2022-11-29

Tropical Semirings

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A general method for declaratively solving graph problems

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Handout

- You may obtain a copy of these notes and code samples on my website
- Code samples are in Haskell and were tested on GHC 9.2.4

Links

- https://simonzeng.com/tropical.pdf
- https://simonzeng.com/tropical.hs

- Greeting
- Spoiler: Functional programming talk
  - Not lambda
  - Not purity
  - Not monads
- Solving real problems declaratively
  - Graph problems!
- We develop an algebraic method that elegantly describes a class of graph problems and their solutions
- These notes are available on my website

2022-11-29

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└─Motivating problem

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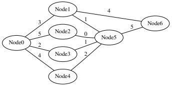


Figure 1: Graph where edge weights represent **number of paths**

- Problem: find the number of paths between two nodes on a graph that traverses n edges?
- Example: there are 3 paths from 0 to 2 that traverse 1 edge

Solving imperatively

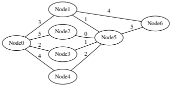


Figure 2: The graph

- DFS, priority queue, etc
- Works, but a bit painful

- Informally go over an imperative DFS
- Do example with $n = 3$

Towards an elegant solution

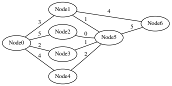
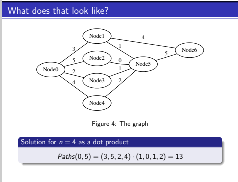


Figure 3: The graph

- Example: What exactly are we doing when we want to solve for paths between 0 and 5 with $n = 2$?
- $Paths(0, 5) = 3 \cdot 1 + 5 \cdot 0 + 2 \cdot 1 + 4 \cdot 2 = 13$

- What does this look like? A linear combination!
- It's actually a dot product!



- Explain where the numbers come from

Expanding it out

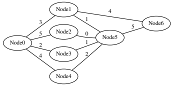


Figure 5: The graph

- Actually, there are 0 paths of length 1 between two unconnected nodes, so the full dot product looks like:

$$Paths(0,5) = (0,3,5,2,4,0,0) \cdot (0,1,0,1,2,0,5) = 13$$

- Explain where the numbers come from again, but with the 0s

The solution for $n = 2$ is a dot product

Our previous exmple

$$Paths(0, 5) = (0, 3, 5, 2, 4, 0, 0) \cdot (0, 1, 0, 1, 2, 0, 5) = 13$$

- If the nodes are labelled $u, v, a, b, c, \cdots$ and edges between nodes u and v are labelled e_{uv}, then we have:

$$Paths(u, v) = (e_{ua}, e_{ub}, e_{uc}, \cdots) \cdot (e_{av}, e_{bv}, e_{cv}, \cdots,)$$

- The first vector is all the edges that start at u
- The second vector is the edges that end at v

So if the answer is just dot products, what if we want to calculate the number of paths between two nodes for the entire graph at once?
A bunch of dot products??? What's that called?

It's a matrix product!

A global answer

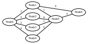


Figure 6: The graph

- Consider the adjacency matrix of the graph:

$$\begin{bmatrix} 0 & 3 & 5 & 2 & 4 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 4 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 & 5 \\ 0 & 4 & 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

- Do you spot the $Path(0,5)$ dot product?

└─A different problem?

A different problem?

swapping semiring to shortest distance

2022-11-29

└─Get Closure

Get Closure

introduce calculation of $1 + a + a\hat{}2 + a\hat{}3 + \ldots$

2022-11-29

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└─Choose your character

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- longest path
- widest flow
- reconstructing path
- dfa->regex
- inverting matrices
- determine graph is bipartite