Learning from Belief Merge
Draft

Zoé Christoff
University of Groningen

Nina Gierasimczuk
Technical University of Denmark

Thomas Løye Skafte
Technical University of Denmark

Abstract

In [1] a method for truth-tracking via identification in the limit was introduced for singular learning agents by using belief revision. In this work we study the options for broadening this concept to multiple agents performing truth-tracking as a collective. The intention of the joint approach is to transfer the information requirements on the individual agent to a shared responsibility. Thereby, the information an individual agent has might not be sufficient to perform truth-tracking on their own and their conjectures will be individually inaccurate but their collective conjecture will be precise. The two main approaches we investigate are (1) merging on the agents conjectures and (2) merging on the agents entire belief ordering as a whole. Merging on conjectures (1) turns out to be prone to information loss with relations to Arrow's Impossibility Result [2, 3], while merging on the entirety of the belief orderings (2) has ties to the epistemic notion of distributed knowledge.

1 Introduction

Intelligent agents are those with the capacity to learn form new information by integrating it into their beliefs and knowledge. No matter how efficient an agent is at learning there must be some set of observations required for the agent to draw the necessary conclusions. This means that for an agent to learn something it is required to collect enough data to make the required inference. One of the pillars of human advancement has been the ability to divide this requirement of knowledge acquirement on intelligent learners such that instead of individual agents only performing inference on their own experience they construct their conjectures on the knowledge of the group as a whole. Through this the probability of a learner to have the necessary knowledge goes from the probability that this specific learner experiences all the necessary encounters to the probability that each event occurs for any member in the group. For this reason collective learning is a hugely important notion in human advancement.

While there has definitely been interesting approaches made to model learning agents, such as [1, 4], they all neglect this very important aspect of collective learning allowing agents to share the burden of required observations. In this work we will examine methods to allow agents to perform collective learning by combining their knowledge using merging methods. The setting that this merging will be done in is that of plausibility spaces [1], where every possible world is considered and the most believed world, the agents conjecture, is the minimum in a partial ordering. The main objective of the work is to see what method of merging is best suited for this goal. The two approaches that we will investigate is on one hand merging on the conjectures of the agents and on the other merging on the entire plausibility spaces.

2 Learning Agents

The purpose of the learning agents introduced in [1] is to identify in the limit a true world from a set of possible worlds. This is primarily done by belief revision methods, but the core concept that is utilised is based on the notion of Occam's razor.

The model used is called a *plausibility space*, $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$, consisting of a set of possible worlds S, a set of all possible *observations* $\mathcal{O}$ and a *plausibility relation* $\preceq \subset S \times S$ which is a total pre-order over S.

An observation is considered as something that can hold in (or about) a world, and is represented extensionally by the set of worlds where that observation holds true, for instance if the observation O is true only in worlds w and v, we write $O = \{w, v\}$. We are only concerned with positive observations, the fact of an observation O not holding in a world w is only derivable from the absence of w in O. $\mathcal{O}_w$ is the set of observations that hold for w.

A data sequence σ is a finite list of observations that the agent is given, and a data stream ρ is a infinite stream of observations. For both σ and ρ it is assumed that they are sound and complete with regards to some true world s and duplicate observations are allowed. The inclusion of streams is to handle the concept of worlds with an infinite amount of observations.

In some situations a world can be distinguished from all other worlds in the plausibility space, by finitely many observations. In such a case we will say that the world has a definite finite tell tale.

Definition 1. Let $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$ be a plausibility space, and let $s \in S$. We say that $D_s \subseteq \mathcal{O}$ is a *definite finite tell tale* of s if the following conditions hold:

1. D_s is finite;
2. $D_s \subseteq \mathcal{O}_s$;

3. for all $w \in S$, if $s \neq w$ then $D_s \not\subseteq \mathcal{O}_w$.

A weaker type of tell tale for a world, simply called a *finite tell-tale* is a finite set of observations that allows to make sure that the less parsimonious worlds than the selected one are eleminainated by it.

