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Tracking the Truth in a Multi-Agent System by Belief Revision

Master Thesis

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Approval

This thesis has been prepared over six months at DTU COMPUTE, Department of Applied Mathematics and Computer Science, at the Technical University of Denmark, DTU, in partial fulfilment for the degree Master of Science in Engineering, Master of Science in Computer Science and Engineering.

It is assumed that the reader has a basic knowledge in the area of classical and epistemic logic.

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Abstract

Learning agents are entities that are capable of broadening their knowledge base on outside stimulus. One design for such agents is with the use of belief revision on external observations in order to navigate an epistemic model of possible worlds that eventually leads the agent to identify the true world. For such a system to work a certain amount of observations is necessary, and one way of lessening the amount required for a given agent is by allowing multiple agents to work together.

In this body of work we show how the classical method (belief merge) for merging information in belief revision does not uphold the requirements we would put on a truth tracking multi-agent system. Even belief merging operators that have been shown to perform well with regard to estimating the truth do not retain enough information to keep the same effectiveness as a single learning agent is capable of. With the goal of preserving more information through the information merging, we introduce the concept of plausibility mergers that given the properties of non-overgeneralising and strong majority maintain the effectiveness on certain classes of problems. It should be noted that finding such a plausibility merger is still an open question.

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Contents

1 Introduction

In various formal approaches to rational behaviour, e.g. in Artificial Intelligence or Game Theory, it is crucial to account for the agents being capable of learning new information and incorporating it into their knowledge. One important concern about such methods is whether or not they are reliable, i.e., whether they guarantee getting to an accurate belief about the underlying reality being learned. This question is addressed in the study of *truth-tracking*.

In what follows we will assume that there is a set of statements that are correct or true and that those are expected to be eventually discovered by an agent, but what is required to take up such a task?

One quality is that it must somehow be able to acquire new information, while keeping its set of truths consistent. This concept is what is commonly known as *learning*. So it seems that if we want to create an agent that can track the truth it must be a *learning agent*.

The ability to learn is thought to be one of the cornerstones of intelligence and throughout the field of Artificial Intelligence the concept of what exactly intelligence entails and how to define it is a topic of much deliberation. One notion that seems to recur is that the ability of adapting to information stemming from an external source is an important element. In [1] Pei Wang defines it as

> *Intelligence is the ability for an information processing system to adapt to its environment with insufficient knowledge and resources. — Pei Wang*

Belief revision is a suggestion of how one could use epistemic logic to model learning. The primary problem of belief revision is, given a new piece of information how should an agent modify its current knowledge of the world such that the new observation is consistent with it, all the while keeping as much information as possible. Using belief revision it is possible to create a learning agent by considering all possible worlds and as observations are found the judgement of which world is most likely changes until we have enough information to identify the true world. In essence this is truth tracking.

There exists some quantity of information that is the minimum amount required to do truth tracking, and the only way of lowering that amount for a singular agent is by having it work together with other agents. The question of this piece is what is required for such collective truth tracking and how would it be done.

First we will recall a way of creating a singular learning agent initially introduced by Baltag, Gierasimczuk and Smets (BGS) in [2] and then go on to show a way of merging conjectures using a merging operator based on integrity constraints which was originally proposed by Konieczny and Pino Pérez in [3]. We will then show how an instance of such a merger, namely the majority belief merger, is not sufficient to merge on learning agents with the goal of collective truth tracking.

In [4] D'Alfonso shows how Konieczny and Pino Pérez's mergers are not a good candidate for merging with truth tracking in mind and instead introduces a new merger, the truthlikeness merger, that is a better option for estimating the truth from multiple belief bases. While it might very well be better at estimating the truth, it turns out that this merger too is not sufficient for guaranteeing truth tracking in the same manner as BGS do.

Lastly we introduce a way to merge on total pre-orders instead of belief bases, with the goal of retaining more information through the merging and then go on to show what kind of properties would be required of such a merger to do collective truth tracking.

2 Learning Agents

In [2] BGS shows a method of creating a *learning agent* that, given a model of uncertainty, can learn everything that is learnable within that model. To understand how this is done we first need to grasp how the models are structured.

2.1 The Model

Considering that we want to work with belief revision, so with potentially false beliefs and under uncertainty, agents should have the capability to model not only the actual world, the one they are in, but also other possible worlds. For this purpose epistemic spaces are used.

Definition 1. Let $\mathbb{S} = (S, \mathcal{O})$ be an *epistemic space*, where S is a set of *possible worlds* and $\mathcal{O}$ is a set of *observable propositions* represented by the worlds for which it holds, $\mathcal{O} \subseteq \mathcal{P}(S)$. Both S and $\mathcal{O}$ are at most countable.

$\mathcal{O}$ is the set of observations that the agent could encounter. Each instance of information, or observable proposition, is depicted as sets of all the worlds in S where the given information holds. Observation $O = \{s, t\}$ then holds for the worlds s and t. The set of all observations that are true in a given world s is noted as $\mathcal{O}_s = \{O \in \mathcal{O} \mid s \in O\}$. Thus $\mathcal{O}$ is a subset of the powerset of S, $\mathcal{O} \subseteq \mathcal{P}(S)$.

Belief revision is based on observations that the agents receives from the real world (sensors or other ways of gathering information). Each observation $O \in \mathcal{O}$ is considered to be acquired one at a time.

Definition 2. Let σ be either an ordered infinite stream or finite sequence of observables, $\sigma = (O, \dots)$.

σ is *sound* w.r.t. world s if every observation in σ is in $\mathcal{O}_s$.

σ is *complete* w.r.t. world s if every observation in $\mathcal{O}_s$ is in $set(\sigma)$.

One can now find the worlds that are supported by each observation in the set $\bigcap \sigma$. If this set is a singleton then we are in the trivial case where we can deduce the correct world. However if it is larger, then we are in a case of uncertainty and something more is required. Epistemic spaces are a good way of modelling which worlds are possible, even considering the information the agent has gained, but when there are multiple candidates they do not provide any way to distinguish between more and less likely worlds. To incorporate belief some form of preference between the sets of worlds is required.

Definition 3. Let $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$ be a *plausibility space*, which is acquired by giving an epistemic space a *binary relation set* which is a total pre-order $\preceq \subseteq S \times S$.

The pair or relation $(s_0, s_1) \in \preceq$ then means that s_0 is at least as likely as s_1 and from $\preceq$ being a pre-order, we get that it is reflexive and transitive. The binary relation set can now be applied to epistemic spaces.

2.2 Belief Revision

In order to have agents work their belief of which world is correct towards a particular one, preferably the true one, the model needs to be changeable and it should happen when the agent processes new information.

Definition 4. Given a plausibility space $\mathbb{B}_{\mathbb{S}}$ and some observation $p \in \mathcal{O}$, $R_1(\mathbb{B}_{\mathbb{S}}, p) = \mathbb{B}'_{\mathbb{S}}$ is a *one-step revision method* that returns a new plausibility space.

Iterating over R_1 with an ordered list of observables σ gives a *belief revision method* $R(\mathbb{B}_{\mathbb{S}}, \sigma) = \mathbb{B}'_{\mathbb{S}}$.

$$R(\mathbb{B}_{\mathbb{S}}, \sigma * p) = R_1(R(\mathbb{B}_{\mathbb{S}}, \sigma), p)$$

In the case where σ is empty the function simply returns the same $\mathbb{B}_{\mathbb{S}}$ as in the input.

2.2.1 Belief Revision Methods

Let us go over three revision methods introduced in [2]. They were initially inspired by similar operations in Dynamic Epistemic Logic (DEL) [5], which we will relate them to in case the reader knows of them already. The revisions methods R will be introduced by their one-step revision methods R_1.

The following plausibility space will be used to help explain the concepts by applying a single belief revision on the plausibility space and see how it changes it.

$S = \{s, t, r\}$
$\mathcal{O} = \{p, q\} = \{\{s, t\}, \{t, r\}\}$
$\preceq = \{(t, s), (r, t), (r, s)\}$

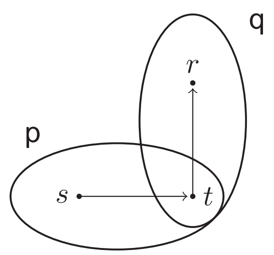

Figure 2.1: Initial plausibility space

The arrows indicate the binary relations of which world is deemed more likely with the world pointed at being preferred.

Conditioning

In DEL the *update* operation takes the new information as fact and removes any world that does not agree with the observation. This leads to shrinking the amount of possible worlds, removing uncertainty. *Conditioning* works the same way.

Definition 5. $Cond_1$ takes a plausibility space $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$ and an observation p and returns a new plausibility space $\mathbb{B}^p_{\mathbb{S}}$ where all worlds support p.

$$Cond_1(\mathbb{B}_{\mathbb{S}}, p) = (S^p, \mathcal{O}, \preceq^p)$$
$$\text{where } S^p = S \cap p,\ \preceq^p = \preceq \cap (S^p \times S^p)$$

Applying the one-step revision operation $Cond_1$ on our example plausibility space together with the observation p returns a plausibility space without any of the worlds where p was not assigned to be true.

