Optimization Theory and Methods

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Chapter1 Optimization Foundations

Introduction to optimization methods

1. Optimality Conditions

2. Introduction to Optimization Methods and Properties

3. Convergence Rate of Iterative Sequence

1. Optimality Conditions

Nonlinear Programming（NP)) problem:

$$(\mathrm{NP}):\begin{cases}\min\ f(x),\\ \text{s.t.}\quad c_i(x)=0, i\in E=\{1,2,\cdots m'\},\\ \qquad\quad c_i(x)\ge 0, i\in I=\{m'+1,\cdots m\}.\end{cases}$$

Denote $D=\{x \mid c_i(x)=0, i\in E, c_i(x)\ge 0, i\in I\}$

-constraint set or feasible domain

Def.1. **Let** $x^* \in D$. **For** $\forall x\in D$

(1) If $f(x)\ge f(x^*)$, **then** x^*

is a global optimizer (minimizer) of NP.

(2) If $f(x)> f(x^*)(x\ne x^*)$ **then** x^*

is a strictly global optimizer (minimizer) of NP.

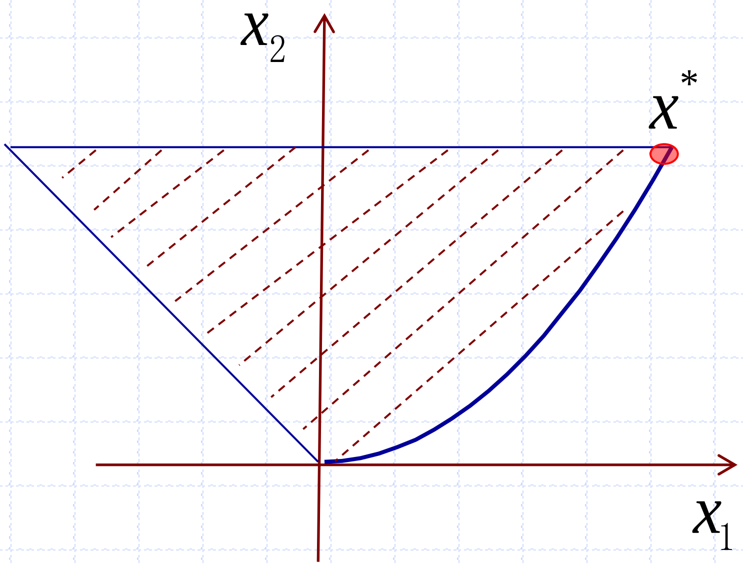
Ex1. Influence of constraints

$$\min\ f(x) = -x_1^3$$

$$s.t.\quad -x_1^2 + x_2 \geq 0 \qquad (1.1)$$

$$x_1 + x_2 \geq 0 \qquad (1.2)$$

$$1 - x_2 \geq 0 \qquad (1.3)$$



Extreme minimizer: $x^* = (1,\ 1)^T$

If constraint(1.2)has a slight perturbation,

$x^* = (1,\ 1)^T$ **is still the extreme minimizer.**

If constraint(1.1) or（1.3）is slightly perturbed,

the extreme minimizer would be changed.

Denote $N_\delta\left(x^*\right)=\left\{x \mid \left\|x-x^*\right\|<\delta\right\}$

Def.2. **Let** $x^* \in D$ **and** $\delta>0.$

(1) **If** $f(x) \geq f\left(x^*\right), \forall x \in D \cap N_\delta\left(x^*\right)$ **then** x^* **is said to be a local optimizer (minimizer)of NP problem.**

(2) **If** $f(x)>f\left(x^*\right), \forall x \in D \cap N_\delta\left(x^*\right), x \neq x^*$ **then** x^* **is said to be a strictly local optimizer (minimizer)of NP.**

Def.3. **Let** $f: R^n \rightarrow R^1$ **be differentiable at** $x \in R^n$.

If there exist a vector $0 \neq d \in R^n$ **and** $\bar{\alpha}>0$

s.t. $f(x+\alpha d)<f(x), \forall \alpha \in(0, \bar{\alpha}),$

then vector d **is named as a descent direction of function** $f(x)$ **at the point** x.

1.1. First-order Necessary Conditions

Th.1. Let $f: R^n \to R^1$ be differentiable at $x_k \in R^n$.

d is a descent direction $\Longleftrightarrow$ $\nabla f(x_k)^{\mathrm{T}} d < 0.$

Proof: $\Longleftarrow$ By means of Taylor's expansion, for $\alpha > 0$,

$$f(x_k + \alpha d) = f(x_k) + \alpha \nabla f(x_k)^{\mathrm{T}} d + o(\|\alpha d\|)$$

For $\varepsilon = -(\nabla f(x_k))^{\mathrm{T}} d$, $\exists \bar{\alpha} > 0$, s.t. for $\alpha < \bar{\alpha}$ we have

$$\frac{o(\|\alpha d\|)}{\alpha} < \frac{1}{2}\varepsilon \quad \text{that is} \quad o(\|\alpha d\|) < \frac{1}{2}\alpha\varepsilon.$$

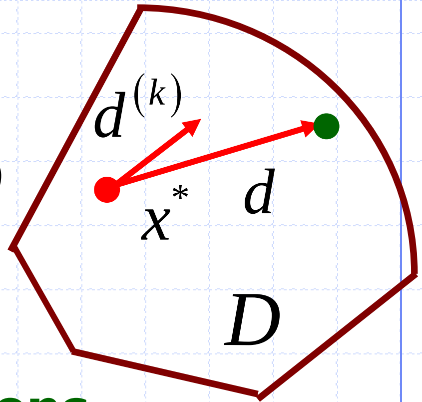
Thus
$$f(x_k + \alpha d) \le f(x_k) - \alpha\varepsilon + \frac{1}{2}\alpha\varepsilon < f(x_k).$$

$\Longrightarrow$ From sign reservation of continuous fcn, obvious.

