# An Exploration of Simple Random Walk through Monte Carlo simulation

PROJECT SUBMITTED FOR PARTIAL FULLFILlMENT
OF BACHELOR'S DEGREE IN STATISTICS HONOURS

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REGN.NO: 1160102

SESSION: 2017-2018

## ACKNOWLEDGEMENT

The success and final outcome of this project required a lot of guidance and assistance from many people and I am extremely privileged to have got this all along the completion of my project. All that I have done is only due to such supervision and assistance and I would not forget to thank them. I respect and thank , for providing me an opportunity to do the project work in department of Statistics, Haldia Government College and giving us all support and guidance which made me complete the project duly. I am extremely thankful to him for providing such a nice support and guidance, although he had busy schedule managing the corporate affairs. I owe my deep gratitude to our project guides Dr. Shyamsundar Sahoo, Mr. Sibsankar Karan, Mr. Tanmay Kumar Maity and Mr. Tuhinsubhra Bhattacharya, who took keen interest on our project work and guided us all along, till the completion of our project work by providing all the necessary information for developing a good system. I would not forget to remember my parents, and also my friends for their encouragement and more over for their timely support and guidance till the completion of our project work. I am thankful to and fortunate enough to get constant encouragement, support and guidance from all Teaching staffs of Statistics department which helped us in successfully completing our project work. Also, I would like to extend our sincere esteems to all staff in laboratory for their timely support.

*CONTENTS*

# 1 Introduction

A random walk is a stochastic process, that describes a path that consists of a succession of random steps on some mathematical space such as the integers. An elementary example of a random walk is the simple random walk on the integer number line, $\mathbb{Z}$, which starts at 0 and at each step moves $+1$ or $-1$ with equal probability. Other examples include the path traced by a molecule as it travels in a liquid or a gas, the search path of a foraging animal, the price of a fluctuating stock and the financial status of a gambler: all can be approximated by random walk models, even though they may not be truly random in reality. As illustrated by those examples, random walks have applications to engineering and many scientific fields including ecology, psychology, computer science, physics, chemistry, biology as well as economics. Random walks explain the observed behaviors of many processes in these fields, and thus serve as a fundamental model for the recorded stochastic activity. The term random walk was first introduced by Karl Pearson in 1905.

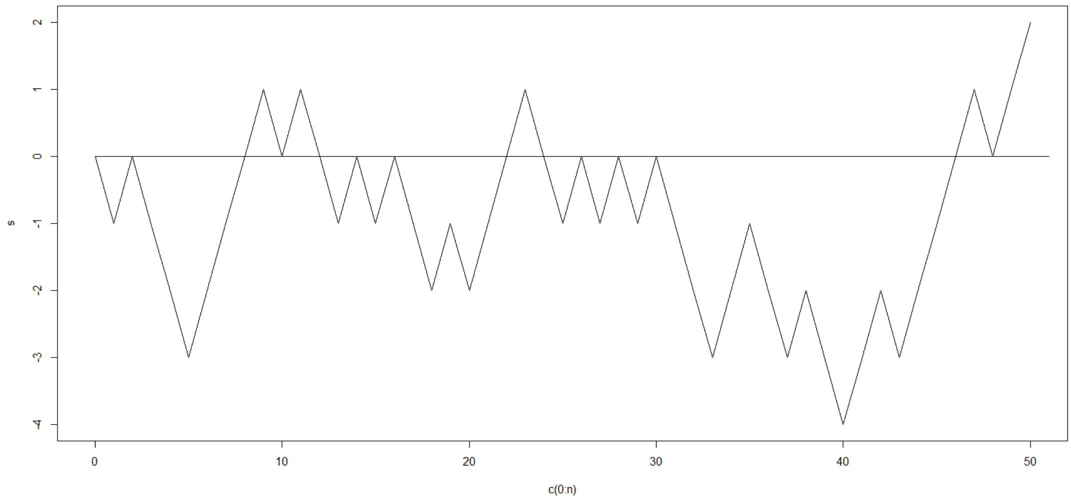
From a formal point of view we shall be concerned with arrangements of finitely many plus ones and minus ones. Consider $n = u + v$ symbols $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ each standing either for $+1$ or for $-1$; suppose that there are $u$ plus ones and $v$ minus ones. The partial sum $S_k = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_k$ represent the difference between the number of pluses and minuses occurring at the first k places. Then

$$S_k - S_{k-1} = \epsilon_k = \pm 1, \quad S_0 = 0, \quad S_n = u - v$$

where $k = 1, 2, \ldots, n$.

A simple description of random walk on integer line can be as follows: Suppose a gambler is playing a game of tossing a coin where he gets 1 rupee if head appears in a given trial and losses 1 rupee if tail appears. Then $S_n$ will represent the cumulative gain of the gambler after $n$ trials. If $X_i$ represents the outcome of $i^{th}$ trial and then $S_n = \sum_{i=1}^{n} X_i$. For $n \in \mathbb{N}$, $(S_n)$ is a stochastic process which is called random walk.

A simulated simple random walk can be as follows:




(R-commands given in appendix-1)

Our objective in this project is to investigate different random events in random walk through Monte Carlo simulation. We wil simulate random walk on $\mathbb{Z}$ and $\mathbb{Z}^2$under different situation to make probability statements about some events in random walk on given space.

## 2 One dimensional simple random walk

The simple random walk on $\mathbb{Z}$ can be described as follows:
Let $X_1, X_2, \ldots$ be independent and identically distributed random variables defined on a given probability space with

$$X_n = \begin{cases} +1 & \text{with probability} \quad p \\ -1 & \text{with probability} \quad 1-p \end{cases}$$

Define $S_1 = X_1$ and $S_n = X_1 + X_2 + \ldots + X_n$. Then $(S_n)_{n\in\mathbb{N}}$ is called simple random walk. If $p = \frac{1}{2}$, then $(S_n)_{n\in\mathbb{N}}$ is symmetric simple random walk. Here $\mathbb{S} = \mathbb{Z}$ is the set of all possible values taken by $(S_n)_{n\in\mathbb{N}}$ which called state space.

Let $x \in \mathbb{S}$ be a state of a random walk. Define $T_x = \min\{n \geq 1 : S_n = x\}$. For every $x \in \mathbb{S}$ $T_x$ is a random variable and is called hitting time of state $\{x\}$.

Define $f_{ij} = P_j(T_i < \infty)$ defines the probability that a random walk started at $j$ will reach state $i$ . A state $i$ is called *recurrent* if $f_{ii} = 1$ and transient if $f_{ii} < 1$.

Let $f_{ij}^n = P\left(S_n = i, S_{n-1} \neq i, \ldots, S_2 \neq i | S_1 = j\right)$ gives the probability that process will reach the state $i$ for the first time at step $n$, given that the process start at state $j$.

Then

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^n = \lim_{n\to\infty} \sum_{k=1}^{n} f_{ij}^k$$

.

