Signal Processing and Linear Systems

Chapter 3
Linear Time-Invariant Systems (Discrete-Time)

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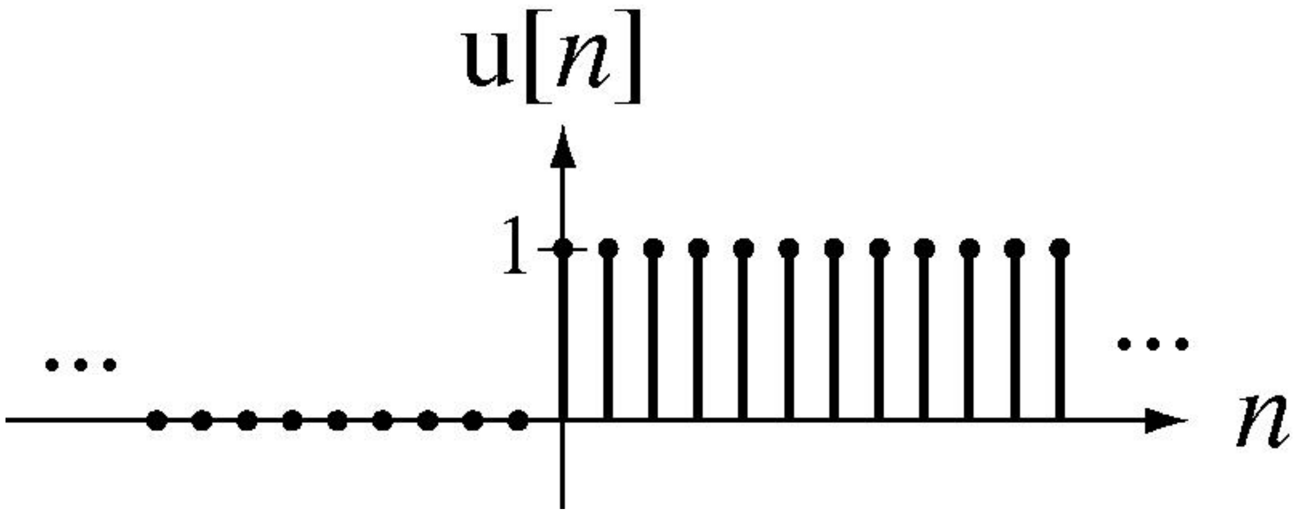
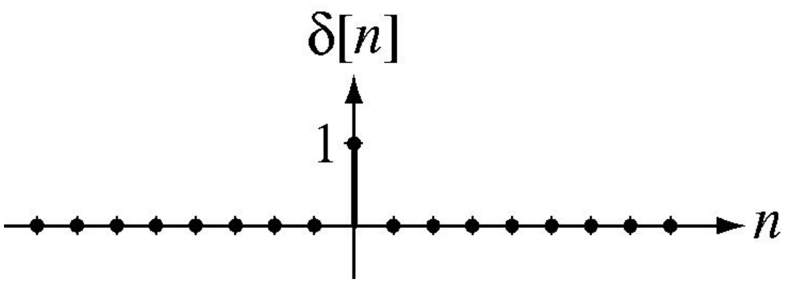
DT Unit Step Sequence

$$\mathrm{u}[n] = \begin{cases} 1 \ , & n \geq 0 \\ 0 \ , & n < 0 \end{cases}$$



DT Unit Impulse Sequence

$$\delta[n]=\begin{cases}1 \ , & n=0 \\ 0 \ , & n\neq 0\end{cases}$$

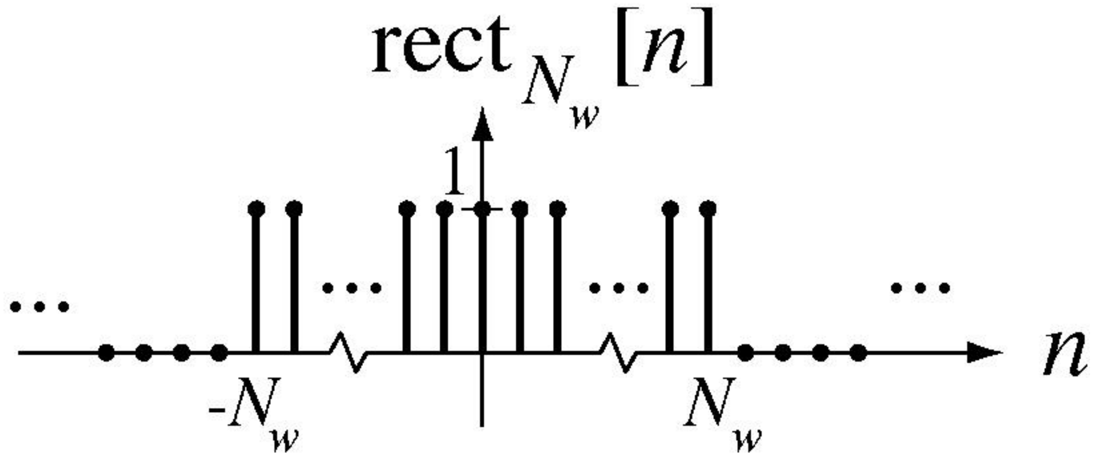


DT unit impulse is a function in the ordinary sense (in contrast with the CT unit impulse). It has a sampling property,

$$\sum_{n=-\infty}^{\infty} A\delta[n-n_0]x[n] = Ax[n_0]$$

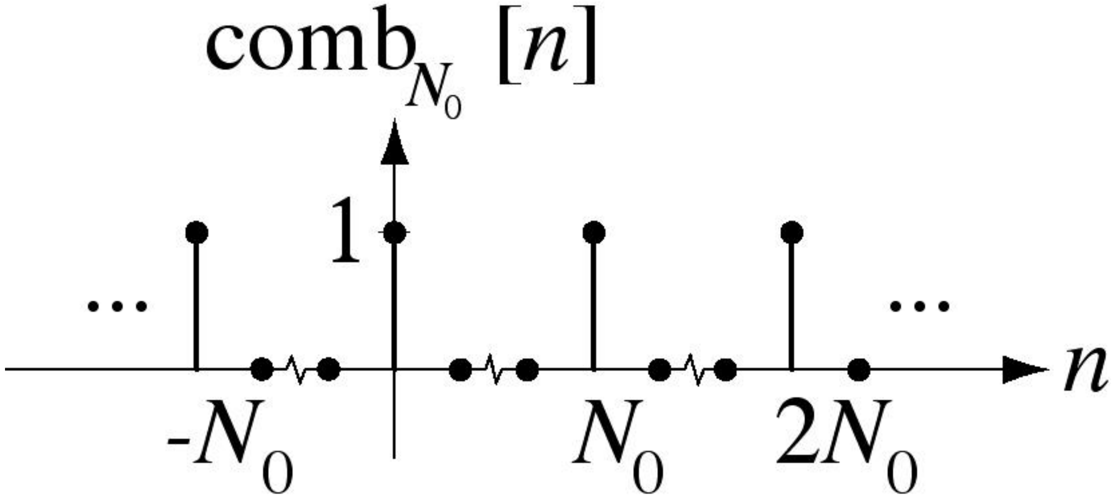
DT Rectangle Sequence

$$\mathrm{rect}_{N_w}[n] = \begin{cases} 1 \ , & |n| \le N_w \\ 0 \ , & |n| > N_w \end{cases} , \ N_w \ge 0 \ , \ N_w \text{ an integer}$$



