# The positivity and integrality of mirror maps

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# Abstract

In this dissertation, we discuss the concept of mirror symmetry, in particular mirror maps on reflexive polytopes. We prove that the power series given by the mirror map of a reflexive polytope has positive integer coefficients for the rank 1 case, and we construct a computer algorithm to test the conjecture on all 2- and 3-dimensional reflexive polytopes.

# Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Sophie Bleau*)

*For Pushkin, Norma and Idgie*

# Contents

# Introduction

> There are two ways of spreading light: to be the candle or the mirror that reflects it.
>
> *Edith Wharton*

To give an exposition on our study of mirror maps, we give some context and history for the subject, we state the theorems and conjectures as we will use them, we give an example to show these in action, and we note the difference between the mirror map we use and the 'real' mirror map.

The following section is purely to provide context for the dissertation, and the concepts brought to attention here will not be defined or addressed elsewhere.

## Context

Coined to explain a phenomenon in string theory connecting pairs of Calabi-Yau manifolds, the term 'mirror symmetry' is of interest in the fields of both mathematics and theoretical physics. In particular, a mirror map relates the complex moduli space of a Calabi-Yau manifold to the Kähler moduli space of its mirror Calabi-Yau manifold, facilitating many unresolved calculations, such as that of the $q$-expansion of the quintic threefold. In this project, we will be examining the coefficients of the mirror map for a Calabi-Yau hypersurface in the toric variety corresponding to various reflexive polytopes.

A trend in the field of mathematics has been the desire to know how many things of a certain type exist. Polyà counted the different colourings of graphs up to symmetry, Catalan numbers count the number of binary trees on $n$ vertices, and the partition function - counting the different ways to sum to a natural number $n$ - is still a mystery. A particular interest for geometers has been the desire to know how many rational curves of degree $i < n$ there are on an $n$-dimensional surface. For instance, there are 27 lines on a cubic surface. A surface that geometers were especially interested in was the quintic threefold in $\mathbb{CP}^4$. This is described as the 3-dimensional locus in $\mathbb{CP}^4$ defined by the vanishing set of a homogeneous polynomial of degree 5. The Calabi-Yau space given by the quintic threefold has a mirror pair given by a modification of the quintic threefold; namely, it is quotiented by a group which removes the singularities. The expansion of the

Yukawa coupling is given in [15] by

$$\kappa_{ttt} = 5 + 2875e^{2\pi it} + 4876875e^{4\pi it} + \dots,$$

where the conjecture is that the coefficients correspond to the count of rational curves on the quintic threefold. Indeed, it was found in [3] that

$$\kappa_{ttt} = 5 + \sum_{k=1}^{\infty} \frac{n_k k^3 e^{2\pi ikt}}{1 - e^{2\pi ikt}}$$

where the $n_k$ count the number of rational curves of degree $k$ on the quintic threefold, giving the number of lines and conics on the quintic threefold as

$$n_1 = 2875, \quad \text{and} \quad n_2 = 609250$$

respectively. However, for higher values of $k$, the calculation difficulty grew exponentially. Here, the physicists stepped in.

Some string theorists noted an interesting correspondence, in [3], between the coefficients that had been found, and some construction of what was to be the mirror map, inspired by Greene-Plesser. The theory was, that one could construct an equation which would allow the transfer from calculations of string interactions on a manifold to calculations on the mirror manifold- or even in Edith Wharton's language, gaining understanding not by seeing the candle itself but by looking in the mirror reflecting it. The calculation via the method of these string theorists, Candelas, Ossa, Green and Parkes, was disputed by many mathematicians at the time - apart from anything else, it was too easy! After spending a year trying to calculate the third coefficient, $n_k$, suddenly physicists were claiming that it was possible to calculate the entire series at once. Not only that, but the methods seemed entirely unrigorous. The mathematical community was almost relieved when at last the long sought-after coefficient yielded a different value ($n_3 = 1,682,549,425$) to that predicted by the string theorists ($n_3 = 317,206,375$). However, after some collaboration with said string theorists, there was a mistake found in the mathematical computation, and with the correction, it agreed with the string theorists' conjecture, forcing the mathematical community to consider mirror symmetry as a relevant and powerful tool. A workshop was arranged for the collaboration of mathematicians and physicists at Berkeley's Mathematical Science Research Institute in May 1991, where there was a significant language gap. However, the resulting collaboration was incredibly fruitful. Shing-Tung Yau said in [17]:

> There was an unusually productive flow of ideas between mathematicians and physicists as each attempted to grasp the vantage point and conceptual framework of the other.

Mirror symmetry, thus, has been allowed to shift and evolve in both realms of study, by string theorists and algebraic geometers alike, proving influential in both fields of study. Digging deeper, the role of reflexive polytopes in this theory became more prominent in articulating the duality between Calabi-Yau pairs, providing a combinatorial framework to study mirror symmetries. In analysing the mirror maps of reflexive polytopes, an intriguing pattern emerges - we find that the power series corresponding to the mirror map generated by a reflexive polytope has all integer coefficients.

This is particularly bizarre, as the mirror map constructed is a solution to a set of differential equations, subject to the exponential function, and division of a power series by an integral power series (a power series with integer coefficients). Indeed, it is not at all automatic that a quotient of a power series by an integral power series is integral; much less the exponential of this quotient. The conjecture is therefore a fascinating topic of study for a mathematician in the area. So, without further ado, let us conject.

# The conjecture

We will follow the Ganatra-Hanlon-Hicks-Pomerleano-Sheridan (GHHPS) conjecture as stated in [7]. Think of a polytope as a multidimensional polygon, so that a lattice polytope in a given space is a polytope whose vertices have integer coordinates. Then a reflexive polytope is a lattice polytope with the origin as its only interior point whose dual polytope is a lattice polytope with the origin as its only interior point.

Suppose $\Delta \subset \mathbb{Z}^d$ is a reflexive polytope, so that its vertices are lattice points. Take $A(\Delta) = \{\mathbb{a}_1, \ldots, \mathbb{a}_e\} \subset \mathbb{Z}^d$ to be the set of lattice points occurring on faces of $\Delta$ of codimension 2. This is to say, $A(\Delta)$ has the lattice points which make up its vertices, and all the lattice points (if there are any) intersecting the edges. We take the harmonic function to be

$$H(k) := \sum_{i=1}^{k} \frac{1}{i},$$

and the function of combinations of $\mathbb{u} \in (\mathbb{Z}_{\geq 0})^e$ to be

$$\operatorname{comb}(u_1, \ldots, u_e) := \frac{(u_1 + \cdots + u_e)!}{u_1! \ldots u_e!}.$$

Let $\mathbb{Z}^e$ be generated by the standard basis vectors $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1)$, and take the function $f : \mathbb{Z}^e \to \mathbb{Z}^d$ which sends the $i$th standard basis vector to $\mathbb{a}_i \in A(\Delta)$. Denote $K := \ker(f) \subset \mathbb{Z}^e$ and $K_{\geq 0} := K \cap (\mathbb{Z}_{\geq 0})^e$. As is standard, for a vector $\mathbb{u} \in (\mathbb{Z}_{\geq 0})^e$ and some vector of $e$ variables, we write

$$\mathbb{z}^{\mathbb{u}} := \prod_{i=1}^{e} z_i^{u_i}. \tag{1}$$

Let $\mathbb{Z}[[z]]$ and $\mathbb{Q}[[z]]$ denote the rings of power series over the integers and rationals respectively.

We define two important functions. The first is $\phi_0^{A(\Delta)}$, which we define

$$\phi_0^{A(\Delta)}(\mathbb{z}) := \sum_{\mathbb{u} \in K_{\geq 0}} \operatorname{comb}(\mathbb{u}) \mathbb{z}^{\mathbb{u}},$$

and the second is $\phi_i^{A(\Delta)}$ for $1 \le i \le e$, which we define

$$\phi_i^{A(\Delta)}(\mathbb{z}) := \sum_{\mathbb{u} \in K_{\ge 0}} \operatorname{comb}(\mathbb{u}) \cdot (H(u_1 + \cdots + u_e) - H(u_i))\mathbb{z}^{\mathbb{u}}.$$

We will often call these $\phi_0$ and $\phi_i$ when our polytope $\Delta$ is obvious from context. Notice that $\phi_i \in \mathbb{Q}[[z_1, \ldots, z_e]]$, whereas $\phi_0 \in \mathbb{Z}[[z_1, \ldots, z_e]]$.

With these two important functions, we can make a new function $\psi_i$ given by

$$\psi_i^{A(\Delta)}(\mathbb{z}) := \exp\left(\frac{\phi_i}{\phi_0}\right) \in \mathbb{Q}[[\mathbb{z}]].$$

Furthermore, we can take a vector $\mathbb{k}$ from the basis vectors $\{\mathbb{k}_1, \ldots, \mathbb{k}_{e-d}\}$ of the kernel $K_{\ge 0}$, and formulate the power series

$$\Psi_{\mathbb{k}}^{A(\Delta)}(\mathbb{z}) := \prod_{i=1}^{e} \psi_i^{\mathbb{k}^{(i)}}.$$

As with $\phi_0$ and $\phi_i$, we will often denote $\psi_i$ and $\Psi_{\mathbb{k}}$ without specifying $A(\Delta)$ if it is clear from the context. The conjecture regards the positivity and integrality of the coefficients of the Taylor series expansion of $\Psi_{\mathbb{k}}^{A(\Delta)}$

**Conjecture A.** For $\Delta$ a reflexive polytope, the power series $\Psi_{\mathbb{k}}^{A(\Delta)}$ have positive integer coefficients.

The conjecture, though not yet proven in the general case, bears a strong resemblance to a theorem proven by Delaygue (Theorem 1, [5]), which we state in Theorem 2.2. Delaygue's theorem is in fact the strongest theorem in the direction of Conjecture A. That said, it does not cover all cases, so it certainly cannot be used to prove the conjecture outright. A counterexample can be found in Example 2.8. In particular, we will show in the proof of Lemma 2.4 that if $\Delta$ is a reflexive polytope with the rank of the kernel $\operatorname{rank}(K_{\ge 0}) = 1$, Delaygue's theorem (Theorem 2.2) is equivalent to Conjecture A.

It is unclear whether Conjecture A is true. In this thesis, we take a step towards the answer by proving the following special cases.

**Theorem B.** For $\Delta$ a reflexive polytope with kernel of rank 1, the power series $\Psi_{\mathbb{k}}^{A(\Delta)}$ have integer coefficients.

This theorem is a variation on a special case of the result given in Conjecture A of [7]), which we prove in Section 3.1.

**Theorem C.** For $\Delta$ a reflexive polytope with kernel of rank 1, the power series $\Psi_{\mathbb{k}}^{A(\Delta)}$ have positive coefficients.

This theorem is a special case of that given in Lemma 2.1 of [12], and is proven in Section 3.2.

**Conjecture D.** For $\Delta$ a reflexive polytope,

1. The power series $\psi_i^{A(\Delta)}$ have integer coefficients.

2. The power series $\psi_i^{A(\Delta)}$ have positive coefficients.

It is worth noting that this conjecture is stronger than Conjecture A, since the integrality/positivity of each $\psi_i^{A(\Delta)}$ implies the integrality/positivity of $\Psi_{\mathbb{k}}^{A(\Delta)}$. To give evidence that this may be true, we prove computationally the following theorem.

**Theorem E.** Conjecture D holds to precision $p(\Delta)$ for all 2- and 3-dimensional reflexive polytopes $\Delta$, where $p(\Delta) = 30$ for $\dim(\Delta) = 2$, and $p(\Delta) = p_k$ as given in

$$(p_1, p_2, \ldots, p_{22}) = (30, 30, 30, 30, 30, 20, 10, 9, 8, 7, 6, 5, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3)$$

when $\dim(\Delta) = 3$, for $k$ the rank of the $A(\Delta)$ matrix kernel.

We checked via Sagemath [16] that all 16 reflexive polytopes of dimension 2 and all 4319 reflexive polytopes of dimension 3 were positive and integral up to a certain precision, where the number of reflexive polytopes of dimension 3 is given in [14]. The code is detailed in Section 4.2 and Chapter A of the Appendix. The precisions ranged from 3 to 30 depending on the rank of the kernel of the polytope, as shown in Chapter 5.

# An example

We show the inner workings of the mirror map with an example that will, in Section 2.1.2, show us the relation of Conjecture A to Theorem 2.2.

**Example 0.1.** Taking $\Delta$ to be the tetrahedron with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-6, -4, -1)$, we find $A(\Delta)$ to have vertices

$$v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (0, 0, 1), \quad v_4 = (-6, -4, -1), \quad v_5 = (-3, -2, 0).$$

To find the kernel $K_{\geq 0}$ of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -4 & -1 \\ -3 & -2 & 0 \end{pmatrix}$$

given by the vertices in $A(\Delta)$, we must find $(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0}$ such that

$$-6a - 3b + e = 0, \quad -4a - 2b + d = 0, \quad -a + c = 0.$$

Since $e = 6a + 3b$, $d = 4a + 2b$ and $c = a$, this kernel definition is equivalent to

$$K_{\geq 0} = \{(a, b, a, 6a + 3b, 4a + 2b) : (a, b) \in \mathbb{Z}^2_{\geq 0}\}.$$

Then $\phi_0$ is given by

$$\begin{aligned}
\phi_0 &= \sum_{\mathbb{u}\in K_{\geq 0}} \mathrm{comb}(\mathbb{u})\mathbb{z}^{\mathbb{u}} \\
&= \sum_{(a,b)\in\mathbb{Z}^2_{\geq 0}} \mathrm{comb}(a,b,a,6a+3b,4a+2b) z_1^a z_2^b z_3^a z_4^{6a+3b} z_5^{4a+2b} \\
&= \sum_{(a,b)\in\mathbb{Z}^2_{\geq 0}} \frac{a+b+a+6a+3b+4a+2b}{a!b!a!(6a+3b)!(4a+2b)!} x_1^a x_2^b \\
&= \sum_{(a,b)\in\mathbb{Z}^2_{\geq 0}} \frac{12a+6b}{a!b!a!(6a+3b)!(4a+2b)!} x_1^a x_2^b,
\end{aligned}$$

where $x_1 = z_1 z_2 z_4^6 z_5^4$ and $x_2 = z_2 z_4^3 z_5^2$. Notice that our kernel can also be written

$$K_{\geq 0} = \{a(1,0,1,6,4) + b(0,1,0,3,2) : (a,b) \in \mathbb{Z}^2_{\geq 0}\}. \tag{2}$$

Let us call the generating vectors of the kernel $\mathbb{k}_1 = (1,0,1,6,4)$ and $\mathbb{k}_2 = (0,1,0,3,2)$, so that any given vector $\mathbb{u}$ in $K_{\geq 0}$ is of the form $\mathbb{u} = a\mathbb{k}_1 + b\mathbb{k}_2$. We find that letting

$$\mathrm{comb}_{a,b} = \frac{12a+6b}{a!b!a!(6a+3b)!(4a+2b)!}$$

for notational ease, we get an idea of what a given $\phi_i$ might look like by finding $\phi_1$. It is given by

$$\begin{aligned}
\phi_1(\mathbb{z}) &= \sum_{\mathbb{u}\in K_{\geq 0}} \mathrm{comb}(\mathbb{u})(H(u_1+\cdots+u_5) - H(u_1)) \\
&= \sum_{(a,b)\in\mathbb{Z}^2_{\geq 0}} \mathrm{comb}_{a,b}\left(H(12a+6b) - H(a)\right)
\end{aligned}$$

whereas $\phi_4$, for instance, is given by

$$\begin{aligned}
\phi_4(\mathbb{z}) &= \sum_{\mathbb{u}\in K_{\geq 0}} \mathrm{comb}(\mathbb{u})(H(u_1+\cdots+u_5) - H(u_4)) \\
&= \sum_{(a,b)\in\mathbb{Z}^2_{\geq 0}} \mathrm{comb}_{a,b}\left(H(12a+6b) - H(6a+3b)\right).
\end{aligned}$$

