Tropical Semirings

A general method for declaratively solving graph problems

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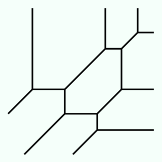

Handout

- You may obtain a copy of these notes and code samples on my website
- Code samples are in Haskell and were tested on GHC 9.2.4

Links

- https://simonzeng.com/tropical.pdf
- https://simonzeng.com/tropical.hs

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Pseudocode

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function Dijkstra(Graph, source):
    dist[source] <- 0
    create vertex priority queue Q
    for each vertex v in Graph.Vertices:
        if v != source
            (dist[v], prev[v]) <- INFINITY, UNDEFINED
        Q.add_with_priority(v, dist[v])
    while Q is not empty:
        u <- Q.extract_min()
        for each neighbor v of u:
            alt <- dist[u] + Graph.Edges(u, v)
            if alt < dist[v]:
                (dist[v], prev[v]) <- alt, u
                Q.decrease_priority(v, alt)
    return dist, prev
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- Very classic, but:
  - Uses lots of state and mutation
  - Hard to tell what's going on from just reading the code

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- Dijkstra's is just node ordering boilerplate around this core operation

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- This is the core of path algorithms!

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Essence of the path algorithm, algebraically

$$dist[v] = dist[v] + dist[u] \cdot Graph.Edges(u, v)$$

What Structure do we Have?