Tropical Semirings

A general method for declaratively solving graph problems

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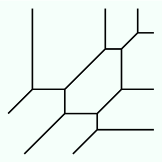


Handout

- You may obtain a copy of these notes and code samples on my website
- Code samples are in Haskell and were tested on GHC 9.2.4

Links

- https://simonzeng.com/tropical.pdf
- https://simonzeng.com/tropical.hs

Motivating problem

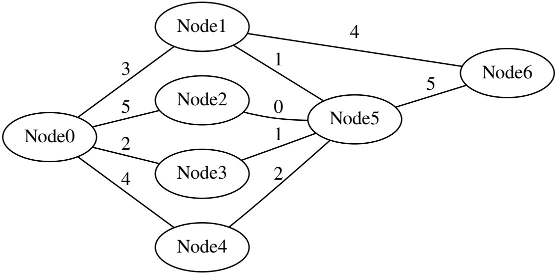


Figure 1: Graph where edge weights represent **number of paths**

- Problem: find the number of paths between two nodes on a graph that traverses n edges?
- Example 1: there are 20 paths from node 1 to node 5 that traverse 2 edges
- Example 2: there are 41 paths from node 6 to node 6 that traverse 2 edges (edge reuse allowed)

Solving imperatively

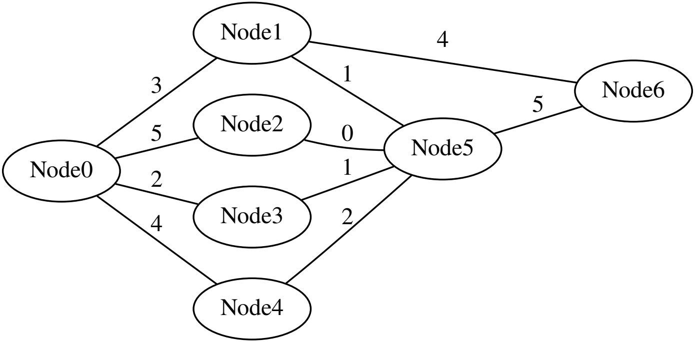


Figure 2: The graph

- DFS, priority queue, etc
- Works, but a bit painful

Towards an elegant solution

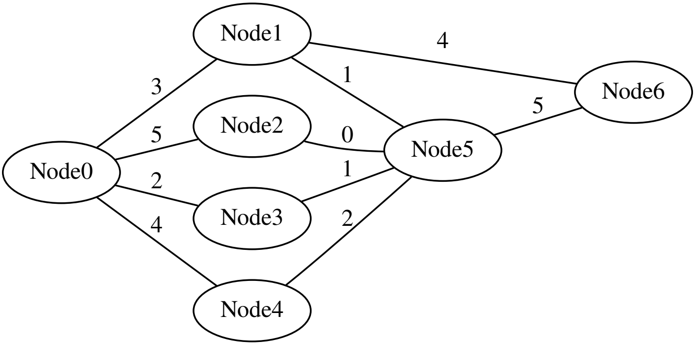


Figure 3: The graph

- Example: What exactly are we doing when we want to solve for paths between 0 and 5 with $n = 2$?
- $Paths(0, 5) = 3 \times 1 + 5 \times 0 + 2 \times 1 + 4 \times 2 = 13$

What does that look like?

