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Tropical Semirings

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A general method for declaratively solving graph problems

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- You may obtain a copy of these notes and code samples on my website
- Code samples are in Haskell and were tested on GHC 9.2.4

Links
- https://simonzeng.com/tropical.pdf
- https://simonzeng.com/tropical.hs

- Greeting
- Functional programming talk
  - Not lambda
  - Not purity
  - Not monads
- Solving real problems declaratively
  - Graph problems!
- We develop an algebraic method that elegantly describes a class of graph problems and their solutions
- These notes are available on my website

Dijkstra's Shortest Path Algorithm

- Q: How do we traditionally get the shortest path between two nodes on a graph?

- Q: read
- (next, then read)
- Everyone love's Dijkstra's
- Set up priority queue, put things into it, take things out of it, stop iterating based on it
- (next, then read)
- It's a very imperative algorithm
- CS 341 shows a few other graph path algorithms
  - from that you'd think they're all inherently imperative
- But: graph problems are **not** inherently imperative!
- Let's zoom in to the meat

Dijkstra's Shortest Path Algorithm

- Q: How do we traditionally get the shortest path between two nodes on a graph?
- A: *Dijkstra's algorithm*

Pseudocode

```
function Dijkstra(Graph, source):
    dist[source] <- 0
    create vertex priority queue Q
    for each vertex v in Graph.Vertices:
        if v != source
            (dist[v], prev[v]) <- INFINITY, UNDEFINED
        Q.add_with_priority(v, dist[v])
    while Q is not empty:
        u <- Q.extract_min()
        for each neighbor v of u:
            alt <- dist[u] + Graph.Edges(u, v)
            if alt < dist[v]:
                (dist[v], prev[v]) <- alt, u
                Q.decrease_priority(v, alt)
    return dist, prev
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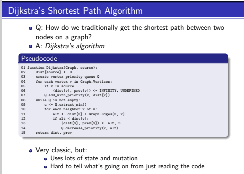

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Core of Dijkstra's Algorithm

- When going from some node u to some node v:

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The Essence of the Path Algorithm

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- Dijkstra's is just node ordering boilerplate around this core operation

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- This is the core of path algorithms!

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Essence of the path algorithm, algebraically

$$dist[v] = dist[v] + dist[u] \cdot Graph.Edges(u, v)$$