# Granular Material

Pieterjan Bartels

An overview of existing simulation techniques



Bournemouth University

May 23, 2015

# Abstract

This report outlines the major existing techniques for the simulation of granular materials, such as sand, snow, sugar, powder, ... It will give a short overview of the physics that these materials obey, which are quite particular. After reviewing the physics, it explains the different ways these materials and their physics are simulated in the computer graphics community, including the interaction between granular material and other materials. While there is a long history of researching and simulating granular matter in physics and engineering, the scope of this report is limited to techniques used in the computer graphics and visual effects community, which has specific demands for its techniques. The report concludes with an overview of the implementation of one of the techniques the author made.

**Keywords:**   Granular Materials, Sand, Simulation, Computer Graphics

# 1  Introduction

Many real-life environments exhibit some form or even multiple forms
of granular materials. As they are ubiquitous in reality, virtual envi-
ronments, such as those in visual effects and animation, should include
these in order to be realistic. In this report, techniques for simulating
this phenomenon are reviewed.

The mechanical behaviour of granular materials is a complex phe-
nomenon that is not yet fully understood and has been a topic of exten-
sive research in physics and engineering. The latter field is interested in
the simulation of this behaviour because sand and other granular ma-
terials play an important role in many manufacturing processes (REF).
While the engineering techniques are very interested in their own right,
this report will not touch upon them, as the main focus of our course is
visual effects and animation. This is an important difference, as visual
effects do not require accuracy but merely visual plausibility, as opposed
to the field of engineering.

Because of this more relaxed requirements, combined with the com-
plexity of the physics of granular material which aren't fully understood
yet JAEGERREF, most visual effect simulation techniques are based
on a simplified mathematical model that mimics some or most of the
peculiarities of granular media, rather than on the actual physical laws
themselves. This report will describe several such models. Although the
research into granular media simulation has been going on for a very long
time, this report will focus on the more recent techniques.

In the next section, granular media are defined, and its physics are
briefly reviewed. Section REF describes the several simulation tech-
niques that were investigated. These techniques are then compared in
the discussion section, section REF. To broaden the scope of the report,
section REF describes some special cases: interaction of granular me-

dia with water and a specialized technique for snow, a particular type of granular material. The report then concludes the findings in section REF.

## 2 Physics

In order to simulate granular materials, it is necessary to define what they are and understand how they physically behave. For this reason, this section will look at how granular materials are described in the physics and engineering literature rather than the computer graphics literature. However, the scope of this report clearly does not allow a thorough discussion of the physics of granular matter. It will give an introduction, geared towards the techniques in the computer graphics community, which are the subject of the next sections. It is interesting to add that the physics of granular matter is still a subject of debate DEGENNES.

As a first step towards understanding, we need to define what granular material is. Referring back to the physics literature, de Gennes (REF) defines a granular material as a material made up out of a large collection of grains or particles, where the size $d$ (for diameter) of the individual grains are larger than one micron. At this scale, the Brownian motion and thermal agitation can be ignored. deGennesREF. This means that entropy can be ignored and the behaviour is determined by dynamical effects. JAEGERREF

Hence, to understand the behaviour, it necessary to investigate the dynamics of the material. The first important property is the friction, both static and dynamic, between the different grains. Coulomb's law is a widely accepted and well-known way of modelling (dry) friction between two surfaces in contact:

$$F_f \leq \mu F_n \tag{1}$$

where $F_f$ is the friction force, tangential to the contact surface, $F_n$ the force perpendicular to the contact surface, and $\mu$ the coefficient of friction. Notice the relation, $\leq$, indicating that the friction force will be as big as necessary to avoid sliding in the case of static friction, i.e. the friction force will exactly oppose sliding and keep the objects in the same position relative to each other. In the case of kinetic sliding, the $\leq$ symbol disappears, making it an equality. Hence, when the objects are sliding relative to each other, kinetic friction results in a constant force opposing the sliding motion. Typically, the friction coefficient $\mu$ is different for static $\mu_s$ and kinetic friction $\mu_k$.

