Chapter 17.
Incomplete Markets Models

17.1. Introduction

In the complete markets model of chapter 8, the optimal consumption allocation is not history dependent: the allocation depends on the current value of the Markov state variable only. This outcome reflects the comprehensive opportunities to insure risks that markets provide. This chapter and chapter 19 describe settings with more impediments to exchanging risks. These reduced opportunities make allocations history dependent. In this chapter, the history dependence is encoded in the dependence of a household's consumption on the household's current asset holdings. In chapter 19, history dependence is encoded in the dependence of the consumption allocation on a continuation value promised by a planner or principal.

The present chapter describes a particular type of incomplete markets model. The models have a large number of ex ante identical but ex post heterogeneous agents who trade a single security. For most of this chapter, we study models with no aggregate uncertainty and no variation of an aggregate state variable over time (so macroeconomic time series variation is absent). But there is much uncertainty at the individual level. Households' only option is to "self-insure" by managing a stock of a single asset to buffer their consumption against adverse shocks. We study several models that differ mainly with respect to the particular asset that is the vehicle for self-insurance, for example, fiat currency or capital.

The tools for constructing these models are discrete-state discounted dynamic programming—used to formulate and solve problems of the individuals; and Markov chains—used to compute a stationary wealth distribution. The models produce a stationary wealth distribution that is determined simultaneously with various aggregates that are defined as means across corresponding individual-level variables.

We begin by recalling our discrete state formulation of a single-agent infinite horizon savings problem. We then describe several economies in which households face some version of this infinite horizon saving problem, and where some of the prices taken parametrically in each household's problem are determined by the *average* behavior of all households.[1]

This class of models was invented by Bewley (1977, 1980, 1983, 1986) partly to study a set of classic issues in monetary theory. The second half of this chapter joins that enterprise by using the model to represent inside and outside money, a free banking regime, a subtle limit to the scope of Friedman's optimal quantity of money, a model of international exchange rate indeterminacy, and some related issues. The chapter closes by describing some recent work of Krusell and Smith (1998) designed to extend the domain of such models to include a time-varying stochastic aggregate state variable. As we shall see, this innovation makes the state of the household's problem include the time-t cross-section distribution of wealth, an immense object.

Researchers have used calibrated versions of Bewley models to give quantitative answers to questions including the welfare costs of inflation (İmrohoroğlu, 1992), the risk-sharing benefits of unfunded social security systems (İmrohoroğlu, İmrohoroğlu, and Joines, 1995), the benefits of insuring unemployed people (Hansen and İmrohoroğlu, 1992), and the welfare costs of taxing capital (Aiyagari, 1995).

17.2. A savings problem

Recall the discrete state saving problem described in chapters 4 and 16. The household's labor income at time t, s_t, evolves according to an m-state Markov chain with transition matrix $\mathcal{P}$. If the realization of the process at t is $\bar{s}_i$, then at time t the household receives labor income $w\bar{s}_i$. Thus, employment opportunities determine the labor income process. We shall sometimes assume that m is 2, and that s_t takes the value 0 in an unemployed state and 1 in an employed state.

We constrain holdings of a single asset to a grid $\mathcal{A} = [0 < a_1 < a_2 < \ldots < a_n]$. For given values of (w, r) and given initial values (a_0, s_0) the household chooses a

[1] Most of the heterogeneous agent models in this chapter have been arranged to shut down aggregate variations over time, to avoid the "curse of dimensionality" that comes into play in formulating the household's dynamic programming problem when there is an aggregate state variable. But we also describe a model of Krusell and Smith (1998) that has an aggregate state variable.

policy for $\{a_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{17.2.1}$$

subject to

$$\begin{aligned} c_t + a_{t+1} &= (1+r)a_t + w s_t \\ a_{t+1} &\in \mathcal{A} \end{aligned} \tag{17.2.2}$$

where $\beta \in (0,1)$ is a discount factor; $u(c)$ is a strictly increasing, strictly concave, twice continuously differentiable one-period utility function satisfying the Inada condition $\lim_{c \downarrow 0} u'(c) = +\infty$; and $\beta(1+r) < 1$.[2]

The Bellman equation, for each $i \in [1, \ldots, m]$ and each $h \in [1, \ldots, n]$, is

$$v(a_h, \bar{s}_i) = \max_{a' \in \mathcal{A}} \{u[(1+r)a_h + w\bar{s}_i - a'] + \beta \sum_{j=1}^{m} \mathcal{P}(i,j) v(a', \bar{s}_j)\}, \tag{17.2.3}$$

where a' is next period's value of asset holdings. Here $v(a,s)$ is the optimal value of the objective function, starting from asset-employment state (a,s). Note that the grid $\mathcal{A}$ incorporates upper and lower limits on the quantity that can be borrowed (i.e., the amount of the asset that can be issued). The upper bound on $\mathcal{A}$ is restrictive. In some of our theortical discussion to follow, it will be important to dispense with that upper bound.

In chapter 16, we described how to solve equation (17.2.3) for a value function $v(a,s)$ and an associated policy function $a' = g(a,s)$ mapping this period's (a,s) pair into an optimal choice of assets to carry into next period.

[2] The Inada condition makes consumption nonnegative, and this fact plays a role in justifying the natural debt limit below.

17.2.1. Wealth-employment distributions

Define the unconditional distribution of (a_t, s_t) pairs, $\lambda_t(a,s) = \text{Prob}(a_t = a, s_t = s)$. The exogenous Markov chain $\mathcal{P}$ on s and the optimal policy function $a' = g(a,s)$ induce a law of motion for the distribution λ_t, namely,

$$\begin{aligned}\text{Prob}(s_{t+1} = s', a_{t+1} = a') = \sum_{a_t}\sum_{s_t} &\text{Prob}(a_{t+1} = a' | a_t = a, s_t = s) \\ &\cdot \text{Prob}(s_{t+1} = s' | s_t = s) \cdot \text{Prob}(a_t = a, s_t = s),\end{aligned}$$

or

$$\lambda_{t+1}(a', s') = \sum_a \sum_s \lambda_t(a,s) \text{Prob}(s_{t+1} = s' | s_t = s) \cdot \mathcal{I}(a', s, a),$$

where we define the indicator function $\mathcal{I}(a', a, s) = 1$ if $a' = g(a,s)$, and 0 otherwise.[3] The indicator function $\mathcal{I}(a', a, s) = 1$ identifies the time-t states a, s that are sent into a' at time $t+1$. The preceding equation can be expressed as

$$\lambda_{t+1}(a', s') = \sum_s \sum_{\{a: a' = g(a,s)\}} \lambda_t(a,s) \mathcal{P}(s,s'). \tag{17.2.4}$$

A time-invariant distribution λ that solves equation (17.2.4) (i.e., one for which $\lambda_{t+1} = \lambda_t$) is called a *stationary distribution.* One way to compute a stationary distribution is to iterate to convergence on equation (17.2.4). An alternative is to create a Markov chain that describes the solution of the optimum problem, then to compute an invariant distribution from a left eigenvector associated with a unit eigenvalue of the stochastic matrix (see chapter 2).

To deduce this Markov chain, we map the pair (a, s) of vectors into a single-state vector x as follows. For $i = 1, \ldots, n$, $h = 1, \ldots, m$, let the jth element of x be the *pair* (a_i, s_h), where $j = (i-1)m + h$. Thus, we denote $x' = [(a_1, s_1), (a_1, s_2), \ldots, (a_1, s_m), (a_2, s_1), \ldots, (a_2, s_m), \ldots, (a_n, s_1), \ldots, (a_n, s_m)]$. The optimal policy function $a' = g(a,s)$ and the Markov chain $\mathcal{P}$ on s induce a Markov chain on x_t via the formula

$$\begin{aligned}&\text{Prob}[(a_{t+1} = a', s_{t+1} = s') | (a_t = a, s_t = s)] \\ &= \text{Prob}(a_{t+1} = a' | a_t = a, s_t = s) \cdot \text{Prob}(s_{t+1} = s' | s_t = s) \\ &= \mathcal{I}(a', a, s) \mathcal{P}(s, s'),\end{aligned}$$

3 This construction exploits the fact that the optimal policy is a deterministic function of the state, which comes from the concavity of the objective function and the convexity of the constraint set.

where $\mathcal{I}(a', a, s) = 1$ is defined as above. This formula defines an $N \times N$ matrix P, where $N = n \cdot m$. This is the Markov chain on the household's state vector x.[4]

Suppose that the Markov chain associated with P is asymptotically stationary and has a unique invariant distribution π_∞. Typically, all states in the Markov chain will be recurrent, and the individual will occasionally revisit each state. For long samples, the distribution π_∞ tells the fraction of time that the household spends in each state. We can "unstack" the state vector x and use π_∞ to deduce the stationary probability measure $\lambda(a_i, s_h)$ over (a, s) pairs, where

$$\lambda(a_i, s_h) = \text{Prob}(a_t = a_i, s_t = s_h) = \pi_\infty(j),$$

and where $\pi_\infty(j)$ is the jth component of the vector π_∞, and $j = (i-1)m + h$.

17.2.2. Reinterpretation of the distribution λ

The solution of the household's optimum saving problem induces a stationary distribution $\lambda(a, s)$ that tells the fraction of time that an infinitely lived agent spends in state (a, s). We want to reinterpret $\lambda(a, s)$. Thus, let (a, s) index the state of a particular household at a particular time period t, and assume that there is a probability distribution of households over state (a, s). We start the economy at time $t = 0$ with a distribution $\lambda(a, s)$ of households that we want to repeat itself over time. The models in this chapter arrange the initial distribution and other things so that the *distribution* of agents over individual state variables (a, s) remains constant over time even though the state of the individual household is a stochastic process. We shall study several models of this type.

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[4] Various Matlab programs to be described later in this chapter create the Markov chain for the joint (a, s) state.

17.2.3. Example 1: A pure credit model

Mark Huggett (1993) studied a pure exchange economy. Each of a continuum of households has access to a centralized loan market in which it can borrow or lend at a constant net risk-free interest rate of r. Each household's endowment is governed by the Markov chain $(\mathcal{P}, \bar{s})$. The household can either borrow or lend at a constant risk-free rate. However, total borrowings cannot exceed $\phi > 0$, where ϕ is a parameter set by Huggett. A household's setting of next period's level of assets is restricted to the discrete set $\mathcal{A} = [a_1, \dots, a_m]$, where the lower bound on assets $a_1 = -\phi$. Later we'll discuss alternative ways to set ϕ, and how it relates to a natural borrowing limit.

The solution of the household's problem is a policy function $a' = g(a, s)$ that induces a stationary distribution $\lambda(a, s)$ over states. Huggett uses the following definition:

DEFINITION: Given ϕ, a *stationary equilibrium* is an interest rate r, a policy function $g(a, s)$, and a stationary distribution $\lambda(a, s)$ for which

a. The policy function $g(a, s)$ solves the household's optimum problem.

b. The stationary distribution $\lambda(a, s)$ is induced by $(\mathcal{P}, \bar{s})$ and $g(a, s)$.

c. The loan market clears

$$\sum_{a,s} \lambda(a, s) g(a, s) = 0.$$

17.2.4. Equilibrium computation

Huggett computed equilibria by using an iterative algorithm. He fixed an $r = r_j$ for $j = 0$, and for that r solved the household's problem for a policy function $g_j(a, s)$ and an associated stationary distribution $\lambda_j(a, s)$. Then he checked to see whether the loan market clears at r_j by computing

$$\sum_{a,s} \lambda_j(a, s) g(a, s) = e_j^*.$$

If $e_j^* > 0$, Huggett raised r_{j+1} above r_j and recomputed excess demand, continuing these iterations until he found an r at which excess demand for loans is zero.

