EXERCISE-2.3

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

(i) {(2, 1), (5, 1), (8, 1), (11, 1), (14,1), (17,1)}

(ii) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)}

(iii) {(1, 3), (1, 5), (2, 5)}

Solution:

(i) {(2, 1), (5, 1), (8, 1), (11, 1), (14,1), (17,1)}

Since, no two ordered pairs have the same first components. So, this relation is a function.

$\therefore$ Domain $= \{2, 5, 8, 11, 14, 17\}$ and Range $= \{1\}$

(ii) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)}

Since, no two ordered pairs have the same first components. So, this relation is a function.

$\therefore$ Domain $= \{2, 4, 6, 8, 10, 12, 14\}$ and Range $= \{1, 2, 3, 4, 5, 6, 7\}$

(iii) {(1, 3), (1, 5), (2, 5)}

Since, the two ordered pairs (1, 3) and (1, 5) have the same first components 1.
So, this relation is not a function.

2. Find the range of each of the following functions.

$(i)\ \ f(x) = 2 - 3x,\ \ x \in \text{R},\ \ x > 0.$

$(ii)\ \ f(x) = x^2 + 2,\ \ x$ is a real number.

$(iii)\ \ f(x) = x,\ \ x$ is a real number

[Ex.-2.3 Q.5]

Solution:

$(iii)\ f(x) = x,\ \ x$ is a real number

Let $y = f(x)$

But, $f(x) = x$

$\therefore y = x = \text{R}$ as x is a real number

So, Range $= \{y : y \in \text{R}\} = \text{R}$

$(ii)\ f(x) = x^2 + 2,\ \ x$ is a real number

Let $y = f(x)$

Given, x is a real number

$\therefore x^2 \geq 0,\ \ \forall\, x \in \text{R}$

$\Rightarrow x^2 + 2 \geq 0 + 2$, [Adding 2 to both sides]

$\Rightarrow f(x) \geq 2$

$\Rightarrow y \geq 2$

So, Range $= \{y : y \in \text{R and } y \geq 2\} = [2,\ \infty)$

$(i)\ f(x) = 2 - 3x\ ,\ \ x > 0$

Let $y = f(x)$

Given,

$x > 0$

$\Rightarrow 3x > 0$

$\Rightarrow -3x < 0$

$\Rightarrow 2 - 3x < 0 + 2\ ,$ [Adding 2 to both sides]

$\Rightarrow f(x) < 2$

$\Rightarrow y < 2$

So, Range $= \{y : y \in \mathrm{R} \text{ and } y < 2\} = (-\infty, 2)$

3. Find the domain and range of the following real functions:

(i) $f(x) = |x - 1|$ [Miscl.Ex.-2 Q.5]

(ii) $f(x) = -|x|$ [Ex.-2.3 Q. 2(i)]

(iii) $f(x) = \sqrt{x - 1}$ [Miscl.Ex.-2 Q.4]

Solution:

$(i)\ \ f(x) = |x - 1|$

Let $y = f(x)$

Since, $|x - 1|$ is well defined for all $x \in \mathrm{R}$

$\therefore$The domain$= \{x : x \in \mathrm{R}\} = \mathrm{R}$

Next, we know that

$|x - 1| \geq 0, \qquad \forall\, x \in \mathrm{R}$

$\Rightarrow f(x) \geq 0, \qquad \forall\, x \in \mathrm{R}$

$\Rightarrow y \geq 0, \qquad \forall\, x \in \mathrm{R}$

So, Range $= \{y : y \in \mathrm{R} \text{ and } y \geq 0\} = [0, \infty)$

$(ii)\ f(x) = -|x|$

Let $y = f(x)$

Since, $-|x|$ is well defined for all $x \in \mathrm{R}$

$\therefore$The domain$= \{x : x \in \mathrm{R}\} = \mathrm{R}$

Next, we know that

$|x| \geq 0,\ \forall\, x \in \mathrm{R}$

$\Rightarrow -|x| \leq 0, \qquad \forall\, x \in \mathrm{R}$

$\Rightarrow f(x) \leq 0, \qquad \forall\, x \in \mathrm{R}$

$\Rightarrow y \leq 0, \qquad \forall\, x \in \mathrm{R}$

So, Range $= \{y : y \in \mathrm{R} \text{ and } y \leq 0\} = (-\infty, 0]$

$(iii)\ f(x) = \sqrt{x-1}$

Let $y = f(x)$

Since, $f(x)$ is well defined when

$x - 1 \geq 0$

$\Rightarrow x \geq 1$

$\therefore$The domain$= \{x : x \in \mathrm{R} \text{ and } x \geq 1\} = [1, \infty)$