Another property of granular material is the inelastic collision between the particles JAEGERREF. This, combined with the friction, makes that granular material are dissipative systems, where a lot of kinetic energy is converted in heat rather than bounces JEAGERREFHowever. It is interesting to note that the some grains might be subject to an infinite amount of collisions in a finite amount of time, a phenomenon called inelastic collapse McNamara and Young REF.

All of this results in some quite particular behaviour. For instance, a well-known result of these physics is that the pressure in a granular material does not linearly increase with height as it does in fluid, but stay constant after a certain depth. REFS This phenomenon is what makes hourglasses work. REFHOURGLASS. The static friction also makes the sand pile up, without the piles collapsing, along as the slope of the pile is less steep than the angle of repose. JAEGERREF. If the slope is steeper, an avalanche will occur, where only the top layer of the granular material flows. ARANSONREF

These physics, and the particular behaviour of granular matter, lead physicists to describe them as a material with both solid-like and fluid-like behaviour. DEGENNESJAEGERARARANSONREF. When approaching the material as a fluid, a continuum model is necessary. This con-

tinuum model is usually formulated using a model based on a variation of the Navier-Stokes equations, adapted to model the granular behavior, and complemented with a elasto-plastic formulation of a yield criterion, for example the Mohr-Coulomb yield criterion. DEGENNESREF The Mohr-Coulomb yield criterion is defined as (notation from REFZAB):

$$\sqrt{3}\frac{|\sigma - \sigma_m\delta|_F}{\sqrt{2}} < sin\phi\sigma_m, \tag{2}$$

where $\sigma$ is the stress tensor, $\sigma_m$ is the mean stress and $\phi$ is the friction angle (angle of repose). The norm taken in the left side is the Frobenius norm, denoted by the subscript. This equation means that the shear stress $(\sigma - \sigma_m\delta)$ in the left side is ignored because of friction when the pressure $\sigma_m$ pushing the material together is high enough, and hence governs the difference between the solid-like and fluid-like behaviour of the granular matter (that is, the material is assumed to start flowing when it 'yields') DEGENNESREF. This law can be seen as a generalization of the classical Coulomb friction to stress tensors, where the friction coefficient is replaced by the sine of the friction angle. By using this rule to determine when the medium starts to flow, ZABREF incorporate static friction into a continuum model.

While discussing the necessary changes to the Navier-Stokes equations to model granular material is interesting in its own rights, this would be out of scope for this report, as the computer graphics techniques usually bring it down to two particular differences from regular fluid simulations REFTOTECH: one needs to account for the friction and a specific form of (in)compressibility: dry granular materials are usually simulated with unilateral incompressibility, while wet granular materials, e.g. snow, are typically compressible. Unilateral incompressibility means that there is no cohesive pressure when the material is diverging REFTECH. The computer graphics techniques based on the continuum approach usually take existing fluid simulators and change these two aspects. The specific formulations for the two differ per technique and will be given while explaining specific techniques in the next section.

4

# 3 Simulation Techniques

This section will present several different simulation techniques for the behaviour of granular material. As explained before, visual effects is the main focus of this report. Consequently, the section only focuses on techniques used in the computer graphics community.

The techniques are split into three categories. First of all there are height fields, described in subsection REF, a simple but limited way of simulating soil interactions. The other two are more involved: discrete element methods and methods based on a continuum definition of granular media, subsection REF and REF respectively.

## 3.1 Height Fields

Earliest research in the simulation of granular material used a technique called height fields REFS. This is a simple geometric technique, loosely based on the physical laws. These techniques focus on simulating the interaction between soil and rigid bodies. Most of them lack 3D effects, such as splashing, completely.

In its most basic form, this technique discretizes the soil into a simple 2D grid (although it is often formulated as a volume, with a second grid under the first, forming small cuboids REFSUM) and keeps track of the height of the middle of each square of the grid. Whenever a rigid body interacts with the grid, the squares in collision with a rigid body are lowered to avoid overlap. A part of the material that gets removed by lowering the elements is then redistributed to adjacent elements. LIREF REFSUM

This obviously creates very steep slopes between elements at the edge of the contact zone REFSUM. As was explained in REFPHYS, if the slope of a pile is steeper than the angle of repose, the material flows down. Hence, the height fields technique uses iterative erosions to smooth out slopes which are too steep. This is an effective way of simulating foot steps or tire tracks. REFSUM, LIREF.