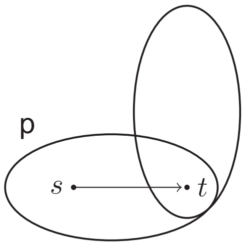

Figure 2.2: The plausibility space after applying $Cond_1(\mathbb{B}_{\mathbb{S}}, p)$

It should be pointed out that this method does not allow us to return to the previous state due to the nature of deleting. This is potentially useful for *hard information*, meaning observations that there can be no doubt of, and there is no point to even consider them.

Lexicographic Revision

First consider this method of combining two distinct binary relation rankings.

Definition 6. Consider $\preceq_i = (S_i \times S_i)$ and $\preceq_j = (S_j \times S_j)$ as two distinct binary relation sets, $S_i \cap S_j = \emptyset$. Let the following operation be a *well behaved union* of two epistemic binary relation sets, meaning the new pre-order $\preceq$ is a total pre-order.

$$\begin{aligned} \preceq &= \preceq_i \cup \preceq_j \cup \{(s,t)\} \mid s \in S_i, t \in S_j \\ &= [\preceq_i ; \preceq_j] \end{aligned}$$

Where in $\preceq$ every state of S_i is preferred to every state in S_j. The relations within $\preceq_i$ and $\preceq_j$ are kept as they were before the well behave union. This produces a total pre-order $\preceq$.

Lexicographic revision, *radical upgrade* in DEL, refrains from deleting worlds and instead simply reorders the binary relation set $\preceq$ in favour of the new information, while keeping previous information whenever possible.

Definition 7. $Lex_1(\mathbb{B}_{\mathbb{S}}, p)$ elevates every world in p to be more preferred than any world from $\overline{p}$ while keeping the binary relations between the worlds in p as well as the relations in $\overline{p}$. Let $\preceq^p$ indicate the relations between the worlds in p.

$$Lex_1(\mathbb{B}_{\mathbb{S}}, p) = (S, \mathcal{O}, [\preceq^p ; \preceq \setminus \preceq^p])$$

where $\preceq \setminus \preceq^p$ is the set subtraction containing every relation in $\preceq$ except the ones in $\preceq^p$.

This time we will see how the operation $Lex_1(\mathbb{B}_{\mathbb{S}}, p)$ changes the plausibility space to keep the ordering within p and reorder such that every world within p is considered more likely than r, which was previously the most believed world.

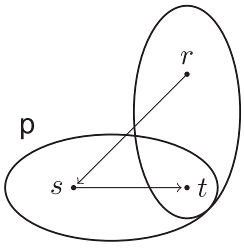

Figure 2.3: Example plausibility space after $Lex_1(\mathbb{B}_{\mathbb{S}}, p)$

Minimal Revision

Instead of elevating all the worlds that satisfy p as in Lex_1, *minimal revision* takes the most preferred worlds in p and promote them to be the most preferred worlds in all of S. In DEL this is known as *conservative upgrade*.

Definition 8. Let $\preceq^p_{min} = (min_{\preceq} S^p \times min_{\preceq} S^p)$ be the epistemic binary relation of the set of worlds that are the most entrenched in S^p.

$$Mini_1(\mathbb{B}_{\mathbb{S}}, p) = (S, \mathcal{O}, [\preceq^p_{min}; \preceq \setminus \preceq^p_{min}])$$

Applying the same observation to the initial example plausibility space we get a different result using $Mini_1$. Because t is the most likely world in p it is moved to be the most likely world overall and the (r, s) binary relation is kept.

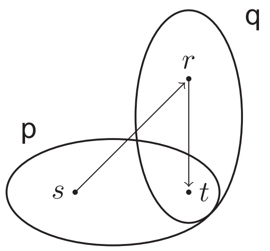

Figure 2.4: Example plausibility space after $Mini_1(\mathbb{B}_{\mathbb{S}}, p)$

2.3 Learning Method

It is obvious that the output of Lex_1 and $Mini_1$ depends heavily upon the preference ordering of the plausibility space used and changing it would result in a different output. Extrapolating this to the iterated versions Lex and $Mini$ means that the initial state of the binary relations, before any belief revision is done, is also of high significance. Arguably this is even more so the case for $Cond$ since the initial ordering of the binary relations are never changed.

In Definition 4 belief revisions are defined to take a plausibility space as argument, and this is obviously necessary, but it means that an opinion on which worlds are preferred is required.

This begs the question of how this opinion should be formed. Considering an epistemic space $\mathbb{S} = (S, \mathcal{O})$ it is easy to argue where we get S and $\mathcal{O}$ from, but from where is the initial binary relation $\preceq$ in a plausibility space acquired? To help answer this some more definitions are necessary.

Definition 9. A *learning agent* is able to use an initial section of a possibly infinite stream of observations to select the world from which the observations stem from.

Definition 10. A mapping of the states in $\mathbb{S}$ to a binary relation set $\preceq$ is called a *prior plausibility assignment*.

$$\mathcal{P}(\mathbb{S}) = (S, \mathcal{O}, \preceq)$$

Definition 11. Let the usage of a belief revision method R and a prior plausibility assignment $\mathcal{P}$ on an epistemic space along with a sequence or stream of data σ be known as a *belief revision based learning method*, L.

$$L^{\mathcal{P}}_R(\mathbb{S}, \sigma) = min\ R(\mathcal{P}(\mathbb{S}), \sigma)$$

Now the question of how the initial binary relations of $\preceq$ should be ordered can be explained by choosing a suitable prior plausibility assignment.

One of the main findings in [2] is a method for selecting such a prior plausibility assignment based on the epistemic space $\mathbb{S}$ that it relates to.

Let's take a moment to consider what we are asking of our learning method. Specifically, the ideal goal is for the learning method to *identify in the limit* the correct world, no matter what epistemic space we throw at it. The concept of *identifying in the limit* touches on the idea that we can only track the truth as well as the input allows us. So the learning method should *identify* the correct world *in the limits* of the input.

Definition 12. Given an epistemic space $\mathbb{S} = (S, \mathcal{O})$, a world $s \in S$ is *identified in the limit by learning method* L if for every sound and complete data stream σ, $L(\mathbb{S}, \sigma)$ eventually outputs only $\{s\}$ henceforth.

Epistemic space $\mathbb{S} = (S, \mathcal{O})$ is *identifiable in the limit by* L if every $s \in S$ is identified in the limit by L

Definition 13. A finite sequence of observations σ_1 is considered a *locking sequence* for learning method L on world s if $L(\sigma_1) = \{s\}$ and appending $(*)$ any finite data sequence σ_2 to σ_1 would still provide the same result, $L(\sigma_1 * \sigma_2) = \{s\}$.

Lemma 2.3.1. Consider an epistemic space $\mathbb{S}$ that is identifiable within the limit by learning method L, then for identified world $s \in S$ there must exist a data sequence σ such that $L(\mathbb{S}, \sigma)$ only outputs $\{s\}$ from then on, no matter what new data that satisfies s occurs afterwards.

Proof can be found in [2].

A locking sequence σ does not need to be the most direct route towards locking the output of the learning method. In fact, appending any data sequence τ, that is sound and complete w.r.t. s, to the end of σ resulting in the data sequence $\sigma * \tau$ is still a locking sequence for L and s due to the nature of locking sequences forcing it to still only return $\{s\}$.

A little bit surprising is how this relates to a different concept known as *tell-tale sets*.

Definition 14. A *finite tell-tale set* D_s is a finite set of observations, where $s \in \bigcap D_s$ and for any $t \in S$ where $t \in \bigcap D_s$ it is the case that $\mathcal{O}_s \subset \mathcal{O}_t$, or else $t = s$.

Any other world $r \notin \bigcap D_s$ disagrees with s, and any other world in D_s, on one or more observation from D_s. We call these tell-tale sets because if the observations are observed, then it is a hint that the state they belong to could be the correct one, but not necessarily since it is possible there exists a world t that differs from s but is still within $\bigcap D_s$.

Tell-tale sets are related to locking sequences because if you take the set of all the elements in a locking sequence σ for world $s \in S$, in order to remove duplicates, then that set will be a finite tell-tale D_s for the world s.

Definition 15. Let there exist a total map $D : S \to \mathcal{P}(\mathcal{O})$, such that $s \mapsto D_s$, where D_s is a finite tell-tale.

Using the previously mentioned method for converting locking sequences into finite tell-tales we get such a total map D.

The goal of generating this mapping is to create a partial ordering from the case when both $s, t \in D_s$, by preferring the one that requires the least assumptions, namely s. To handle this formally we introduce the *ordering tell-tale map* D'.

Definition 16. An *ordering tell-tale map* is a total map, $D' : S \to \mathcal{P}(\mathcal{O})$, such that $s \mapsto D'_s$, where D'_s is a finite tell-tale.

There is also a related injective map $i : S \to \mathbb{N}$, where if $t \in \bigcap D'_s$ and $\mathcal{O}_s \not\subseteq \mathcal{O}_t$, then $i(s) < i(t)$.

From this definition there seems to be little differentiating D and D', but the difference comes in having to uphold the the comparison requirement on the injective map. The way in which D' is created is by first finding D and then handling each specific case that goes against the injective map requirement by adding new information to D_s.

Consider $s, t \in D_s$, where $s \neq t$. i being an injective map means that it must induce an ordering. Take s and t such that $i(s) > i(t)$ and $\mathcal{O}_s \not\subseteq \mathcal{O}_t$, then we do not uphold the requirement. However, since $\mathcal{O}_s \not\subseteq \mathcal{O}_t$ we know that there is some observation $p \in \mathcal{O}_s$ that is not in $\mathcal{O}_t$. If we add p to D_s then we uphold the requirement again, since they are not both in D_s. The new finite tell-tale that is the combination of D_s and p is D'_s.