Then, to give an idea of what $\psi_i$ might look like, let us use this expression for $\phi_4(\mathbb{z})$ to find $\psi_4$. It is given by

$$\psi_4(\mathbb{z}) = \exp\left(\frac{\sum_{(a,b)\in\mathbb{Z}^2_{\geq 0}} \mathrm{comb}_{a,b}\left(H(12a+6b) - H(6a+3b)\right)}{\sum_{(a,b)\in\mathbb{Z}^2_{\geq 0}} \mathrm{comb}_{a,b}\, x_1^a x_2^b}\right).$$

Then since there are two basis vectors $\mathbb{k}_1 = (1,0,1,6,4)$ and $\mathbb{k}_2 = (0,1,0,3,2)$ for the kernel, we have two power series $\Psi_{\Bbbk_1}$ and $\Psi_{\Bbbk_2}$ given by

$$\begin{aligned}
\Psi_{\Bbbk_1}(\mathbf{z}) &= \psi_1\psi_3\psi_4^6\psi_5^4 \\
&= \exp\left(\left(\frac{\phi_1}{\phi_0}\right)\left(\frac{\phi_3}{\phi_0}\right)\left(\frac{\phi_4}{\phi_0}\right)^6\left(\frac{\phi_5}{\phi_0}\right)^4\right) \\
&= \exp\left(\frac{\phi_1+\phi_3+6\phi_4+4\phi_5}{\phi_0}\right) \\
&= \exp\left(\frac{\sum_{a,b\in\mathbb{Z}_{\geq 0}}\mathrm{comb}_{a,b}(12H(12a+6b)-2H(a)-6H(6a+3b)-4H(4a+2b)x_1^a x_2^b}{\sum_{a,b\in\mathbb{Z}_{\geq 0}}\mathrm{comb}_{a,b}\, x_1^a x_2^b}\right),
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{\Bbbk_2}(\mathbf{z}) &= \psi_2\psi_4^3\psi_5^2 \\
&= \exp\left(\left(\frac{\phi_3}{\phi_0}\right)\left(\frac{\phi_4}{\phi_0}\right)^3\left(\frac{\phi_5}{\phi_0}\right)^2\right) \\
&= \exp\left(\frac{\phi_2+3\phi_4+2\phi_5}{\phi_0}\right) \\
&= \exp\left(\frac{\sum_{a,b\in\mathbb{Z}_{\geq 0}}\mathrm{comb}_{a,b}(6H(12a+6b)-H(b)-3H(6a+3b)-2H(4a+2b)x_1^a x_2^b}{\sum_{a,b\in\mathbb{Z}_{\geq 0}}\mathrm{comb}_{a,b}\, x_1^a x_2^b}\right).
\end{aligned}$$

We will look at this example a bit more in 2.3, where we strike up a correspondence between the language used in Delaygue's theorem (Theorem 1, [5]) and that used in the GHHPS conjecture (Conj. A, [7]).

## The real mirror map

It should be noted that the mirror map as researched for the last three decades differs very slightly from the mirror map as we have presented it in this thesis. Due to a slight confusion consisting of multiple different assertions of the conjecture in [4], it was incorrectly copied from the source. As stated in [7], $\psi_i^{A(\Delta)}$ is given by

$$\psi_i^{A(\Delta)} = \exp\left(\frac{\phi_i(\mathbf{z})+\gamma_i(\mathbf{z})}{\phi_0(\mathbf{z})}\right)$$

for some function $\gamma_i(\mathbf{z})$ which vanishes if and only if, for every point $p \in A(\Delta)$ there is a hyperplane passing through the origin which separates $p$ from the other points in $A(\Delta)$. In this case, $\Delta$ is a reflexive polytope for which the only lattice points on $\Delta$ are its vertices. This mistake was discovered months into the dissertation, so that the code was already running and returning the truth of the integrality of $\Psi_{\Bbbk}^{A(\Delta)}$- and going a step further. We also had that each $\psi_i^{A(\Delta)}$ was positive and integral. This gives us an entirely new conjecture to work with. Furthermore, by including the error term, the mirror map calculations seemed to return power series with integrality but not positivity, so that in fact the mistaken mirror map formula outputs something stronger than the 'real' mirror map. This begs the question, which I do not address in this dissertation, but is nonetheless a fascinating idea: do the positive integer coefficients of this new mirror map count something?

# Acknowledgements

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# Chapter 1

# Polytopes

A broad understanding of mirror maps is available to us through applying mirror maps to reflexive polytopes, which are a special kind of polytope with one interior lattice point. We use polytopes to represent hypersurfaces in a toric variety, which in turn can correspond in many cases to Calabi-Yau spaces. In doing so, we can understand the relationship between the complex moduli space of a manifold and the Kähler moduli space of its mirror pair by looking at the properties of reflexive polytopes and their duals. As stated in [1], if $X$ and $Y$ are a mirror pair of Calabi-Yau spaces then the complexified Kähler moduli space of $X$ is isomorphic to the complex structure moduli space of $Y$.

Now that we have motivated their study, let us commence our polytopic discussion by first making some basic definitions.

**Definition 1.1.** (p26, [9])

- A **polytope** is the convex hull of finitely many points in $\mathbb{R}^d$, or equivalently, the bounded intersection of a finite set of affine half spaces, denoted

  $$P = \text{conv}\{x \in \mathbb{R}^d : \langle \mathbf{y}_i, \mathbf{x} \rangle \geq c_i \text{ for } 1 \leq i \leq k\},$$

  where each $\langle \mathbf{y}_i, \mathbf{x} \rangle \geq c_i$ defines an affine half space.

- A polytope has **dimension** $d$ if $d$ is the smallest natural number such that the polytope can be embedded into $\mathbb{R}^d$.

- A **lattice polytope** is a polytope whose vertices have integer coordinates.

**Definition 1.2.** (§2, p3, [8]) The **dual of a polytope** $P$ with the origin in the interior is the polytope $P^\vee = \text{conv}\{\mathbf{v} \in (\mathbb{R}^d)^\vee : \langle \mathbf{v}, \mathbf{w} \rangle \leq 1 \; \forall \, \mathbf{w} \in P\}$.

In our definition of the polytope, it is of note that each $\mathbf{y}_i$ is a primitive element of the dual lattice $(\mathbb{Z}^d)^\vee$,

**Definition 1.3.** (Definition 1.1, [8]) A lattice polytope $P$ with one interior lattice point $x_0$ is a **reflexive polytope** if for $P = \text{conv}\{x \in \mathbb{R}^d : \langle \mathbf{y}_i, \mathbf{x} \rangle \geq c_i \text{ for } 1 \leq i \leq k\}$ we have $\langle \mathbf{y}_i, \mathbf{x} \rangle - c_i = 1$ for all $1 \leq i \leq k$.

When the unique interior point of the lattice polytope is the origin, this condition is equivalent to the condition that the dual is a lattice polytope with only one interior point.

In two dimensions, there are 16 reflexive polytopes. They are exactly the lattice polytopes with one interior lattice point, as shown in Figure 1.1.

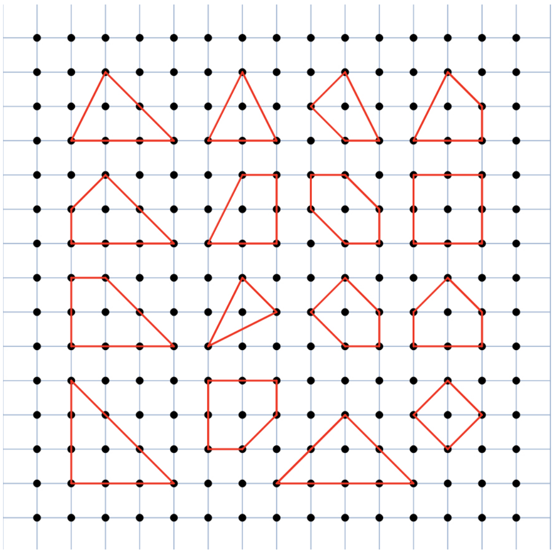

Figure 1.1: The figure, adapted from p2 of [8], shows all of the 2-dimensional polytopes up to rotation or reflection.

**Example 1.4** (Hexagon)**.** Let's look at the hexagon polytope, as shown in Figure 1.2.

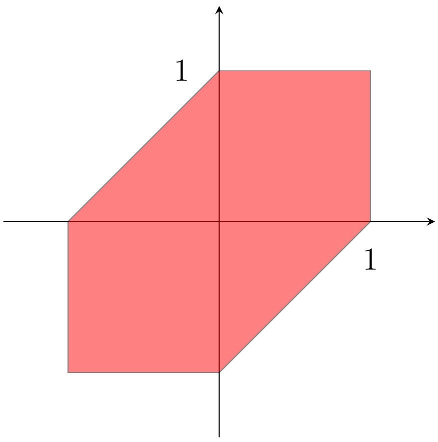


Figure 1.2: The figure shows the polytope whose vertices are given by the vectors $a = (0, 1)$, $b = (1, 1)$, $c = (1, 0)$, $d = (0, -1)$, $e = (-1, -1)$, $f = (-1, 0)$.

When we work with reflexive polytopes like this one, the mirror map will be a computation on the vectors in the kernel of the matrix whose rows are coordinates of the vertices of the polytope. For instance in this case the lattice generated by combinations of these lattice points is $\mathbb{Z}^6$, and each vector is given by a linear combination of the points $a, \dots, f$. For instance, a linear combination of one lot of $a$ and zero lots of the other generators can be symbolised as $e_a = (1,0,0,0,0,0)$. Embedding this vector space in $\mathbb{Z}^2$, we see that, adding elementwise, we have

$$a + c + e = b + d + f = 0,$$

so that both $(1,0,1,0,1,0)$ and $(0,1,0,1,0,1)$ are in the kernel of the map $\mathbb{Z}^6_{\geq 0} \mapsto \mathbb{Z}^2$. Furthermore,

$$a + d = b + e = d + f = 0,$$

so that $(1,0,0,1,0,0), (0,1,0,0,1,0), (0,0,1,0,0,1)$ are also in the kernel of this map. The kernel is therefore generated by the five vectors

$$v = (0,1,0,0,1,0),\ w = (0,0,1,0,0,1),\ x = (1,0,1,0,1,0),$$
$$y = (0,1,0,1,0,1), z = (1,0,0,1,0,0).$$

Notice that we can examine the dimensions of our spaces, and observe that by rank-nullity, we expect our kernel to have dimension 4. Indeed, the space generated by $v, \dots, z$ is 4-dimensional since we have the relation $v + w + x = y + z$. This relation has a complicated effect: it means that we cannot index over each of $v, \dots, z$ in our sum because the variables are not linearly independent. That is to say, if we have performed a calculation for the kernel element $v + w + x$, and also for the element $y + z$, we will be overcounting. Formally, the intersection of the kernel with $\mathbb{Z}^6_{\geq 0}$ is not isomorphic, as a monoid, to $\mathbb{Z}^k_{\geq 0}$ for any $k$. The way we solve this is finding the sum over $\mathbb{u} = (u_1, \dots, u_6)$ in the kernel by taking

$$\begin{aligned}\phi_0(\mathbb{z}) &= \sum_{\mathbb{u} \in K_{\geq 0}} \mathrm{comb}(\mathbb{u}) \cdot \mathbb{z}^{\mathbb{u}} \\ &= \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \cdots \sum_{u_6=0}^{\infty} \delta(u_1, \dots, u_6) \cdot \mathrm{comb}(u_1, \dots, u_6) \cdot z_1^{u_1} z_2^{u_2} \dots z_6^{u_6}\end{aligned}$$

where

$$\delta(\mathbb{u}) = \begin{cases} 1 & \text{if } \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix} \mathbb{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\delta$ is identifying what is in the kernel without overcounting. Using this, we can also calculate $\phi_i$ for each $1 \leq i \leq 6$ and eventually find $\psi_i$ for each $i$ and $\Psi_{\mathbb{u}}$ for each $\mathbb{u}$ in the basis of the kernel. We show that this mirror map is indeed integral via computation by Sagemath in Section 5.1.

## The moment polytope of $\mathbb{CP}^n$

Moment polytopes are extremely useful objects defined on basis vectors in a given space. Their study is of interest throughout the fields of symplectic geometry, algebraic geometry and theoretical physics due to their connections with symplectic manifolds with Hamiltonian action. We start with an example before generalising the definition.

**Example 1.5.** Two basic examples of reflexive polytopes are

- the reflexive polygon, denoted $\nabla_2$, given by vertices $(-1,-1), (2,-1), (-1,2)$, and
- its dual, denoted $\Delta_2$, given by vertices $(-1,0), (0,-1), (1,1)$.

The former, $\nabla_2$, is an example of a 'moment polytope of $\mathbb{CP}^2$'. This nomenclature comes from the physics' 'momentum'[1].

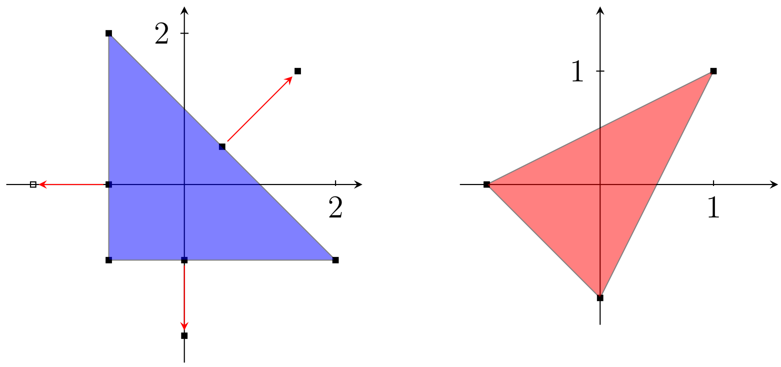


Figure 1.3: The moment polytope of $\mathbb{CP}^2$, denoted $\nabla_2$, is shown on the left, where the normal vectors are shown in red, and its dual, denoted $\Delta_2$, is shown on the right. We may observe that the vertices of $\Delta_2$ correspond to the orthogonal vectors to the edges of $\nabla_2$.

**Remark 1.6.** As helpful as Figure 1.3 is, it is worth noting that this example is slightly misleading. Notice in particular that the lattices on the two diagrams above are not on the same scale. We should not imagine the dual of a polytope as living in the same space as said polytope, as the dual space does not necessarily have the same lattice points or angle of skew. Checking whether a polytope has lattice vertices - or indeed, what lattice actually means in the dual space - sometimes requires us to think from a skewed point of view.

We may define the dual formally as per our definition (Definition 1.2) of the dual of a polytope

$$\Delta^2 := \operatorname{conv}\{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{u} \geq -1 \forall \mathbf{u} \in \nabla^2\}.$$

We can define this also in terms of the free abelian group $M$ of rank $n$, canonically isomorphic to $\mathbb{Z}^n$. Then $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ is the lattice space of integer coordinates in

[1]The reference to momentum is an homage to Noether's Theorem for the conservation of momentum in a physical system invariant under translation; as Noether suggests the correspondence between translational symmetry and the conservation of linear momentum, in the same way, we are to observe the moment polytope as conservation of certain properties under the action of a Lie group on a symplectic manifold.

$\mathbb{R}^n$, which is to say that $\nabla^2$ lives in $M_{\mathbb{R}}$. Then the dual must live in $\mathrm{Hom}(M_{\mathbb{R}}, \mathbb{R})$. Indeed, the expression can also be written

$$\Delta^2 = \mathrm{conv}\{v \in \mathrm{Hom}(M_{\mathbb{R}}, \mathbb{R}) : v(u) \geq -1 \forall u \in \nabla^2\}.$$

That these two polytopes are dual and we care about them is a fact we will not delve into. For now, we will care only about the dual of the moment polytope. The driving message should be the following theorem.

**Theorem 1.7.** For a vector space $V$ with basis vectors $\{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1)\}$, the dual $\Delta^n$ of the moment polytope given by

$$\mathrm{conv}\{-e_1, -e_2, \dots, -e_n, \sum_{i=1}^{n} e_i\}$$

is a reflexive polytope.

*Proof.* Clearly, the dual $\Delta^n$ is a lattice polytope as all of its vertices have integer coefficients:

$$\Delta^n = \mathrm{conv}\{(-1, 0, \dots, 0), (0, -1, \dots, 0), \dots, (0, 0, \dots, -1), (1, 1, \dots, 1)\}$$

Furthermore, we observe that its only interior point is the origin.

