Tropical Semirings

A general method for declaratively solving graph problems

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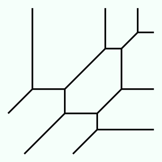

Handout

- You may obtain a copy of these notes and code samples on my website
- Code samples are in Haskell and were tested on GHC 9.2.4

Links

- https://simonzeng.com/tropical.pdf
- https://simonzeng.com/tropical.hs

Motivating problem

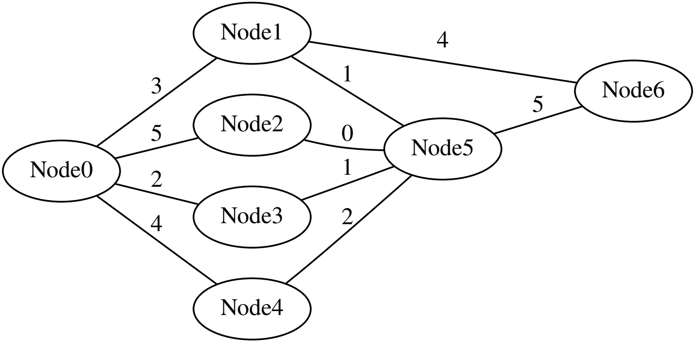

Figure 1: Graph where edge weights represent **number of paths**

- Problem: find the number of paths between two nodes on a graph that traverses n edges?
- Example: there are 3 paths from 0 to 2 that traverse 1 edge

Solving imperatively

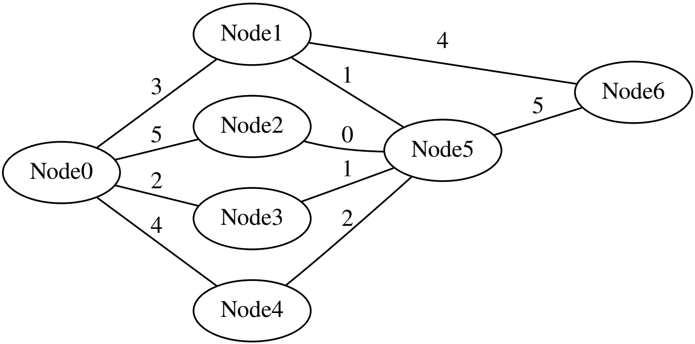

Figure 2: The graph

- DFS, priority queue, etc
- Works, but a bit painful

Towards an elegant solution

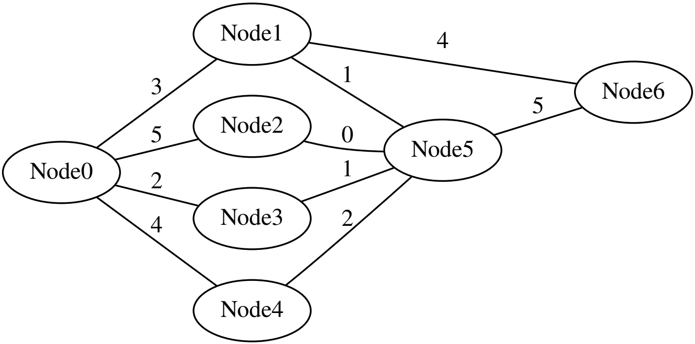

Figure 3: The graph

- Example: What exactly are we doing when we want to solve for paths between 0 and 5 with $n = 2$?
- $Paths(0, 5) = 3 \cdot 1 + 5 \cdot 0 + 2 \cdot 1 + 4 \cdot 2 = 13$

What does that look like?

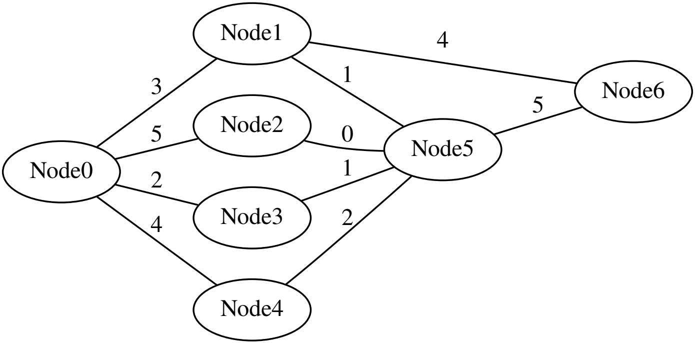

Figure 4: The graph

Solution for $n = 4$ as a dot product

$$Paths(0, 5) = (3, 5, 2, 4) \cdot (1, 0, 1, 2) = 13$$

Expanding it out

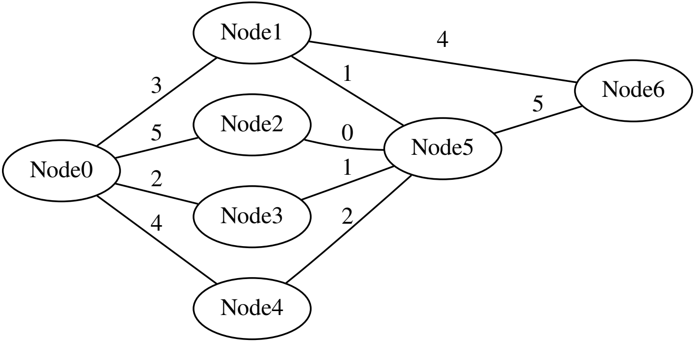

Figure 5: The graph

- Actually, there are 0 paths of length 1 between two unconnected nodes, so the full dot product looks like:

$$Paths(0,5) = (0,3,5,2,4,0,0) \cdot (0,1,0,1,2,0,5) = 13$$

The solution for $n = 2$ is a dot product

Our previous exmple

$$Paths(0,5) = (0,3,5,2,4,0,0) \cdot (0,1,0,1,2,0,5) = 13$$

- If the nodes are labelled $u, v, a, b, c, \cdots$ and edges between nodes u and v are labelled e_{uv}, then we have:

$$Paths(u,v) = (e_{ua}, e_{ub}, e_{uc}, \cdots) \cdot (e_{av}, e_{bv}, e_{cv}, \cdots,)$$

- The first vector is all the edges that start at u
- The second vector is the edges that end at v

A global answer

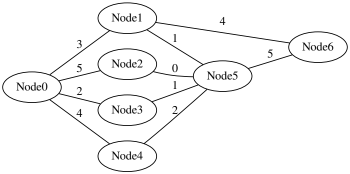

Figure 6: The graph

- Consider the adjacency matrix of the graph:

$$\begin{bmatrix} 0 & 3 & 5 & 2 & 4 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 4 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 & 5 \\ 0 & 4 & 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

- Do you spot the $Path(0,5)$ dot product?

A different problem?

swapping semiring to shortest distance

Get Closure

introduce calculation of $1 + a + a\hat{}2 + a\hat{}3 + \ldots$

Choose your character

- longest path
- widest flow
- reconstructing path
- dfa->regex
- inverting matrices
- determine graph is bipartite