**Theorem 1** *For a simple random walk the state* 0 *is recurrent if its symmetric.*

We illustrated this theorem by Monte carlo simulation in the following table. The follwing table gives the probability that a random walk started at 0 will again visit the state 0 at $n = 50, 100, 250, 500$ steps.

| p | .1 | .2 | .3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| $\sum_{k=1}^{50} f_{00}^k$ | 0.20 | 0.37 | 0.59 | 0.7 | 0.82 | 0.64 | 0.63 | 0.38 | 0.25 |
| $\sum_{k=1}^{100} f_{00}^k$ | 0.17 | 0.34 | 0.63 | 0.67 | 0.88 | 0.85 | 0.64 | 0.46 | 0.23 |
| $\sum_{k=1}^{250} f_{00}^k$ | 0.19 | 0.40 | 0.61 | 0.79 | 0.94 | 0.80 | 0.6 | 0.4 | 0.19 |
| $\sum_{k=1}^{500} f_{00}^k$ | 0.18 | 0.36 | 0.53 | 0.69 | 0.99 | 0.74 | 0.72 | 0.43 | 0.16 |

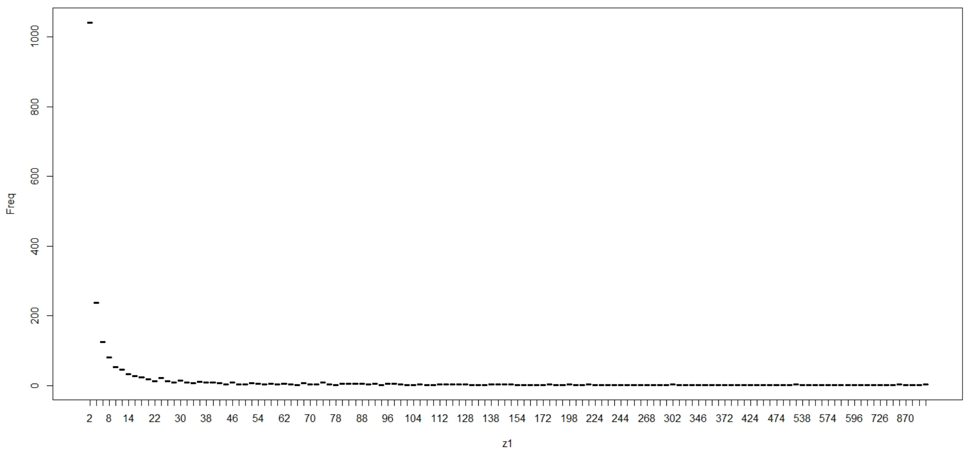
(R-commands given in appendix-2)

We can see that under $p = 0.5$, $\sum_{k=1}^{n} f_{ij}^k$ increases to 1 as $n$ increases which does not hold for other choices to $p$. From the table we can see that the probability that the random walk will visit state 0 at $n^{th}$ step after starting at 0 state is decreasing as $n$ increases for $p \neq 0.5$.

### 2.1 Distribution of $T_0$

$T_0 = \min\{n \in \mathbb{N} : S_n = 0\}$ gives the first hitting time $T_0' = \max\{n \in \mathbb{N} : S_n = 0\}$ gives the last hitting time for state 0 of random walk. Both $T_0$ and $T_0'$ takes only even values. The following are the simulated distributions of $T_0$ and $T_0'$

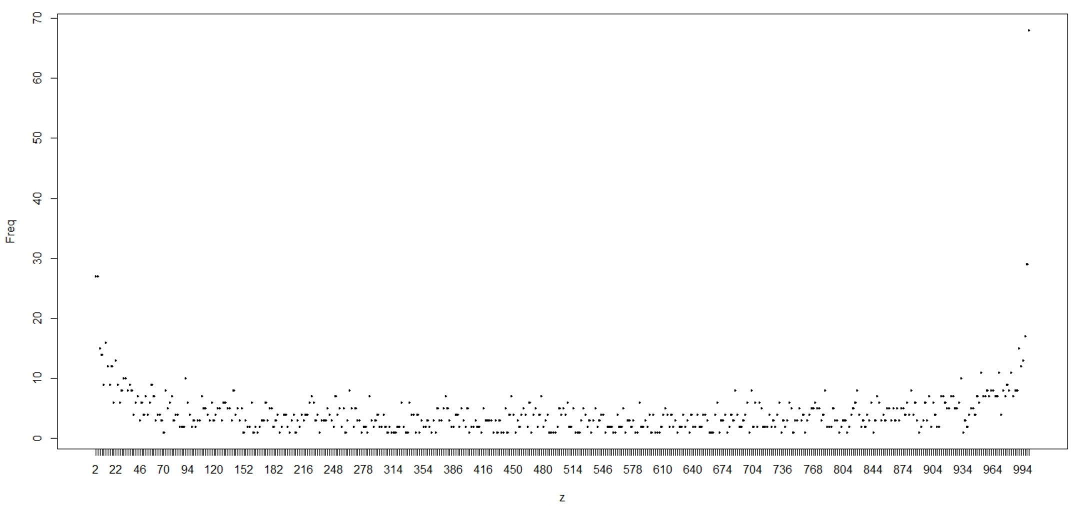
| Serial Number | $T_0$ | Frequency |
|---|---|---|
| 1 | 2 | 1041 |
| 5 | 10 | 53 |
| 10 | 20 | 18 |
| 15 | 30 | 14 |
| 20 | 40 | 8 |
| 25 | 50 | 3 |
| 30 | 60 | 3 |
| 40 | 80 | 5 |
| 50 | 102 | 1 |
| 60 | 130 | 1 |
| 70 | 170 | 1 |
| 80 | 228 | 1 |
| 90 | 300 | 1 |
| 100 | 378 | 1 |
| 120 | 602 | 1 |



For $T_0$ we tried to fit many well known distribution but the p-value always gets less then 0.01 . From picture we can say that it's a positively skwed distribution.
(R-commands given in appendix-3)

| Serial Number | $T_0'$ | Frequency |
|---|---|---|
| 2 | 2 | 33 |
| 6 | 10 | 18 |
| 11 | 20 | 14 |
| 16 | 30 | 12 |
| 21 | 40 | 9 |
| 26 | 50 | 6 |
| 31 | 60 | 6 |
| 36 | 70 | 6 |
| 41 | 80 | 8 |
| 46 | 90 | 5 |
| 51 | 100 | 3 |
| 56 | 110 | 7 |
| 61 | 120 | 4 |
| 66 | 130 | 4 |
| 71 | 140 | 2 |
| 90 | 178 | 2 |
| 100 | 200 | 1 |
| 130 | 266 | 1 |
| 150 | 308 | 2 |
| 170 | 354 | 4 |
| 190 | 396 | 4 |
| 210 | 440 | 4 |
| 230 | 482 | 3 |
| 250 | 526 | 3 |
| 270 | 566 | 2 |
| 290 | 612 | 9 |
| 310 | 654 | 2 |
| 330 | 698 | 2 |

| | | |
|---|---|---|
| 340 | 720 | 5 |
| 360 | 762 | 2 |
| 380 | 806 | 2 |
| 400 | 850 | 3 |
| 420 | 890 | 4 |
| 440 | 930 | 7 |
| 450 | 950 | 7 |
| 456 | 962 | 10 |
| 460 | 970 | 3 |
| 463 | 976 | 14 |
| 465 | 980 | 5 |
| 468 | 986 | 13 |
| 471 | 992 | 16 |
| 473 | 996 | 23 |
| 474 | 998 | 24 |