However, this technique only simulates the ground, and no granular material can splash or stay attached to the rigid objects. In order to achieve such effects ONOREF introduce multi-valued height fields, which are able to simulate granular material on concave objects. However this still doesn't provide splashing effects. SUMREF adds a simple particle simulation to his height field to create splashes, which is a first step towards discrete element techniques, the subject of the next subsection.

## 3.2   Discrete Element Techniques

A simple idea for simulating materials that are made out of large numbers of small grains is simulating each grain separately as a discrete element. Several such techniques have been proposed in the research literature REF. Two key issues need to be addressed when simulating grains individually, already introduced in section 2: the grains collide with each other, in an inelastic and therefore dissipative manner, and there is friction between grains, both static and dynamic friction.

A first option is to create a large number of rigid bodies and simulate their dynamics. This is the approach taken by Milenkovic REF, amongst others REF (same guy and Guendelman REF). Due the small size of sand grains, some authors use particles rather than rigid bodies. They do however use rigid body collisions, avoiding overlap between particles, effectively making it into a rigid body simulation with spheres (REF). The form of the of the rigid bodies implicity adds the necessary friction.

It is however clear that a rigid body collisions are infeasible for the scale most granular material simulations require (i.e. thousands and millions of grains). As was shown by McNamara and Young (REF) and mentioned earlier in section 2, granular materials are subject to inelastic collapse, where some grains undergo an infinite amount of collisions in a finite amount of time. For this reason, a lot of authors prefer a different approach to the collision handling in discrete element methods: Molecular Dynamics. All of these techniques use a particle-system, but rather than using a typical Event-Driven collision handling (REF), they use Molecular Dynamics (MD).

In MD, small overlap between particles is allowed and overlapping particles get a contact force assigned to push them apart. There are several examples of this principle used for granular material: (REFS). We introduce this idea with (typical) spherical particles, based on the work (and notation) of Nathan Bell (REF). Note that we assume that overlapping particles have been found. Obviously it is necessary to have an efficient way of finding neighbouring or overlapping particles to have an efficient algorithm. This is however a recurring problem in computer graphics and solutions based on spatial hashing are well-known, and are out of the scope of this report.

We start by defining the necessary variables. Given two particles' positions $x_1$ and $x_2$ and their radius $r_1$ and $r_2$, we can define the overlap $\xi$ between the two:

$$\xi = max(0, r_1 + r_2 - ||x_2 - x_1||). \tag{3}$$

This is the distance of overlap along the line of the two centers. The normalized direction $N$ of this line of centers is:

$$N = \frac{x_2 - x_1}{||x_2 - x_1||}. \tag{4}$$

For the dynamic friction, we will need the relative velocity $V$ as well:

$$V = v_1 - v_2 \tag{5}$$

$$\xi_v = V \cdot N \tag{6}$$

$$V_t = V - \xi_v N \tag{7}$$

In these equations, $V$ is the relative velocity, found from $v_1$ and $v_2$, the velocity of the particles. The relative velocity is split into a normal component and a tangential. The normal component is represented by $\xi_v$, which is the magnitude of the relative velocity in the normal direction N. The tangential velocity is represented by the vector $V_t$.

In this MD concept, the particles are then subjected to forces based on these variables. In order to handle collisions, normal forces are applied, while the friction is off course a tangential force. Starting with the first, we take a look at the normal forces introduced by Bell REF. The normal force is aligned with the direction $N$, as in eq. (4), and the magnitude $f_n$ is computed using following model:

$$f_n + k_d \xi_v \xi^\alpha + k_r \xi^\beta = 0, \tag{8}$$

with $k_d$ the viscous damping coefficient and $k_r$ the elastic restoration, which controls particle stiffness. Due to the dissipative nature of the collisions, explained in section 2 and JAEGERREF, the damping term is essential.