17.2.5. Example 2: A model with capital

The next model was created by Rao Aiyagari (1994). He used a version of the saving problem in an economy with many agents and interpreted the single asset as homogeneous physical capital, denoted k. The capital holdings of a household evolve according to

$$k_{t+1} = (1-\delta)k_t + x_t$$

where $\delta \in (0,1)$ is a depreciation rate and x_t is gross investment. The household's consumption is constrained by

$$c_t + x_t = \tilde{r}k_t + ws_t,$$

where $\tilde{r}$ is the rental rate on capital and w is a competitive wage, to be determined later. The preceding two equations can be combined to become

$$c_t + k_{t+1} = (1+\tilde{r}-\delta)k_t + ws_t,$$

which agrees with equation (17.2.2) if we take $a_t \equiv k_t$ and $r \equiv \tilde{r} - \delta$.

There is a large number of households with identical preferences (17.2.1) whose distribution across (k,s) pairs is given by $\lambda(k,s)$, and whose average behavior determines (w,r) as follows: Households are identical in their preferences, the Markov processes governing their employment opportunities, and the prices that they face. However, they differ in their histories $s_0^t = \{s_h\}_{h=0}^t$ of employment opportunities, and therefore in the capital that they have accumulated. Each household has its own history s_0^t as well as its own initial capital k_0. The productivity processes are assumed to be independent across households. The behavior of the collection of these households determines the wage and interest rate (w,r).

Assume an initial distribution *across* households of $\lambda(k,s)$. The average level of capital per household K satisfies

$$K = \sum_{k,s} \lambda(k,s) g(k,s),$$

where $k' = g(k,s)$. Assuming that we start from the invariant distribution, the average level of employment is

$$N = \xi_\infty' \bar{s},$$

where ξ_∞ is the invariant distribution associated with $\mathcal{P}$ and $\bar{s}$ is the exogenously specified vector of individual employment rates. The average employment rate is exogenous to the model, but the average level of capital is endogenous.

There is an aggregate production function whose arguments are the average levels of capital and employment. The production function determines the rental rates on capital and labor from the marginal conditions

$$w = \partial F(K, N)/\partial N$$
$$\tilde{r} = \partial F(K, N)/\partial K$$

where $F(K, N) = AK^\alpha N^{1-\alpha}$ and $\alpha \in (0, 1)$.

We now have identified all of the objects in terms of which a stationary equilibrium is defined.

Definition of Equilibrium: A *stationary equilibrium* is a policy function $g(k, s)$, a probability distribution $\lambda(k, s)$, and positive real numbers $(K, \tilde{r}, w)$ such that

a. The prices (w, r) satisfy

$$\begin{aligned} w &= \partial F(K, N)/\partial N \\ r &= \partial F(K, N)/\partial K - \delta. \end{aligned} \tag{17.2.5}$$

b. The policy function $g(k, s)$ solves the household's optimum problem.

c. The probability distribution $\lambda(k, s)$ is a stationary distribution associated with $[g(k, s), \mathcal{P}]$; that is, it satisfies

$$\lambda(k', s') = \sum_s \sum_{\{k : k' = g(k, s)\}} \lambda(k, s) \mathcal{P}(s, s').$$

d. The average value of K is implied by the average the households' decisions

$$K = \sum_{k, s} \lambda(k, s) g(k, s).$$

17.2.6. Computation of equilibrium

Aiyagari computed an equilibrium of the model by defining a mapping from $K \in I\!R$ into $I\!R$, with the property that a fixed point of the mapping is an equilibrium K. Here is an algorithm for finding a fixed point:

1. For fixed value of $K = K_j$ with $j = 0$, compute (w, r) from equation (17.2.5), then solve the household's optimum problem. Use the optimal policy $g_j(k, s)$ to deduce an associated stationary distribution $\lambda_j(k, s)$.
2. Compute the average value of capital associated with $\lambda_j(k, s)$, namely,

$$K_j^* = \sum_{k,s} \lambda_j(k,s) g_j(k,s).$$

3. For a fixed "relaxation parameter" $\xi \in (0,1)$, compute a new estimate of K from method[5]

$$K_{j+1} = \xi K_j + (1-\xi) K_j^*.$$

4. Iterate on this scheme to convergence.

Later, we shall display some computed examples of equilibria of both Huggett's model and Aiyagari's model. But first we shall analyze some features of both models more formally.

[5] By setting $\xi < 1$, the relaxation method often converges to a fixed point in cases in which direct iteration (i.e., setting $\xi = 0$) fails to converge.

17.3. Unification and further analysis

We can display salient features of several models by using a graphical apparatus of Aiyagari (1994). We shall show relationships among several models that have identical household sectors but make different assumptions about the single asset being traded.

For convenience, recall the basic savings problem. The household's objective is to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{17.3.1a}$$

$$c_t + a_{t+1} = w s_t + (1+r) a_t \tag{17.3.1b}$$

subject to the borrowing constraint

$$a_{t+1} \geq -\phi. \tag{17.3.1c}$$

We now temporarily suppose that a_{t+1} can take any real value exceeding $-\phi$. Thus, we now suppose that $a_t \in [-\phi, +\infty)$. We occasionally find it useful to express the discount factor $\beta \in (0,1)$ in terms of a discount *rate* ρ as $\beta = \frac{1}{1+\rho}$. In equation (17.3.1b), w is sometimes a given function $\psi(r)$ of the net interest rate r.

17.4. Digression: the nonstochastic savings problem

It is useful briefly to study the nonstochastic version of the savings problem when $\beta(1+r) < 1$. For $\beta(1+r) = 1$, we studied this problem in chapter 16. To get the nonstochastic savings problem, assume that s_t is fixed at some positive level s. Associated with the household's maximum problem is the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \theta_t \left[(1+r) a_t + w s - c_t - a_{t+1} \right] \right\}, \tag{17.4.1}$$

where $\{\theta_t\}_{t=0}^{\infty}$ is a sequence of nonnegative Lagrange multipliers on the budget constraint. The first-order conditions for this problem are

$$u'(c_t) \geq \beta(1+r) u'(c_{t+1}), \quad = \text{if } a_{t+1} > -\phi. \tag{17.4.2}$$

When $a_{t+1} > -\phi$, the first-order condition implies

$$u'(c_{t+1}) = \frac{1}{\beta(1+r)} u'(c_t), \tag{17.4.3}$$

which because $\beta(1+r) < 1$ in turn implies that $u'(c_{t+1}) > u'(c_t)$ and $c_{t+1} < c_t$. Thus, consumption is declining during periods when the household is not borrowing constrained. Thus, $\{c_t\}$ is a monotone decreasing sequence. If it is bounded below, either because of an Inada condition on $u(\cdot)$ at 0 or a nonnegativity constraint on c_t, then c_t will converge as $t \to +\infty$. When it converges, the household will be borrowing constrained.

We can compute the steady level of consumption when the household eventually becomes permanently stuck at the borrowing constraint. Set $a_{t+1} = a_t = -\phi$. This and (17.3.1b) gives

$$c_t = \bar{c} = ws - r\phi. \tag{17.4.4}$$

This is the level of labor income left after paying the net interest on the debt at the borrowing limit. The household would like to shift consumption from tomorrow to today but can't.

If we solve the budget constraint forward, we obtain the present-value budget constraint

$$a_0 = (1+r)^{-1} \sum_{t=0}^{\infty} (1+r)^{-t} (c_t - ws). \tag{17.4.5}$$

Thus, when $\beta(1+r) < 1$, the household's consumption plan can be found from solving equations (17.4.5), (17.4.4), and (17.4.3) for an initial c_0 and a date T after which the debt limit is binding and c_t is constant.

If consumption is required to be nonnegative,[6] equation (17.4.4) implies that the debt limit must satisfy

$$\phi \leq \frac{ws}{r}. \tag{17.4.6}$$

We call the right side the *natural debt limit.* If $\phi < \frac{ws}{r}$, we say that there is an *ad hoc* debt limit.

We have deduced that when $\beta(1+r) < 1$, if a steady-state level exists, consumption is given by equation (17.4.4) and assets by $a_t = -\phi$.

Now turn to the case that $\beta(1+r) = 1$. Here equation (17.4.3) implies that $c_{t+1} = c_t$ and the budget constraint implies $c_t = ws + ra$ and $a_{t+1} = a_t = a_0$. So when $\beta(1+r) = 1$, *any* a_0 is a stationary value of a. It is optimal forever to roll over the initial asset level.

[6] Consumption must be nonnegative, for example, if we impose the Inada condition discussed earlier.

In summary, in the deterministic case, the steady-state demand for assets is $-\phi$ when $(1+r) < \beta^{-1}$ (i.e., when $r < \rho$); and it equals a_0 when $r = \rho$. Letting the steady-state level be $\bar{a}$, we have

$$\bar{a} = \begin{cases} -\phi, & \text{if } r < \rho; \\ a_0, & \text{if } r = \rho, \end{cases}$$

where $\beta = (1+\rho)^{-1}$. When $r = \rho$, we say that the steady state asset level $\bar{a}$ is indeterminate.

17.5. Borrowing limits: "natural" and "ad hoc"

We return to the stochastic case and take up the issue of debt limits. Imposing $c_t \geq 0$ implies the emergence of what Aiyagari calls a "natural" debt limit. Thus, imposing $c_t \geq 0$ and solving equation (17.3.1b) forward gives

$$a_t \geq -\frac{1}{1+r} \sum_{j=0}^{\infty} w s_{t+j} (1+r)^{-j}. \tag{17.5.1}$$

Since the right side is a random variable, not known at t, we have to supplement equation (17.5.1) to obtain the borrowing constraint. One possible approach is to replace the right side of equation (17.5.1) with its conditional expectation, and to require equation (17.5.1) to hold in expected value. But this expected value formulation is incompatible with the notion that the loan is risk free, and that the household can repay it for sure. If we insist that equation (17.5.1) hold almost surely, for all $t \geq 0$, then we obtain the constraint that emerges by replacing s_t with $\min s \equiv s_1$, which yields

$$a_t \geq -\frac{s_1 w}{r}. \tag{17.5.2}$$

Aiyagari (1994) calls this the "natural debt limit." To accommodate possibly more stringent debt limits, beyond those dictated by the notion that it is feasible to repay the debt for sure, Aiyagari specifies the debt limit as

$$a_t \geq -\phi, \tag{17.5.3}$$

where

$$\phi = \min\left[b, \frac{s_1 w}{r}\right], \tag{17.5.4}$$

and $b > 0$ is an arbitrary parameter defining an "ad hoc" debt limit.