The reasoning for needing D' is that if $s, t \in D_s$ and $\mathcal{O}_s \not\subseteq \mathcal{O}_t$, where s is the true world, then since any information gained from σ supports s, and therefore t as well, σ will never provide a way to differentiate between s and t. The solution in D' is akin to Occam's razor in that we choose the world that requires the least amount of assumptions as the more likely option. Using D' we can create a partial order by introducing the comparison method $\preceq^1_{D'}$ for $s, t \in S$,

$$\text{iff } t \in \bigcap D'_s \text{ then } \preceq^1_{D'} = (s, t)$$

From which we can get $\preceq_{D'}$ by transitive closure of the relation $\preceq^1_{D'}$. With use of the Order-Extension Principle [6] we can extend $\preceq_{D'}$ to a total pre-order that in turn can function as the initial binary relation $\preceq$ for $\mathbb{B}_{\mathbb{S}}$.

Hereby we have a procedure that can be used as a prior plausibility assignment $\mathcal{P}$, given we know a locking sequence for each world.

Lemma 2.3.2. A binary relation $\preceq$ can be generated from the epistemic space $\mathbb{S}$ by applying the following prior plausibility assignment $\mathcal{P}$

1. Acquire a locking sequence σ_i for each world.
2. Calculate each worlds ordering tell-tale D'_i from $D_i = set(\sigma_i)$.
3. From D' find a partial ordering and extend it to a total pre-order.

Definition 17. A learning method L is *universal* if it can identify within the limit any epistemic space that is identifiable within the limit.

Theorem 2.3.3. Using $\mathcal{P}$ together with either *Lex* or *Cond* produces a *universal learning method*.

Mini does decidedly not providing a *universal learning method* with the prior plausibility assignment $\mathcal{P}$.

Proofs for both can be found in [2].

3 Multiple Agent Belief Revision

The usual goal of multi-agent systems (MAS) is to exploit the multiple perspectives to either enhance the performance, or even solve new problems with this advantage. The idea with MAS in belief revision is handling the situation of individual agents not having enough information to complete the task alone, but combined they do.

One solution to this problem is simply forwarding the observations of each agent to a trusted agent that can apply singular agent belief revision, as per [2], on the collectively gathered information. This centralised concept we will call *multiple source belief revision*, MSBR.

Since the model can only handle a single observation at a time, it is required that the observations are ordered in some manner. As long as the observations are non-erroneous the way in which the serialisation occurs does not matter, and while we do not require them to be complete, they must still be sound individually. The interesting part is that the combined observations must still provide a sound and complete data sequence with regard to the correct world in order to be identifiable within the limit. This is the same requirement we have for a normal single agent belief revision model.

Using MSBR we can handle multiple sources with the Baltag et al. model and learning methods that follow it. But there are still reasons to investigate other solutions. One limitation of MSBR is that it does not allow for any privacy between agents since each agent is sharing all observations they make. Say an observation provides some side information besides the goal of the truth tracking, that the agent would prefer to keep private, then the centralised agent would be just as capable of calculate that information from the observation. In [7] the concept of sharing information with belief revision while keeping privacy is explored deeper.

The objective behind MSBR is that the information that each agent has needs to be combined in some fashion, and the purest method of doing this is by combining the observations. However the currently most believed result of the learning methods output still contains the information obtained from the observations. So another possible point in the chain of truth tracking where information merging could occur is with the currently believed result. It turns out that this method has its own problems but we will address those later.

3.1 Belief Merge

In order to combine the information from the outputs of the learning agents an efficient method of aggregating multiple beliefs into a singular collective belief is required. To the pursuit of this purpose the classical notion of *belief merge* from Konieczny and Pino Pérez [3] discussed in [8] by Pigozzi is introduced, allowing for merging of n agents presenting their belief as belief bases.

Konieczny and Pino Pérez define belief bases as sets of propositions and then for any belief base there exists a set of worlds that satisfy the belief base. They then do the operations on the set of worlds that satisfy the belief base, arguing that these worlds represent the belief base. Our situation is a little different in that the learning agents provide us with a set of worlds, *conjecture*, they believe are candidates for the actual world. This means that we can apply our operations on the conjecture directly, without having to convert the belief base into worlds first.

Definition 18. Let a *conjecture* B be a set of worlds.

$$B = \{w_1, \ldots, w_m\}$$

And let a *belief profile*, E, be a multi-set of conjectures.

$$E = \{B_1, \ldots, B_n\}$$

Then a *belief merger*, Δ_{IC}, is a aggregation function that takes a belief profile and outputs a new conjecture B_c which satisfies a set of integrity constraints IC (see below). B_c then represents the collective conjecture.

$$\Delta_{IC}(E) = B_c$$

The standard set of integrity constraints that are required for Δ_{IC} to be a belief merge operator, originally from [3], are the following.

(IC0)	$\Delta_{IC}(E) \models IC$
(IC1)	If IC is consistent, then $\Delta_{IC}(E)$ is consistent.
(IC2)	If $\bigwedge E$ is consistent with IC, then $\Delta_{IC}(E) \equiv \bigwedge E \wedge IC$
(IC3)	IF $E_1 \equiv E_2$, and $IC_1 \equiv IC_2$, then $\Delta_{IC_1}(E_1) \equiv \Delta_{IC_2}(E_2)$
(IC4)	If $B_1 \models IC$ and $B_2 \models IC$, then $\Delta_{IC}(\{B_1, B_2\}) \wedge B_1$ is consistent if and only if $\Delta_{IC}(\{B_1, B_2\}) \wedge B_2$ is consistent.
(IC5)	$\Delta_{IC}(E_1) \wedge \Delta_{IC}(E_2) \models \Delta_{IC}(E_1 \sqcup E_2)$
(IC6)	If $\Delta_{IC}(E_1) \wedge \Delta_{IC}(E_2)$ is consistent, then $\Delta_{IC}(E_1 \sqcup E_2) \models \Delta_{IC}(E_1) \wedge \Delta_{IC}(E_2)$
(IC7)	$\Delta_{IC_1}(E) \wedge IC_2 \models \Delta_{IC_1 \wedge IC_2}(E)$
(IC8)	If $\Delta_{IC_1}(E) \wedge IC_2$ is consistent, then $\Delta_{IC_1 \wedge IC_2}(E) \models \Delta_{IC_1}(E) \wedge IC_2$

(IC0) ensures that the result follows the integrity constraints. (IC1) if the constraints are consistent then so is the result. (IC2) states that when its possible the result of the merge is the conjunction of the profile and the integrity constraints. (IC3) irrelevancy of syntax; if two conjectures are logically equivalent and two integrity constraints are equivalent, then the result of the two will be logically equivalent. (IC4) is the principle of fairness; when merging two or more conjectures the merger must not give preference towards any of them. (IC5) if two groups agree on some assignment of a proposition, then when joining the two the assignment must be chosen. From (IC5) and (IC6) together we gain that if it is possible to find two subgroups that agree on an assignment, then the result of the merge will include that assignment. (IC7) and (IC8) provide the notion of closeness is well-behaved; given that merging E under IC_1 and IC_2 is consistent, then IC_1 will remain fulfilled if more restrictions, such as IC_2 is added. This also means that if a merge results in B_c then adding more restrictions will not change the result, so long as the new restrictions don't remove B_c. Intuitively this can be understood as; adding restrictions can only remove options and removing some option does not degrade the comparative value of B_c or increase the comparative value of a third option. For example, say there are three options that a merger Δ_{IC_1} could choose from, namely B_a, B_b and B_c. If $\Delta_{IC_1}(E) = B_c$, then re-running the merging with additional restrictions IC_2, that removes B_b as an option, will always result in $\Delta_{IC_1+IC_2}(E) = B_c$.

These were the standard postulates that are required, but alone they do not provide much of a merger. To gain anything of use we must add additional postulates. This is the design choice that comes with integrity constraint based mergers, and it gives a lot of freedom.

The merger that we will be looking into is the *majority merger*, for which the goal is that the chosen result must be the most supported one. This does not mean that the merger only picks between the conjectures in the profile, but rather it picks the sound conjecture that is closest to the profile as a whole.

Definition 19. If, for every integer n, E^n expresses the multi-set containing n times E, then a merging operator from the majority merger class Maj satisfies the following postulate (Maj) in addition to (IC0-8).

$$\text{(Maj)} \mid \exists n \Delta_{IC}(E_1 \sqcup E_2^n) \models \Delta_{IC}(E_2)$$

Integrity constraints are also used to enforce specific restrictions, such as if observations O_1 implies that O_2 cannot be true, then an IC_μ could enforce that by $O_1 \rightarrow \neg O_2$. We call these the *mundane integrity constraints*, and an instance can be seen in Example 3.2.1.

3.2 Distance Based Merging

With what we have discussed so far integrity constraint mergers are nothing more than an aggregation function with a set of postulates that are desirable, however it is not too difficult to implement procedures that uphold them.

