Pic}(X)$ *if and only if*

$$\sum_{i=1}^{n} \Phi(P_i) = \sum_{i=1}^{n} \Phi(Q_i)$$

in $\mathrm{Jac}(X)$*.*

3 Jacobian Varieties

Definition 3.1 (Jacobian Variety)**.** *The **Jacobian variety** of a smooth projective curve X of genus g is the group $\mathrm{Pic}^0(X)$ of divisor classes of degree 0. It is denoted $\mathrm{Jac}(X)$.*

Proposition 3.2. *The Jacobian $\mathrm{Jac}(X)$ satisfies the following properties:*

1. *It is a smooth projective variety.*
2. *It has a natural structure as an abelian group.*
3. *The group operations are morphisms of varieties.*
4. *Its dimension equals the genus g of X.*
5. *Its tangent space at the identity is canonically isomorphic to $H^1(X, \mathcal{O}_X)$.*

In particular $\mathrm{Jac}(X)$ is an abelian variety of dimension g.

Example 3.3. *For a curve X of genus $g = 0$ (e.g., $\mathbb{P}^1$), the Jacobian $\mathrm{Jac}(X)$ is trivial, consisting of just the identity element.*

Example 3.4. *For a curve X of genus $g = 1$ (an elliptic curve), the Jacobian $\mathrm{Jac}(X)$ is isomorphic to X itself as both a variety and an abelian group.*

Example 3.5. *For curves of genus $g > 1$, the Jacobian $\mathrm{Jac}(X)$ is a g-dimensional abelian variety distinct from X.*

Theorem 3.6 (Torelli's Theorem)**.** *Two smooth projective curves are isomorphic if and only if their Jacobians are isomorphic as principally polarized abelian varieties.*

Proposition 3.7. *A consequence of Example 1.4.3: Let X be a non-singular projective curve of genus g over an algebraically closed field. Then the Picard group $\mathrm{Pic}(X)$ of divisor classes on X is not finitely generated when $g > 0$.*

3.1 The Theta Divisor

Definition 3.8 (Theta Divisor)**.** *Let X be a curve of genus $g \geq 2$. The image of*

$$\Phi_{g-1} : \mathrm{Sym}^{g-1}(X) \to \mathrm{Jac}(X)$$

*defines an effective divisor Θ on $\mathrm{Jac}(X)$, called the **theta divisor**.*

Theorem 3.9. *The theta divisor Θ is ample and defines a principal polarization on $\mathrm{Jac}(X)$.*