17.5.1. A candidate for a single state variable

For the special case in which s is i.i.d., Aiyagari showed how to cast the model in terms of a single state variable to appear in the household's value function. To synthesize a single state variable, note that the "disposable resources" available to be allocated at t are $z_t = ws_t + (1+r)a_t + \phi$. Thus, z_t is the sum of the current endowment, current savings at the beginning of the period, and the maximimal borrowing capacity ϕ. This can be rewritten as

$$z_t = ws_t + (1+r)\hat{a}_t - r\phi$$

where $\hat{a}_t \equiv a_t + \phi$. In terms of the single state variable z_t, the household's budget set can be represented recursively as

$$c_t + \hat{a}_{t+1} \leq z_t \tag{17.5.5a}$$

$$z_{t+1} = ws_{t+1} + (1+r)\hat{a}_{t+1} - r\phi \tag{17.5.5b}$$

where we must have $\hat{a}_{t+1} \geq 0$. The Bellman equation is

$$v(z_t, s_t) = \max_{\hat{a}_{t+1} \geq 0} \left\{ u(z_t - \hat{a}_{t+1}) + \beta E v(z_{t+1}, s_{t+1}) \right\}. \tag{17.5.6}$$

Here s_t appears in the state vector purely as an information variable for predicting the employment component s_{t+1} of next period's disposable resources z_{t+1}, conditional on the choice of $\hat{a}_{t+1}$ made this period. Therefore, it disappears from both the value function and the decision rule in the i.i.d. case.

More generally, with a serially correlated state, associated with the solution of the Bellman equation is a policy function

$$\hat{a}_{t+1} = A(z_t, s_t). \tag{17.5.7}$$

17.5.2. Supermartingale convergence again

Let's revisit a main issue from chapter 16, but now consider the possible case $\beta(1+r) < 1$. From equation (17.5.5a), optimal consumption satisfies $c_t = z_t - A(z_t, s_t)$. The optimal policy obeys the Euler inequality:

$$u'(c_t) \geq \beta(1+r)E_t u'(c_{t+1}), \quad = \text{ if } \hat{a}_{t+1} > 0. \tag{17.5.8}$$

We can use equation (17.5.8) to deduce significant aspects of the limiting behavior of mean assets as a function of r. Following Chamberlain and Wilson (2000) and others, to deduce the effect of r on the mean of assets, we analyze the limiting behavior of consumption implied by the Euler inequality (17.5.8). Define

$$M_t = \beta^t (1+r)^t u'(c_t) \geq 0.$$

Then $M_{t+1} - M_t = \beta^t(1+r)^t[\beta(1+r)u'(c_{t+1}) - u'(c_t)]$. Equation (17.5.8) can be written

$$E_t(M_{t+1} - M_t) \leq 0, \tag{17.5.9}$$

which asserts that M_t is a supermartingale. Because M_t is nonnegative, the supermartingale convergence theorem applies. It asserts that M_t converges almost surely to a nonnegative random variable $\bar{M}$: $M_t \rightarrow_{\text{a.s.}} \bar{M}$.

It is interesting to consider three cases: (1) $\beta(1+r) > 1$; (2) $\beta(1+r) < 1$, and (3) $\beta(1+r) = 1$. In case 1, the fact that M_t converges implies that $u'(c_t)$ converges to zero almost surely. If $u(\cdot)$ is unbounded (has no satiation point), this fact then implies that $c_t \to +\infty$ and that the consumer's asset holdings must be diverging to $+\infty$. Chamberlain and Wilson (2000) show that such results also characterize the borderline case (3) (see chapter 16). In case 2, convergence of M_t leaves open the possibility that $u'(c)$ does not converge a.s., that it remains finite and continues to vary randomly. Indeed, when $\beta(1+r) < 1$, the average level of assets remains finite, and so does the level of consumption.

It is easier to analyze the borderline case $\beta(1+r) = 1$ in the special case that the employment process is independently and identically distributed, meaning that the stochastic matrix $\mathcal{P}$ has identical rows.[7] In this case, s_t provides no information about z_{t+1}, and so s_t can be dropped as an argument of both $v(\cdot)$ and $A(\cdot)$. For the case in which s_t is i. i. d., Aiyagari (1994) uses the following argument by contradiction

[7] See chapter 16 for a closely related proof.

to show that if $\beta(1+r)=1$, then z_t diverges to $+\infty$. Assume that there is some upper limit $z_{\max}$ such that $z_{t+1} \leq z_{\max} = ws_{\max} + (1+r)A(z_{\max}) - r\phi$. Then when $\beta(1+r)=1$, the strict concavity of the value function, the Benveniste-Scheinkman formula, and equation (17.5.8) imply

$$\begin{aligned} v'(z_{\max}) &\geq E_t v'\left[ws_{t+1} + (1+r)A(z_{\max}) - r\phi\right] \\ &> v'\left[ws_{\max} + (1+r)A(z_{\max}) - r\phi\right] = v'(z_{\max}), \end{aligned}$$

which is a contradiction.

17.6. Average assets as function of r

In the next several sections we use versions of a graph of Aiyagari (1994) to analyze several models. The graph plots the average level of assets as a function of r. In the model with capital, the graph is constructed to incorporate the equilibrium dependence of the wage w on r. In models without capital, like Huggett's, the wage is fixed. We shall focus on situations where $\beta(1+r)<1$. We consider cases where the optimal decision rule $A(z_t, s_t)$ and the Markov chain for s induce a Markov chain jointly for assets and s that has a unique invariant distribution. For fixed r, let $Ea(r)$ denote the mean level of assets a and let $E\hat{a}(r) = Ea(r) + \phi$ be the mean level of $a+\phi$, where the mean is taken with respect to the invariant distribution. Here it is understood that $Ea(r)$ is a function of ϕ; when we want to make the dependence explicit we write $Ea(r;\phi)$. Also, as we have said, where the single asset is capital, it is appropriate to make the wage w a function of r. This approach incorporates the way different values of r affect average capital, the marginal product of labor, and therefore the wage.

The preceding analysis applying supermartingale convergence implies that as $\beta(1+r)$ goes to 1 from below (i.e., r goes to ρ from below), $Ea(r)$ diverges to $+\infty$. This feature is reflected in the shape of the $Ea(r)$ curve in Fig. 17.6.1.[8]

Figure 17.6.1 assumes that the wage w is fixed in drawing the $Ea(r)$ curve. Later, we will discuss how to draw a similar curve, making w adjust as the function of r that is induced by the marginal productivity conditions for positive values of K.

[8] As discussed in Aiyagari (1994), $Ea(r)$ need not be a monotonically increasing function of r, especially because w can be a function of r.

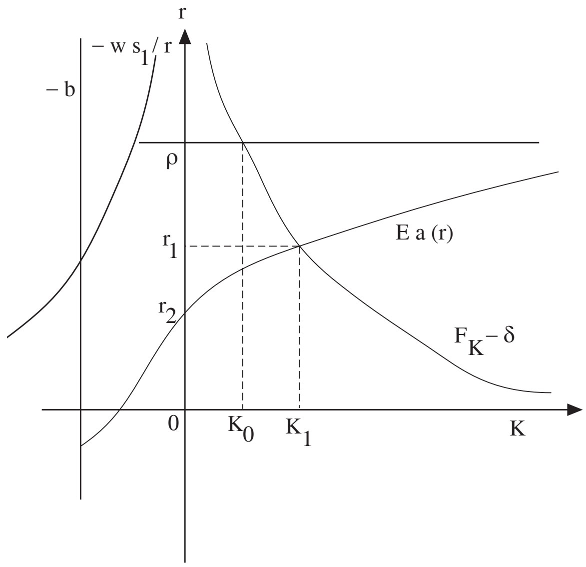

Figure 17.6.1: Demand for capital and determination of interest rate.

For now, we just assume that w is fixed at the value equal to the marginal product of labor when $K = K_1$, the equilibrium level of capital in the model. The equilibrium interest rate is determined at the intersection of the $Ea(r)$ curve with the marginal productivity of capital curve. Notice that the equilibrium interest rate r is lower than ρ, its value in the nonstochastic version of the model, and that the equilibrium value of capital K_1 exceeds the equilibrium value K_0 (determined by the marginal productivity of capital at $r = \rho$ in the nonstochastic version of the model.)

For a pure credit version of the model like Huggett's, but the same $Ea(r)$ curve, the equilibrium interest rate is determined by the intersection of the $Ea(r)$ curve with the r axis.

For the purpose of comparing some of the models that follow, it is useful to note the following aspect of the dependence of $Ea(0)$ on ϕ:

PROPOSITION 1: When $r = 0$, the optimal rule $\hat{a}_{t+1} = A(z_t, s_t)$ is independent of ϕ. This implies that for $\phi > 0$, $Ea(0; \phi) = Ea(0; 0) - \phi$.

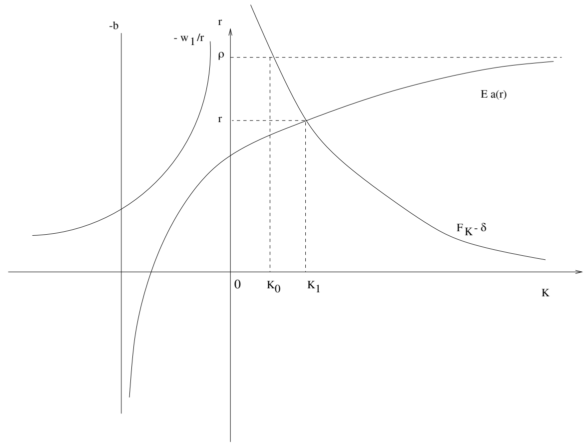

Figure 17.6.2: Demand for capital and determination of interest rate. The $Ea(r)$ curve is constructed for a fixed wage that equals the marginal product of labor at level of capital K_1. In the nonstochastic version of the model with capital, the equilibrium interest rate and capital stock are (ρ, K_0), while in the stochastic version they are (r, K_1). For a version of the model without capital in which w is fixed at this same fixed wage, the equilibrium interest rate in Huggett's pure credit economy occurs at the intersection of the $Ea(r)$ curve with the r axis.

Proof: It is sufficient to note that when $r = 0$, ϕ disappears from the right side of equation $(17.5.5b)$ (the consumer's budget constraint). Therefore, the optimal rule $\hat{a}_{t+1} = A(z_t, s_t)$ does not depend on ϕ when $r = 0$. More explicitly, when $r = 0$, add ϕ to both sides of the household's budget constraint to get

$$(a_{t+1} + \phi) + c_t \leq (a_t + \phi) + ws_t.$$

If the household's problem with $\phi = 0$ is solved by the decision rule $a_{t+1} = g(a_t, z_t)$, then the household's problem with $\phi > 0$ is solved with the same decision rule evaluated at $a_{t+1} + \phi = g(a_t + \phi, z_t)$. ∎

Thus, it follows that at $r = 0$, an increase in ϕ displaces the $Ea(r)$ curve to the left by the same amount. See Figure 17.6.3. We shall use this result to analyze several models.

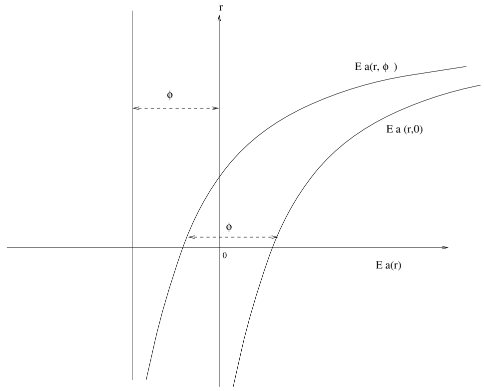

Figure 17.6.3: The effect of a shift in ϕ on the $Ea(r)$ curve. Both $Ea(r)$ curves are drawn assuming that the wage is fixed.