Definition 2. Let $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$ be a plausibility space, and let $s \in S$. We say that $F_s \subseteq \mathcal{O}$ is a *finite tell tale* of s if the following conditions hold:

1. F_s is finite;
2. $F_s \subseteq \mathcal{O}_s$;
3. for all $w \in S$, if $s \neq w$ and $F_s \subseteq \mathcal{O}_w$ then $\mathcal{O}_w \not\subset \mathcal{O}_s$.

A learning method is said to identify in the limit if there exists a plausibility relation $\preceq$ such that it will eventually output the true world s and only s from then on. If the initial plausibility relations $\preceq$ is build in accordance with the *Occam's razor property*, which follows the natural ordering by finite-tell tales (worlds with fewer observations will always be prioritised over any world with more observations), then both conditioning and lexicographic revision have been proven to identify in the limit the true world s, given the two assumptions of the data sequence being sound and complete, [1].

Belief revision methods are used to update the current plausibility space $\mathbb{B}_{\mathbb{S}}$ as observations are given from the data sequence. The two methods used are *conditioning*, where $\mathbb{B}_{\mathbb{S}}$ is updated by removing inconsistent worlds from S, and *lexicographic revision* where the plausibility relation $\preceq$ is changed by making consistent worlds the most plausible and inconsistent worlds the least plausible, while keeping internal relations. These two belief revision methods have been shown to generate universal single-agent learning methods in [1]. The third kind, *minimal revision*, works by making only the most-preferred worlds (consistent with the incoming information) the best in the resulting plausibility space. The popular *minimal revision* has been shown to be deficient as a learning method. We might also investigate it below in the multi-agent context.

For conditioning this is easy to argue for because only the set of worlds satisfying F_s will remain in $\mathbb{B}_{\mathbb{S}}$ after some sufficient amount of updates and knowing that s is guaranteed to be the world in $\cap F_s$ with the fewest observations it will also be the most prioritised world in $\cap F_s$ due to how $\preceq$ is built.

The argument for lexicographic revision builds largely on the *non-overgeneralising property* which is a consequence of the Occam's razor property and the data sequence being sound and complete w.r.t. s.

Definition 3. A plausibility space $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$ satisfies the *non-overgeneralising property* if the plausibility relation $\preceq$ is built in accordance with the Occam's razor property.

Observation 1. If a plausibility space $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$ satisfies the *non-overgeneralising property*, then an $s \in S$ will always (after any number of belief revision updates using observations from $\mathcal{O}_s$) be prioritised over world w, if $\mathcal{O}_s \subset \mathcal{O}_w$.[1]

First note that if $\mathcal{O}_s \subset \mathcal{O}_w$ then $w \in \cap F_s$, meaning w satisfy the finite tell tale of s. Given that σ is sound w.r.t. s it is enforced that any observation from σ will satisfy both w and s, thereby any lexicographic revision operation using σ will keep the relation between w and s intact. This together with the fact that s is the world with the fewest observations in $\cap F_s$ (otherwise it would not be the finite tell tale for s) means that in $\preceq$ the world s will always be the most prioritised world from the $\cap F_s$ set. Making s the most prioritised world in $\preceq$ from the set S simply requires σ to contain every observation from $\mathcal{O}_s$, which is why σ is assumed to be complete w.r.t. s.

2.1 Properties of learning methods

To discuss the differences and similarities between learning methods, properties of them can be very useful. Learning method L is:

- *weakly data-retentive* iff $L(\mathbb{S}, \sigma) \neq \emptyset$ implies $L(\mathbb{S}, \sigma) \subseteq \sigma_n$ for any $\mathbb{S}$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$;
- *strongly data-retentive* iff $L(\mathbb{S}, \sigma) \neq \emptyset$ implies $L(\mathbb{S}, \sigma) \subseteq \bigcap_{i \in \{1,\ldots,n\}} \sigma_i$, for any $\mathbb{S}$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$;
- *conservative* iff $\emptyset \neq L(\mathbb{S}, \sigma) \subseteq p$ implies $L(\mathbb{S}, \sigma) = L(\mathbb{S}, \sigma * p)$, for any $\mathbb{S}$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$;
- *data-driven* if it is both conservative and weakly data-retentive;
- *memory-free* iff $L(\mathbb{S}, \sigma) = L(\mathbb{S}, \sigma')$ implies $L(\mathbb{S}, \sigma * p) = L(\mathbb{S}, \sigma' * p$, for any $\mathbb{S}$ and every two data sequences σ, σ' and every $p \in \mathcal{O}$.