We now examine the moment polytope $\nabla^n$, of which $\Delta^n$ is dual. By the definition of the dual polytope, $\nabla^n$ is given by

$$\nabla^n = \mathrm{conv}\{\mathbf{x} \in (\mathbb{R}^n)^\vee : \langle \mathbf{x}, \mathbf{v}_1 \rangle \leq 1, \dots, \langle \mathbf{x}, \mathbf{v}_{n+1} \rangle \leq 1\},$$

so that any $x \in \nabla^n$ satisfies

$$\begin{gathered} \langle (-1, 0, \dots, 0), (x_1, x_2, \dots, x_n) \rangle \leq 1, \\ \vdots \\ \langle (0, 0, \dots, -1), (x_1, x_2, \dots, x_n) \rangle \leq 1, \end{gathered}$$

and

$$\langle (1, 1, \dots, 1), (x_1, x_2, \dots, x_n) \rangle \leq 1.$$

The boundaries of this polytope occur when these inequalities become equalities. So for example, if

$$\langle (-1, 0, \dots, 0), (x_1, x_2, \dots, x_n) \rangle = 1$$

we have $-x_1 = 1$ at one boundary hyperplane of the polytope. Similarly, $-x_i = 1$ forms a boundary hyperplane, for all $1 \leq i \leq n$. Also if

$$\langle (1, 1, \dots, 1), (x_1, x_2, \dots, x_n) \rangle = 1$$

we must have $\sum_{i=1}^{n} x_i = 1$ forming a boundary hyperplane. We can find the defining $n+1$ vertices of a polytope by finding the intersection of $n$ hyperplanes bounding the polytope. Taking the intersections of these hyperplanes, we find that these vertices are

$$(-1,\dots,-1),(n,-1,\dots,-1),(-1,n,\dots,-1),\dots,(-1,-1,\dots,n),$$

since in $(-1,-1,\dots,-1)$ we have $-x_i = 1$ for all $i$, so that the vertex is an intersection of $n$ boundary hyperplanes, and in a given $(-1,\dots,-1,n,-1,\dots,-1)$ we have $-x_i = 1$ for all $i$ except one, and $\sum_{i=1}^{n} x_i = 1$, so that this vertex also is an intersection of $n$ boundary hyperplanes. This is to say that $\nabla^n$ is the convex hull

$$\operatorname{conv}\{(-1,\dots,-1),(n,-1,\dots,-1),(-1,n,\dots,-1),\dots,(-1,-1,\dots,n)\}.$$

Since all of these are integer coordinates, $\nabla^\vee$ is a lattice polytope. We can also see that the origin is the only interior lattice point of the polytope with these vertices. This is to say that $\Delta^n$ is indeed a reflexive polytope. □

Indeed, the dual of the moment polytope of $\mathbb{CP}^2$ as we have stated it is equivalent to one of this form; taking $-e_1 = (-1,0)$ and $-e_2 = (0,-1)$.

A yet more trivial example exists than the triangle. Via the simplicity of this example, we unearth the difficult question of where product polytopes live.

**Definition 1.8.** For $P$ a convex polytope of $\mathbb{R}^n$ and $Q$ a convex polytope of $\mathbb{R}^m$, the product polytope $P * Q$ lives in $\mathbb{R}^{n+m}$ and is given by

$$P * Q := \operatorname{conv}\left(P \times \{0\}_{\in\mathbb{R}^m} \cup \{0\}_{\in\mathbb{R}^n} \times Q\right)$$

where $\{0\}_{\in\mathbb{R}^i}$ is the zero space in $\mathbb{R}^i$.

Furthermore, we see in [10] that the dual $(P * Q)^\vee$ of the product polytope $P * Q$ is the cartesian product

$$(P * Q)^\vee = P^\vee \times Q^\vee.$$

We can use this fact to prove the following lemma.

**Lemma 1.9.** If $P$ and $Q$ are reflexive then $P * Q$ is reflexive.

*Proof.* To prove that $P * Q$ is reflexive, we must show that $(P * Q)^\vee$ is a lattice polytope with unique lattice point at the origin. Indeed, if $P$ and $Q$ are reflexive with the origin as the unique interior point then $0 \in \operatorname{Int}(P^\vee)$ and $0 \in \operatorname{Int}(Q^\vee)$, so by the definition of the product polytope, $0 \in \operatorname{Int}((P * Q)^\vee)$. Furthermore, by reflexivity of $P$ and $Q$, the vertices of $P^\vee$ and $Q^\vee$ are lattice points, so the vertices of $(P * Q)^\vee$ will be lattice points. Now we prove that the origin is the only interior lattice point of $(P * Q)^\vee$.

We know by the reflexivity of $P$ and $Q$ that

$$\operatorname{Int}(P^\vee) \cap \mathbb{Z}^m = 0 \quad \text{and} \quad \operatorname{Int}(Q^\vee) \cap \mathbb{Z}^n = 0.$$

Now we check the interior of $(P * Q)^\vee$. The intersection of the product polytope

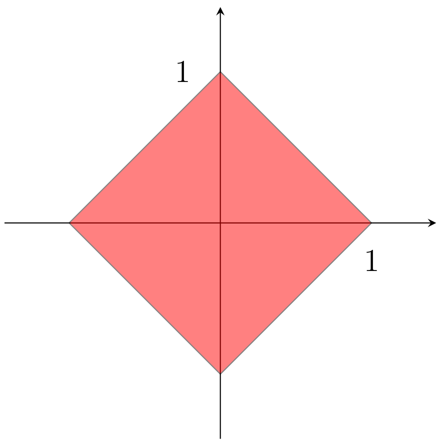


Figure 1.4: The figure shows the product polytope of the interval in $\mathbb{R}$ with itself.

with the lattice is

$$\begin{aligned}\mathrm{Int}((P * Q)^\vee) \cap \mathbb{Z}^{m+n} &= \mathrm{Int}(P^\vee \times Q^\vee) \cap \mathbb{Z}^{m+n} \\ &= (\mathrm{Int}(P^\vee) \times \mathrm{Int}(Q^\vee)) \cap (\mathbb{Z}^m \times \mathbb{Z}^n) \\ &= (\mathrm{Int}(P^\vee) \cap \mathbb{Z}^m) \times (\mathrm{Int}(Q^\vee) \cap \mathbb{Z}^n) \\ &= 0 \times 0 \\ &= 0,\end{aligned}$$

so that the origin is the only lattice point in the interior of $P * Q$. □

**Example 1.10.** For $P = [-1,1] \subset \mathbb{R}$ and $Q = [-1,1] \subset \mathbb{R}$, the product polytope $P * Q$ lives in $\mathbb{R}^2$ and is given by

$$\mathrm{conv}\left(P \times \{0\}_{\in\mathbb{R}} \cup \{0\}_{\in\mathbb{R}} \times Q\right)$$

so that the resulting polytope of $[-1,1] \subset \mathbb{R}$, $[-1,1] \subset \mathbb{R}$ would be that as shown in Figure 1.4.

In general, for polytopes $(P_i \subset \mathbb{R}^{n_i})_{1 \geq i \geq k}$, we have the product polytope $P_1 * \cdots * P_k \subseteq (\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k})^k$ given by

$$\mathrm{conv}\left(P_1 \times \{0\} \times \ldots \times \{0\} \cup \{0\} \times P_2 \times \{0\} \times \ldots \times \{0\} \cup \cdots \cup \{0\} \times \ldots \times \{0\} \times P_k\right).$$

Extending this, for some vector $d \in \mathbb{Z}^n_{>0}$, let $\Delta_{d_1,\ldots,d_k} \subset \mathbb{Z}^{d_1+\cdots+d_k}$ be the polytope with vertices

$$\left\{\begin{array}{lllll} e_1, & e_2, & \ldots, & e_{d_1}, & -e_1 - \cdots - e_{d_1}, \\ e_{d_1+1}, & e_{d_1+2}, & \ldots, & e_{d_1+d_2}, & -e_{d_1+1} - \cdots - e_{d_1+d_2}, \\ & \vdots, & & & \\ e_{d_1+\cdots+d_{k-1}+1}, & e_{d_1+\cdots+d_{k-1}+2}, & \ldots, & e_{d_1+\cdots+d_k}, & -e_{d_1+\cdots+d_{k-1}+1} - \cdots - e_{d_1+\cdots+d_k}. \end{array}\right\}$$

This is not the same as the Cartesian product $\Delta_1 \times \ldots \times \Delta_k$. Rather, it is the convex hull of

$$(\Delta_1 \times 0 \times \ldots \times 0, 0 \times \Delta_2 \times 0 \times \ldots \times 0, \ldots 0 \times 0 \times \ldots \times 0 \times \Delta_k)$$

in $\mathbb{Z}^{d_1+\cdots+d_k}$. A vector in the kernel $K_{\geq 0}$ will therefore have the form

$$(j_1, \ldots, j_1, j_2, \ldots, j_2, \ldots, j_k, \ldots, j_k),$$

where $j_i$ is a nonnegative integer repeated $d_i + 1$ times. This allows us to write $\phi_0$ and $\phi_i$ by summing over each of $j_1, j_2, \ldots, j_k$, as shown in the following example.

**Example 1.11.** Another example of the above is the 4-dimensional polytope $\Delta_{2,2}$ with vertices

$$\begin{aligned}&(1,0,0,0), (0,1,0,0), (-1,-1,0,0),\\ &(0,0,1,0), (0,0,0,1), (0,0,-1,-1).\end{aligned}$$

We can see that a vector $(u_1, \ldots, u_6)$ is in the kernel of the map $\mathbb{Z}^6 \to \mathbb{Z}^4$ exactly when

$$\begin{aligned}&(u_1,0,0,0) + (0,u_2,0,0) + (-u_3,-u_3,0,0)\\ &+ (0,0,u_4,0) + (0,0,0,u_5) + (0,0,-u_6,-u_6)\\ &= (0,0,0,0),\end{aligned}$$

so we are looking for vectors with $(u_1 - u_3, u_2 - u_3, u_4 - u_6, u_5 - u_6) = (0,0,0,0)$. This is to say we wish to have $u_1 = u_2 = u_3$ and $u_4 = u_5 = u_6$, so that any vector in the kernel is of the form $(i,i,i,j,j,j)$ for $i, j \in \mathbb{Z}_{\geq 0}$. As shown in (p30, [2]) and (p6, [13]), we find the mirror map calculations in Delaygue's language, giving the $F$-function

$$F(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(3m+3n)!}{m!^3 n|^3} z_1^m z_2^n$$

corresponding to $\phi_0$ in our notation and the $G$-functions

$$G_1(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(3m+3n)!}{m!^3 n!^3} (3H_{3m+3n} - 3H_m) z_1^m z_2^n$$

and

$$G_2(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(3m+3n)!}{m!^3 n!^3} (3H_{3m+3n} - 3H_n) z_1^m z_2^n,$$

corresponding to $\phi_1$ and $\phi_2$ in our notation, by following a similar procedure to that shown in Example 2.3. The reader is encouraged to follow this calculation through to check the functions given.

# Chapter 2

# Comparison of results and conjectures

The mirror map whose integrality we are so concerned about is a Taylor series of a computation performed on the lattice points of the kernel of the map from the vertices of a reflexive polytope to the basis vectors. This is quite a mouthful, so let us deconstruct it a little bit, starting from the other end.

The kernel of the map from a polytope's vertices to the basis vectors is the kernel of the matrix of the vertex coordinates. If a lattice point is in this kernel, then we apply our computation of $\phi_i/\phi_0$ to it, and add that to the sum of the computations performed on the other lattice points in the kernel. After dividing by the sum of a few more computations and taking the exponential of the result, we take the product of these Taylor series with some vector power as shown in Equation 1. The fact that such a product, with exponentiation and power series in the denominator, returns a Taylor series with all integer coefficients is... surprising, to say the least.

Indeed, this project is dedicated to exploring the proofs of why this is true, and analysing some examples with the use of computational power (with the Sagemath software). One of the most general proofs is given in [5].

## 2.1 Integrality theorems

The notation surrounding the integrality of mirror maps is involved, cumbersome, and somewhat unintuitive, so you can refer to the notation page for any reminders. But without further ado, let us introduce the necessary colloquialisms of the field.

Take $\mathbb{m} = (m^{(1)}, m^{(2)}, \dots, m^{(d)}), \mathbb{n} = (n^{(1)}, n^{(2)}, \dots, n^{(d)})$ to be vectors in $\mathbb{R}^d$ for $d \in \mathbb{N}$, where for the $k$th entry of vector $\mathbb{m}$ we write $\mathbb{m}^{(k)}$. The elementwise sum and scalar multiple are denoted $\mathbb{m} + \mathbb{n} = (m^{(1)} + n^{(1)}, m^{(2)} + n^{(2)}, \dots, m^{(d)} + n^{(d)})$ and $\lambda\mathbb{m} = (\lambda m^{(1)}, \lambda m^{(2)}, \dots, \lambda m^{(d)})$ respectively, for $\lambda \in \mathbb{R}$. The inequality $\mathbb{m} \geq \mathbb{n}$ indicates that $m^{(i)} \geq n^{(i)}$ for $1 \leq i \leq d$. The scalar product is denoted $\mathbb{m} \cdot \mathbb{n} = m^{(1)}n^{(1)} + m^{(2)}n^{(2)} + \dots + m^{(d)}n^{(d)}$ and the norm is given by $|\mathbb{m}| = m^{(1)} + m^{(2)} + \dots + m^{(d)}$. For a sequence $\mathbf{e} = (\mathbb{e}^{(1)}, \dots, \mathbb{e}^{(q_1)})$ of vectors in $\mathbb{N}^d$ we give the norm $|\mathbf{e}| = \sum_{i=1}^{q_1} \mathbb{e}^{(i)}$, such that $1 \leq k \leq d$ gives the index $|\mathbf{e}|^{(k)} = \sum_{i=1}^{q_1} \mathbb{e}_i^{(k)}$. We use this to state the theorem of integrality on $\Psi_{\mathbb{k}}^{A(\Delta)}$, as given and proven in [5].

### 2.1.1 Delaygue's theorem

If $\mathbf{e}$ is a sequence of vectors of length $q_1$ and $\mathbf{f}$ is one of length $q_2$, then we can define the combinations function $Q_{\mathbf{e},\mathbf{f}}$ on $\mathbb{N}^d$ as

$$Q_{\mathbf{e},\mathbf{f}}(\mathbb{n}) = \frac{(\mathbb{e}_1 \cdot \mathbb{n})! \dots (\mathbb{e}_{q_1} \cdot \mathbb{n})!}{(\mathbb{f}_1 \cdot \mathbb{n})! \dots (\mathbb{f}_{q_1} \cdot \mathbb{n})!}. \tag{2.1}$$

Delaygue also defined the combinations function with respect to $\mathbb{z}$, a vector in $d$ variables, given by

$$F_{\mathbf{e},\mathbf{f}}(\mathbb{z}) = \sum_{\mathbb{n} \geq 0} \frac{(\mathbb{e}_1 \cdot \mathbb{n})! \dots (\mathbb{e}_{q_1} \cdot \mathbb{n})!}{(\mathbb{f}_1 \cdot \mathbb{n})! \dots (\mathbb{f}_{q_1} \cdot \mathbb{n})!} \mathbb{z}^{\mathbb{n}}, \tag{2.2}$$

and the harmonic combinations function on $\mathbb{z}$ given by

$$G_{\mathbf{e},\mathbf{f},k}(\mathbb{z}) = \sum_{\mathbb{n} \geq 0} \frac{(\mathbb{e}_1 \cdot \mathbb{n})! \dots (\mathbb{e}_{q_1} \cdot \mathbb{n})!}{(\mathbb{f}_1 \cdot \mathbb{n})! \dots (\mathbb{f}_{q_1} \cdot \mathbb{n})!} \left( \sum_{i=1}^{q_1} \mathbb{e}_i^{(k)} H_{\mathbb{e}_i \cdot \mathbb{n}} - \sum_{i=1}^{q_2} \mathbb{f}_i^{(k)} H_{\mathbb{f}_i \cdot \mathbb{n}} \right) \mathbb{z}^{\mathbb{n}}. \tag{2.3}$$

Then for each $k \in \{1, \dots, d\}$ we have a function

$$q_{\mathbf{e},\mathbf{f},k}(\mathbb{z}) = z_k \exp(G_{\mathbf{e},\mathbf{f},k}(\mathbb{z}) / F_{\mathbf{e},\mathbf{f}}(\mathbb{z}))$$

representing the canonical coordinates of mirror maps.

