Covering Spaces of the Figure of Eight

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Abstract

This paper will investigate the question "What topological spaces cover the figure of eight?" We begin by introducing the concepts of topological spaces. We explore the question for covers on Euclidean space, culminating with all results found. We then walk the reader through multiple further examples of spaces to cover.

Contents

1 Introduction

The figure of eight, also known as the infinity symbol (depending on one's perspective) is renowned for its distinctive and continuous shape. Unlike a circle, it has a twist at its centre, or a **node** as we will address it in the discussion of its space and the associated covering spaces. At any point in a circle there are two legs — one on either side — that extend from that point, but on a figure of eight (which will henceforth be referred to as ∞) we have exactly one point — our node (often denoted as x_0) — with four legs. This will give us plenty to think about in the coming pages.

Acknowledgements

With thanks to Professor Jon Pridham for his supervision and guidance, and to Lilian Morrison for her illustrations, technical support and endless patience.

2 Background

Recall from the Fundamentals of Pure Mathematics course (FPM) that a graph is a finite set of vertices joined by edges. We assume that there is at most one edge joining two given vertices and no edge joins a vertex to itself. Since the definition of a graph only refers to the set of vertices, and to which pairs of vertices are joined by an edge, it is possible to draw the same graph in many different ways. This means that two graphs with different shapes or dimensions can be considered isomorphic if they have the same vertices with the same relationships between them.

Definition (Isomorphism between two graphs)**.** An **isomorphism** between two graphs is a bijection between them that preserves all edges. More precisely, if Δ_1 and Δ_2 are graphs, with sets of vertices V_1 and V_2, respectively, then an *isomorphism* from Δ_1 to Δ_2 is a bijection

$$f : V_1 \to V_2$$

such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are joined by an edge.

We say that Δ_1 and Δ_2 are isomorphic if there exists an isomorphism $f : \Delta_1 \to \Delta_2$.

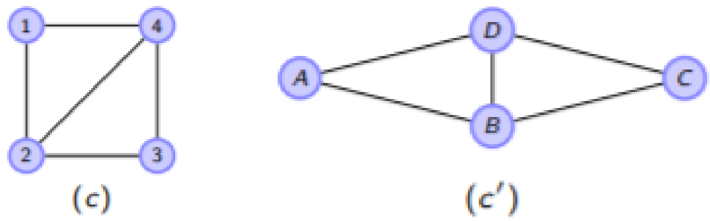


Figure 1: Two isomorphic graphs.

Now to go beyond the graph theory studied in this course, I now explore independently the concept of Topological Spaces.

The ∞ figure is an example of a topological space.

Definition (Topological Space)**.** A topological space (X, τ) is a set of points, with a set of neighbourhoods defined for each point, satisfying a set of axioms relating points and neighbourhoods.

In this notation, X defines the set of points in the space, and τ a collection of subsets of this set. There are various requirements on what τ, the set of subsets (and a topology of X), must contain, but proving that the spaces in question are indeed topological spaces is beyond the scope of this project. We will therefore define the space by (X, x_0), where X is, as before, the set of all points in the topological space, and x_0 is one such node (from which we choose to start paths).

The definition above is present more to give us the idea that the main point of topological spaces is to give an intrinsic definition of continuous functions.

A defining feature of topological spaces is that dimensions and shape have no effect on how a space is defined. For instance, the left loop of the ∞ could be much smaller and the right loop could be square, but the space would still be seen topologically as identical to the figure of eight.

Notation. Let (X, x_0) be a topological space.

1. The subsets $V \subset X$ are **open sets**. These are sets that, with every point P, contain all points that are sufficiently near to P.

2. A subset $F \subset X$ is **closed** if its complement F^c is open. In this case we might write $F \sqsubset X$.

3. An **open neighbourhood** of a point $x \in X$ is an open set $V \subset X$ such that $x \in V$. We let $\tau_x = \{V \subset X : x \in V\}$ denote the collection of open neighbourhoods of x.

4. A subset $W \subset X$ is a neighbourhood of x if there exists $V \subset \tau_x$ such that $V \subset W$.

5. A collection $\eta \subset \tau_x$ is called a **neighbourhood base** at $x \in X$ if for all $V \in \tau_x$ there exists $W \in \eta$ such that $W \subset V$.

If we have two topological spaces whose points have similar neighbourhoods, we can find a continuous function, or a "map", from one to the other.

Definition (Map)**.** For two topological spaces X and $\widetilde{X}$, a map $p : \widetilde{X} \to X$ is a continuous function such that for all open sets in $V \subseteq X$, the preimage $f^{-1}(V)$ is open in $\widetilde{X}$.

Henceforth we will equate the terms "map" and "continuous function" as synonymous. The point of the term topology is to give a notion of continuous functions, and so the topologies will be clear in all the examples we see from here on.

In the Fundamentals of Pure Mathematics course we discussed isomorphisms between graphs, as previously mentioned. We can also define isomorphisms between groups in general using group homomorphisms.

Definition (Isomorphism between groups)**.** There is a group homomorphism $\phi : G \to H$ between two groups G and H if $\phi(x)\phi(y) = \phi(xy)$ for all $x, y \in G$.

A group homomorphism $\phi : G \to H$ that is also injective (one-to-one) and surjective (every element $h \in H$ has a preimage) is called an isomorphism of groups. In this case we say that G and H are isomorphic and we write $G \cong H$.

To explore the topological equivalent, we introduce homeomorphisms.

Definition (Homeomorphism)**.** A homeomorphism is an isomorphism between two topological spaces. We consider it as a continuous function with a continuous inverse.

Here we define what it means for a space to cover another space with a given map.

Definition (Covering Space)**.** A covering space of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying the following condition: Each point $x \in X$ has an open neighbourhood U in X evenly covered by p, such that $p^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by p.

Example 2.1 (Finite Lines cover Finite Lines)*.* To explain the concept of a covering space, let us take a simple example. A set of n finite lines map to another finite line trivially:

$$(0,1) \times \{1,2,\ldots,n\} \to (0,1).$$

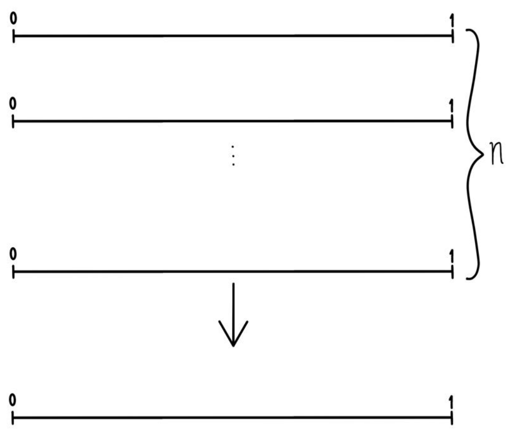


Figure 2: n versions of x, where x is the interval $(0,1)$, map to x trivially.

Likewise, a set of lines stretching infinitely in both directions maps to another infinite line trivially. In general, $X \times \{1,2,\ldots,n\}$ maps to X.

Example 2.2 (Circles cover Circles)*.* A simple covering space of a circle is another circle. If a circle, A, is scaled by a multiple of 9 to produce another space B, the map $p : A \to B$ would take each full rotation of A to be equivalent to a $\frac{2\pi}{9}$ rotation of B. We can justify that this is a covering space by recognising that the preimage of any point $x \in B$ has an open neighbourhood $(p^{-1}(x) - \epsilon, p^{-1}(x) + \epsilon)$ on which A is evenly covered by p.

An interval of any kind, let us say $[a, b]$, is not a covering space of the circle because at the edge points $p(a) = n$ there is not an open neighbourhood around a on which $[a, b]$ is evenly covered. The map does not cover the edge cases.

Example 2.3 (The Real Number Line)*.* A covering space can be much bigger than the space it covers, for example the real number line covers a circle by coiling it round to form a helix. This is the largest covering space of a circle, where the map $p : \mathbb{R} \to A$ sends each $x \in [0, 2\pi]$ to $p(x + 2\pi n) = p(x)$.

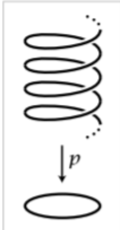


Figure 3: At each point on the space there are two legs, like with ∞. The **Deck Transformation** of p, a term which we are about to define, is simply to add integer multiples of 2π to $x \in A$.

We can recognise different properties in covering spaces, like the number of Deck Transformations of the map.