3 Multiple Learning Agents

We will extend the above setting to the multi-agent case of n agents by combining their individual data streams into a multi-set of n data streams $P = \{\rho^1, \ldots, \rho^n\}$, which will be called a *data stream profile*. To allow modelling of agents receiving information at different times the empty observation $\emptyset$ is allowed. Formally it can be thought of as an observation that holds in every world, so receiving it provides no information.

At *learning step* m there is a finite m-sized segment of P, denoted $P[m]$, that can be found by taking the initial segment of each data stream in the profile, $P[m] = (\rho^1[m], \ldots, \rho^n[m])$. Since $\rho[m]$ is a finite data sequence we will also reference them as σ and this is also how we will denote *data sequence profiles* $\Sigma = (\sigma^1, \ldots, \sigma^n)$.

Definition 4. A *plausibility profile* E is a multi-set of n plausibility spaces $\mathbb{B}_{\mathbb{S}}$ that statically represents n agents current beliefs.

For any plausibility space $\mathbb{B}_{\mathbb{S}}$, data stream profile P, learning step m and revision method R there is a unique E that expresses each individual agents current beliefs after m up-

[1] This is due to s being prioritised over w in the initial plausibility relation, and any update that is sound w.r.t. s cannot change that relation.

dates from their own data stream.

$$\begin{aligned} E &= (R(\mathbb{B}_{\mathbb{S}}, \rho^1[m]), \ldots, R(\mathbb{B}_{\mathbb{S}}, \rho^n[m])) \\ &= (R(\mathbb{B}_{\mathbb{S}}, \sigma^1), \ldots, R(\mathbb{B}_{\mathbb{S}}, \sigma^n)) \\ &= (\mathbb{B}^1_{\mathbb{S}}, \ldots, \mathbb{B}^n_{\mathbb{S}}) \end{aligned}$$

Note how any plausibility profile will always be based on a data sequence profile. This is the case because plausibility spaces are a static representation of the agent's belief at learning step m, so even if an agent is operating on a stream any plausibility space that the agent produces will always be based on a finite amount of observations $\rho[m]$.

Let the class of all plausibility spaces $\mathbb{B}_{\mathbb{S}}$ be denoted $\mathcal{B}$, and the class of all plausibility profiles be $\mathcal{E}$.

Definition 5. A *plausibility aggregation function* is a function that takes a plausibility profile of size n and returns a plausibility space.

$$f : \mathcal{E} \to \mathcal{B}$$

Definition 6. Given a plausibility aggregation function f and a selection function γ, the *collective learning method* Λ^γ_f assigns a set of worlds to each plausibility profile by first aggregating to a single plausibility space and applying the selection function after.

$$\Lambda^\gamma_f : \mathcal{E} \to \mathcal{P}(\bigcup_{i=1}^{n} S_i)$$

An agent applying a collective learning method is a *collective learning agent*.

The selection function is a modular part, but for this work we take it to be the $\min$ of the plausibility relation $\preceq$ of $f(E)$, and so $\Lambda^{\min}_f(E) = \min(f(E))$. The result of the $\min$ selection function on an individual plausibility space is called the *conjecture* and can consist of multiple worlds. Whenever specifically referring to the multi-agent case we will call the output of Λ^γ_f a *collective conjecture*.

We want to study collective learning, and so we want to make the above setting dynamic. There are many ways to achieve that. The path we want to pursue in this paper is characterised by the following principles:

We make the following assumptions:

1. the agents are homogeneous, they all revise their belief according to the same belief revision method (either of conditioning, lexicographic, or minimal revision);
2. the agents start with the same plausibility space, $\mathbb{B}_{\mathbb{S}} = (S, \mathcal{O}, \preceq)$, with one world $s \in S$ (unknown to the agents) is designated to be the actual one;
3. each agent i is inductively given a stream ρ_i that is sound with respect to the real world, which is an $s \in S$;
4. collectively their data streams sum up to a sound and complete data stream for s, i.e., $set(\Sigma) = \mathcal{O}_s$;

Assumption 4 is a relaxation of the requirements for a singular agent to identify in the limit that is possible because the completeness burden can now be shared between all of the agents involved.

