# $S_n$ as a Coxeter group

Sophie Bleau

## Introduction

A Coxeter group is an object in representation theory generated by some set of reflections on a vector space. In this report, we aim to discuss how a reflection is defined in relation to a vector space, and some properties of reflections defined in this way. We aim to define a system of roots in a vector space which generate a specific set of reflections, and define a total ordering on this system in order to define a set of simple roots that generate the Coxeter group. Using this we will discuss how to encode the information we have about these simple roots into first graph, and second matrix form. Using this we will define some different types of Coxeter graphs. Throughout this report, we will apply the theory to the symmetric group $S_n$.

## 1 The reflection equation

We give a formula for calculating the reflection of a vector, but let us first try to see how it might be constructed. Any introductory course in linear algebra will tell you that in any inner product space with some vector $\lambda$ and some subspace $V$ generated by a collection of vectors $\{\alpha_1, \ldots, \alpha_n\}$, the vector we get by projecting $\lambda$ onto $V$ is given by

$$\text{proj}_V \lambda = \frac{(\lambda, \alpha_1)}{(\alpha_1, \alpha_1)}\alpha_1 + \cdots + \frac{(\lambda, \alpha_n)}{(\alpha_n, \alpha_n)}\alpha_n.$$

To illustrate, let us take the inner product space $(\mathbb{R}^2, \langle \cdot, \cdot \rangle_{std}$ with the standard inner product. We take $V$ to be a 1 dimensional subspace with nonzero generator $\alpha$.

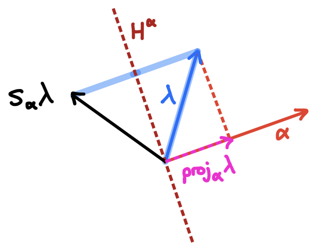


Figure 1: The figure shows the vector $\alpha$ generating the reflection, its orthogonal complement $H^\alpha$, the projection $\text{proj}_\alpha \lambda$ of $\lambda$ onto the subspace generated by $\alpha$, and the reflection $s_\alpha \lambda$ of $\lambda$ in $H^\alpha$.

If we want then to construct a reflection that sends $\alpha$ to its negative and fixes its orthogonal complement $H^\alpha$, we can understand this geometrically as reflecting in the "line" of symmetry given by $H^\alpha$. Generally $H^\alpha$ is a hyperplane, but in $\mathbb{R}^2$ it will conveniently just be a line. So take $\lambda$ and project it onto $H^\alpha$ by subtracting $\text{proj}_\alpha \lambda = (\lambda, \alpha)/(\alpha, \alpha)$ once. Subtracting it twice, we have reflected $\lambda$ in the axis of $H^\alpha$. This gives us our formula for the reflection $s_\alpha$:

$$s_\alpha \lambda = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

As noted, if we live in anything with more than 2 dimensions, $H^\alpha$ will be a hyperplane of dimension $n - 1$ where $\dim V = n$.

It is crucial to note that, as we might expect, reflection is a transformation of order 2 since applying it twice returns the identity. We can also see that $s_\alpha$ is an orthogonal transformation in the following claim.

**Claim 1.1.** (§1.1, [3]) The reflection $s_\alpha$ is an orthogonal transformation, which is to say that $(s_\alpha\lambda, s_\alpha\mu) = (\lambda, \mu)$.

*Proof.* We can expand the expressions in the inner product to find that

$$\begin{aligned}(s_\alpha\lambda, s_\alpha\mu) &= \left(\lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \mu - \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\alpha\right) \\ &= (\lambda,\mu) - \left(\lambda, \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\alpha\right) - \left(\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \mu\right) + \left(\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\alpha\right) \\ &= (\lambda,\mu) - 2\frac{(\mu,\alpha)}{(\alpha,\alpha)}(\lambda,\alpha) - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}(\mu,\alpha) + 4\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\frac{(\mu,\alpha)}{(\alpha,\alpha)}(\alpha,\alpha) \\ &= (\lambda,\mu),\end{aligned}$$

by noticing that the last three terms cancel. Therefore $s_\alpha$ is an orthogonal transformation in $O(V)$. □

The group generated by reflections $\{s_\alpha\}_{\alpha\in\Lambda}$ is a finite subgroup of the group $O(V)$ of orthogonal transformations of $V$. To see an example of how reflections can generate a group, take an $n$-gon centred at the origin and observe its group of symmetries, $D_n$. If we take a vector to be that from the centre of the polygon to some vertex, then we can generate the reflection in the orthogonal complement of this vector in the traditional way. In fact, as the product of two unequal reflections gives a rotation, the reflections generate the whole group $D_n$, such that $D_n$ is what we call a reflection group.