In the following sections, we use a version Figure 17.6.1 to compute equilibria of various models. For models without capital, the figure is drawn assuming that the wage is fixed. Typically, the $Ea(r)$ curve will have the same shape as Figure 14.1. In Huggett's model, the equilibrium interest rate is determined by the intersection of the $Ea(r)$ curve with the r-axis, reflecting that the asset (pure consumption loans) is available in zero net supply. In some models with money, the availability of a perfect substitute for consumption loans (fiat currency) creates positive net supply.

17.7. Computed examples

We used some Matlab programs that solve discrete-state dynamic programming problems to compute some examples.[9] We discretized the space of assets from $-\phi$ to a parameter $a_{\max} = 16$ with step size $.2$.

The utility function is $u(c) = (1-\mu)^{-1} c^{1-\mu}$, with $\mu = 3$. We set $\beta = .96$. We used two specifications of the Markov process for s. First, we used Tauchen's (1986)

[9] The Matlab programs used to compute the $Ea(r)$ functions are `bewley99.m`, `bewley99v2.m`, `aiyagari2.m`, `bewleyplot.m`, and `bewleyplot2.m`. The program `markovapprox.m` implements Tauchen's method for approximating a continuous autoregressive process with a Markov chain. A program `markov.m` simulates a Markov chain.

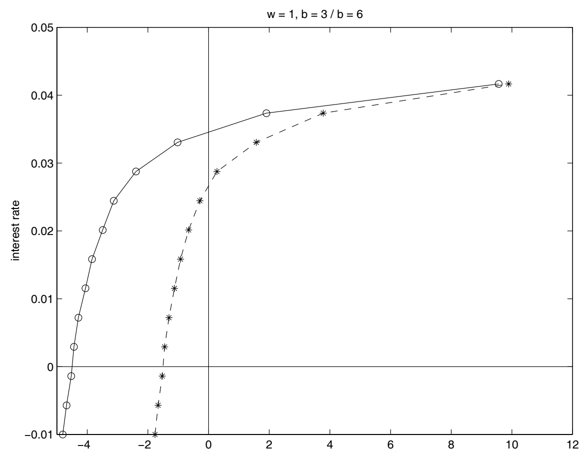

Figure 17.6.4: Two $Ea(r)$ curves, one with $b = 6$, the other with $b = 3$, with w fixed at $w = 1$. Notice that at $r - 0$, the difference between the two curves is 3, the difference in the b's.

method to get a discrete-state Markov chain to approximate a first-order autoregressive process

$$\log s_t = \rho \log s_{t-1} + u_t,$$

where u_t is a sequence of i.i.d. Gaussian random variables. We set $\rho = .2$ and the standard deviation of u_t equal to $.4\sqrt{(1-\rho)^2}$. We used Tauchen's method with $N = 7$ being the number of points in the grid for s.

For the second specification, we assumed that s is i.i.d. with mean 1.0903. For this case, we compared two settings for the variance: .22 and .68. Figures 17.6.4 and 17.7.1 plot the $Ea(r)$ curves for these various specifications. Figure 17.7.1 plots $Ea(r)$ for the first case of serially correlated s. The two $E[a(r)]$ curves correspond to two distinct settings of the ad hoc debt constraint. One is for $b = 3$, the other for $b = 6$. Figure 17.7.2 plots the invariant distribution of asset holdings for the case in which $b = 3$ and the interest rate is determined at the intersection of the $Ea(r)$ curve and the r axis.

Figure 17.7.1 summarizes a precautionary savings experiment for the i.i.d. specification of s. Two $Ea(r)$ curves are plotted. For each, we set the ad hoc debt limit $b = 0$. The $Ea(r)$ curve further to the right is the one for the higher variance of the

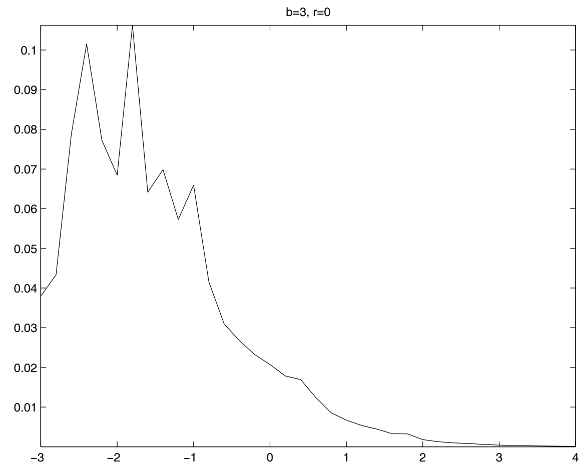

Figure 17.7.2: The invariant distribution of capital when $b = 3$.

endowment shock s. Thus, a larger variance in the random shock causes increased savings.

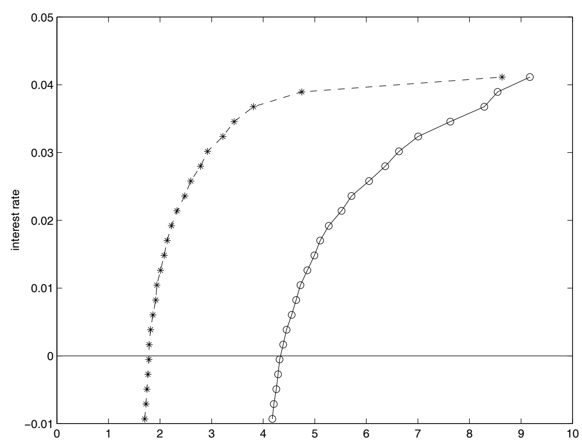

Figure 17.7.1: Two $Ea(r)$ curves when $b = 0$ and the endowment shock s is i.i.d. but with different variances; the curve with circles belongs to the economy with the higher variance.

Keep these graphs in mind as we turn to analyze some particular models in more detail.

17.8. Several Bewley models

We consider several models in which a continuum of households faces the same problem. Their behavior generates the asset demand function $Ea(r;\phi)$. The models share the same family of $Ea(r;\phi)$ curves, but differ in their settings of ϕ and in their interpretations of the supply of the asset. The models are (1) Aiyagari's (1994, 1995) model in which the risk-free asset is either physical capital or private IOUs, with physical capital being the net supply of the asset; (2) Huggett's model (1993), where the asset is private IOUs, available in zero net supply; (3) Bewley's model of fiat currency; (4) modifications of Bewley's model to permit an inflation tax; and (5) modifications of Bewley's model to pay interest on currency, either explicitly or implicitly through deflation.

17.8.1. Optimal stationary allocation

Because there is no aggregate risk and the aggregate endowment is constant, a stationary optimal allocation would have consumption constant over time for each household. Each household's consumption plan would have constant consumption over time. The implicit risk-free interest rate associated with such an allocation would be $r = \rho$. In the version of the model with capital, the stationary aggregate capital stock solves

$$F_K(K,N) - \delta = \rho. \tag{17.8.1}$$

Equation (17.8.1) restricts the stationary optimal capital stock in the nonstochastic optimal growth model of Cass (1965) and Koopmans (1965). The stationary level of capital is K_0 in Figure 17.6.1, depicted as the ordinate of the intersection of the marginal productivity net of depreciation curve with a horizontal line $r = \rho$. As we saw before, the horizontal line at $r = \rho$ acts as a "long-run" demand curve for savings for a nonstochastic version of the savings problem. The stationary optimal allocation matches the one produced by a nonstochastic growth model. We shall use the risk-free interest rate $r = \rho$ as a benchmark against which to compare some alternative incomplete market allocations. Aiyagari's (1994) model replaces the horizontal line

$r = \rho$ with an upward sloping curve $Ea(r)$, causing the stationary equilibrium interest rate to fall and the capital stock to rise relative to the risk-free model.

17.9. A model with capital and private IOUs

Figure 17.6.1 can be used to depict the equilibrium of Aiyagari's model described above. The single asset is capital. There is an aggregate production function $Y = F(K, N)$, and $w = F_N(K, N)$, $r + \delta = F_K(K, N)$. We can invert the marginal condition for capital to deduce a downward-sloping curve $K = K(r)$. This is drawn as the curve labelled $F_K - \delta$ in Figure 17.6.1. We can use the marginal productivity conditions to deduce a factor price frontier $w = \psi(r)$. For fixed r, we use $w = \psi(r)$ as the wage in the savings problem and then deduce $Ea(r)$. We want the equilibrium r to satisfy

$$Ea(r) = K(r). \tag{17.9.1}$$

The equilibrium interest rate occurs at the intersection of $Ea(r)$ with the $F_K - \delta$ curve. See Figure 17.6.1.[10]

It follows from the shape of the curves that the equilibrium capital stock K_1 exceeds K_0, the capital stock required at the given level of total labor to make the interest rate equal ρ. There is capital overaccumulation in the stochastic version of the model.

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10 Recall that Figure 17.6.1 was drawn for a fixed wage w, fixed at the value equal to the marginal product of labor when $K = K_1$. Thus, the new version of Figure 17.6.1 that incorporates $w = \psi(r)$ has a new curve $Ea(r)$ that intersects the $F_K - \delta$ curve at the same point (r_1, K_1) as the old curve $Ea(r)$ with the fixed wage. Further, the new $Ea(r)$ curve would not be defined for negative values of K.

17.10. Private IOUs only

It is easy to compute the equilibrium of Mark Huggett's (1993) model with Figure 17.6.1. We recall that in Huggett's model, the one asset consists of risk-free loans issued by other households. There are no "outside" assets. This fits the basic model with a_t being the quantity of loans owed to the individual at the beginning of t. The equilibrium condition is

$$Ea(r, \phi) = 0, \tag{17.10.1}$$

which is depicted as the intersection of the $Ea(r)$ curve in Figure 17.6.1 with the r-axis. There is a family of such curves, one for each value of the "ad hoc" debt limit. Relaxing the ad hoc debt limit (by driving $b \to +\infty$) sends the equilibrium interest rate upward toward the intersection of the furthest to the left $Ea(r)$ curve, the one that is associated with the natural debt limit, with the r-axis.

17.10.1. Limitation of what credit can achieve

The equilibrium condition (17.10.1) and $\lim_{r \nearrow \rho} Ea(r) = +\infty$ imply that the equilibrium value of r is less than ρ, for all values of the debt limit respecting the natural debt limit. This outcome supports the following conclusion:

PROPOSITION 2: (Suboptimality of equilibrium with credit) The equilibrium interest rate associated with the "natural debt limit" is the highest one that Huggett's model can support. This interest rate falls short of ρ, the interest rate that would prevail in a complete market world.[11]

[11] Huggett used the model to study how tightening the ad hoc debt limit parameter b would reduce the risk-free rate far enough below ρ to explain the "risk-free rate" puzzle.

