The trivial square of the differential: a Morse theoretic perspective

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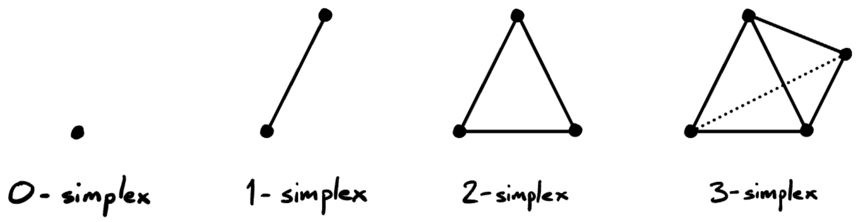
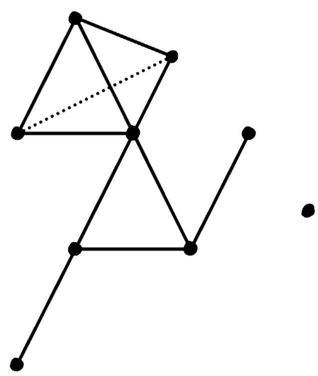
The differential is an interesting beast in many respects. It is a map which has been, ironically, integral to the theory of calculus, and has the fascinating property that its square disappears. Last summer I had a go at proving why this happens using a combinatorial version of an existing proof by Michele Audin, Mihai Damian and Reinie Erné. The goal was to utilise the subtle force that is Morse theory (sadly no relation to the code of dots and dashes).

Morse theory is all about analysing the topology on a manifold by looking at how differentiable functions act on it. To translate this into a combinatorial language, we can make the manifolds be simplicial complexes. If you don't know what a simplicial complex is, the name can look a bit intimidating but it is in reality just a graph, where each 'vertex' of the graph is a simplex. An n-simplex has n+1 vertices all connected to each other, so that a 0-simplex is a vertex, a 1-simplex is an edge, a 2-simplex is a triangle and a 3-simplex is a tetrahedron, et cetera.



A simplicial complex is a gluing together of some simplices. For instance, in the image we have a 3-simplex glued to a 2-simplex, which is glued to two 1-simplices, and a disconnected zero simplex.



So what do functions on these objects look like?

In general, we want it to feel natural to slide down dimension, for instance, from a triangle to one of the edges on its boundary, as in Figure 1, and occasionally have the opportunity to jump up a dimension, for instance, from a boundary edge of a triangle to the triangle itself, as in Figure 2.

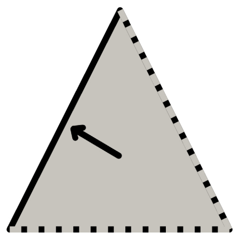


Figure 1: Slide down

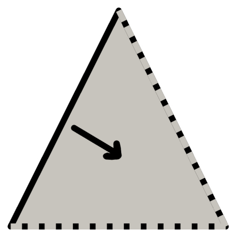


Figure 2: Jump up

A Morse function takes an n-simplex either up a dimension or down a dimension, and the rare instances of being able to jump up dimension we call Morse arrows. The distinguishing rule is that there can only ever be at most one Morse arrow associated to any simplex. If a simplex doesn't have any Morse arrows belonging to it, we call it a critical simplex. I like to think that it is critical of all the other simplices jumping about and being rowdy.

A nice way of displaying the information of 'where each simplex can travel' is a Hasse diagram.

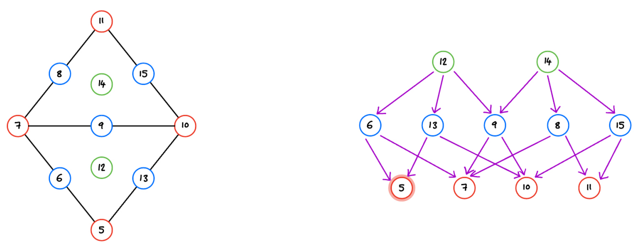
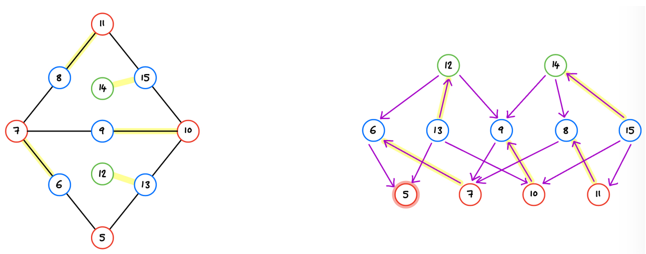


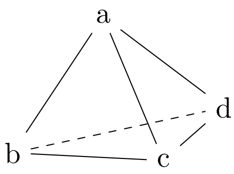
Figure 3: Left: a simplicial complex of two two-simplices glued together along an edge. Right: the Hasse diagram associated with the simplicial complex.

