ECE 590/CSC 591:Quantum Computing Project Report
Combinatorial Optimization: Subset Sum Problem

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A. ***PROJECT WEBPAGE LINK***

B. *What is Subset Sum Problem?*

The **Subset Sum Problem** belongs to category of decision problems. The problem states that given a set (or multi-set) of integers, is it possible to construct a non-empty subset whose sum is equal to zero? For example, given the set { -40, -2, 1, 2, 3, 4 ,5 ,6 } , the answer is ”yes” because the subset { -2 ,2 } sums to zero.

The Subset Sum Problem belongs to complexity class of NP-complete, meaning it is easier to evaluate whether the final result obtained is correct or not. Also, this checking can be done in polynomial time. The problem itself takes non-deterministic polynomial time to find the solution. The subset sum problem is used in the field of complexity theory and cryptography.

In computational complexity theory, a problem is NP-complete when it can be solved by a restricted class of brute force search algorithms and it can be used to simulate any other problem with a similar algorithm. More precisely, each input to the problem should be associated with a set of solutions of polynomial length, whose validity can be tested quickly (in polynomial time), such that the output for any input is ”yes” if the solution set is non-empty and ”no” if it is empty [1]

The **Subset Sum Problem** can be thought of as a modification to **knapsack problem** where the limit of the bag is infinite and the assumption that the weight of the elements can be negative.

C. *Variations of Subset Sum Problem*

1) Subset Sum Problem where the sum is equal to some constant. This variation is extension of the question such that the sum instead of 0 can be any other integer k Example - Suppose the parent set is {-2 ,0 ,1 ,3}. and we are find subset such that sum equals 2. The answer is ”yes” the subset {-2 ,1 ,3} gives the sum=2.
2) Subset Sum Problem where it is possible to subdivide this set into two strict subsets such that the sum of elements of one set is equal to another set. Example - Suppose the parent set is {-2 ,0 ,1 ,3}. The answer is ”yes” the subsets are {-2 ,3} and {0 ,1 } where the sum of elements is 1

For simplicity we will just be working with the simple variation where the subset sum has to be equal to 0.

D. *Project Approach*

For our project we will try to implement the Subset Sum Problem problem using Quantum Computers. We will try to run it in Qiskit . So suppose we have ’x’ elements in our set and we are supposed to find the subset with ’m’ elements such that m¡=x and the sum of ’m’ elements =0. So to work with Quantum Circuits we will have ’x’ Qubits as our input to the circuit and then the output will be in terms of qubit value {0,1}. If the i^{th} line’s output is ’0’ it signifies that the i^{th} element is not included in our answer. Similarly, If the i^{th} line’s output is ’1’ it signifies that the i^{th} element is included in our answer.

In our poject we will be working on both the QAOA algorithm in Qiskit and the Quantum Annealing algorithm in DWAVE system. We will try to compare the results obtained from both the methods and do an analysis on it.

E. *Theory of QAOA*

For combinatorial optimization, the Quantum Approximate Optimization Algorithm (QAOA)gives better results than classical algorithms. The heart of the QAOA relies on the use of unitary operators dependent on 2 p angles, where p ¿ 1 is an input integer. These operators are iteratively applied on a state that is an equal-weighted quantum superposition of all the possible states in the computational basis. In each iteration, the state is measured in the computational basis and C (z) is calculated. After a sufficient number of repetitions, the value of C (z) is almost optimal, and the state being measured is close to being optimal as well.

Combinatorial optimization problems are specied by n bits and m clauses. Each clause is a constraint on a subset of the bits which is satised for certain assignments of those bits and unsatised for the other assignments. The objective function, dened on n bit strings, is the number of satised clauses,

$$\sum_{j=1}^{m} C_{\alpha}$$

where $z = z_1, z_2, ...z_n$ is the bit string and $C_\alpha(z) = 1$ if z satises clause α and 0 otherwise. Typically C_α depends on only a few of the n bits. Satisability asks if there is a string that satises every clause. In the QAOA algorithm our motive is to find Approximate Optimization(To find a string (z) for which C(z) is close to the maximum).

F. QAOA(Qiskit) in our Problem

To implement QAOA we will be using Qiskit Aqua as it has well pre-defined libraries to make our task simpler. The Qiskit itself provides implementation examples of TSP problem,Vertex cut problem which act as the foundation for our implementation.The next step in designing the QAOA is to create a cost function for our problem. This will be really helpful because using this we can also design the QUBO for our DWAVE simulation.

One assumption we have made for our problem is that we have the sum m in the subset that we are asking our code to find.