17.10.2. Proximity of r to ρ

Notice how in figure 14.3 the equilibrium interest rate r gets closer to ρ as the borrowing constraint is relaxed. How close it can get under the natural borrowing limit depends on several key parameters of the model: (1) the discount factor β, (2) the curvature of $u(\cdot)$, (3) the persistence of the endowment process, and (4) the volatility of the innovations to the endowment process. When he selected a plausible β and $u(\cdot)$, then calibrated the persistence and volatility of the endowment process to U.S. panel data on workers' earnings, Huggett (1993) found that under the natural borrowing limit, r is quite close to ρ and that the household can achieve substantial self-insurance.[12] We shall encounter an echo of this finding when we review Krusell and Smith's (1998) finding that under their calibration of idiosyncratic risk, a real business cycle with complete markets does a good job of approximating the prices and the aggregate allocation of the same model in which only a risk-free asset can be traded.

17.10.3. Inside money or 'free banking' interpretation

Huggett's can be viewed as a model of pure "inside money," or of circulating private IOUs. Every person is a "banker" in this setting, entitled to issue "notes" or evidences of indebtedness, subject to the debt limit (17.5.3). A household has issued notes whenever $a_{t+1} < 0$.

There are several ways to think about the "clearing" of notes imposed by equation (17.10.1). Here is one: In period t, trading occurs in subperiods as follows: First, households realize their s_t. Second, some households who choose to set $a_{t+1} < a_t \leq 0$ issue new IOUs in the amount $-a_{t+1} + a_t$. Other households with $a_t < 0$ may decide to set $a_{t+1} \geq 0$, meaning that they want to "redeem" their outstanding notes and possibly acquire notes issued by others. Third, households go to the market and exchange goods for notes. Fourth, notes are "cleared" or "netted out" in a centralized clearing house: positive holdings of notes issued by others are used to retire possibly negative initial holdings of one's own notes. If a person holds positive amounts of notes issued by others, some of these are used to retire any of his own notes outstanding.

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[12] This result depends sensitively on how one specifies the left-tail of the endowment distribution. Notice that if the minimum endowment $\bar{s}_1$ is set to zero, then the natural borrowing limit is zero. However, Huggett's calibration permits positive borrowing under the natural borrowing limit.

This clearing operation leaves each person with a particular a_{t+1} to carry into the next period, with no owner of IOUs also being in the position of having some notes outstanding.

There are other ways to interpret the trading arrangement in terms of circulating notes that implement multilateral long-term lending among corresponding "banks": notes issued by individual A and owned by B are "honored" or redeemed by individual C by being exchanged for goods.[13] In a different setting, Kocherlakota (1996b) and Kocherlakota and Wallace (1998) describe such trading mechanisms.

Under the natural borrowing limit, we can think of this pure consumption loans or inside money model as possible a model of 'free banking'. In the model, households' ability to issue IOU's is restrained only by the requirement that all loans be of risk-free and of one period in duration. Later, we'll use the equilibrium allocation of this 'free banking' model as a benchmark against which to judge the celebrated Friedman rule in a model with outside money and a severe borrowing limit.

We now tighten the borrowing limit enough to make room for some "outside money."

17.10.4. Bewley's basic model of fiat money

This version of the model is set up to generate a demand for fiat money, an inconvertible currency supplied in a fixed nominal amount by the government (an entity outside the model). Individuals can hold currency, but not issue it. To map the individual's problem into problem (17.3.1), we let $m_{t+1}/p = a_{t+1}, b = \phi = 0$, where m_{t+1} is the individual's holding of currency from t to $t+1$, and p is a constant price level. With a constant price level, $r = 0$. With $b = \phi = 0$, $\hat{a}_t = a_t$. Currency is the only asset that can be held. The fixed supply of currency is M. The condition for a stationary equilibrium is

$$Ea(0) = \frac{M}{p}. \tag{17.10.2}$$

This equation is to be solved for p. The equation states a version of the quantity theory of money.

Since $r = 0$, we need *some* ad hoc borrowing constraint (i.e., $b < \infty$) to make this model have a stationary equilibrium. If we relax the borrowing constraint from

13 It is possible to tell versions of this story in which notes issued by one individual or group of individuals are "extinguished" by another.

$b = 0$ to permit some borrowing (letting $b > 0$), the $Ea(r)$ curve shifts to the left, causing $Ea(0)$ to fall and the stationary price level to rise.

Let $\bar{m} = Ea(0, \phi = 0)$ be the solution of equation (17.10.2) when $\phi = 0$. Proposition 1 tells how to construct a set of stationary equilibria, indexed by $\phi \in (0, \bar{m})$, which have identical allocations but different price levels. Given an initial stationary equilibrium with $\phi = 0$ and a price level satisfying equation (17.10.2), we construct the equilibrium for $\phi \in (0, \bar{m})$ by setting $\hat{a}_t$ for the new equilibrium equal to $\hat{a}_t$ for the old equilibrium for each person for each period.

This set of equilibria highlights how expanding the amount of "inside money," by substituting for "outside" money, causes the value of outside money (currency) to fall. The construction also indicates that if we set $\phi > \bar{m}$, then there exists no stationary monetary equilibrium with a finite positive price level. For $\phi > \bar{m}$, $Ea(0) < 0$ indicating a force for the interest rate to rise and for private IOUs to dominate currency in rate of return and to drive it out of the model. This outcome leads us to consider proposals to get currency back into the model by paying interest on it. Before we do, let's consider some situations more often observed, where a government raises revenues by an inflation tax.

17.11. A model of seigniorage

The household side of the model is described in the previous section; we continue to summarize this in a stationary demand function $Ea(r)$. We suppose that $\phi = 0$, so individuals cannot borrow. But now the government augments the nominal supply of currency over time to finance a fixed aggregate flow of real purchases G. The government budget constraint at $t \geq 0$ is

$$M_{t+1} = M_t + p_t G, \tag{17.11.1}$$

which for $t \geq 1$ can be expressed

$$\frac{M_{t+1}}{p_t} = \frac{M_t}{p_{t-1}} \left(\frac{p_{t-1}}{p_t} \right) + G.$$

We shall seek a stationary equilibrium with $\frac{p_{t-1}}{p_t} = (1 + r)$ for $t \geq 1$ and $\frac{M_{t+1}}{p_t} = \bar{a}$ for $t \geq 0$. These guesses make the previous equation become

$$\bar{a} = \frac{G}{-r}. \tag{17.11.2}$$

For $G > 0$, this is a rectangular hyperbola in the southeast quadrant. A stationary equilibrium value of r is determined at the intersection of this curve with $Ea(r)$ (see Figure 14.6). Evidently, when $G > 0$, the equilibrium net interest rate $r < 0$; $-r$ can be regarded as an inflation tax. Notice that if there is one equilibrium value, there is typically more than one. This is a symptom of the Laffer curve present in this model. Typically if a stationary equilibrium exists, there are at least two stationary inflation rates that finance the government budget. This conclusion follows from the fact that both curves in Figure 14.6 have positive slopes.

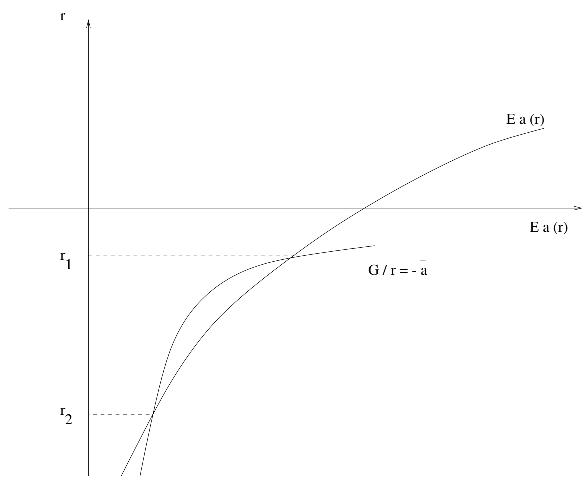

Figure 17.11.1: Two stationary equilibrium rates of return on currency that finance the constant government deficit G.

After r is determined, the initial price level can be determined by the time-0 version of the government budget constraint (17.11.1), namely,

$$\bar{a} = M_0/p_0 + G.$$

This is the version of the quantity theory of money that prevails in this model. An increase in M_0 increases p_0 and all subsequent prices proportionately.

Since there are generally multiple stationary equilibrium inflation rates, which one should we select? We recommend choosing the one with the highest rate of return to currency, that is, the lowest inflation tax. This selection gives "classical" comparative statics: increasing G causes r to fall. In distinct but related settings, Marcet and Sargent (1989) and Bruno and Fischer (1990) give learning procedures that select the same equilibrium we have recommended. Marimon and Sunder (1993) describe experiments with human subjects that they interpret as supporting this selection.

Note the effects of alterations in the debt limit ϕ on the inflation rate. Raising ϕ causes the $Ea(r)$ curve to shift to the left, and *lowers* r. It is even possible for such an increase in ϕ to cause all stationary equilibria to vanish. This experiment indicates why governments intent on raising seigniorage might want to restrict private borrowing. See Bryant and Wallace (1984) for an extensive theoretical elaboration of this and related points. See Sargent and Velde (1995) for a practical example from the French Revolution.

17.12. Exchange rate indeterminacy

We can adapt the preceding model to display a version of Kareken and Wallace's (1980) theory of exchange rate indeterminacy. Consider a model consisting of two countries, each of which is a Bewley economy with stationary money demand function $Ea_i(r)$ in country i. The same single consumption good is available in each country. Residents of both countries are free to hold the currency of either country. Households of either country are indifferent between the two currencies as long as their rates of return are equal. Let p_{it} be the price level in country i, and let $p_{1t} = e_t p_{2t}$ define the time-t exchange rate e_t. The gross return on currency i between $t-1$ and t is $(1+r) = \left(\frac{p_{i,t-1}}{p_{i,t}}\right)$ for $i = 1, 2$. Equality of rates of return implies $e_t = e_{t-1}$ for all t and therefore $p_{1,t} = e p_{2,t}$ for all t, where e is a *constant* exchange rate to be determined.

Each of the two countries finances a fixed expenditure level G_i by printing its own currency. Let $\bar{a}_i$ be the stationary level of real balances in country i's currency. Stationary versions of the two countries' budget constraints are

$$\bar{a}_1 = \bar{a}_1(1+r) + G_1 \tag{17.12.1}$$

$$\bar{a}_2 = \bar{a}_2(1+r) + G_2 \tag{17.12.2}$$

Sum these to get

$$\bar{a}_1 + \bar{a}_2 = \frac{(G_1 + G_2)}{-r}.$$

Setting this curve against $Ea_1(r) + Ea_2(r)$ determines a stationary equilibrium rate of return r. To determine the initial price level and exchange rate, we use the time-0 budget constraints of the two governments. The time-0 budget constraint for country i is

$$\frac{M_{i,1}}{p_{i,0}} = \frac{M_{i,0}}{p_{i,0}} + G_i$$

or

$$\bar{a}_i = \frac{M_{i,0}}{p_{i,0}} + G_i. \tag{17.12.3}$$

Add these and use $p_{1,0} = ep_{2,0}$ to get

$$(\bar{a}_1 + \bar{a}_2) - (G_1 + G_2) = \frac{M_{1,0} + eM_{2,0}}{p_{1,0}}.$$

This is one equation in two variables $(e, p_{1,0})$. If there is a solution for some $e \in (0, +\infty)$, then there is a solution for any other $e \in (0, +\infty)$. In this sense, the equilibrium exchange rate is indeterminate.