When working on the more practical merging method that is distance based merging, we will need to handle worlds in a more concrete and practical way. For this purpose we will consider a world as a list of truth assignments to each observation in $\mathcal{O}$ where any observation that is missing in $\mathcal{O}_w$ will be considered to have $\neg\mathcal{O}_w$. So for world w we will assign 1 to any observation that is present in $\mathcal{O}_w$ and 0 otherwise. This will be called the propositions of the world, $prop(w) = \{0,1\}^{\mathcal{O}}$.

As mentioned earlier the majority merger selects the conjecture that is the closest to the whole profile. The notion of distance imply that it is possible to measure some value that represents the difference between worlds, and that is exactly what is done here.

There are all of two requirements for the chosen distance function, d. It must be symmetric and if the distance is zero the two worlds must be the same world. Otherwise any arbitrary distance function can be chosen, for example Hamming distance.

Distance Function, $d(w, w')$
$d : W \times W \rightarrow R^+$, s.t. for all $w, w' \in W$:

$$d(w, w') = d(w', w)$$
$$d(w, w') = 0 \text{ iff } w = w'$$

The distance function is then used in pursuit of applying a total ordering on the set of sound worlds, where the minimum world would then be the closest world to the belief profile. To do that we need a way of comparing a world w to a conjecture B.

Conjecture Distance Function, $D_B(w, B)$
Aggregation function, $D_B : R^{+m} \rightarrow R^+ \mid m = |B|$

$$D_B(w, B) = D_B(d(w, w_1), d(w, w_2), \ldots, d(w, w_m)) \mid w_1, \ldots, w_m \in B$$

D_B is then some method of aggregating the different distances between w and $w_j \in B$ to a single output. The solution that Konieczny and Pino Pérez use is taking the minimum distance.

Being capable of calculating the distance between a world and a conjecture lets us find the distance from a world w to each conjecture B in the profile E, however it is not obvious how the distance from w to E should be determined. This is where the *profile distance function* $D_P(w, E)$ comes in, which is yet another modular part of the IC merging system.

Profile Distance Function, $D_P(w, E)$
Aggregation function, $D_P : R^{+n} \to R^+ \mid n = |E|$

$$D_P(w, E) = D_P(D_B(w, B_1), D_B(w, B_2), \ldots, D_B(w, B_n))$$

The conceptual idea behind D is that it should provide a means to combine the distances w has to each conjecture in E, such that we can compare it to other worlds $w' \in W$. Simple example is to sum each distance to a total distance. The only requirements there are for D are

1. If $x \geq y$, then $D_P(x_1, \ldots, x, \ldots, x_n) \geq D_P(x_1, \ldots, y, \ldots, x_n)$
2. $D_P(x_1, \ldots, x, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$
3. $D_P(x) = x$

1. says that given a choice to insert two different distances into an already populated set of distances the comparative relation of the aggregated results will follow the comparative relations between the two options. 2. is rather simple and it results in aggregation operations such as minimum is not an option. 3. ensures that D does not inflate or deflate the distances.

$D_P(w, E)$ lets us compare worlds by their distance to E which in turn provides a method to acquire a total pre-order by calculating $D_P(w, E)$ for all $w \in W$. The result of the merging is then found by selecting the worlds with the smallest distance to the profile, which is the same as selecting the smallest worlds in the ordering.

Selecting a combination of distance function d, conjecture distance function D_B and profile distance function D_P provides a way of introducing new integrity constraints on the merger, such as (Maj).

3.2.1 Minisum Belief Merger

The *minisum* merger $\Delta_{\sum}$ is an instance of such a integrity constraint belief merger that is identified by a specific distance function d, conjecture distance function D_B and profile distance function D_P.

$$\begin{aligned} d(w, w') &= \text{Ham}(w, w') \\ D_B(w, B) &= D_B(d(w, w_1), \ldots, d(w, w_m)) = \min_i^m d_i \\ D_P(w, E) &= D_P(D_{B1}, \ldots, D_{Bn}) = \sum_i^n D_{Bi} \end{aligned}$$

Hamming Distance, Ham(w, w')
The hamming distance between two worlds w and w' is calculated by the amount of propositions the two disagree on. One way of calculating this is performing a XOR $\oplus$ operation on the binary representation of the propositions and counting the amount of 1's in the result.

$$\text{Ham}(w, w') = count(prop(w) \oplus prop(w'))$$

Minisum satisfies both the standard integrity constraint postulates IC0-8 as well as the majority postulate Maj, and as such is considered a *majority merger*. Proof can be found in [3].

Example 3.2.1. Five children K_1, K_2, K_3, K_4 and K_5 are shown an image of a giraffe and are later asked questions about three features of the giraffe; if it had a long neck (N), long legs (L) and if it had spots (S), all three being required for it to be a giraffe. They are also told that any animal can only have a long neck if it has long legs ($N \to L$).

They answer the questions as followed:

	Answers	conjecture
K_1	{N,L}	{{1,1,1}, {1,1,0}}
K_2	{L,S}	{{1,1,1}, {0,1,1}}
K_3	{N,S}	{{1,1,1}, {1,0,1}}
K_4	{N,L}	{{1,1,1}, {1,1,0}}
K_5	{S}	{{0,0,1}, {0,1,1}, {1,0,1}, {1,1,1}}

Clearly K_3 was not listening when the rule of long necks was explained. Using minisum $\Delta_{\sum}$ they try to combine their information with the mundane integrity constraint for animals with a long neck $IC_\mu : N \to L$.

(N,L,S)	K_1	K_2	K_3	K_4	K_5	$\sum$
(0,0,0)	2	2	2	2	1	9
(0,0,1)	2	1	1	2	0	6
(0,1,0)	1	1	2	1	1	6
(0,1,1)	1	0	1	1	0	3
(1,0,0)	1	2	1	1	1	6
(1,0,1)	1	1	0	1	0	3
(1,1,0)	0	1	1	0	1	3
(1,1,1)	0	0	0	0	0	0

Table 3.1: $\Delta_{\sum}$ calculations. Greyed out rows are inconsistent with the integrity rule IC_μ, and are not an option.

After the calculations they agree that it was a giraffe in the picture.

3.2.2 Leximax Belief Merger

$\Delta_{leximax}$ also uses the hamming distance for d, but instead of doing a summation for D it orders the distances in reverse lexicographical order, also known as descending order.

$$d(w, w') = \text{Ham}(w, w')$$

$$D_B(w, B) = D_B(d(w, w_1), \ldots, d(w, w_m)) = \min_i^m d_i$$

$$D_P(w, E) = D_P(D_{B1}, \ldots, D_{Bn}) = lex(D_{B1}, \ldots, D_{Bn})$$

The worlds that $\Delta_{leximax}$ then crowns the result is the worlds with the smallest distance in the first index in the lex result. What this equates to is selecting the world w_c that minimising the distance between w_c and the conjecture $B \in E$ that is the furthest from w_c.

This is known as a *arbitration merger* and it differs from the majority merger by selecting the result on a different basis. While the majority merger selects the most supported worlds, an arbitration merger selects the worlds that minimises the incorrectness of the least correct conjecture. This can also be thought of as minimising the dissatisfaction of the most dissatisfied agent.

$$\left.\begin{array}{l}\Delta_{IC_1}(B_1) \leftrightarrow \Delta_{IC_2}(B_2) \\ \Delta_{IC_1 \leftrightarrow \neg IC_2}(B_1 \sqcup B_2) \leftrightarrow (B_1 \leftrightarrow \neg B_2) \\ IC_1 \not\models IC_2 \\ IC_2 \not\models IC_1\end{array}\right\} = \Delta_{IC_1 \wedge IC_2}(B_1 \sqcup B_2) \leftrightarrow \Delta_{IC_1}(B_1) \qquad \text{(Arb)}$$

When we introduced the D function it was said to be an aggregation function, $D : R^{+n} \rightarrow R^{+}$, but the D that $\Delta_{leximax}$ uses does not strictly follow this. It still works because we also change how the total order is devised. This shows the flexibility and modular nature of integrity constraint mergers.

4 Merging Learning Methods

The goal of introducing the integrity constraint merging concept is to have a way of combining the results that different learning agents output, the idea being that the information that the individual agent's acquire from their data sequences persists through the learning methods. The intention is then to combine the learning agents output and thereby making a multi-agent system that can identify the true world in more situations than the agents would individually. For ease of reference we will be addressing single agent belief revision as SBR and multi-agent belief revision as MABR.

The output of learning methods being a set of worlds, moreover the set of worlds that the agent believes to be the most likely ones, fits nicely with the idea of belief mergers taking a set of conjectures. The merging operations are considered static operations, in the sense that there is no temporal factor to them. They simply take a set of conjectures and output a combined conjecture. However the act of learning has a temporal aspect to it. Learning something means going from a state of being unaware of something to a state of knowledge about the said thing. This temporal aspect is present in the learning agents since they learn what is learn-able from the list of observations, each belief revision operation on an observation can be considered as a temporal unit. The static nature of merging is then expected as no learning is actually attempted in the merging operation, but rather an attempt at combining already learned knowledge.

In an epistemic space with many possible observations there are most likely some combinations of observations that are inconsistent with which worlds are possible. Here we can use the so called mundane integrity constraints to handle such situations, just as was done in Elxample 3.2.1. That way we ensure that only possible worlds are returned from the merging operation.

Consider now the situation of trying to track the truth in $\mathbb{S}$ with n agents each with their own data sequence σ_i. Let $L_R^{\mathcal{P}}(\mathbb{S}, \sigma_i)$ be the set of worlds that agent i outputs from its learning method, meaning

$$B_i = L_R^{\mathcal{P}}(\mathbb{S}, \sigma_i)$$

is the conjecture of agent i.