Since, the domain$= [1, \infty)$

$\therefore x \geq 1,$

$\Rightarrow x - 1 \geq 1 - 1,$ [Subtracting 1 from both sides]

$\Rightarrow x - 1 \geq 0$

$\Rightarrow \sqrt{x-1} \geq \sqrt{0}$ [Taking square root of both sides]

$\Rightarrow f(x) \geq 0$

$\Rightarrow y \geq 0,$

So, Range $= \{y : y \in \mathrm{R} \text{ and } y \geq 0\} = [0, \infty)$

4. Find the domain and range of the real function $f(x) = \sqrt{9 - x^2}$.

[Ex.-2.3 Q.2 (ii)]

Solution:

Let

$y = f(x) = \sqrt{9 - x^2}$

Clearly, $f(x)$ is well defined when

$9 - x^2 \geq 0$

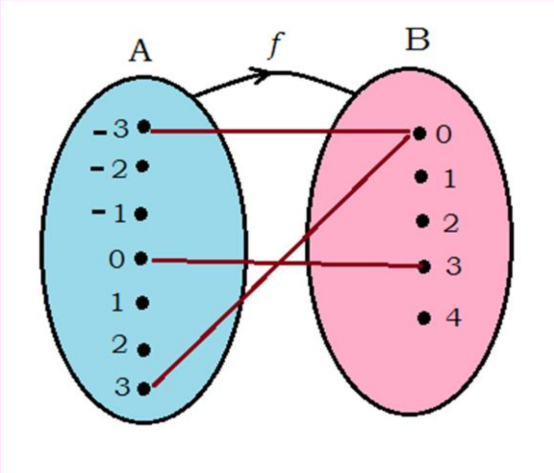
$\Rightarrow 9 \geq x^2$

$\Rightarrow x^2 \leq 9$

$\Rightarrow x^2 \leq 3^2$

$\Rightarrow -3 \leq x \leq 3$

$\therefore$ The domain$= \{x : x \in \mathrm{R} \text{ and } -3 \leq x \leq 3\} = [-3, 3]$



We have

$y = \sqrt{9 - x^2}$ ……………….(1)

$y^2 = 9 - x^2$ [Squaring]

$\Rightarrow x^2 = 9 - y^2$ [Subtracting 1 from both sides]

$\Rightarrow x = \sqrt{9 - y^2}$ [Taking square root of both sides]

Here , it is clear that x is well defined when

$9 - y^2 \geq 0$

$\Rightarrow 9 \geq y^2$

$\Rightarrow y^2 \leq 9$

$\Rightarrow -3 \leq y \leq 3 \dots\dots\dots\dots\dots\dots\dots\dots.(2)$

But, from (1) , it is clear that

$y \geq 0, \ \forall \ x \in [-3, 3]$ ....(3)

Thus, from (2) and (3), we find,

$0 \leq y \leq 3$

So, Range = $\{y : y \in \mathrm{R} \text{ and } 0 \leq y \leq 3\} = [0, 3]$

Thus, Domain= $[-3, \ 3]$ and Range = $[0, \ 3]$

MISCELLANEOUS EXERCISE- Ch-2

1. The function f is defined by

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 3 \\ 3x, & 3 \leq x \leq 10 \end{cases}$$

The relation g is defined by

$$g(x) = \begin{cases} x^2, & 0 \leq x \leq 2 \\ 3x, & 2 \leq x \leq 10 \end{cases}$$

Show that f is a function and g is not a function

[Miscl.Ex. Q.1]

Solution:

Given,

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 3 \\ 3x, & 3 \leq x \leq 10 \end{cases}$$

When $x = 3$, then

$f(x) = x^2$ gives $f(3) = 3^2 = 9$

$f(x) = 3x$ gives $f(3) = 3 \times 3 = 9$

Since, $x = 3$ has the unique image 9 under the function f.