Corollary1.

$-\nabla f(x)$ is a descent direction of $f(x)$ at x.

Def.4. (Feasible direction) Let $x^* \in D, 0 \neq d \in R^n$, d is called as a feasible direction at x^* w.r.t set D if $\exists \bar{\alpha} > 0$ s.t. $x^* + \alpha d \in D, \forall \alpha \in [0, \bar{\alpha}]$.



Denote $FD(x^*, D)$ as the set of all feasible directions at x^* w.r.t set D.

Def.5. (Sequential feasible direction) Let $x^* \in D, d \in R^n$, d is called a sequential feasible direction at x^* w.r.t the set D if there exist sequences $d^{(k)} \neq 0\ (k = 1, 2, \cdots)$ and $\alpha_k > 0\ (k = 1, 2, \cdots)$ s.t. $x^* + \alpha_k d^{(k)} \in D, k = 1, 2, \cdots$, $\lim\limits_{k\to\infty} d^{(k)} = d$, and $\lim\limits_{k\to\infty} \alpha_k = 0$.

Denote $SFD(x^*, D)$ as the set of all sequential feasible directions at x^* w.r.t the set D.

Def.6 . Effective constraint index set

For NP:
$$\begin{cases} \min\ f(x), \\ \text{s. t.}\quad c_i(x)=0, i\in E=\{1,2,\cdots m'\}, \\ \qquad\quad c_i(x)\ge 0, i\in I=\{m'+1,\cdots m\}. \end{cases}$$

Index set $I^*=\{i \mid i\in I=\{m'+1,\cdots m\}, c_i(x^*)=0\}$ **is called an active constraint index set at** x^*.

Def.7. Linearized feasible direction

Let $x^*\in D$. d **is called a linearized feasible direction at** x^*

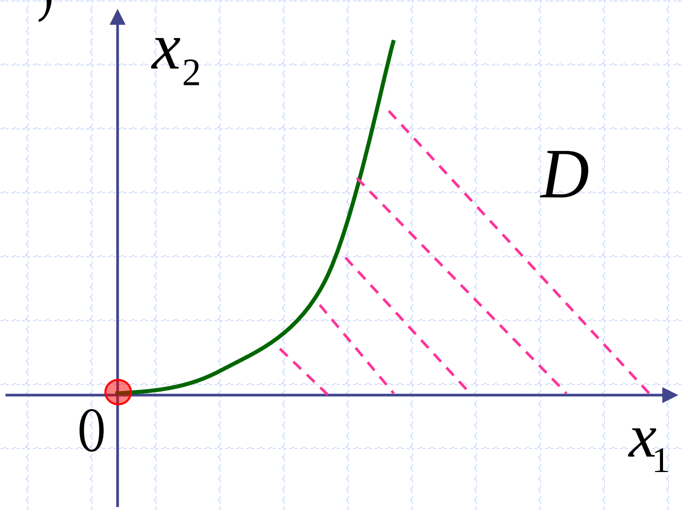
if $d^{\mathrm{T}}\nabla c_i(x^*)=0, i\in E$ **and** $d^{\mathrm{T}}\nabla c_i(x^*)\ge 0, i\in I^*$.

Denote $LFD(x^*,D)$ **as the set of all sequential feasible directions at** x^* **w.r.t the set** D.

Ex2. **Let** $D=\{(x_1,x_2)\mid x_1^3-x_2\geq 0, x_2\geq 0\},\ x^*=[0,0]^{\mathrm{T}}.$

Solve $SFD(x^*,D),\quad LFD(x^*,D).$

Solution: $SFD(x^*,D)=\left\{[a,0]^{\mathrm{T}},a>0\right\}$



Because $\nabla c_1(x^*)=[0,-1]^{\mathrm{T}},$

$\nabla c_2(x^*)=[0,1]^{\mathrm{T}}.$

Therefore

$LFD(x^*,D)=\left\{[b,0]^{\mathrm{T}},b\neq 0\right\}$

Obviously $SFD(x^*,D)\neq LFD(x^*,D),$

But $SFD(x^*,D)\subset LFD(x^*,D)$

Lemma1. Let all constraints fcns $c_i(x)(i \in E \bigcup I)$ be continuously differentiable at $x^* \in D$.

Then $FD(x^*,D) \subset SFD(x^*,D) \subset LFD(x^*,D)$.

Denote $D^* = D(x^*) = \{d \mid d^{\mathrm{T}} \nabla f(x^*) < 0, d \in R^n\}$

Th.2.（Optimality condition）

Let $x^* \in D$ be a local minimizer of NP and fcns $f(x)$ and $c_i(x)(i \in E \bigcup I)$ be continuously differentiable at x^*. Then

$$SFD(x^*,D) \bigcap D^* = \varnothing.$$

Proof:

$\forall d \in SFD(x^*,D), \exists x^{(k)} = x^* + \alpha_k d^{(k)}, s.t. \alpha_k \to 0, d^{(k)} \to d$ and

$$f(x^{(k)}) = f(x^* + \alpha_k d^{(k)}) = f(x^*) + \alpha_k d^{(k)\mathrm{T}} \nabla f(x^*) + o(\|\alpha_k d^{(k)}\|) \geq f(x^*)$$

Then $d^{\mathrm{T}} \nabla f(x^*) \geq 0$. That is $d \notin D(x^*)$.