We can show this too with the symmetric group. Indeed, if our vector space has basis $\mathcal{B} = \{e_1, \dots, e_n\}$ and the symmetric group acts on $V$ by permuting the basis elements, a transposition $(i, j)$ sends $e_i - e_j \mapsto e_j - e_i$ and keeps all other basis elements fixed. In this way, the orthogonal complement is fixed pointwise, and everything else is reflected in the line through which the coefficients $c_i$ and $c_j$ of $e_i$ and $e_j$ are equal. Indeed, this transposition is a reflection, and in fact all reflections in the symmetric group are transpositions. As all permutations can be generated by a composition of transpositions, the symmetric group is also a reflection group.

A reflection fixes a hyperplane $H^\alpha$ pointwise and negates vectors in the line $L_\alpha = \{c\alpha : c \in \mathbb{R}\}$ containing $\alpha$. We wish now to demonstrate the way in which the group $W$ acts on our vector space $V$.

**Theorem 1.2** (§1.2, [3])**.** *If $t \in O(V)$ and $\alpha$ is any nonzero vector in $V$, then $ts_\alpha t^{-1} = s_{t\alpha}$. In particular, if $w \in W$ then $s_{wa}$ belongs to $W$ whenever $s_\alpha$ does.*

*Proof.* We follow the proof given in [3], seeing that $ts_\alpha t^{-1}(t\alpha) = ts_\alpha(\alpha) = -t\alpha$, so $ts_\alpha t^{-1}$ reflects $t\alpha$, and the orthogonality $(\lambda, \alpha) = (t\lambda, t\alpha)$ implies that $\lambda \in H_\alpha$ if and only if $t\lambda \in H_{t\alpha}$, so that $(ts_\alpha t^{-1})(t\lambda) = ts_\alpha\lambda = t\lambda$ for $\lambda \in H$. Therefore $ts_\alpha t^{-1} = s_{t\alpha}$. □

## 2 Systems of roots

Let us now define a set of vectors whose reflections we care about.

**Definition 2.1.** (§1.2, [3]) Let $\Phi$ be a finite set of nonzero vectors in $V$ where for any $\alpha \in \Phi$ we notate $L_\alpha = \langle\alpha\rangle$ and require that

- $\Phi \cap L_\alpha = \{\alpha, -\alpha\}$ and
- $s_\alpha\Phi = \Phi$ for all $\alpha \in \Phi$.

Then we call $\Phi$ a **root system** and take $W$ to be the group generated by all reflections $s_\alpha$ for $\alpha \in \Phi$. Then $W$ is a finite reflection group, with $W \le O(V)$.

Note that the vectors in $\Phi$ need not be unit vectors.
For example we may take the following set of roots.

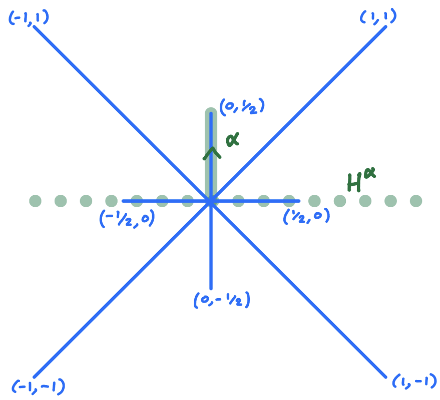


Figure 2: The figure shows a root system with highlighted root $\alpha$ to show that the system is fixed by the reflection in $\alpha$ and $L_\alpha$ intersects only at $(0, 1/2)$ and $(0, 1/2)$.

Clearly these are not unit vectors, yet they form a root system.

**Claim 2.2.** (§1.2, [3]) Any reflection group $W$ generated over a root system is finite.

*Proof.* Each reflection $s_\alpha \in W$ fixes only the $x \in H_\alpha$. Therefore the identity is the only reflection that fixes all points. This is to say that the natural homomorphism $\varphi : W \to \Phi$ has trivial kernel, so $\varphi$ is injective and $W$ is finite. $\square$

We can specify that we want all $\alpha \in \Delta$ to be positive and find a minimal 'simple' system generating our root system, but for this we need to define an ordering to give an idea of what 'positive' actually means. Positivity is easy to define in a one-dimensional space, but for the complex plane, for instance, we do not have an explicit ordering. For this, we choose an ordered basis $\lambda_1, \lambda_2, \ldots, \lambda_n$ of roots of $V$ and let $\sum a_i\lambda_i < \sum b_i\lambda_i$ if $a_k < b_k$ for $k$ the least index such that $a_i \neq b_i$. This is to say that $\lambda_1$ has a heavier weighting than $\lambda_2$ and so forth. This is a total ordering, which means that it satisfies the following constraints. For any $\lambda, \mu, v \in V$ and $c \neq 0 \in \mathbb{R}$.

- We must have one of $\lambda < \mu$, $\lambda = \mu$, or $\mu < \lambda$.
- For $\mu < v$ we must have $\lambda + \mu < \lambda + v$.
- Let $\mu < v$. If $c > 0$ then $c\mu < cv$. If $c < 0$ then $cv < c\mu$.