We fitted $beta(1/2, 1/2)$ to the distribution of $T_0'$. And the p-value is 0.5347 .
(R-commands given in appendix-4)

## 2.2 THE GAMBLER'S RUIN PROBLEM

We shall state the classical gambler's ruin problem in general at the opening of this chapter. Let a fictitious gambler enters into a game where he/she wins 1 rupee with probability $p$ or losses 1 rupee with probability $q$. Let $z$ be the initial capital of the gambler and his counterpart has an initial capital $a - z$. The game continues until the gambler's capital either reduced to zero or has increased to $a$, that is until one of the two player is ruined.

Let $q_z$ be the probability of the gambler's ultimate ruin and $p_z$ the probability of winning. In random-walk terminology $q_z$ and $p_z$ are the probabilities that a particle starting at $z$ will be absorbed at 0 and $a$, respectively. We shall assume that $p_z + q_z = 1$,so that we need not consider the possibilities of an unending game.

After the first trial the gambler's fortune is either $z - 1$ or $z + 1$, and therefore we must have

$$q_z = pq_{z+1} + qq_{z-1}$$

provided $1 < z < a - 1$. We define $q_0 = 1, q_a = 0$.

**EXPECTED DURATION OF THE GAME**

Let $D_z$ be the expected duration time of this problem. So if the first trial results in success, the game continues as if the initial position had been z+1. The conditional expectation of the duration conditioned on success at the first trial is therefore $D_{z+1}$+1. Likewise if the first trial results in a loss, the duration conditioned on the loss at the first trial is $D_{z-1}$+1.

This argument shows that the expected duration satisfies the difference equation, obtained by expectation by conditioning

$$D_z = pD_{z+1} + qD_{z-1} + 1$$

with the boundary conditions $D_0 = 0, D_a = 0$.

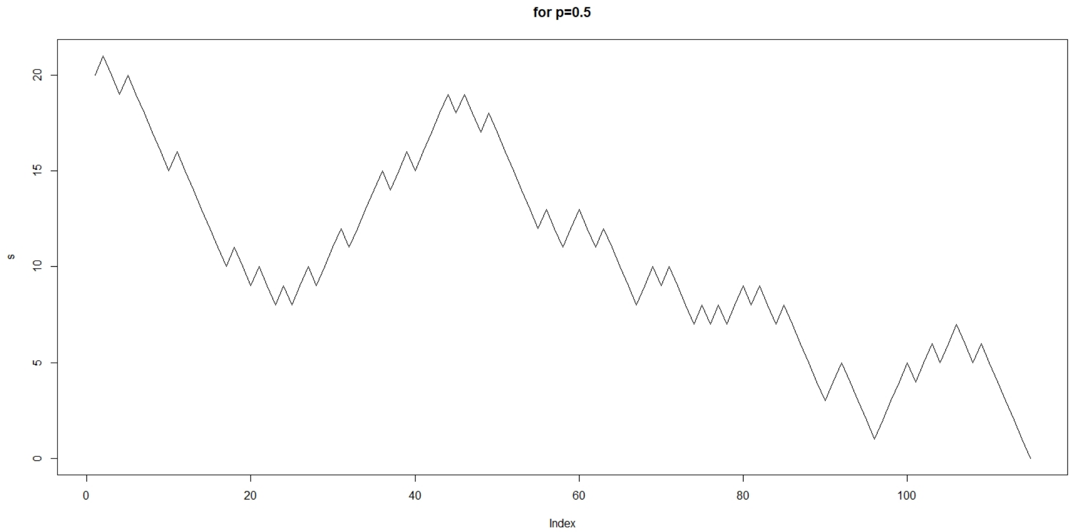
We simulate the probability $q_z$ and expected duration time $D_z$ using Monte-carlo simulation for different combination of p and q.

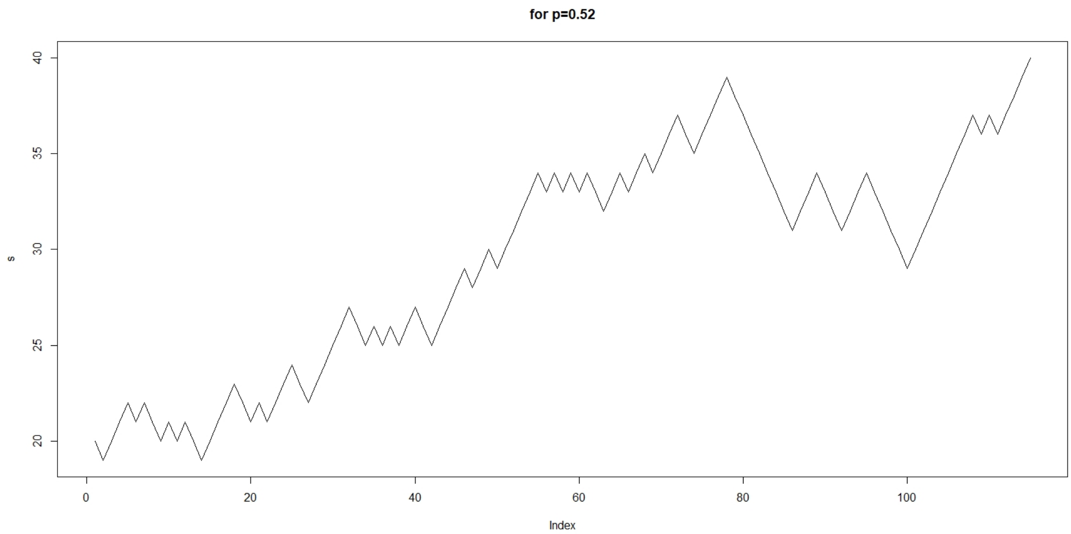
If a gambler start with $z = 20$ rupees and he or she wants to get $a = 40$ rupees the following table gives the ruin probability and expected duration time of the game