The simplest way to use this model is to choose $\alpha = 0$ and $\beta = 1$, which turns (8) into a damped linear spring force. The other two variables; $k_r$ and $k_d$, can then be chosen by the user or determined based on a real-life measured normal restitution coefficient $\epsilon$ and contact duration

$t_c$ BELLREF:

$$k_d = 2m_e \frac{-\ln\epsilon}{t_c}, \tag{9}$$

$$k_r = \frac{m_e}{t_c^2}(\ln^2\epsilon + \pi^2). \tag{10}$$

where $m_e$ is defined as:

$$\frac{1}{m_e} = \frac{1}{m_1} + \frac{1}{m_2} \tag{11}$$

with $m_1$ and $m_2$ the mass of the particles. More complex models can be used as well, such as one based on Hertz' model for contact forces between spheres REF. Here $\beta$ is 1.5, and $k_r$ is determined with the following relationships:

$$\frac{1}{R_{eff}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{12}$$

$$\frac{1}{E_{eff}} = \frac{1 - \mu_1^2}{E_1} + \frac{1 - \mu_2^2}{E_2} \tag{13}$$

$$k_r = \frac{4}{3}E_{eff}\sqrt{R_{eff}} \tag{14}$$

Here $E_1$ and $E_2$ are the Young modulus of the particles, $\mu_1$ and $\mu_2$ the Poisson ratios. $R_1$ and $R_2$ are still the radii of the particles. $\alpha$ then still needs to be chosen. Complex models might use a value that depends on the relative velocity $V$ REFSEEBELL. Nathan Bell chooses $\frac{1}{2}$ in his work REFBELL.

These normal forces are meant to handle the collisions, but frictional, i.e. tangential, forces are needed for plausible simulation of granular materials, as was earlier explained in section 2. However, many authors (REFEXS) only introduce an explicit force term for dynamic friction, i.e. friction between moving particles. Static friction is handled in a different way. The next paragraph will show the dynamic friction model, after the static friction is explained.

The simplest form of dynamic friction is a damping force on the tangential velocity. However, as explained in section 2, friction is determined by the law of Coulomb, eq. (1). Hence a better option is using following model for the frictional force $F_t$:

$$F_t = -\mu f_n \frac{V_t}{||V_t||},$$

(15)

which incorporates the law of Coulomb and the friction coefficient $\mu$. Care must be taken with the discontinuity that occurs when $V_t$ equals zero REFBELL.

So far, coming up with a model for a MD particle-based granular material simulation has been straightforward. However the model still misses static friction. As explained in the physics overview, section 2, static friction is essential for reproducing some of the most peculiar effects in granular material, such as the forming of piles. Using Coulomb's law as a threshold is infeasible in practice BELLREF, and some different methods have been introduced SEVERALREFS.

Some authors choose to use another explicit force term, e.g. a spring, for this type of friction. For instance, REFLH93 introduces a spring between two particles that collide. The spring gets introduced when the particles first touch. REFLH93. As long the particles remain in contact (i.e. $\xi > 0$), the total displacement from the original contact point is forms the basis of the tangential force:

$$F_t = -\min(\mu f_n, k_t ||D|| \frac{D}{||D||}),$$

(16)

where all the variables are as previously defined.

Another idea is to rely on particle geometry itself. This approach is adopted by SEVERALREFS (at least 4). In this case, the simulation happens with non-spherical particles, rather than the typical spherical ones. In this approach, no explicit frictional term is necessary to account

10

for the static friction: the normal forces between the non-spherical structures provide the same behaviour static friction does (i.e. resistance to relative tangential motion).

These non-spherical particles are usually built by connecting several spherical ones together. REFPB93 compounded spheres together with springs. Others, such BELLREF, REFWB93, REFNEWER, use rigid structures of several spheres. While this avoids the instability of spring systems, it adds the necessity to keep track of the orientation of the particles, i.e. we need to calculate torque and angular velocity as well. This happens by using the classic formula for the torque $t$ by force $F$:

$$t = (p - x) \times F, \tag{17}$$

where $p$ is the point where the force works and $x$ is the rotational center of the particle. Basic dynamics teaches us that integrating the sum of the torques (one for each force) gives the angular velocity, which contributes to the rotation of the particle.