The Hasse diagram above has only critical simplices, since none of the arrows point up. We can chose to make this slightly less trivial by changing the Morse function. This means reversing some of the arrows, like so.

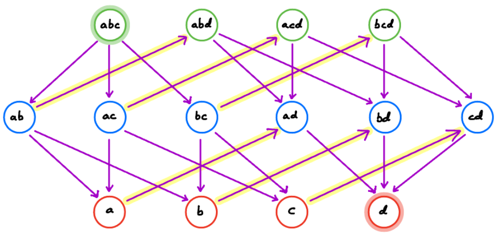


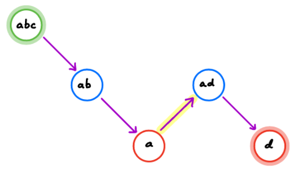
Notice that this satisfies the rule that any simplex can only have one Morse arrow (red) attached. Can you find the critical simplices in this Hasse diagram?

A directed path through a Hasse diagram between two critical simplices two dimensions apart is what we call a flowline, and forms the pivotal concept of this article. To see a flowline, let's look at the tetrahedron example.



If the 2-simplex 'abc' is a critical maximum, and the 0-simplex 'd' is a critical minimum, the tetrahedron has the following Hasse diagram (left), and an example of a flowline (right).

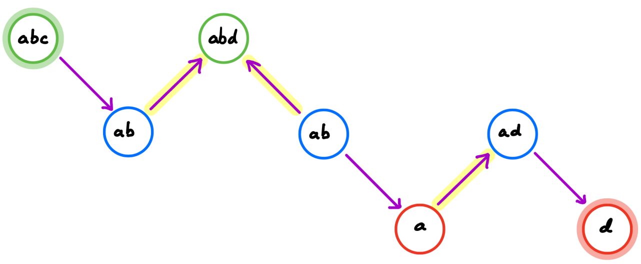




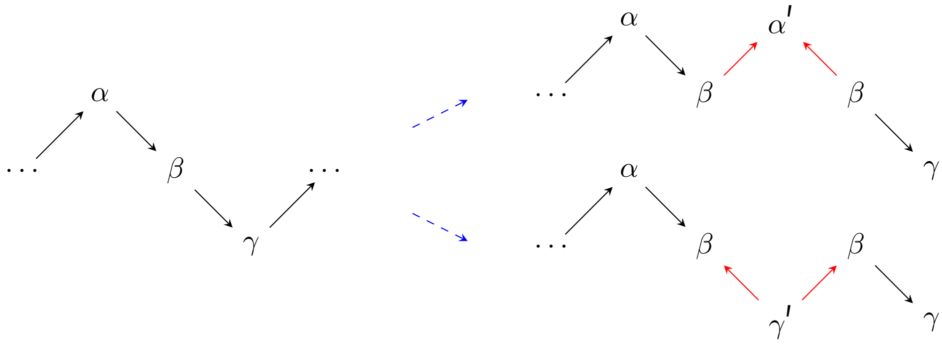
You see, we can define an algorithm which changes a flowline bit by bit in a small way to become another flowline, seeking out a flowline with a particular property. You may notice that a flowline travels down and up a dimension almost in alternation, but to get to the level below it must drop down twice through an 'intermediate simplex'. The intermediate simplex in this flowline, for instance, is the 1-simplex 'ab'. The property we want is that the intermediate simplex of the flowline is critical. If the flowline has this property, we say it is a critical flowline.

Using this concept of flowline criticality, we define the square differential of a critical $(n+1)$-simplex α in Morse theory to be the signed count of critical flowlines to critical $(n-1)$-simplices. In particular, if there are no critical n-simplices, then there are no critical flowlines, so that $\partial^2\alpha = 0$. Similarly, if there are no critical $(n-1)$-simplices γ, then there are no flowlines to any such γ, so that $\partial^2\alpha = 0$ here too. But usually this is not the case, so we have to do a bit more to definitively prove our case.