DT Comb Sequence

$$\mathrm{comb}_{N_0}[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN_0]$$

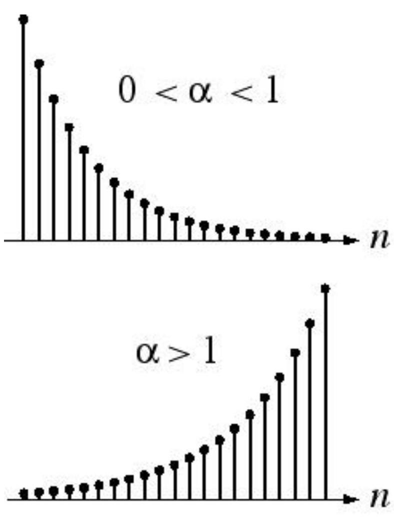


DT Exponential Sequences

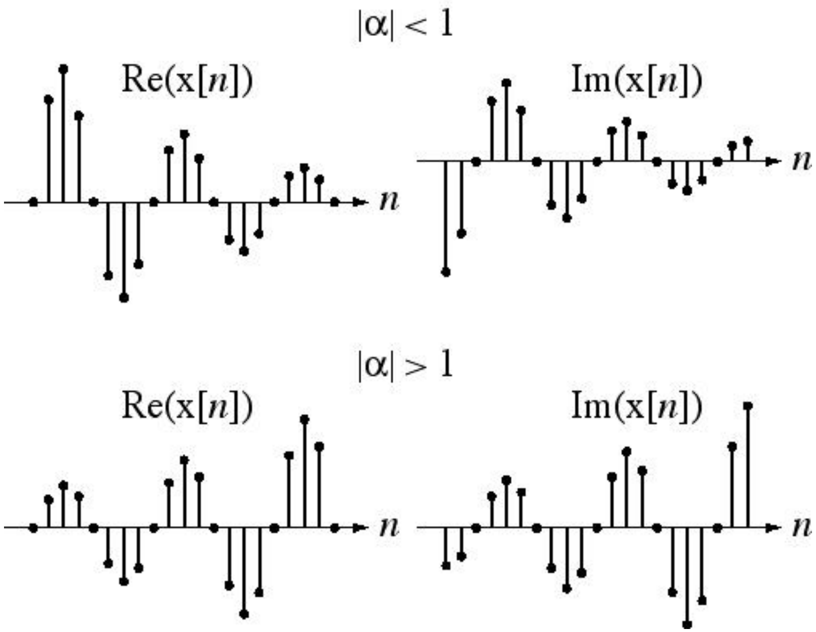
$$g[n] = A\alpha^n \quad \text{or} \quad g[n] = Ae^{\beta n}$$