Notice then that a vector $v = \sum_i c_i\lambda_i$ is positive if, and only if, the least $i$ such that $c_i \neq 0$ has $c_i > 0$. Relative to this ordering, we can define a positive system.

**Definition 2.3.** (§1.3, [3]) A **positive system** $\Pi$ is a subset of $\Phi$ containing only the roots which are 'positive' relative to this ordering, which is to say that for the least nonnegative $\lambda_i$, its coefficient $\alpha_i > 0$.

A negative system $-\Pi$ consists of the negative roots of $\Phi$, and we have that $\Pi \sqcup -\Pi = \Phi$.

**Definition 2.4.** (§1.3, [3]) Let $\Phi$ be a root system and $W$ the associated reflection group. Take some positive system $\Pi$ in $\Phi$. A **simple system** $\Delta \subset \Pi$ is a system of roots which act as a vector space basis for the $\mathbb{R}$-span of $\Phi$ in $V$, such that every $\lambda \in \Pi$ is a linear combination of roots in $\Delta$ with nonnegative coefficients.

The above definition implies that the elements of $\Delta$ are linearly independent, which is to say that if $\alpha \in \Delta$ then $-\alpha \notin \Delta$, so we do not need to worry about containment in $\Pi$.

**Theorem 2.5.** *(§1.3, [3]) If $\Delta$ is a simple system in $\Phi$ then there is a unique positive system containing $\Delta$. Conversely, every positive system $\Pi$ in $\Phi$ contains a unique simple system.*

Since we always have an ordering, there will always exist a positive system, which by the above theorem, implies the existence of a simple system in $\Phi$.

In the example of the symmetric group, we know that the roots in the root system are transpositions, but to find the simple system, we need to find a linearly independent set of these transpositions which span $S_n$. Given the ordering of the basis vectors $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$ for our $n$ dimensional vector space $V$, we can choose only the adjacent transpositions, $(i, i+1)$. As we saw in Honours Algebra, this set generates the whole of $S_n$, so the span is equal to the span of $\Phi$, the set of all transpositions.

We now prove a lemma which relies on the fact that the elements of $\Pi$ are a positive combination of roots in $\Delta$.

**Lemma 2.6.** (§1.3, [3]) For $\alpha, \beta$ two distinct roots in a simple system $\Delta$ of $\Phi$, $(\alpha, \beta) \leq 0$.

*Proof.* We again follow the proof as given in [3]. Assume falsely that $(\alpha, \beta) > 0$. Then for $s_\alpha\beta = \beta - c\alpha$ we must have that $c = 2(\beta, \alpha)/(\alpha, \alpha)$ is positive. As a root reflection, either $s_\alpha\beta$ or $-s_\alpha\beta$ is in $\Pi$. For some $c_\gamma \geq 0$, take

$$\sum_{\gamma \in \Delta} c_\gamma \gamma. \tag{1}$$

If we have $c_\beta < 1$, then combining the equation of the reflection and Equation 1, we procure

$$s_\alpha\beta = \beta - c\alpha = c_\beta\beta + \sum_{\gamma \neq \beta} c_\gamma \gamma \tag{2}$$

which one may rearrange to get $(1 - c_\beta)\beta = c\alpha + \sum_{\gamma \neq \beta} c_\gamma \gamma$. Since the right hand side is not dependent on $\beta$, we need not have $\beta \in \Delta$, but this contradicts the assumption that $\Delta$ is minimal. So we must have $c_\beta \geq 1$. This leads us to the equation $(c_\beta - 1)\beta + c\alpha + \sum_{\gamma \neq \beta} c_\gamma \gamma = 0$, which by linear dependence can be true only if all coefficients are zero, which is another contradiction. We conclude that $s_\alpha\beta$ is not positive.

Now we know that $s_\alpha\beta \leq 0$, we want to show that $s_\alpha\beta$ cannot be negative, thereby showing that any pair of roots with positive inner product cannot both be members of the simple system. Similarly to our previous course, we let

$$s_\alpha\beta = \beta - c\alpha = c_\alpha\alpha + \sum_{\gamma \neq \alpha} c_\gamma \gamma \tag{3}$$

so that $(c + c_\alpha)\alpha = \beta - \sum_{\gamma \neq \alpha} c_\gamma \gamma$. If $c + c_\alpha > 0$ we can discard $\alpha$ from $\Delta$, which again contradicts the assumption that $\Delta$ is minimal. If $c + c_\alpha \leq 0$ then

$$\beta - \sum_{\gamma \neq \alpha} c_\gamma \gamma - (c + c_\alpha)\alpha = 0 \tag{4}$$

has at least one positive coefficient, and so the roots are not linearly independent. Therefore $s_\alpha\beta = 0$, giving us that any reflection with positive '$c$' coefficient cannot be a member of $\Pi$, which is to say that any $\alpha, \beta \in \Delta$ must have $(\alpha, \beta) \leq 0$, as required. □

To give an idea of the behaviour of $\Pi$, we have the following lemma (which we will not prove here).