If $\lfloor \cdot \rfloor$ denotes the floor function and $\{\cdot\}$ denotes the fractional part found by subtracting $\lfloor \cdot \rfloor$ from the whole, then for all $\mathbb{z} \in \mathbb{R}^d$ we can define as in [5] the Landau's function to be that given by

$$\Delta_{\mathbf{e},\mathbf{f}}(\mathbb{z}) = \sum_{i=1}^{q_1} \lfloor \mathbb{e}_i \cdot \mathbb{z} \rfloor - \sum_{i=1}^{q_2} \lfloor \mathbb{f}_i \cdot \mathbb{z} \rfloor.$$

**Definition 2.1.** We say $\mathbb{x} \in [0,1)^d$ is a member of the semi-algebraic set $D_{\mathbf{e},\mathbf{f}}$ if and only if there is some vector $\mathbb{b} \in \{\mathbb{e}_1, \dots, \mathbb{e}_{q_1}, \mathbb{f}_1, \dots, \mathbb{f}_{q_2}\}$ such that $\mathbb{b} \cdot \mathbb{x} \geq 1$.

Delaygue in [5] states the following theorem.

**Theorem 2.2.** (Delaygue's theorem) For $e, f$ two disjoint sequences of nonzero vectors in $\mathbb{N}^d$ where $Q_{\mathbf{e},\mathbf{f}}$ is a family of integers with $|\mathbf{e}| = |\mathbf{f}|$, then if $\mathbb{x} \in D_{\mathbf{e},\mathbf{f}}$ we have $\Delta_{\mathbf{e},\mathbf{f}}(\mathbb{x}) \geq 1$ then for all $k \in \{1, 2, \dots, d\}$, $q_{\mathbf{e},\mathbf{f},k}(\mathbb{z}) \in z_k \mathbb{Z}[[\mathbb{z}]]$.

That is, $q_{\mathbf{e},\mathbf{f},k}$ is a power series of coefficients in $\mathbb{z}$, multiplied by the variable $z_k$. So, like magic, we have a set of constraints for what we need for our mirror map power series to have integer coefficients. Let us see how it relates to the conjecture given in the introduction.

### 2.1.2 The GHHPS conjecture versus Delaygue's

It is not obvious that Conjecture A and Theorem 2.2 relate to each other; in particular, the notation itself is distinctly disjoint. To give an idea of why the two are comparable, let us walk through a familiar example.

**Example 2.3.** We return to Example 0.1 from the introduction. Taking $\Delta$ the reflexive polytope with $A(\Delta)$ of vertices

$$v_1 = (1,0,0), \quad v_2 = (0,1,0), \quad v_3 = (0,0,1), \quad v_4 = (-6,-4,-1), \quad v_5 = (-3,-2,0),$$

recall that the kernel $K_{\geq 0}$ is given by

$$K_{\geq 0} = \{a(\mathbb{k}_1) + b(\mathbb{k}_2) : (a,b) \in \mathbb{Z}^2_{\geq 0}\} \tag{2.4}$$

for $\mathbb{k}_1 = (1,0,1,6,4)$ and $\mathbb{k}_2 = (0,1,0,3,2)$. We can now, to gear the writing towards Delaygue's style, write $\phi_0$ as

$$\begin{aligned}\phi_0(\mathbb{z}) &= \sum_{\mathbb{u} \in K_{\geq 0}} \operatorname{comb}(\mathbb{u}) \mathbb{z}^{\mathbb{u}} \\ &= \sum_{(a,b) \in \mathbb{Z}^2_{\geq 0}} \frac{a+b+a+6a+3b+4a+2b}{a!b!a!(6a+3b)!(4a+2b)!} x_1^a x_2^b, \\ &= \sum_{(a,b) \in \mathbb{Z}^2_{\geq 0}} \frac{((12,6)\cdot(a,b))!}{((1,0)\cdot(a,b))! \cdot \dots \cdot ((4,2)\cdot(a,b))!} x_1^a x_2^b,\end{aligned}$$

where, as before, $x_1 = z_1 z_2 z_4^6 z_5^4$ and $x_2 = z_2 z_4^3 z_5^2$. Then by letting $e_1 = (12,6)$ and $f_1 = (1,0)$, $f_2 = (0,1)$, $f_3 = (1,0)$, $f_4 = (6,3)$, $f_5 = (4,2)$, we have successfully translated $\phi_0$ in the language of Conjecture A to $F_{\mathbf{e},\mathbf{f}}$ in Delaygue's language! This gives us an idea in general how to translate back and forth between Delaygue's language and ours. To continue this, let us figure out what $q_{\mathbf{e},\mathbf{f},k}(\mathbb{z})$ from Delaygue looks like in the language of Ganatra-Hanlon-Hicks-Pomerleano-Sheridan in [7]. We find that using our assignments of $\mathbf{e} = (\mathbb{e}_1)$ and $\mathbf{f} = (\mathbb{f}_1, \dots, \mathbb{f}_5)$ given above, and notating

$$\operatorname{comb}_{a,b} = \frac{12a+6b}{a!b!a!(6a+3b)!(4a+2b)!},$$

we have that

$$\begin{aligned}q_{\mathbf{e},\mathbf{f},1}(\mathbb{z}) &= \exp\left(\frac{G_{\mathbf{e},\mathbf{f},1}(\mathbb{z})}{F_{\mathbf{e},\mathbf{f}}(\mathbb{z})}\right) \\ &= \exp\left(\frac{\sum_{\mathbb{n}\geq 0} \frac{(\mathbb{e}_1\cdot\mathbb{n})!\dots(\mathbb{e}_{q_1}\cdot\mathbb{n})!}{(\mathbb{f}_1\cdot\mathbb{n})!\dots(\mathbb{f}_{q_1}\cdot\mathbb{n})!}\left(\sum_{i=1}^{q_1} \mathbb{e}_i^{(k)} H_{\mathbb{e}_i\cdot\mathbb{n}} - \sum_{i=1}^{q_2} \mathbb{f}_i^{(1)} H_{\mathbb{f}_i\cdot\mathbb{n}}\right)\mathbb{z}^{\mathbb{n}}}{\sum_{\mathbb{n}\geq 0} \frac{(\mathbb{e}_1\cdot\mathbb{n})!\dots(\mathbb{e}_{q_1}\cdot\mathbb{n})!}{(\mathbb{f}_1\cdot\mathbb{n})!\dots(\mathbb{f}_{q_1}\cdot\mathbb{n})!}\mathbb{z}^{\mathbb{n}}}\right) \\ &= \exp\left(\frac{\sum_{a,b=0}^{\infty} \operatorname{comb}_{a,b}\left(12H_{12a+6b} - H_a - H_a - 6H_{6a+3b} - 4H_{4a+2b}\right) x_1^a x_2^b}{\sum_{a,b=0}^{\infty} \operatorname{comb}_{a,b} x_1^a x_2^b}\right)\end{aligned}$$

and likewise

$$q_{\mathbf{e},\mathbf{f},2}(\mathbb{z}) = \exp\left(\frac{\sum_{a,b=0}^{\infty} \operatorname{comb}_{a,b}\left(6H_{12a+6b} - H_b - 3H_{6a+3b} - 2H_{4a+2b}\right) x_1^a x_2^b}{\sum_{a,b=0}^{\infty} \operatorname{comb}_{a,b} x_1^a x_2^b}\right).$$

We notice that

$$12H_{12a+6b} - H_a - H_a - 6H_{6a+3b} - 4H_{4a+2b} = \sum_{i=1}^{5} H_{\mathbb{e}_1\cdot(a,b)} - \mathbb{f}_i^{(1)} H_{\mathbb{f}_i\cdot(a,b)}$$

and

$$6H_{12a+6b} - H_b - 3H_{6a+3b} - 2H_{4a+2b} = \sum_{i=1}^{5} H_{\mathbb{e}_1\cdot(a,b)} - \mathbb{f}_i^{(2)} H_{\mathbb{f}_i\cdot(a,b)}.$$

Then since

$$H_{\mathbb{e}_1\cdot(a,b)} - \mathbb{f}_i^{(1)} H_{\mathbb{f}_i\cdot(a,b)} \xrightarrow{\text{becomes}} H(u_1 + \dots + u_e) - H(u_i)$$

and

$$\mathrm{comb}_{a,b} = \frac{12a+6b}{a!b!a!(6a+3b)!(4a+2b)!} \xrightarrow{\text{becomes}} \mathrm{comb}(\mathbb{u}),$$

via change of notation, we can write

$$\sum_{a,b=0}^{\infty} \frac{12a+6b}{a!b!a!(6a+3b)!(4a+2b)!} \left(H_{12a+6b} - H_{f_i\cdot(a,b)}\right) = \phi_i,$$

for any $1 \le i \le 5$. This is equivalent to saying that

$$q_{\mathbf{e},\mathbf{f},1}(\mathbb{x}) = \exp\left(\frac{\phi_1 + \phi_3 + 6\phi_4 + 4\phi_5}{\phi_0}\right) \quad \text{and} \quad q_{\mathbf{e},\mathbf{f},2}(\mathbb{x}) = \exp\left(\frac{\phi_2 + 3\phi_4 + 2\phi_5}{\phi_0}\right),$$

so that

$$\begin{aligned} q_{\mathbf{e},\mathbf{f},1}(\mathbb{x}) &= \exp\left(\frac{\phi_1}{\phi_0}\right)\exp\left(\frac{\phi_3}{\phi_0}\right)\exp\left(\frac{\phi_4}{\phi_0}\right)^6\exp\left(\frac{\phi_5}{\phi_0}\right)^4 \\ &= \psi_1\psi_3\psi_4^6\psi_5^4 \\ &= \Psi_{\mathbb{k}_1} \end{aligned}$$

and

$$\begin{aligned} q_{\mathbf{e},\mathbf{f},2}(\mathbb{x}) &= \exp\left(\left(\frac{\phi_2}{\phi_0}\right)\left(\frac{\phi_4}{\phi_0}\right)^3\left(\frac{\phi_5}{\phi_0}\right)^2\right) \\ &= \psi_2\psi_4^3\psi_5^2 \\ &= \Psi_{\mathbb{k}_2}, \end{aligned}$$

for $\mathbb{k}_1$ and $\mathbb{k}_2$ the basis vectors of $K_{\ge 0}$, shown in Equation 2.4. As a result, we are led to believe that Delaygue's theorem is equivalent to Conjecture A; that $q_{\mathbf{e},\mathbf{f},k} \in \mathbb{Z}[[\mathbb{x}]]$ if and only if $\Psi_{\mathbb{k}_k} \in \mathbb{Z}[[\mathbb{x}]]$, where $\mathbb{k}_k$ are basis vectors for $K_{\ge 0}$.

The question of mirror maps of reflexive polytopes is a special case of Delaygue's theorem, in which $q_1 = 1$, so that the sequence $\mathbb{e}$ consists only of one vector, and $\mathbb{e} = \sum_{i=1}^{q_2} \mathbb{f}_i$. We notice that Example 2.3 was such an example, and showed us that we can mould Conjecture A into a form to which Delaygue's theorem can be applied. We now generalise this, to see exactly what Delaygue's theorem is equivalent to in the language of Conjecture A.

**Lemma 2.4.** For $\Delta$ a reflexive polytope with the rank of the kernel $\mathrm{rank}(K_{\geq 0}) = 1$, Delaygue's theorem (Theorem 2.2) is equivalent to Conjecture A.

*Proof.* Let $\Delta$ be such a reflexive polytope. Furthermore, let us call $(\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{Z}^{d+1}_{\geq 0}$ a **weight system** if $\sum_{j=1}^{d+1} \frac{1}{\alpha_j} = 1$ (we will motivate this definition in the proof of Theorem 3.1). Since the kernel has rank 1, $\Delta$ must be a simplex, so that

$$e_1 = k = \sum_{i=1}^{d} k_i, \quad \text{and} \quad f_i = k_i$$

for some values $k_i$ given by a weight system $(\alpha_1, \dots, \alpha_d)$ with $\sum_{i=1}^{d} \frac{1}{\alpha_i} = 1$ and $k_i = \frac{\mathrm{lcm}(\alpha_j)}{\alpha_i}$. Then we express $F_{\mathbf{e},\mathbf{f}}$ in Equation 2.2 by

$$F_{\mathbf{e},\mathbf{f}}(z) = \sum_{n\geq 0} \frac{(kn)!}{\prod_{i=1}^{e}(k_i n)!} z^n$$

and $G_{\mathbf{e},\mathbf{f},1}$ in Equation 2.3 by

$$G_{\mathbf{e},\mathbf{f},1}(z) = \sum_{n\geq 0} \frac{(kn)!}{\prod_{i=1}^{e}(k_i n)!} z^n,$$

where $G_{\mathbf{e},\mathbf{f},k}$ only exists for $k = 1$ since $K_{\geq 0}$ has only one basis vector, say $\mathbb{u} = (k_1, \dots, k_e)$. Notice also that $z$ is a single entry vector for the same reason. We write

$$\phi_i(z) = \sum_{n\geq 0} \frac{(kn)!}{\prod_{j=1}^{e}(k_j n)!} \left(H_{nk} - H_{k_i n}\right) z^n.$$

Then

$$\begin{aligned}
\Psi_{\mathbb{u}} &= \psi_1^{k_1} \psi_2^{k_2} \dots \psi_e^{k_e} \\
&= \exp\left(\frac{\phi_1}{\phi_0}\right)^{k_1} \exp\left(\frac{\phi_2}{\phi_0}\right)^{k_2} \dots \exp\left(\frac{\phi_e}{\phi_0}\right)^{k_e} \\
&= \exp\left(\frac{k_1\phi_1 + k_2\phi_2 + \dots + k_e\phi_e}{\phi_0}\right) \\
&= \exp\left(\frac{\left[\sum_{n\geq 0} \frac{(kn)!}{\prod_{j=1}^{e}(k_j n)!}\right] \cdot \left(\left(\sum_{i=1}^{e} k_i H_{nk}\right) - k_1 H_{k_1 n} - k_2 H_{k_2 n} - \dots - k_e H_{k_e n}\right)}{F_{\mathbf{e},\mathbf{f}}}\right) \\
&= \exp\left(\frac{\left[\sum_{n\geq 0} \frac{(kn)!}{\prod_{j=1}^{e}(k_j n)!}\right] \cdot \left(\sum_{i=1}^{e} k_i H_{nk} - \sum_{i=1}^{e} k_i H_{k_i n}\right)\right)}{F_{\mathbf{e},\mathbf{f}}}\right) \\
&= \exp\left(\frac{G_{\mathbf{e},\mathbf{f},1}}{F_{\mathbf{e},\mathbf{f}}}\right) \\
&= q_{\mathbf{e},\mathbf{f},1},
\end{aligned}$$

giving the required correspondence between $\Psi_{\mathbb{u}}$ and $q_{\mathbf{e},\mathbf{f},1}$. $\square$

Hence, we can explicitly use the results of Delaygue to infer results concerning Conjecture A. However, we might ask ourselves whether this is always the case;

whether we always have this explicit correspondence between the two statements. We find our answer in the following lemma.

**Lemma 2.5.** We can apply Theorem 2.2 if there is some injective linear map $i : \mathbb{Z}^k \hookrightarrow \mathbb{Z}^e$ where $\mathrm{im}(i) = K$ and $i(\mathbb{Z}^k_{\geq 0}) = K_{\geq 0}$.

*Proof.* Assume such an $i$ exists, and express it in matrix form. Let

$$\mathbb{e}_1 = i^T(1, \ldots, 1) \quad \text{and} \quad \mathbb{f}_j = i^T(\delta^1_j, \ldots, \delta^e_j).$$

Then for any vector $\mathbf{v} \in \mathbb{Z}^k$ we have

$$\begin{aligned}\mathbb{e}_1 \cdot \mathbf{v} &= \mathbb{e}_1^T \mathbf{v} \\ &= (1, \ldots, 1)^T i \mathbf{v} \\ &= (1, \ldots, 1) \cdot (iv) \\ &= \sum_{k=1}^{e} (i(v)^{(k)}),\end{aligned}$$

which is to say it is the sum of the coordinates of $i(v)$.

