CS5321 Numerical Optimization Homework 4

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1. (20%) Consider the problem

$$\begin{aligned} \min_{x_1,x_2} \quad & 0.1 * (x_1 - 3)^2 + x_2^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 - 1 \le 0 \end{aligned} \tag{1}$$

   (a) Write down the KKT conditions for (1).

   Answer. $\vec{x} = (x_1, x_2)$
   The Lagrangian function

$$\mathcal{L}(\vec{x}, \lambda) = 0.1(x_1 - 3)^2 + x_2^2 - \lambda(-x_1^2 - x_2^2 + 1)$$

   The KKT conditions (First-Order Necessary Conditions)
   Suppose that $\vec{x}^*$ is a local solution, and that the LICQ holds at $\vec{x}^*$.
   Stationarity

$$\begin{aligned} \nabla_{\vec{x}^*}\mathcal{L}(\vec{x}^*, \lambda^*) &= 0 \\ \Rightarrow \begin{bmatrix} (2\lambda + 0.2)x_1^* - 0.6 \\ (2\lambda + 2)x_2^* \end{bmatrix} &= 0 \end{aligned}$$

   Primal feasibility

$$-x_1^{*2} - x_2^{*2} + 1 \ge 0$$

   Dual feasibility

$$\lambda^* \ge 0$$

   Complementary slackness

$$\lambda^*(-x_1^{*2} - x_2^{*2} + 1) = 0$$

   Strict Complementarity
   Exactly one of λ^* and $(-x_1^{*2} - x_2^{*2} + 1)$ is zero. □

(b) Solve the KKT conditions and find the optimal solutions, including the Lagrangian parameters.

Answer. $\vec{x}^* = (1, 0)$ and $\lambda^* = 0.2$. □

(c) Compute the reduced Hessian and check the second order conditions for the solution.

Answer. Second-Order Necessary Conditions

$$
\begin{aligned}
w^T \nabla^2_{\vec{x}\vec{x}} \mathcal{L}(\vec{x}^*, \lambda^*) w \geq 0 \\
\text{where } \begin{bmatrix} -2x_1^* \\ -2x_2^* \end{bmatrix}^T w = 0 \text{ with } \lambda^* > 0 \\
\text{or } \begin{bmatrix} -2x_1^* \\ -2x_2^* \end{bmatrix}^T w \geq 0 \text{ with } \lambda^* = 0
\end{aligned}
$$

Hessian matrix

$$
\nabla^2_{\vec{x}\vec{x}} \mathcal{L}(\vec{x}^*, \lambda^*) = \begin{bmatrix} 2\lambda^* + 0.2 & 0 \\ 0 & 2\lambda^* + 2 \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ 0 & 2.4 \end{bmatrix}
$$

Because $\nabla^2_{\vec{x}\vec{x}} \mathcal{L}(\vec{x}^*, \lambda^*)$ is positive-definite, satisfy the second-order conditions. □

2. (20%) The trust region method (for unconstrained optimization problem) needs to solve a local model in each step

$$
\begin{aligned}
\min_{\vec{p}} \quad & m(\vec{p}) = \frac{1}{2}\vec{p}^T A \vec{p} + \vec{g}^T \vec{p} \\
\text{s.t.} \quad & \vec{p}^T \vec{p} \leq \Delta^2.
\end{aligned}
$$

Prove that the optimal solution $\vec{p}^*$ of the local model satisfies

$$
\begin{aligned}
(A + \lambda I)\, \vec{p}^* = -\vec{g} \\
\lambda\, (\Delta - \|\vec{p}^*\|) = 0 \\
(A + \lambda I) \text{ is positive semidefinite.}
\end{aligned}
$$

(Hint: to prove the last statement, you only need to consider the directions in the *critical cone.*)

Answer. The Lagrangian function

$$
\begin{aligned}
\mathcal{L}(\vec{p}, \lambda) &= m(\vec{p}) - \frac{\lambda}{2}(\Delta^2 - \vec{p}^T \vec{p}) \\
&= m(\vec{p}) - \frac{\lambda}{2}\Delta^2 + \frac{1}{2}\vec{p}^T (\lambda I) \vec{p} \\
&= \frac{1}{2}\vec{p}^T (A + \lambda I)\vec{p} + \vec{g}^T \vec{p} - \frac{\lambda}{2}\Delta^2
\end{aligned}
$$

Suppose that $\vec{p}^*$ is a local solution, and that the LICQ holds at $\vec{p}^*$. Then KKT conditions are hold.

$$\begin{aligned} \text{Stationarity } & \nabla_{\vec{p}^*} \mathcal{L}(\vec{p}^*, \lambda^*) = 0 \\ \text{Complementary slackness } & \lambda(\Delta^2 - \vec{p}^T \vec{p}) = 0 \end{aligned}$$

Equivalently,

$$\begin{aligned} (A + \lambda I)\, \vec{p}^* &= -\vec{g}, \\ \lambda\, (\Delta - \|\vec{p}^*\|) &= 0. \end{aligned}$$

Also, the second-order necessary conditions are satisfied.

$$w^T \nabla^2_{\vec{p}\vec{p}} \mathcal{L}(\vec{p}^*, \lambda^*) w \geq 0 \text{ for all } w \in \mathcal{C}(\vec{p}^*, \lambda^*).$$

Equivalently,

$$w^T (A + \lambda I) w \geq 0 \text{ for all } w \in \mathcal{C}(\vec{p}^*, \lambda^*).$$

Hence $(A + \lambda I)$ is positive semidefinite. $\square$

3. (60%) Implement the Interior Point Method (IPM), as shown in Figure 1, to solve linear programming problem.

$$\begin{aligned} \min_{\vec{x}} \quad & \vec{c}^T \vec{x} \\ \text{s.t.} \quad & A\vec{x} - \vec{s} = \vec{b} \\ & \vec{s} \geq 0 \end{aligned}$$

You can assume $\vec{x}_0 = 0$ is a feasible interior point.

(1) Given $\vec{x}_0$, $\vec{\lambda}_0$, and $\vec{s}_0$, in which $\vec{\lambda}_0, \vec{s}_0 \geq 0$.
(2) For $k = 0, 1, \ldots$

(3) Choose $\sigma_k \in [0, 1]$ and solve

$$\begin{pmatrix} 0 & -A^T & 0 \\ -A & 0 & I \\ 0 & S^k & \Lambda^k \end{pmatrix} \begin{pmatrix} \Delta x_k \\ \Delta \lambda_k \\ \Delta s_k \end{pmatrix} = \begin{pmatrix} A^T \vec{\lambda}_k - \vec{c} \\ A\vec{x}_k - \vec{s}_k - \vec{b} \\ \sigma_k \mu_k e - \Lambda^k S^k e \end{pmatrix},$$

where $\mu_k = \dfrac{\vec{\lambda}_k^T \vec{s}_k}{m}, \Lambda^k = \mathrm{diag}(\vec{\lambda}_k), S^k = \mathrm{diag}(\vec{s}_k)$.
(4) Compute α_k such that

$$(\vec{x}_{k+1}, \vec{\lambda}_{k+1}, \vec{s}_{k+1}) = (\vec{x}_k, \vec{\lambda}_k, \vec{s}_k) + \alpha(\Delta x_k, \Delta \lambda_k, \Delta s_k)$$

is in the region $N(\gamma) = \{(\vec{x}, \vec{\lambda}, \vec{s}) | \lambda_i s_i \geq \gamma \mu_k, \forall i = 1, 2, \ldots, n\}$ for some $\gamma = 10^{-3}$.

Figure 1: The interior point method for solving linear programming