where $\alpha = e^{\beta}$

Real α

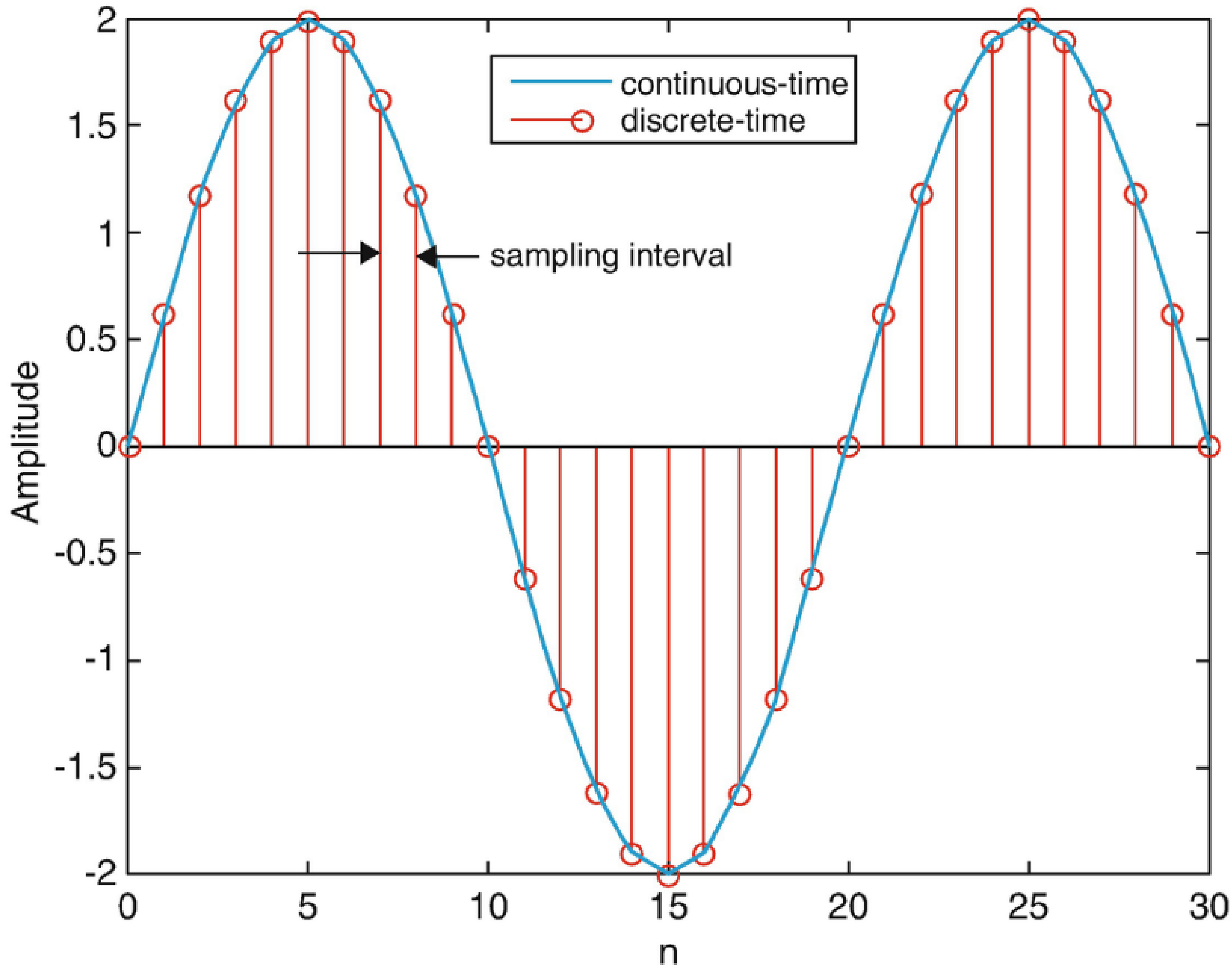


Complex α



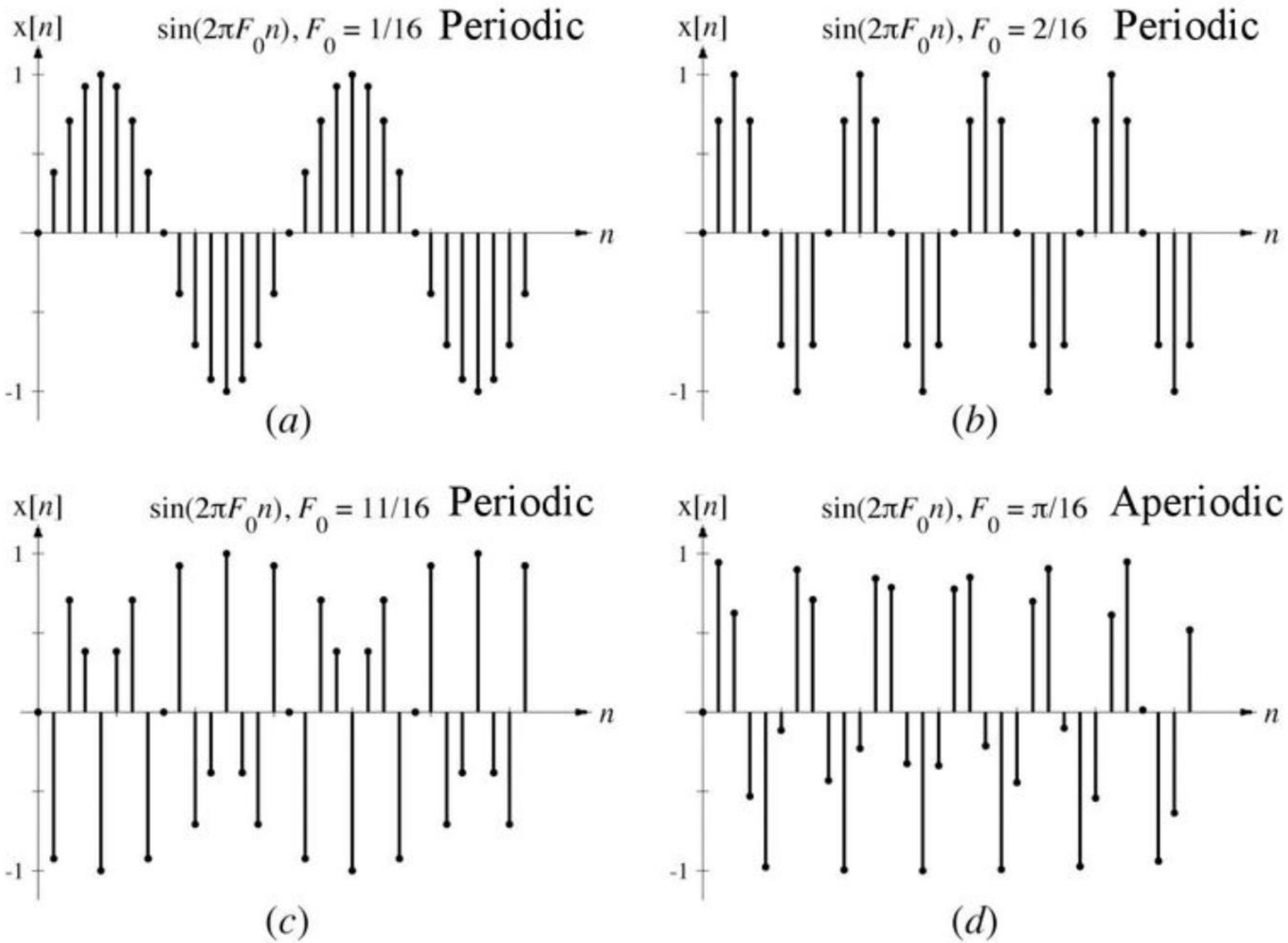
CT and DT Sinusoids

Periodic Sinusoid



DT Sinusoids

A sinusoid is not necessarily periodic in DT domain



Condition to be Periodic in DT Domain

- Let $x(t)$ be the complex exponential signal $x(t) = e^{j\omega_0 t}$ with fundamental frequency $f_0 = \frac{\omega_0}{2\pi}$ and fundamental period $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$.
- Consider the discrete-time sequence $x[n]$ obtained by uniformly sampling of $x(t)$ with sampling interval Ts.
- That is, $x[n] = x(nT_s) = e^{j\omega_0 n T_s}$
- Find the condition of Ts value, so that $x[n]$ is periodic.
- If $x[n]$ is periodic with fundamental period N_0, then

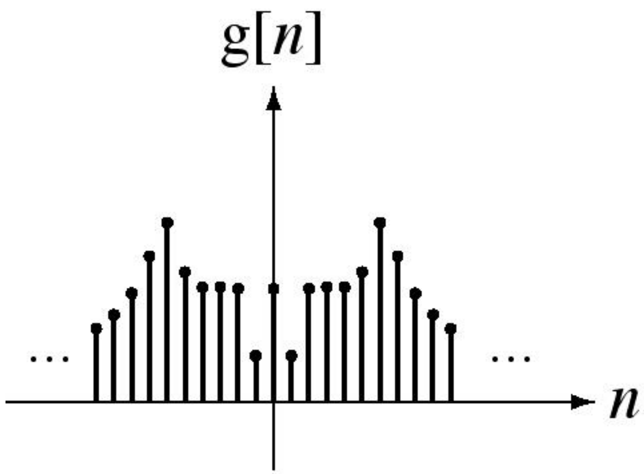
$$e^{j\omega_0 n T_s} = e^{j\omega_0 (n+N_0) T_s} = e^{j\omega_0 n T_s} \cdot e^{j\omega_0 N_0 T_s}$$

- Thus, we must have $e^{j\omega_0 N_0 T_s} = 1$
- Therefore, $\omega_0 N_0 T_s = 2\pi f_0 N_0 T_s = \frac{2\pi}{T_0} N_0 T_s = m2\pi$, m=positive integer
- Finally, we have $\frac{T_s}{T_0} = \frac{m}{N_0}$ = rational number
- Conclusion = $x[n]$ is periodic if the ratio $\frac{T_s}{T_0}$ of the sampling interval and the fundamental period of $x(t)$ is a rational number.
- Note that this condition is also true for sinusoidal signals.