Definition (Deck Transformations)**.** The Deck Transformations of a covering space $p : \widetilde{X} \to X$ are all the homeomorphisms which preserve p; specifically all θ such that $p \circ \theta = p$.

Example 2.4 (Deck Transformations from the Real Number Line)*.* The Deck Transformations of the map from the real number line ($\widetilde{X}$) to the circle (X), where the map sends each number in $\widetilde{X}$ to itself modulo 2π, would simply be to add integer multiples of 2π to the specified point on $\widetilde{X}$.

Example 2.5 (Deck Transformations from a Circle to a Circle)*.* It is also possible to have three loops mapping down to a circle if we have them joined as shown in Figure 4.

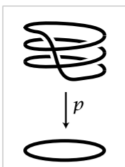


Figure 4: Finite loops of any size map to a circle.

Since this loop is ultimately just another circle 3 times the size of the downstairs circle, we can map each loop of the downstairs circle to a third of a loop of the upstairs circle, with Deck Transformations as rotations of $\frac{2\pi}{3}$.

Example 2.6 (Deck Transformations of the Trivial Map)*.* The smallest cover of ∞ is ∞ itself. This map has only the identity element in the group of Deck Transformations.

Showing a cover visually can be done by using its Cayley graph: a graph that encodes the abstract structure of a group. It uses the specified set of Deck Transformation generators for the group and shows the relationships visually.

Example 2.7 (∞'s Cayley graph). In the case of the figure of eight, the Cayley graph would look as we might expect:

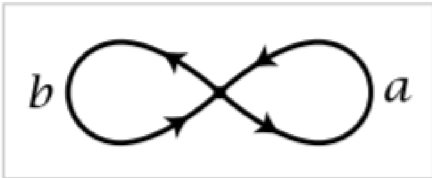


Figure 5: The Cayley graph of the figure of eight.

Since there are two edges, it is helpful to give them names: a and b. We also give each edge a direction. Although the direction of a seems unnecessary here, in other spaces we will see that the directions will influence the relations. In particular, notice that at the node there is an a edge and b edge pointing into x_0, and an a and a b pointing out from x_0. This will be an intuitive property we will observe in all the covering spaces we will consider.

Example 2.8 (Infinite Grid Cayley Graph). The grid given by $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subset \mathbb{R}^2$ i.e. $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$ is also a covering space of ∞. We will denote this as the infinite grid, or $\mathfrak{z}$ (said "yog").

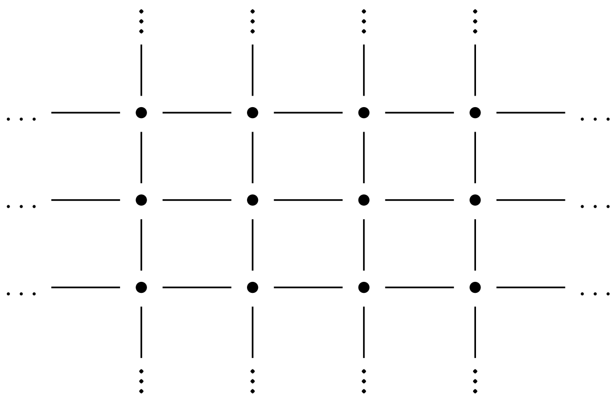


Figure 6: $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subset \mathbb{R}^2$.

The Deck Transformations of the map from $\mathfrak{z}$ to ∞ correspond to vertices of the infinite grid itself, and thus includes any word composed of a and b. This is an **abelian** cover, and in fact we find that it is the largest example of an abelian cover of ∞. An abelian cover has words whose letters commute, meaning that if we apply a then b it is equivalent to swapping the order. There are infinite Deck Transformations of this cover since it is an abelian cover with infinite nodes. Each time we apply a we move a step right along the x-axis, and every time we apply b we move a step up along the y-axis (and vice versa for a^{-1} and b^{-1}). If we map our path on the grid, it is clear that the order in which we take the steps from the start node to the end node does not matter.

The largest cover of the ∞ figure is the fractal cross.

Example 2.9 (The *Free Group* on 2 variables). The **Universal Cover** (hereto denoted by $\uplus$, pronounced “u cross”), also known as the non-abelian fractal cross, is the largest covering space of ∞. Since the space is non-abelian, we have that the order of the components of a word is unique; $ab \neq ba$, so we get a branch-like leg in each direction, which we will call a tree, or fractal tree. The free group on two variables, F_2, is the group of Deck Transformations of $\uplus$ over the figure of eight.

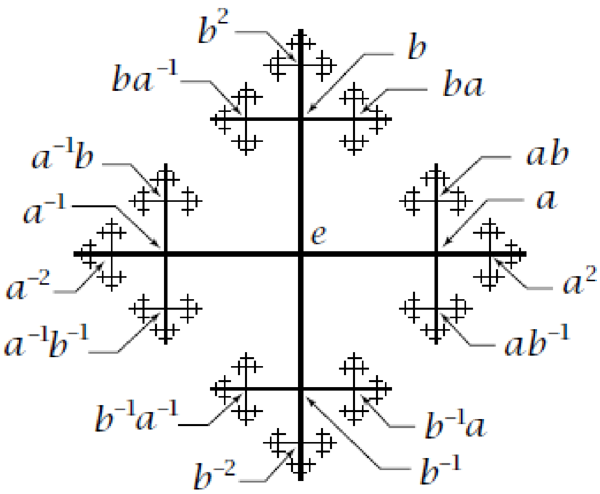


Figure 7: The fractal cross ($\uplus$), shown by scaling down the lengths of the farther out branches.

Another way to express the fractal cross with the lines not intersecting is to curve the lines away from each other.

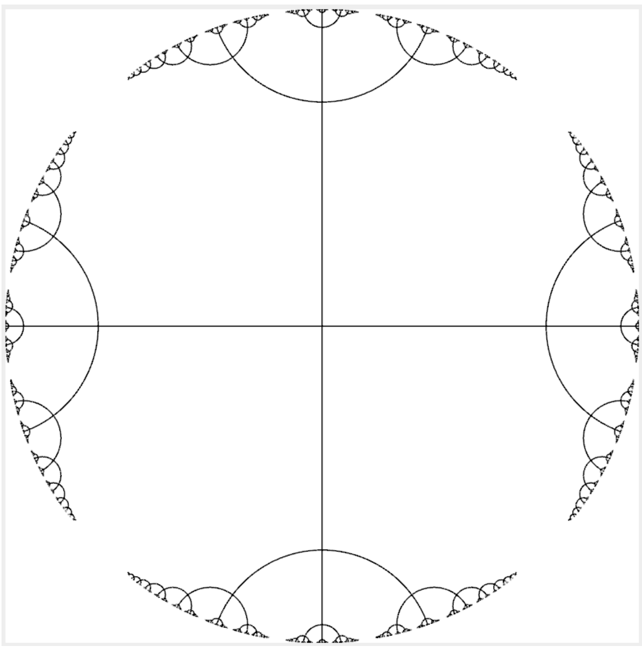


Figure 8: The circular version of the fractal cross.

We can express a on a grid as one step right along the x-axis and b as one step up along the y-axis. Since a and b are non-commutative in this covering space, the word ab takes us to a different point from the word ba. This quality of the space gives it a fractal shape. The map from the fractal cross to itself has no non-trivial Deck Transformations.

We can get ʒ from $\uplus$ by gluing together all words of the same order of a and b: for instance ab to ba.

We will define each cover by their Deck Transformations. Later we will find that the number of generators of the group of Deck Transformations is equal to the number of holes in the Cayley Graph. For instance in the case of the ʒ Cayley graph, there are infinitely many loops and therefore infinitely many generators. For instance

$$abab = ab^2a = ba^2b = a^2b^2 = b^2a^2 = baba\,.$$

The implication that there will be infinitely many Deck Transformations here makes intuitive sense, since for every permutation of the letters $abab$ we glue to a^2b^2 creating loops, and for every permutation of the letters $ababab$ we glue to a^3b^3. Ultimately every possible permutation of the same number of a's and b's must be equated, for every combination of a's and b's. Generalising this gives infinitely many loops with infinitely many Deck Transformations.

We can describe the map we use to take a cover to another cover as a Group Homomorphism.

Definition (Group Homomorphism)**.** Let G, H be groups. A map $\phi : G \to H$ is called a group homomorphism if

$$\phi(xy) = \phi(x)\phi(y)\,\forall x, y \in G$$

We define the Kernel of a Group Homomorphism to be the group of actions that return the identity.