Taking these kind of particle simulations to a larger scale is difficult. Simulating a beach with discrete elements is obviously infeasible. To somewhat counter these problems, some authors have proposed multiscale approaches, e.g. LucianiREF, ALDUANREF. Picking out the more recent technique, AlduanRef build on the work of BELLREF, using the same force models and non-spherical particles made out of rigid compositions of spheres for the (static friction). However, AlduanREF computes the internal forces (i.e. normal forces and dynamic friction) on coarse, low resolution particles. These results are then distributed down to finer, high resolution particles using interpolation between the values of the closest low resolution particles. The external forces are applied to both high and low resolution particles. They report speed-ups of over 11x for simulations of 2500 low resolution and 50000 high resolution particles.

A completely different discrete element approach can be found in po-

sition based dynamics (PBD). These particle-based methods avoid the instability of integrating force models by working directly on the position of the particles REFMULLER06. These positions are subjects to constraints mimicking the dynamic behaviour, and the different constraints are solved one-by-one in Gauss-Seidel or Jacobi iterations. REF-MULLER06, 07?, MACKLIN. Because of the large time-steps allowed by the stability and the possibility to parallellize the constraint solving, these algorithms are popular in real-time applications and computer games. MACKLINREF.

MACKLINREF is most the recent publication involving position based dynamics, and it includes a way of simulating granular materials. Where MullerREF is unable to simulate static friction, Macklin introduces a constraint for it, including static friction in the constraint solve loop. This allows PBD to simulate a sand pile at rest, one of the classic results of static friction, section 2.

For Granular materials in particular, PBD as in MACKLINREF first computes new positions for all particles based on other constraints, including collision constraints. It then uses these new positions to calculate a relative frictional position delta of any two particles that were in contact:

$$\delta x_\perp = ((x_i^* - x_i) - (x_j^* - x_j) \perp n, \tag{18}$$

where $x^*$ are the newly computed particle positions and $x$ the positions at the start of the time-step. The contact normal is represented as $n$, and $\perp$ means that the result is project to be perpendicular to $n$. The algorithm uses this value to determine whether to remove all tangential movement (static friction) or part of it (dynamic friction). The static one is used when $|\delta x_\perp| < \mu_s d$, where $\mu_s$ is the static friction coefficient and $d$ is the overlap between the particles. In conclusion, Macklin uses Coulomb's law, equation (1), but replaces the forces by positions and displacements MACKLINREF.

Another issue that's worth mentioning is two way coupling between the granular media and rigid bodies. These interactions are subject to friction as well, and so contact between them has influence on both the granular material and the rigid body. Both BELLREF and NEWER-REF use the simple idea of covering the rigid body surfaces with their non-spherical particles. The forces on these particles can then be used in the rigid body simulation and the friction is automatically taken into account because of the non-spherical particles and the dynamic friction, equation (15), between the particles.

In MACKLINREF, all behaviour is particle-based, including rigid bodies. Hence two-way coupling comes automatically when the friction constraints are applied to the particles of the rigid bodies in contact with the granular material as well.

## 3.3   Continuum-based Techniques

Granular material exhibits peculiar physics: while they consists of huge amounts of small grains, the material shows both solid and fluid-like behaviour. This was explained earlier, in section 2. The previous section gave an overview of techniques based on individual grain behaviour, this section looks at methods that approach granular material from its fluid-like behaviour, i.e. they see granular media as a continuum and simulate it as such. As was mentioned in the physics section, all of the techniques in this category start from a standard fluid simulation technique and add friction and some form unilateral incompressibility.

The first to do this were Zhu and Bridson REF. They build upon what they call the standard approach in soil mechanics, which is to simulate granular materials as a continuum in terms of plastic yielding, as was explained in section 2. Hence, the Mohr-Coulomb law, equation (2), is used to determine when the materials "yields", i.e. starts flowing RE-FZAB. Note that a very similar Drucker-Prager model could be used as

well NARREF. By using this rule to determine when the medium starts to flow, ZABREF incorporate static friction into a continuum model.

The standard approach in soil mechanics engineering, according to ZABREF, is to add an elasticity model and simulate the material with a finite element solver. However, as was noted earlier, the computer graphics society doesn't want accuracy as much as plausibility. ZABREF therefore choose to simplify the model, without losing any visual clues. These simplifications allow them to simulate granular material using a modified fluid simulation algorithm.