Definition 7. A plausibility profile E is sound and complete w.r.t. s if for the data sequence profile Σ it is the case that $set(\Sigma)$ is sound and complete w.r.t. s.

The following are the two interesting cases of their group communication:

1. at each stage their individual plausibility spaces are aggregated, corresponding to them submitting their full individual plausibility spaces;
2. at each stage their individual conjectures are combined, corresponding to them individually submitting their sets of most plausible worlds, or equivalently, communicating the conjunction of all believed propositions.

4 Merging Perspectives

There exists suggestions of frameworks for convergence of multiple sources of belief bases into a single belief base, commonly called *belief merge*. One notable framework is the *integrity constraint belief mergers* by Konieczny and Pino Pérez, [5] [6]. The standard objective of belief mergers is to combine a profile of belief bases into a consistent belief base that best represents the whole profile. This concept somewhat deviates from the goal of truth tracking since if the profile agrees on some incorrect observation then the incorrect observation should be in the output. With the assumption that the information given is sound w.r.t. the true world this difference does not matter as such an incorrect observation can never occur, however we would like to let go of that assumption if possible, at least to some degree.

Synthetic View: From the synthetic perspective the goal is to find a collective belief base that best represents the information of the profile, even if some of it is incorrect.

Epistemic View: From the epistemic perspective the goal is to identify the true world, even if given incorrect information.

Therefore a belief merger of synthetic view is not satisfactory and frameworks of such a perspective are not adequate, including the integrity constraint belief mergers. Another argument is that belief mergers tend to work on profiles of belief bases, where as we are working on profiles of plausibility spaces. One could take the conjecture of the agent as the belief base and use it for merging, however this introduces uncertainty into the system as the conjecture can contain observations that are not sound w.r.t. the true world, even if all the observations in σ are. This occurs because going from a plausibility space to its conjecture with insufficient information can allow the conjecture to contain observations that have yet to be shown inconsistent with the true world. Applying traditional aggregation functions from the synthetic view, such as majority, on the conjectures allows for the possibility of incorrect observations to out weigh the correct observations. For example if a single agent is given observation p, but every other agent's conjecture contains

$\neg p$, then the majority merger will select $\neg p$ and the observations of p will be ignored. This *information loss* is an immediate consequence of the synthetic view.

4.1 Preaggregation

A well known result from social choice theory is *Arrow's impossibility theorem*, [2], for which one interpretation is that when working with a profile of preference relations represented by total preorders there is no aggregation function that successfully merges the profile without being a dictatorship, [7]. While preference aggregation is another example of the synthetic view the result about aggregating total preorders is very relevant given that the plausibility relations $\preceq$ are total preoders. In fact in [7] a theorem generalising Arrows impossibility theorem describes the problem to be more of a mathematical nature, by explaining the problem with the perspective of graph aggregation.

In a paper by Vincke, [3], they divide aggregation functions into two steps. The first is regarded as the preaggregation function, for which the point is to combine the information present in the profile to more precisely find the best possible candidates for the final result of the aggregation function. The second part is then to select the result of the aggregation function between the best candidates from the preaggregation function. What is notable here is that the problem of Arrow's impossibility theorem occurs during the selection after the preaggregation has concluded. Consider an example of four agents merging their total preorders on three possible worlds a, b and c.