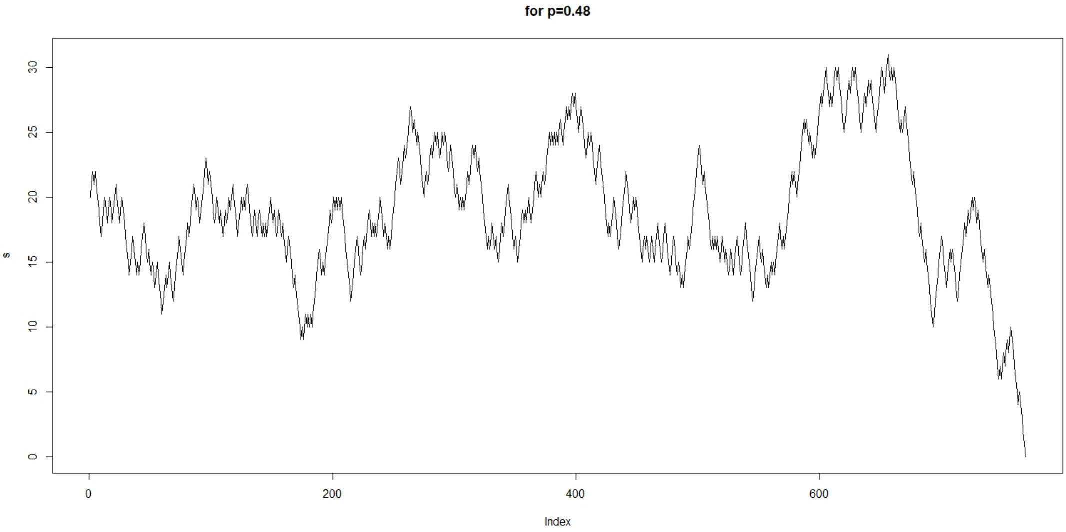
| p | .1 | .2 | .3 | 0.4 | 0.45 | 0.47 | 0.49 | 0.5 |
|---|---|---|---|---|---|---|---|---|
| $q_z$ | 1 | 1 | 1 | 0.996 | 0.9813 | 0.9155 | 0.6992 | 0.5021 |
| $D_z$ | 24.9964 | 33.3818 | 50.0284 | 100.1244 | 193.0258 | 276.5176 | 379.3484 | 396.0942 |
| p | 0.51 | 0.52 | 0.55 | 0.6 | 0.7 | 0.8 | 0.9 | |
| $q_z$ | 0.3177 | 0.1733 | 0.0154 | 0 | 0 | 0 | 0 | |
| $D_z$ | 380.8192 | 331.1874 | 192.2162 | 100.46 | 49.9518 | 33.359 | 25.011 | |

(R-commands given in appendix-5)

Simulated pictures of Gambler's game for different probability






(R-commands given in appendix-6)

# 3 Two dimensional simple random walk

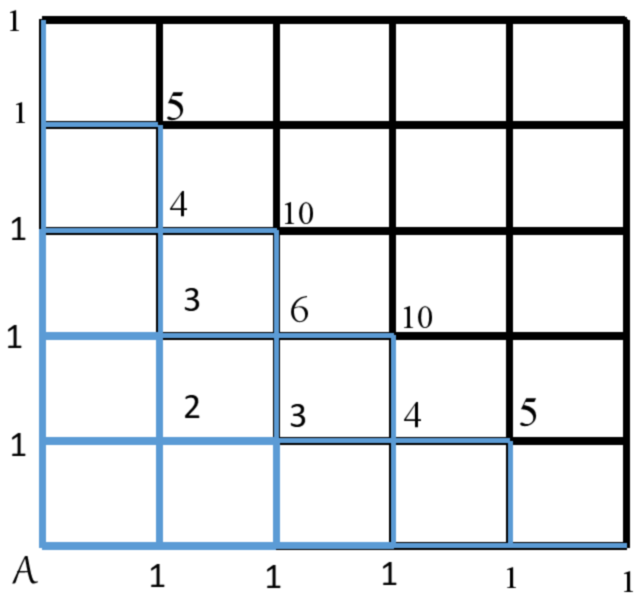
In the previous sections we have considered the random walk in a line and a popular problem called "Gambler's ruin problem" based on that. Now, we will try to extend the notion of random walk in two dimensions. We shall do that two steps and by considering a specific problem based of random walk on two dimension.

In two-dimensional random walk the particle moves in unit steps in one of the four direction parallel to $x$ and $y$ axes. For a particle starting at the origin the possible position are all points of the plane with integer-valued coordinates. Each positions has four neighbors. At first we will consider the situation the movement of the particles restricted for eaxmple it can move in forward direction and at the next we shall consider the unrestricted movement of particle where they can move in four directions. The complexity of problem is considerably grater than one-dimension, for now the domain to which the particle is restricted may have arbitrary shapes and complicated boundaries .

### 3.1 Random Walk in Two Dimension with Restricted Movement

We shall start with finite sample space. Suppose in a $n \times n$ grid two particles start their random walk from two furthest corners say $A$ and $B$. We assume that particles can move only in two directions. The first particle say $A_1$ starts at point $A$, at each step it can go one right ot up with equal probability 1/2 and it continues to move until it reaches the opposite corner $B$. At the same time the second particle $B_1$ starts at point $B$ and at each step it goes either one edge left or down with equal probability 1/2 in order to reach point $A$. Our problem is to calculate the probability that two particles meet (occupy the same point at the same time) during their random walk.

If the particles take equal number of steps in their respective random walk then they have a chance to meet. This means they can only meet in diagonal points. Each particle needs 10 steps to reach opposite corner. Now as the sample space is finite we can calculate the probability of meet only by using combinatorics which can be done as follows: To understand the problem we first take a $5 \times 5$ grid and count the number of ways that particle $A_1$ reaches at any of the diagonal points-




We can say that, the number of ways in which the particle $A_1$ can reach any point after $k_th$ step taking $m$ up steps and $(k-m)$ right steps is the coefficient of $U^m.R^{k-m}$ in $(U+R)^k$. Similarly, the number of ways in which the particle $B_1$ can reach any point after $k_th$ step taking $m$ down steps and $(k-m)$ left steps is the coefficient of $D^m.L^{k-m}$ in $(D+l)^k$.

Let $l$ represent number of ways in which $A_1$ can reach a particular point where it can meet $B_1$. Thus there are a total of $l \cdot l = l^2$ ways for both to meet at particular point. The number of ways one can reach any diagonal point in

$$\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}=2^5$$

possible choices. For two particles , there are then $2^5.2^5 = 4^5$ choices.

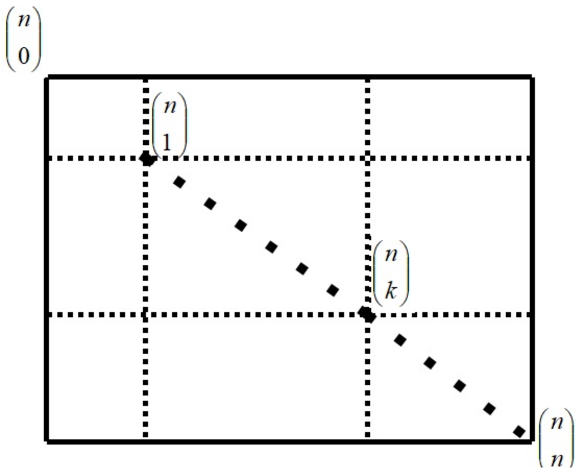
and number of ways to meet is

$$1^2+5^2+10^2+10^2+5^2+1^2=252$$

Now because of symmetry of there random walk all the possible choices are equally probable and hence $P(\text{meet}) = 0.246$

Now let us consider the main problem with $n \times n$ grid.