ZABREF ignore the elastic deformation and the volume changes when flow starts and stops, as they are not perceivable by eye. Ignoring these makes the simulation easier: it allows the granular media to be split up in regions of rigid behaviour (e.g. a pile) and regions of fluid-like behavior (shearing flow). A final assumption is that the pressure in the material is similar to that in an incompressible fluid. As explained in section 2, this is untrue: in granular material there is an upper limit to the pressure. Furthermore, this also introduces unwanted cohesion forces in the material when it is diverging (i.e. they do not incorporate unilateral incompressibility) REFNAR. ZABREF will hence produce inaccurate results in some cases. Again, accuracy is not of the utmost importance here.

ZABREF then use following models for the frictional stress in the flowing areas $\sigma_f$:

$$\sigma_f = -\sin\phi p \frac{D}{\sqrt{\frac{1}{3}|D|_F}} \tag{19}$$

for the parts where the material is flowing, with $p$ the pressure and $D$ the strain rate. All others variables have been defined earlier. For the frictional stress $\sigma_r$ in the rigid parts, ZABREF use:

$$\sigma_r = -\frac{\rho D \delta x^2}{\delta t}, \tag{20}$$

14

where $\rho$ is the density. This is derived from the forces needed to stop a cell with side length $\delta x$ from sliding during a timestep $\delta t$ ZABREF. They use $\sigma_r$ to check the yield condition (Mohr-Coulomb) for flow, eq. (2). Sometimes a constant is added to the right side of this equation, to simulate cohesion. This allows for sticky granular materials, such as snow. If the cell does not yield, this $\sigma_r$ is stored as the stress and the cell is marked as rigid. Otherwise $\sigma_f$ is computed for the cell.

These steps of computing the stress and rigidity or fluidity are incorporated in a fluid simulator: all usual steps are carried out as if it were a fluid (adding gravity, applying boundary conditions, computing the pressure and ensuring incompressibility). After that $\sigma_r$ is computed, and $\sigma_f$ if necessary. The velocity field in neighbouring groups of rigid cells are then projected to make the group move as a rigid body. The velocities in the flowing cells are updated with the computed stress ZABREF. As was the case for particle-based methods in the previous subsection, friction at the boundaries of the material is necessary as well. ZABREF use the friction model of BRIDSONREF2002 for this, and apart from that two-way coupling is handled as it is for fluids. Zhu and Bridson REF then introduce the FLIP method, a hybrid between Eulerian and Lagrangian, for fluids and make the mentioned changes to it to simulate sand.

Continuing on ZABREF, NARREF note that ZAB's work has flaws due to the lack of unilateral incompressibility, the granular material has too much fluid-like properties, such as cohesive forces when diverging NARREF. Cohesion is indeed not present in dry granular materials, although it is in wet sand and snow, see section 2. NARREF again use FLIP, a hybrid of Eulerian and Langrangian fluid simulation methods.

Working from the same ideas as ZABREF, i.e. the continuum granular media formulation from section ZABREF and a plastic yield criterion for the friction response, eq. (2), NARREF then introduces a new way of computing the pressure in granular materials, different from the

way it's computed in incompressible fluids. To do so, two simplifying assumptions are made: They assume there is a max density $\rho_{max}$ for the granular material, beyond which it cannot be compacted more, and that if the current density $\rho$ is lower than the critical one, particles only interact through intermittent collisions, which are ignored. They then require the pressure $p$ to avoid densities higher than $\rho_{max}$, without providing internal forces when the material is diverging, achieving unilateral incompressibility.

NARREF expresses this as a constraint on the volume fraction $\phi$:

$$\phi = \frac{\rho}{\rho_{max}} \leq 1. \tag{21}$$

From this constraint the corresponding pressure $p$ can be computed, as happens in NARREF09CROWD, using a minimization problem.