So let us assume that the cost function for our problem is C.

$$C = \sum_{j=1}^{m} x_i * a_i$$

where x_i belongs to $1, 0$

1: if that element is taken in our final solution set

0: if not taken in our final solution set and a_i is the i^{th} element in our initial set.

G. Challenges

1) The cost function is known but it's QUBO formulation is still unknown.
   Reason: The above Cost function we have written is not in the form QUBO. The QUBO format should be in terms of $x^T Qx$ which is not the case with us. This is essential because in QAOA or any other optimization algorithm, the first step is to generate the QUBO and then try to minimize or maximize it.
2) Let us for a while assume that the cost function we established is correct. The problem arises is to determine what the minimum energy level will be for the given sum.
   Reason:We dont know whether minimizing or maximizing the cost function will yield lowest energy level or not.

So the above two challenges are crucial for the given problem. We are still working to overcome these challenges but if time does not permit then we are planning to implement a special variation of Subset Sum Problem which is Set Partition Problem using both QAOA and Dwave system. In the Set Partiton Problem we divide the parent set into subsets such that the sum of these two subsets are equal for this problem we can device the QUBO in the following manner

Let, $S = s_1, s_2, s_3.............s_m$ So our cost function (C) = difference between

$$\sum_{j=1}^{m} x_i * s_i$$

$$\sum_{j=1}^{m} s_i - \sum_{j=1}^{m} x_i * s_i$$

where x_i belongs to $1, 0$

1: if that element is taken in our final solution set

0: if not taken in our final solution set and a_i is the i^{th} element in our initial set.

On further simplification, the equation reduces to

$$c^2 + 4x^T Qx$$

where

$$C = \sum_{j=1}^{m} s_i$$

Hence this problem of ours can be implemented on QAOA.

H. Challenges in Set Partition Problem

1) Try to figure out the Data Structure to store the QUBO. All the problems described in the Qiskit are for graps and our example does not deal with graphs.
2) How many elements can we have in our set?

I. DWAVE

Quantum annealing (QA) is a metaheuristic for finding the global minimum of a given objective function over a given set of candidate states (candidate solutions), by a process using quantum tunneling. Quantum annealing is used mainly for problems where the search space is discrete (combinatorial optimization problems) with many local minima; such as finding the ground state of a spin glass.This system uses a 128 qubits.D-Wave's architecture differs from traditional quantum computers. It is not known to be polynomially equivalent to a universal quantum computer and, in particular, cannot execute Shor's algorithm because Shor's algorithm is not a hill climbing process. Shor's algorithm requires a universal quantum computer. D-Wave claims only to do quantum annealing. From the implementation of QAOA we can extract the QUBO and plug it into the DWAVE program.

The QUBO Formulation :

QUBO: minimize $Y = x^T Qx$

To convert it into symmetric form we can replace q_{ij} by $(q_{ij} + q_{ji})/2$ where i is not equal to j.

J. Classical Approach

Using Dyamic programming the best solution is in terms of $O(2^n)$. This is an exponential complexity which increases heavily with larger n. There are various code available online for the following subset sum problem.

Timeline and individual Contribution

NAME/WEEK	ATHARVA GUPTA	KUSHAL BATRA	COMMENTS
Week 1	Understand the problem. Select the algorithm to work on	Understand the problem. Select the algorithm to work on	
Oct 28	---------------------	---------------------	Feedback received from professor
Week 2	Implementation using D-Wave(Quantum Annealing) + identify the challenges	Implementation using Qiskit(QAOA) + identify the challenges	discuss challenges and look for alternate solutions
Week 3	Figure out the QUBO for Subset Sum Problem + Work on Special case of Subset Sum Problem (Set Partiton problem) in DWave(Quantum Annealing)	Figure out the QUBO for Subset Sum Problem + Work on Special case of Subset Sum Problem (Set Partiton problem) in Qiskit(QAOA)	Discuss with TA or professor on how to formulate QUBO for Subset Sum Problem and keep working on special case as a backup

References

[1] https://en.wikipedia.org/wiki/NP-completeness

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[3] Bernstein, Daniel Jeffery, Stacey Lange, Tanja Meurer, Alexander. (2013). Quantum Algorithms for the Subset-Sum Problem. 10.1007/978-3-642-38616-9-2.

[4] Daskin, Ammar. (2017). A Quantum Approach to Subset-Sum and Similar Problems.

[5] Flaxman, Abraham & Przydatek, Bartosz. (2005). Solving Medium-Density Subset Sum Problems in Expected Polynomial Time. Lecture Notes in Computer Science. 3404. 305-314. 10.1007/978-3-540-31856-9_25.

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