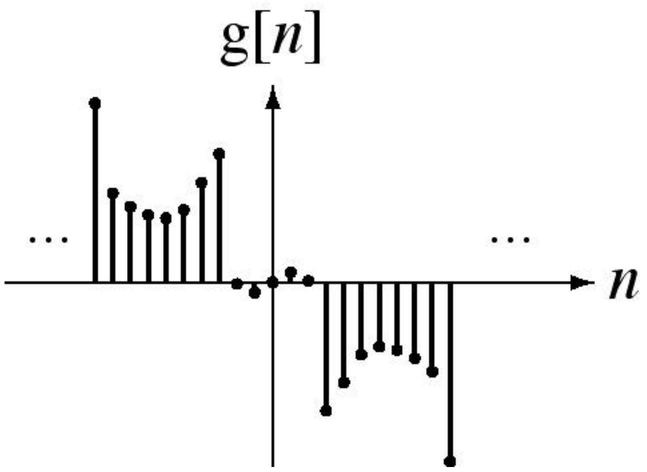
Even and Odd DT Sequences

$$g[n] = g[-n] \qquad\qquad g[n] = -g[-n]$$

Even sequence



Odd sequence

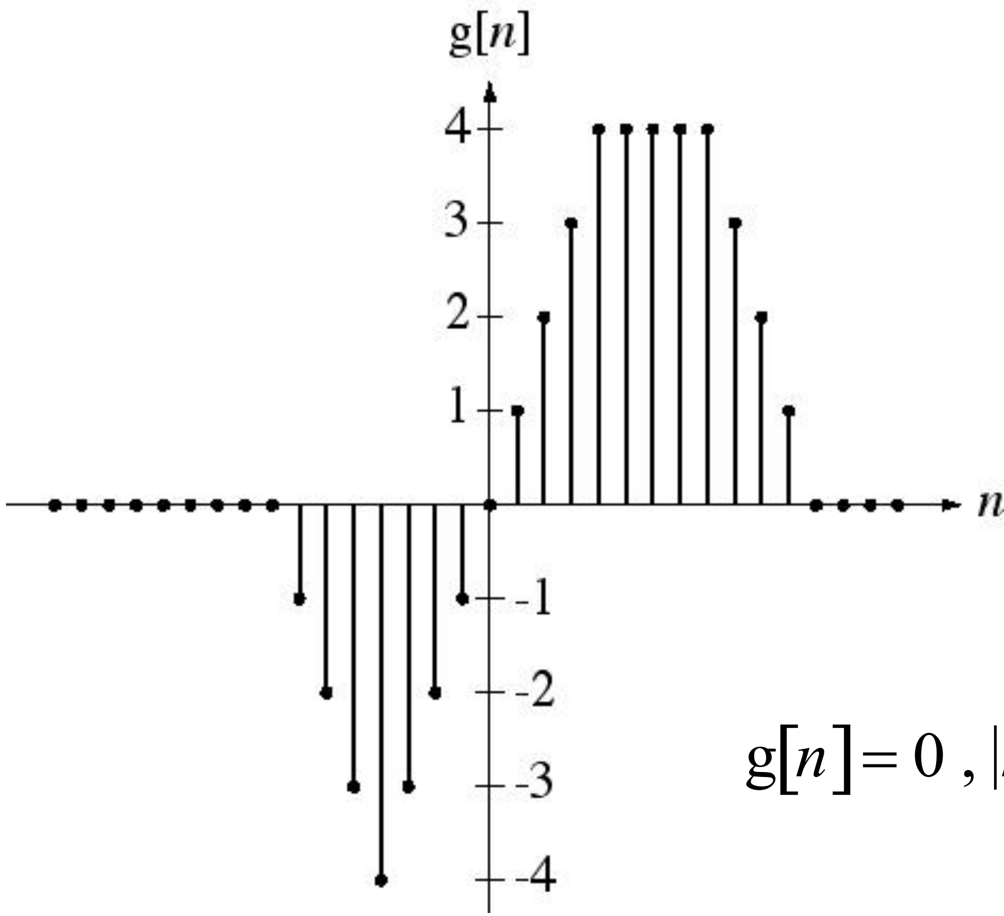


Making of even sequence and odd sequence from general sequence

$$g_e[n] = \frac{g[n] + g[-n]}{2} \qquad\qquad g_o[n] = \frac{g[n] - g[-n]}{2}$$

A Sample of DT Signal

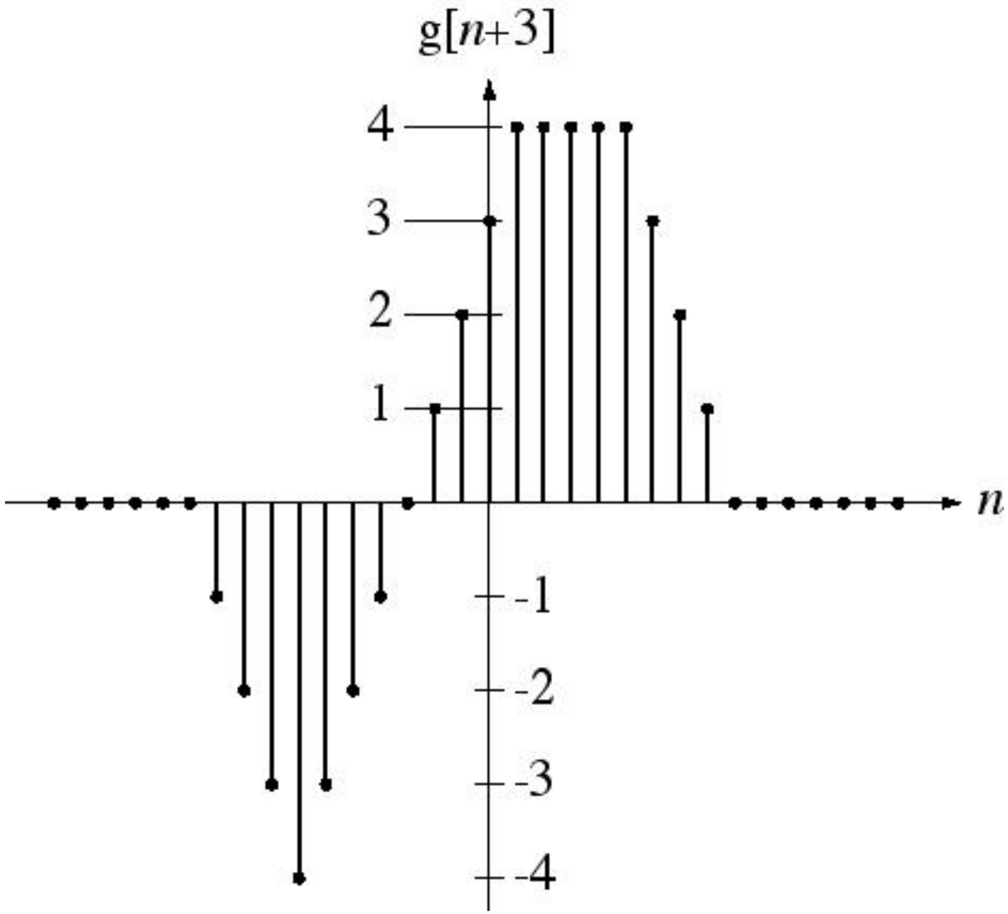
Let $g[n]$ be defined by



$$g[n] = 0 \ , \ |n| > 15$$

Transformation of DT Signal

Time shift $\quad n \rightarrow n+n_0,\ n_0$ an integer

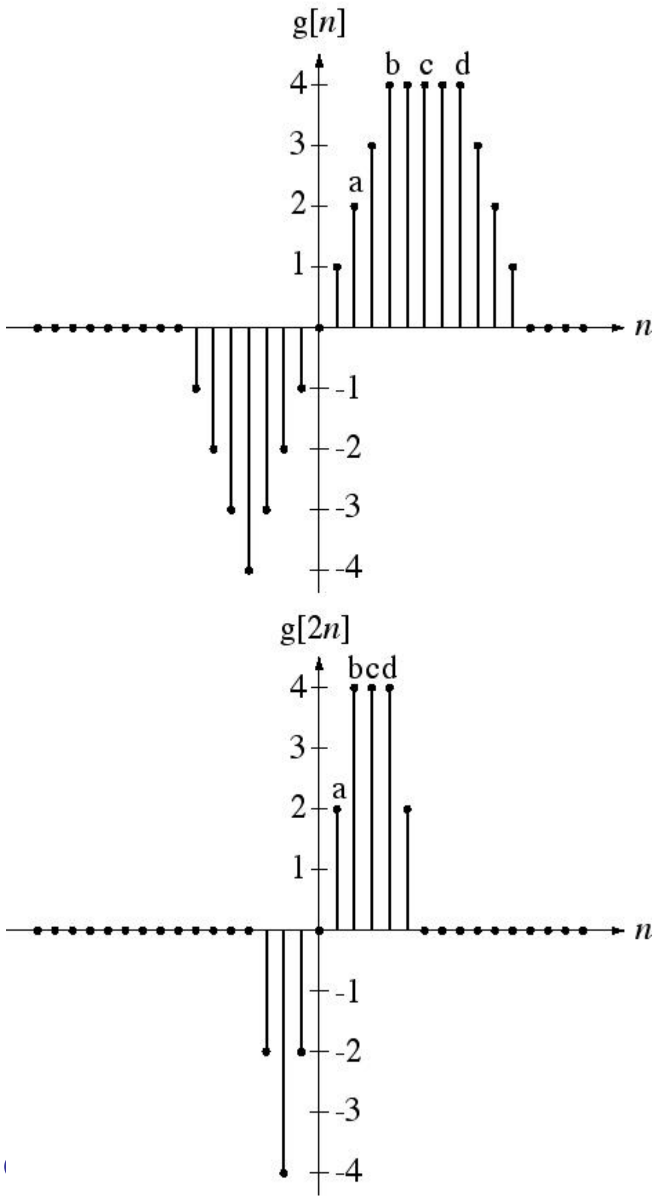


Transformation of DT Signal

Time scaling (compression)