Any kind of additional information that the SBR agent infers from doing the belief revision is then kept in B_i since if it is inferred it must be present in the most likely world. For instance say an observation O_1 is observed and for every world where O_1 is present O_2 is as well, then the agent knows that O_2 must hold as well. This inference is kept in B_i and thereby transferred to the merger.

Definition 20. A world $s \in S$ is *identified by merger* Δ_{IC} if Δ_{IC} selects only s as the output for all belief profiles $E = \{B_1, \dots, B_n\}$, for which it holds that

1. $\forall B_i \in E$, B_i is based on SBR on data sequence σ_i from agent i
2. Every σ_i is sound w.r.t. s
3. The content of all data sequences, $\mathcal{O}_A = \bigcup_{i=0}^{n} set(\sigma_i)$, is complete w.r.t. s.

$$\Delta_{IC}(B_1, \dots, B_n) = \{s\}$$

Definition 21. A merger Δ_{IC} is a *learning merger on class $\mathcal{E}$ of belief profiles* if it can for any belief profile $E \in \mathcal{E}$ identify the world s that E is built from. A *universal learning merger* is a learning merger on the class of all belief profiles.

Let us examine the case where each σ_i is sound, but not complete, w.r.t. s. Now every agent is incapable of identifying the true world by itself, because some crucial information is missing. In the same vein, if that essential observation is missing from every agents σ_i, then the merger has no hope of ever acquiring that information. This is obvious considering the nature of a merger is to combine information and not acquiring new information.

Lemma 4.0.1. In order for $s \in S$ to be identified by majority merger Δ_{Maj} from the profile $E = \{B_1, \ldots, B_n\}$ the content of the observations $\mathcal{O}_A = \bigcup_{i=0}^{n} set(\sigma_i)$ must be sound and complete w.r.t. s.

Proof. Say s is identified by merger Δ_{Maj} on E, but $\mathcal{O}_A$ is either not sound or not complete w.r.t. s.

$\mathcal{O}_A$ not being sound w.r.t s means some observation O goes against s, and nothing is stopping this to occur for every agent. This leads to a majority not supporting O, which is required for s so the merger will pick some other world. The possibility of not picking s means we do not have a guarantee of picking s and s is therefore not identified, contradiction.

$\mathcal{O}_A$ not being complete w.r.t. s means there is some observation O in $\mathcal{O}_s$ that is not found in any of the data sequences σ_i. Each learning agents decision on O is either decided by the prior plausibility assignment, which could go against the truth assignment in s or there is no commitment to an opinion on O, so s is not the only world returned by the merger. No guarantee. □

Lemma 4.0.2. For s to be identified by majority merger Δ_{Maj} from the profile $E = \{B_1, \ldots, B_n\}$, s must also be identified in the limit by a SBR agent using a universal learning method on a data sequence σ_A that contains every observation in $\mathcal{O}_A$ and is therefore sound and complete w.r.t. s.

$$\Delta_{Maj}(B_1, \ldots, B_n) = \{s\}$$
$$\Longrightarrow$$
$$L_R^{\mathcal{P}}(\mathbb{S}, \sigma_A) = \{s\}$$

Proof. From Lemma 4.0.1 we know that $\mathcal{O}_A$ must be sound and complete w.r.t. s for s to be identifiable by majority merger Δ_{Maj} and from theorem 2.3.3 we know that for SBR if a data sequence is sound and complete w.r.t. s then s is identifiable in the limit by a universal learning method $L_R^{\mathcal{P}}$. Given that σ_A must be sound and complete w.r.t. s if $\mathcal{O}_A$ is, then we also know that if $\Delta_{Maj}(B_1, \ldots, B_n)$ can identify s then $L_R^{\mathcal{P}}(\mathbb{S}, \sigma_A)$ can in the limit.

□

The implications of this is that any truth tracking done with a majority merger can be performed by a SBR agent, if that agent is provided with the same information in the form of σ_A.

4.1 Information Drowning

There is still one problem that we have yet to touch on. Notice that each learning agent is applying their learning method on the same epistemic space $\mathbb{S}$. Since the prior plausibility assignment $\mathcal{P}$ is solely based on $\mathbb{S}$ the agents will all have the same binary relation $\preceq$ to start from, and then the difference in observations makes them diverge.[1] This concept shows its importance when considering the case of a single agent a_1 having all the observations required for $\mathcal{O}_A$ to be complete w.r.t. s, while the rest $a_2, \dots, a_n$ have no observations. First we notice that $\mathcal{O}_A$ is still sound and complete w.r.t. s and that a_1 can identity in the limit the true world alone as per theorem 2.3.3. Secondly, it is obvious that agents $a_2, \dots, a_n$ will return a conjecture that is not very suited to tracking the truth, but they will all agree on the same one. This is an issue because the conjecture from a_1 that contains all of the information will be in the minority and the conjectures based solely on $\mathcal{P}$ will rule the majority merging operation. This is the concept of *information drowning*.

Information drowning does not only occur in that extreme situation. For arguments sake consider an epistemic space where each agent has exactly one observation, $|\sigma_i| = 1$, but $\mathcal{O}_A$ is still complete w.r.t. s. Before applying any belief revision method each agent holds the conjecture $B_{\mathcal{P}}$ that is solely based on the prior plausibility assignment. If the observation of agent i, O_i, is independent then B_i will agree with $B_{\mathcal{P}}$ on every other observation than O_i. If this is the case for every agent, then the observations that are made will be drowned out since there is only ever one agent that believes O_i.[2]

The root of the problem is that the prior plausibility assignment can have an impact on the conjecture when the amount of observations is not sufficient, so part of the conjecture is not solely created by the observations. Considering that every agent starts with $B_{\mathcal{P}}$ means that a majority towards $B_{\mathcal{P}}$ is very likely if not handled. The influence of $B_{\mathcal{P}}$ is not a problem in SBR because there we require that the data sequence is complete w.r.t s and therefore the important information needed to revise the agents belief from $B_{\mathcal{P}}$ must be present. Changing the completeness requirement from individual agents data sequences to the content of the data sequences introduced this problem. The argument for moving the completeness requirement still holds up because otherwise the agents would be capable of tracking the truth by themselves.

Definition 22. An observation O is *majority represented* in profile E when O is present in more than half of the data sequences σ_i used for generating E.

A profile E is *majority represented w.r.t.* s when each observation in $\mathcal{O}_s$ is majority represented in E.

Any majority represented belief profile's data sequences $\sigma_1, \dots, \sigma_n$ provides a content set $\mathcal{O}_A$ that is sound and complete w.r.t. s due to every observation being majority represented. This means that majority representation is a stronger requirement that encapsulates the completeness requirement on $\mathcal{O}_A$.

Majority representation is quite restrictive in that having such a requirement increases the amount of observations that are necessary to do truth tracking. One of the concepts of why information drowning happens is that there are not enough observations present

[1] You could also contemplate the case where agents have differing initial binary relations, maybe from previous truth tracking objectives, but it is not immediately obvious how that would work considering the epistemic spaces would have to be different between the two. We will consider this a different problem for another time.

[2] There is a small exception in when $O_i \in B_{\mathcal{P}}$, but the choice of choosing O_i in the merge operation is still due to $B_{\mathcal{P}}$ and not the observation.

to remove the influence that the initial total pre-order has on the conjecture. Majority representation simply says that the sufficient amount of observations must be present.

Theorem 4.1.1. A majority merger Δ_{Maj} is **not** a learning merger on the class of belief profiles $\mathcal{E}_m$ where every $E \in \mathcal{E}_m$ is majority represented w.r.t. s and every $B_i \in E$ is sound w.r.t. s.

Proof. First we will discuss the problem conceptually and then go on to show a counter example that highlights the concept.

Since the belief profile E is in $\mathcal{E}_m$ we know that any observation $O_s \in \mathcal{O}_s$ will have majority representation and thus they are guaranteed to be part of anything that the merger outputs. This means that for every world $w \in \Delta_{Maj}(E)$, $\mathcal{O}_s \subseteq \mathcal{O}_w$. However there is no restrictions on the observations outside of $\mathcal{O}_s$, let us call them $\mathcal{O}_{\overline{s}}$, and in such a situation we do not know if $\Delta_{Maj}(E)$ will output s or w or both.

For the counter example consider this plausibility space $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$[3], where s is the true world.

$S = \{r, s, t, u\}, \quad \text{true world} = s$
$\mathcal{O} = \{p, q, z\} = \{\{r, s, t\}, \{r, s, u\}, \{r, t, u\}\}$
$\preceq = \{(u, t), (t, s), (s, r)\}$

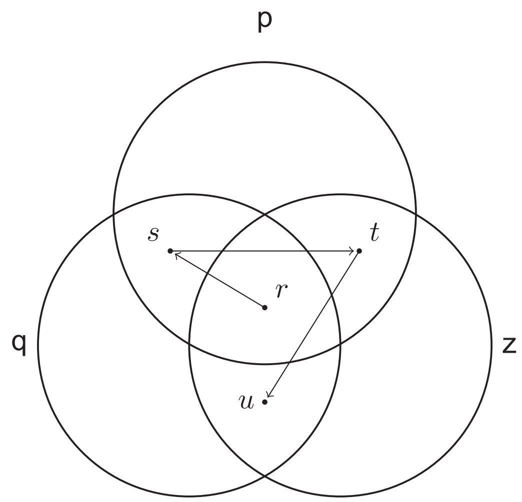

Figure 4.1: Initial plausibility space

The propositional representation of the worlds with the format (p, q, z) are;

$$r = (1, 1, 1) \quad s = (1, 1, 0) \quad t = (1, 0, 1) \quad u = (0, 1, 1)$$

Now consider three agents a_1, a_2, a_3 with this initial setup. Their data sequences, that are sound w.r.t. s, are as followed;

$$\sigma_1 = (p) \quad \sigma_2 = (q) \quad \sigma_3 = (p, q)$$

[3]Transitive relations in $\preceq$ are omitted in the listing for ease of reading.