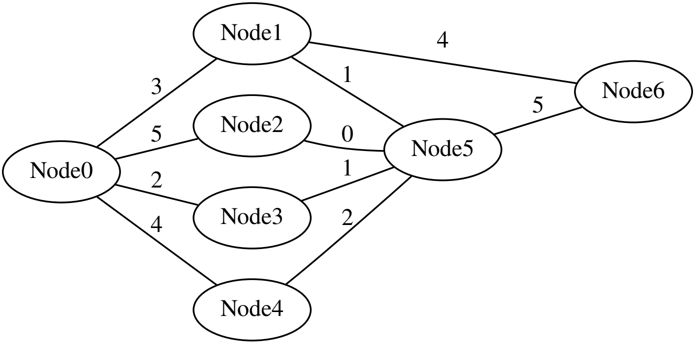


Figure 4: The graph

Solution for $n = 4$ as a dot product

$$Paths(0,5) = (3,5,2,4) \times (1,0,1,2) = 13$$

Expanding it out

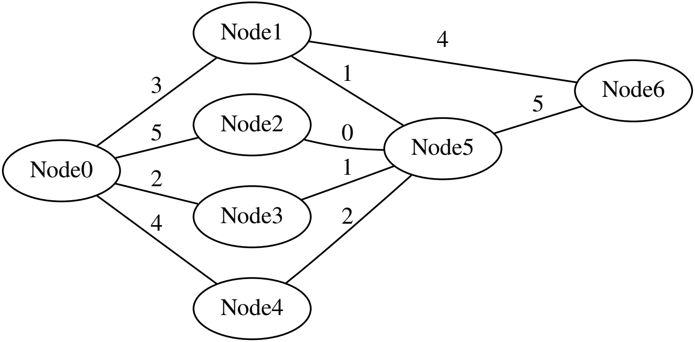


Figure 5: The graph

- Actually, there are 0 paths of length 1 between two unconnected nodes, so the full dot product looks like:

$$Paths(0,5) = (0,3,5,2,4,0,0) \times (0,1,0,1,2,0,5) = 13$$

The solution for $n = 2$ is a dot product

Our previous exmple

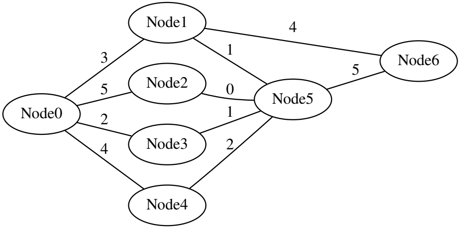
$$Paths(0,5) = (0,3,5,2,4,0,0) \times (0,1,0,1,2,0,5) = 13$$

- If the nodes are labelled $u, v, a, b, c, \cdots$ and edges between nodes u and v are labelled e_{uv}, then we have:

$$Paths(u,v) = (e_{ua}, e_{ub}, e_{uc}, \cdots) \times (e_{av}, e_{bv}, e_{cv}, \cdots)$$

- The first vector is all the edges that start at u
- The second vector is the edges that end at v

A global answer

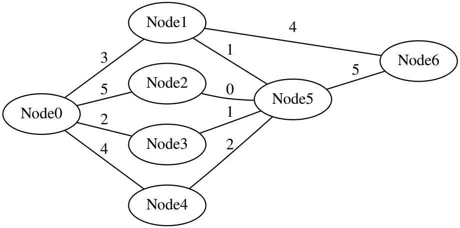


- Adjacency Matrix: matrix where ith row and jth column is weight of edge ij
- Consider the adjacency matrix of the graph:

$$a = \begin{bmatrix} 0 & 3 & 5 & 2 & 4 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 4 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 & 5 \\ 0 & 4 & 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

- Do you spot the $Path(0, 5)$ dot product?

A global answer



To get the correct dot products for every pair of nodes, we can simply **square the matrix**:

$$a^2 = \begin{bmatrix} 0 & 3 & 5 & 2 & 4 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 4 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 & 5 \\ 0 & 4 & 0 & 0 & 0 & 5 & 0 \end{bmatrix}^2 = \begin{bmatrix} 54 & 0 & 0 & 0 & 0 & 13 & 12 \\ 0 & 26 & 15 & 7 & 14 & 20 & 5 \\ 0 & 15 & 25 & 10 & 20 & 0 & 0 \\ 0 & 7 & 10 & 5 & 10 & 0 & 5 \\ 0 & 14 & 20 & 10 & 20 & 0 & 10 \\ 13 & 20 & 0 & 0 & 0 & 31 & 4 \\ 12 & 5 & 0 & 5 & 10 & 4 & 41 \end{bmatrix}$$