$$n \rightarrow Kn$$

K an integer > 1



Transformation of DT Signal

Time expansion $n \to \dfrac{n}{K}$ where, $K > 1$

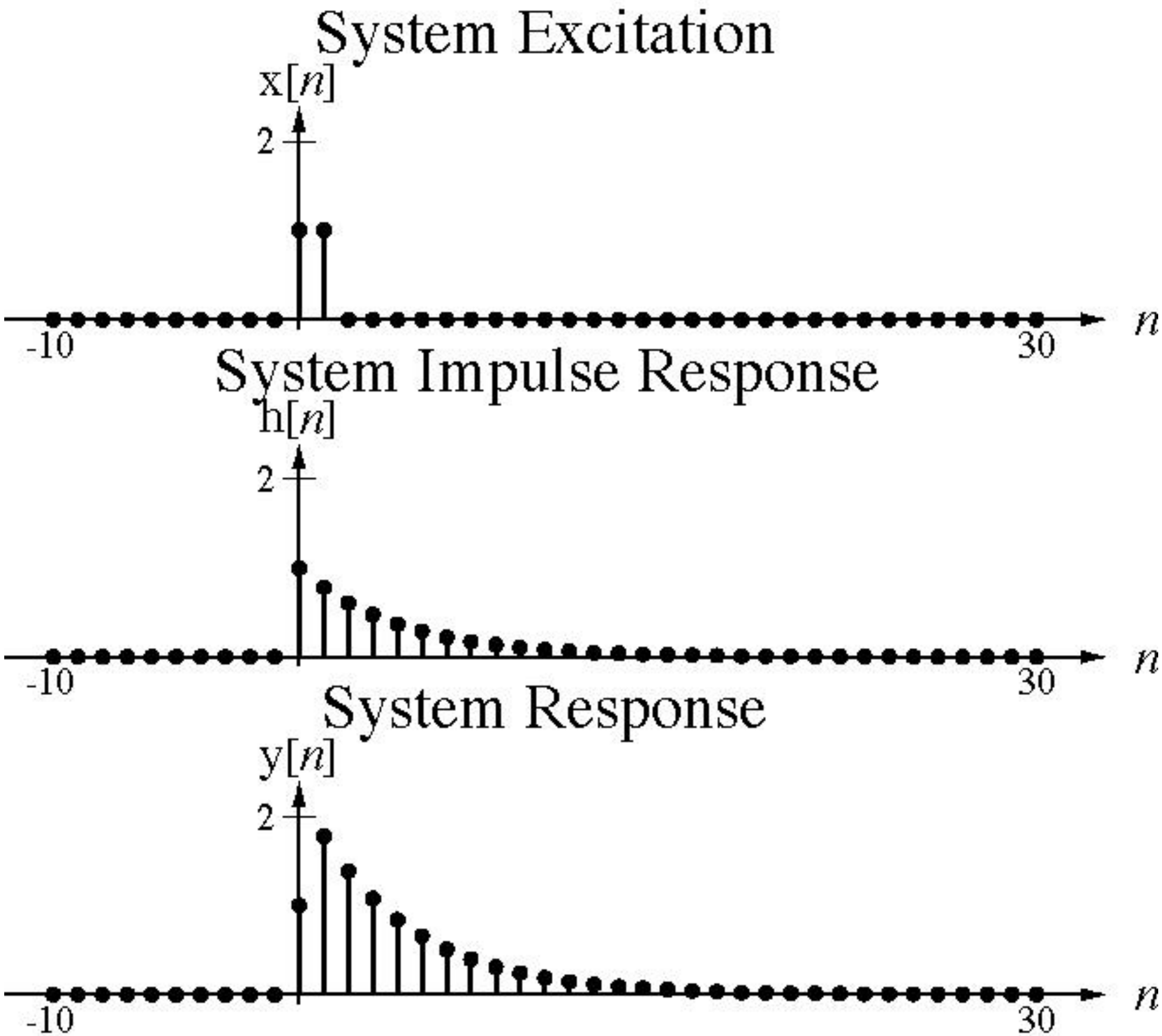
For all n such that n/K is an integer, $\mathrm{g}\left[\dfrac{n}{K}\right]$ is defined.

For all n such that n/K is not an integer, $\mathrm{g}\left[\dfrac{n}{K}\right]$ is not defined.

System Response

- For any discrete-time LTI system:
  - If the *response to a unit impulse* (or *impulse response*) is known, the response to any arbitrary excitation can be found
  - Any arbitrary excitation (input sequence) is simply a sequence of amplitude-scaled and time-shifted DT *unit impulses*
  - The response (output sequence) is simply a sequence of amplitude-scaled and time-shifted DT *impulse response*