Definition (Kernel)**.** For any group homomorphism $\phi : G \to H$, the kernel is defined as

$$\mathrm{Ker}(\phi) := \{g : g \in G \mid \phi(g) = 0\},$$

so $\mathrm{Ker}(p) < G$.

The group of Deck Transformations of a map is closely linked to the Kernel of the Group Homomorphism given by the map. We know that subgroups G of F_2 give rise to covering spaces $\tilde{X}$ of X. If the subgroup G is normal, then there is a group homomorphism ϕ from F_2 to the deck transformations of $\tilde{X}$, and the kernel of ϕ is G.

3 Covering Spaces of ∞

To find the covering spaces of the ∞ symbol, we treat it like two loops (or edges) with a node, x_0, connecting them. So, now we must find mappings in a similar fashion to when we found covers of the circle, with the added complication that at x_0 there must be an open neighbourhood in four directions. To illustrate this, see Figure 5.

If edge a and edge b join at node x_0, then for some ϵ there must be an open neighbourhood on the upstairs space such that both $q^{-1}(a-\epsilon, a+\epsilon)$ and $q^{-1}(b-\epsilon, b+\epsilon)$ are contained in the open neighbourhood, and the space maps evenly on the interval.

An intuitive rule to consider when finding covers of ∞ is that at each node of a covering space there must be two arrows going into the node and two arrows coming out of it, as in Figure 5. This is true in the figure of eight and all covers of it, satisfying the open-neighbourhood condition specified in the definition of **covering space**.

3.1 Non-Abelian Symmetry Mappings

As previously discussed, the map from the grid of $\mathfrak{z}$ to ∞ has abelian Deck Transformations.

However, non-abelian spaces also map to ∞; in fact most of the covering spaces we find will be non-abelian. The biggest such space is the non-abelian cross (the free space on two variables), which we will denote $\uplus$. This space is similar to a grid in that one moves through the space in steps, but since it is non-abelian, the point at ab is distinct from the point at ba. To visualize this grid, see Figure 7. F_2 is the group of deck transformations of the map $p : \uplus \to \infty$. The fractal cross is the Universal Cover of ∞ because it is a covering space of every other connected covering space of the ∞ figure.

We denote as p the map from the universal cover to the covering space, and q the map from the covering space to the figure of eight.

$$\uplus \xrightarrow{p} \widetilde{\infty} \xrightarrow{q} \infty$$

To describe each quotient space $\widetilde{\infty}$ that covers ∞, we write the group, $\mathfrak{D}$, that acts on $\uplus$ to produce that space.

$\mathfrak{D}$ is the subgroup of Deck Transformations of the universal cover, $\uplus$.

We will call this the **Deck Transformation group**, or $\mathfrak{D}_p$ where $p : \uplus \to \widetilde{\infty}$ is the map in question. If $r : \uplus \to \infty$ is the map from the universal cover to the figure of eight, then $F_2 = \mathfrak{D}_r(\uplus)$.

Then the group $\mathfrak{D}_p(\uplus)$ is a subgroup of $\mathfrak{D}_r(\uplus) = F_2$, and the covering space $\widetilde{\infty}$ can be described as the quotient space $\widetilde{\infty} = \uplus/\mathfrak{D}_p$.

Example 3.1 ($\mathfrak{D}_p = \langle a, b^2, bab^{-1}\rangle$). This space is the first example given in Hatcher's Table. There are three Deck Transformation generators (one defined for each loop).

The first generator acts on $\uplus$ to produce the quotient space given in Figure 9, with Deck Transformation group $\mathfrak{D}_p = \langle a \rangle$. This first space shows that if b is not defined as a generator, then we get a pair of ⊎ shapes where the b loop was. This is because the only path that we know takes us back to x_0 from $b(x_0)$ is the trivial path; $b^{-1}b(x_0)$. So, a and a^{-1} act trivially, whereas both b and b^{-1} take us to fractal trees.

The second generator lets b^{2n} ($\forall n \in \mathbb{Z}$) return to x_0, giving the second space in Figure 9. Note that we still have fractal crosses at $b(x_0)$ where a has no defined path.

The third generator disposes of the fractal trees, gluing ba to b and giving the first covering space of Hatcher's table.

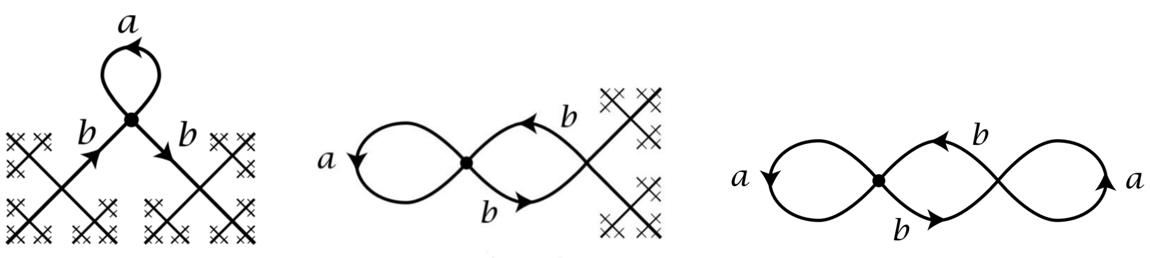


Figure 9: The covering spaces with Deck Transformation groups $\mathfrak{D}_p = \langle a \rangle$ (left), $\mathfrak{D}_p = \langle a, b^2 \rangle$ (middle), $\mathfrak{D}_p = \langle a, b^2, bab^{-1} \rangle$ (right).

The set in the angle brackets generates the Deck Transformation group: of all words that can be arranged with these elements. In a basic example, ∞ is the quotient space of $\langle a, b \rangle$ acting on ⊎.

Another example of the progression of restrictions from fractal cross to the figure of eight is as follows:

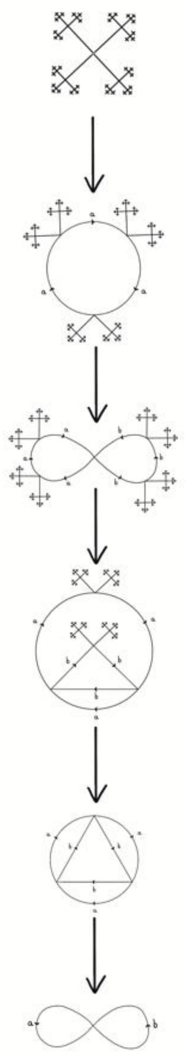


Figure 10: The transitions from the fractal cross to each cover, and ultimately to ∞.

The fractal cross will have no elements in the Deck Transformation group.

The second cover has only $\mathfrak{D}_p = \langle a^3 \rangle$.

The third cover, beginning to look more like the figure of eight, further restricts the space with $\mathfrak{D}_p = \langle a^3, b^3 \rangle$.

The fourth cover has $\mathfrak{D}_p = \langle a^3, b^3, ab^{-1} \rangle$, such that now two vertices are bound to both a edges and b edges.

The fifth and penultimate cover has $\mathfrak{D}_p = \langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$, binding the third vertex and eradicating the last of the fractal crosses.

The final space is the figure of eight itself.

This process demonstrates that each Deck Transformation generator ties edges of the fractal cross together, restricting it further until it is but an ∞ symbol.

