Numerical Optimization 2015 Homework 1

Due Oct 16

1. (20%) Show that Newton's method for optimization (for single variable) is using a quadratic model, which means in each step it
   (a) builds a tangent quadratic model at x_k;
   (b) finds the minimum (or the maximum) of the quadratic model x_{k+1}; and
   (c) walks from x_k to x_{k+1}.

2. (30%) Besides Taylor series, we can use interpolation to build polynomial models. For example, if we know $f(x_1), f(x_2)$ and $f(x_3)$, we can build $m(x) = ax^2 + bx + c$ where (a, b, c) are the solution of
$$\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{pmatrix}. \tag{1}$$
   We can design an algorithm to find $\min_x f(x)$.
   - For x_k, build $m(x)$ with $f(x_k)$, $f(x_k + \alpha)$ and $f(x_k - \alpha)$ for some small $0 < \alpha$.
   - Find $\min_x m(x)$.

   (a) Give an example to show this method can be failed.
   (b) Compare the pros/cons of Newton's method and this method.
   (c) We can also combine derivative method and interpolation method to build models. Suppose we know $f(x_1), f'(x_1), f(x_2)$ and $f'(x_2)$. Given an equation, like (1), to build a cubic model $m(x) = ax^3 + bx^2 + cx + d$.

3. (50%) Let $f(x, y) = \frac{1}{2}x^2 + \frac{9}{2}y^2$. This is a positive definite quadratic with minimizer at $(x^*, y^*) = (0, 0)$.
   (a) Derive the gradient g and the Hessian H of f.
   (b) Write Matlab codes to implement the steepest descent method and Newton's method with $\vec{x}_0 = (9, 1)$, and compare their convergent results. The formula of the steepest descent method is
$$\vec{x}_{k+1} = \vec{x}_k - \frac{\vec{g}_k^T \vec{g}_k}{\vec{g}_k^T H_k \vec{g}_k} \vec{g}_k,$$

and the formula of Newton's method is

$$\vec{x}_{k+1} = \vec{x}_k - H_k^{-1}\vec{g}_k,$$

where $\vec{g}_k = g(\vec{x}_k)$ and $H_k = H(\vec{x}_k)$.

(c) Draw the trace of $\{\vec{x}_k\}$ for the steepest descent method and Newton's method. Below is an example code for trace drawing.

```
function draw_trace()

% draw the contour of the function z = (x*x+9*y*y)/2;
step = 0.1;
X = 0:step:9;
Y = -1:step:1;

n = size(X,2);
m = size(Y,2);

Z = zeros(m,n);
for i = 1:n
    for j = 1:m
        Z(j,i) = f(X(i),Y(j));
    end
end

contour(X,Y,Z,100)

% plot the trace
%   You can record the trace of your results and use the following
%   code to plot the trace.
xk = [9 8 8 7 7 6 6 5 5 4 4 3 3 2 2];
yk = [.5 .5 -.5 -.5 .5 .5 -.5 -.5 .5 .5 -.5 -.5 .5 .5 -.5];
hold on;   % this is important!! This will overlap your plots.
plot(xk,yk,'-','LineWidth',3);
hold off;

% function definition
    function z = f(x,y)
        z = (x*x+9*y*y)/2;
    end
end
```