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PGT (MATHS)

ALGEBRA OF COMPLEX NUMBERS

1. ADDITION OF TWO COMPLEX NUMBERS:

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then the sum $z_1 + z_2$ is defined as follows:

$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d) = \mathrm{A} + i\mathrm{B} =$ a complex number

Example:

$z_1 = 2 + 3i$ and $z_2 = -5 + 7i$

$\therefore z_1 + z_2 = (2 + 3i) + (-5 + 7i) = 2 + 3i - 5 + 7i = (2 - 5) + i(3 + 7) = -3 + 10i$

2. PROPERTIES OF ADDITION COMPLEX NUMBERS:

(a) CLOSURE LAW:

The sum of two complex numbers is a complex number.

$(z_1 + z_2)$ is a complex for all complex numbers z_1 and z_2.

(b) COMMUTATIVE LAW:

For any two complex numbers z_1 and z_2.

$z_1 + z_2 = z_2 + z_1$

(c) ASSOCIATIVE LAW:

For any three complex numbers z_1, z_2 and z_3

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(d) EXISTENCE OF ADDITIVE IDENTITY:

There exists the complex number $0 + i\,0$ (denoted as 0), called the *additive identity* or the *zero complex number,* such that, for every complex number *z*,

$z + 0 = 0 + z = z$

(e) THE EXISTENCE OF ADDITIVE INVERSE:

To every complex number $z = a + ib$, we have the complex number

$-z = -a + i(-b)$ called the additive inverse or negative of z. We observe that

$z + (-z) = (-z) + z = 0$

3. DIFFERENCE OF TWO COMPLEX NUMBERS:

Given any two complex numbers z_1 and z_2, the difference is defined as follows:

$z_1 - z_2 = z_1 + (-z_2)$

Examples:

(i) $(6 + 3i) - (2 + 5i) = (6 + 3i) + (-2 - 5i) = (6 - 2) + i(3 - 5) = 4 - 2i$

$(ii)(2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = (2 - 6) - i\,(1 + 3) = -4 - 4i$

4. MULTIPLICATION OF TWO COMPLEX NUMBERS:

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers.
It is clear that
$\mathrm{Re}(z_1) = a$, $\mathrm{Im}(z_1) = b$, $\mathrm{Re}(z_2) = c$, $\mathrm{Im}(z_2) = d$
Then the sum z_1z_2 is defined as follows:

$z_1z_2 = (a + ib)(c + id)$

$= ac + iad + ibc + i^2bd$

$= ac + iad + ibc - bd \qquad [\because i^2 = -1]$

$= (ac - bd) + i(ad + bc) = \mathrm{A} + i\mathrm{B} =$ a complex number

Here, we observe that

$\mathrm{Re}(z_1z_2) = ac - bd = \mathrm{Re}(z_1)\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\mathrm{Im}(z_2)$ [Miscl. Ex. Q. −2]

$\mathrm{Im}(z_1z_2) = ad - bc = \mathrm{Re}(z_1)\mathrm{Im}(z_2) - \mathrm{Im}(z_1)\mathrm{Re}(z_2)$

Example:

Let $z_1 = 2 - 3i$ and $z_2 = -5 + 7i$, then

$\therefore z_1z_2 = (2 - 3i)(-5 + 7i)$

$= 2(-5 + 7i) - 3i(-5 + 7i)$

$= -10 + 14i + 15i - 21i^2$

$= -10 + 14i + 15i + 21 \qquad [\because i^2 = -1]$

$= (-10 + 21) + i(14 + 15)$

$= 11 + 29i$

5. PROPERTIES OF MULTIPLICATION COMPLEX NUMBERS:

(a) CLOSURE LAW:

The product of two complex numbers is a complex number.
z_1z_2 is a complex for all complex numbers z_1 and z_2.

(b) COMMUTATIVE LAW:

For any two complex numbers z_1 and z_2.

$z_1\,z_2 = z_2z_1$

(c) ASSOCIATIVE LAW:

For any three complex numbers z_1, z_2 and z_3.

$(z_1\,z_2)\,z_3 = z_1(z_2z_3)$

(d) EXISTENCE OF MULTIPLICATIVE IDENTITY:

There exists the complex number $1 + i\,0$ (denoted as 1), called the *multiplicative identity,* such that, for every complex number z,

$$z\,.1 = 1.z = z$$

(e) THE EXISTENCE OF ADDITIVE INVERSE:

For every non-zero complex number $z = a + ib$, there exists a complex number $z^{-1} = \frac{1}{z}$ called the multiplicative inverse such that $z.z^{-1} = 1$

(f) DISTRIBUTIVE LAW:

For any three complex numbers z_1, z_2 and z_3

(i) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

(ii) $(z_1 + z_2)\,z_3 = z_1z_3 + z_2z_3$

Example:

Find the multiplicative inverse of $4 - 3i$.

Solution:

The multiplicative inverse of $z = 4 - 3i$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{4-3i}$$

$$= \frac{1}{4-3i} \times \frac{4+3i}{4+3i}$$

$$= \frac{4+3i}{4^2 - (3i)^2}$$

$$= \frac{4+3i}{16 - 9i^2}$$

$$= \frac{4+3i}{16 - 9\,(-1)}$$

$$= \frac{4+3i}{16+9}$$

$$= \frac{4+3i}{25}$$

$$= \frac{4}{25} + i\frac{3}{25} \quad \text{Ans.}$$

6. DIVISION OF TWO COMPLEX NUMBERS

Give two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined as

$$\frac{z_1}{z_2} = z_1\frac{1}{z_2}$$

Example - 2

Divide $z_1 = 2 - 7i$ by $z_2 = 3 + 4i$

Solution:

$$\frac{z_1}{z_2} = \frac{2 - 7i}{3 + 4i}$$

$$= \frac{2 - 7i}{3 + 4i} \times \frac{3 - 4i}{3 - 4i}$$

$$= \frac{(2 - 7i)(3 - 4i)}{3^2 - (4i)^2}$$

$$= \frac{6 - 8i - 21i + 28i^2}{9 - 16i^2}$$

$$= \frac{6 - 29i - 28}{9 - 16\,(-1)} \qquad [\because i^2 = -1]$$

$$= \frac{-22 - 29i}{9 + 16}$$

$$= \frac{-22 - 29i}{25}$$

$$= \frac{-22}{25} + i\frac{-29}{25} \quad \text{Ans.}$$