Furthermore, $\mathbb{f}_j \cdot \mathbf{v} = i(v)^{(j)}$, the $j$th coordinate of $i(\mathbf{v})$. Using this, we find that

$$\begin{aligned}\mathrm{comb}(i(v)) &= \frac{(\sum_{k=1}^{e}(i(v)^{(k)})!}{\prod_{j=1}^{e} i(v)^{(j)}!} \\ &= \frac{(\mathbb{e}_1 \cdot \mathbf{v})!}{\prod_{j=1}^{e}(\mathbb{f}_j \cdot \mathbf{v})!}.\end{aligned}$$

We also see that the map $i$ induces a map

$$\begin{array}{rcl}\phi_i : \mathbb{Q}[z_1, \ldots, z_k] & \to & \mathbb{Q}[x_1, \ldots, x_e] \\ z_1^{\mathbf{v}^{(1)}} z_2^{\mathbf{v}^{(2)}} \ldots z_k^{\mathbf{v}^{(k)}} & \mapsto & x_1^{\mathbb{f}_1 \cdot \mathbf{v}} x_2^{\mathbb{f}_2 \cdot \mathbf{v}} \ldots x_e^{\mathbb{f}_e \cdot \mathbf{v}}.\end{array}$$

Then

$$\begin{aligned}\phi_0(x_1, \ldots x_e) &= \varphi_i(F_{\mathbf{e},\mathbf{f}}(z_1, \ldots, z_k)), \text{ and} \\ \phi_j(x_1, \ldots, x_e) &= \varphi_i(G_{e,f,j}(z_1, \ldots, z_k)),\end{aligned}$$

giving us a relation which allows us to translate between the existing notation in 2.2 and the notation we have introduced. □

**Remark 2.6.** Although we do not prove it here, the converse also holds. That is, if we can apply Theorem 2.2 then there must exist an injective linear map $i$ satisfying the given constraints.

Let us examine how this might work with an example.

**Example 2.7.** We examine the case of the reflexive simplex with vertices

$$x_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, x_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, x_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, x_4 = \begin{pmatrix}-2\\-2\\-1\end{pmatrix}.$$

There is another lattice point on the boundary of the simplex given by coordinates

$$x_5 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

As we must include all lattice points on codimension 2 faces of the polytope, the vectors for our mirror map will be those in the kernel

$$K_{\geq 0} := \ker \begin{pmatrix} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

That is, $K_{\geq 0}$ is the set of $x_1, \ldots, x_5 \in \mathbb{Z}_{\geq 0}$ satisfying

1. $x_1 - 2x_4 - x_5 = 0$,
2. $x_2 - 2x_4 - x_5 = 0$,
3. $x_3 - x_5$.

The kernel space is therefore equal to $i(\mathbb{Z}^2_{\geq 0})$ defined on the basis such that

$$i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

so there does exist an injective homomorphism $i$ with $\mathrm{im}(i) = K$ and $i(\mathbb{Z}^2_{\geq 0}) = K_{\geq 0}$. Therefore we can apply Delaygue's theorem in this case.

The next question to which one might be led is whether such an injective homomorphism $i$ exists for the mirror map of any reflexive polytope. The answer is no; there exist reflexive polytopes for which we cannot apply Delaygue's theorem, such as the following.

**Example 2.8.** We consider the reflexive simplex whose vertices are

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, x_4 = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}.$$

Then the lattice points on the boundary of the simplex also include the points

$$x_5 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad x_6 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

as midpoints of edges $x_1x_4$ and $x_2x_3$ respectively. The resulting kernel is given by

$$K_{\geq 0} := \ker \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -2 & -1 & 1 \\ 0 & 0 & 2 & -2 & -1 & 1 \end{pmatrix} \cap \mathbb{Z}^6_{\geq 0}.$$

That is, $K_{\geq 0}$ is the set of $x_1, \ldots, x_6 \in \mathbb{Z}_{\geq 0}$ satisfying

1. $x_1 - x_4 = 0$,
2. $x_2 + x_3 - 2x_4 - x_5 + x_6 = 0$,
3. $2x_3 - 2x_4 - x_5 + x_6 = 0$.

Via elimination techniques, we find that $x_1 = x_4$ and $x_2 = x_3$, and we can eliminate $x_6$ as a scalar multiple of $x_5$, so that we have

$$K_{\geq 0} = \{(x_3, x_4, x_5) \in \mathbb{Z}^3_{\geq 0} \mid -2x_3 + 2x_4 + x_5 \geq 0\}.$$

We observe that $\mathbb{Z}^3_{\geq 0}$ is a cone with three edges given by the positive axis, and so $K_{\geq 0}$ is a cone with these edges as well as an additional edge given by $-2x_3 + 2x_4 + x_5 = 0$ as per our expression of $K_{\geq 0}$, so that $K^3_{\geq 0}$ is a cone with four edges. In particular, the mismatch in the number of edges means that there cannot exist a homomorphism $i : \mathbb{Z}^3_{\geq 0} \to K_{\geq 0}$.

## 2.2 Positivity theorems

In Krattenthaler and Rivoal's research (see [12]), they found various connections between certain mirror maps and the positivity of the coefficients.

**Theorem 2.9.** (Lemma 2.1, [12] and Satz 3, [11]) Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ where $f_0 = 1$ such that $f_1 > 0$ and $f_n^2 \leq f_{n-1} f_{n+1}$ for all $n \geq 1$. If $g(z) = \sum_{n=0}^{\infty} h_n f_n z^n$ such that $h_{n+1} \geq h_n > 0$ for all $n \geq 0$, then the Taylor coefficients of $g(z)/f(z)$ are nonnegative.

*Proof.* We first prove that if $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $f_0 = 1$ where $f_1 > 0$ and $f_{n+1} f_{n-1} \geq f_n^2$ for $n \geq 1$ then the Taylor coefficients of

$$1 - \frac{1}{f(z)}$$

are nonnegative.

Let us denote $q(z) = -\frac{1}{f(z)} = \sum_{n=0}^{\infty} q_n z^n$, where $q_0 = -1$. This is to say that

$$1 + q(z) = 1 - \frac{1}{f(z)}.$$

Let us denote $\widetilde{f}(z) = 1 + q(z)$, where $\sum_{n=1}^{\infty} \widetilde{f}_n z^n = 1 + \sum_{n=0}^{\infty} q_n z^n$, so that $\widetilde{f}_n = q_n$ for $n \geq 1$. In [11], we acknowledge that $\widetilde{f}_1 = f_1$, $\widetilde{f}_2 = f_2 - f_1 \widetilde{f}_1, \ldots, \widetilde{f}_n = f_n - \sum_{k=1}^{n-1} f_{n-k} \widetilde{f}_k$. For the proof of positivity of the coefficients $\widetilde{f}_n$, let us first observe that by assumption we have $f_0 = 1 > 0$ and $f_1 > 0$. Furthermore, since $f_0 f_2 \geq f_1^2$ we must have $f_2 > 0$, and since $f_1 f_3 \geq f_2^2$ we must have $f_3 > 0$. Continuing in this way, we see that $f_n > 0$ for all $n \in \mathbb{N}$. Since $f_0 = 1$, we also have that for all such $n$,

$$f_n - \sum_{k=1}^{n} \widetilde{f}_k f_{n-k} = 0. \tag{2.5}$$

Going up a level, we have

$$f_{n+1} - \sum_{k=1}^{n} \widetilde{f}_k f_{n-k+1} = \widetilde{f}_{n+1}. \tag{2.6}$$

Multiplying 2.5 by $-f_{n+1}$ and 2.6 by $f_n$ and adding them together, we see that

$$\widetilde{f}_{n+1} f_n = \sum_{k=1}^{n} (f_{n-k} f_{n+1} - f_{n-k+1} f_n) \widetilde{f}_k. \tag{2.7}$$

Furthermore, since

$$\frac{f_0}{f_1} \geq \frac{f_1}{f_2} \geq \frac{f_2}{f_3} \geq \dots,$$

each of the terms $(f_{n-k} f_{n+1} - f_{n-k+1} f_n)$ are nonnegative. Also, since $\widetilde{f}_1 = f_1 > 0$, and we have a recursive formula to determine each $\widetilde{f}_n$ as the sum of previous nonnegative $\widetilde{f}_i$, then $\widetilde{f}_n > 0$ for all $n \in \mathbb{N}$.

We now prove that if the Taylor coefficients of $1 - 1/f(z)$ are nonnegative then the Taylor coefficients of

$$\frac{g(z)}{f(z)}$$

are nonnegative. Indeed, if

$$\widetilde{f}(z) = 1 - \frac{1}{f(z)}$$

then we can write the Taylor expansions $\widetilde{f}(z) = \sum_{n=1}^{\infty} \widetilde{f}_n z^n$ and $f(z) = \sum_{n=0}^{\infty} f_n z^n$ such that

$$f_n - \sum_{n=1}^{\infty} \widetilde{f}_k f_{n-k} = \delta_n^0. \tag{2.8}$$

Indeed, we can see for $n = 0$ that $f_0 = 1$, which is true by assumption, and for $n = 1$ that $f_1 = \widetilde{f}_1 f_0$. Using this, we may also describe

$$g(z)\widetilde{f}(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{n} \widetilde{f}_k f_{n-k} f_{n-k} z^n,$$

which is to say that we can redescribe $g(z)/f(z)$ in terms of $\widetilde{f}$, in the following way:

$$\begin{aligned} \frac{g(z)}{f(z)} &= g(z)(1 - \widetilde{f}(z)) \\ &= \sum_{n=0}^{\infty} \left( h_n f_n - \sum_{k=1}^{n} \widetilde{f}_k f_{n-k} h_{n-k} \right) z^n. \end{aligned}$$

Furthermore, we have assumed that $(h_n)_{n\geq 0}$ is nondecreasing and nonnegative. Using this fact, we observe that

$$\begin{aligned} h_n f_n - \sum_{k=1}^{n} \widetilde{f}_k f_{n-k} h_{n-k} &\geq h_n f_n - \sum_{k=1}^{n} \widetilde{f}_k f_{n-k} h_n \\ &= h_n \left( f_n - \sum_{k=1}^{n} \widetilde{f}_k f_{n-k} \right) \\ &= h_n \delta_n^0 \geq 0 \end{aligned}$$

due to Equation 2.8. Therefore we have positivity, as required. □

**Corollary 2.10.** If the assertions of Theorem 2.9 hold, then $\exp(g(z)/f(z))$ are nonnegative.

*Proof.* As the exponent of a power series with nonnegative coefficients is itself a power series with nonnegative coefficients, the result follows immediately. □

We have discussed the proofs of mirror map integrality and positivity, as given in [5], [12] and [11]. Using these theorems, we will now prove that for rank 1 reflexive polytopes, the mirror map $\Psi_{\mathbb{k}}^{A(\Delta)}$ will have positive integral coefficients for all basis vectors $\mathbb{k}$ for the kernel of $A(\Delta)$.

# Chapter 3

# Proving the rank 1 case

In this chapter, we prove that a given reflexive polytope with rank 1 kernel will have a positive and integral mirror map. For this, we will be using Delaygue's theorem to show that the rank 1 case gives us specific conditions which satisfy the assertions required to apply 2.2.

Consider the case where the kernel of the map $\mathbb{Z}^e \to \mathbb{Z}^d$ has rank 1. In particular, $e - d = 1$, and the polytope is a simplex $\Delta$, so that $A(\Delta)$ consists only of the vertices of $\Delta$.

## 3.1 Integrality

In this section, we prove that $\Psi_{\Bbbk}^{A(\Delta)}$ is integral for $\Delta$ a reflexive polytope whose kernel has rank 1. Before proceeding further with the question of integrality of $\Delta$'s mirror map, we need to address the use of a certain tool for the proof: the inner product. This sesquilinear device can take many forms, but when taking the product of a vector in one space with a vector from another space, it can be tricky to know how to define what the inner product needs to do. In particular, if $P$ is a reflexive polytope in $M_{\mathbb{R}}$ and $P^\vee$ is its dual polytope then $P$ and $P^\vee$ live in different spaces, as we discussed in Remark 1.6. Say $x \in P$ is a vertex. Then we will have a corresponding face $F_x$ in $P^\vee$. Then for any vertex $y \in F_x$, as a 'dual' vertex to $x$, we might expect their inner product to yield $\langle x, y\rangle = -1$. Indeed, this seems to agree with the definition of a dual polytope,

$$P^\vee := \{y \in M_{\mathbb{R}}^\vee : \langle x, y\rangle \geq -1 \forall x \in P\}.$$

In particular, if $P$ is a reflexive simplex with vertices $x_i$ and $P^\vee$ is its dual with vertices $y_i$, then

$$\langle x_i, y_j\rangle = -1 \text{ for } i \neq j.$$

Now, we want to find out the value of the inner product on diagonal entries, given the constraints of the inner product and using what we know about all the other entries. Let us notate $\langle x_i, y_i\rangle = B_i$ in the meantime. Define the kernel

$$K = \left\{(k_1, \ldots, k_n) : \sum_{i=1}^{n} k_i x_i = 0\right\},$$

so that $\langle \sum_{i=1}^n k_i x_i, y_i \rangle = 0$, which infers that $\sum_{i=1}^n k_i \langle x_i, y_i \rangle = 0$, so that $k_i$ is in the kernel of $\langle x_i, y_i \rangle$. In particular,

$$k_i \in \ker \begin{pmatrix} B_1 & -1 & -1 & \dots & -1 \\ -1 & B_2 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & B_n \end{pmatrix}.$$

Then by taking the properties of a kernel, we have that $k_i B_i - \sum_{j \neq i} k_j = 0$, so that $k_i(B_i + 1) = \sum_{j=1}^n k_j$. Then

$$k_i = \frac{\sum_{j=1}^n k_j}{B_i + 1} \quad \text{and} \quad \sum_{i=1}^n k_i = \sum_{i=1}^n \frac{\sum_{j=1}^n k_j}{B_i + 1} = \left( \sum_{j=1}^n k_j \right) \sum_{i=1}^n \frac{1}{B_i + 1},$$

so we have $\sum_{i=1}^n \frac{1}{B_i+1} = 1$.

Therefore, if we want our inner product to satisfy the constraints of the kernel, we need the diagonal entries of our matrix to be of form

$$\langle x_i, y_i \rangle = \alpha_i - 1$$

where $\sum_{i=1}^n \frac{1}{\alpha_i} = 1$.

**Theorem 3.1.** For a polytope $\Delta$ with $d+1$ vertices in $d$ entries, such that the kernel of the map $\mathbb{Z}^{d+1} \to \mathbb{Z}^d$ has rank 1, the mirror maps $(\psi_1^{A(\Delta)}, \dots, \psi_e^{A(\Delta)})$ have integer coefficients.

*Proof.* Let $\mathbf{v}_1, \dots, \mathbf{v}_{d+1}$ be the vertices of $\Delta$, and $\mathbf{w}_1, \dots, \mathbf{w}_{d+1}$ be the vertices of $\Delta^\vee$, the dual polytope of $\Delta$. Then

$$\begin{aligned} \langle \mathbf{v}_i, \mathbf{w}_j \rangle &= \begin{cases} -1 \text{ if } i \neq j \\ \alpha_i - 1 \text{ if } i = j \end{cases} \\ &= -1 + \alpha_i \delta_j^i, \end{aligned}$$

for a collection of $\alpha_i$ satisfying $\sum_{i=1}^n \frac{1}{\alpha_i} = 1$, so that $(\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1}$ is a weight system. Let $K$ be the kernel of the map $\mathbb{Z}^{d+1} \to \mathbb{Z}^d$. Then for $(k_1, \dots, k_{d+1}) \in K_{\geq 0}$, we have that

$$\sum_{i=1}^{d+1} k_i \mathbf{v}_i = 0.$$

This is to say that

$$\begin{aligned}
&\left\langle \sum_{i=1}^{d+1} k_i \mathbf{v}_i, \mathbf{w}_j \right\rangle = 0\\
&\Rightarrow \sum_{i=1}^{d+1} k_i(-1 + \alpha_i \delta_j^i) = 0\\
&\Rightarrow k_j \alpha_j = \sum_{i=1}^{d+1} k_i\\
&\Rightarrow \sum_{j=1}^{d+1} \frac{1}{\alpha_j} = \sum_{j=1}^{d+1} \frac{k_j}{\sum_{i=1}^{d+1} k_i} = 1.
\end{aligned}$$

This somewhat motivates the definition of a weight system $(\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1}$; they are the weights necessary to balance the system of inner products.