Th.3.(1st-order Kuhn-Tucker Necessity conditions)

Let x^* **be a local minimizer of NP.** $f(x)$ **and** $c_i(x)(i \in E \cup I)$ **are continuously differentiable at** x^*. **If** $SFD(x^*, D) = LFD(x^*, D)$, **then there exists such a vector** $\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_m^*)^{\mathrm{T}}$ **that**

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla c_i(x^*) = 0, i \in E \cup I, \quad \lambda_i^* \geq 0, i \in I, \quad \lambda_i^* c_i(x^*) = 0, i \in I.$$

Proof: $d^{\mathrm{T}} \nabla f(x^*) \geq 0, \forall d \in SFD(x^*, D) = LFD(x^*, D).$

Let $A_1 = \left[\nabla c_1(x^*), \cdots, \nabla c_{m'}(x^*)\right]$, $A_2 = \left[-\nabla c_1(x^*), \cdots, -\nabla c_{m'}(x^*)\right]$,

$A_3 = \left[-\nabla c_{i_1}(x^*) \cdots, -\nabla c_{i_s}(x^*)\right]$, $i_1, \cdots, i_s \in I^*$, $b = -\nabla f(x^*)$. **Then**

$$\begin{cases} [A_1, A_2, A_3]^{\mathrm{T}} d \leq 0, \\ b^T d > 0. \end{cases} \text{ no solution } \Longleftrightarrow \begin{cases} \nabla c_i(x^*)^{\mathrm{T}} d = 0, i \in E, \\ \nabla c_i(x^*)^{\mathrm{T}} d \geq 0, i \in I^*, \\ b^T d > 0. \end{cases} \text{ no solution}$$

From Farkas Lemma, $\exists y = \begin{bmatrix} \mu^{*-} \\ \mu^{*+} \\ \omega^* \end{bmatrix} \geq 0$ so that

$$\left[A_1, A_2, A_3\right] y = b = -\nabla f\left(x^*\right).$$

$\left(x^{*T}, \lambda^{*T}\right)$

Kuhn-Tucker pair

That is $-\nabla f\left(x^*\right) = A_1\mu^{*-} + A_2\mu^{*+} + A_3\omega^*$

Kuhn-Tucker point

$$= \sum_{i\in E} \mu_i^{*-}\nabla c_i\left(x^*\right) - \sum_{i\in E} \mu_i^{*+}\nabla c_i\left(x^*\right) - \sum_{i\in I^*} \omega_i^*\nabla c_i\left(x^*\right)$$

i.e. $\nabla f\left(x^*\right) = \sum_{i\in E}\left(\mu_i^{*+} - \mu_i^{*-}\right)\nabla c_i\left(x^*\right) + \sum_{i\in I^*} \omega_i^*\nabla c_i\left(x^*\right)$

K-T conditions

Let $\lambda_i^* = \mu_i^{*+} - \mu_i^{*-}, i \in E;\quad \lambda_i^* = \omega_i^*, i \in I^*;\quad \lambda_i^* = 0, i \in I \setminus I^*.$

Then

$$\nabla f\left(\boxed{x^*}\right) - \sum_{i=1}^{m} \lambda_i^*\nabla c_i\left(x^*\right) = 0, i \in E \cup I,$$

$$\lambda_i^* \geq 0, i \in I, \quad \boxed{\lambda_i^* c_i\left(x^*\right) = 0, i \in I.}$$

Complementary relaxed conds.

Remarks:

1. If $SFD(x^*, D) = LFD(x^*, D)$ then x^* is Kuhn-Tucker pnt.

But not easy to testify. Can be replaced by simple condition later.

If $SFD(x^*, D) \neq LFD(x^*, D)$ then x^* is not definitely a K-T pnt.

2. Define $n+m$ -variable fcn $L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i c_i(x)$

Then $L(x, \lambda)$ is called Lagrangian fcn of NP.

Thus, Eq in K-T cond $\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla c_i(x^*) = 0, i \in E \cup I,$

is expressed as $\nabla_x L(x^*, \lambda^*) = 0.$ λ^* is Lagrangian multiplier

3. Let $f(x), x \in N(x^*)$ be continuously differentiable.

If $f(x^*) = \min_{x \in R^n} f(x)$ then $\nabla f(x^*) = 0.$

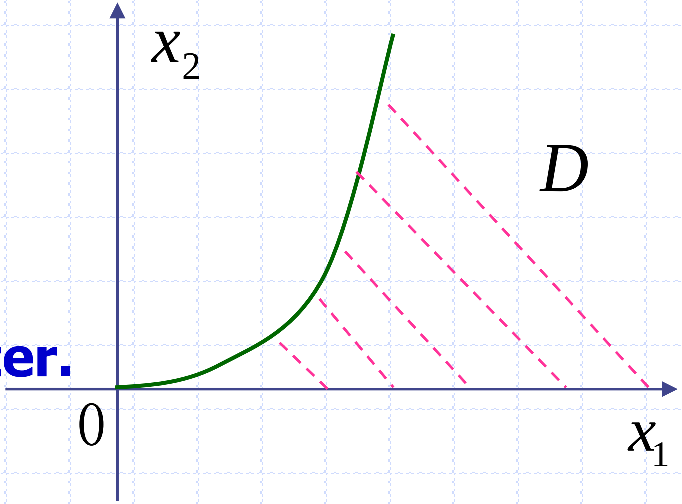
The constraint conditions are sufficient but not necessary.

Ex3.

$$\min\ x_1$$

$$s.t.\ x_1^3 - x_2 \geq 0$$

$$x_2 \geq 0$$



Obviously, $x^* = [0,0]^{\mathrm{T}}$ **is the minimizer.**

As $\nabla f(x^*) = [1,0]^{\mathrm{T}}$,

$$\nabla c_1(x^*) = [0,-1]^{\mathrm{T}}, \quad \nabla c_2(x^*) = [0,1]^{\mathrm{T}}.$$

Therefore,

no matter what values $\lambda_1^*,\ \lambda_2^*$ **are chosen, we have**

$$\nabla f(x^*) \neq \lambda_1^* \nabla c_1(x^*) + \lambda_2^* \nabla c_2(x^*)$$

Th.4. **At the local minimizer x^* if any of the following cond. holds**

(1) $c_i(x)\left(i\in E\bigcup I^*\right)$ **are linear.**

(2) $\nabla c_i\left(x^*\right)\left(i\in E\bigcup I^*\right)$ **are linearly independent.**