A System Response Example



Convolution Sum

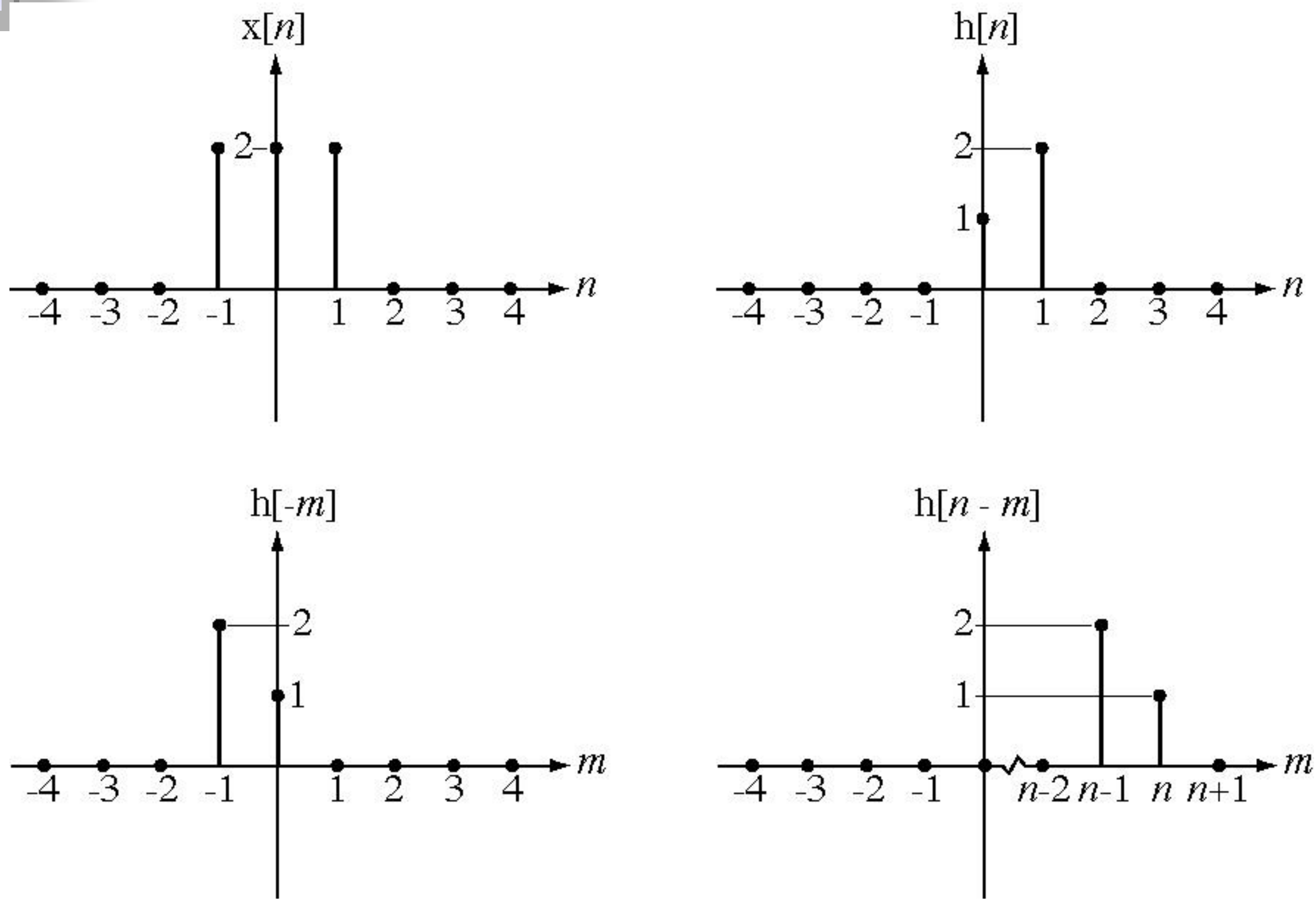
The response, $y[n]$, to an arbitrary excitation, $x[n]$, is of the form,

$$\mathrm{y}[n]=\cdots\mathrm{x}[-1]\mathrm{h}[n+1]+\mathrm{x}[0]\mathrm{h}[n]+\mathrm{x}[1]\mathrm{h}[n-1]+\cdots$$

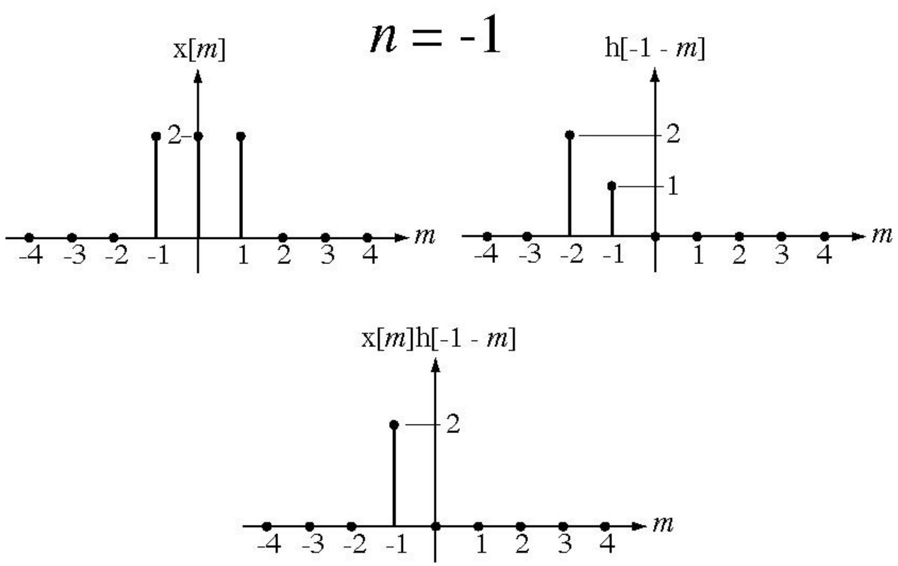
where $h[n]$ is the impulse response. This can be written in a more compact form, or *convolution sum*