In the discussion above we see that that $K_{\geq 0}$ is generated by $(k_1, \dots, k_{d+1})$, where $k_i = \frac{\sum_{j=1}^{d+1} k_j}{\alpha_i}$. Therefore we have an injective group homomorphism $i : \mathbb{Z} \to \mathbb{Z}^{d+1}$ with matrix $[k_1, \dots, k_{d+1}]$ where $\mathrm{im}(i) = K$ and $i(\mathbb{Z}_{\geq 0}) = K \geq 0$, satisfying the constraints given in Lemma 2.5, allowing us to use Delaygue's theorem.

By earlier observation, Delaygue applies to the mirror maps of reflexive polytopes when $q_1 = 1$, so there is only one $e_i$ vector and $e_1 = \sum_{i=1}^{d+1} f_i$. Therefore we have no choice but to assign

$$f_i = i^T(0, \dots, 0, 1, 0, \dots, 0) = k_i \quad \text{and} \quad e_1 = \sum_{i=1}^{d+1} k_i.$$

By Delaygue's theorem, we know that the mirror map is integral if for $x \in [0, 1)$ we have the implication

$$e_1 x \geq 1 \Rightarrow \lfloor e_1 x \rfloor - \sum_{i=1}^{d+1} \lfloor f_i x \rfloor \geq 1.$$

We claim it is enough to show the implication

$$x \in \left[\frac{1}{\sum_{i=1}^{d+1} k_i}, 1\right) \Rightarrow \sum_{i=1}^{d+1} \{k_i x\} \geq 1.$$

First, we see that $x \in \left[\frac{1}{\sum_{i=1}^{d+1} k_i}, 1\right)$ is equivalent to stating that $e_1 x = \sum_{i=1}^{d+1} k_i x$ is in the interval

$$\left[\frac{\sum_{i=1}^{d+1} k_i}{\sum_{i=1}^{d+1} k_i}, \sum_{i=1}^{d+1} k_i\right) = \left[1, \sum_{i=1}^{d+1} k_i\right),$$

so that $e_1 x \geq 1$ if and only if $x \in \left[\frac{1}{\sum_{i=1}^{d+1} k_i}, 1\right)$.

Now we want to show that if

$$x \in \left[\frac{1}{\sum_{i=1}^{d+1} k_i}, 1\right) \Rightarrow \sum_{i=1}^{d+1} \{k_i x\} \geq 1$$

then

$$e_1 x \geq 1 \Rightarrow \lfloor e_1 x \rfloor - \sum_{i=1}^{d+1} \lfloor f_i x \rfloor \geq 1,$$

which is to say that we need the implication

$$\sum_{i=1}^{d+1} \{k_i x\} \geq 1 \Rightarrow \lfloor e_1 x \rfloor - \sum_{i=1}^{d+1} \lfloor f_i x \rfloor \geq 1.$$

Indeed, we find that if $\sum_{i=1}^{d+1} \{k_i x\} \geq 1$ then

$$\begin{aligned}\left\lfloor \sum_{j=1}^{q_2} f_j x \right\rfloor - \sum_{j=1}^{q_2} \lfloor f_j x \rfloor &= \left\lfloor \sum_{i=1}^{d+1} k_i x \right\rfloor - \sum_{j=1}^{d+1} \lfloor k_j x \rfloor \\ &= \left\lfloor \sum_{i=1}^{d+1} k_i x \right\rfloor - \sum_{j=1}^{d+1} k_j x + \sum_{j=1}^{d+1} \{k_j x\} \\ &= \sum_{j=1}^{d+1} \{k_j x\} - \left\{ \sum_{i=1}^{d+1} k_i x \right\}.\end{aligned}$$

Then since $\sum_{j=1}^{d+1} \{k_j x\} \geq 1$ by assumption, and $\{\sum_{i=1}^{d+1} k_i x\} < 1$ by definition of $\{\cdot\}$, we use the fact that $\lfloor \sum_{j=1}^{q_2} f_j x \rfloor - \sum_{j=1}^{q_2} \lfloor f_j x \rfloor$ must be an integer greater than $1 - \{\sum_{i=1}^{d+1} k_i x\} > 0$, so that

$$\left\lfloor \sum_{j=1}^{q_2} f_j x \right\rfloor - \sum_{j=1}^{q_2} \lfloor f_j x \rfloor \geq 1.$$

Therefore the implication

$$x \in \left[ \frac{1}{\sum_{i=1}^{d+1} k_i}, 1 \right) \Rightarrow \sum_{i=1}^{d+1} \{k_i x\} \geq 1$$

will indeed prove that the mirror map is integral.

So let us assume that $x \in \left[\frac{1}{\sum_{i=1}^{d+1} k_i}, 1\right)$. We claim that for $x \geq 1$, if we write $x = \frac{a}{42}$ for some $a \in \mathbb{R}$, $x$ is smallest when $a \in \mathbb{Z}$. To see why this claim might be true, let us examine an example.

**Example 3.2.** Let our weight system be given by $\alpha = (2, 3, 7, 42)$. This verifies $\sum_{i=1}^{d+1} \frac{1}{\alpha_i} = 1$, and we find $\mathrm{lcm}(\alpha_j) = 42$ giving us $k$-values $k = (21, 14, 6, 1)$ which satisfy $\sum_{i=1}^{d+1} k_i = \mathrm{lcm}(\alpha_j)$, the answer to life, the universe, and everything[1]. We wish to observe the following implication:

$$42x \geq 1 \quad \Rightarrow \quad \{21x\} + \{14x\} + \{6x\} + \{x\} \geq 1.$$

Let us take a few calculations to see where we stand.

[1]As stated in [6]

- For $x = \frac{1}{42}$, we find that

$$\begin{aligned}\{21x\} + \{14x\} + \{6x\} + \{x\} &= \left\{\frac{1}{2}\right\} + \left\{\frac{1}{3}\right\} + \left\{\frac{1}{4}\right\} + \left\{\frac{1}{42}\right\} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} \\ &= 1,\end{aligned}$$

  as per our previous observation that $\sum_{i=1}^{d+1} \frac{1}{\alpha_i} = 1$. We observe that this fractional part will continue to increase as $x$ increases, but only until one of the fractional parts reaches a whole number, at which point the value will drop.

- For $x = \frac{2}{42}$, we notice that the fractional part of $\frac{21}{21}$ disappears, so we expect to observe a drop in the sum of fractional parts. Indeed, we see that

$$\begin{aligned}\{21x\} + \{14x\} + \{6x\} + \{x\} &= \left\{\frac{21}{21}\right\} + \left\{\frac{14}{21}\right\} + \left\{\frac{6}{21}\right\} + \left\{\frac{1}{21}\right\} \\ &= \frac{14 + 6 + 1}{21} \\ &= 1,\end{aligned}$$

  so that the sum of fractional parts has not dropped lower than 1.

- For $x = \frac{3}{42}$, the fractional part of $\frac{14}{14}$ disappears, so that we obtain

$$\begin{aligned}\{21x\} + \{14x\} + \{6x\} + \{x\} &= \left\{\frac{21}{14}\right\} + \left\{\frac{14}{14}\right\} + \left\{\frac{6}{14}\right\} + \left\{\frac{1}{14}\right\} \\ &= \frac{7 + 6 + 1}{14} \\ &= 1.\end{aligned}$$

- For $x = \frac{4}{42}$, we have

$$\begin{aligned}\{21x\} + \{14x\} + \{6x\} + \{x\} &= \left\{\frac{42}{21}\right\} + \left\{\frac{28}{21}\right\} + \left\{\frac{12}{21}\right\} + \left\{\frac{2}{21}\right\} \\ &= \frac{7 + 12 + 2}{21} \\ &= 1.\end{aligned}$$

- For $x = \frac{5}{42}$, we notice that 5 is not a factor of 42, so we expect that there is no drop here. Indeed, we have

$$\begin{aligned}\{21x\} + \{14x\} + \{6x\} + \{x\} &= \left\{\frac{21 \cdot 5}{42}\right\} + \left\{\frac{14 \cdot 6}{42}\right\} + \left\{\frac{6 \cdot 5}{42}\right\} + \left\{\frac{5}{42}\right\} \\ &= \frac{21 + 28 + 30 + 5}{42} \\ &= 2.\end{aligned}$$

Therefore at this point, the graph continues to increase.

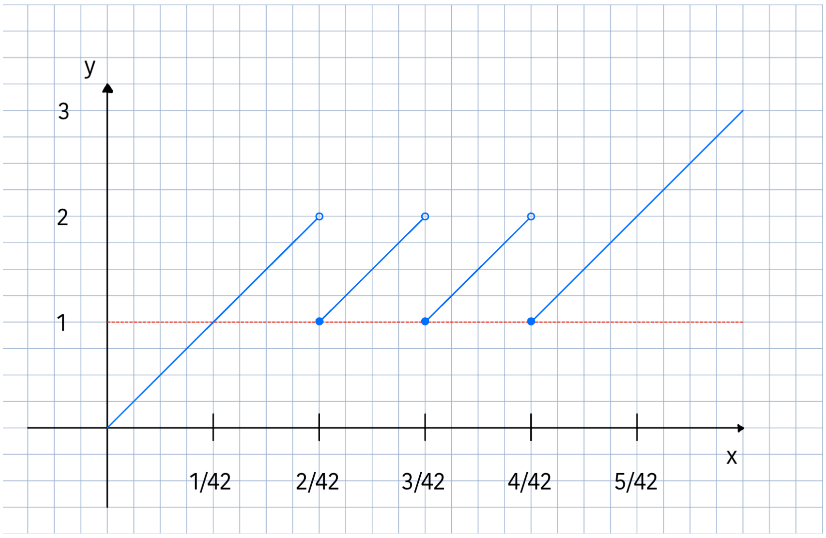


Figure 3.1: The figure shows the graph of $y = \{21x\} + \{14x\} + \{6x\} + \{x\}$ for small values of $x$. The dotted red line indicates the points where $y = 1$, and the figure gives us the impression that the graph might occur entirely above this line for $x \geq \frac{1}{42}$.

The punchline is that the drop points in the graph look like they occur only when $x = \frac{a}{42}$, for $a \in \mathbb{Z}$.

We claim that the function $\{\cdot\}$ is constantly increasing except at $x = \frac{a}{42}$ for some integer values $a$. To see this, note that as $x$ increases infinitesimally, we expect $\sum_{i=1}^{d+1}\{k_i x\}$ to do so too, except if one of the $\frac{k_j a}{\sum_{i=1}^{d+1} k_i}$ reaches an integer value, which is only possible if $a$ is an integer. Therefore the minima occur exactly when $a$ is an integer.

So we wish to now show that for $x = \frac{a}{\sum_{i=1}^{d+1} k_i}$ with $a \in \mathbb{Z}$, we have

$$\sum_{i=1}^{d+1} \frac{k_i a}{\sum_{j=1}^{d+1} k_j} \geq 1.$$

For this, we first observe that

$$\sum_{i=1}^{d+1} \frac{k_i a}{\sum_{j=1}^{d+1} k_j} = a \in \mathbb{Z},$$

so that $\sum_{i=1}^{d+1} \left\{ \frac{k_i a}{\sum_{j=1}^{d+1} k_j} \right\}$ is also an integer. So to prove that $\sum_{i=1}^{d+1} \frac{k_i a}{\sum_{j=1}^{d+1} k_j} \geq 1$, all we need to show is that the integer it is equal to is not zero.

Suppose, for contradiction, that it is zero. Then $\frac{k_i a}{\sum_{j=1}^{d+1} k_j} \in \mathbb{Z}$ for all $1 \leq i \leq d+1$. Then

$$\sum_{j=1}^{d+1} k_j \;\Big|\; k_i a$$

for all such $i$. Then since all $k_i$ are coprime, it must be that $\sum_{j=1}^{d+1} k_j$ divides $a$. This is a contradiction since our assumption stated that

$$x = \frac{a}{42} \in \left[\frac{1}{\sum_{i=1}^{d+1} k_i}, 1\right);$$

in particular we require that $a < \sum_{i=1}^{d+1} k_i$. As we have derived a contradiction, we have that

$$\sum_{i=1}^{d+1} \frac{k_i a}{\sum_{j=1}^{d+1} k_j} \geq 1.$$

Therefore, we have the required implication to apply Delaygue's theorem, and the mirror map is integral. □

## 3.2 Positivity

In this section, we prove that $\Psi_{\mathbb{k}}^{A(\Delta)}$ has positive coefficients when $\Delta$ is a reflexive polytope whose kernel has rank 1. In the simple case of the dual of the moment polytope with standard basis vectors as vertices, i.e.

$$\Delta^N = \{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1),(-1,-1,\ldots,-1)\},$$

we have that

$$f(z) = \sum_{j=0}^{\infty} \frac{(Nj)!}{(j!)^N} z^j \quad \text{and} \quad g(z) = \sum_{j=1}^{\infty} \frac{(Nj)!}{(j!)^N} \left(\sum_{i=j+1}^{Nj} \frac{1}{i}\right) z^j.$$

Then using the results from Krattenthaller and Rivoal we would like to prove the following.

**Lemma 3.3.** For

$$f(z) = \sum_{j=0}^{\infty} \frac{(Nj)!}{(j!)^N} z^j \quad \text{and} \quad g(z) = \sum_{j=1}^{\infty} \frac{(Nj)!}{(j!)^N} \left(\sum_{i=j+1}^{Nj} \frac{1}{i}\right) z^j,$$

we have that $\exp(g(z)/f(z)) \in \mathbb{N}[[z]]$, which is to say that the coefficients are not only integral but also positive.

*Proof.* By Satz 3 of [11] and Lemma 2.1 of [12], it is enough to show that if we express

$$f(z) = \sum_{j=0}^{\infty} f_j z^j, \quad \text{and} \quad g(z) = \sum_{j=1}^{\infty} f_i h_i z^j$$

then $h_n$ is a nondecreasing sequence, $f_1 > 0$ and $f_n^2 \leq f_{n-1} f_{n+1}$ (as we proved in Theorem 2.9).

Indeed, for this example we write $f_n = \frac{(Nn)!}{(n!)^N}$ and $h_n = \sum_{i=n+1}^{Nn} \frac{1}{i}$, so that $f_1 = N! > 0$ and

$$h_{j+1} - h_j = \sum_{i=j+2}^{Nj+N} \frac{1}{i} - \sum_{i=j+1}^{Nj} \frac{1}{i} = \frac{1}{Nj+1} + \frac{1}{Nj+2} + \frac{1}{Nj+N} - \frac{1}{j+1} > 0,$$

so that $h_n$ is nondecreasing. Now, to show that $f_n^2 \leq f_{n-1}f_{n+1}$, we first recognise that for $a > b > n$, we have $ab - an < ab - nb$ so that $a(b-n) < b(a-n)$ and $\frac{a}{b} < \frac{a-n}{b-n}$. We wish to show that

$$\left(\frac{(Nj)!}{(j)!^N}\right)^2 < \frac{(N(j-1))!}{(j-1)!^N}\frac{(N(j+1))!}{(j+1)!^N}.$$

This is true if and only if

$$\frac{(Nj)(Nj-1)\dots(Nj-N+1)}{j^N} < \frac{(Nj+N)(Nj+N-1)\dots(Nj+1)}{(j+1)^N},$$

which in turn is true if and only if

$$\left(\frac{j+1}{j}\right)^N < \frac{Nj+N}{Nj}\frac{Nj+N-1}{Nj-1}\dots\frac{Nj+1}{Nj-N+1},$$

where we may note that both sides have $N$ terms in the product, and $\frac{Nj+N}{Nj} = \frac{j+1}{j}$, so we can prove the inequality by showing that $\frac{Nj+N}{Nj}$ is the smallest term on the RHS. Indeed, as we remarked before,

$$\frac{Nj+N}{Nj} < \frac{Nj+N-n}{Nj-n}$$

for all $1 \leq n \leq N$, so we know that the inequality holds and that $f_n^2 < f_{n-1}f_{n+1}$ for all $n$.