The particles $A_1$ and $B_1$ could possibly meet after exactly $n$ steps. As before there are




$l$ ways for each particle to meet. Thus there are total of $l \cdot l = l^2$ ways for both for a particular point. After $n$ steps, each particle could make $2^n$ possible choices. For two particles, there are then $2^n.2^n = 4^n$ choices.

Number of ways of meeting of the particles is

$$\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{k}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}$$

Because of symmetry of their random walks all the possible choices are equally probable. Hence

$$Pr(\text{meet}) = \frac{\binom{2n}{n}}{4^n} \approx \frac{1}{\sqrt{n\pi}} \qquad \text{(using Stirling's approximation)}$$

Now we have calculated the above probability using Monte Carlo simulation. The algorithm of the simulation is follows

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Let $X_i$ be a random variable which denote

$$X_i = \begin{cases} 1 & \text{if } A_1 \text{ goes right at } i^{th} \text{ step} \\ 0 & \text{if } A_1 \text{ goes up at } i^{th} \text{ step} \end{cases}$$

Let $Y_i$ be a random variable which denote

$$Y_i = \begin{cases} 1 & \text{if } B_1 \text{ goes down at } i^{th} \text{ step} \\ 0 & \text{if } B_1 \text{ goes left at } i^{th} \text{ step} \end{cases}$$

As each steps are equally probable for both particles we have $X_i \sim Ber(0.5)$ and $Y_i \sim Ber(0.5)$ for every $i$. Now we generate two samples of size $n$ from $Bernoulli(0.5)$. If $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are the two samples of size $n$.

Let the samples of size $n$ are drawn $'m'$ times. Define $D$ as the number of times we get $\left(\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i\right)$.

Therefore Monte Carlo estimate of probability of meet is

$$\hat{P}(\text{meet}) = \frac{D}{m}$$

The following compares $P(\text{meet})$ and $\hat{P}(\text{meet})$.

| Serial Number | n | $\hat{P}$(meet) | $P$(meet) |
|---|---|---|---|
| 1 | 5 | 0.24479 | 0.24609375 |
| 2 | 15 | 0.14459 | 0.14446445 |
| 3 | 25 | 0.11178 | 0.11227517 |
| 4 | 35 | 0.09669 | 0.09502547 |
| 5 | 45 | 0.08400 | 0.08387112 |
| 6 | 55 | 0.07580 | 0.07590261 |
| 7 | 65 | 0.06989 | 0.06984466 |
| 8 | 75 | 0.06588 | 0.06503851 |
| 9 | 85 | 0.06173 | 0.06110503 |
| 10 | 95 | 0.05795 | 0.05780852 |

| | | | |
|---|---|---|---|
| 11 | 105 | 0.05545 | 0.05499376 |
| 12 | 115 | 0.05177 | 0.05255380 |
| 13 | 125 | 0.05032 | 0.05041221 |
| 14 | 135 | 0.04824 | 0.04851277 |
| 15 | 145 | 0.04729 | 0.04681302 |
| 16 | 155 | 0.04481 | 0.04528027 |
| 17 | 165 | 0.04301 | 0.04388884 |
| 18 | 175 | 0.04196 | 0.04261827 |
| 19 | 185 | 0.04104 | 0.04145203 |
| 20 | 195 | 0.04055 | 0.04037656 |
| 21 | 205 | 0.03920 | 0.03938069 |
| 22 | 215 | 0.03781 | 0.03845505 |
| 23 | 225 | 0.03688 | 0.03759175 |
| 24 | 235 | 0.03553 | 0.03678410 |
| 25 | 245 | 0.03644 | 0.03602636 |
| 26 | 255 | 0.03520 | 0.03531361 |
| 27 | 265 | 0.03462 | 0.03464155 |
| 28 | 275 | 0.03429 | 0.03400645 |
| 29 | 285 | 0.03429 | 0.03340505 |
| 30 | 295 | 0.03225 | 0.03283447 |
| 31 | 305 | 0.03356 | 0.03229216 |
| 32 | 315 | 0.03191 | 0.03177587 |
| 33 | 325 | 0.03123 | 0.03128357 |
| 34 | 335 | 0.03104 | 0.03081347 |
| 35 | 345 | 0.03156 | 0.03036394 |
| 36 | 355 | 0.02948 | 0.02993353 |
| 37 | 365 | 0.02991 | 0.02952092 |
| 38 | 375 | 0.02923 | 0.02912491 |
| 39 | 385 | 0.02906 | 0.02874443 |
| 40 | 395 | 0.02950 | 0.02837848 |
| 41 | 405 | 0.02754 | 0.02802615 |
| 42 | 415 | 0.02752 | 0.02768664 |
| 43 | 425 | 0.02782 | 0.02735917 |

| | | | |
|---|---|---|---|
| 44 | 435 | 0.02729 | 0.02704305 |
| 45 | 445 | 0.02709 | 0.02673764 |
| 46 | 455 | 0.02724 | 0.02644235 |
| 47 | 465 | 0.02676 | 0.02615663 |
| 48 | 475 | 0.02607 | 0.02587998 |
| 49 | 485 | 0.02465 | 0.02561193 |
| 50 | 495 | 0.02424 | 0.02535203 |
| 51 | 505 | 0.02488 | 0.02509989 |

(R-commands given in appendix-7)

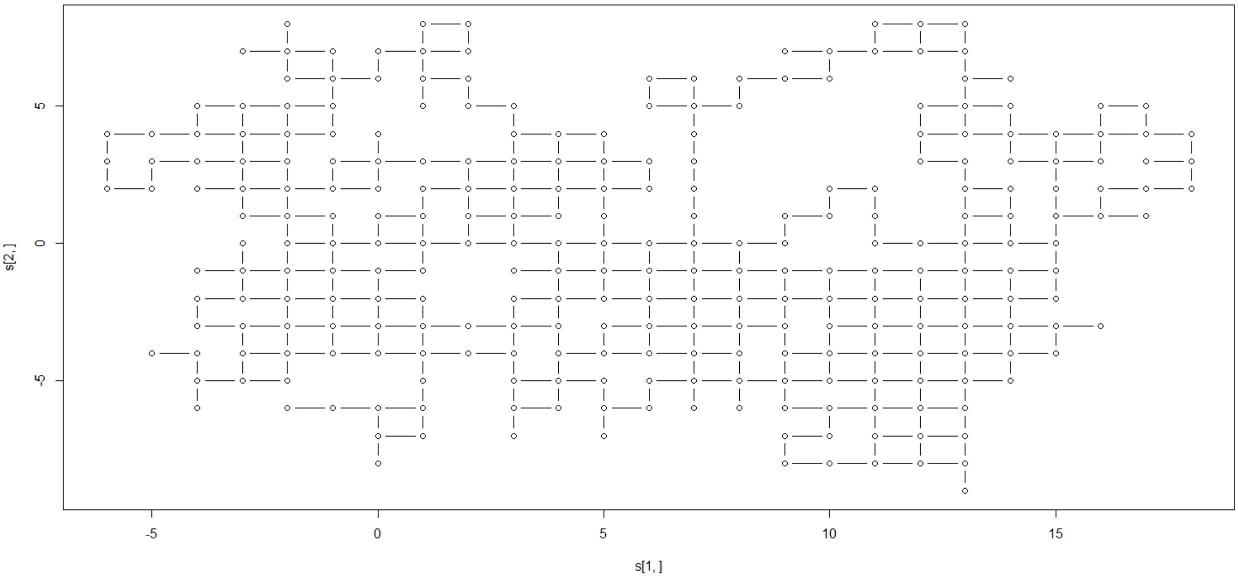
## 3.2 Random Walk in Two Dimensions with Unrestricted Movement

