1. (30%) For the linear least square problem, most people use polynomials as the basis, $1, x, x^2, \cdots$. If we use the trigonometric functions as basis, we have Fourier approximation. For example, the model function is

$$f(x) = \frac{a_0}{2} + a_1\cos(x) + \cdots + a_m\cos(mx) + b_1\sin(x) + \cdots + b_m\sin(mx),$$

and the measured data are (x_k, y_k), where $x_k = \frac{k2\pi}{n}$ for $k = 0, \cdots, n-1$.

(a) Write down it as an optimization problem $\min_{\vec{x}} \|A\vec{x} - \vec{b}\|$, where $\vec{x} = (a_0, a_1, \ldots, a_m, b_1, \ldots, b_m)^T$, and $\vec{b} = (y_0, y_1, \ldots, y_n - 1)^T$.

(b) Show that each column vector of A is orthogonal to each other.

(c) Use the normal equation to find the solution of a_i and b_i.

(d) Comparing to the polynomial basis, what is the advantage of using trigonometric basis?

(a)

$$\min_X \|AX - b\| \propto \min_X \|AX - b\|^2$$

$$A = \begin{bmatrix} 1/2 & \cos(x_0) & \cos(2x_0) & \cdots & \cos(mx_0) & \sin(x_0) & \cdots & \sin(mx_0) \\ 1/2 & \cos(x_1) & \cos(2x_1) & \cdots & \cos(mx_1) & \sin(x_1) & \cdots & \sin(mx_1) \\ & \vdots & & & \vdots & & & \vdots \\ 1/2 & \cos(x_{n-1}) & \cos(2x_{n-1}) & \cdots & \cos(mx_{n-1}) & \sin(x_{n-1}) & \cdots & \sin(mx_{n-1}) \end{bmatrix}$$

X = $[a_0\ a_1 \ldots a_m\ b_1 \ldots b_m]$ ^T

b = $[y_1\ y_2 \ldots y_k]$ ^T

(b)

$$\forall m \neq n \quad m, n = 0,1,2,\ldots \ , \qquad ; X_i = \frac{i * (2\pi)}{k} \quad i = 0,1,\ldots,k-1 \quad 令\mathbf{k}逼近於\infty$$

$$< \cos mX_i, \cos nX_i > = \sum_{i=1}^{k} (\cos(mX_i))\,(cos\,(nX_i)) = \int_0^{2\pi} \frac{1}{2}\left[\cos((m+n)X) + \cos((m-n)X)\right]dX$$

$$= \frac{1}{2(m+n)}\sin((m+n)X)|_0^{2\pi} + \frac{1}{2(m-n)}\sin((m-n)X)|_0^{2\pi} = 0$$

$$< \sin mX_i, \sin nX_i > = \sum_{i=1}^{k} (\sin(mX_i))\,(\sin(nX_i)) = \int_0^{2\pi} \frac{1}{2}\left[-\cos((m+n)X) + \cos((m-n)X)\right]dX$$

$$= \frac{-1}{2(m+n)}\sin((m+n)X)|_0^{2\pi} + \frac{1}{2(m-n)}\sin((m-n)X)|_0^{2\pi} = 0$$

$$\forall m = 1,2,\ldots \quad n = 0,1,2,\ldots \ , \qquad ; X_i = \frac{i * (2\pi)}{k} \quad i = 0,1,\ldots,k-1 \quad 令\mathbf{k}逼近於\infty$$

$$< \sin mX_i, \cos nX_i > = \sum_{i=1}^{k} (\sin(mX_i))\,(\cos(nX_i)) = \int_0^{2\pi} \frac{1}{2}\left[-\sin((m+n)X) + \sin((m-n)X)\right]dX$$

$$= \frac{1}{2(m+n)}\cos((m+n)X)|_0^{2\pi} + \frac{-1}{2(m-n)}\cos((m-n)X)|_0^{2\pi} = 0$$

$\therefore$ each column vector of A is orthogonal to each other.

(c)