∴ f is a function.

Next,

$$g(x) = \begin{cases} x^2, & 0 \leq x \leq 2 \\ 3x, & 2 \leq x \leq 10 \end{cases}$$

When x = 2, then

$g(x) = x^2$ gives $g(2) = 2^2 = 4$

$g(x) = 3x$ gives $g(2) = 3 \times 2 = 6$

Since, $x = 2$ has the different images 4 and 6 under the function g.

∴ g is a relation.

2. If $f(x) = x^2$, find $\frac{f(1.1) - f(1)}{(1.1 - 1)}$ [Miscl.Ex. Q.2]

Solution:

Given,

$f(x) = x^2$

$\therefore f(1.1) = (1.1)^2 = 1.21$

and $f(1) = (1)^2 = 1$

$$\therefore \frac{f(1.1) - f(1)}{(1.1 - 1)} = \frac{1.21 - 1}{0.1} = \frac{0.21}{0.1} = \frac{21}{10} = 2.1$$

3. Find the domain of the function

$$f\ (x) = \frac{x^2 + \ 2x\ + \ 1}{x^2 - 8x - 12}$$

[Miscl.Ex. Q.3]

Solution:

Given,

$$f\ (x) = \frac{x^2 + \ 2x\ + \ 1}{x^2 - 8x - 12}$$

It is clear that the function f(x) is not defined when

Denominator $= 0$

$\Rightarrow x^2 - 8x - 12 = 0$

$\Rightarrow x^2 - 2x - 6x - 12 = 0$

$\Rightarrow x(x-2) - 6(x-2) = 0$

$\Rightarrow (x-2)(x-6) = 0$

$\Rightarrow x = 2, 6$

$\therefore$ Domain $= R - \{2, 6\}$

4. Find the domain and the range of the real function f defined by $f\ (x) = \sqrt{(x-1)}$ [Miscl.Ex. Q.4]

Solution:

Let $y = f(x) = \sqrt{x-1}$

Since, $f(x)$ is well defined when

$x - 1 \geq 0$

$\Rightarrow x \geq 1$

$\therefore$The domain$= \{x : x \in \text{R and } x \geq 1\} = [1, \infty)$

Since, the domain$= [1, \infty)$

$\therefore x \geq 1,$

$\Rightarrow x - 1 \geq 1 - 1,$ [Subtracting 1 from both sides]

$\Rightarrow x - 1 \geq 0$

$\Rightarrow \sqrt{x-1} \geq \sqrt{0}$ [Taking square root of both sides]

$\Rightarrow y \geq 0,$

So, Range $= \{y : y \in \text{R and } y \geq 0\} = [0, \infty)$

5. Find the domain and the range of the real function f defined by $f\ (x) = |x - 1|$ [Miscl.Ex. Q.5]

Solution:

Let $y = f(x) = |x - 1|$

Since, $|x-1|$ is well defined for all $x \in \mathrm{R}$

$\therefore$The domain$= \{x: x \in \mathrm{R}\} = \mathrm{R}$

Next, we know that

$|x-1| \geq 0, \quad \forall\, x \in \mathrm{R}$

$\Rightarrow y \geq 0, \quad \forall\, x \in \mathrm{R}$

So, Range $= \{y : y \in \mathrm{R}$ and $y \geq 0\} = [0, \infty)$

6. Let $f(x) = \{(1,1), (2,3), (0,-1), (-1,-3)\}$ be a function from Z to Z defined by $f(x) = ax + b$ for some integers a and b. Find a and b. [Miscl.Ex. Q.8]

Solution:

Given,

$f = \{(1,1), (2,3), (0,-1), (-1,-3)\}$

And the function is

$f(x) = ax + b$ .... (1)

Since, , $(1,1) \in f$

$\therefore x = 1$ and $f(x) = 1$

Therefore, from equation (1)

$1 = a + b$ ....(2)

Similarly, $(2,3) \in f$

$\therefore x = 2$ and $f(x) = 3$

Therefore, from equation (1)

$3 = 2a + b$ ....(3)

Solving equations (2) and (3), we get,

$a = 2$ and $b = -1$

7. Let $\mathrm{A} = \{1,2,3,4\}, \mathrm{B} = \{1,5,9,11,15,16\}$ and $f = \{(1,5), (2,9), (3,1), (4,5)\}$

Are the following true ?