Example 1.
$A = \{a, b, c\}, \quad n = 4$

$$(\preceq_1, \preceq_2, \preceq_3, \preceq_4) = \begin{cases} a \preceq_1 b \preceq_1 c, \\ a \preceq_2 b \preceq_2 c, \\ b \preceq_3 a \preceq_3 c, \\ b \preceq_4 a \preceq_4 c. \end{cases}$$

For $\preceq$ to aggregate $(\preceq_1, \preceq_2, \preceq_3, \preceq_4)$ we must have $a \preceq c$ and $b \preceq c$ since every agent agrees, but there is no way to decide between $(a \prec b)$ or $(b \prec a)$ or $(a \preceq b$ and $b \preceq a)$. It is in selecting between the candidates from the preaggregation function that a dictatorship arises, or at least an oligarchy. From the synthetic perspective all three candidates perform equally as a final option, so selecting either of them as the main result works. From the epistemic perspective there is only one correct answer, the true world, which makes it more accurate to have the merger give the set of all three candidates rather than selecting one for which there is no proof is better. In fact, the learning agents of [1] work in similar fashion. They output a conjecture consisting of the best candidates for a result which is guaranteed to be of size one when the conditions of identifying in the limit are met.

5 Collective learning

For sound and complete data streams using conditioning for learning agents is a very elegant solution that fits simulating learning. Removing the worlds that don't comply with the observations is a very simple solution to truth tracking as long as the information never goes against the true world. For singular learning agents working on sound and complete sequences it will eventually remove any world that disagrees with the true world, and then the non-overgeneralising property will ensure that the correct world is selected from the set of worlds that satisfy the finite tell tale. To have this same concept work in a multi agent setting means the information that is split over all of the agents would have to be combined in such a way that the set of worlds satisfying the finite tell tale can be found, from which the non-overgeneralising property will again ensure selecting the correct world. This also implies that the non-overgeneralising property has to persist through the merging.

Definition 8. An aggregation function f is *collectively rational* w.r.t. graph property p if $f(E)$ satisfy p when all $\preceq \in E$ do. [7].

Combining the information from each agent in the plausibility profile turns out to be very simple due to the simplicity of conditioning.

Observation 2. Intersection on a set A of learning agents remaining worlds S_i will output the set S_A of worlds that every agent in A still considers plausible.

$$\bigcap_{i=1}^{n} S_i = S_A$$

After intersection on a plausibility profile that is sound and complete w.r.t. the true world s, only worlds that satisfy every observation in $\mathcal{O}_s$ will remain. In other words, the result of the intersection will be exactly the worlds that satisfy the finite tell tale F_s, which is $\cap F_s$. The method that the singular learning agents of [1] use to identify the true world in the limit is by finding $\cap F_s$ and then use the non-overgeneralising property to select the true world from $\cap F_s$. In the following section we will define a collective learning method $\Lambda_\cap$ that finds $\cap F_s$ by intersection and uses it to identify in the limit the same way the individual learning agents do.

5.1 Collective learning method with intersection

A collective learning method requires an aggregation function that provides a single plausibility space which contains the entire plausibility profiles information and then some selection function that selects the correct world from the collective plausibility space.

Aggregation Function

$$f_\cap(E) = (\bigcap_{i=1}^{n} S_i, \mathcal{O}, \bigcap_{i=1}^{n} \preceq_i)$$

Since the initial plausibility space of each agent is the same and no conditioning operation can alter the individual relation directions finding the correct relations for the aggregated plausibility space is as simple as figuring out what worlds remain after the aggregation and then keep the relations where both worlds are kept. A faster way of doing this is to think of the relations as simple elements, by enumeration for example, and apply intersection.

Observation 3. Conditioning can be defined in terms of intersection.

$$cond(\mathbb{B}_{\mathbb{S}}, p) = (S \cap p, \mathcal{O}, \preceq^{S \cap p})$$

Where $\preceq^{S \cap p}$ are the relations between the worlds that remain in $S \cap p$.

Proposition 1. The aggregation function $f_{\cap}$ is a generalisation of conditioning.

$$cond(\mathbb{B}_{\mathbb{S}}, p) = f_{\cap}(\{\mathbb{B}_{\mathbb{S}}, \mathbb{B}_p\})$$

Where $\mathbb{B}_p = (p, \mathcal{O}, \preceq^p)$ is the plausibility space that can be created around the observation p. Since both plausibility relations $\preceq^p$ and $\preceq^S$ are built following the Occam's razor property they will agree on any relation between two worlds that are in both plausibility spaces.