□

More generally, if $(\alpha_1, \dots, \alpha_d)$ is a weight system, where $k = \operatorname{lcm}(\alpha_j)$ and $k_i = k/\alpha_i$, we wish to prove the same positivity conjecture with

$$f(z) = \sum_{k=0}^{\infty} \frac{(kn)!}{\prod_{i=0}^{d}(k_i n)!} z^n \quad \text{and} \quad g_i(z) = \sum_{k=0}^{\infty} \frac{(kn)!}{\prod_{i=1}^{d}(k_i n)!}\left[H_{kn} - H_{k_i n}\right] z^n.$$

**Lemma 3.4.** For

$$f(z) = \sum_{k=0}^{\infty} \frac{(kn)!}{\prod_{i=0}^{d}(k_i n)!} z^n \quad \text{and} \quad g_i(z) = \sum_{k=0}^{\infty} \frac{(kn)!}{\prod_{i=1}^{d}(k_i n)!}\left[H_{kn} - H_{k_i n}\right] z^n,$$

we have that $\exp(g(z)/f(z)) \in \mathbb{N}[[z]]$.

*Proof.* By our definition of $f$ and $g$, we want some $h$ such that

$$f(z) = \sum_{j=0}^{\infty} f_j z^j, \quad \text{and} \quad g(z) = \sum_{j=1}^{\infty} f_i h_i z^j,$$

so for fixed $i$ we write

$$h_n = \left[H_{kn} - H_{k_i n}\right] = \sum_{j=1}^{kn}\frac{1}{j} - \sum_{j=1}^{k_i n}\frac{1}{j}.$$

To use Krattenthaler and Rivoal's results (Theorem 2.9), we must check that $h_n$ is nondecreasing, that $f_1 > 0$ and that $f_n^2 \le f_{n-1}f_{n+1}$.

__Nondecreasing $h_n$__: We wish to show that for all $n$, $h_{n+1} > h_n$. This is equivalent to requiring that

$$\left(\sum_{j=1}^{k(n+1)} \frac{1}{j} - \sum_{j=1}^{k_i(n+1)} \frac{1}{j}\right) - \left(\sum_{j=1}^{kn} \frac{1}{j} - \sum_{j=1}^{k_in} \frac{1}{j}\right) > 0,$$

which, since

$$\left(\sum_{j=1}^{k(n+1)} \frac{1}{j} - \sum_{j=1}^{k_i(n+1)} \frac{1}{j}\right) - \left(\sum_{j=1}^{kn} \frac{1}{j} - \sum_{j=1}^{k_in} \frac{1}{j}\right) = \sum_{j=kn+1}^{kn+k} \frac{1}{j} - \sum_{k_in+1}^{k_in+k_i} \frac{1}{j}$$

holds, is equivalent to showing that

$$\sum_{j=kn+1}^{kn+k} \frac{1}{j} > \sum_{k_in+1}^{k_in+k_i} \frac{1}{j}.$$

Setting $k = k_id_i$ for each $i$, we can rewrite this as

$$\sum_{a=1}^{k_i} \left(\sum_{j=(kn+(a-1)d_i+1)}^{kn+ad_i} \frac{1}{j}\right) > \sum_{a=1}^{k_i} \frac{1}{k_in + a},$$

since $a = 1$ gives $kn + (a-1)d_i + 1 = kn + 1$ and $a = k_i$ gives $kn + ad_i = kn + k$, so that we are taking the sum of $1/j$ over the same range of values. Using this, we see that it suffices to prove that

$$\sum_{j=kn+(a-1)d_i+1}^{kn+ad_i} \frac{1}{j} > \frac{1}{k_in + a}$$

holds for each $i$. Indeed, since

$$\frac{1}{k_in + a} = \frac{d_i}{d_ik_in + d_ia} = \frac{d_i}{kn + ad_i},$$

we recognise that

$$\frac{1}{kn + (a-1)d_i + 1} > \frac{1}{kn + ad_i},$$

so that each term $1/j$ from $j = kn + (a-1)d_i + 1$ to $j = kn + ad_i$ is bigger than $\frac{1}{kn+ad_i}$. Since there are $d_i$ terms in the sum on the left-hand side, we have

$$\sum_{j=kn+(a-1)d_i+1}^{kn+ad_i} \frac{1}{j} > d_i\left(\frac{1}{kn + (a-1)d_i + 1}\right) > \frac{d_i}{kn + ad_i},$$

as required. Therefore $h_n$ is increasing in $n$.

**Positivity of $f_1$**: We wish to show that $f_1 > 0$. Indeed,

$$f_1 = \frac{k!}{\prod_{i=1}^{d}(k_i)!},$$

and as a quotient of a positive number by a positive number, $f_1 > 0$.

**The bound of the square**: We wish to show that $f_n^2 \leq f_{n-1}f_{n+1}$. This is to say that we wish to prove

$$\left(\frac{(kn)!}{\prod_{i=1}^{d}(k_i n)!}\right)^2 \leq \left(\frac{(kn-k)!}{\prod_{i=1}^{d}(k_i n - k_i)!}\right)\left(\frac{(kn+k)!}{\prod_{i=1}^{d}(k_i n + k_i)!}\right).$$

This is equivalent to proving that

$$\frac{\prod_{i=1}^{d}(k_i n + k_i)(k_i n + k_i - 1)\ldots(k_i n + 2)(k_i n + 1)}{\prod_{i=1}^{d}(k_i n)(k_i n - 1)\ldots(k_i n - k_i + 2)(k_i n - k_i + 1)} \leq \frac{\prod_{j=0}^{k-1}(kn+k-j)}{\prod_{j=0}^{k-1}(kn-j)},$$

or equivalently

$$\prod_{i=1}^{d}\prod_{j=1}^{k_i}\frac{k_i n + j}{k_i n - k_i + j} \leq \frac{\prod_{j=0}^{k-1}(kn+k-j)}{\prod_{j=0}^{k-1}(kn-j)}. \tag{3.1}$$

We define $\sigma$ to be an auxiliary function

$$\sigma\left(\frac{j}{k}\right) := \frac{n+x}{n-1+x}.$$

Then

$$\sigma\left(\frac{j}{k_i}\right) = \frac{k_i n + j}{k_i n - k_i + j}, \quad \text{and} \quad \sigma\left(\frac{\ell}{k}\right) = \frac{kn+\ell}{kn-k+\ell},$$

so that we can rewrite our goal in Equation 3.1 to be

$$\prod_{i=1}^{d}\prod_{j=1}^{k_i}\sigma\left(\frac{j}{k_i}\right) \leq \prod_{\ell=1}^{k}\sigma\left(\frac{\ell}{k}\right). \tag{3.2}$$

Noticing that as $\sigma$ is a decreasing function, we can set $d_i = \frac{k}{k_i}$ to get

$$\sigma\left(\frac{j}{k_i}\right) = \sigma\left(\frac{jd_i}{k}\right) \leq \sigma\left(\frac{jd_i - s}{k}\right)$$

for $0 \leq s \leq d_i - 1$. Taking the product of this inequality over all $s$, we obtain the inequality

$$\sigma\left(\frac{j}{k_i}\right)^{d_i} \leq \sigma\left(\frac{jd_i-(d_i-1)}{k}\right)\cdot\sigma\left(\frac{jd_i-(d_i-2)}{k}\right)\cdot\cdots\cdot\sigma\left(\frac{jd_i-1}{k}\right)\cdot\sigma\left(\frac{jd_i-0}{k}\right).$$

Then the product of this over all $0 \le j \le k_i$ gives us

$$\prod_{j=0}^{k_i} \sigma\left(\frac{j}{k_i}\right)^{d_i} \le \prod_{j=1}^{k_i} \prod_{s=0}^{d_i-1} \sigma\left(\frac{jd_i - s}{k}\right).$$

We can then see that

$$\begin{aligned}
\prod_{j=1}^{k_i} \prod_{s=0}^{d_i-1} \sigma\left(\frac{jd_i - s}{k}\right) &= \prod_{j=1}^{k_i} \sigma\left(\frac{jd_i - 0}{k}\right) \cdot \sigma\left(\frac{jd_i - 1}{k}\right) \cdot \ldots \cdot \sigma\left(\frac{jd_i - d_i + 2}{k}\right) \cdot \sigma\left(\frac{jd_i - d_i + 1}{k}\right) \\
&= \prod \left\{ \begin{array}{cccc}
\sigma\left(\frac{d_i-0}{k}\right) & \sigma\left(\frac{d_i-1}{k}\right) & \ldots & \sigma\left(\frac{d_i-d_i+1}{k}\right) \\
\sigma\left(\frac{2d_i-0}{k}\right) & \sigma\left(\frac{2d_i-1}{k}\right) & \ldots & \sigma\left(\frac{d_i+1}{k}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma\left(\frac{(k_i-1)d_i-0}{k}\right) & \sigma\left(\frac{(k_i-1)d_i-1}{k}\right) & \ldots & \sigma\left(\frac{(k_i-1)d_i-d_i+1}{k}\right) \\
\sigma\left(\frac{k_id_i-0}{k}\right) & \sigma\left(\frac{k_id_i-1}{k}\right) & \ldots & \sigma\left(\frac{k_id_i-d_i+1}{k}\right)
\end{array} \right\} \\
&= \prod \left\{ \begin{array}{cccc}
\sigma\left(\frac{d_i-0}{k}\right) & \sigma\left(\frac{d_i-1}{k}\right) & \ldots & \sigma\left(\frac{1}{k}\right) \\
\sigma\left(\frac{2d_i-0}{k}\right) & \sigma\left(\frac{2d_i-1}{k}\right) & \ldots & \sigma\left(\frac{d_i+1}{k}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma\left(\frac{(k_i-1)d_i-0}{k}\right) & \sigma\left(\frac{(k_i-1)d_i-1}{k}\right) & \ldots & \sigma\left(\frac{(k_i-1)d_i-d_i+1}{k}\right) \\
\sigma\left(\frac{k}{k}\right) & \sigma\left(\frac{k-1}{k}\right) & \ldots & \sigma\left(\frac{(k_i-1)d_i+1}{k}\right)
\end{array} \right\} \\
&= \sigma\left(\frac{1}{k}\right) \cdot \sigma\left(\frac{2}{k}\right) \cdot \cdots \cdot \sigma\left(\frac{k-1}{k}\right) \cdot \sigma\left(\frac{k}{k}\right) \\
&= \prod_{\ell=1}^{k} \sigma\left(\frac{\ell}{k}\right)
\end{aligned}$$

giving us that

$$\prod_{j=0}^{k_i} \sigma\left(\frac{j}{k_i}\right)^{d_i} \le \prod_{j=1}^{k_i} \prod_{s=0}^{d_i-1} \sigma\left(\frac{jd_i - s}{k}\right) = \prod_{\ell=1}^{k} \sigma\left(\frac{\ell}{k}\right).$$

Taking the $d_i$th root, we obtain

$$\prod_{j=0}^{k_i} \sigma \frac{j}{k_i} \le \left(\prod_{\ell=1}^{k} \sigma\left(\frac{\ell}{k}\right)\right)^{\frac{1}{d_i}}.$$

Taking the product over all $i$, we obtain

$$\prod_{i=1}^{d} \prod_{j=0}^{k_i} \sigma \frac{j}{k_i} \le \left(\prod_{\ell=1}^{k} \sigma\left(\frac{\ell}{k}\right)\right)^{\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_{d-1}}+\frac{1}{d_d}}.$$

Also, since $\frac{1}{d_i} = \frac{k_i}{k}$ then

$$\sum_{i=1}^{d} \frac{1}{d_i} = \frac{\sum_{i=1}^{d} k_i}{k} = \frac{k}{k} = 1,$$

so that our inequality becomes

$$\prod_{i=1}^{d}\prod_{j=0}^{k_i}\sigma\left(\frac{j}{k_i}\right) \leq \prod_{\ell=1}^{k}\sigma\left(\frac{\ell}{k}\right),$$

thus obtaining Equation 3.2. $\square$

As a result, we have proved that for reflexive polytopes with rank 1 kernel, the mirror map $\Psi_{\Bbbk}^{A(\Delta)}$ will have positive integral coefficients.

# Chapter 4

# Code and commentary

In this chapter, we discuss the computations used to check the validity of Conjecture D on reflexive polytopes using Sagemath (see [16]).

## 4.1 Precision

As the calculation deals with power series, we must combat the intrinsic flaw of computation: computers have finite computational power, but a power series has infinitely many terms. Therefore to generate coefficients manually, we need to define a precision which is small enough to be calculated with reasonable computational power, but large enough to be accurate for our means. Say, for example, one wants to find whether coefficients are all integers, but checking up from the coefficient of the constant term to that of the $x^7$ term is enough to put one's mind at ease. We want to know that approximating up to this precision earlier in the calculations will not be problematic for the resulting calculations.

For this, we first make our definition of precision . . . precise.

**Definition 4.1.** Given vectors $\mathbb{u}, \mathbb{v} \in (\mathbb{Z}_{>0})^e$, let's say $\mathbb{u} < \mathbb{v}$ if $u^{(i)} < v^{(i)}$ for all $i$. Given power series $C(\mathbb{z}) = \sum \mathbb{c}_{\mathbb{u}} \mathbb{z}^{\mathbb{u}}$ and $D(\mathbb{z}) = \sum \mathbb{d}_{\mathbb{u}} \mathbb{z}^{\mathbb{u}}$, we say that $C = D$ to precision $P$, denoted $C \sim_p D$, if $\mathbb{c}_{\mathbb{u}} = \mathbb{d}_{\mathbb{u}}$ for all $\mathbb{u} < \mathrm{p} := (p, \dots, p)$.

**Lemma 4.2.** If $C_1 \sim_p C_2$, and $D_1 \sim_p D_2$, then $C_1/D_1 \sim_p C_2/D_2$.

For the proof of this lemma, we introduce some new notation. We denote by $\mathbb{d}^{\mathbb{v}}$ the product $\prod_{i=0}^{|\mathbb{v}|} d_{\mathbb{v}_i}$. We also write $\Omega(\mathbb{z}^{p+1})$ for the sum of terms of $\mathbb{z}$ with power greater than $p$.