**Lemma 2.7.** (§1.4, [3]) Let $\Delta$ be a simple system in $\Pi$. If $\alpha \in \Delta$ then $s_\alpha(\Pi \backslash \{\alpha\}) = \Pi \backslash \{\alpha\}$.

Since $s_\alpha$ fixes its orthogonal complement $H^\alpha$, and the positive system contains all positive roots except $\alpha$, then all we need to prove is that every $\lambda \in \Pi \backslash \{\alpha\}$ is sent to another positive root $\mu \neq \alpha \in \Pi \backslash \{\alpha\}$.

## 3 Coxeter graphs

If $W$ is the group generated by reflections $s_\alpha$ in $\Phi$, we can ask ourselves whether this generating set is minimal. We know that $\forall\, \alpha \in \Phi$, we must have $-\alpha \in \Phi$, so we can at least half the number of generators needed. A concrete answer to this question is the following.

**Theorem 3.1.** *(§1.5, [3]) $W$ is generated by simple reflections $s_{\alpha\in\Delta}$.*

*Proof.* Assume otherwise. Then there is some $w \in W$ such that $w = \prod_i s_i$ for $s_i \in S = \{s_{alpha} : \alpha \in \Phi\}$ but $w$ cannot be written as a product of $s_j \in S' = \{s_\alpha : \alpha \in \Delta\}$. However, since $\Delta$ spans the $\mathbb{R}$-span of $\Phi$ we must have $s_i \in S$ has $s_i = \prod_j s_j$ for $s_j \in S'$, so that $w = \prod_i \prod_j s_j$, so that every element can be written as a product of reflections $s_\alpha$ for $\alpha \in \Delta$, giving us that $W$ is generated by simple reflections. □

Using this, we can state a stronger assertion about $W$. For roots $\alpha, \beta \in \Phi$, let $m(\alpha, \beta)$ denote the order of the composition $s_\alpha s_\beta$.

**Theorem 3.2.** *(§1.5, [3]) Let $\Delta$ be a simple system in $\Phi$. Then $W$ is generated by the set $S := \{s_\alpha : \alpha \in \Delta\}$, subject only to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$$

*for $\alpha, \beta \in \Delta$.*

**Definition 3.3.** (§1.9, [3]) A group $W$ generated by reflections $S = \{s_\alpha : \alpha \in \Delta\}$ is a **Coxeter group**, and the pair $(W, S)$ is a **Coxeter system**.

As you might have guessed, Coxeter groups coincide with finite reflection groups.

To efficiently convey the information of a Coxeter system, we would like to specify the simple roots in $\Delta$ and the order of their compositions. We can encode this information in a graph. Let our graph $\Gamma$ have $|\Delta|$ vertices, where each vertex $v_{\alpha\in\Delta}$ stands for a simple root. We note that if $m(\alpha, \beta) = 1$ then $s_\alpha s_\beta = 1$ so $s_\alpha = s_\beta$. We also note that if $m(\alpha, \beta) = 2$ then $(s_\alpha s_\beta)^2 = 1$. These two cases are more trivial than those on which we wish to focus, so let us only consider compositions of order $\geq 3$. In this way, for any pair of vertices $v_\alpha, v_\beta$ on the graph $\Gamma$, connect them by an edge if, and only if, $m(\alpha, \beta) \geq 3$. We label the edge between $v_\alpha$ and $v_\beta$ by its order $m(\alpha, \beta)$ except for $m(\alpha, \beta) = 3$, which is such a common occurrence that we leave it blank to emphasise the presence of an edge of higher order if there is one. Any distinct vertices without a connecting edge we assume to have composition order 2, and every vertex has composition order 1 with itself.

**Definition 3.4.** (Definition 2.2, [1]) The **Coxeter graph** (or a Coxeter-Dynkin diagram) of $W$ is the graph $\Gamma$ constructed with a vertex $v_\alpha$ per simple root $\alpha \in \Delta$ and an edge per nontrivial pair $m(\alpha, \beta) \geq 3$.

In the case of the symmetric group, the generating reflections are the transpositions $(i, i+1)$, so there will be $n-1$ vertices. The composition of $(i-1, i)$ with $(i, i+1)$ is an order 3 permutation given by $(i-1, i, i+1)$, so there will be edges of order 3 between adjacent vertices. All other combinations have either $i = j$, giving $(i, i+1)(i, i+1) = \mathrm{id}$, or $|i-j| > 1$, giving $m((i, i+1), (j, j+1)) = 2$, so there will be no other edges. The graph of $S_n$ is therefore given by:


Figure 3: The figure shows the Coxeter graph of the group $S_n$, with $n-1$ vertices and an edge of order 3 between each adjacent vertex.