Now we consider the case random walk in two dimension where a particle can move in four directions at each step in the grid. The random walk is defined by specifying the corresponding four probabilities. For simplicity we shall consider only the symmetric case where all direction have same probability.

Without loss of generality we can consider a particle start random walk from $(0,0)$. From this point it can in all four direction, namely, upward $C_1 = (0,1)$ , downward $C_2 = (0,-1)$, left $C_3 = (-1,0)$ or right $C_4 = (1,0)$. Let us define a random variable $R_i$ which denotes the position of the particle at $i^{th}$ step if position of the particle ,

$$R_i(x,y) = \begin{cases} (x, y+1) & \text{with probability} \quad 0.25 \\ (x, y-1) & \text{with probability} \quad 0.25 \\ (x+1, y) & \text{with probability} \quad 0.25 \\ (x-1, y) & \text{with probability} \quad 0.25 \end{cases}$$

A simulated two-dimensional simple random walk can be as follows
(R-commands given in appendix-8)




Let us consider a random variable $U_i$ such that

$$U_i = \begin{cases} C_1 & \text{with probability} \quad 0.25 \\ C_2 & \text{with probability} \quad 0.25 \\ C_3 & \text{with probability} \quad 0.25 \\ C_4 & \text{with probability} \quad 0.25 \end{cases}$$

After first step the particle reach a point $R_1 = U_1$. And after $n$ step it reaches at the point $R_n = U_1 + U_2 + \ldots + U_n$.

Here again we will consider the same problem as previous section. In a $n \times n$ grid two particles $A_1$ and $B_1$ start a symmetric random walk at the same time. Our aim to is to calculate the probability of meet of the two particles. The difference is now the two particles can move in all four directions at each step in a plane. The movement of the particles can be explained through the random variables $R_i$.

Now in this problem they have a chance to meet anywhere after $n$ steps for $n \times n$ grid.

ALGORITHM

In this problem particles have different probability at a coordinate for any direction. For any corner they have two way to go,for other four boundary points ( at up, at down, at left and at right) of the grid they have three direction to go and at points within the grid they have four choice to go.

As before we define

$$U_i = \begin{cases} C_1 & \text{with probability} \quad p_1 \\ C_2 & \text{with probability} \quad p_2 \\ C_3 & \text{with probability} \quad p_3 \\ C_4 & \text{with probability} \quad p_4 \end{cases}$$

and

$$V_i = \begin{cases} C_1 & \text{with probability} \quad p_1 \\ C_2 & \text{with probability} \quad p_2 \\ C_3 & \text{with probability} \quad p_3 \\ C_4 & \text{with probability} \quad p_4 \end{cases}$$

For the down-left corner $p_1$=0.5, $p_2$=0, $p_3$=0, $p_4$=0.5

For the down-right corner $p_1$=0.5, $p_2$=0, $p_3$=0.5, $p_4$=0

For the up-left corner $p_1$=0, $p_2$=0.5, $p_3$=0, $p_4$=0.5

For the up-right corner $p_1$=0, $p_2$=0.5, $p_3$=0.5, $p_4$=0

For the down boundary $p_1$=1/3, $p_2$=0, $p_3$=1/3, $p_4$=1/3

For the up boundary $p_1$=0, $p_2$=1/3, $p_3$=1/3, $p_4$=1/3

For the left boundary $p_1$=1/3, $p_2$=1/3, $p_3$=0, $p_4$=1/3

For the right boundary $p_1$=1/3, $p_2$=1/3, $p_3$=1/3, $p_4$=0

For into the grid $p_1$=0.25, $p_2$=0.25, $p_3$=0.25, $p_4$=0.25

For above any case $\sum_{i=1}^{4} p_i = 1$

Now we generate two samples of size $n$ from four point distribution which satisfy above conditions. If $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ are the two samples of size $n$.

Let the samples of size $n$ are drawn $'m'$ times. Define $D$ as the number of times we get $\left(\sum_{i=1}^{n} U_i = \sum_{i=1}^{n} V_i\right)$.

Monte Carlo estimate of probability of meet is

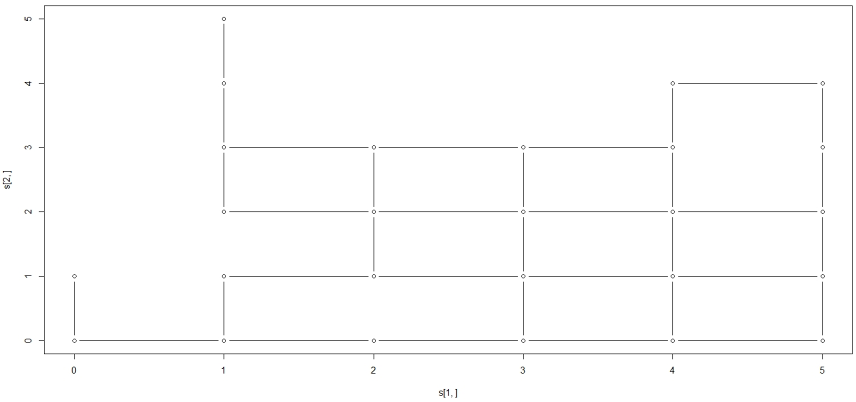
$$\hat{P} = \frac{D}{m}$$

We have calculated the probability of meet in the grid after any step using MonteCarlo simulation. Following the example in the previous section where the two particle meet at $5^{th}$ step, here we obtained an Monte Carlo estimate of the $\hat{P}$(meet) after $5^{th}$ in a

5 × 5 grid which is

$$\hat{P}(\text{meet at } 5^{th} \text{ step for } 5 \times 5 \text{ grid}) = 0.0040423$$

(R-commands given in appendix-9) (taking 100 steps in 5×5 grid, R-commands given




in appendix-10 )

# 4 Future direction of work

Many different experiments and simulations have been left for the future due to lack of time . The future extension of this work may be the following

- In gambler's ruin problem also we can calculate ruin probability and duration of the game which can include $n$ gamblers.

- We can examine the recurrence of the events in two dimensional random walk.