Proof. Both conditioning and the intersection aggregation function removes worlds from S and then removes relations in $\preceq$ that are no longer necessary. This means that we can solely look at S as everything else follows from what is done to S. Since both functions in the formula can be expressed through intersection on the set of plausible worlds, we can see if they equate when transforming both of them.

As per observation 3 we know that the conditioning can be written as $S \cap p$ and as for the aggregation function all that is done to the set plausible worlds is an intersection between the two.

$$\begin{aligned} cond(\mathbb{B}_{\mathbb{S}}, p) &= f_{\cap}(\{\mathbb{B}_{\mathbb{S}}, \mathbb{B}_p\}) \\ (S \cap p) &= f_{\cap}(\{\mathbb{B}_{\mathbb{S}}, \mathbb{B}_p\}) \\ &= (S \cap p) \end{aligned}$$

□

Proposition 2. If for data sequence σ and plausibility profile E their data sets are the same, $set(\sigma) = set(\Sigma)$, and their starting points are equal, $S = \bigcap_i^n S_i$, then applying conditioning on the whole σ provides the same plausibility space as the aggregation function on E.

$$cond(\mathbb{B}_{\mathbb{S}}, \sigma) = f_{\cap}(\{cond(\mathbb{B}^1_{\mathbb{S}}, \sigma_1), \dots, cond(\mathbb{B}^n_{\mathbb{S}}, \sigma_n)\})$$

$$\text{when} \quad set(\sigma) = set(\Sigma) \text{ and } S = \bigcap_i^n S_i,\ S \in \mathbb{B}_{\mathbb{S}}, S_i \in \mathbb{B}^i_{\mathbb{S}}$$

Proof. Both conditioning and the intersection aggregation function are done by intersection and thus follow the distributive law saying that serial operations can be expressed by applying the operations individually and then on the results, [8]. Shown abstractly with intersection in our system this can be expressed as

$$((S \cap p) \cap q) \dots = (S \cap p) \cap (S \cap q) \cap \dots$$

Together with the requirement that $S = \bigcap_i^n S_i$ we get

$$((S \cap p) \cap q) \dots = (S_1 \cap p) \cap (S_2 \cap q) \cap \dots$$

And then converted back to the operations

$$\begin{aligned} &cond(\dots cond(cond(\mathbb{B}_{\mathbb{S}}, p), q), \dots) \\ &\qquad = f_{\cap}(\{cond(\mathbb{B}^1_{\mathbb{S}}, p), cond(\mathbb{B}^2_{\mathbb{S}}, q), \dots\}) \end{aligned}$$

Writing the sequences of observations as data sequences

$$cond(\mathbb{B}_{\mathbb{S}}, \sigma) = f_{\cap}(\{cond(\mathbb{B}_{\mathbb{S}}, \sigma_1), \dots, cond(\mathbb{B}_{\mathbb{S}}, \sigma_n)\})$$

□

Lemma 1. If E is sound and complete w.r.t. s then it is the case that $\bigcap_{i=1}^n S_i = \bigcap F_s$ when $S_i \in \mathbb{B}^i_{\mathbb{S}}$.

Proof. Since the plausibility profile E is sound w.r.t. s the possible world sets S_i will all contain every world w that agrees with s on every observation in $\mathcal{O}_s$ including s.

That E is collectively complete w.r.t. s necessarily implies that for any world v that is not in $\bigcap \mathcal{O}_s$ there has to be at least one S_i that doesn't contain v, and thus the intersection $\bigcap_{i=1}^n S_i$ will not contain v either.

From these two arguments we know that the intersection $\bigcap_{i=1}^n S_i$ will only contain worlds that agree with s on every observation in $\mathcal{O}_s$, therefore $\bigcap_{i=1}^n S_i = \bigcap \mathcal{O}_s$. By definition the set of worlds that satisfy the finite tell tale of s is exactly the worlds that satisfy every observation in $\mathcal{O}_s$, formally written $\bigcap \mathcal{O}_s = \bigcap F_s$.

$$\bigcap_{i=1}^n S_i = \bigcap \mathcal{O}_s = \bigcap F_s$$

□

Selection Function

$$\gamma_{\cap}(\mathbb{B}_{\mathbb{S}}) = \min(\preceq)$$

From Proposition 2 we know that the resulting plausibility space of the aggregation function will be the same as if it was done by a single learning agent given identical information, so the same selection function that the learning agent uses can be used. Said selection function utilises the non-overgeneralising property, so we need to ensure that the property follows through the merging, i.e. it is required to be collectively rational with respect to this property.