**Definition 3.5.** (§2.2, [3]) A Coxeter system is **irreducible** if the Coxeter graph $\Gamma$ is connected.

A common theme in mathematics is the method of taking an object and breaking it up into units with which we are familiar.

If we fix a simple system $\Delta$ and take $S = \{s_\alpha : \alpha \in \Delta\}$, we can look at some subset $I \subseteq S$, where $W_I$ is the subgroup generated by reflections in $I$. If we change $\Delta$ to some other simple system which differs from $\Delta$ by some vector $w$, we replace $\Delta$ by $w\Delta$ and replace $W$ by its conjugate $wW_Iw^{-1}$. Subgroups $W_I$ are called **parabolic subgroups** of $W$. With this in mind, we state the following.

**Theorem 3.6.** *(§2.2, [3]) Let $(W, S)$ have Coxeter graph $\Gamma$ with connected components $\Gamma_1, \dots, \Gamma_r$ and let $S_1, \dots, S_r$ be the corresponding subsets of $S$. Then $W$ is the direct product of the parabolic subgroups $W_{S_1}, \dots, W_{S_r}$, and each Coxeter system $(W_{S_i}, S_i)$ is irreducible.*

**Example 3.7.** Let $W$ be the dihedral group $D_6$ with Coxeter generator set $S = \{s, s'\}$. Drawing the Coxeter graph $\Gamma$, we find that the Coxeter graph is connected and therefore the Coxeter system $(W, S)$ is irreducible.
Alternatively, taking the generating set $S' = \{s, (s's)^3, s(s's)^2\}$, we claim that the Coxeter system $(W, S')$ is reducible. Taking the compositions, we find that since $s$ and $s(s's)^2$ are reflections and $(s's)^3$ is an order 2 rotation, the composition of $(s's)^3$ with either reflection yields a reflection, so $m((s's)^3, s) = m((s's)^3, s(s's)^2) = 2$, so the vertex $v_{(s's)^3}$ is isolated and the graph is disconnected. Therefore, the Coxeter system $(W, S')$ is not irreducible.

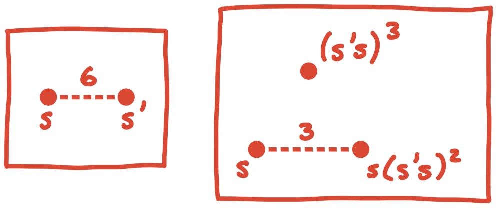


Figure 4: The figure shows the Coxeter graph of the group $D_6$, with two different generating sets, where the first gives an irreducible Coxeter graph, and the second does not.

This example shows us that $D_6$ is a Coxeter group. Indeed, it is shown in [2] that any dihedral group $D_m$ with appropriate generating set forms a Coxeter system. Furthermore, the punchline of this example is that the Coxeter group is invariant of the choice of the generating set, and of the reducibility of the Coxeter graph.

*Remark* 3.8. Note that vectors to the vertices of an $n$-gon provide a root system $\Phi$ only when $n$ is even, given that $\Phi \cap L_\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

The information encoded in a Coxeter graph $\Gamma$ with $n$ vertices can also be shown in an $n \times n$ matrix.

**Definition 3.9.** (Definition 2.2, [3]) For a Coxeter graph $\Gamma$ of some Coxeter system $(W, S)$, the **Coxeter matrix** $A$ is an $|S| \times |S|$ matrix whose $i$th row/column stands for the relation of the vertex $v_i$ with the other vertices, and

$$A_{i,j} = -\cos\left(\frac{\pi}{m(\alpha_i, \alpha_j)}\right).$$

We remark that sources vary on how to define a Coxeter matrix. In some sources the Coxeter matrix has entries given by $A_{i,j} = m(\alpha_i, \alpha_j)$, but we will take the matrix defined in [3] as our primary source. Note also that the **Cartan matrix** (Definition 2.2, [1]), whose entries are given by

$$A_{i,j} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j) = -2\frac{|\alpha_i|}{|\alpha_j|}\cos(\pi/m(\alpha_i, \alpha_j)),$$

differs from a Coxeter matrix by a factor of $2|\alpha_i|/|\alpha_j|$. In any case, the Coxeter matrix we have defined in Definition 3.9 is symmetric by construction, which is to say that it will have all real eigenvalues. Left and right composition of $A$ by some vector $x$ and its transpose $x^\top$ respectively gives the quadratic form associated to $A$. Notice also that the $i,j$th entry of the matrix is positive if, and only if, $\pi/2 < \pi/m(\alpha_i,\alpha_j) < 3\pi/2$. This is to say that the roots are more than an angle of $\pi/2$ radians apart, which agrees with our earlier claim in Lemma 2.6.