$$y[n]=x[n]*h[n]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$$

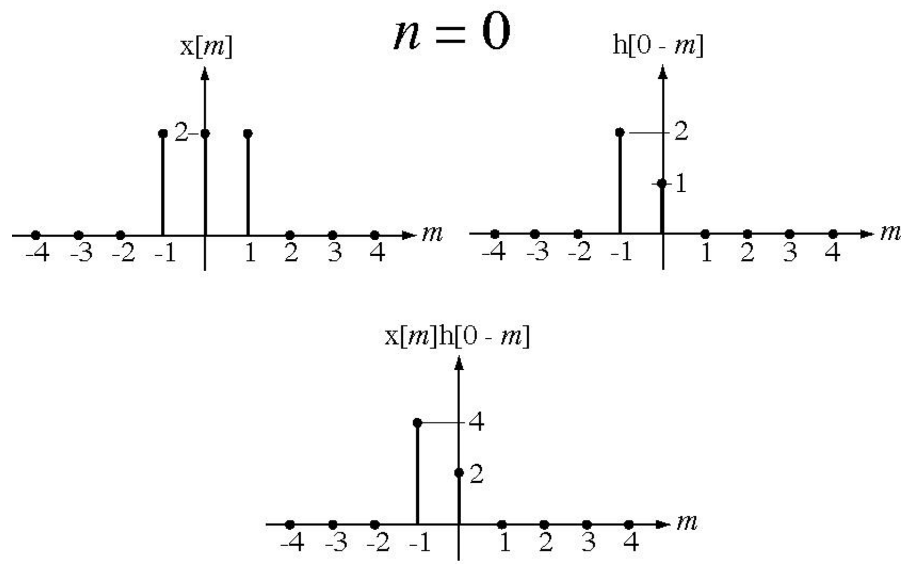
A Convolution Sum Example



A Convolution Sum Example



y[-1] = 2

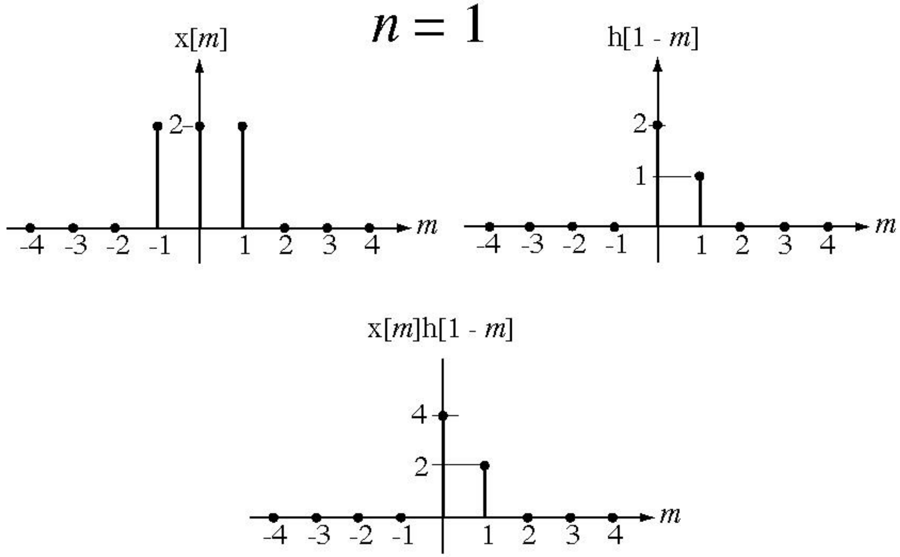


y[0] = 6

$$y[-1] = \sum_{m=-\infty}^{\infty} x[m]h[-1-m]$$

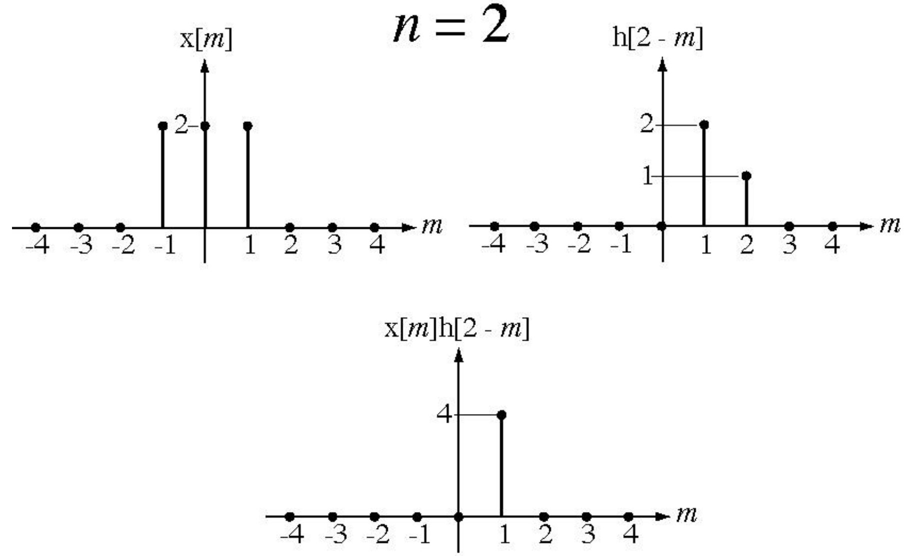
A Convolution Sum Example

$n = 1$



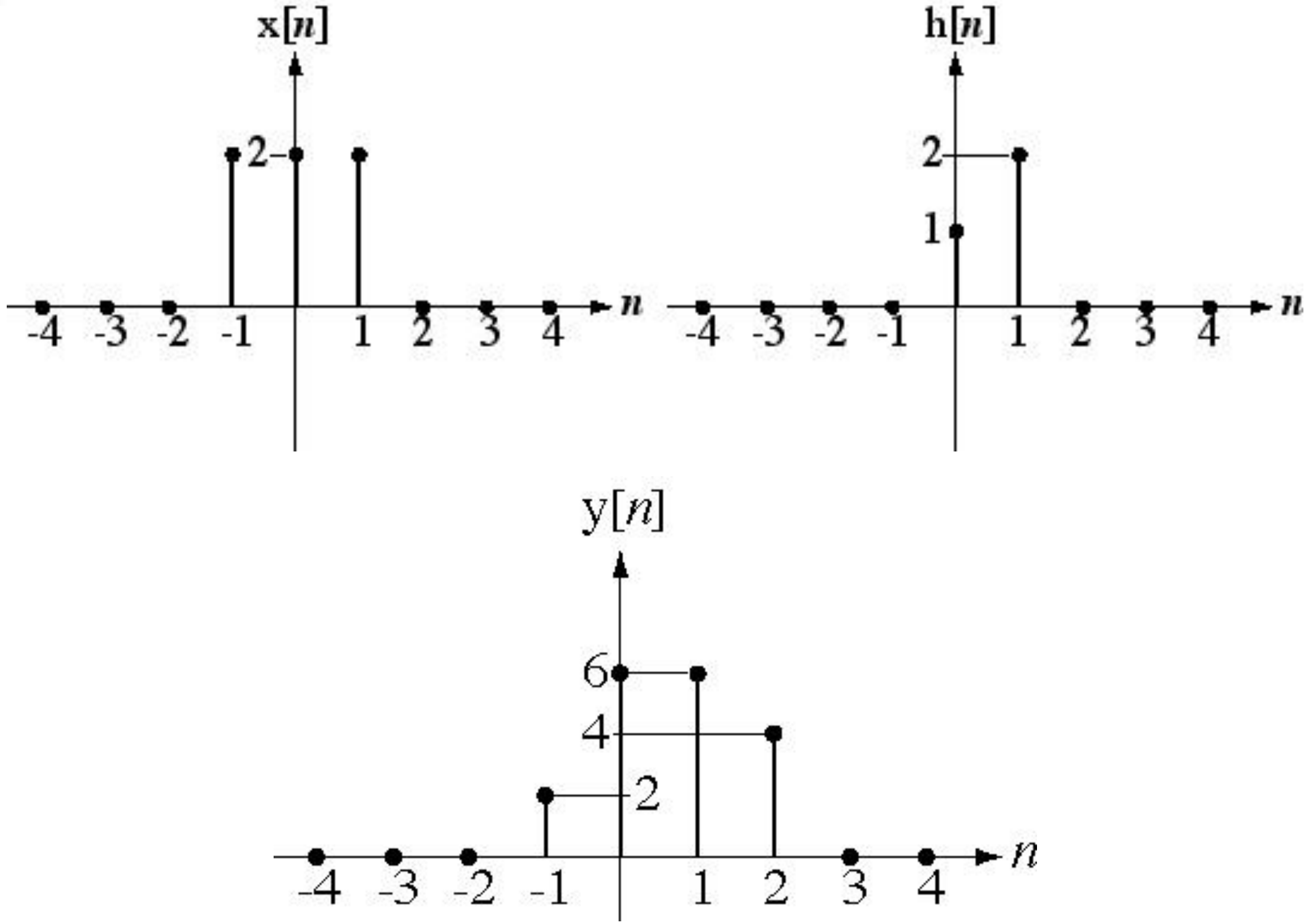
$y[1] = 6$

$n = 2$



$y[2] = 4$

A Convolution Sum Example



Convolution Sum Properties

Commutativity
可換性

$$x[n]*y[n]=y[n]*x[n]$$

Associativity
結合性

$$(x[n]*y[n])*z[n]=x[n]*(y[n]*z[n])$$

Distributivity
分配性

$$(x[n]+y[n])*z[n]=x[n]*z[n]+y[n]*z[n]$$

System Interconnections

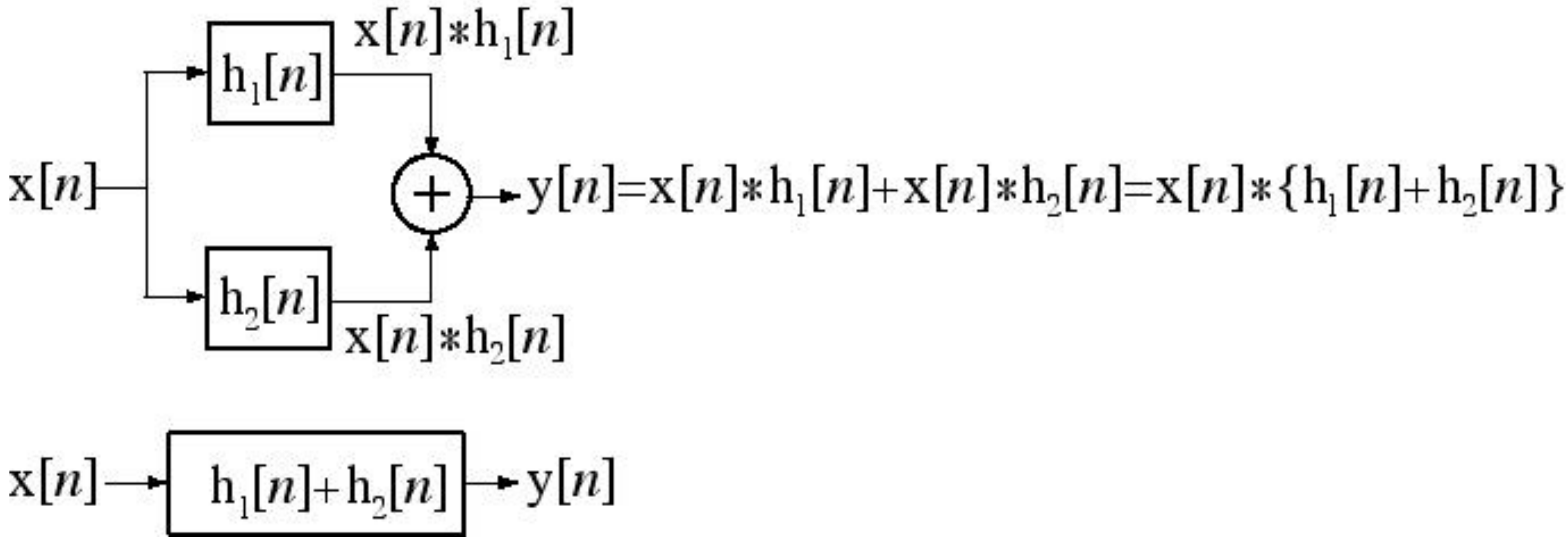
If the response of one system is the excitation of another system, the two systems are said to be *cascade* connected

$$x[n] \rightarrow \boxed{h_1[n]} \rightarrow x[n] * h_1[n] \rightarrow \boxed{h_2[n]} \rightarrow y[n] = \{x[n] * h_1[n]\} * h_2[n]$$

$$x[n] \rightarrow \boxed{h_1[n] * h_2[n]} \rightarrow y[n]$$

System Interconnections

If two systems are excited by the same excitation signal and their responses are added, they are said to be *parallel* connected.



Stability and Impulse Response

It can be shown that a BIBO-stable DT system has an impulse response that is absolutely summable below.

$$\sum_{n=-\infty}^{\infty} |h[n]| \leq k$$

k is a finite real constant.

Unit Step Response $s[n]$

$$u[n] \longrightarrow \boxed{h[n]} \longrightarrow s[n]$$

The unit step response of a discrete-time LTI system is the convolution of the unit step with the impulse response:

$$s[n] = u[n] * h[n] = h[n] * u[n]$$

This means that $s[n]$ is the response to the input $h[n]$ of a discrete-time LTI system with unit impulse response $u[n]$.

$$h[n] \longrightarrow \boxed{u[n]} \longrightarrow s[n]$$

Relationship between h[n] and $s[n]$

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k]$$

Since $u[n\text{-}k]$ is 0 for $n\text{-}k<0$, i.e. $k>n$ and 1 for $n\text{-}k>0$, i.e. $k<=n$.
Therefore,

$$s[n] = \sum_{k=-\infty}^{n} h[k] \qquad s[n-1] = \sum_{k=-\infty}^{n-1} h[k]$$

$$s[n] - s[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k] = h[n] + \sum_{k=-\infty}^{n-1} h[k] - \sum_{k=-\infty}^{n-1} h[k]$$

$$h[n] = s[n] - s[n-1]$$

$h[n]$ can be obtained from $s[n]$.
The impulse response is the first difference of its step response.