Proposition 3. $f_{\cap}$ is collectively rational w.r.t. the non-overgeneralising property.

Proof. For a plausibility relation to follow the non-overgeneralising property the observations used for belief revision operations must be sound and complete w.r.t. the true world s and the relations must follow the Occam's razor property, stating that the worlds with the least amount of observations are prioritised. Since soundness and completeness are assumed all that is needed to show that the non-overgeneralising property persists through the merging is that Occam's razor property does.

Intersection will by definition only ever remove relations. Since the plausibility relation is total and transitive removing a world and all the relations tied to it will not have an impact on the relations between the other worlds. As such the Occam's razor property will be upheld and therefore the non-overgeneralising property will as well. □

Intersection learning method

$$\Lambda_\cap(E) = \gamma_\cap(f_\cap(E))$$

The intersection learning method applies the selection function on the output of the aggregation function.

Proposition 4. Given a plausibility space E that is sound and complete w.r.t. s, the intersection learning method $\Lambda_\cap(E)$ will produce the singleton set $\{s\}$.

Proof. From Lemma 1 we know that an intersection on the set of possible worlds from a plausibility profile E that is sound and complete w.r.t. s will provide the set of worlds that satisfy the finite tell tale of s, namely F_s. So $f_\cap(E_j) = (\bigcap F_s, \mathcal{O}, \preceq_{\cap F_s})$. Proposition 3 says that $f_\cap$ is collectively rational w.r.t. the non-overgeneralising property, so since the plausibility spaces are built such that they are non-overgeneralising (from assumption) then so is $f_\cap(E)$. Since $f_\cap(E)$ is both the plausibility space for the finite tell tale of s and non-overgeneralising we know that the most prioritised world must be s, and therefore $\Lambda_\cap(E) = \{s\}$ □

Proposition 5. A learning agent using conditioning on σ equates to applying the intersection learning method on plausibility profile E with data profile Σ when $set(\sigma) = set(\Sigma)$ and $S = \bigcap_i^n S_i$.

$$L_{cond}(\mathbb{S}, \sigma) = \Lambda_\cap(\{cond(\mathbb{B}^1_\mathbb{S}, \sigma_1), \ldots, cond(\mathbb{B}^n_\mathbb{S}, \sigma_n)\})$$
$$\text{when} \quad set(\sigma) = set(\Sigma) \text{ and } S = \bigcap_i^n S_i,\ S \in \mathbb{S}, S_i \in \mathbb{B}^i_\mathbb{S}$$

Proof. From Proposition 2 we know that the plausibility space from $f_\cap(E)$ and $cond(\mathbb{B}_\mathbb{S}, \sigma_A)$ is the same when $set(\sigma) = set(\Sigma)$ and $S = \bigcap_i^n S_i$.

$$\begin{aligned}\Lambda_\cap(E) &= \gamma_\cap(f_\cap(E)) \\ &= \gamma_\cap(cond(\mathbb{B}_\mathbb{S}, \sigma_A))\end{aligned}$$

The selection function $\gamma_\cap = \min(\preceq)$ is exactly the method used by the learning agents to produce the conjecture, and since the two methods produce the same plausibility space the selection function will also select the same conjecture.

$$\begin{aligned}\gamma_\cap(f_\cap(E)) &= \min(cond(\mathbb{B}_\mathbb{S}, \sigma_A)) \\ &= L_{cond}(\mathbb{S}, \sigma_A)\end{aligned}$$

□

The output of an agent doing the intersection conjecture function on a profile E cannot be distinguished from a singular learning agent performing conditioning on σ when $set(\sigma) = set(\Sigma)$ and $S = \bigcap_i^n S_i$.