**Definition 3.10.** (§2.8, [3]) A root system is **crystallographic** if it satisfies

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$$

for all $\alpha,\beta \in \mathbb{Z}$. These integers are called Cartan integers.

**Theorem 3.11.** *(§2.8, [3]) If $W$ is crystallographic then each integer $m(\alpha,\beta)$ must be 2, 3, 4 or 6 when $\alpha \neq \beta$ in $\Delta$.*

We also define 'indecomposability' of a Coxeter matrix, which we will see is equivalent to the Coxeter graph being irreducible as defined in Definition 3.5.

**Definition 3.12.** (§2.6, [3]) An $n \times n$ matrix $A$ is **decomposable** if there are nonempty subsets $I, J$ such that $A_{i,j} = 0$ for $i \in I$, $j \in J$.

This is to say that a decomposable matrix can be re-indexed to produce a block diagonal matrix. An **indecomposable** matrix is one that is not decomposable. We will not prove the following.

**Theorem 3.13.** *(§2.6, [3]) A Coxeter matrix is indecomposable if, and only if, the associated Coxeter graph is connected.*

Since the connectedness of a Coxeter graph defines whether the graph is irreducible, we say that indecomposability is equivalent to irreducibility.

Using the fact that a Coxeter matrix $A$ is symmetric, we may use it to define a quadratic form in the following definition.

**Definition 3.14.** (§2.3, [3]) For all $x \neq 0$, a Coxeter matrix $A$ is **positive definite** if $x^\top Ax > 0$, and **positive semi-definite** if $x^\top Ax \geq 0$.

We call a Coxeter graph $\Gamma$/system $(W,S)$ positive (respectively semi-) definite if the associated matrix is positive (respectively semi-) definite. Alert readers will notice that the above definition is equivalent to the statement that $A$ is positive (respectively semi-) definite if, and only if, all eigenvalues $\lambda_i$ of $A$ have $\lambda_i > 0$ (respectively $\lambda_i \geq 0$).

Another way to determine positive (semi-) definiteness is with the use of 'principal minors'.

**Definition 3.15.** (§2.3, [3]) The $k$th principal minor of $A$ is the determinant of the $k \times k$ submatrix obtained by removing the last $n-k$ rows and columns.

Using the concept of a principal minor, we can give an equivalent constraint for positive (semi-) definite matrices.

**Theorem 3.16.** *(§2.3, [3]) $A$ is positive (semi-) definite if, and only if, all of its principal minors are positive (respectively nonnegative).*

Due to the common occurrence of 2 in the denominator, it is standard to check the value of $\det(2A)$ instead of $\det(A)$, so calculations hereon will use this convention. For a Coxeter system of a finite reflection group, $\Gamma$ is always positive definite, as the bilinear form $x^\top Ax$ is the standard inner product, and therefore always positive definite.

**Example 3.17.** For $S_n$, our Coxeter graph is composed of $n-1$ vertices with order 3 edges between the $i$th and the $i+1$th, as shown in Figure 3. In calculating $2A$, first notice that $m(i,i)=1$, so that $-\cos\left(\frac{\pi}{m(\alpha_i,\alpha_i)}\right)=1$, and therefore the diagonal will be populated with twos. As we assume any two vertices $v_i, v_j$ with no connecting edge must have $m(\alpha_i,\alpha_j)=2$, the entry $A_{i,j}$ for $|i-j|>1$ must be given by

$$A_{i,j} = -\cos\left(\frac{\pi}{m(\alpha_i,\alpha_j)}\right) = 0.$$

Finally, for adjacent vertices $v_i$ and $v_{i+1}$, we have $m(i,i+1)=3$, giving us $-\cos\left(\frac{\pi}{m(i,i+1)}\right)=-1/2$ on the off diagonals, so that our final $(n-1)\times(n-1)$ matrix has the following form.

$$2A_{n-1} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

# 4 Graphs of positive type

A Coxeter system has **positive type** if it is either positive definite or positive semi-definite. In this section, we wish to show that the Coxeter system of the symmetric group is positive definite, and to classify finite reflection groups in terms of Coxeter systems of positive type.

## 4.1 The positive definite symmetric group

We claim that the symmetric group $S_n$ is positive definite, and aim to prove this by showing that all of its principal minors are positive. On the way to proving that this is indeed true, we must first prove a lemma.