- Also we can find the distribution of first meet or last meet in grid(unrestricted movement) for different values of probability.

# References

[1] Feller,W. *An Introduction to Probability Theory and Its Applications*-Vol-I. John Wiley & Sons

[2] A. K. Basu *Introduction to Stochastic Process*-Alpha Science (2003)

# APPENDIX-1

```
n<-200
p<-.5
b<-0
s<-NULL
s[1]<-0
for(i in 1:n)
{
a<-sample(c(1,-1),1,replace = TRUE, prob =c(p,1-p) )
s[i+1]<-a+s[i]
}
plot(c(0:n),s,type="l")
lines(c(0,n+1),c(0,0))
length(s)
```

# APPENDIX-2

```
####probability of meeting at zero for different p
n<-seq(2,500,2) #number of steps
m<-300 #number of repitions
n1<-NULL
aq<-matrix(0:0,m,n)
n1[1]<-0
p<-seq(.1,.9,.1)###different probabilities
mx<-matrix(0:0,length(n),length(p))
for(l in 1:length(n))
{
for(k in 1:length(p))
{
b=0

for(j in 1:m)
{
```

```
s<-NULL
q1<-0
s[1]<-sample(c(1,-1),1,replace = TRUE, prob =c(p[k],1-p[k]) )
for(i in 2:(n[l]))
{
a<-sample(c(1,-1),1,replace = TRUE, prob =c(p[k],1-p[k]) )
s[i]<-a+s[i-1]
}
for(i in 1:n[l])
{
if(s[i]==0)
{
q1<-i
break
}
}
if(q1==n[l]){b=b+1}
}
##plot(s,type="l")
##lines(c(1,n[l]+1),c(0,0))
mx[l,k]<-b/m
}
###n1[l+1]<-n[l]
}
colSums(mx)
###data.frame(n1,mx)
```

# APPENDIX-3

```
n<-1000 #number of steps
m<-2000#number of repitions
p1<-0
p2<-0
z1<-NULL
```

```
z<-NULL
for(j in 1:m)
{
a<-sample(c(1,-1),n,replace = TRUE, prob = c(.5,.5))
s<-NULL
s[1]<-a[1]
for(i in 2:n)
{
s[i]<-a[i]+s[i-1]
if(s[i]==0)
{
p1<-i
break
}
}
for(i in 2:n)
{
s[i]<-a[i]+s[i-1]
if(s[i]==0)
{
p2<-i
}
}
z1[j]<-p1
z[j]<-p2
}
##plot(s,type="l")
##lines(c(1,n+1),c(0,0))
w1<-data.frame(table(z1))
w<-data.frame(table(z))

x1<-NULL
nn<-length(strtoi(w$Freq))
for(i in 1:nn-1){
```

```
x1[i]<-strtoi(w$Freq)[i]
}
yy<-strtoi(w$z)
yy<-yy[yy<n]

plot(w)
##cdf1<-cumsum(w$Freq)/m
##cdf2<-cumsum(w1$Freq)/m
PDF1<-dbeta(yy/n,.5,.5)
PDF2<-dbeta(strtoi(w1$z1)/n,.5,.5)
##w2<-data.frame(table(z),cdf1,CDF1)
##w3<-data.frame(w1,cdf2,CDF2)
 chisq.test(x2,PDF1)
```

# APPENDIX-4

```
n<-1000 #number of steps
m<-2000#number of repitions
p1<-0
p2<-0
z1<-NULL
z<-NULL
for(j in 1:m)
{
a<-sample(c(1,-1),n,replace = TRUE, prob = c(.5,.5))
s<-NULL
s[1]<-a[1]
for(i in 2:n)
{
s[i]<-a[i]+s[i-1]
if(s[i]==0)
{
p1<-i
break
```

```
}
}
for(i in 2:n)
{
s[i]<-a[i]+s[i-1]
if(s[i]==0)
{
p2<-i
}
}
z1[j]<-p1
z[j]<-p2
}
##plot(s,type="l")
##lines(c(1,n+1),c(0,0))
w1<-data.frame(table(z1))
w<-data.frame(table(z))

x1<-NULL
nn<-length(strtoi(w$Freq))
for(i in 1:nn-1){
x1[i]<-strtoi(w$Freq)[i]
}
yy<-strtoi(w$z)
yy<-yy[yy<n]

plot(w)
##cdf1<-cumsum(w$Freq)/m
##cdf2<-cumsum(w1$Freq)/m
PDF1<-dbeta(yy/n,.5,.5)
PDF2<-dbeta(strtoi(w1$z1)/n,.5,.5)
##w2<-data.frame(table(z),cdf1,CDF1)
##w3<-data.frame(w1,cdf2,CDF2)
 chisq.test(x1,PDF1)
```

# APPENDIX-5

```
m<-10000 ##repetition
r<-0
z<-NULL
ru<-20 ##starting money
lb<-40 ##want to get this money
p<-c(.5,.5) ##probabilities
for(j in 1:m)
{
i=1
s<-NULL
s[1]<-ru
repeat
{
a<-sample(c(1,-1),1,replace=TRUE, prob =p)
s[i+1]<-a+s[i]
i=i+1
if(s[i]==0||s[i]==lb){break}
z1<-i
}
r=r+ifelse(s[i]==0,1,0)
z[j]<-z1
}
p1<-r/m ## probability of reaching at 0
p1
mean(z) ##expected duration spend in game
```

# APPENDIX-6

```
m<-10000 ##repetition
r<-0
z<-NULL
ru<-20 ##starting money
```

```
lb<-40 ##want to get this money
p<-c(.5,.5) ##probabilities
for(j in 1:m)
{
i=1
s<-NULL
s[1]<-ru
repeat
{
a<-sample(c(1,-1),1,replace=TRUE, prob =p)
s[i+1]<-a+s[i]
i=i+1
if(s[i]==0||s[i]==lb){break}
z1<-i
}
r=r+ifelse(s[i]==0,1,0)
z[j]<-z1
}
p1<-r/m ## probability of reaching at 0
p1
mean(z) ##expected duration spend in game
plot(s,main="p=.5",type="l")
```

# APPENDIX-7

```
 n=seq(5,1000,10)
 m=100000
 p<-NULL
 p1<-NULL
 set.seed(123)
 for(j in 1:length(n))
{
 d=0
 for(i in 1:m)
```

```
 {
 x=rbinom(n[j],1,.5)
 y=rbinom(n[j],1,.5)
 d=d+ifelse(sum(x)==sum(y),1,0)
 }
 d
 p[j]=d/m;
 p1[j]<-choose(2*n[j],n[j])/4^n[j]
 }
```

# APPENDIX-8

```
n<-1000
a<-matrix(c(-1,0,0,1,0,-1,1,0),2,4)
p<-c(.25,.25,.25,.25)
s<-matrix(0:0,2,n+1)
for(i in 1:n)
{
z<-sample(1:4,1,replace = TRUE,prob=p)
s[,i+1]<-s[,i]+a[,z]
}
plot(s[1,],s[2,],type="b")
```