Extending the global answer

- What if we want to find the answer for other n?
- Simply **take a to the nth power!**
- Example: a^5 represents the number of paths between nodes that traverse exactly 5 edges:

$$a^5 = \begin{bmatrix} 1560 & 15608 & 16145 & 7611 & 15222 & 6580 & 7045 \\ 15608 & 4480 & 1600 & 1800 & 3600 & 7181 & 12084 \\ 16145 & 1600 & 0 & 300 & 600 & 5765 & 5960 \\ 7611 & 1800 & 300 & 280 & 560 & 3852 & 2928 \\ 15222 & 3600 & 600 & 560 & 1120 & 7704 & 5856 \\ 6580 & 7181 & 5765 & 3852 & 7704 & 4680 & 12370 \\ 7045 & 12084 & 5960 & 2928 & 5856 & 12370 & 5080 \end{bmatrix}$$

- Lends to efficient implementation via square-and-multiply!

A more useful answer

- What if we want to know the number of paths that use **less or equal to** n moves, rather than exactly n?
- Solution: **replace the diagonals of** a **with 1**
- Then there will always be a "no-op" action, where staying at the same node counts as a move
- So a^n will give all the number of paths between nodes that use less or equal to n moves

$$a^2 = \begin{bmatrix} 1 & 3 & 5 & 2 & 4 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 4 \\ 5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 & 5 \\ 0 & 4 & 0 & 0 & 0 & 5 & 1 \end{bmatrix}^2 = \begin{bmatrix} 55 & 6 & 10 & 4 & 8 & 13 & 12 \\ 6 & 27 & 15 & 7 & 14 & 22 & 13 \\ 10 & 15 & 26 & 10 & 20 & 0 & 0 \\ 4 & 7 & 10 & 6 & 10 & 2 & 5 \\ 8 & 14 & 20 & 10 & 21 & 4 & 10 \\ 13 & 22 & 0 & 2 & 4 & 32 & 14 \\ 12 & 13 & 0 & 5 & 10 & 14 & 42 \end{bmatrix}$$

A different problem?

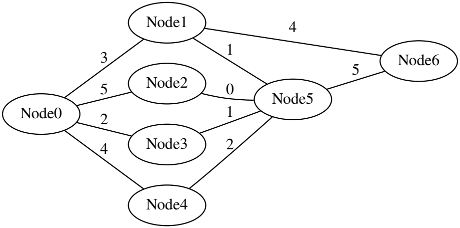


Figure 6: Graph where edge weights represent **distance between nodes**

- Problem: find the **shortest** path between two nodes on a graph
- Traditional ways: Dijkstra, Bellman-Ford, etc.
    - Lots of state, not the most elegant
- Can we do better (or at least cleaner)?

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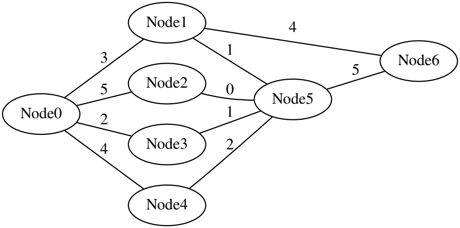

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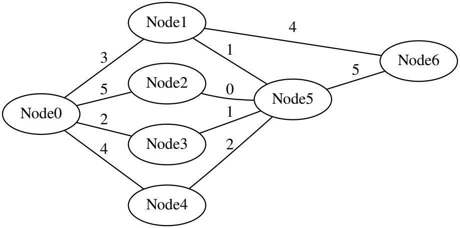

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- Traditional ways: Dijkstra, Bellman-Ford, etc.
  - Lots of state, not the most elegant
- Can we do better (or at least cleaner)?**Yes**
- Solution: **the exact same solution as the number of paths problem!**

Recall the solution

The Algorithm

1. Get the adjacency matrix a of the graph
   a. Row i, Col j has weight of edge ij
   b. Row i, Col j has additive identity 0 if node i and j are unconnected
   c. The diagonals are filled with the multiplicative identity 1
2. Take the nth power of a, for whatever value of n we want to calculate up to

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- A: **Changing the algebraic structure we work in!**

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- In the shortest path problem, we were working in $\mathbb{N}$
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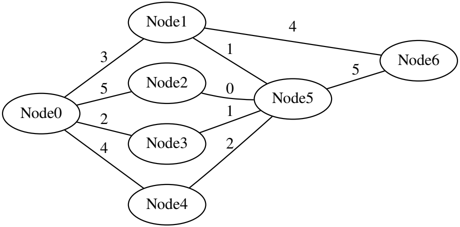
  - Related laws (i.e. associativity, distributivity, etc)
- An algebraic structure that has these properties is called a **semiring**
- Matrices over a semiring also form a semiring
- We can run our graph algorithm with **any** semiring

So what semiring will help us solve shortest path?