Consider a set of n singular learning agents whose data streams together make up a data stream profile $P = \{\rho^1, \ldots, \rho^n\}$. Each ρ^i is sound w.r.t. the world s and collectively the streams are complete w.r.t. s, meaning the profile is sound and complete w.r.t. s. For each step in the streams there is a plausibility profile E that represents the agents static belief at that stage in the learning. The different iterations of E will be denoted E_0 for the initial profile before any conditioning is applied, E_j for learning step j and E_m for the last iteration.

For each step every stream in the profile must include another observation, however since the empty observation $\emptyset$ is allowed we can still model the case of an agent not receiving any new observation.

Step	Profile		Conjecture
0	$E_0 = \{\mathbb{B}^1_\mathbb{S}, \ldots, \mathbb{B}^n_\mathbb{S}\}$	$\rightarrow$	$\Lambda_\cap(E_0)$
1	$E_1 = \{\mathbb{B}^1_\mathbb{S}, \ldots, \mathbb{B}^n_\mathbb{S}\}$	$\rightarrow$	$\Lambda_\cap(E_1)$
$\vdots$	$\vdots$		$\vdots$
m	$E_m = \{\mathbb{B}^1_\mathbb{S}, \ldots, \mathbb{B}^n_\mathbb{S}\}$	$\rightarrow$	$\Lambda_\cap(E_m)$

Definition 9. A collective learner is *monotonic* if $\Lambda(E_j) \subseteq \Lambda(E_i)$ when $i < j$.

Proposition 6. An agent using the intersection learning method $\Lambda_\cap$ is a monotonic learner.

Proof. Both functions, conditioning and intersection learning method, operate by removing worlds that have been shown to be inconsistent with s. Therefore any application of the two functions will either produce the same plausibility space as previous, if the information is already known, or reduce the state space in some fashion, so $\Lambda(E_j) \subseteq \Lambda(E_i)$ when $i < j$.

□

Since conditioning based learning agents and intersection merging agents behave interchangeably when provided the same information and starting point we can consider merging agents as learning agents and thereby also apply the same concept of singular learning agents identifying in the limit from [1] to collective learners.

Definition 10. Given an epistemic space $\mathbb{S} = (S, \mathcal{O})$, a world $s \in S$ is *collectively identified in the limit by learning pair* (R, Λ) if, for every sound and complete data stream profile for s, there exists a learning step j after which $\Lambda(R(\mathbb{B}_\mathbb{S}, \rho^1[j]), \ldots, R(\mathbb{B}_\mathbb{S}, \rho^n[j]))$ outputs the singleton $\{s\}$.

The epistemic space $\mathbb{S}$ is *identified in the limit by* (R, Λ) iff all its worlds are identified in the limit by (R, Λ).

An epistemic space $\mathbb{S}$ is *identifiable in the limit* (learnable) if there exists a learning pair that can identify it in the limit.

Lemma 2. Given a data stream profile P that is sound and complete w.r.t. s the collective learning pair $(L_{cond}, \Lambda_\cap)$ will collectively identify in the limit the world s.

Proof. If the reader has read [1] this is easily to argue with proposition 5 considering that the collective learner $\Lambda_\cap$ is indistinguishable from a singular learning agent L_{cond}, which has been showed in [1] to identify s in the limit for such a situation. However let us still provide proof without relying on this relation.

Since the data stream profile P is sound and complete w.r.t. s there must be some learning step j where the resulting plausibility profile will be sound and complete w.r.t. s.

In proposition 4 it is shown that the intersection learning method on a plausibility profile that is sound and complete w.r.t. s produces the singleton set $\{s\}$. Once the intersection learning method has produced the singleton $\{s\}$ it will stay on $\{s\}$ forever since any observation O that any agent could update its individual plausibility space with would never remove s from its S_i due to $O \in \mathcal{O}_s$. So we know that the learning pair will by some learning step j return the singleton $\{s\}$ and continually do so forever, thereby having collectively identified in the limit the world s. □

Definition 11. A learning method pair are *universal on a class C of epistemic spaces* if they can identify in the limit every epistemic space in C that is identifiable in the limit. A *universal learning method pair* is one that is universal on the class of all epistemic spaces.

Proposition 7. The learning method pair $(L_{cond}, \Lambda_\cap)$ are universal.

Proof. Work in progress. □

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