Then $SFD\left(x^*,D\right)=LFD\left(x^*,D\right)$

By Lemma1, $SFD\left(x^*,D\right)\subset LFD\left(x^*,D\right).$

Proof: **(1)** $\forall d\in LFD\left(x^*,D\right)$ **then** $d\neq 0$

and $\nabla c_i\left(x^*\right)^{\mathrm{T}}d=0,i\in E,\quad \nabla c_i\left(x^*\right)^{\mathrm{T}}d\geq 0,i\in I^*.$ **Let** $x^{(k)}=x^*+\alpha_k d^{(k)}$

where $d^{(k)}=d\to d\left(k\to\infty\right),\quad \forall 0<\alpha_k\to 0\left(k\to\infty\right).$

Then $c_i\left(x^{(k)}\right)=c_i\left(x^*+\alpha_k d\right)=c_i\left(x^*\right)+\alpha_k d^{\mathrm{T}}\nabla c_i\left(x^*\right).$ **That is**

$$c_i\left(x^{(k)}\right)=\alpha_k d^{\mathrm{T}}\nabla c_i\left(x^*\right)=0,i\in E,\qquad c_i\left(x^{(k)}\right)=\alpha_k d^{\mathrm{T}}\nabla c_i\left(x^*\right)\geq 0,i\in I^*,$$

$$c_i\left(x^{(k)}\right)=c_i\left(x^*\right)+\alpha_k d^{\mathrm{T}}\nabla c_i\left(x^*+\theta\alpha_k d\right)\geq 0,\left(0<\theta<1,\alpha_k<<1\right),\ i\in I\setminus I^*.$$

Thus $d\in SFD\left(x^*,D\right)$ **Therefore** $LFD\left(x^*,D\right)\subset SFD\left(x^*,D\right).$

(2) $\nabla c_i\left(x^*\right)\left(i \in E \bigcup I^*\right)$ **are linearly independent.**

Proof: $\forall d \in LFD\left(x^*, D\right)$ **Then** $d \neq 0$ **and** $\nabla c_i\left(x^*\right)^{\mathrm{T}} d = 0, i \in E,$

$\nabla c_i\left(x^*\right)^{\mathrm{T}} d \geq 0, i \in I^*.$ **Denote** $E \bigcup I^* = \{1, 2, \cdots, l\}, m' \leq l \leq n.$

From $\nabla c_i\left(x^*\right)\left(i \in E \bigcup I^*\right)$ **independence,** $\exists b_{l+1}, \cdots, b_n \in R^n$ **s.t.**

$\nabla c_1\left(x^*\right), \cdots, \nabla c_l\left(x^*\right), b_{l+1}, \cdots, b_n$ **are independent.**

Then the Eq

$$\begin{cases} c_i\left(x\right) - d^{\mathrm{T}} \nabla c_i\left(x^*\right) \theta = 0, i = 1, 2, \cdots, l; \\ \left(x - x^*\right)^{\mathrm{T}} b_i - d^{\mathrm{T}} b_i \theta = 0, i = l+1, \cdots, n. \end{cases} \qquad (**)$$

has a solution $\begin{bmatrix} x^* \\ 0 \end{bmatrix}$ **w.r.t variables** $\left[x^{\mathrm{T}}, \theta\right]^{\mathrm{T}} \in R^{n+1}$

and Jacobian matrix is nonsingular at x^*. **Thus Eq (**) can**

determine fcn $x = x(\theta), \theta \in N(0)$ **at neighborhood of** $\begin{bmatrix} x^* \\ 0 \end{bmatrix}$

and $x^* = x(0).$

Differentiating both sides of Eq (**) w.r.t θ yields

$$\begin{cases} \nabla c_i\left(x^*\right)^{\mathrm{T}}\left(\left.\dfrac{dx}{d\theta}\right|_{\theta=0}-d\right)=0, i=1,2,\cdots,l; \\ b_i^{\mathrm{T}}\left(\left.\dfrac{dx}{d\theta}\right|_{\theta=0}-d\right)=0, i=l+1,\cdots,n. \end{cases}$$

has a unique solution $\left.\dfrac{dx}{d\theta}\right|_{\theta=0}=d.$

By mean theorem $x=x(\theta)=x(0)+\left.\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\right|_{\theta=0+a(\theta)\theta}\theta=x^*+\theta d(\theta),\quad 0<a(\theta)<1.$

Specifically, choose

$$0<\theta_k\to 0(k\to\infty),\quad x^{(k)}=x^*+\theta_k d(\theta_k)=x^*+\theta_k d_k$$ **where**

$$d_k=d(\theta_k)=\left.\frac{\mathrm{d}x(\theta)}{\mathrm{d}\theta}\right|_{\theta=a(\theta_k)\theta_k}\to\left.\frac{\mathrm{d}x(\theta)}{\mathrm{d}\theta}\right|_{\theta=0}=d.$$ **Substituting** $\left[x^{(k)\mathrm{T}},\theta_k\right]^{\mathrm{T}}$ **into Eq makes**

$$c_i\left(x^{(k)}\right)=\theta_k d_k^{\mathrm{T}}\nabla c_i\left(x^*\right)=0, i\in E,\ c_i\left(x^{(k)}\right)=\theta_k d_k^{\mathrm{T}}\nabla c_i\left(x^*\right)\geq 0, i\in I^*,$$

$$c_i\left(x^{(k)}\right)=c_i\left(x^*\right)+\theta_k d_k^{\mathrm{T}}\nabla c_i\left(x^*\right)+o\left(\left\|\theta_k d_k\right\|\right)\geq 0,\left(i\in I\setminus I^*\right)$$

Thus $d\in SFD\left(x^*,D\right)$ **Therefore** $LFD\left(x^*,D\right)\subset SFD\left(x^*,D\right).$

Th.5. At the local minimizer x^* if any of the following cond. holds

(1) $c_i(x)\left(i \in E \cup I^*\right)$ are linear.