3.2 Hatcher's Covering Spaces

Fourteen covering spaces (with their defining generating sets) of the figure are given in Allen Hatcher's *Algebraic Topology.*

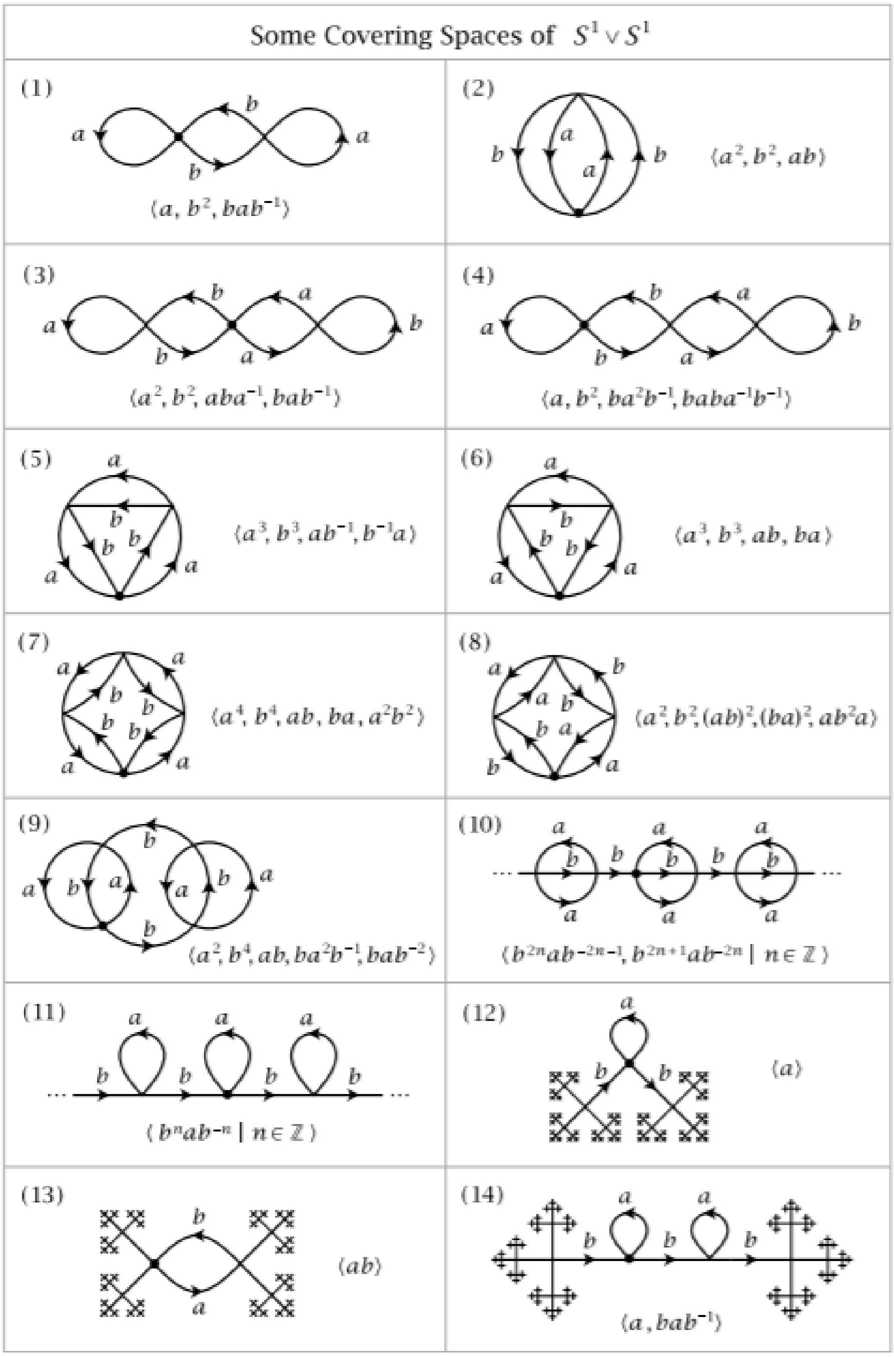


Figure 11: The set that generates the group of Deck Transformations from the universal cover, $\mathfrak{U}$, to each cover are given in the angular brackets. If $\mathfrak{D}_p$ is normal, any word made with these generators or their inverses maps to the identity in $\mathfrak{D}_q$.

One thing we can observe from this is that if one space covers another and neither have fractal trees (they are normal) then there will be non-trivial Deck Transformations.

We also see that there are as many generators in each example as there are loops in the Cayley graph.

If a subgroup is normal then every Deck Transformation of the cross is a Deck Transformation of the cover.

If $\mathfrak{D}_p(\uplus)$ is normal (i.e. $\mathfrak{D}_p(\uplus) \lhd F_2$) then $\mathfrak{D}_q(\widetilde{\infty}) = F_2/\mathfrak{D}_p$.

If $\mathfrak{D}_p(\uplus)$ is not normal, $\mathfrak{D}_q(\widetilde{\infty}) = N_{F_2}(\mathfrak{D}_p)/\mathfrak{D}_p$, where $N_{F_2}(\mathfrak{D}_p)$ is the Normaliser of $\mathfrak{D}_p$.

The **Normaliser** of H is the biggest subgroup G of F_2 in which the subgroup H is normal. In other words, the normaliser is the set of elements $g \in G$ such that $gH = Hg$.

If a subgroup is **normal** then every Deck Transformation of F_2 is a Deck Transformation of the cover. In the case of ∞, we have that the Deck Transformations are the trivial group.

We also have that every path represents a unique a Deck Transformation; there are as many Deck Transformations as there are vertices.

If a subgroup is **not normal**, not every Deck Transformation of the non-abelian cross gives a Deck Transformation of the cover. This is to say that not every path has a Deck Transformation associated with it.

Theorem 3.1. *There is never more than one Deck Transformation between two vertices in* $\widetilde{\infty}$*.*

Proof: The vertices of the quotient space $\widetilde{\infty}$ correspond to cosets of the subgroup in F_2. If there exist two Deck Transformations that have the same action on the coset H, i.e. $\exists x, y$ such that $xH = yH$ where H is the subgroup in question, then the corresponding elements in F_2 must differ by an element of the subgroup, i.e. $x^{-1}xH = x^{-1}yH$, and so $x^{-1}y \in H$, indicating that $x^{-1}y$ is trivial, and x gives the same Deck Transformation as y. $\square$

Definition (Betti Numbers)**.** In simple terms, the kth Betti Number - named after Enrico Betti Glaoui - is the number of k-dimensional holes in a topological surface. The 0-dimensional Betti Number, b_0, is the number of connected components, where b_1 is the number of 1-dimensional holes (or circles), and b_2 is the number of 2-dimensional holes (or voids/cavities).

Example 3.2 (Second Betti Number of a Torus)*.* A torus, for example, has two circles on its surface, so $b_1 = 2$.

3.3 Nielsen-Schreier Theorem on Free Groups

The subgroup H of any free group G is a free group. In other words, there exists a set S of elements that generate H with no relations among the elements of S.

Furthermore, the Schreier Index Formula states that if G is free on $|S_G| = n$ generators (i.e. G has rank n), and H has finite index $[G : H] = e$, then H is a free group of rank $1 + e(n - 1)$.

An example of this lets G be the free group on two variables a and b, and $H < G$ be every even composition of a and b, generated by $u = aa$, $v = ab$, $w = ba$, $x = bb$, $y = ab^{-1}$, $z = a^{-1}b$. However this does not satisfy the condition that there are no relations among S_H; $uz = v$. Hence we reduce the group of generators S_H to the smallest set that generates H. There are multiple possibilities here, but let us take $S_H = \{u, v, w\}$. We find that H is therefore free on 3 variables. This cover is given in Hatcher's Table:

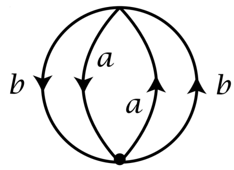


Figure 12: $\mathfrak{D}_p = \langle a^2, ab, ba\rangle$.

To verify that this is what we should expect, we use the index formula: G is free on $n = 2$ generators, the index of H is $e = 2$, so H should have rank $1 + e(n - 1) = 1 + 2(2 - 1) = 3$, which agrees with the results of our example.

Proof: The free group G of rank n can be shown graphically as a bouquet of n circles sharing one vertex. Let us assume that the subgroup H has finite index $[G : H] = e$. Then the first Betti Number of H will be the number of edges minus the number of vertices plus the number of connected components. We have that the cover Y of X shown by the group H has en edges and e vertices. Therefore the rank of H is $h(Y) = en - e + 1 = e(n - 1) + 1$.

3.4 Change of Basepoint

If we change the point from which we are defining our covering space (for example in space 5 in Figure 11 if we let the x_0 be the top left point instead of the bottom point) we will procure an equivalent generating set. In other words, the map $\beta_p : (\widetilde{X}, \tilde{x_0}) \to (\widetilde{X}, \tilde{x_1})$ is an isomorphism.

Example 3.3 ($\mathfrak{D}_p = \langle a, b\rangle$). We can imagine ∞ as two circles joined by a node x_0, as referenced earlier. The covering space with both a and b in the set of Deck Transformation generators is the trivial covering space of ∞. Since both journeys lead us straight back to x_0, they describe the figure of eight itself, which maps trivially to ∞. Another way to say this is that ∞ is the quotient space of $\langle a, b\rangle$ acting on $\uplus$.