**Lemma 4.1.** (§2.4, [3]) For a symmetric matrix $2A$, where $A$ is the Coxeter matrix associated to some Coxeter graph $\Gamma$, and $d_{n-1}, d_{n-2}$ are the first and second largest principal minors respectively of $2A$, if $\Gamma$ has a vertex connected by an order $m=3$ (or $m=4$) edge to exactly one vertex, we must have

$$\det(2A) = \begin{cases} 2d_{n-1} - d_{n-2} & \text{for } m = 3 \\ 2d_{n-1} - 2d_{n-2} & \text{for } m = 4. \end{cases}$$

*Proof.* We know a few things about our matrix $2A$. Firstly, that the diagonal is populated with twos. Secondly, that there exists at least one 'reclusive' vertex: one that is connected to only one other vertex. Without loss of generality, let our reclusive vertex be the last vertex, say $v_n$, connected only to $v_{n-1}$, giving

$$A_{n,i} = \begin{cases} 2 & \text{for } i = n \\ -2\cos\left(\frac{\pi}{m(n,i)}\right) & \text{for } i = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

So our matrix is of the following form.

$$
\begin{aligned}
\det(2A) &= \det \begin{pmatrix} 2 & * & \dots & * & * & 0 \\ * & 2 & \dots & * & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \dots & * & 2 & -2\cos\left(\frac{\pi}{m(n,n-1)}\right) \\ 0 & 0 & \dots & 0 & -2\cos\left(\frac{\pi}{m(n,n-1)}\right) & 2 \end{pmatrix} \\
&= 2\det \begin{pmatrix} 2 & * & \dots & * & * \\ * & 2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & 2 & * \\ * & * & \dots & * & 2 \end{pmatrix} - 4\cos^2\left(\frac{\pi}{m(n,n-1)}\right) \det \begin{pmatrix} 2 & * & \dots & * \\ * & 2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & 2 \end{pmatrix} \\
&= 2d_{n-1} - 4\cos^2\left(\frac{\pi}{m(n,n-1)}\right) d_{n-2}.
\end{aligned}
$$

As posed in the assumptions of the lemma, $m(n,n-1) \in \{3,4\}$. If $m(n,n-1) = 3$ then $4\cos^2\left(\frac{\pi}{m(n,n-1)}\right) = 1$, giving

$$\det(2A) = 2d_{n-1} - d_{n-2},$$

and if $m(n,n-1) = 4$ then $4\cos^2\left(\frac{\pi}{m(n,n-1)}\right) = 2$, giving

$$\det(2A) = 2d_{n-1} - 2d_{n-2},$$

as required. □

Now, we want to use this lemma to show that every principal minor of the Coxeter matrix $A$ of the symmetric group is positive definite. For this, we construct some small examples and build up by induction. Notice that $S_2$ is generated by a single transposition, and $S_3$ by two, giving our Coxeter matrices

$$2A_1 = \begin{pmatrix} 2 \end{pmatrix} \quad \text{and} \quad 2A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

with $\det(2A_1) = 2$ and $\det(2A_2) = 3$. We also know that for any symmetric matrix, all edges of the Coxeter graph will have order 3, so by Lemma 4.1 we must have

$$\det(2A_n) = 2d_{n-1} - d_{n-2}. \tag{5}$$

Let us now invoke induction. Taking $d_{n-1} = k-1$, $d_{n-2} = k-2$, as in the previous examples, we use Equation 5 to get

$$
\begin{aligned}
\det(2A_n) &= 2d_{n-1} - d_{n-2} \\
&= 2(k-1) - (k-2) \\
&= k.
\end{aligned}
$$

Extrapolating from the $S_2$ and $S_3$ examples, we see that the Coxeter matrix $2A_{n-1}$ of $S_n$ will have determinant $n$. This is to say that the principal minors are all positive, so the Coxeter matrix $A_{n-1}$ is positive definite.

## 4.2 The entire collection

We claim that all of the graphs in Figure 5 are positive definite. Furthermore, we claim that they are the only connected positive definite graphs.

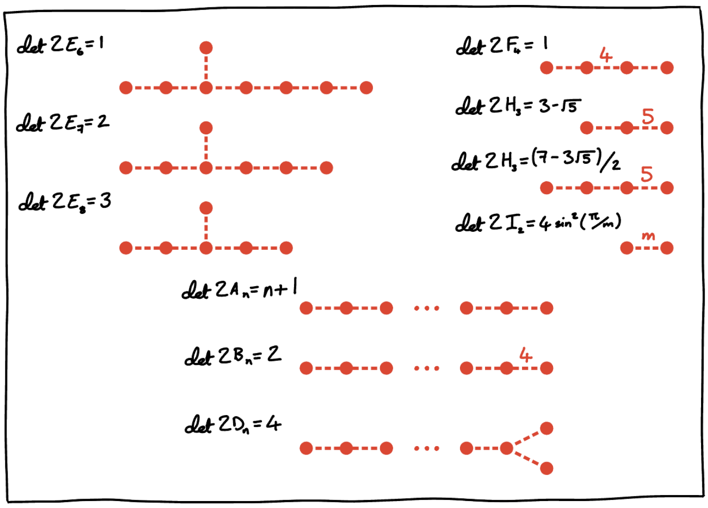


Figure 5: The figure shows all possible positive definite Coxeter graphs, with the determinant of their associated Coxeter matrix.