(2) $\nabla c_i\left(x^*\right)\left(i \in E \cup I^*\right)$ are linearly independent.

Then there exists such a vector $\lambda^* = \left(\lambda_1^*, \lambda_2^*, \cdots, \lambda_m^*\right)^{\mathrm{T}}$ that

$$\nabla f\left(x^*\right) - \sum_{i=1}^{m} \lambda_i^* \nabla c_i\left(x^*\right) = 0, i \in E \cup I,$$

$$\lambda_i^* \geq 0, i \in I, \quad \lambda_i^* c_i\left(x^*\right) = 0, i \in I.$$

Th.6.（1st-order sufficiency cond. of an optimizer）

$x^* \in D$ is a strictly local minimizer if $f(x)$ and $c_i(x)(i \in E \cup I)$ are 1st-order continuously differentiable at x^* and

$$d^{\mathrm{T}} \nabla f\left(x^*\right) > 0, \forall d \in SFD\left(x^*, D\right) \subset LFD\left(x^*, D\right)$$

1.2. 2nd-order Optimization Conditions

The sufficiency requires the 2-nd-order derivatives of the objective fcn and the constraints fcns.

Reminder: Let x^* be K--T point. Then Lagrangian multiplier

satisfying $\nabla f\left(x^*\right)-\sum_{i=1}^{m}\boxed{\lambda_i^*}\nabla c_i\left(x^*\right)=0, i\in E\bigcup I,\quad \lambda_i^*\geq 0, i\in I,$

and $\boxed{\lambda_i^*}c_i\left(x^*\right)=0, i\in I.$ Then $\lambda_i^*\geq 0, i\in I^*.$

Thus for some index *i* $\boxed{\lambda_i^*=0}$ or $\boxed{\lambda_i^*>0}$ $\Rightarrow$ effective on cond.

$\Downarrow$

no effect on cond.

Denote $I_+^*=I_+\left(x^*\right)=\left\{i\middle| i\in I^*,\lambda_i^*>0\right\}$ Then $A_+(x^*)=E\bigcup I_+(x^*)$

is called a strongly effective constraints index set at x^*.

or a strongly effective set.

Def.8. Sequential nullity constraints direction

$$G\left(x^*,\lambda^*\right)\subseteq LFD\left(x^*,D\right)$$
$$S\left(x^*,\lambda^*\right)\subseteq SFD\left(x^*,D\right)$$
$$S\left(x^*,\lambda^*\right)\subseteq G\left(x^*,\lambda^*\right)$$

Let $\left(x^{*\mathrm{T}},\lambda^{*\mathrm{T}}\right)$ **be Kuhn-Tucker pair of NP.**

d **is a sequential nullity constraints direction at** x^* **if there exist sequences**

$$d^{(k)}\neq 0\left(k=1,2,\cdots\right) \quad \textbf{and} \quad \alpha_k>0\left(k=1,2,\cdots\right).$$

so that $x^{(k)}=x^*+\alpha_k d^{(k)}\in D, k=1,2,\cdots$ **satisfying**

$$c_i\left(x^{(k)}\right)=0, i\in E\bigcup I_+^* \quad \textbf{and} \quad c_i\left(x^{(k)}\right)\geq 0, i\in I\setminus I_+^*$$

together with $\lim\limits_{k\to\infty} d^{(k)}=d$ **and** $\lim\limits_{k\to\infty}\alpha_k=0.$

Denote $S\left(x^*,\lambda^*\right)$ **as a set of all SNCDs at** x^* **and**

$$G\left(x^*,\lambda^*\right)=\left\{d \,\middle|\, d\neq 0, d^{\mathrm{T}}\nabla c_i\left(x^*\right)=0\left(i\in E\bigcup I_+^*\right), d^{\mathrm{T}}\nabla c_i\left(x^*\right)\geq 0\left(i\in I^*\setminus I_+^*\right)\right\}$$

as a set of all linearized nullity constraints directions at x^*

Th.7. (2nd-order Kuhn-Tucker Necessity Conditions)

Let x^* **be a local minimizer of NP.** $f(x)$ **and** $c_i(x)(i \in E \bigcup I)$ **are 2nd-order continuously differentiable at** x^*.

If $SFD(x^*, D) = LFD(x^*, D)$, **then there exists such a vector** $\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_m^*)^{\mathrm{T}}$ **that**

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla c_i(x^*) = 0, i \in E \bigcup I,$$

$$\lambda_i^* \geq 0, i \in I, \qquad \lambda_i^* c_i(x^*) = 0, i \in I.$$

$$d^{\mathrm{T}} \nabla_{xx}^2 L(x^*, \lambda^*) d \geq 0, \forall d \in S(x^*, \lambda^*).$$

where $L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i c_i(x)$

$$\nabla_{xx}^2 L(x^*, \lambda^*) = \nabla^2 f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla^2 c_i(x^*)$$

Lemma2:

At a local optimizer x^* if any of the following holds

(1) $c_i(x)\left(i \in E \cup I^*\right)$ **are linear.**

(2) $\nabla c_i\left(x^*\right)\left(i \in E \cup I^*\right)$ **are linearly independent.**

Then $G\left(x^*, \lambda^*\right) = S\left(x^*, \lambda^*\right).$

Th.8. (2nd-order Necessity Conditions)

Let x^* be a local minimizer of NP. $f(x)$ **and** $c_i(x)(i \in E \cup I)$ **are 2nd-order continuously differentiable at** x^*. **If**

(1) $c_i(x)(i \in E \cup I^*)$ **are linear. or**

(2) $\nabla c_i(x^*)(i \in E \cup I^*)$ **are linearly independent.**

Then there exists a vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_m^*)^{\mathrm{T}}$ **such that**

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla c_i(x^*) = 0, i \in E \cup I,$$

$$\lambda_i^* \geq 0, i \in I, \qquad \lambda_i^* c_i(x^*) = 0, i \in I.$$

$$d^{\mathrm{T}} \nabla_{xx}^2 L(x^*, \lambda^*) d \geq 0, \forall d \in S(x^*, \lambda^*).$$

where

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i c_i(x)$$

$$\nabla_{xx}^2 L(x^*, \lambda^*) = \nabla^2 f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla^2 c_i(x^*)$$

Th.9. (2nd-order Sufficiency Conditions)

Let $\left(x^{*\mathrm{T}}, \lambda^{*\mathrm{T}}\right)$ be Kuhn-Tucker pair of NP.