NARREF then computes the (essential!) frictional stress in each cell based on the maximum dissipation principle, introduced by SIMOREF, by computing the frictional stress that takes out the most kinetic energy (which implies maximum dissipation, as the behaviour of granular material is dictated by dynamic principles JAEGERREF). They then minimize a weighted version of the energy $E$ to have a well-conditioned problem REFNAR:

$$E = \frac{1}{2\rho_{max}} \int \rho^n ||v_i||^2 dV \tag{22}$$

where $v_i$ is the intermediate velocity defined as:

$$v_i = v^n + \frac{\delta t}{\rho^n}(f_{ext} - \delta p + \delta * s). \tag{23}$$

Here $s$ is the frictional stress we are looking for, $f_{ext}$ are the external forces and $p$ is the pressure (NARREF split the stress $\sigma$ in pressure $p$ and frictional stress $s$). $\delta t$ is the timestep and $v$ the velocity field. The superscript $n$ stands for the current timestep. The frictional stress used

16

is the one that minimizes $E$ NARREF, with the constraint that $s$ is limited by the Mohr-Coulomb law in EQREF.

In conclusion, NARREF use FLIP to simulate granular materials, much like Zhu and Bridson. However Narain uses a different way of calculating the frictional stress and changes the way the pressure is computed to avoid cohesion effects.

Very much in the same style of thinking, ALD11REF adapt a fluid technique to simulate granular material. In this case, it is the smoothed particle hydrodynamics (SPH) technique, more specifically the variant with predictive-corrective incompressibility (PCISPH) as introduced by REF. Just like in NARREF, in order to change the technique from incompressible fluid to granular materials, they alter the way the incompressibility is computed so it becomes unilateral incompressibility, as in EQREF, and they add a frictional stress.

For the unilateral incompressibility, ALD11REF keeps PCISPH almost the same: the jacobi-style iterations to find the fluid density $\rho_{max}$ are kept the same, but a corrective pressure increment is only applied when the current density $\rho$ is larger than $\rho_{max}$. If it is smaller, the granular material is allowed to flow freely. The corrective pressure term stays the same as in PCISPH, REFPCISPH, for particle $i$:

$$\delta p_i = \delta \cdot (\rho_i - \rho_{max}) \tag{24}$$

where the corrective factor $\delta$ is computed as:

$$\delta = \frac{\rho_0^2}{2m^2(\Sigma_j \nabla W_{ij}^T \Sigma_j \nabla W_{ij} + \Sigma_j \nabla W_{ij}^T \nabla W_{ij})} \tag{25}$$

where $m$ is the mass of the particle, $\delta t$ is the time step, and $\nabla W_{ij}$ is the kernel function between particle $i$ and $j$. A full description of the SPH algorithm is out of scope for this report. We refer to PCISPHREF.

For the friction, ALD11REF doesn't minimize the kinetic energy as in NARREF, but minimize the strain rate, as this is a measure of relative velocity rather than absolute velocity and hence better suited. This is however still based on the principle of maximal dissipation REF. The friction is limited by a Mohr-Coulomb type of plastic yield law once again EQREF. The minimization problem is

$$\min ||\epsilon||, s.t. ||s_{ij}|| \leq \alpha \cdot p. \tag{26}$$

Here $\epsilon$ is the strain rate, $alpha = \sqrt{\frac{2}{3}} sin\theta$, with $\theta$ the angle of repose and $p$ the pressure.

Ihmsen REF use SPH, using the unilateral incompressibility and friction from ALD11REF. However to provide better performance, the authors perform the internal force calculations on a coarser scale than the external ones, very in much in the spirit of ALDREFPARTICLES. This gives the advantages of the continuum approaches with the added benefits of high resolution details.

# 4  Discussion

# 5  Special Cases

While the simulation of granular media on its own is an interesting topic, things truly get interesting when it is coupled with other types of simulations. Coupling with rigid body simulations is usually covered when a new technique for granular media is introduced, and was covered in section REF, but coupling with other types of simulation is less common.

To broaden the scope of this report this section looks at coupling of granular material with fluid simulations. Important here is the changes the fluid causes to the granular media: water turns dry sand into mud. this is the subject of subsection 5.1.

Furthermore this section also looks briefly at a very recent technique that is focused on one particular type of granular material: snow. Snow has some special properties, and a new technique incorporating these properties was developed for Disneys frozen REF. This technique and the aforementioned properties are described in subsection 5.2.