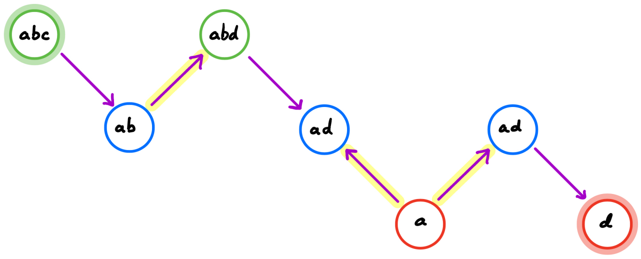
So what are the actions we can perform on a flowline? Actually, there are 3. We can 'Insert' a Morse arrow to the intermediate simplex, like so:



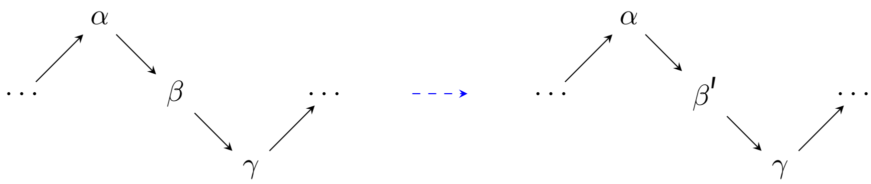
In general, insert adds in two Morse arrows either above the intermediate simplex, or below the intermediate simplex, depending on where its Morse arrow is (if it has one at all), as show in the following.



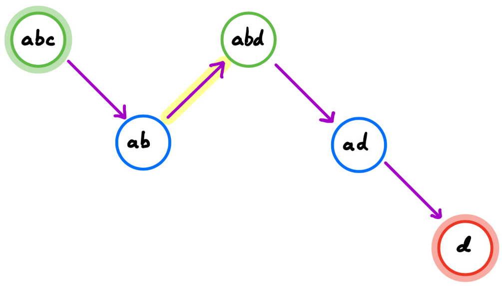
If β is our intermediate simplex, we can find the other intermediate simplex β' that is a boundary of α and has γ on its boundary (this β' is unique), and 'Flop' to this other path, like so:



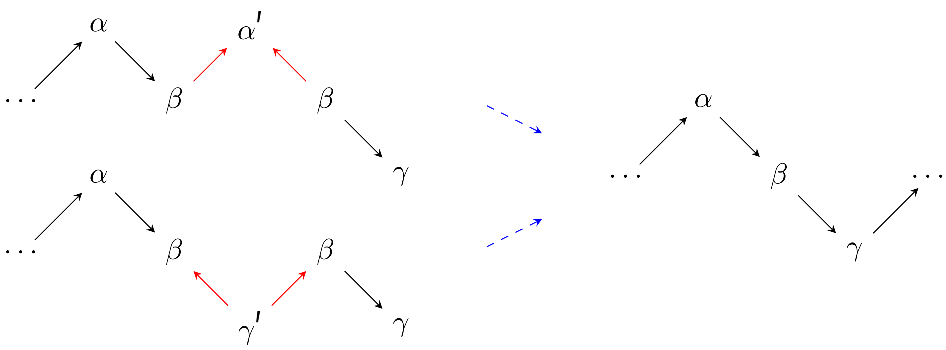
In general, this flop action is always possible, and is shown as follows:



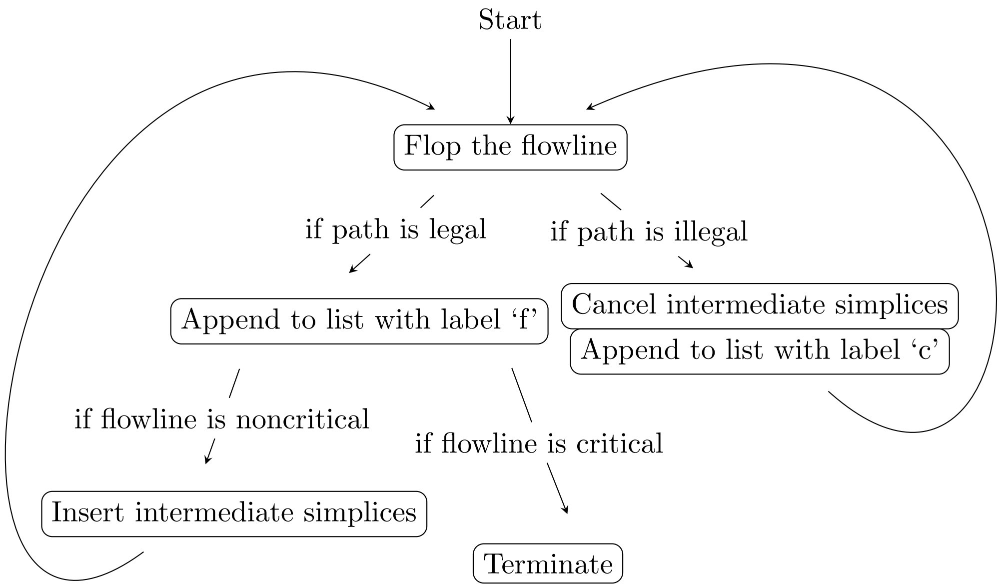
We can 'Cancel' out the Morse arrow at an intermediate simplex by removing the redundant path adjacent to the intermediate simplex like so:



In general, we can cancel whenever there is a backwards Morse arrow, like so:

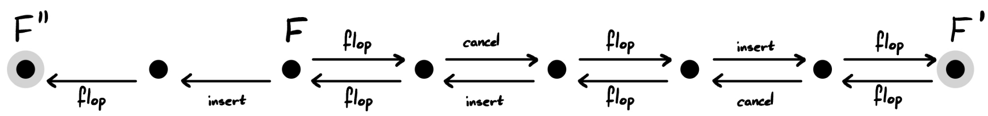


It's worth noting that Cancel is inverse to Insert, so if we divide an algorithm it doesn't make any sense to apply one after the other. Similarly, applying Flop twice will get us back to the same flowline, so it doesn't make sense to ever apply two Flops in a row. Ultimately, the algorithm we find looks like this:

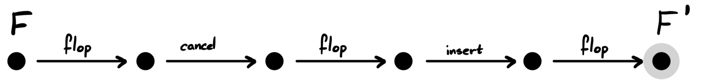


Notice that no matter how many times we cycle around each wing of the diagram, we'll always have an odd number of actions in our algorithm. This is going to be important later when we look at the signs of a flowline. But the punchline of this algorithm is that it terminates at a critical flowline. Not only that, but when we apply the algorithm to a critical flowline, the algorithm is involutive. This means that applying it once to some critical F' gives us a distinct critical flowline F'', and applying it twice gives us F' again. It is possible to find a flowline through the simplicial complex which isn't touched by this algorithm, but in this case when we apply the algorithm once and twice to this flowline, we will get a different pair of critical flowlines. In this way, we can partition all flowlines into equivalence classes, where two flowlines are equivalent if some number of steps in the algorithm can take you from one flowline to another. A consequence of this is that every critical flowline belongs to one equivalence class and there is a unique distinct flowline that is also in that equivalence class.

Another aspect of flowline we haven't talked about yet is the concept of a 'sign'. A flowline can be seen as positive or as negative, and each of the actions of Flop, Insert and Cancel negates the sign of the flowline. For instance, we could have a flowline F, where applying the algorithm gives F', giving a sequence of flowlines like that shown below:



and applying it twice gives F'', with a possible algorithm step-by-step shown below:



In particular, we pass through the same flowlines to get from F'' to F' as we do to get from the F' to F''. You may be able to see, now, why the algorithm is involutive.

Now, if F has sign $+$ then we can find the sign of F' and F'' by looking at the number of actions between them. In particular, we can see that since there is always an odd number of actions (starting with Flop and ending with Flop) between F' and F'', they must have opposite signs.

Putting all this information together, we can find an expression for the squared differential in terms of flowlines through a simplicial complex. The squared differential of alpha, some $(n+1)$-simplex, is the signed sum of flowlines alpha to beta times the number of flowlines beta to gamma for each critical gamma and each critical beta.

$$\partial^2\alpha = \sum_{\substack{\text{critical } \gamma \\ \dim(\gamma)=n-1}} \sum_{\substack{\text{critical flowlines } F \\ F:\alpha\to\gamma}} \mathrm{sign}(F)$$

This is equivalent to taking the signed count of the two critical flowlines in each equivalence class of flowlines from alpha to gamma for each critical gamma, since each critical flowline occurs in a unique equivalence class.

$$\partial^2\alpha = \sum_{\substack{\text{critical } \gamma \\ \dim(\gamma)=n-1}} \sum_{\substack{\text{eq. classes } [F] \\ F:\alpha\to\gamma}} \mathrm{sign}(F') + \mathrm{sign}(F'')$$

This is equivalent to taking the sum of the sign of a critical flowline minus the sign of the critical flowline for each equivalence class of flowlines alpha to gamma, for each critical gamma.

$$\partial^2\alpha = \sum_{\substack{\text{critical } \gamma \\ \dim(\gamma)=n-1}} \sum_{\substack{\text{eq. classes } [F] \\ F:\alpha\to\gamma}} \mathrm{sign}(F') - \mathrm{sign}(F') = \sum 0$$

As each term of this sum cancels out, the result will equate to a very nice and simple zero. And therefore we have proved that $\partial^2 = 0$.