Example 3.4 (Torus Cover). We can cover the figure of eight with any cover whose normal subgroup is generated by the elements $\{a^m, b^n, aba^{-1}b^{-1}\}$. To do this, we make an m by n grid and glue the sides of length n together to make a cylinder. Folding the top circle inside the cylinder and gluing it to the bottom circle, the space we get is a Torus. This applies trivially in the case of $\langle a, b\rangle$, as shown in Figure 13.

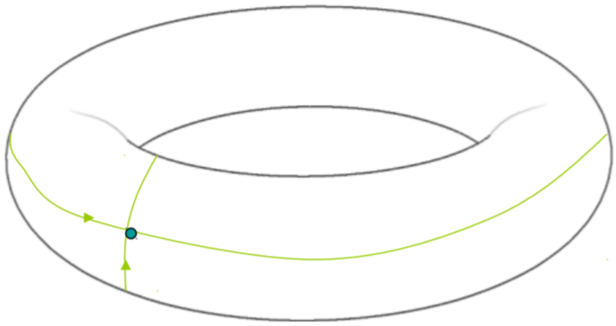


Figure 13: One loop a corresponds to an edge around the outer circumference of the torus, and one loop b corresponds to a edge starting on the outside and wrapping inside and back again.

4 Finding new covers

4.1 Celtic Knots

Ultimately, there are infinitely many covering spaces. A particularly intricate example might be DaVinci's Celtic knot.

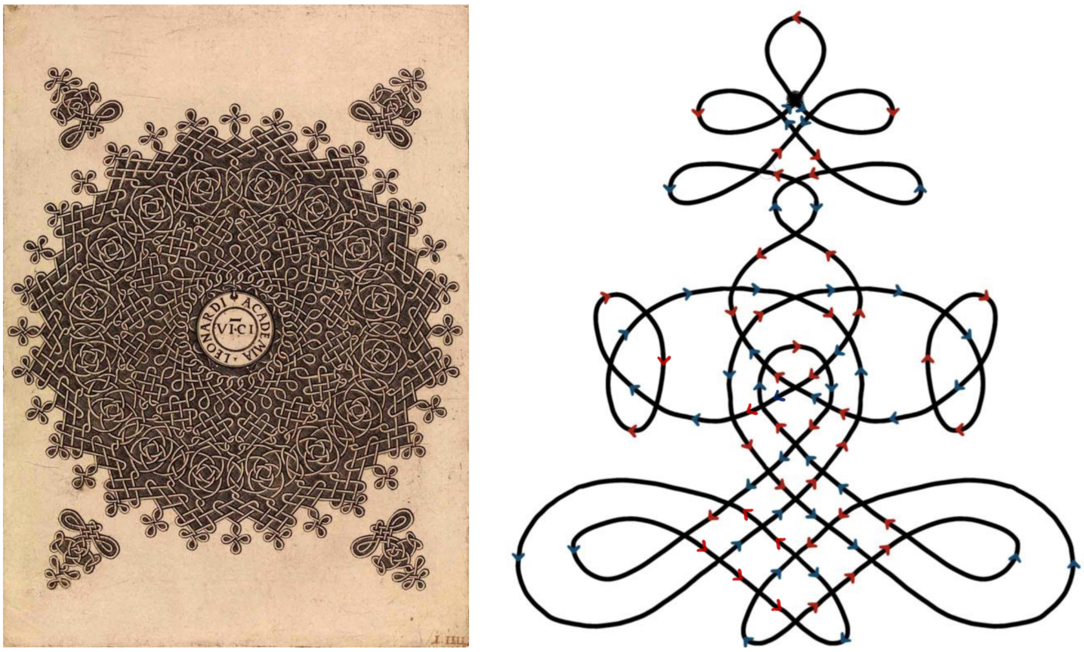


Figure 14: Looking at each knot, we find that they are each made up of only one line with many nodes. The corner piece, for example has $\mathfrak{D}_p = \langle a, b^4, bab^{-1}, b^{-1}ab, b^2a^4b^2, ...\rangle$. See Appendix A for the full set of Deck Transformation generators.

This shape can be made into a covering space by first defining a starting node, x_0, then labelling each edge a and b and giving them direction. In labelling and directing these edges, the rules are that at each node, there must be a red (a) edge entering, a blue (b) edge entering, a red (a) edge exiting and a blue (b) edge exiting. For example, with the knot given, we can choose arbitrarily for the top loop to be a red edge with any direction. This forces the other two lines protruding from x_0 to be blue edges. We can choose their direction, but one has to point toward x_0 and the other away from. Given that the next loops provide an outward and inward edge, they cannot be blue because then the nodes at which they are centred have three blue edges, which does not satisfy the constraints. By the same logic, let the loops below both be red. Then whenever there is a choice about direction or colour, choose arbitrarily. If one encounters a contradiction, return to the last point of forced choice and start the section again from that checkpoint. Trial and error seems to be the only method of correct labelling.

As you may have guessed, this way of labelling and directing the edges on this particular Celtic knot is not unique. For example, the single loops all have unimportant direction, and the two lowermost blue loops can be switched with the parallel red lines.

We expect that this procedure would work for any Celtic knot, leading to the following conjecture.

Conjecture (Celtic knots)**.** *All Celtic knots can form a cover of the figure of eight.*

In a Celtic knot, we have that one line weaves over and under every line it passes in alternation. As in all covers of the figure of eight, we have that every crossing point has 4 lines leading out from it. Simple loops prevent us from finding an obvious relation between the crossing and the colourings of the edges - like saying, perhaps, let all "over"s be blue and "under"s be red.

It seems a coincidence that Celtic knots always have this over and under weaving property, without fail. It is also feels coincidental that this Celtic knot has all the requirements for covering the figure of eight. As it is my experience that such coincidences usually have something to do with one another, I would not be surprised if they were related.

4.2 More Covers

More simplistic covers can be found. Let us look at common groups, and model these as the group of Deck Transformations in each case, finding the associated Cayley graph and discussing its properties.

Example 4.1 ($\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$). If a is a clockwise rotation of $\frac{2\pi}{3}$ and b swaps between the inner and outer triangles, we have a cover where the Deck Transformation group from the map $q : \widetilde{\infty} \to \infty$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_2$. This group is cyclic, and can be generated by ab. Note that this is different from the map $p : \uplus \to \widetilde{\infty}$, whose Deck Transformation group would be

$$\mathfrak{D}_p = \langle a^3, b^2, aba^{-1}b, a^{-1}bab, baba^{-1}, ba^{-1}ba, ababa \rangle.$$

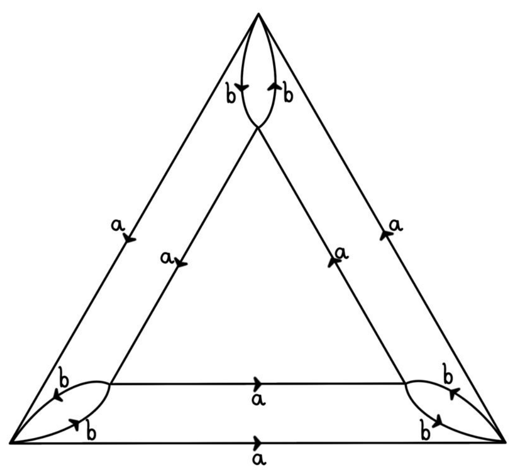


Figure 15: The Cayley graph of the cover whose Deck Transformations form the group Z_6. Note that the direction of the arrows on the inner triangle is the same as that of the outer triangle.

Deck Transformations send x_0 to other vertices with similar properties. In the previous example, the rotational Deck Transformations are all clockwise, regardless of the basepoint we set. This is because we are looking at a commutative group. Let us now look at a non-commutative group.

Example 4.2 ($D_3 \cong S_3$). If a is a clockwise rotation of $\frac{2\pi}{3}$, and b swaps (between the inner and outer triangles) and reflects (about the vertical axis), we have a cover where the Deck Transformation group from the map $q : \widetilde{\infty} \to \infty$ is isomorphic to D_3. The quotient map $p : \uplus \to \widetilde{\infty}$ has Deck Transformation group $\mathfrak{D}_p = \langle a^3, b^2, ab^{-1}ab, abab, bab^{-1}a, baba, aba^{-1}ba \rangle$. As this cover represents a non-commutative group, we find that moving the basepoint changes the Deck Transformations. For example, if x_0 were to be placed at the vertex above (the uppermost point of the outer triangle), the rotational Deck Transformations would be in the other direction; a now corresponds to an **anti-clockwise** rotation of $\frac{2\pi}{3}$.