Equation (17.12.3) is a quantity theory of money stated in terms of the initial "world money supply" $M_{1,0} + eM_{2,0}$.

17.12.1. Interest on currency

Bewley (1980, 1983) studied whether Friedman's recommendation to pay interest on currency could improve outcomes in a stationary equilibrium, and possibly even support an optimal allocation. He found that when $\beta < 1$, Friedman's rule could improve things but could not implement an optimal allocation for reasons we now describe.

As in the earlier fiat money model, there is one asset, fiat currency, issued by a government. Households cannot borrow ($b = 0$). The consumer's budget constraint is

$$m_{t+1} + p_t c_t \leq (1 + \tilde{r}) m_t + p_t w s_t - \tau p_t$$

where $m_{t+1} \geq 0$ is currency carried over from t to $t+1$, p_t is the price level at t, $\tilde{r}$ is nominal interest on currency paid by the government, and τ is a real lump-sum tax. This tax is used to finance the interest payments on currency. The government's budget constraint at t is

$$M_{t+1} = M_t + \tilde{r} M_t - \tau p_t,$$

where M_t is the nominal stock of currency per person at the beginning of t.

There are two versions of this model: one where the government pays explicit interest, while keeping the nominal stock of currency fixed; another where the government pays no explicit interest, but varies the stock of currency to pay interest through deflation.

For each setting, we can show that paying interest on currency, where currency holdings continue to obey $m_t \geq 0$, can be viewed as a device for weakening the impact of this nonnegativity constraint. We establish this point for each setting by showing that the household's problem is isomorphic with Aiyagari's problem of expressions (17.3.1), (17.5.3), and (17.5.4).

17.12.2. Explicit interest

In the first setting, the government leaves the money supply fixed, setting $M_{t+1} = M_t \ \forall t$, and undertakes to support a constant price level. These settings make the government budget constraint imply

$$\tau = \tilde{r}M/p.$$

Substituting this into the household's budget constraint and rearranging gives

$$\frac{m_{t+1}}{p} + c_t \leq \frac{m_t}{p}(1+\tilde{r}) + ws_t - \tilde{r}\frac{M}{p}$$

where the choice of currency is subject to $m_{t+1} \geq 0$. With appropriate transformations of variables, this matches Aiyagari's setup of expressions (17.3.1), (17.5.3), and (17.5.4). In particular, take $r = \tilde{r}$, $\phi = \frac{M}{p}$, $\frac{m_{t+1}}{p} = \hat{a}_{t+1} \geq 0$. With these choices, the solution of the household's saving problem living in an economy with aggregate real balances of $\frac{M}{p}$ and with nominal interest $\tilde{r}$ on currency can be read from the solution of the savings problem with the real interest rate $\tilde{r}$ and a borrowing constraint parameter $\phi \equiv \frac{M}{p}$. Let the solution of this problem be given by the policy function $a_{t+1} = g(a, s; r, \phi)$. Because we have set $\frac{m_{t+1}}{p} = \hat{a}_{t+1} \equiv a_{t+1} + \frac{M}{p}$, the condition that the supply of real balances equals the demand $E\frac{m_{t+1}}{p} = \frac{M}{p}$ is equivalent with $E\hat{a}(r) = \phi$. Note that because $a_t = \hat{a}_t - \phi$, the equilibrium can also be expressed as $Ea(r) = 0$, where as usual $Ea(r)$ is the average of a computed with respect to the invariant distribution $\lambda(a, s)$.

The preceding argument shows that an equilibrium of the money economy with $m_{t+1} \geq 0$, equilibrium real balances $\frac{M}{p}$, and explicit interest on currency r therefore is isomorphic to a pure credit economy with borrowing constraint $\phi = \frac{M}{p}$. We formalize this conclusion in the following proposition:

PROPOSITION 3: A stationary equilibrium with interest on currency financed by lump-sum taxation has the same allocation and interest rate as an equilibrium of Huggett's free banking model for debt limit ϕ equaling the equilibrium real balances from the monetary economy.

To compute an equilibrium with interest on currency, we use a "backsolving" method.[14] Thus, even though the spirit of the model is that the government names $\tilde{r} = r$ and commits itself to set the lump-sum tax needed to finance interest payments on whatever $\frac{M}{p}$ emerges, we can compute the equilibrium by naming $\frac{M}{p}$ *first*, then finding an r that makes things work. In particular, we use the following steps:

1. Set ϕ to satisfy $0 \leq \phi \leq \frac{ws_1}{r}$. (We will elaborate on the upper bound in the next section.) Compute real balances and therefore p by solving $\frac{M}{p} = \phi$.
2. Find r from $E\hat{a}(r) = \frac{M}{p}$ or $Ea(r) = 0$.
3. Compute the equilibrium tax rate from the government budget constraint $\tau = r\frac{M}{p}$.

This construction finds a constant tax that satisfies the government budget constraint and that supports a level of real balances in the interval $0 \leq \frac{M}{p} \leq \frac{ws_1}{r}$. Evidently, the largest level of real balances that can be supported in equilibrium is the one associated with the natural debt limit. The levels of interest rates that are associated with monetary equilibria are in the range $0 \leq r \leq r_{FB}$ where $Ea(r_{FB}) = 0$ and r_{FB} is the equilibrium interest rate in the pure credit economy (i.e., Huggett's model) under the natural debt limit.

[14] See Sims (1989) and Diaz-Giménez, Prescott, Fitgerald, and Alvarez (1992) for an explanation and application of backsolving.

17.12.3. The upper bound on $\frac{M}{p}$

To interpret the upper bound on attainable $\frac{M}{p}$, note that the government's budget constraint and the budget constraint of a household with zero real balances imply that $\tau = r\frac{M}{p} \leq ws$ for all realizations of s. Assume that the stationary distribution of real balances has a positive fraction of agents with real balances arbitrarily close to zero. Let the distribution of employment shocks s be such that a positive fraction of these low-wealth consumers receive income ws_1 at any time. Then for it to be feasible for the lowest wealth consumers to pay their lump-sum taxes, we must have $\tau \equiv \frac{rM}{p} \leq ws_1$ or $\frac{M}{p} \leq \frac{ws_1}{r}$.

In a figure like Figure 17.6.1 or 17.6.2, the equilibrium real interest rate r can be read from the intersection of the $Ea(r)$ curve and the r-axis. Think of a graph with two $Ea(r)$ curves, one with the "natural debt limit" $\phi = \frac{s_1 w}{r}$, the other one with an "ad hoc" debt limit $\phi = \min[b, \frac{s_1 w}{r}]$ shifted to the right. The highest interest rate that can be supported by an interest on currency policy is evidently determined by the point where the $Ea(r)$ curve for the "natural" debt limit passes through the r-axis. This is higher than the equilibrium interest rate associated with any of the ad hoc debt limits, but must be below ρ. Note that ρ is the interest rate associated with the "optimal quantity of money." Thus, we have Aiyagari's (1994) graphical version of Bewley's (1983) result that the optimal quantity of money (Friedman's rule) cannot be implemented in this setting.

We summarize this discussion with a proposition:

PROPOSITION 4: *Free Banking and Friedman's Rule* The highest interest rate that can be supported by paying interest on currency equals that associated with the pure credit (i.e., the pure inside money) model with the natural debt limit.

If $\rho > 0$, Friedman's rule—to pay real interest on currency at the rate ρ—cannot be implemented in this model. The most that can be achieved by paying interest on currency is to eradicate the restriction that prevents households from issuing currency in competition with the government and to implement the free banking outcome.

17.12.4. A very special case

Levine and Zame (1999) have studied a special limiting case of the preceding model in which the free banking equilibrium, which we have seen is equivalent to the best stationary equilibrium with interest on currency, is optimal. They attain this special case as the limit of a sequence of economies with $\rho \downarrow 0$. Heuristically, under the natural debt limits, the $Ea(r)$ curves converge to a horizontal line at $r = 0$. At the limit $\rho = 0$, the argument leading to Proposition 4 allows for the optimal $r = \rho$ equilibrium.

17.12.5. Implicit interest through inflation

There is another arrangement equivalent to paying explicit interest on currency. Here the government aspires to pay interest through deflation, but abstains from paying explicit interest. This purpose is accomplished by setting $\tilde{r} = 0$ and $\tau p_t = -gM_t$, where it is intended that the outcome will be $(1+r)^{-1} = (1+g)$, with $g < 0$. The government budget constraint becomes $M_{t+1} = M_t(1+g)$. This can be written

$$\frac{M_{t+1}}{p_t} = \frac{M_t}{p_{t-1}}\frac{p_{t-1}}{p_t}(1+g).$$

We seek a steady state with constant real balances and inverse of the gross inflation rate $\frac{p_{t-1}}{p_t} = (1+r)$. Such a steady state implies that the preceding equation gives $(1+r) = (1+g)^{-1}$, as desired. The implied lump-sum tax rate is $\tau = -\frac{M_t}{p_{t-1}}(1+r)g$. Using $(1+r) = (1+g)^{-1}$, this can be expressed

$$\tau = \frac{M_t}{p_{t-1}}r.$$

The household's budget constraint with taxes set in this way becomes

$$c_t + \frac{m_{t+1}}{p_t} \leq \frac{m_t}{p_{t-1}}(1+r) + ws_t - \frac{M_t}{p_{t-1}}r \tag{17.12.4}$$

This matches Aiyagari's setup with $\frac{M_t}{p_{t-1}} = \phi$.

With these matches the steady-state equilibrium is determined just as though explicit interest were paid on currency. The intersection of the $Ea(r)$ curve with the r-axis determines the real interest rate. Given the parameter b setting the debt limit, the interest rate equals that for the economy with explicit interest on currency.

17.13. Precautionary savings

As we have seen in the production economy with idiosyncratic labor income shocks, the steady-state capital stock is larger when agents have no access to insurance markets as compared to the capital stock in a complete-markets economy. The "excessive" accumulation of capital can be thought of as the economy's aggregate amount of *precautionary savings*—a point emphasized by Huggett and Ospina (2000). The precautionary demand for savings is usually described as the extra savings caused by future income being random rather than determinate.

In a partial-equilibrium savings problem, it has been known since Leland (1968) and Sandmo (1970) that precautionary savings in response to risk are associated with convexity of the marginal utility function, or a positive third derivative of the utility function. In a two-period model, the intuition can be obtained from the Euler equation, assuming an interior solution with respect to consumption:

$$u'[(1+r)a_0 + w_0 - a_1] = \beta(1+r)E_0 u'[(1+r)a_1 + w_1],$$

where $1+r$ is the gross interest rate, w_t is labor income (endowment) in period $t = 0, 1$; a_0 are initial assets and a_1 is the optimal amount of savings between periods 0 and 1. Now compare the optimal choice of a_1 in two economies where next period's labor income w_1 is either determinate and equal to $\bar{w}_1$, or random with a mean value of $\bar{w}_1$. Let a_1^n and a_1^s denote the optimal choice of savings in the nonstochastic and stochastic economy, respectively, that satisfy the Euler equations:

$$\begin{aligned}
u'[(1+r)a_0 + w_0 - a_1^n] &= \beta(1+r)u'[(1+r)a_1^n + \bar{w}_1] \\
u'[(1+r)a_0 + w_0 - a_1^s] &= \beta(1+r)E_0 u'[(1+r)a_1^s + w_1] \\
&> \beta(1+r)u'[(1+r)a_1^s + \bar{w}_1],
\end{aligned}$$

where the strict inequality is implied by Jensen's inequality under the assumption that $u''' > 0$. It follows immediately from these expressions that the optimal asset level is strictly greater in the stochastic economy as compared to the nonstochastic economy, $a_1^s > a_1^n$.

