(20%) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & -2 \end{pmatrix}$

(a) Compute the LDLT decomposition of $A = LDL^T$.

(b) Compute the eigenvalues of A.

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U$, $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$$A = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{L^T}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\lambda = \begin{matrix} -2.3794 \\ 0.0982 \\ 4.2812 \end{matrix}$

(b) $\det(A - xI) = 0 \Rightarrow \begin{vmatrix} 1-x & 2 & 1 \\ 2 & 3-x & 0 \\ 1 & 0 & -2-x \end{vmatrix} = -x^3 + 2x^2 + 10x - 1 \doteqdot -(x+2.3794)(x-0.0982)(x-4.2812)$

use Matlab

2. (20%) The matrix norm can be defined by vector norm as follows

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

For vector 2-norm, which means $\|x\| = \sqrt{x^T x}$, show that

(a) If Q is an orthogonal matrix, $Q^T Q = I$, $\|Qx\| = \|x\|$.

(b) If A is symmetric, $\|A\| = \max_i |\lambda_i|$, where λ_i are A's eigenvalues

(a) $\because$ Q is orthogonal matrix $\therefore$ $Q^T Q = I$ (by def)

$\therefore \langle Q\bar{x}, Q\bar{y} \rangle = \langle Q^T Q \bar{x}, \bar{y} \rangle = \langle \bar{x}, \bar{y} \rangle$ 保內積

$\therefore \|Q\bar{x}\|^2 = \langle Q\bar{x}, Q\bar{x} \rangle = \langle \bar{x}, \bar{x} \rangle = \|\bar{x}\|^2$ 保長度

(b) 依 Def $\|A\| = \max_{x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2}$ (operator norm)

$\|A\|^2 = \max_{x \neq 0} \left(\dfrac{\|Ax\|_2}{\|x\|_2} \right)^2 = \max_{x \neq 0} \dfrac{\|Ax\|_2^2}{\|x\|_2^2} = \max_{x \neq 0} \dfrac{\langle Ax, Ax \rangle}{\langle x, x \rangle} = \max_{x \neq 0} \dfrac{\langle A^H A x, x \rangle}{\langle x, x \rangle} = P(x)$

$A^H A$ 的 Rayleigh 商式

依 Rayleigh 商式, $\max_{x \neq 0} \dfrac{\|Ax\|_2^2}{\|x\|_2^2} = \max_{x \neq 0} P(x) = \lambda_{max}(A^T A)$

$\therefore \|A\|_2 = [\lambda_{max}(A^T A)]^{\frac{1}{2}}$

$\because A^T A = A^2 \Rightarrow \lambda_1, \lambda_2, \cdots, \lambda$ 為 A 之 eigenvalue $\Rightarrow \lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2$ 為 $A^T A$ 之 eigenvalue

$\therefore \|A\|_2 = (\lambda_1^2)^{\frac{1}{2}} = \lambda_1$, (令 λ_1 為 max λ)

(下一頁)

3. (60%) The Rosenbrock function $f(x, y) = (1-x)^2 + 100(y-x^2)^2$ has minimizer at $(1, 1)$. [1]

(a) Derive the gradient and the Hessian of $f(x, y)$.

(b) Implement the backtracking line-search method, and make it a function. Your input should at least include

- The current solution $\vec{x}_k$.
- The search direction $\vec{p}_k$.
- The function that can evaluate $f(\vec{z})$ for a given $\vec{z}$.
- The function that can evaluate $\nabla f(\vec{z})$ for a given $\vec{z}$.

(c) Implement (1) the steepest descent method (2) Newton's method (3) CG and (4) BFGS. Use your line search algorithm to find the best step length. Use $(x_0, y_0) = (-1.2, 1.0)$ as the initial guess and compare their results. Draw the convergence figures of those four methods, whose y-axis is $\log(|\nabla f|)$ and x-axis is the number of iterations. You can use Matlab's `semilogy` for the plot.

(Do not use any symbolic computation of Matlab, like sub or diff, or eval. Write your own function, gradient, and Hessian subroutines, and use them in your code.

(a)

$$g.=\begin{bmatrix} 400x^3 - 400xy + 2x - x \\ -200x^2 + 200y \end{bmatrix}$$

$$H=\begin{bmatrix} 1200x^2 - 400y + 2 & -400x \\ -400x & 200 \end{bmatrix}$$

(b)

(c)

Hw2-3 為程式起始點