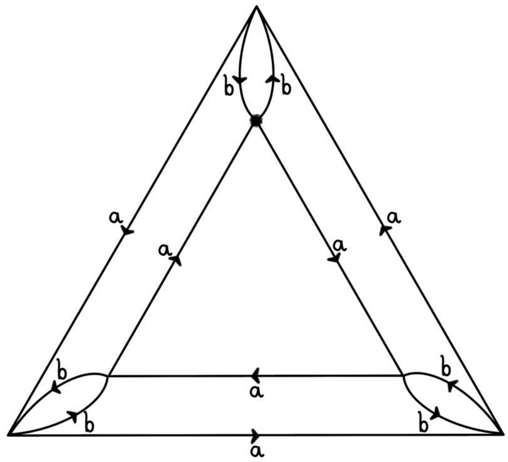


Figure 16: The Cayley graph of the cover whose Deck Transformations form the group D_3. Note that the direction of the arrows on the inner triangle is **opposite** from that of the outer triangle, unlike in the space represented by the group $\mathbb{Z}_6$.

Example 4.3 (D_n). We use the same way of thinking when finding a cover with the group of Deck Transformations isomorphic to D_n. Take two n-gons, where a is a clockwise rotation of $\frac{2\pi}{n}$ on the n-gon and b swaps (between the inner and outer triangles) and reflects (about the vertical axis). This cover will have $2n + 1$ generators; one for each b^2 loop, one for each $abab$ loop, and one for the centre a^n loop. We can also prove this with the Schreier Index Formula: $|S_G| = 2$, $e = 2n$, and therefore rank $= 1 + 2n(2 - 1) = 2n + 1$.

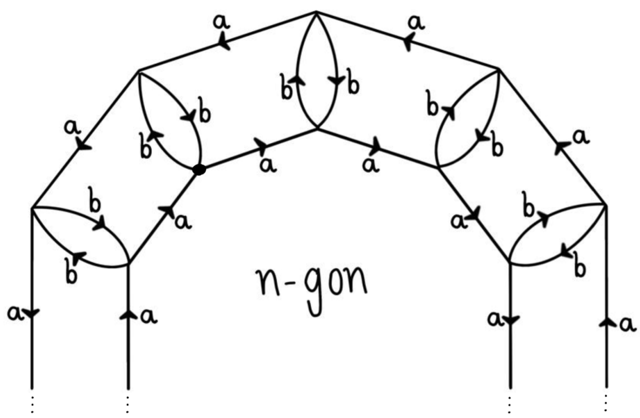


Figure 17: The Cayley graph of the cover whose Deck Transformations form the group D_n. Similarly to the D_3 example, the inner n-gon has edges that are directed oppositely to the outer n-gon.

Example 4.4 (The m by n grid). If we take the infinite number grid, cut it so that we have an m by n grid, join the two edges of length m together to form a cylinder and then join the circles with circumference n together to make a torus, then we have a cover whose Deck Transformations are generated by an element of order m and an element of order n. This cover will have $m \times n + 1$ Deck Transformation generators as that is how many distinct loops will appear in the Cayley graph - the plus 1 is the loop in the middle of the torus. An example of this is $\mathbb{Z}_6$ from Example 4.2.

$$
\begin{array}{ccccccccc}
(0,m) & — & (1,m) & — & \dots & — & (n-1,m) & — & (n,m) \\
| & & | & & & & | & & | \\
\vdots & & \vdots & & \dots & & \vdots & & \vdots \\
| & & | & & & & | & & | \\
(0,1) & — & (1,1) & — & \dots & — & (n-1,1) & — & (n,1) \\
| & & | & & & & | & & | \\
(0,0) & — & (1,0) & — & \dots & — & (n-1,0) & — & (n,0)
\end{array}
$$

Figure 4.4: The $m \times n$ grid.

Example 4.5 (S_n). To find the covering space with a symmetric group of Deck Transformations, we must find a way to generate the symmetric group S_n with two generators. This can be done with generators (12) and $(123\dots n)$.

Example 4.6 (A Circle with Decoration (n-gons)). If we treat ∞ like two circles, we can consider only one circle as before; every 2π we wrap once around circle A, and at each loop interval there is a node connecting the helix to a circle.

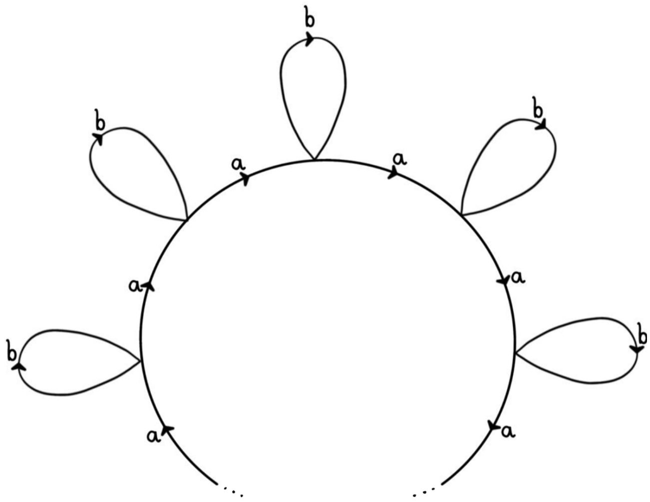


Figure 19: The decorated circle as a cover of $\widetilde{\infty}$, with Deck Transformation group $\mathfrak{D}_p = \langle a^n, b, a^k b a^{-k} \mid k \in \mathbb{Z} \rangle$.

As $n \to \infty$, this can be expressed as $\mathbb{R} \cup (\mathbb{Z} \times S^1)$, where S^1 is a single loop and $S^1 \vee S^1$ defines ∞, which gives the 11th example in Hatcher's Table.

There are finitely many, say "n", b loops of this helix situated at n nodes, with the larger circle a as an n-gon of sorts.

Example 4.7 (Quaternions (Q8)). The Quaternion number system (Sir William Rowan Hamilton) extends the complex numbers such that

$$i^2 = j^2 = k^2 = ijk = -1$$

Here we will look at the finite group of Quaternions:

$$\{1, -1, i, -i, j, -j, k, -k\}.$$

We see that i and j can generate the group where $ij = k$ and $ji = -k = -ij$.

This can cover ∞ by looking at i and j as diagonal relations, rather than perpendicular. This is to account for the specific non-abelian property of Quaternions.

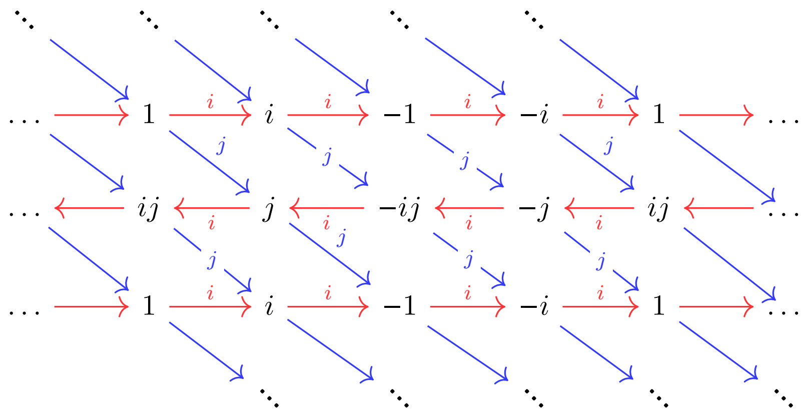


Figure 20: A graphical representation of the Quaternions (with red arrows symbolising the left application of i and blue symbolising that of j).

The graph is infinite and has a repeating patch of 8 parallelograms. As mentioned earlier, the number of loops is the same as the number of Deck Transformation generators. Let us use these 8 loops to find the defining group for the Quaternions.

If we let x_0 be the node at 1, and then first trace the loop bounded by i, $-ij$ and j, the steps we take travel i right then j down, followed by i left and $-j$ up. We write this as $j^{-1}iji$. We can do the same for the next loop bounded by i, -1, $-j$ and $-ij$ using the same method. We find that the generator for this loop is $ij^{-1}iji^2$.