Versions of precautionary savings have been analyzed by Miller (1974), Sibley (1975), Zeldes (1989), Caballero (1990), Kimball (1990, 1993), and Carroll and Kimball (1996), just to mention a few other studies in a vast literature. Using numerical methods for a finite-horizon savings problem and assuming a constant relative risk-aversion utility function, Zeldes (1989) found that introducing labor income uncertainty made the optimal consumption function concave in assets. That is, the marginal propensity to consume out of assets or transitory income declines with the level of assets. In contrast, without uncertainty and when $\beta(1+r)=1$ (as assumed by Zeldes), the marginal propensity to consume depends only on the number of periods left to live, and is neither a function of the agent's asset level nor the present-value of lifetime wealth.[15] Here we briefly summarize Carroll and Kimball's (1996) analytical explanation for the concavity of the consumption function that income uncertainty seemed to induce.

In a finite-horizon model where both the interest rate and endowment are stochastic processes, Carroll and Kimball cast their argument in terms of the class of hyperbolic absolute risk-aversion (HARA) one-period utility functions. These are defined by $\frac{u'''u'}{u''^2}=k$ for some number k. To induce precautionary savings, it must be true that $k>0$. Most commonly used utility functions are of the HARA class: quadratic utility has $k=0$, constant absolute risk-aversion (CARA) corresponds to $k=1$, and constant relative risk-aversion (CRRA) utility functions satisify $k>1$.

Carroll and Kimball show that if $k>0$, then consumption is a concave function of wealth. Moreover, except for some special cases, they show that the consumption function is *strictly* concave; that is, the marginal propensity to consume out of wealth declines with increases in wealth. The exceptions to strict concavity include two well-known cases: CARA utility if all of the risk is to labor income (no rate-of-return risk), and CRRA utility if all of the risk is rate-of-return risk (no labor-income risk).

In the course of the proof, Carroll and Kimball generalize the result of Sibley (1975) that a positive third derivative of the utility function is inherited by the value function. For there to be precautionary savings, the third derivative of the value function with respect to assets must be *positive*; that is, the marginal utility of assets must be a convex function of assets. The case of quadratic one-period utility is an

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15 When $\beta(1+r)=1$ and there are T periods left to live in a nonstochastic economy, consumption smoothing prescibes a constant consumption level c given by $\sum_{t=0}^{T-1}\frac{c}{(1+r)^t}=\Omega$, which implies $c=\frac{r}{1+r}\left[1-\frac{1}{(1+r)^T}\right]^{-1}\Omega\equiv \mathrm{MPC}_T\,\Omega$, where Ω is the agent's current assets plus the present value of her future labor income. Hence, the marginal propensity to consume out of an additional unit of assets or transitory income, MPC_T, is only a function of the time horizon T.

example where there is no precautionary saving. Off corners, the value function is quadratic, and the third derivative of the value function is zero.[16]

Where precautionary saving occurs, and where the marginal utility of consumption is always positive, the consumption function becomes approximately linear for large asset levels.[17] This feature of the consumption function plays a decisive role in governing the behavior of a model of Krusell and Smith (1998), to which we now turn.

17.14. Models with fluctuating aggregate variables

That the aggregate equilibrium state variables are constant helps makes the preceding models tractable. This section describes a way to extend such models to situations with time-varying stochastic aggregate state variables.[18]

Krusell and Smith (1998) modified Aiyagari's (1994) model by adding an aggregate state variable z, a technology shock that follows a Markov process. Each household continues to receive an idiosyncratic labor-endowment shock s that averages to the same constant value for each value of the aggregate shock z. The aggregate shock causes the size of the state of the economy to expand dramatically because every household's wealth will depend on the history of the *aggregate* shock z, call it z^t, as well as the history of the household-specific shock s^t. That makes the joint histories of z^t, s^t correlated across households, which in turn makes the cross-section distribution of (k, s) vary randomly over time. Therefore, the interest rate and wage will also vary randomly over time.

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16 In linear quadratic models, decision rules for consumption and asset accumulation are independent of the variances of innovations to exogenous income processes.

17 Roughly speaking, this follows from applying the Benveniste-Scheinkman formula and noting that, where v is the value function, v'' is increasing in savings and v'' is bounded.

18 See Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) for a general formulation and equilibrium existence theorem for such models. These authors cast doubt on whether in general the current distribution of wealth is enough to serve as a complete description of the history of the aggregate state. Kubler?? (XXXXX) See Marcet and Singleton (1999) for a computational strategy for incomplete markets models with a finite number of heterogeneous agents.

One way to specify the state is to include the cross-section distribution $\lambda(k,s)$ *each period* among the state variables. Thus, the state includes a cross-section probability distribution of (capital, employment) pairs. In addition, a description of a recursive competitive equilibrium must include a law of motion mapping today's distribution $\lambda(k,s)$ into tomorrow's distribution.

17.14.1. Aiyagari's model again

To prepare the way for Krusell and Smith's way of handling such a model, we recall the structure of Aiyagari's model. The household's Bellman equation in Aiyagari's model is

$$v(k,s) = \max_{c,k'} \{u(c) + \beta E[v(k',s')|s]\} \tag{17.14.1}$$

where the maximization is subject to

$$c + k' = \tilde{r}k + ws + (1-\delta)k, \tag{17.14.2}$$

and the prices $\tilde{r}$ and w are fixed numbers satisfying

$$\tilde{r} = \tilde{r}(K,N) = \alpha \left(\frac{K}{N}\right)^{\alpha-1} \tag{17.14.3a}$$

$$w = w(K,N) = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}. \tag{17.14.3b}$$

Recall that aggregate capital and labor K, N are the average values of k, s computed from

$$K = \int k\lambda(k,s)dkds \tag{17.14.4}$$

$$N = \int s\lambda(k,s)dkds. \tag{17.14.5}$$

Here we are following Aiyagari by assuming a Cobb-Douglas aggregate production function. The definition of a stationary equilibrium requires that $\lambda(k,s)$ be the stationary distribution of (k,s) across households induced by the decision rule that attains the right side of equation (17.14.1).

17.14.2. Krusell and Smith's extension

Krusell and Smith (1998) modify Aiyagari's model by adding an aggregate productivity shock z to the price equations, emanating from the presence of z in the production function. The shock z is governed by an exogenous Markov process. Now the state must include λ and z too, so the household's Bellman equation becomes

$$v(k,s;\lambda,z) = \max_{c,k'} \{u(c) + \beta E[v(k',s';\lambda',z')|(s,z,\lambda)]\} \tag{17.14.6}$$

where the maximization is subject to

$$c + k' = \tilde{r}(K,N,z)k + w(K,N,z)s + (1-\delta)k \tag{17.14.7a}$$

$$\tilde{r} = \tilde{r}(K,N,z) = z\alpha \left(\frac{K}{N}\right)^{\alpha-1} \tag{17.14.7b}$$

$$w = w(K,N,z) = z(1-\alpha)\left(\frac{K}{N}\right)^{\alpha} \tag{17.14.7c}$$

$$\lambda' = H(\lambda, z) \tag{17.14.7d}$$

where (K,N) is a stochastic processes determined from[19]

$$K_t = \int k\lambda_t(k,s)dkds \tag{17.14.8}$$

$$N_t = \int s\lambda_t(k,s)dkds. \tag{17.14.9}$$

Here $\lambda_t(k,s)$ is the distribution of k,s across households at time t. The *distribution* is itself a random function disturbed by the aggregate shock z_t.

Krusell and Smith make the plausible guess that $\lambda_t(k,s)$ is enough to complete the descrition of the state.[20, 21] The Bellman equation and the pricing functions

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19 In our simplified formulation, N is actually constant over time. But in Krusell and Smith's model, N too can be a stochastic process, because leisure is in the one-period utility function.

20 However, in general settings, this guess remains to be verified. Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) give an example of an incomplete markets economy in which it is necessary to keep track of a longer history of the distribution of wealth.

21 Loosely speaking, that the individual moves through the distribution of wealth as time passes indicates that his implicit Pareto weight is fluctuating.

induce the household to want to forecast the average capital stock K, in order to forecast future prices. That desire makes the household want to forecast the cross-section distribution of holdings of capital. To do so it consults the law of motion (17.14.7d).

DEFINITION: A recursive competitive equilibrium is a pair of price functions $\tilde{r}, w$, a value function, a decision rule $k' = f(k,s;\lambda,z)$, and a law of motion H for $\lambda(k,s)$ such that (a) given the price functions and H, the value function solves the Bellman equation (17.14.6) and the optimal decision rule is f; and (b) the decision rule f and the Markov processes for s and z imply that today's distribution $\lambda(k,s)$ is mapped into tomorrow's $\lambda'(k,s)$ by H.

The curse of dimensionality makes an equilibrium difficult to compute. Krusell and Smith propose a way to approximate an equilibrium using simulations. First, they characterize the distribution $\lambda(k,s)$ by a finite set of moments of capital $m = (m_1,\ldots,m_I)$. They assume a parametric functional form for H mapping today's m into next period's value m'. They assume a form that can be conveniently estimated using least squares. They assume initial values for the parameters of H. Given H, they use numerical dynamic programming to solve the Bellman equation

$$v(k,s;m,z) = \max_{c,k'} \{u(c) + \beta E[v(k',s';m',z')|(s,z,m)]\}$$

subject to the assumed law of motion H for m. They take the solution of this problem and draw a single long realization from the Markov process for $\{z_t\}$, say, of length T. For that particular realization of z, they then simulate paths of $\{k_t, s_t\}$ of length T for a large number M of households. They assemble these M simulations into a history of T empirical cross-section distributions $\lambda_t(k,s)$. They use the cross-section at t to compute the cross-section moments $m(t)$, thereby assembling a time series of length T of the cross-section moments $m(t)$. They use this sample and nonlinear least squares to estimate the transition function H mapping $m(t)$ into $m(t+1)$. They return to the beginning of the procedure, use this new guess at H, and continue, iterating to convergence of the function H.

Krusell and Smith compare the aggregate time series $K_t, N_t, \tilde{r}_t, w_t$ from this model with a corresponding representative agent (or complete markets) model. They find that the statistics for the aggregate quantities and prices for the two types of models are very close. Krusell and Smith interpret this result in terms of an "approximate aggregation theorem" that follows from two properties of their parameterized model. First, consumption as a function of wealth is concave but close to linear for moderate-to-high wealth levels. Second, most of the saving is done by the high-wealth people. These two properties mean that fluctuations in the distribution of wealth have only a small effect on the aggregate amount saved and invested. Thus, distribution effects are small. Also, for these high-wealth people, self-insurance works quite well, so aggregate consumption is not much lower than it would be for the complete markets economy.

Krusell and Smith compare the distributions of wealth from their model to the U.S. data. Relative to the data, the model with a constant discount factor generates too few very poor people and too many rich people. Krusell and Smith modify the model by making the discount factor an exogenous stochastic process. The discount factor switches occasionally between two values. Krusell and Smith find that a modest difference between two discount factors can bring the model's wealth distribution much closer to the data. Patient people become wealthier; impatient people eventually become poorer.