If $f(x)$ and $c_i(x)(i \in E \bigcup I)$ are 2nd-order continuously differentiable at x^*

and $d^{\mathrm{T}} \nabla_{xx}^2 L\left(x^*, \lambda^*\right) d > 0, \forall d \in G\left(x^*, \lambda^*\right)$.

Then x^* is a strictly local optimizer.

Corollary2.(2nd-order unconstraint sufficiency cond.)

If $f(x)$ is 2nd-order continuously differentiable in nbhd $N\left(x^*\right)$ along with $\nabla f\left(x^*\right) = 0$ and $\nabla^2 f\left(x^*\right)$ positive definite.

Then x^* is a strictly local optimizer of $\min\limits_{x \in R^n.} f(x)$.

1.3. Convex Programming (CP)

For Nonlinear Programming（NP)):

$$\min f(x), \text{s. t.} \begin{cases} c_i(x)=0, i\in E=\{1,2,\cdots m'\}, \\ c_i(x)\ge 0, i\in I=\{m'+1,\cdots m\}. \end{cases}$$

$$D=\{x \mid c_i(x)=0, i\in E, c_i(x)\ge 0, i\in I\}$$

If $f(x)$ **and** D **are convex, then NP is called CP.**

D **is convex if** $c_i(x)(i\in E)$ **linear and** $c_i(x)(i\in I)$ **concave.**

The course addresses CP on Convex Domain.

Ex4. **（NP) :**

$$\begin{cases} \min f(x,y,z)=2x^2+3y^2+4z^2, \\ \text{s. t.} \quad c_1(x,y,z)=x+y-z+\frac{2}{3}=0, \\ \qquad c_2(x,y,z)=2-(x^2+y^2+z^2)\ge 0. \end{cases}$$

Th.10. A local minimizer of CP is a global minimizer.

Proof: **Let** x^* **be a local minimizer but not a global one.**

Then there exists a neighborhood $N_\delta(x^*) = \{x \mid \|x - x^*\| < \delta\}$

and a point $\bar{x} \neq x^*, \bar{x} \in D$ **but** $\bar{x} \notin N_\delta(x^*)$

such that $f(x^*) \leq f(x), \forall x \in N_\delta(x^*)$ **but** $f(\bar{x}) < f(x^*)$.

Choose $0 < \alpha < \dfrac{\delta}{\|\bar{x} - x^*\|}$ **and** $\tilde{x} = \alpha \bar{x} + (1-\alpha) x^* \in D,$

Then $\|\tilde{x} - x^*\| = \alpha \|\bar{x} - x^*\| < \delta.$ **Thus** $\tilde{x} \in N_\delta(x^*).$

From convexity

$$f(\tilde{x}) = f\left(\alpha\bar{x} + (1-\alpha)x^*\right) \leq \alpha f(\bar{x}) + (1-\alpha) f(x^*) \leq f(x^*).$$

Th.11. Let $f(x)$ and $c_i(x)(i \in E \cup I)$ of CP be continuously differentiable and x^* be K-T point. Then x^* must be a global minimizer.

Proof: Let $\left(x^{*\mathrm{T}}, \lambda^{*\mathrm{T}}\right)$ be Kuhn-Tucker pair. Then

$$f(x) \geq f(x^*) + \nabla f(x^*)^{\mathrm{T}}(x - x^*), \quad c_i(x) = c_i(x^*) + \nabla c_i(x^*)^{\mathrm{T}}(x - x^*)(i \in E)$$

$$-c_i(x) \geq -c_i(x^*) - \nabla c_i(x^*)^{\mathrm{T}}(x - x^*)(i \in I)$$

Thus

$$\begin{aligned} f(x) &\geq f(x) - \sum_{i=1}^{m} \lambda_i^* c_i(x) \\ &\geq f(x^*) + \nabla f(x^*)^{\mathrm{T}}(x - x^*) - \sum \lambda_i^* \left(c_i(x^*) + \nabla c_i(x^*)(x - x^*)\right) \\ &= f(x^*) - \sum_{i=1}^{m} \lambda_i^* c_i(x^*) + \left[\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla c_i(x^*)\right]^{\mathrm{T}}(x - x^*) \\ &= f(x^*) - \sum_{i=1}^{m} \lambda_i^* c_i(x^*) = f(x^*) \end{aligned}$$

Corollary3.

If $f(x)$ of CP is strictly convex, the minimizer is unique.

Slater condition: There exists $\bar{x} \in R^n$ such that

$$c_i(\bar{x}) = 0, i \in E; \text{ and } c_i(\bar{x}) > 0, i \in I.$$

Th.12. Let $f(x)$ and $c_i(x)(i \in E \cup I)$ of CP is continuously differentiable and satisfy Slater condition.

Then

A feasible point x^* is a minimizer

iff x^* is a Kuhn-Tucker point.

Ex5.

NP:

$$\min \; f(x) = f(x_1, x_2) = \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}x_2^{\,2}$$

$$s.t. \quad c(x) = -x_1 + bx_2^{\,2} \geq 0$$ where ***b*** is const.