$A^T A X = A^T b$

T

$X=(A^T A)^{-1} A^T b=$

$$\left\{\begin{bmatrix} 1/2 & 1/2 & \cdots & 1/2 \\ \cos(x_0) & \cos(x_1) & \cdots & \cos(x_{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \cos(mx_0) & \cos(mx_1) & \cdots & \cos(mx_{n-1}) \\ \sin(x_0) & \sin(x_1) & \cdots & \sin(x_{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \sin(mx_0) & \sin(mx_1) & \cdots & \sin(mx_{n-1}) \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & \cdots & 1/2 \\ \cos(x_0) & \cos(x_1) & \cdots & \cos(x_{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \cos(mx_0) & \cos(mx_1) & \cdots & \cos(mx_{n-1}) \\ \sin(x_0) & \sin(x_1) & \cdots & \sin(x_{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \sin(mx_0) & \sin(mx_1) & \cdots & \sin(mx_{n-1}) \end{bmatrix}\right\}^{-1} \begin{bmatrix} 1/2 & 1/2 & \cdots & 1/2 \\ \cos(x_0) & \cos(x_1) & \cdots & \cos(x_{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \cos(mx_0) & \cos(mx_1) & \cdots & \cos(mx_{n-1}) \\ \sin(x_0) & \sin(x_1) & \cdots & \sin(x_{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \sin(mx_0) & \sin(mx_1) & \cdots & \sin(mx_{n-1}) \end{bmatrix}^T \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{4}{k} * 1/2(y_0 + y_1 + \cdots + y_{n-1}) \\ \frac{1}{(\cos {x_1}^2 + \cdots + \cos {x_k}^2)} * [y_0 \cos(x_0) + y_1 \cos(x_1) + \cdots + y_{n-1} \cos(x_{n-1})] \\ \vdots \\ \vdots \\ \vdots \\ \frac{1}{(\sin {x_1}^2 + \cdots + \sin {x_k}^2)} y_0 \sin(mx_0) + y_1 \sin(mx_1) + \cdots + y_{n-1} \sin(mx_{n-1}) \end{bmatrix}$$

(d)

2. (70%) Implement the simplex method for linear programming. The pseudo code is in Figure 1. The calling interface will be like

```
[x, case] = mysimplex(c, A, b, x0)
```

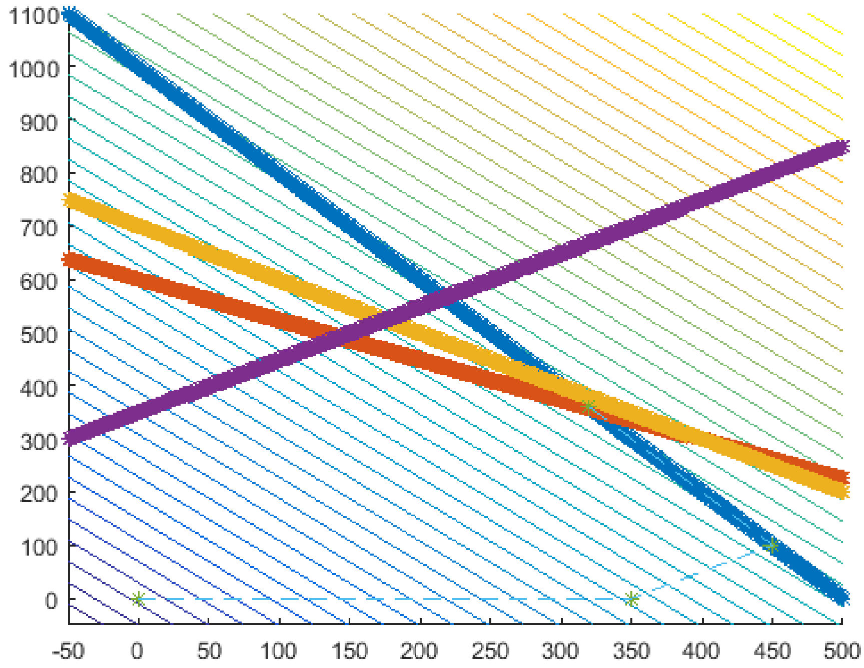
which solves

$$\begin{aligned} \min_{\vec{x}} \quad & \vec{c}^T \vec{x} \\ \text{subject to} \quad & A\vec{x} = \vec{b} \\ & \vec{x} \geq 0 \end{aligned}$$

The return value `case` should be 0, 1, or 2, which means (0) solved, (1) unbounded, (2) infeasible. You can assume $\vec{x}_0$ is a feasible point. Also,

print out each $\vec{x}_i$ during the computation. Use it to solve the following problem.

$$\begin{aligned} \max_{x_1, x_2} \quad & z = 8x_1 + 5x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 1000 \\ & 3x_1 + 4x_2 \leq 2400 \\ & x_1 + x_2 \leq 700 \\ & x_1 - x_2 \leq 350 \\ & x_1, x_2 \geq 0 \end{aligned}$$



很粗的線是 限制式 (不知道怎麼做出限制方向 <= ,>= …)
背景圖是 object function contour
紅 ‘*’ 虛線路徑 是解最佳解的路徑
最終解(320,360) MAX=4360

```
X =

         0  350.0000  450.0000  320.0000
         0         0  100.0000  360.0000


Z =

           0        2800        4100        4360
```