Using Lemma 4.1, we find that the corresponding determinants for these graphs are shown in the following table.

| $A_n$ | $B_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $H_3$ | $H_4$ | $I_2(m)$ |
|---|---|---|---|---|---|---|---|---|---|
| $n+1$ | 2 | 4 | 3 | 2 | 1 | 1 | $3-\sqrt{5}$ | $(7-3\sqrt{5})/2$ | $4\sin^2(\pi/m)$ |

where $A_n$ is the Coxeter matrix of the symmetric group and $I_2(m)$ is the Coxeter matrix of the dihedral group $D_m$.

We can use these values to find the determinant of graphs containing these graphs. To fully take advantage of this, we must define a subgraph.

**Definition 4.2.** (§2.6, [3]) A **subgraph** $\Gamma'$ of a Coxeter graph $\Gamma$ is obtained by omitting some vertices (and adjacent edges), and/or by decreasing the labels on one or more edges.

We say that $\Gamma$ 'contains' a subgraph $\Gamma'$. To see some subgraphs in action, let us take the collection of positive definite Coxeter graphs and compare to the collection of positive semi-definite Coxeter graphs.

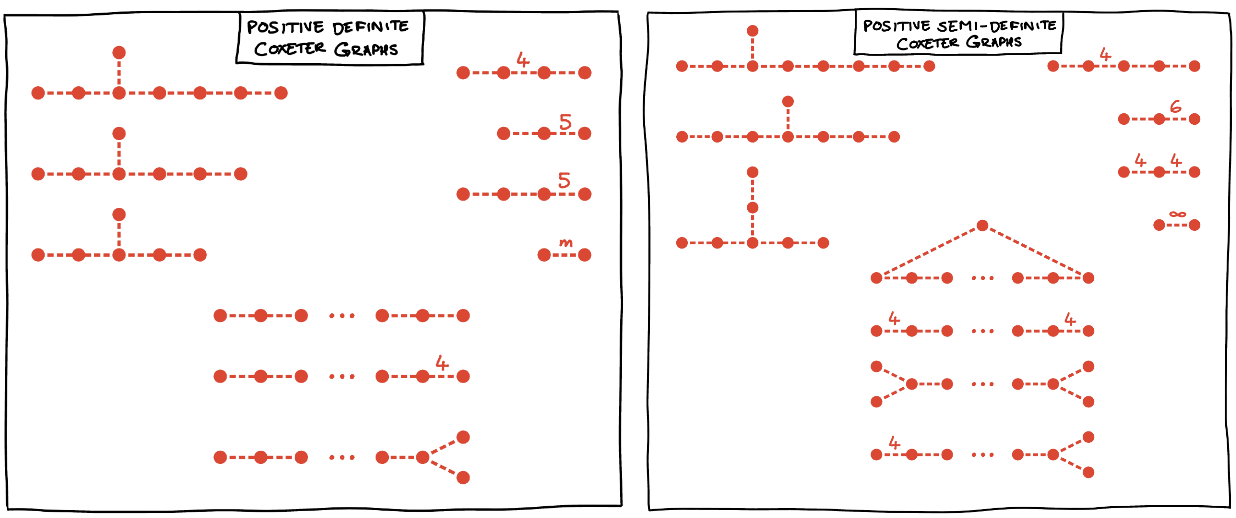


Figure 6: The figure shows the positive definite Coxeter matrices (left), and the positive semi-definite Coxeter matrices (right).

In fact, one can get from a positive semi-definite Coxeter graph to a positive definite graph by either removing a vertex or decreasing the order of an edge. This is equivalent to the observation that every positive definite Coxeter graph is a subgraph of a positive semi-definite Coxeter graph. The classification of graphs of positive type in all its grandeur is given in §2.7 of [3], stating and proving that all Coxeter graphs of positive type are those shown in Figure 6. The proof assumes the existence of some other graph $\Gamma$ of positive type with size $n$, and takes 20 precise steps to restrict the possibilities of $\Gamma$'s form, before reaching the irrevocable conclusion that it cannot exist.

## Conclusion

In this report, we have given an exposition of Chapters 1 and 2 in [3] with specific reference to the symmetric group. In this way, we have discussed finite reflection groups, their Coxeter graphs including presentation in matrix format, and a classification of finite reflection groups of positive type. Furthermore, we have proved that $S_n$ is a positive definite Coxeter system.

## References

[1] P. P. Dechant. *Clifford Algebra is the Natural Framework for Root Systems and Coxeter Groups. Group Theory: Coxeter, Conformal and Modular Groups*. 2017.

[2] H. Hu. *Representations of Coxeter groups and homology of Coxeter graphs*. 2022. arXiv: `2112.03771 [math.RT]`.

[3] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. 1990.