How to choose constant *b* so that $x^* = [0,0]^{\mathrm{T}}$ **is a local minimizer.**

Solution:

$$D = \{(x_1, x_2)^{\mathrm{T}} \mid c(x) = -x_1 + bx_2^{\,2} \geq 0\}$$

$$c(x^*) = 0, \qquad \nabla f(x^*) = [-1, 0]^{\mathrm{T}}, \qquad \nabla c(x^*) = [-1, 0]^{\mathrm{T}}.$$

$$L(x, \lambda) = \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}x_2^{\,2} - \lambda(-x_1 + bx_2^{\,2})$$

K-T conds:

$$\nabla_x L(x^*, \lambda) = [-1, 0]^{\mathrm{T}} - \lambda[-1, 0]^{\mathrm{T}} = 0$$

$$c(x^*) = 0$$

$$\lambda \geq 0$$

Solving yields $\lambda^* = 1 > 0$. **Thus** $x^* = [0,0]^{\mathrm{T}}$ **is a K-T point.**

Hessian matrix of Lagrangian fcn at *x is** $W^* = \begin{bmatrix} 1 & 0 \\ 0 & 1-2b \end{bmatrix}$

Kinds of feasible directions sets are

$$S(x^*,\lambda^*) = G(x^*,\lambda^*) = \{d = [0,a]^{\mathrm{T}}, a \neq 0\}$$

Note $SFD(x^*,D) = \{d \mid [-1,0]^{\mathrm{T}} d \geq 0, d \neq 0\} = \{[a_1,a_2]^{\mathrm{T}}, a_1 < 0\}$

Therefore

$$[0,a]W^*\begin{bmatrix} 0 \\ a \end{bmatrix} = [0,a]\begin{bmatrix} 1 & 0 \\ 0 & 1-2b \end{bmatrix}\begin{bmatrix} 0 \\ a \end{bmatrix} = a^2(1-2b)$$

Thus

If $b > 0.5$ x^* **is not a local minimizer from 2nd-order necessity.**

If $b < 0.5$ x^* **is a local minimizer from 2nd-order sufficiency.**

If $b = 0.5$

As $f(x) = (1 + \frac{1}{2}x_1^{\ 2}) - x_1 + \frac{1}{2}x_2^{\ 2}$

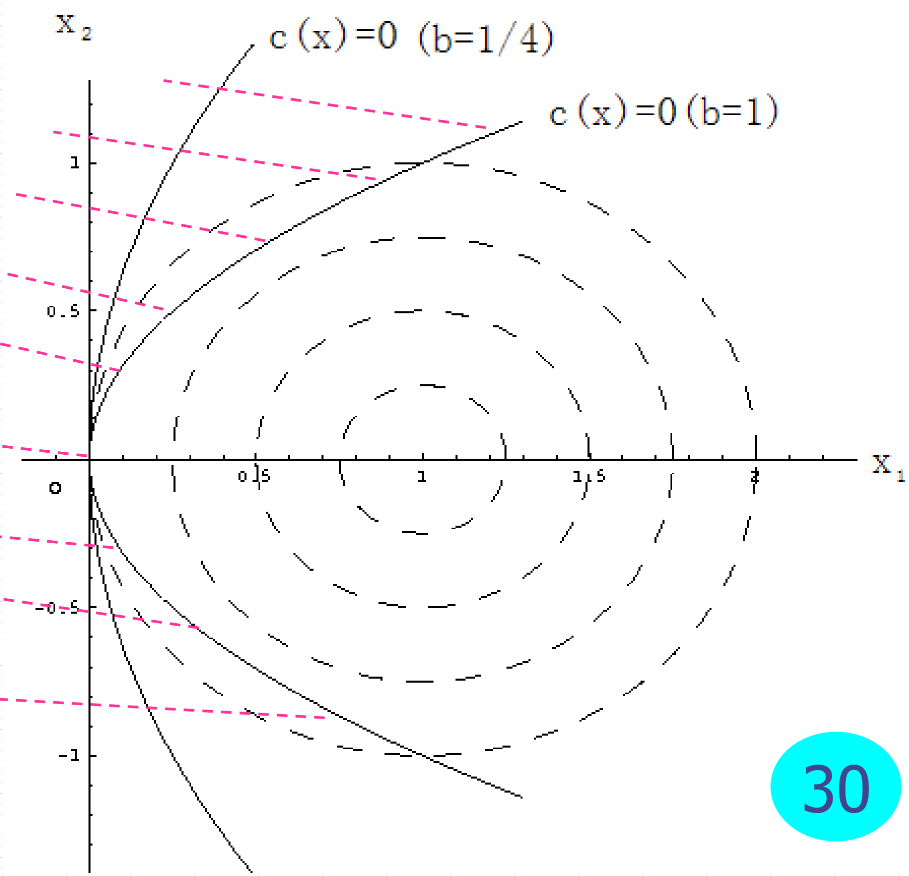
s.t. $c(x) = -x_1 + \frac{1}{2}x_2^{\ 2} \geq 0$

i.e. $f(x) \geq (1 + \frac{1}{2}x_1^{\ 2})$

Then

x^* **is a local minimizer.**

$b - x^*$ **relationship as figured.**



2. Brief Introduction to Optimization Methods and Properties

For Nonlinear Programming（NP）:

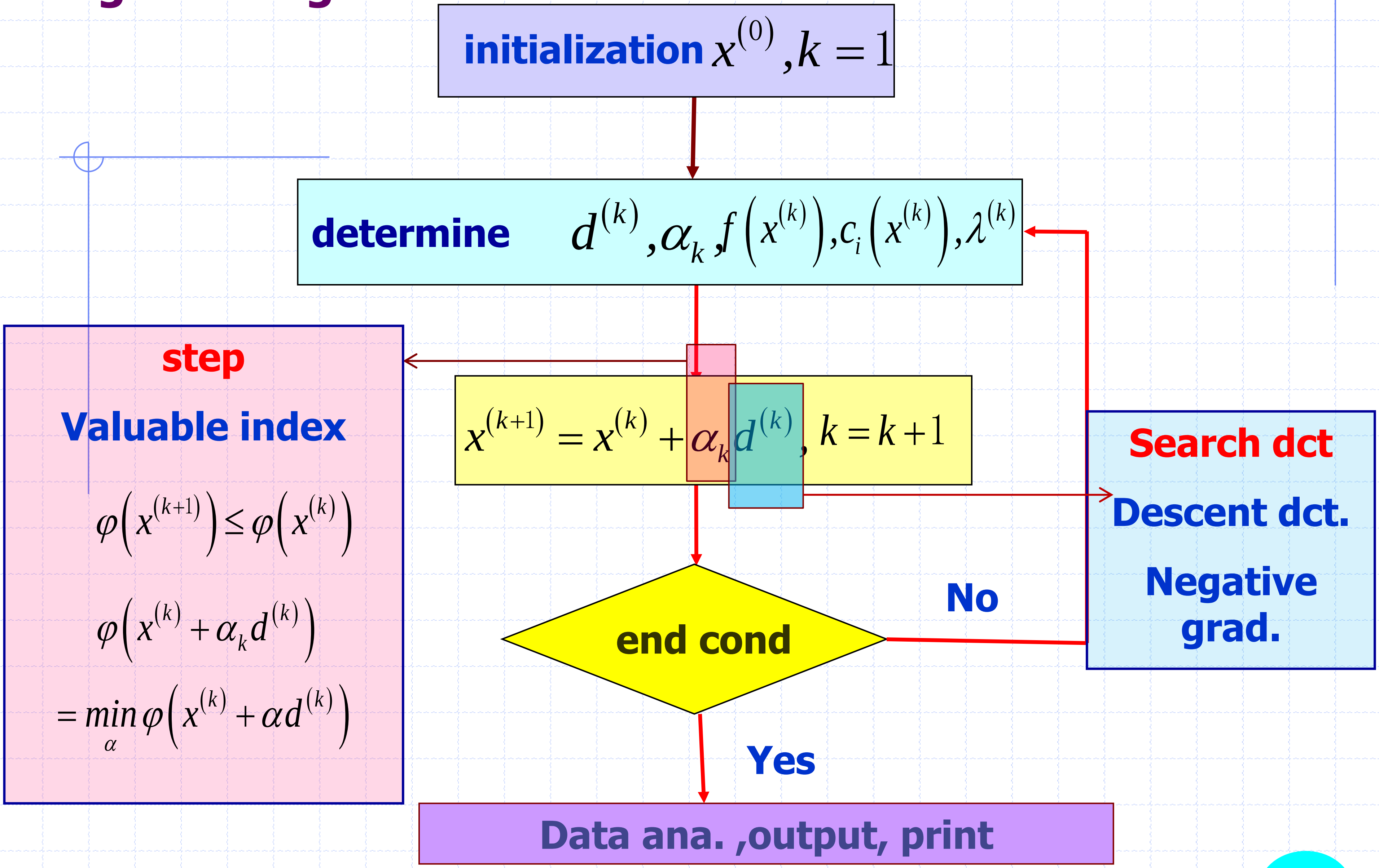
$$(\text{NP}):\quad \begin{cases} \min\ f(x), \\ \text{s. t.}\quad c_i(x)=0, i\in E=\{1,2,\cdots m'\}, \\ \qquad\quad c_i(x)\ge 0, i\in I=\{m'+1,\cdots m\}. \end{cases}$$

Usual method is to solve a Kuhn-Tucker point of following Eqs. by iterative algorithm

$$\nabla f(x^*)-\sum_{i=1}^{m}\lambda_i^*\nabla c_i(x^*)=0, i\in E\cup I,$$

$$\lambda_i^*\ge 0, i\in I, \qquad \lambda_i^* c_i(x^*)=0, i\in I.$$

Programming flow



3. Convergence Rate of Iterative Sequence

Let $\left\{x^{(k)}\right\}, e\left(x^{(k)}\right) = \left\| x^{(k)} - x^{*} \right\| \to 0 \left(k \to \infty\right)$

$$Q_p = \limsup_{k\to\infty} \frac{e\left(x^{(k+1)}\right)}{e\left(x^{(k)}\right)^p}, \quad p \in [1, +\infty) \qquad Q-\textbf{factor}$$

is called Quotient convergence factor of sequence $\left\{x^{(k)}\right\}$.

If $Q_1 = 0, \left\{x^{(k)}\right\}$ **is said to Q super-linearly converges to** x^{*}

If $0 < Q_1 < 1, \left\{x^{(k)}\right\}$ **is said to Q linearly converges to** x^{*}

If $Q_1 = 1, \left\{x^{(k)}\right\}$ **is said to Q sub-linearly converges to** x^{*}

Analogously $Q_2 = 0, 0 < Q_2 < 1, Q_2 = 1$

are respectively Q super-square, square and sub-square convergent.

$$R_p = \begin{cases} \limsup\limits_{k\to\infty} e\left(x^{(k)}\right)^{\frac{1}{k}}, p = 1, \\ \limsup\limits_{k\to\infty} e\left(x^{(k)}\right)^{\frac{1}{p^k}}, p > 1. \end{cases}$$

$R-$ **factor**

is called Root convergence factor of sequence $\left\{x^{(k)}\right\}$.

If $R_1 = 0$, $\left\{x^{(k)}\right\}$ **is said to R superlinearly converges to** x^*

If $0 < R_1 < 1$, $\left\{x^{(k)}\right\}$ **is said to R linearly converges to** x^*

If $R_1 = 1$, $\left\{x^{(k)}\right\}$ **is said to R sublinearly converges to** x^*

Analogously **If** $R_2 = 0, 0 < R_2 < 1, R_2 = 1,$ **sequence** $\left\{x^{(k)}\right\}$

are respectively R super-quadratically, quadratically and sub-quadratic ally convergent.

THANK YOU FOR ATTENDING