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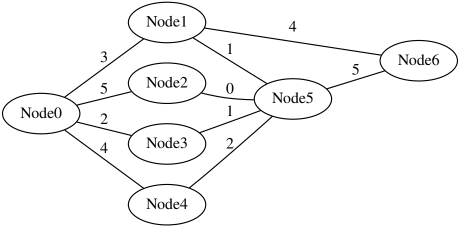

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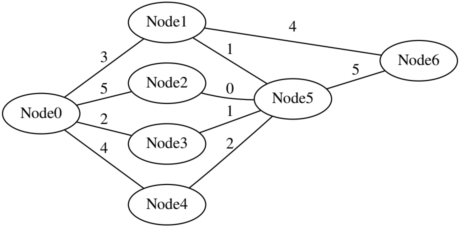

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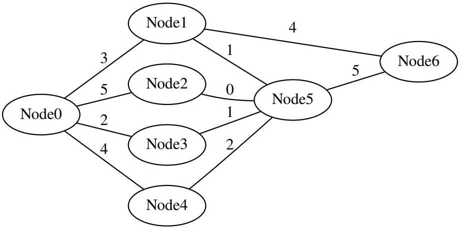

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- Example: how do we get the shortest distance between node 0 and node 5?
- Answer: $\mathit{Shortest}(0,5) = \mathit{min}(3+1, 5+0, 2+1, 4+2) = 3$
- Compare to the number of paths problem: $\mathit{Paths}(0,5) = 3 \times 1 + 5 \times 0 + 2 \times 1 + 4 \times 2 = 13$
- If you replace multiplication with addition, and addition with *min*, then you can convert between these!

Introducing the tropical semiring

Defining a semiring

Definition A **semiring** is a 3-tuple $(S, +, \times)$ of a set S and binary operators $+, \times$ such that $\forall a, b, c \in S$ the following laws hold:

1. Commutivity of $+$: $a + b = b + a$
2. Associativity of $+$: $a + (b + c) = (a + b) + c$
3. Additive identity: $\exists 0 \in S, a + 0 = a$
4. Associativity of $\times$: $a \times (b \times c) = (a \times b) \times c$
5. Multiplicative identity: $\exists 1 \in S, a \times 1 = 1 \times a = 1$
6. Multiplying by additive identity: $a \times 0 = 0 \times a = 0$
7. Distributivity 1: $a \times (b + c) = a \times b + a \times c$
8. Distributivity 2: $(a + b) \times c = a \times c + b \times c$

The **Tropical Semiring** is the semiring $(\mathbb{N} \cup \{\infty\}, min, +)$

- Additive identity is ∞
- Multiplicative identity is $0 \in \mathbb{N}$

The Tropical Semiring Solves Shortest Path

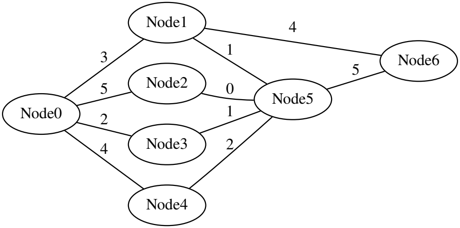

Figure 8: The graph

- Recall: $\mathit{Shortest}(0,5) = \mathit{min}(3+1, 5+0, 2+1, 4+2) = 3$

The Tropical Semiring Solves Shortest Path

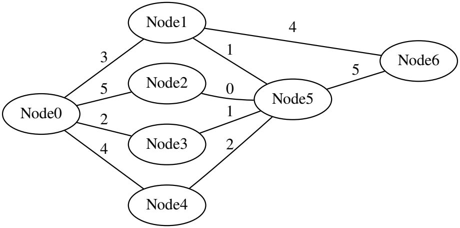

Figure 8: The graph

- Recall: $Shortest(0,5) = min(3+1, 5+0, 2+1, 4+2) = 3$
- Written with the tropical semiring:

$$Shortest(0,5) = 3 \times 1 + 5 \times 0 + 2 \times 1 + 4 \times 2 = 3$$

$$= (3,5,2,4) \cdot (1,0,1,2)$$

- **Dot products just work!**
- Hence, powers of the adjacency matrix just work as well!

