UNBOUNDED CONNECTED ALGEBRAIC SUBGROUPS OF GROUPS BIRATIONAL TRANSFORMATIONS

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1. Introduction

For a variety X we define $\mathrm{Bir}(X)$ to be the group of birational self maps. In general, $\mathrm{Bir}(X)$ is an enormous group, without any reasonable algebraic structure: it need not be an algebraic group or even homeomorphic to an ind-group (see [BF13]). One approach to understanding the structure of $\mathrm{Bir}(X)$ is by studying its *algebraic subgroups*, as well as the *inclusion relations* among them (up to conjugation). Note that the most interesting phenomena appear when X is uniruled, which will be precisely our case. We also restrict to the case when the base field $\mathbf{k}$ is algebraically closed, of characteristic 0.

Studying connected algebraic subgroups. The modern framework for dealing with this problem relies on two main tools: *regularization theorems* and the *(Equivariant) Minimal Model Program*, or *(E)MMP* for short. An algebraic subgroup $G \leq \mathrm{Bir}(X)$ is called *regularizable* if there exists a G-variety Y, together with a G-equivariant birational map $f: X \dashrightarrow Y$. Two notable classes of regularizable groups are *finite groups* and *connected groups*. The main body of work on the field restricts to the cases when $G \leq \mathrm{Bir}(X)$ lies in one of these two classes. Here we will restrict to the connected case. For the finite case see [Dol10] and [Pro23] for a background and further references.

Given a connected algebraic subgroup $G \leq \mathrm{Bir}(X)$, the standard procedure is to regularize it to a smooth variety Y, and then run an EMMP on Y to reduce to the case when G acts on a Mori fiber space: a variety, together with a fibration, having a relatively easy structure. This essentially reduces the problem to studying regular actions on relatively simple varieties with a nice structure.

Controlling inclusion relations. The *G-Sarkisov program* is a (pseudo-)algorithm that decomposes any G-equivariant birational maps between Mori fiber spaces into simpler ones, called *G-Sarkisov links*. The most commonly utilized technique for controlling inclusion relations among connected subgroups $G \leq \mathrm{Bir}(X)$ is by, first reducing to the case when X is a Mori fiber space and then understanding the G-Sarkisov links. This approach relies on the ideas of [Flo20]. More precisely:

Proposition 1 ([Flo20, Corollary 1.4])**.** *Let W be an uniruled variety and let G be a connected algebraic group acting rationally on W. Then G is maximal among the connected groups acting rationally on W if and only if $G = \mathrm{Aut}^\circ(X)$, where X is a Mori fiber space, and for every Mori fiber space Y which is related to X by a finite sequence of G-Sarkisov links we have $G = \mathrm{Aut}^\circ(Y)$.*

Essentially, if G acts on a Mori fiber space X, then G is contained (up to conjugation) in another connected algebraic subgroup of $\mathrm{Bir}(X)$ if and only if there exists a G-Sarkisov link from (a Mori fiber space birational to) X to Y with $G \not\leq \mathrm{Aut}^\circ(Y)$.

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As a guiding principle, maximal connected algebraic subgroups usually show up as automorphism groups of highly symmetrical Mori fiber spaces, with fewer and larger orbits.

2. Classification Results

The first result in the field is due to Enriques [Enr93] and dates all the way back to the 19th century:

Theorem 2. *The maximal connected algebraic subgroups of* $\mathrm{Bir}(\mathbb{P}^2)$ *are the groups* $\mathrm{Aut}^\circ(S)$ *with* S *isomorphic to* $\mathbb{P}^2$ *or* $\mathbb{F}_n$*,* $n \neq 1$*.*

A corollary of the classification and its proof is the following:

Corollary 3. *Any connected algebraic subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *is contained in a maximal one.*

A surprising feature is that even though every subgroup is contained in maximal one we still have infinite non-stationary increasing sequences:

Example 4. *Let* $n \geq d \geq 2$*. Define the connected algebraic groups*

$$G_d = \left\{\mathbb{A}^2 \to \mathbb{A}^2, (x,y) \mapsto (x, y + p(x)), p \in \boldsymbol{k}[x]_{\leq d}\right\},$$

acting regularly on $\mathbb{A}^2$*, and then birationally on* $\mathbb{P}^2$ *via any embedding* $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$*. Then* $G_d \subsetneq G_{d+1}$ *for all* d*. On the other hand, using an explicit description of* $\mathrm{Aut}^\circ(\mathbb{F}_n)$ *from [Bla09, §4.2], we get that* G_d *is a subgroup of* $\mathrm{Aut}^\circ(\mathbb{F}_n)$ *for all* $n \geq d$*.*

In the 3-dimensional case Umemura [Ume80, Ume82a, Ume82b, Ume85], and later Blanc-Fanelli-Terpereau [BFT22, BFT23] gave a classification of maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$: they again appear as automorphism groups of 13 families of Mori fiber spaces. Similar to the 2-dimensional case, the classification implies that every connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^3)$ is contained in a maximal one.

A connected algebraic subgroup of $\mathrm{Bir}(X)$ is called *unbounded* if it is not contained in a maximal one. The previous results on $\mathrm{Bir}(\mathbb{P}^2)$ and $\mathrm{Bir}(\mathbb{P}^3)$ motivate the following question:

Question 5. *Does there exist a uniruled variety* W*, such that* $\mathrm{Bir}(W)$ *admits unbounded connected algebraic subgroups?*

3. Unbounded Subgroups

The first examples of unbounded subgroups were found by Fong in [Fon24], when W is birational to $C \times \mathbb{P}^1$, with C a smooth curve of genus $\mathrm{g}(C) \geq 1$. His initial proof relied on the classification of maximal connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$; in fact, he showed that such subgroups can be of dimension at most 4; thus finding non-stationary increasing sequences, like the one of Example 4, is enough to demonstrate unboundedness.

In collaboration with Fong in [FZ23], we provided an alternative proof of the unboundedness of the subgroups exhibited in [Fon24], independent of any classification. This allowed us to generalize the result to higher dimensions:

Theorem 6. *Let* $n \geq 1$ *and* C *be a curve of genus* $\mathrm{g}(C) \geq 1$*. Then* $\mathrm{Bir}(C \times \mathbb{P}^n)$ *admits unbounded connected algebraic subgroups.*

Sketch of Proof. We first treat the case $n = 1$. Start with a $\mathbb{P}^1$-bundle $S_0 \coloneqq \mathbb{P}(\mathcal{E}_0) \to C$, where $\mathcal{E}_0$ is *sufficiently unstable*, and $G_0 \coloneqq \mathrm{Aut}^\circ(S_0)$. Then we can show that *all* G_0-Sarkisov links have target some $\mathbb{P}^1$-bundle $S_1 \coloneqq \mathbb{P}(\mathcal{E}_1) \to C$ with $\mathcal{E}_1$ being again *sufficiently* *unstable*; moreover G_0 is strictly contained in $G_1 \coloneqq \mathrm{Aut}^\circ(S_1)$. Continuing this process gives us a increasing sequence

$$G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_k \subsetneq G_{k+1} \subsetneq \ldots$$

Using Proposition 1 we deduce that none of the G_i is contained in a maximal connected algebraic subgroup.

As for the higher dimensional case, we start with $S_0 \times \mathbb{P}^n \to S_0$ and show that all $\mathrm{Aut}^\circ(S_0 \times \mathbb{P}^n)$-Sarkisov links are of product type, i.e. of the form: $\phi \times \mathrm{id}\colon S_0 \times \mathbb{P}^n \dashrightarrow S_1 \times \mathbb{P}^n$. Again using Proposition 1 we reduce to the $n = 1$ case, and get unboundedness from there. □

While Theorem 6 shows the existence of unbounded subgroups for uncountably many rationally connected varieties and in all dimensions, it still leaves the question open in one of most interesting cases: that of $\mathbb{P}^n$. Using again projective bundles, this time over a stably rational, non-rational 3-fold, Fanelli-Floris-Zimmermann in [FFZ24] exhibited unbounded connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$, with $n \geq 5$.

Finally, in [Kol24], Kollár provides a different framework, not reliant in birational geometry, for identifying unbounded algebraic subgroups. Along the way and among many new examples, he also shows the existence of such subgroups in $\mathrm{Bir}(\mathbb{P}^4)$.

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