## 5.1 Interaction with water

The difference between wet and dry granular materials has been mentioned before REFSEC. Adding water to dry sand changes it: the fluid flows into the sand, and the mixing turns the sand into mud, which has different properties than dry sand LADREF. Obviously, the behaviour of the sand also influences the way the water behaves. Hence, a two-way coupling is necessary.

There are not a lot of authors that have addressed this problem. One interesting example however is the work of Lenaerts and Dutré REF. To simulate this phenomenon they use SPH for both sand and water. The SPH method for sand has been explained in section SECREF. Lenaerts uses the plastic yielding from ZABREF.

In order to achieve the two-way coupling they look at the granular material as a porous material, with the holes between grains the pore space. They use the porous particles of LAD08 to simulate the porous flow. A full description of this technique is out of scope for this report. The saturation of the pores is used to alter the cohesion constant in the plastic yield function as mixing a fluid in the granular material makes it into a more rigid structure. The cohesion is determined by interpolation between a lower cohesion constant for dry material $c_{dry}$ and a higher one $c_{wet}$ for wet material, based on the saturation $S$ of the pores:

$$
c = \begin{cases} c_{dry}(1 - \frac{S}{S_i}) + c_{wet}\frac{S}{S_i} & \text{if } S \le S_i \\ c_{wet}\frac{1-S}{1-S_i} & \text{if } S > S_i \end{cases}
\tag{27}
$$

where $S_i$ is the ideal saturation level. This $c$ is added to the rightside of the Mohr-Coulomb law, as was already mentioned earlier, in section REF. Using the notation from LADREF:

$$\sqrt{3}\sigma_s < \mu_f \sigma_m + c \tag{28}$$

with $\sigma_s$ the shear stress.

Additionally, LADREF add a viscosity term to the velocity field of the granular material, since it behaves more like a viscous fluid when enough water is added. In order to make this viscosity dependent on the amount of water added, they scale the viscosity coefficient $\mu$ once again with the saturation level:

$$\mu_{sand} = \mu \frac{S - S_i}{1 - S_i} \tag{29}$$

and they bring down the friction coefficient $\mu_f$, used in equation (??), as well:

$$\mu_f^{sand} = /mu_f \frac{1 - S}{1 - S_i}. \tag{30}$$

In conclusion, LADREF present an algorithm that allows a dry granular material to turn into a viscous fluid such as mud (and back).

## 5.2   Snow Simulation

Snow is a complicated phenomenon. Many authors avoid talking about snow, as it so complex REFPHYSICS. It sometimes behaves as solid, sometimes as a fluid, and is therefore often seen as a granular material. However, due to the continuously varying phase changes of snow and its compressibility (snow can easily be compressed), there are some differences REFMPM.

In order to simulate this, REFMPM use a elasto-plastic constitutive model to model the snow dynamics, in addition to the conservation of

mass and momentum usually seen in fluids REFPHYS:

$$\sigma = \frac{1}{J}\frac{\delta\Psi}{\delta F_E}F_E^T, \tag{31}$$

where $F$ is the deformation gradient, $J$ is the determinant of $F$ and $F_E$ is the elastic part of $F$. $\Psi$ is the elasto-plastic potential energy function and $\sigma$ is the Cauchy stress REFMPM.

In order to solve for these dynamics, REFMPM use the material point method (MPM). MPM was introduced in the engineering literature SUL-SKYREF, and is related to FLIP, which was also originally an engineering method, and was introduced into the graphics community by ZABREF. Both methods are a hybrid between the Eulerian (grid) and Lagrangian (particle-based) approach.

Where ZABREF claimed MPM was too slow, REFMPM do adapt it to the computer graphics community's needs, using semi-implicit time integration, famously faster than explicit BARAFFREF, and parallelism. MPM is more versatile than FLIP as it can deal with different dynamic models, such as the one in (31), whereas ZABREF had to incorporate different stress, strain and a rigid grouping technique into FLIP. Consequently, MPMREF call MPM a generalization of FLIP.

# 6　Conclusion

# References