We also note that the content set of the data sequences, $\{p, q\}$, is sound and complete w.r.t. s. Applying the lexicographical learning method[4] $Lex(\mathbb{B}_{\mathbb{S}}, \sigma)$ the three agents outputs;

$$B_1 = Lex(\mathbb{B}_{\mathbb{S}}, \sigma_1) = \{t\} = \{(1,0,1)\}$$
$$B_2 = Lex(\mathbb{B}_{\mathbb{S}}, \sigma_2) = \{u\} = \{(0,1,1)\}$$
$$B_3 = Lex(\mathbb{B}_{\mathbb{S}}, \sigma_3) = \{s\} = \{(1,1,0)\}$$

The belief profile $E = \{B_1, B_2, B_3\}$ is sound w.r.t. s and there is majority representation w.r.t. s. Applying $\Delta_{\sum}$ on the belief profile we get the following table;

(p,q,z)	B_1	B_2	B_3	$\sum$
(0,0,0)	2	2	2	6
(0,0,1)	1	1	3	5
(0,1,0)	3	1	1	5
(0,1,1)	2	0	2	4
(1,0,0)	1	3	1	5
(1,0,1)	0	2	2	4
(1,1,0)	2	2	0	4
(1,1,1)	1	1	1	3

Table 4.1: $\Delta_{\sum}$ calculations. Greyed out rows are worlds that do not exist in the epistemic model and are therefore ruled out by mundane integrity constraints.

Here it is seen that $\Delta_{\sum}$ selects $(1,1,1)$ as the output which is the world r and indeed not the true world.

□

The reasoning for why r is selected in the counter example is that even with majority representation we still do not have any information about the observations that are outside of $\mathcal{O}_s$, $\mathcal{O}_{\overline{s}}$, so there can exist cases where the merger selects the wrong truth assignment on those observations.

Majority representation is about as good as the conditions get for a majority merger and since it is still incapable of identifying the true world it seems that it is unfit for the purpose.

In a paper by D'Alfonso [4] another problem of majority mergers have is pointed out and it stems from the design choice of how they are structured. The goal of Konieczny and Pino Pérez was to find the belief base that is the closest to every agents belief base, looking for the result satisfy the most amount of agents. The problem here is that the world that satisfy the agents most appropriately is an excellent choice for preference or opinion merging, but it is not guaranteed to be the true world. Just because you believe something doesn't make it true.

Recall the distance based approach of majority merge for which we have a three layered

[4] Conditioning provides the same results.

distance measure,

$$
\begin{aligned}
d(w, w') &= \mathsf{Ham}(w, w') \\
D_B(w, B) &= D_B(d(w, w_1), \ldots, d(w, w_m)) = \min_i^m d_i \\
D_P(w, E) &= D_P(D_{B1}, \ldots, D_{Bn}) = \sum_i^n D_{Bi}
\end{aligned}
$$

The main culprit here is the conjecture distance function D_B, which says that the distance between a world w and a conjecture B is the minimum distance between w and any world from B. This works really well when you want to find the world that is the closest to every conjecture but not for truth tracking. To see why let us examining what happens if we add worlds to a conjecture when in the context of truth tracking. Say world w_2 is added to conjecture $B_{before} = \{w_1\}$, then $B_{after} = \{w_1, w_2\}$.

$$
\begin{aligned}
&S = \{w_1, w_2, w_3, w_4\} \\
&\mathcal{O} = \{O_1, O_2\} = \{\{w_3, w_4\}, \{w_2, w_4\}\} \\
&B_{before} = \{w_1\} \\
&B_{after} = \{w_1, w_2\}
\end{aligned}
$$

Conceptually that would increase the uncertainty of the agent, meaning the 'weight' given to w_1 by being in B_{before} is decreased since it is split between the two worlds now. However when a world is added to a conjecture that is used for majority merging the distance from some world w to B can only ever go down since we are using the minimum.

	(O_1, O_2)	B_{before}	B_{after}
w_1	(0,0)	0	0
w_2	(0,1)	1	0
w_3	(1,0)	1	1
w_4	(1,1)	2	1

This goes against the concept that the importance of w_1 being in B should decrease when more worlds are added. Exactly that is the problem of D_B being the minimum aggregation function.

4.2 Truthlikeness Merge

in [4] D'alfonso introduces a different merger for the purpose of *truthlikeness*, which is suppose to emphasise that the merger is good at estimating the true world. The only difference between the majority merger and D'alfonso's *truthlikeness merger* is that D_B aggregates by calculating the average instead of taking the minimum.

A simple example of how this could look on two conjectures B_1 and B_2 (formed as triangles in this example) with the two possible worlds w_1 and w_2.

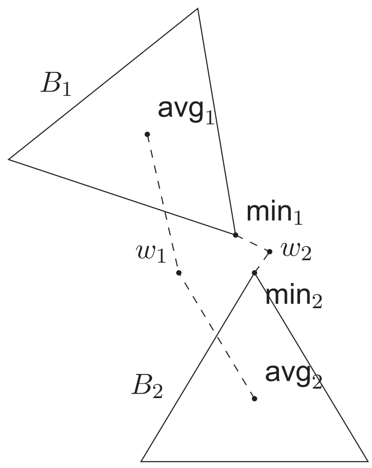

Figure 4.2: Comparing the minimum aggregation function and the average aggregation function.

What should be noted here is that there is room to change B_1 and B_2 without changing the minimum distances, and while its possible for the average distance as well they better represent the full conjecture.

Definition 23. If avg$(d_1, \ldots, d_m) : R^{+m} \to R^+$ is an operation that calculates the average of the input,

$$\overset{m}{\underset{i}{\text{avg}}}\, d_i = \frac{\sum_i^m d_i}{m}$$

then the *truthlikeness merger* Δ_{avg} is an integrity constraint merger that only satisfy (IC0-1), (IC3) and (IC5-8), by use of the following distance measures. (Proofs can be found in [4])

$$d(w, w') = \text{Ham}(w, w')$$

$$D_B(w, B) = D_B(d(w, w_1), \ldots, d(w, w_m)) = \overset{m}{\underset{i}{\text{avg}}}\, d_i$$

$$D_P(w, E) = D_P(D_{B1}, \ldots, D_{Bn}) = \sum_i^n D_{Bi}$$

It turns out that failure to satisfy (IC2) and (IC4) is not a problem for truth tracking. Let us refresh what the two constraints uphold.

(IC2) If $\bigwedge E$ is consistent with IC, then $\Delta_{IC}(E) \equiv \bigwedge E \wedge IC$

(IC4) If $B_1 \models IC$ and $B_2 \models IC$, then $\Delta_{IC}(\{B_1, B_2\}) \wedge B_1$ is consistent if and only if $\Delta_{IC}(\{B_1, B_2\}) \wedge B_2$ is consistent.

(IC2) says that when possible the merging should be solved by conjunction. When considering the goal of satisfying the agents with regards to merging different agents opinion or proposals, then this constraint makes a lot of sense. If there are one or multiple worlds that all satisfy everyone optimally, then it does not matter which one of those optimal solutions you pick. However for truth tracking it is a little different in that there is always only one correct solution, so simply taking any world that is optimal w.r.t. the input is not good enough. With this argument in mind we declare that Δ_{avg} not satisfying (IC2) is acceptable.

(IC4) is the fairness postulate. Its purpose is to ensure that no agent is being unfairly under-represented. Again, this makes a lot of sense for opinion merging, however it is of little importance for truth tracking.

Theorem 4.2.1. The truthlikeness merger Δ_{avg} is **not** a universal merger on the class of belief profiles $\mathcal{E}_m$ where every $E \in \mathcal{E}_m$ is majority represented w.r.t. s and every $B_i \in E$ is sound w.r.t. s.

Proof. If we consider the same counter example as in theorem 4.1.1 and calculate the distances using the average merger instead, then we get the exact same distances since the conjectures consist of only one world so the minimum is equal to the average.

(p,q,z)	B_1	B_2	B_3	$\sum$
(0,0,0)	2	2	2	6
(0,0,1)	1	1	3	5
(0,1,0)	3	1	1	5
(0,1,1)	2	0	2	4
(1,0,0)	1	3	1	5
(1,0,1)	0	2	2	4
(1,1,0)	2	2	0	4
(1,1,1)	1	1	1	3

Table 4.2: Δ_{avg} calculations. Greyed out rows are worlds that do not exist in the epistemic model and are therefore ruled out by mundane integrity constraints.

□

The problem here is that Δ_{avg} still does not handle the observations that we do not have any information on such as z in the example.