Ultimately, performing this process for each loop we get that

$$\mathfrak{D}_p = \left\langle j^{-1}iji,\ ij^{-1}iji^2,\ i^2j^{-1}iji^3,\ ij^{-1}ij,\ jij^{-1}i,\ i^{-1}jij^{-1}i^2,\ i^{-2}jij^{-1}i^3,\ ijij^{-1} \right\rangle.$$

Since the set of Deck Transformation generators in the case do little to help us understand what space the Quaternions make in order to cover $\widetilde{\infty}$.

To understand the graph better, we can look at one section of it:

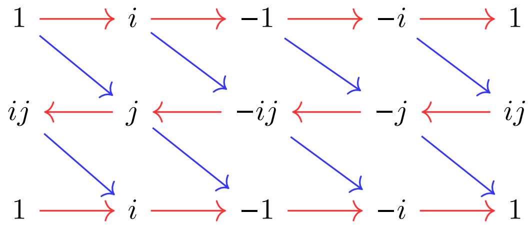


Figure 21: A section of the infinite Cayley graph of the Quaternions.

We see here that the top and bottom lines are identical, so we can glue them together to make a torus like so:

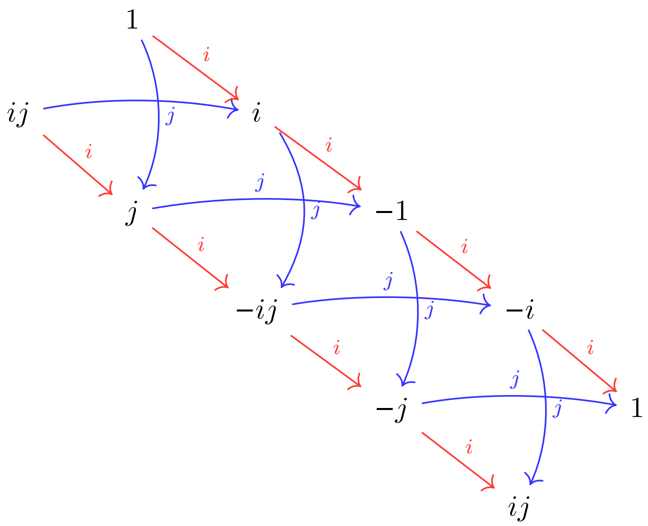


Figure 22: The cylinder version of the Quaternions Cayley graph.

Another Cayley graph of the quaternions is:

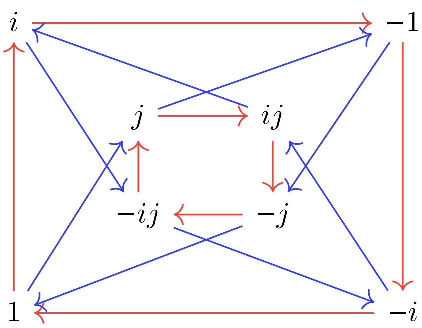


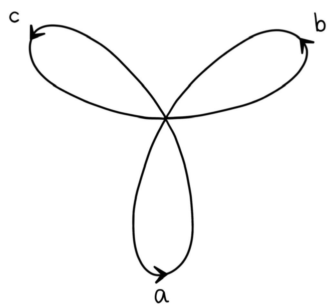
Figure 23: The unit version of the Quaternions Cayley graph.

In fact, we find that any word made up from a and b (such as $a^5b^{-1}a^{-3}$) generates a space that covers ∞.

5 Roses

5.1 3 Petalled Rose

The covering spaces of a three petal rose such as:



can be found in much the same way as those of $\widetilde{\infty}$, except at each node we require 3 edges going into and 3 edges coming out of each node.

In the same way that the fractal cross is the largest covering space of the figure of eight, we have an equivalent fractal for the 3 petalled rose.

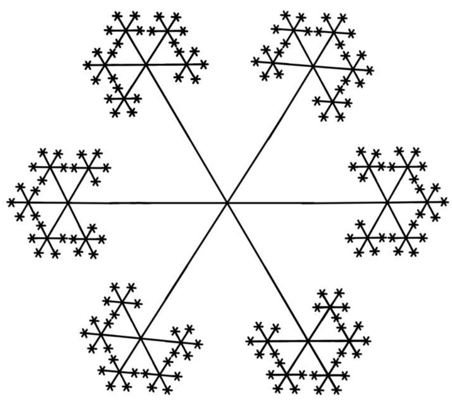


Figure 24: The fractal cross with three variables. At each node there are six legs; three going in and three going out.

We can use Quaternions in the same way as before, but with the third operator, k (given by the green arrows), also included.

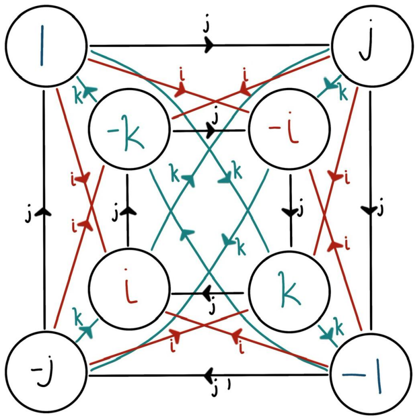


Figure 25: The Cayley Graph for Quaternions with all three actions i, j and k included. Each node has an i, j and k going in and an i, j and k pointing out from it.

Other covering spaces of the three petalled rose are:

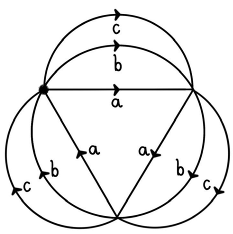


Figure 26: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle a^3, ab^{-1}, ac^{-1}, aba, aca, a^2b, a^2c\rangle$

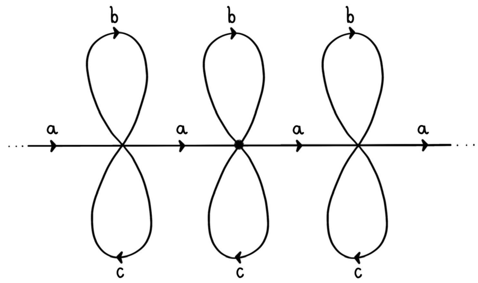


Figure 27: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle a^p b a^{-p}, a^p c a^{-p} \mid p \in \mathbb{Z}\rangle$

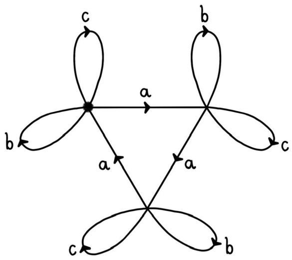


Figure 28: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle a^3, b, c, aba^{-1}, aca^{-1}, a^{-1}ba, a^{-1}ca \rangle$

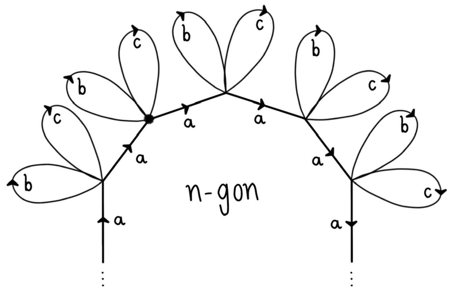


Figure 29: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle a^n, a^k ba^{-k}, a^k ca^{-k} \mid k \in \mathbb{Z},\ 0 \le k < n \rangle$.

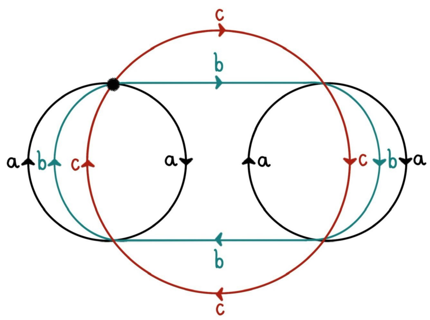


Figure 30: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle a^2, ab, ac, b^4, baba, ba^{-1}ba, bcbc, bc^3, c^4 \rangle$.

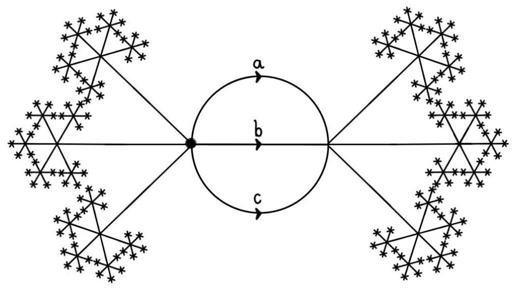
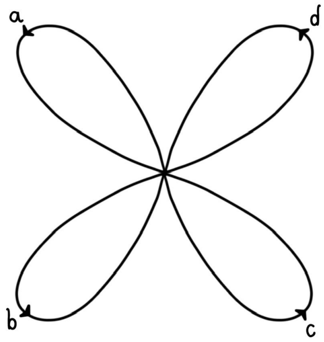


Figure 31: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle ab^{-1}, ac^{-1} \rangle$.