(i) f is a relation from A to B. (ii) f is a function from A to B.

Justify your answer in each case.

Solution:

Given,

$A = \{1, 2, 3, 4\}$, and $B = \{1, 5, 9, 11, 15, 16\}$

$\therefore A \times B = \{1, 2, 3, 4\} \times \{1, 5, 9, 11, 15, 16\}$

$\therefore A \times B = \{(1, 1), (1, 5), (1, 9), (1, 11), \ldots (4, 16)\}$

and $f = \{(1, 5), (2, 9), (3, 1), (4, 5)\}$

(i) Since, every element of f is also an element $A \times B$.

$\therefore f \subset (A \times B)$

Hence, f is relation from the set A to the set B.

(ii) Given, $f = \{(1, 5), (2, 9), (3, 1), (4, 5)\}$

Since, no two ordered pair of f have the first component same, so f is function from the set A to the set B.

8. Let $f = \left\{\left(x, \frac{x^2}{1+x^2}\right) : x \in R\right\}$ be function from R to R. Determine the range of f.

Solution:

The given function is

$$f = \left\{\left(x, \frac{x^2}{1+x^2}\right) : x \in R\right\}$$

Putting $x = 0, \pm 1, \pm 2, \pm 3 \ldots$

$$f = \left\{(0, 0), \left(\pm 1, \frac{1}{2}\right), \left(\pm 2, \frac{4}{5}\right), \left(\pm 3, \frac{9}{10}\right), \left(\pm 4, \frac{16}{17}\right), \ldots\right\}$$

The range of the function f is the set of all second components.

It is clear that all the second components are greater or equal to 0 but less than 1 as denominators are greater than numerators.

$\therefore$ Range $= [0,1)$

Alternate Solution

Here, $f(x) = \dfrac{x^2}{1+x^2}$

Let,

$y = f(x)$

$y = \dfrac{x^2}{1+x^2}$

$\Rightarrow y(1 + x^2) = x^2$

$\Rightarrow y + x^2 y = x^2$

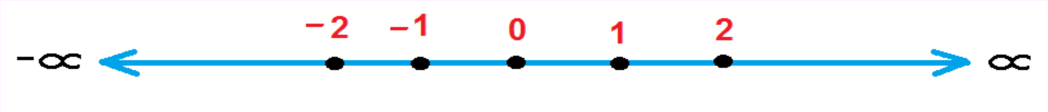
$\Rightarrow y = x^2 - x^2 y$

$\Rightarrow y = x^2(1 - \mathrm{y})$

$\Rightarrow x^2 = \dfrac{y}{1-y}$

$\Rightarrow x = \sqrt{\dfrac{y}{1-y}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Let us now find those possible values of y for which x is well defined.



When

$y = -0.5$, $\Rightarrow x = \sqrt{\dfrac{-0.5}{1+0.5}} = \sqrt{\dfrac{-0.5}{1.5}} = \sqrt{\dfrac{-5}{15}} = \sqrt{\dfrac{-1}{3}} = \sqrt{-ve}$ = Not Real

$y = 0$, $\Rightarrow x = \sqrt{\dfrac{0}{1-0}} = \sqrt{\dfrac{0}{1}} = \sqrt{0} = 0 =$ Real

$y = 0.5$, $\Rightarrow x = \sqrt{\dfrac{0.5}{1-0.5}} = \sqrt{\dfrac{0.5}{0.5}} = \sqrt{1} = 1 =$ Real

$y = 1$, $\Rightarrow x = \sqrt{\dfrac{1}{1-1}} = \sqrt{\dfrac{1}{0}} = \sqrt{\infty} = \infty ==$ Not Real

$y = 1.5$, $\Rightarrow x = \sqrt{\dfrac{1.5}{1-1.5}} = \sqrt{\dfrac{1.5}{-0.5}} = \sqrt{\dfrac{15}{-5}} = \sqrt{-3} = \sqrt{-ve}$ = Not Real

Thus, we observed that x is well defined for all those real values of y for which $0 \leq y < 1$

$\therefore$ Range $= [0,1)$