# APPENDIX-9

```
n<-50
z<-5 ### grid size
a<-matrix(c(-1,0,0,1,0,-1,1,0),2,4)
s<-matrix(0:0,2,n+1)
for(i in 1:n)
{
if(s[1,i]==0&&s[2,i]==0){
```

```
s[,i+1]<-s[,i]+a[,sample(c(2,4),1,replace = TRUE,prob=c(.5,.5))]}
if(s[1,i]==z&&s[2,i]==z){
s[,i+1]<-s[,i]+a[,sample(c(1,3),1,replace = TRUE,prob=c(.5,.5))]}
if(s[1,i]==z&&s[2,i]==0){
s[,i+1]<-s[,i]+a[,sample(c(1,2),1,replace = TRUE,prob=c(.5,.5))]
}
if(s[1,i]==0&&s[2,i]==z){
s[,i+1]<-s[,i]+a[,sample(c(3,4),1,replace = TRUE,prob=c(.5,.5))]}
if(1<=s[1,i]&&s[1,i]<=(z-1)&&s[2,i]==0){
s[,i+1]<-s[,i]+a[,sample(c(1,2,4),1,replace = TRUE,prob=c(1/3,1/3,1/3))]}
if(s[1,i]==0&&1<=s[2,i]&&s[2,i]<=(z-1)){
s[,i+1]<-s[,i]+a[,sample(c(2,3,4),1,replace = TRUE,prob=c(1/3,1/3,1/3))]
}
if(s[1,i]==z&&1<=s[2,i]&&s[2,i]<=(z-1)){
s[,i+1]<-s[,i]+a[,sample(c(1,2,3),1,replace = TRUE,prob=c(1/3,1/3,1/3))]}
if(s[2,i]==z&&1<=s[1,i]&&s[1,i]<=(z-1)){
s[,i+1]<-s[,i]+a[,sample(c(1,3,3),1,replace = TRUE,prob=c(1/3,1/3,1/3))]}
if(1<=s[1,i]&&s[1,i]<=(z-1)&&1<=s[2,i]&&s[2,i]<=(z-1)){
s[,i+1]<-s[,i]+a[,sample(1:4,1,replace = TRUE,prob=c(.25,.25,.25,.25))]
}}
plot(s[1,],s[2,],type="b")
```

# APPENDIX-10

```
n<-5
m<-10000000
z<-5
a<-matrix(c(-1,0,0,1,0,-1,1,0),2,4)
s<-matrix(0:0,2,n+1)
s1<-matrix(0:0,2,n+1)
s1[,1]<-c(z,z)
p<-0
```

```
for(j in 1:m){
for(i in 1:n)
{
if(s[1,i]==0&&s[2,i]==0){
s[,i+1]<-s[,i]+a[,sample(c(2,4),1,replace = TRUE,prob=c(.5,.5))]}
if(s[1,i]==z&&s[2,i]==z){
s[,i+1]<-s[,i]+a[,sample(c(1,3),1,replace = TRUE,prob=c(.5,.5))]}
if(s[1,i]==z&&s[2,i]==0){
s[,i+1]<-s[,i]+a[,sample(c(1,2),1,replace = TRUE,prob=c(.5,.5))]
}
if(s[1,i]==0&&s[2,i]==z){
s[,i+1]<-s[,i]+a[,sample(c(3,4),1,replace = TRUE,prob=c(.5,.5))]}
if(1<=s[1,i]&&s[1,i]<=(z-1)&&s[2,i]==0){
s[,i+1]<-s[,i]+a[,sample(c(1,2,4),1,replace = TRUE,prob=c(1/3,1/3,1/3))]}
if(s[1,i]==0&&1<=s[2,i]&&s[2,i]<=(z-1)){
s[,i+1]<-s[,i]+a[,sample(c(2,3,4),1,replace = TRUE,prob=c(1/3,1/3,1/3))]
}
if(s[1,i]==z&&1<=s[2,i]&&s[2,i]<=(z-1)){
s[,i+1]<-s[,i]+a[,sample(c(1,2,3),1,replace = TRUE,prob=c(1/3,1/3,1/3))]}
if(s[2,i]==z&&1<=s[1,i]&&s[1,i]<=(z-1)){
s[,i+1]<-s[,i]+a[,sample(c(1,3,3),1,replace = TRUE,prob=c(1/3,1/3,1/3))]}
if(1<=s[1,i]&&s[1,i]<=(z-1)&&1<=s[2,i]&&s[2,i]<=(z-1)){
s[,i+1]<-s[,i]+a[,sample(1:4,1,replace = TRUE,prob=c(.25,.25,.25,.25))]
}
######
if(s1[1,i]==0&&s1[2,i]==0){
s1[,i+1]<-s1[,i]+a[,sample(c(2,4),1,replace = TRUE,prob=c(.5,.5))]}
if(s1[1,i]==z&&s1[2,i]==z){
s1[,i+1]<-s1[,i]+a[,sample(c(1,3),1,replace = TRUE,prob=c(.5,.5))]}
if(s1[1,i]==z&&s1[2,i]==0){
s1[,i+1]<-s1[,i]+a[,sample(c(1,2),1,replace = TRUE,prob=c(.5,.5))]
}
if(s1[1,i]==0&&s1[2,i]==z){
s1[,i+1]<-s1[,i]+a[,sample(c(3,4),1,replace = TRUE,prob=c(.5,.5))]}
```

```
if(1<=s1[1,i]&&s1[1,i]<=(z-1)&&s1[2,i]==0){
s1[,i+1]<-s1[,i]+a[,sample(c(1,2,4),1,replace = TRUE,prob=c(1/3,1/3,1/3))]}
if(s1[1,i]==0&&1<=s1[2,i]&&s1[2,i]<=(z-1)){
s1[,i+1]<-s1[,i]+a[,sample(c(2,3,4),1,replace = TRUE,prob=c(1/3,1/3,1/3))]
}
if(s1[1,i]==z&&1<=s1[2,i]&&s1[2,i]<=(z-1)){
s1[,i+1]<-s1[,i]+a[,sample(c(1,2,3),1,replace = TRUE,prob=c(1/3,1/3,1/3))]}
if(s1[2,i]==z&&1<=s1[1,i]&&s1[1,i]<=(z-1)){
s1[,i+1]<-s1[,i]+a[,sample(c(1,3,3),1,replace = TRUE,prob=c(1/3,1/3,1/3))]}
if(1<=s1[1,i]&&s1[1,i]<=(z-1)&&1<=s1[2,i]&&s1[2,i]<=(z-1)){
s1[,i+1]<-s1[,i]+a[,sample(1:4,1,replace = TRUE,prob=c(.25,.25,.25,.25))]
}
}
q<-0
##plot(s[1,],s[2,],type="b")
for(i in 1:n+1){
if(s[1,i]==s1[1,i]&&s[2,i]==s1[2,i]){q<-q+1} }
if(q>=1){p=p+1}}


p/m ### probability of meet
```