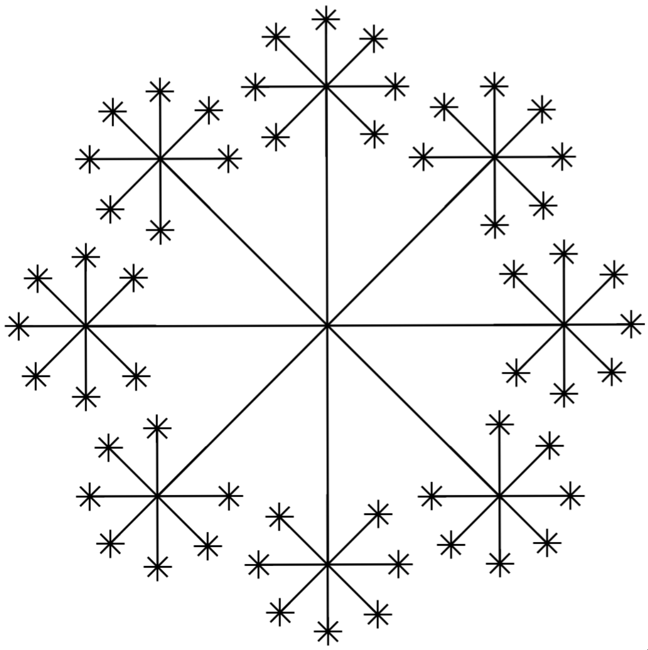
5.2 4 Petalled Rose

A four petalled rose, as you might have predicted, looks like so:



.

Its Universal Cover is the fractal cross on 4 variables:



.

Some examples of covering spaces of the four petalled rose are:

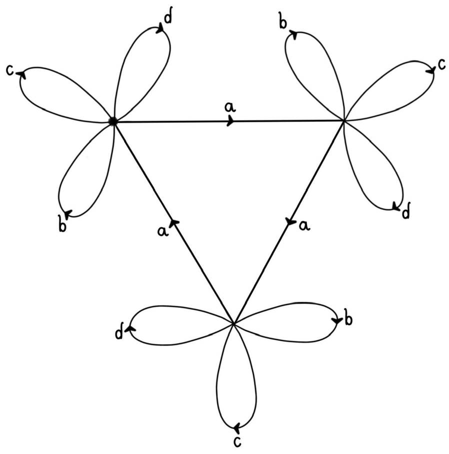


Figure 32: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle a^3, b, c, d, aba^{-1}, aca^{-1}, ada^{-1}, a^{-1}ba,$
$a^{-1}ca, a^{-1}da\rangle$.

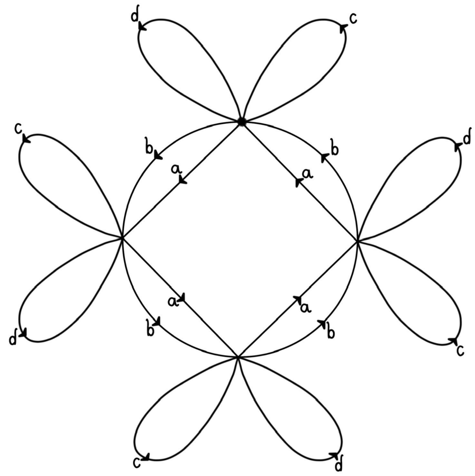


Figure 33: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle a^4, c, d, a^{-1}b, a^{-1}ca, a^{-1}da, ab^{-1}, aca^{-1},$
$ada^{-1}, a^{-2}ba, aba^{-2}, a^2ca^2, a^2da^2\rangle$.

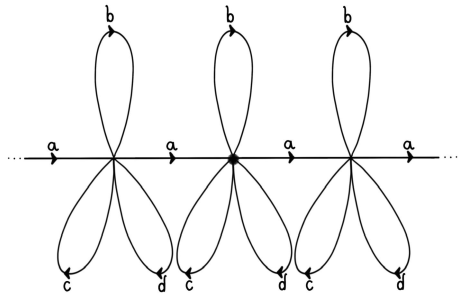


Figure 34: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle a^nba^{-n}, a^nca^{-n}, a^nda^{-n} \mid n \in \mathbb{Z}\rangle$.

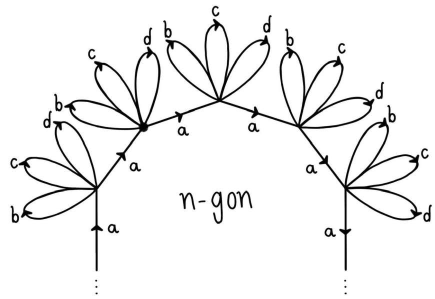


Figure 35: The cover associated with the Deck Transformation group:
$\mathfrak{D}_p = \langle a^n, a^kba^{-k}, a^kca^{-k}, a^kda^{-k} \mid k \in \mathbb{Z}\rangle$.

Appendices

A The Deck Transformations of the Da Vinci Knot

The Deck Transformation generators for the corner piece are

$$
\begin{aligned}
\mathfrak{D}_p = \langle & a, b^4, bab^{-1}, b^{-1}ab, b^2a^4b^2, b^2aba^{-1}b^2, b^2a^{-1}bab^2, b^2a^2b^2a^2b^2, b^2a^2bab^2aba^2b^2, b^2a^2baba^3ba^2b^2, \\
& b^2a^2ba^3baba^2b^2, b^2a^2ba^2b^{-1}ab^{-1}a^2ba^2b^2, b^2a^2ba^2b^2a^{-1}ba^{-1}ba^2b^2, b^2a^2ba^4b^{-3}a^{-1}b^{-1}aba^2b^2, \\
& b^2a^2ba^2b^3aba^{-1}ba^2b^2, b^2a^2ba^2b^{-1}a^{-2}b^{-1}a^2ba^2b^2, b^2a^2ba^4b^{-5}aba^2b^2, b^2a^2ba^2b^5a^{-1}ba^2b^2, \\
& b^2a^2ba^2b^{-1}a^{-1}b^{-3}a^2ba^2b^2, b^2a^2ba^2b^{-3}abababa^2b^2, b^2a^2ba^4b^{-2}ab^{-1}aba^2b^2, \\
& b^2a^2ba^2ba^{-1}b^2a^{-2}ba^2b^2, b^2a^2ba^4b^2a^{-1}ba^2ba^2b^2, b^2a^2ba^4b^{-2}a^{-2}b^{-1}aba^2b^2, \\
& b^2a^2ba^2b^2a^2ba^{-1}ba^2b^2, b^2a^2ba^2ba^{-4}ba^2ba^2b^2, b^2a^2ba^4b^2ab^{-1}aba^2ba^2b^2, \\
& b^2a^2ba^2bab^{-1}ab^2a^{-2}ba^2b^2, b^2a^2ba^2ba^4ba^2ba^2b^2, b^2a^2ba^2ba^2ba^{-1}b^2ab^2a^{-2}ba^2b^2, \\
& b^2a^2ba^2baba^{-1}ba^2ba^2ba^2b^2, b^2a^2ba^2ba^2ba^2ba^2ba^2ba^2b^2, b^2a^2ba^2ba^2b^2a^2baba^2ba^2b^2, \\
& b^2a^2ba^2baba^2b^2a^2ba^2ba^2b^2, b^2a^2ba^2ba^2b^2abababa^2ba^2b^2, \\
& b^2a^2ba^2bababa^{-1}b^{-1}a^3ba^2ba^2b^2, b^2a^2ba^2ba^2b^4aba^2ba^2b^2, b^2a^2ba^2bab^4a^2ba^2ba^2b^2, \\
& b^2a^2ba^2ba^2bab^{-1}a^3baba^2ba^2b^2, b^2a^2ba^2baba^3b^{-1}aba^2ba^2ba^2b^2, b^2a^2ba^2baba^8baba^2ba^2b^2, \\
& b^2a^2ba^2baba^4ba^3baba^2ba^2b^2, b^2a^2ba^2baba^3ba^4baba^2ba^2b^2 \rangle
\end{aligned}
$$

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