Constructing our adjacency matrix

The Algorithm

1. Get the adjacency matrix a of the graph
   a. Row i, Col j has weight of edge ij
   b. Row i, Col j has additive identity O if node i and j are unconnected
   c. The diagonals are filled with the multiplicative identity I
2. Take the nth power of a, for whatever value of n we want to calculate up to, **under whatever semiring solves our problem**

$$a = \begin{bmatrix} I & 3 & 5 & 2 & 4 & O & O \\ 3 & I & O & O & O & 1 & 4 \\ 5 & O & I & O & O & 0 & O \\ 2 & O & O & I & O & 1 & O \\ 4 & O & O & O & I & 2 & O \\ O & 1 & 0 & 1 & 2 & I & 5 \\ O & 4 & O & O & O & 5 & I \end{bmatrix} = \begin{bmatrix} 0 & 3 & 5 & 2 & 4 & \infty & \infty \\ 3 & 0 & \infty & \infty & \infty & 1 & 4 \\ 5 & \infty & 0 & \infty & \infty & 0 & \infty \\ 2 & \infty & \infty & 0 & \infty & 1 & \infty \\ 4 & \infty & \infty & \infty & 0 & 2 & \infty \\ \infty & 1 & 0 & 1 & 2 & 0 & 5 \\ \infty & 4 & \infty & \infty & \infty & 5 & 0 \end{bmatrix}$$

Solving shortest path

$$a^{\text{num edges}+1} = \begin{bmatrix} 0 & 3 & 5 & 2 & 4 & \infty & \infty \\ 3 & 0 & \infty & \infty & \infty & 1 & 4 \\ 5 & \infty & 0 & \infty & \infty & 0 & \infty \\ 2 & \infty & \infty & 0 & \infty & 1 & \infty \\ 4 & \infty & \infty & \infty & 0 & 2 & \infty \\ \infty & 1 & 0 & 1 & 2 & 0 & 5 \\ \infty & 4 & \infty & \infty & \infty & 5 & 0 \end{bmatrix}^{11}$$

$$= \begin{bmatrix} 0 & 3 & 3 & 2 & 4 & 3 & 7 \\ 3 & 0 & 1 & 2 & 3 & 1 & 4 \\ 3 & 1 & 0 & 1 & 2 & 0 & 5 \\ 2 & 2 & 1 & 0 & 3 & 1 & 6 \\ 4 & 3 & 2 & 3 & 0 & 2 & 7 \\ 3 & 1 & 0 & 1 & 2 & 0 & 5 \\ 7 & 4 & 5 & 6 & 7 & 5 & 8 \end{bmatrix}$$

Choose your character

By selecting different semirings, we can use this same algorithm to solve multiple different problems:

- Connectedness: $(\mathbb{Z}_2, +, *)$
- Longest path: $(\mathbb{N} \cup \{-\infty\}, \mathit{max}, +)$
- Widest flow: $(\mathbb{N} \cup \{-\infty, \infty\}, \mathit{max}, \mathit{min})$
- Dfa $\rightarrow$ Regex: $(\text{Strings}, \text{Or}, \text{concat})$

The following are left as an exercise:

- Determine graph is bipartite
- Inverting matrices
- Knapsack problem

The end

Thanks for listening!

Extra note: getting closure

Closure

Definition The **closure** of semiring element a, written as a^*, is defined as the fixed point:

$$a^* = 1 + a \times a^*$$

You can think of it as:

$$a^* = 1 + a + a^2 + a^3 + a^4 + a^5 + \cdots$$

There exists a closed form solution for closure of a semiring matrix

For DFA to regex and a few other problems on the last list, you need to calculating the closure of a matrix instead of a big power.