Complex Exponential Response

Let a DT LTI system be excited by a complex exponential of the form,

$$x[n] = z^n$$

The response is the convolution of the excitation with the impulse response

$$y[n] = \sum_{m=-\infty}^{\infty} z^m h[n-m] = \sum_{m=-\infty}^{\infty} z^{n-m} h[m]$$

which can be written as

$$y[n] = z^n \underbrace{\sum_{m=-\infty}^{\infty} h[m] z^{-m}}_{\text{complex constant}}$$

Independent on n

Complex Exponential Response

The response of a DT LTI system to a complex exponential excitation is another complex exponential of the same functional form but multiplied by a complex constant. That complex constant is

$$\sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

This is the *z transform* of the impulse response, one of the important transform methods.
It will be discussed later.

Difference Equation

$$y[n] = \boldsymbol{T}\{x[n]\}$$

DT LTI systems are described mathematically by difference equations of the general form,

$$a_n y[n] + a_{n-1} y[n-1] + \ldots + a_{n-M} y[n-M] =$$
$$b_n x[n] + b_{n-1} x[n-1] + \ldots + b_{n-N} x[n-N]$$

An example of a DT LTI system

$$y[n] = x[n] - 3y[n-1] + 2y[n-2]$$

Block Diagrams

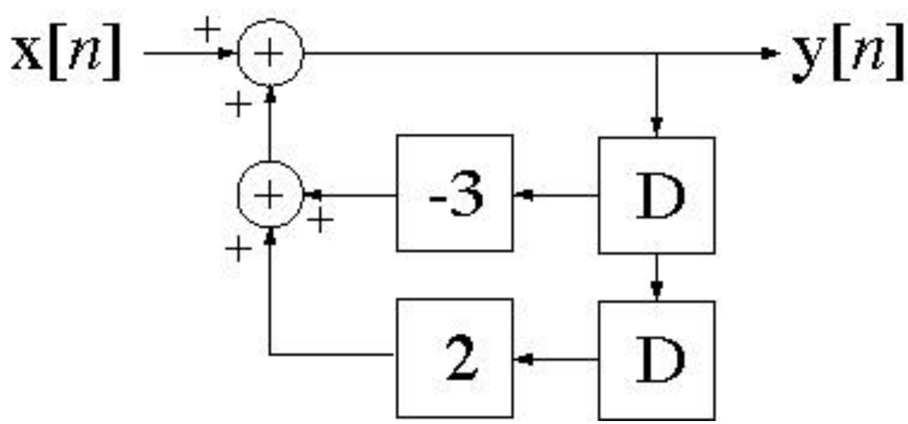
A very useful method for describing and analyzing systems is the *block diagram*. It can be drawn directly from the difference equation which describes the system.

For example, if the system is described by

$$y[n]+3y[n-1]-2y[n-2]=x[n]$$

or $$y[n]=x[n]-3y[n-1]+2y[n-2]$$

It can also be described by the block diagram below in which "D" represents a delay of one unit discrete time.



Remarks

- The convolution method for finding the response of a system to an input/excitation takes advantage of the *linearity* and *time-invariance* of the system
- Input/excitation is represented as a linear combination of an *impulse sequence*
- Output/response is represented as a linear combination of *impulse responses*
- Block diagram is used for describing and analyzing systems

Review of Signals & Systems

Impulse input &
Impulse response

Input signal $x[n]$ → System $h[n]$ → Output signal $y[n]$

Convolution

Decompose into
simple *terms*

"transform"

Combination of
impulses

Combination of
complex sinusoids
(Fourier series)

Fourier
Transform

Laplace
Transform

- This is the big picture of our course
- We will re-visit this figure later