4.3 Plausibility Mergers

Lets take a step back and consider what it is the mergers are trying to accomplish. Through distance measures they attempt to make a total pre-order that can guarantee the true world is prioritised, and so far we have been using the sets of worlds that each SBR agent believes to be the most likely, their conjectures. What has been neglected so far is that each of these conjectures are found by the agents own internal total pre-order $\preceq$. The information that lies in the conjectures simply is not the whole story and more information would be retained through the merging if instead the merging operator is applied on the agents total pre-orders.

The main argument for this change is that the total pre-orders of the agents are initially built from the ordering tell-tale map D', definition 16, which follows the concept of Occam's razor when creating the ordering. For the following consider an agent using *Lex* as their learning method. The concept of Occam's razor means that for two worlds s and w if their initial binary relation is (s, w) then there is some observation $O \in \mathcal{O}_w$ that is not in $\mathcal{O}_s$. If s is the true world then because the data sequence σ is sound w.r.t. s any observation used for the belief revision operation will either (1) support only s or (2) support both s and w. If (1) then (s, w) still holds as only the position of s in the ordering improves, and if (2) then both worlds position improves, but they keep their relative ordering as *Lex* requires. This leads to the binary relation between s and w will stay as (s, w) no matter what. Lets call this the *Occam's razor property*.

Definition 24. Let a *plausibility profile* $E_{\preceq} = \{\preceq_1, \ldots, \preceq_n\}$ be a multi-set of total pre-orders on epistemic space $\mathbb{S} = (S, \mathcal{O})$ from n agents. Each total pre-order represents the agents ordering of relative plausibility between the worlds.

A *plausibility aggregation function* $D_{\preceq}(E_{\preceq})$ takes a plausibility profile and outputs a new total pre-order $\preceq_c$ that represents the collective plausibility assignment.

Then a *plausibility merger*, $\Lambda(E_{\preceq})$, uses such a plausibility aggregation function to find the most plausible world from the plausibility profile.

Let us take a moment to discuss the importance of integrity constraints in belief merge and how they relate to plausibility mergers. The integrity constraints were introduced in order to have a set of postulates that makes belief merging fair for every agent when using belief merge for opinion merging. The problem with them with regards to truth tracking is that we do not care about how much each agent is represented in the merging output, but rather that it is in fact the true world that the operation selects. A thorough explanation for each constraint is in order.

For (IC2) and (IC4) we have already explained earlier that they are not of interest for truth tracking. (IC1) says that the output must be consistent, which is trivially true in plausibility merging because every world in the total pre-orders are from S and therefore consistent. (IC3) requires that syntax is irrelevant and that is also trivially true in plausibility merging, since any total pre-order only has one binary relation set that represents it. The intention of (IC5) and (IC6) is that if two subgroups can agree on a proposition then it should be used, because it is an easy proposition to satisfy everyone on. Obviously satisfying agents is not relevant for learning, but if majority is used as the selection method towards truth tracking then these constraints end up holding. To allow the possibility of as many selection methods as possible (IC5) and (IC6) should not be required. The same thought process goes into (IC7) and (IC8) that state adding additional restrictions should not change the selected output so long as the restrictions don't disallow the chosen world.

With the interest of having different classes of plausibility mergers, for ease of reference, we now introduce three relevant properties that a plausibility merger could have.

Definition 25. A *non-overgeneralising plausibility merger* $\Lambda(E_{\preceq})$ retains the Occam's razor property through the merge operation. If every total pre-order $\preceq$ in $E_{\preceq}$ satisfy the Occam's razor property, then the resulting total pre-order $\preceq_c$ also satisfy it.

$$\forall \preceq \in E_{\preceq} \models \text{Occam's razor property}$$
$$\Longrightarrow$$
$$\Lambda(E_{\preceq}) \models \text{Occam's razor property}$$

Any total pre-order from a SBR learning agent will satisfy the Occam's razor property due to the design of the ordering tell-tale map D' that is used to generate the total pre-order.

Definition 26. If, for every integer k, E^k expresses the multi-set containing n times $E_{\preceq}$, then the *weak majority plausibility merger* satisfy the following

$$\exists k \Lambda(E_{\preceq 1} \sqcup E_{\preceq 2}^{k}) \models \Lambda(E_{\preceq 2})$$

and the *strong majority plausibility merger* satisfy

$$\text{if } l < k \text{, then}$$
$$\Lambda(E_{\preceq 1}^{l} \sqcup E_{\preceq 2}^{k}) \models \Lambda(E_{\preceq 2})$$

The intuition behind the weak majority merger is that given infinite amounts of total pre-orders that prefer w_1 to w_2 will steer the merger to priorities w_1 over w_2, whereas for the strong majority merger any majority of total pre-orders that prefer w_1 over w_2 is enough.

As was done with belief mergers we need to define what it means to identify a world in this setting, as well as what a learning merger is and what it means to have majority representation.

Definition 27. A world $s \in S$ is *identified by merger* Λ if Λ selects only s as the output for all plausibility profiles $E_{\preceq} = \{\preceq_1, \ldots, \preceq_n\}$, for which it holds that

1. $\forall \preceq_i \in E_{\preceq}$, $\preceq_i$ is based on SBR on data sequence σ_i from agent i
2. Every σ_i is sound w.r.t. s
3. The content of all data sequences, $\mathcal{O}_A = \bigcup_{i=0}^{n} set(\sigma_i)$, is complete w.r.t. s.

$$\Lambda(\preceq_1, \ldots, \preceq_n) = \{s\}$$

Definition 28. A merger Λ is a *learning merger on class $\mathcal{E}_{\preceq}$ of plausibility profiles* if it can for any plausibility profile $E_{\preceq} \in \mathcal{E}_{\preceq}$ identify the world s that $E_{\preceq}$ is built from. A *universal learning merger* is a learning merger on the class of all plausibility profiles.

Definition 29. An observation O is *majority represented* in plausibility space $E_{\preceq}$ when O is present in more than half of the data sequences σ_i used for generating $E_{\preceq}$.

A plausibility profile is *majority represented* w.r.t. s when every observation in $\mathcal{O}_s$ is majority represented in $E_{\preceq}$.

Theorem 4.3.1. Any strong majority plausibility merger Λ that is non-overgeneralising is a learning merger on the class $\mathcal{E}_{\preceq}$ of plausibility profiles where every $E_{\preceq} \in \mathcal{E}_{\preceq}$ is majority represented w.r.t. s and every $\preceq_i \in E_{\preceq}$ satisfy the Occam's razor property, while the σ_i used for $\preceq_i$ is sound w.r.t. s.

Proof. Consider the true world as its truth assignment on propositions representing the observations. For the merger to select the correct world it must find the correct truth assignment for every observation, meaning $\top$ or 1 for any observation in $\mathcal{O}_s$ and $\bot$ or 0 for any observation in $\mathcal{O}$ but not in $\mathcal{O}_s$.

Since Λ is a strong majority merger it will select the worlds that agree with s on the observations in $\mathcal{O}_s$ because there is a majority of total pre-orders supporting those worlds due to the majority representation on $E_{\preceq}$. From this we know that the set of worlds that Λ selects must agree with $\mathcal{O}_s$, so the truth assignment of the result from $\Lambda(E_{\preceq})$ will have selected $\top$ for the correct observations.

The remaining assignments are all observations that can never be in any data sequence since they are outside of $\mathcal{O}_s$. Since Λ is non-overgeneralising we know that any such observation is assumed to be $\bot$ due to the Occam's razor property. $\square$

Noticeably majority representation is still required as information drowning is a continuous problem. If we do the same thought experiment as we did earlier, of having one agent with all of the observations while the rest have none, then we again run into the problem of the agents without information drown out the lone knowledgeable agent. This is a natural problem that comes with using the majority as a selection method.

4.3.1 Ranksum

If $rank(w, \preceq) : w \times \preceq \rightarrow R^+$ is a function that returns world w's position in $\preceq$, then *Ranksum* $\Lambda_{r,\Sigma}(E_\preceq)$ is a plausibility merger that uses $rank(w, \preceq)$ to compare each worlds ranking in all of the total pre-orders.

Calculate each worlds summation of ranks and use it as a numbering on the worlds to provide a total pre-order from which the most plausible world is returned.

$$\begin{aligned} d_\preceq(w, E_\preceq) &= \sum_i^n rank(w, \preceq_i) \mid n = |E_\preceq| \\ D_\preceq(E_\preceq) &= LexOrder(d(w_1, E_\preceq), \ldots, d(w_m, E_\preceq)) \mid m = |S| \\ \Lambda_{r,\Sigma}(E_\preceq) &= min(D_\preceq) \end{aligned}$$

Here $LexOrder$ is used as the standard lexicographical ordering function and not the Lex learning method.

Theorem 4.3.2. $Ranksum$ is a non-overgeneralising plausibility merger.

Proof. Consider $Ranksum$ being applied on a set of total pre-orders that satisfy the Occam's razor property. Let the world s be the true world and for world w it holds that $\mathcal{O}_s \subseteq \mathcal{O}_w$ then according to the Occam's razor property every total pre-order $\preceq_i$ in $E_\preceq$ will priorities s over w no matter which observations are in σ_i, so long as it is sound w.r.t. s.

$$rank(s, \preceq_i) < rank(w, \preceq_i)$$

This leads to the same relation when the summation of the ranks are calculated.

$$d_\preceq(s, E_\preceq) < d_\preceq(w, E_\preceq)$$

So the collective total pre-order will also satisfy the Occam's razor property.

$$rank(s, \preceq_c) < rank(w, \preceq_c)$$

□

Theorem 4.3.3. $Ranksum$ is a weak majority plausibility merger.