17.15. Concluding remarks

The models in this chapter pursue some of the adjustments that households make when their preferences and endowments give a motive to insure but markets offer limited opportunities to do so. We have studied settings where households' savings occurs through a single risk-free asset. Households use the asset to "self-insure," by making intertemporal adjustments of the asset holdings to smooth their consumption. Their consumption rates at a given date become a function of their asset holdings, which in turn depend on the histories of their endowments. In pure exchange versions of the model, the equilibrium allocation becomes individual-history specific, in contrast to the history-independence of the corresponding complete markets model.

The models of this chapter arbitrarily shut down or allow markets without explanation. The market structure is imposed, its consequences then analyzed. In chapter 19, we study a class of models for similar environments that, like the models of this chapter, make consumption allocations history dependent. But the spirit of the models in chapter 19 differs from those in this chapter in requiring that the trading structure be more firmly motivated by the environment. In particular, the models in chapter 19 posit a particular reason that complete markets do not exist, coming from enforcement or information problems, and then study how risk sharing among people can best be arranged.

Exercises

Exercise 17.1 **Stochastic discount factor** (Bewley-Krusell-Smith)

A household has preferences over consumption of a single good ordered by a value function defined recursively according to $v(\beta_t, a_t, s_t) = u(c_t) + \beta_t E_t v(\beta_{t+1}, a_{t+1}, s_{t+1})$, where $\beta_t \in (0,1)$ is the time-t value of a discount factor, and a_t is time-t holding of a single asset. Here v is the discounted utility for a consumer with asset holding a_t, discount factor β_t, and employment state s_t. The discount factor evolves according to a three-state Markov chain with transition probabilities $P_{i,j} = \text{Prob}(\beta_{t+1} = \bar{\beta}_j | \beta_t = \bar{\beta}_i)$. The discount factor and employment state at t are both known. The household faces the sequence of budget constraints

$$a_{t+1} + c_t \leq (1+r)a_t + ws_t$$

where s_t evolves according to an n-state Markov chain with transition matrix $\mathcal{P}$. The household faces the borrowing constraint $a_{t+1} \geq -\phi$ for all t.

Formulate Bellman equations for the household's problem. Describe an algorithm for solving the Bellman equations. *Hint:* Form three coupled Bellman equations.

Exercise 17.2 **Mobility costs** (Bertola)

A worker seeks to maximize $E \sum_{t=0}^{\infty} \beta^t u(c_t)$, where $\beta \in (0,1)$ and $u(c) = \frac{c^{1-\sigma}}{(1-\sigma)}$, and E is the expectation operator. Each period, the worker supplies one unit of labor inelastically (there is no unemployment) and either w^g or w^b, where $w^g > w^b$. A new "job" starts off paying w^g the first period. Thereafter, a job earns a wage governed by the two-state Markov process governing transition between good and bad wages on all jobs; the transition matrix is $\begin{bmatrix} p & (1-p) \\ (1-p) & p \end{bmatrix}$. A new (well-paying) job is always available, but the worker must pay mobility cost $m > 0$ to change jobs. The mobility cost is paid at the beginning of the period that a worker decides to move. The worker's period-t budget constraint is

$$A_{t+1} + c_t + mI_t \leq RA_t + w_t,$$

where R is a gross interest rate on assets, c_t is consumption at t, $m > 0$ is moving costs, I_t is an indicator equaling 1 if the worker moves in period t, zero otherwise, and w_t is the wage. Assume that $A_0 > 0$ is given and that the worker faces the no-borrowing constraint, $A_t \geq 0$ for all t.

a. Formulate the Bellman equation for the worker.

b. Write a Matlab program to solve the worker's Bellman equation. Show the optimal decision rules computed for the following parameter values: $m = .9, p = .8, R = 1.02, \beta = .95, w^g = 1.4, w^b = 1, \sigma = 4$. Use a range of assets levels of $[0, 3]$. Describe how the decision to move depends on wealth.

c. Compute the Markov chain governing the transition of the individual's state (A, w). If it exists, compute the invariant distribution.

d. In the fashion of Bewley, use the invariant distribution computed in part **c** to describe the distribution of wealth across a large number of workers all facing this same optimum problem.

Exercise 17.3 **Unemployment**

There is a continuum of workers with identical probabilities λ of being fired each period when they are employed. With probability $\mu \in (0,1)$, each unemployed worker receives one offer to work at wage w drawn from the cumulative distribution function $F(w)$. If he accepts the offer, the worker receives the offered wage each period until he is fired. With probability $1-\mu$, an unemployed worker receives no offer this period. The probability μ is determined by the function $\mu = f(U)$, where U is the unemployment rate, and $f'(U) < 0, f(0) = 1, f(1) = 0$. A worker's utility is given by $E\sum_{t=0}^{\infty}\beta^t y_t$, where $\beta \in (0,1)$ and y_t is income in period t, which equals the wage if employed and zero otherwise. There is no unemployment compensation. Each worker regards U as fixed and constant over time in making his decisions.

a. For fixed U, write the Bellman equation for the worker. Argue that his optimal policy has the reservation wage property.

b. Given the typical worker's policy (i.e., his reservation wage), display a difference equation for the unemployment rate. Show that a stationary unemployment rate must satisfy

$$\lambda(1-U) = f(U)\big[1 - F(\bar{w})\big]U,$$

where $\bar{w}$ is the reservation wage.

c. Define a *stationary equilibrium.*

d. Describe how to compute a stationary equilibrium. You don't actually have to compute it.

Exercise 17.4 **Asset insurance**

Consider the following setup. There is a continuum of households who maximize

$$E\sum_{t=0}^{\infty}\beta^t u(c_t),$$

subject to

$$c_t + k_{t+1} + \tau \leq y + \max(x_t, g)k_t^{\alpha}, \quad c_t \geq 0,\ k_{t+1} \geq 0,\ t \geq 0,$$

where $y > 0$ is a constant level of income not derived from capital, $\alpha \in (0,1)$, τ is a fixed lump sum tax, k_t is the capital held at the beginning of t, $g \leq 1$ is an "investment insurance" parameter set by the government, and x_t is a stochastic household-specific gross rate of return on capital. We assume that x_t is governed by a two-state Markov process with stochastic matrix $\mathcal{P}$, which takes on the two values $\bar{x}_1 > 1$ and $\bar{x}_2 < 1$. When the bad investment return occurs, ($x_t = \bar{x}_2$), the government supplements the household's return by $\max(0, g - \bar{x}_2)$.

The household-specific randomness is distributed identically and independently across households. Except for paying taxes and possibly receiving insurance payments from the government, households have no interactions with one another; there are no markets.

Given the government policy parameters τ, g, the household's Bellman equation is

$$v(k,x) = \max_{k'}\{u\big[\max(x,g)k^{\alpha} - k' - \tau\big] + \beta\sum_{x'} v(k',x')\mathcal{P}(x,x')\}.$$

The solution of this problem is attained by a decision rule

$$k' = G(k,x),$$

that induces a stationary distribution $\lambda(k,x)$ of agents across states (k,x).

The average (or per capita) physical output of the economy is

$$Y = \sum_k \sum_x (x \times k^{\alpha})\lambda(k,x).$$

The average return on capital to households, *including* the investment insurance, is

$$\nu = \sum_k \bar{x}_1 k^\alpha \lambda(k, x_1) + \max(g, \bar{x}_2) \sum_k k^\alpha \lambda(k, x_2),$$

which states that the government pays out insurance to all households for which $g > \bar{x}_2$.

Define a stationary equilibrium.

Exercise 17.5 **Matching and job quality**

Consider the following Bewley model, a version of which Daron Acemoglu and Robert Shimer (2000) calibrate to deduce quantitative statements about the effects of government supplied unemployment insurance on equilibrium level of unemployment, output, and workers' welfare. Time is discrete. Each of a continuum of *ex ante* identical workers can accumulate nonnegative amounts of a single risk-free asset bearing gross one-period rate of return R; R is exogenous and satisfies $\beta R < 1$. There are good jobs with wage w_g and bad jobs with wage $w_b < w_g$. Both wages are exogenous. Unemployed workers must decide whether to search for good jobs or bad jobs. (They cannot search for both.) If an unemployment worker devotes h units of time to search for a good job, a good job arrives with probability $m_g h$; h units of time devoted to searching for bad jobs makes a bad job arrive with probability $m_b h$. Assume that $m_g < m_b$. Good jobs terminate exogenously each period with probability δ_g, bad jobs with probability δ_b. Exogenous terminations entitle an unemployed worker to unemployment compensation of b, which is independent of the worker's lagged earnings. However, each period, an unemployed worker's entitlement to unemployment insurance is exposed to an i.i.d. probability of ϕ of expiring. Workers who quit are not entitled to unemployment insurance.

Workers choose $\{c_t, h_t\}_{t=0}^\infty$ to maximize

$$E_0 \sum_{t=0}^\infty \beta^t (1-\theta)^{-1} (c_t(\bar{h} - h_t)^\eta)^{1-\theta},$$

where $\beta \in (0,1)$, and θ is a coefficient of relative risk aversion, subject to the asset accumulation equation

$$a_{t+1} = R(a_t + y_t - c_t)$$

and the no-borrowing condition $a_{t+1} \geq 0$; η governs the substitutability between consumption and leisure. Unemployed workers eligible for u.i. receive income $y_t = b$, while those not eligible receive 0. Employed workers with good jobs receive after tax income of $y_t = w_g h(1-\tau)$, and those with bad jobs receive $y_t = w_b h(1-\tau)$. In equilibrium, the flat rate tax is set so that the government budget for u.i. balances. Workers with bad jobs have the option of quitting to search for good jobs.

Define a worker's composite *state* as his asset level, together with one of four possible employment states: (1) employed in a good job, (2) employed in a bad job, (3) unemployed and eligible for u.i.; (4) unemployed and ineligible for u.i.

a. Formulate value functions for the four types of employment states, and describe Bellman equations that link them.

b. In the fashion of Bewley, define a stationary stochastic equilibrium, being careful to define all of the objects composing an equilibrium.

c. Adjust the Bellman equations to accommodate the following modification. Assume that every period that a worker finds himself in a bad job, there is a probability $\delta_{upgrade}$ that the following period, the bad job is upgraded to a good job, conditional on not having been fired.

d. Acemoglu and Shimer calibrate their model to U.S. high school graduates, then perform a 'local' analysis of the consequences of increasing the unemployment compensation rate b. For their calibration, they find that there are substantial benefits to raising the unemployment compensation rate and that this conclusion prevails despite the presence of a 'moral hazard problem' associated with providing u.i. benefits in their model. The reason is that too many workers choose to search for bad rather than good jobs. They calibrate β so that workers are sufficiently impatient that most workers with low assets search for bad jobs. If workers were more fully insured, more workers would search for better jobs. That would put a larger fraction of workers in good jobs and raise average productivity. In equilibrium, unemployed workers with high asset levels *do* search for good jobs, because their assets provide them with the 'self-insurance' needed to support their investment in search for good jobs. Do you think that the modification suggested in part in part (c) would affect the outcomes of increasing unemployment compensation b?