Proof. Given two total pre-orders $\preceq_1$ and $\preceq_2$ where $\preceq_1$ priorities world w_1 over w_2 and $\preceq_2$ priorities w_2 over w_1, then;

$$\begin{aligned} d_\preceq(w_1, \preceq_1) < d_\preceq(w_2, \preceq_1) \\ d_\preceq(w_1, \preceq_2) > d_\preceq(w_2, \preceq_2) \end{aligned}$$

While it is possible for the ranks to have very big numeric differences, which gives more weight to one of the total pre-orders, the concept of infinite repetitions dwarfs any big number. Example,

$$\begin{aligned} d_\preceq(w_1, \preceq_1) = 1 < d_\preceq(w_2, \preceq_1) = 10 \\ d_\preceq(w_1, \preceq_2) = 2 > d_\preceq(w_2, \preceq_2) = 1 \end{aligned}$$

	$\preceq_1$	$\preceq_2$	$\sum$
w_1	1	10	11
w_2	2	1	3

Here having only one of each pre-order means that the 10 given to w_1 by $\preceq_1$ outweighs the 2 given to w_2 given by $\preceq_2$. However if we have infinitely many $\preceq_1$, each providing one value more to w_2 than w_1, then the merger will select w_1. □

Theorem 4.3.4. $Ranksum$ is **not** a strong majority plausibility merger.

Proof. By counter example.
Consider the initial plausibility space $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$[5]

$S = \{w_1, w_2, w_3, w_4\}$, true world $= w_4$
$\mathcal{O} = \{O_1, O_2, O_3, O_4\} = \{\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}\}$
$\preceq = \{(w_1, w_2), (w_2, w_3), (w_3, w_4)\}$

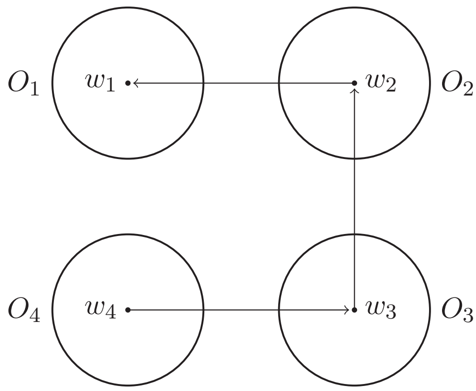

Figure 4.3: Initial plausibility space

There are three agents a_1, a_2 and a_3 with this initial setup. Their data sequences, that are sound w.r.t. s, are as followed;

$$\sigma_1 = (\emptyset) \quad \sigma_2 = (O_4) \quad \sigma_3 = (O_4)$$

Applying the lexicographical learning method $Lex(\mathbb{B}_{\mathbb{S}}, \sigma)$ the three agents pre-order after the belief revision operations are

$$\preceq_1 = \{(w_1, w_2), (w_2, w_3), (w_3, w_4)\}$$
$$\preceq_2 = \{(w_4, w_1), (w_1, w_2), (w_2, w_3)\}$$
$$\preceq_3 = \{(w_4, w_1), (w_1, w_2), (w_2, w_3)\}$$

Should be noted that there is a majority towards w_4, so if $Ranksum$ is a strong majority merger, then it should pick w_4.

[5]Transitive relations in $\preceq$ are omitted in the listing for ease of reading.

	$\preceq_1$	$\preceq_2$	$\preceq_3$	$\sum$
w$_1$	1	2	2	5
w$_2$	2	3	3	8
w$_3$	3	4	4	11
w$_4$	4	1	1	6

Table 4.3: Λ_Σ calculations. Only possible worlds are represented.

Here we see that it selects w_1 even though there is a majority of total pre-orders that priorities w_4. $\square$

$Ranksum$ not being a strong majority merger means that we cannot call it a learning merger on class $\mathcal{E}_{\preceq}$ using Theorem 4.3.1. However it is still an open question if a non-generalising weak majority plausibility merger is a learning merger on class $\mathcal{E}_{\preceq}$.

5 Conclusion

In the endeavour of finding a reasonable merging solution for a set of learning agents, as per [2], that individually do not have sufficient information for truth tracking, but together they do, we have shown that Konieczny and Pino Pérez's majority integrity constraint mergers on conjectures [3] are not sufficient and neither are D'Alfonso's attempt at modifying Konieczny and Pino Pérez's mergers into a truthlikeness merger [4].

The result that has been found is that when trying to combine belief revision agents to improve their scope of learning it is necessary to consider every aspect of the information that each individual agent has. In the case of Baltag, Gierasimczuk and Smets's learning agents from [2] it is necessary to merge on the agents total pre-orders for it is within them that all of the information from their observations are stored. In an attempt to merge the total pre-orders we have introduced plausibility mergers, where two properties have been found required to achieve a so called *learning merger* that can combine learning agents to identify the true world when they cannot individually. The two properties are *non-overgeneralisation*, which is important in order to assume observations that are outside the real world are false, and the other is that of *strong majority plausibility mergers* which is one way of ensuring that when all the necessary information is present and in majority between all of the learning agents, then the merger will select worlds that follow that majority.

5.1 Future Work

Is $Ranksum$ a learning merger on $\mathcal{E}_{\preceq}$? In section 4.3.1 we show how we cannot use Theorem 4.3.1 to argue that $Ranksum$ is a learning merger on class $\mathcal{E}_{\preceq}$, but this does not necessarily mean that it isn't. While $Ranksum$ is not a strong majority merger, we do prove that it is a weak majority merger in Theorem 4.3.3 and it is still an open question whether or not the weak majority property is enough to be a learning merger on $\mathcal{E}_{\preceq}$.

Does there exist a strong majority plausibility merger? If it turns out that strong majority property is required to nullify the problem from information drowning then the next natural question is there a non-overgeneralising plausibility merger that is a strong majority merger. As has been shown in 4.3.1 such a merger would be a learning merger on the class $\mathcal{E}_{\preceq}$.

Is it possible to make a merger that does not require majority representation? If there exists a selection method besides the majority method then that solution would, probably, remove the requirement of majority representation. It is, however, rare to find alternatives that have zero drawbacks, but maybe there exists solutions with less severe ones.

It would be amiss to not talk about our assumption of truth, in that we expect that there is a harsh line between true and false worlds, and even that there is only one world that is true. In the pragmatic theory of truth [9], introduced to me as the ideas of William James although I think there are more people that should be credited, they talk about the idea that what is true is more so a question of what is useful to you rather than which specific world is true. There might be multiple worlds where if you believe it to be true it is of use to you, and which one you pick might not matter to you in this instance. This idea works very well with the fact that scientific beliefs are constantly changing, and theories we believe to be true might be refuted at any moment.

So instead of thinking that there is only one true world s, maybe it would benefit us to say that there is a set of true worlds instead. This idea only really works when there is some goal with the truth tracking. Say that the propositions p and q are the only important propositions with regards to the goal of the truth tracking, but there are more propositions available for consideration. Then the set of true worlds would be those that satisfy p and q, and the rest are of little importance. An interesting relation here is that the majority belief merger and truthlikeness merger's main problem was there were observations that we did not have information for. However if we adopt the concept of having a set of true worlds, then it could be that those observation we did not have information for are exactly those we don't care about.

Merging on total pre-orders As far as I can tell there has not been very much research into the possibility of merging total pre-orders, and it may be that to further the topic of collective turth tracking more work in that field is required. It would be very interesting to consider that problem in more detail.

Probability based mergers (Condorcet's Jury Theorem) In Judgement Aggregation one of the main strengths is the property of Condorcet's Jury Theorem, whose central point is that as a set of agents that are better than average at selecting the true world grows towards infinity the probability of selecting the true world tends towards 1. If we can show that each learning agent is better than average at selecting the true world, then merging their conjectures would follow Condorcet's Jury Theorem. While this notion of tending towards infinite agents tends the result towards being correct is very desirable, one would still be interested in using a merger that retain as much information as possible as infinite agents are not an option in the real world.

In general Condorcet's Jury Theorem is a very powerful concept and is probably the approach that has the highest likelihood of providing an applicable solution in the real world. Probabilistic solutions have proved to be very effective when converting theoretical ideas into something that works when implemented.

Erroneous Data In the real world observations can never really be expected to be a hundred percent certain to be sound with regard to the true world and any implemented system should have at least some margin for error. This is a notion that is often ignored in theoretical works, where it is expected to be handled later. In [2] Baltag et al. show a way to handle erroneous data sequences by expecting them to be *fair*, meaning any error will eventually be corrected later in the data sequence.

Combining multiple agents information becomes a little more complicated when the data sequences are only fair with regard to the true world instead of sound. Since we are working with a static operation on a specific time in the learning process of individual agents it becomes harder to argue that the merger will identify the true world. In Baltag et al.'s paper they have the luxury of working on a continuous process so as long as it eventually receives the correct data it is not a problem. If we wanted to use the same argument for our static merger it would require that the identification of the true world can only happen when the observations are sound, so the resolution is to simply add an additional restriction on the data sequences, which does not seem satisfactory. Probably a better solution is to transform the merging operation to be a continuous operation as Baltag et al. do.

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