*Proof.* First we claim that if $P \sim_p Q$ then $\frac{1}{P} \sim_p \frac{1}{Q}$. Let us say that

$$P(\mathbb{x}) = \sum_{\mathbb{u} \in \mathbb{Z}^d_{\geq 0}} \mathbb{d}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}} \quad Q(\mathbb{x}) = \sum_{\mathbb{u} \in \mathbb{Z}^d_{\geq 0}} \mathbb{e}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}$$

where $\mathbb{d}_{\mathbb{u}} = \mathbb{e}_{\mathbb{u}}$ for $(0, \dots, 0) \leq \mathbb{u} \leq (p, \dots, p)$. Then by the geometric series formula, we have that

$$
\begin{aligned}
\frac{1}{P(\mathbb{x})} &= \frac{1/d_0}{P(\mathbb{x})/d_0}\\
&= \frac{1/d_0}{1+\sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\frac{\mathbb{d}_\mathbb{u}}{d_0}\mathbb{x}^\mathbb{u}}\\
&= \frac{1}{d_0}\left(\frac{1}{1-\left(-\sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\frac{\mathbb{d}_\mathbb{u}}{d_0}\mathbb{x}^\mathbb{u}\right)}\right)\\
&= \frac{1}{d_0}\left(\sum_{j=0}^{\infty}\left(-\sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\frac{\mathbb{d}_\mathbb{u}}{d_0}\mathbb{x}^\mathbb{u}\right)^j\right)\\
&= \frac{1}{d_0}\left(\left(-\sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\frac{\mathbb{d}_\mathbb{u}}{d_0}\mathbb{x}^\mathbb{u}\right)^0+\left(-\sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\frac{\mathbb{d}_\mathbb{u}}{d_0}\mathbb{x}^\mathbb{u}\right)^1+\left(-\sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\frac{\mathbb{d}_\mathbb{u}}{d_0}\mathbb{x}^\mathbb{u}\right)^2+\dots\right)\\
&= \frac{1}{d_0}\left(1-\sum_{i=1}^{p}\frac{\mathbb{d}_\mathbb{u}}{d_0}\mathbb{x}^\mathbb{u}+\left(\sum_{\mathbb{u}\leq\mathbb{p}}\frac{\mathbb{d}_\mathbb{u}}{d_0}\mathbb{x}^\mathbb{u}\right)^2+\left(-\sum_{\mathbb{u}\leq\mathbb{p}}\frac{\mathbb{d}_\mathbb{u}}{d_0}\mathbb{x}^\mathbb{u}\right)^3+\Omega(\mathbb{x}^{p+1})\dots\right)\\
&= \frac{1}{d_0}\left(1-\sum_{i=1}^{p}\frac{\mathbb{e}_\mathbb{u}}{e_0}\mathbb{x}^\mathbb{u}+\left(\sum_{\mathbb{u}\leq\mathbb{p}}\frac{\mathbb{e}_\mathbb{u}}{e_0}\mathbb{x}^\mathbb{u}\right)^2+\left(-\sum_{\mathbb{u}\leq\mathbb{p}}\frac{\mathbb{e}_\mathbb{u}}{e_0}\mathbb{x}^\mathbb{u}\right)^3+\Omega(\mathbb{x}^{p+1})\dots\right)\\
&\sim_p \quad \frac{1}{Q(\mathbb{x})}.
\end{aligned}
$$

Now we claim that for $C_1 \sim_p C_2$ and $D_1 \sim_p D_2$, we have $C_1D_1 \sim_p C_2D_2$. Indeed, writing

$$
C_1(\mathbb{x}) = \sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\mathbb{a}_\mathbb{u}\mathbb{x}^\mathbb{u},\quad C_2 = \sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\mathbb{b}_\mathbb{u}\mathbb{x}^\mathbb{u},\quad D_1(\mathbb{x}) = \sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\mathbb{e}_\mathbb{u}\mathbb{x}^\mathbb{u},\quad D_2(\mathbb{x}) = \sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\mathbb{f}_\mathbb{u}\mathbb{x}^\mathbb{u},
$$

where

$$
\mathbb{a}_\mathbb{u} = \mathbb{b}_\mathbb{u} \quad\text{and}\quad \mathbb{e}_\mathbb{u} = \mathbb{f}_\mathbb{u} \quad \text{for } (0,\dots,0)\leq\mathbb{u}\leq(p,\dots,p),
$$

we can write

$$
\begin{aligned}
C_1D_1(\mathbb{x}) &= \left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\mathbb{a}_\mathbb{u}\mathbb{x}^\mathbb{u}\right)\left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\geq 0}}\mathbb{e}_\mathbb{u}\mathbb{x}^\mathbb{u}\right)\\
&= \left(\sum_{\mathbb{u}\leq\mathbb{p}}\mathbb{a}_\mathbb{u}\mathbb{x}^\mathbb{u}+\sum_{\mathbb{u}>\mathbb{p}}\mathbb{a}_\mathbb{u}\mathbb{x}^\mathbb{u}\right)\left(\sum_{\mathbb{u}\leq\mathbb{p}}\mathbb{e}_\mathbb{u}\mathbb{x}^\mathbb{u}\sum_{\mathbb{u}>\mathbb{p}}\mathbb{e}_\mathbb{u}\mathbb{x}^\mathbb{u}\right)\\
&= \left(\sum_{\mathbb{u}\leq\mathbb{p}}\mathbb{a}_\mathbb{u}\mathbb{x}^\mathbb{u}\right)\left(\sum_{\mathbb{u}\leq\mathbb{p}}\mathbb{e}_\mathbb{u}\mathbb{x}^\mathbb{u}\right)+\Omega(\mathbb{x}^{p+1})\\
&= \left(\sum_{\mathbb{u}\leq\mathbb{p}}\mathbb{b}_\mathbb{u}\mathbb{x}^\mathbb{u}\right)\left(\sum_{\mathbb{u}\leq\mathbb{p}}\mathbb{f}_\mathbb{u}\mathbb{x}^\mathbb{u}\right)+\Omega(\mathbb{x}^{p+1})\sim_p C_2D_2(\mathbb{x}),
\end{aligned}
$$

as required.

Now, since $C_1 \sim_p C_2$ and $D_1 \sim_p D_1$ imply that $\frac{1}{D_1} \sim_p \frac{1}{D_2}$ and $C_1D_1 \sim_p C_2D_2$,

we then have that
$$\frac{C_1}{D_1} \sim_p \frac{C_2}{D_2},$$
as required. □

**Corollary 4.3.** If $\phi_i \sim_p \phi_i'$ for $i = 0, 1, \ldots, e$, and $\psi_i = \exp\left(\frac{\phi_i}{\phi_0}\right)$, $\psi_i' = \exp\left(\frac{\phi_i'}{\phi_0'}\right)$, then $\psi_i \sim_p \psi_i'$ for all $0 \le i \le e$.

*Proof.* First, we see that if $\phi_i \sim_p \phi_i'$ and $\phi_0 \sim_p \phi_0'$ then $\frac{\phi_i}{\phi_0} \sim_p \frac{\phi_i'}{\phi_0'}$ by the previous proof. Let us write

$$\frac{\phi_i}{\phi_0}(\mathbb{x}) = \sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{a}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}, \quad \frac{\phi_i'}{\phi_0'}(\mathbb{x}) = \sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{b}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}$$

where $\mathbb{a}_{\mathbb{u}} = \mathbb{b}_{\mathbb{u}}$ for $(0, \ldots, 0) \le \mathbb{u} \le (p, \ldots, p)$. Then

$$\begin{aligned}
\exp\left(\frac{\phi_i}{\phi_0}(\mathbb{x})\right) &= \exp\left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{a}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}\right) \\
&= \sum_{j=0}^{\infty} \frac{\left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{a}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}\right)^j}{j!} \\
&= 1 + \left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{a}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}\right) + \frac{\left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{a}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}\right)^2}{2} + \cdots + \frac{\left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{a}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}\right)^p}{p!} + \Omega(\mathbb{x}^{p+1}) \\
&= 1 + \left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{b}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}\right) + \frac{\left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{b}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}\right)^2}{2} + \cdots + \frac{\left(\sum_{\mathbb{u}\in\mathbb{Z}^d_{\ge 0}} \mathbb{b}_{\mathbb{u}} \mathbb{x}^{\mathbb{u}}\right)^p}{p!} + \Omega(\mathbb{x}^{p+1}) \\
&\sim_p \exp\left(\frac{\phi_i'}{\phi_0'}(\mathbb{x})\right),
\end{aligned}$$

which gives us that $\exp\left(\frac{\phi_i}{\phi_0}\right) = \exp\left(\frac{\phi_i'}{\phi_0'}\right)$, as required. □

In particular, we can find $\psi_i$ up to precision $p$ without having to worry about the accuracy of anything more than the first $p$ coefficients, and any product of $\psi_i$ for some collection of $i \in I$ is also accurate to precision $p$.

## 4.2 Method

To translate the calculation of the mirror map of an arbitrary reflexive polytope into a computational one, we will use several useful tools built into Sagemath (see [16]). Multiplication by inverse power series can be truncated to a polynomial of certain precision to look only at the relevant terms. Although Sagemath is capable of dividing one polynomial by another, to maximize computational efficiency we decide instead to calculate and multiply by the inverse of a polynomial. As $|\mathbf{z}| < 1$, we can express the inverse as a geometric series, and calculate only the coefficients of $\mathbb{z}^{\mathbb{u}}$ for $\mathbb{u} < (p, p, \ldots, p)$. If our polynomial $p$ has constant coefficient $a_0$ and $P(z) - a_0 = r(z)$, then we calculate

$$P^{-1}(z) = \sum_{k=0}^{p+1} a_0^{-k+1} r^k(z).$$

Similarly, the exponential of a power series $P$ was calculated up to precision $p$ with `exp prec` by finding

$$\sum_{k=0}^{p+1} \frac{P(z)^k}{k!}.$$

The function `A list` takes a list of lattice points and finds and appends all lattice points on codimension 2 faces of the polytope given by these points. A brief check is conducted to make sure each lattice point in the output of `A list` only appears once. To find the kernel of a matrix of vertices bounded below by the axes and above by the hyperplane $|u| = p$ (where $p$ is the chosen precision), we used the `Polyhedron` function from Sagemath. We input first the inequalities specifying that each $x_i$ in a given vector $\mathbb{x}$ should be greater than or equal to zero and should sum to less than $p$. We then input the equalities specifying that a given $\mathbb{x}$ should be in the kernel of the matrix of vertices. The `Polyhedron` function finds all vectors satisfying these constraints.

Putting all of these together, we make the function which computes the mirror maps $\psi_i^{A(\Delta)}$ of a given polytope $\Delta$. It takes a natural number as the precision and a list of vertices as the polyhedron's defining points. We use the `PolynomialRing` function to generate $e_m$ many variables and allow them to be composed and to have rational coefficients. The code finds $\phi_0$ and $\phi_i$ for $0 < i \leq e_m$, and subsequently calculates $\psi_i$ for $0 < i \leq e_m$. The function then checks, for each $\psi_i$, whether there are any noninteger or negative coefficients. The function, if all coefficients are positive integers, returns the list of coefficients with the tag `true`. Otherwise, it returns the list of coefficients with the tag `false`. To test the integrality and positivity of mirror maps, we then downloaded the data from `PALPreader` to extract reflexive polytopes of dimensions 2 and 3.

## 4.3 Computational ease

The required computations were computationally heavy and therefore took a substantial amount of time to test the conjecture on more than 20 polytopes at a time. The mirror maps of reflexive polytopes with kernel of low rank had fewer lattice points to cycle through in the computation and were calculated to a higher precision of 30. Those of higher rank have more lattice vertices in the kernel, so required much more time and memory, and were therefore calculated to a lower precision of 3. The computation was conducted using CoCalc. This used up to 96 CPUs and 16GB RAM.

# Chapter 5

# Results

In this chapter, we will discuss the results obtained through the course of running the code discussed in the previous chapter. We found that all reflexive polytopes of dimension 2 and 3 are positive and integral up to all precisions calculated.

Let us write the vector

$$(p_1, p_2, \ldots, p_{22}) = (30, 30, 30, 30, 30, 20, 10, 9, 8, 7, 6, 5, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3).$$

We calculated the mirror map power series of $\Delta$ up to precision $p_k$ when $\Delta$ had kernel $K_{\geq 0}$ of rank $k$, where 22 is the highest possible rank for the kernel of a 2- or 3-dimensional polytope. Up to these precisions, it was found that all of the mirror map power series for reflexive polytopes of dimension 2 and 3 were both positive and integral, giving persuasive evidence for the truth of Conjecture D.

## 5.1 The hexagon mirror map

The mirror map of the reflexive polytope shown in Example 1.4, which we called the hexagon, gives six power series, the first of which we calculate to precision 9 below.

$$\begin{aligned}
&4757z_1^3z_2^3z_3^3 + 1392z_0^3z_1z_2z_3^4 + 13726z_0^2z_1^2z_2z_3^3z_4 + 21836z_0z_1^3z_2z_3^2z_4^2+\\
&6708z_1^4z_2z_3z_4^3 + 13726z_0^2z_1z_2^2z_3^3z_5 + 67069z_0z_1^2z_2^2z_3^2z_4z_5 + 992z_0^4z_3^3z_4z_5+\\
&41364z_1^3z_2^2z_3z_4^2z_5 + 9614z_0^3z_1z_3^2z_4^2z_5 + 14150z_0^2z_1^2z_3z_4^3z_5 + 3528z_0z_1^3z_4^4z_5+\\
&21836z_0z_1z_2^3z_3^2z_5^2 + 41364z_1^2z_2^3z_3z_4z_5^2 + 9614z_0^3z_2z_3^2z_4z_5^2 + 43804z_0^2z_1z_2z_3z_4^2z_5^2+\\
&22314z_0z_1^2z_2z_4^3z_5^2 + 6708z_1z_2^4z_3z_5^3 + 14150z_0^2z_2^2z_3z_4z_5^3 + 22314z_0z_1z_2^2z_4^2z_5^3+\\
&1188z_0^3z_4^3z_5^3 + 3528z_0z_2^3z_4z_5^4 + 2287z_0z_1^2z_2^2z_3^3 + 14z_0^4z_3^4+\\
&4457z_1^3z_2^2z_3^2z_4 + 561z_0^3z_1z_3^3z_4 + 2063z_0^2z_1^2z_3^2z_4^2 + 1438z_0z_1^3z_3z_4^3+\\
&163z_1^4z_4^4 + 4457z_1^2z_2^3z_3^2z_5 + 561z_0^3z_2z_3^3z_5 + 8623z_0^2z_1z_2z_3^2z_4z_5+\\
&13876z_0z_1^2z_2z_3z_4^2z_5 + 2830z_1^3z_2z_4^3z_5 + 2063z_0^2z_2^2z_3^2z_5^2 + 13876z_0z_1z_2^2z_3z_4z_5^2+\\
&6488z_1^2z_2^2z_4^2z_5^2 + 1022z_0^3z_3z_4^2z_5^2 + 1520z_0^2z_1z_4^3z_5^2 + 1438z_0z_2^3z_3z_5^3+\\
&2830z_1z_2^3z_4z_5^3 + 1520z_0^2z_2z_4^2z_5^3 + 163z_2^4z_5^4 + 297z_0^2z_1z_2z_3^3+\\
&1578z_0z_1^2z_2z_3^2z_4 + 1022z_1^3z_2z_3z_4^2 + 1578z_0z_1z_2^2z_3^2z_5 + 3126z_1^2z_2^2z_3z_4z_5+\\
&199z_0^3z_3^2z_4z_5 + 996z_0^2z_1z_3z_4^2z_5 + 524z_0z_1^2z_4^3z_5 + 1022z_1z_2^3z_3z_5^2+\\
&996z_0^2z_2z_3z_4z_5^2 + 1632z_0z_1z_2z_4^2z_5^2 + 524z_0z_2^2z_4z_5^3 + 215z_1^2z_2^2z_3^2 + 5z_0^3z_3^3+\\
&114z_0^2z_1z_3^2z_4 + 199z_0z_1^2z_3z_4^2 + 42z_1^3z_4^3 + 114z_0^2z_2z_3^2z_5+\\
&842z_0z_1z_2z_3z_4z_5 + 409z_1^2z_2z_4^2z_5 + 199z_0z_2^2z_3z_5^2 + 409z_1z_2^2z_4z_5^2+\\
&68z_0^2z_4^2z_5^2 + 42z_2^3z_5^3 + 60z_0z_1z_2z_3^2 + 130z_1^2z_2z_3z_4+\\
&130z_1z_2^2z_3z_5 + 36z_0^2z_3z_4z_5 + 64z_0z_1z_4^2z_5 + 64z_0z_2z_4z_5^2+\\
&2z_0^2z_3^2 + 21z_0z_1z_3z_4 + 11z_1^2z_4^2 + 21z_0z_2z_3z_5+\\
&47z_1z_2z_4z_5 + 11z_2^2z_5^2 + 11z_1z_2z_3 + 5z_0z_4z_5+\\
&z_0z_3 + 3z_1z_4 + 3z_2z_5 + 1
\end{aligned}$$

We see that the coefficients are indeed positive integers.

## 5.2 Sanity checks

Our theorem is proved for reflexive polytopes, but we can see how the output polynomial compares with polytopes that are not reflexive.

A polytope which is certainly not reflexive (as it is a lattice polytope with more than one interior lattice point) is that given by the convex hull of points $(0,1)$, $(1,0)$ and $(-10,-10)$. We can see with precision 25 that the three polynomials given by the code are positive but not integral.

$$\frac{166770367}{60}z_0^{10}z_1^{10}z_2 + 1, \quad \frac{166770367}{60}z_0^{10}z_1^{10}z_2 + 1, \quad \frac{41054655}{4}z_0^{10}z_1^{10}z_2 + 1$$

However, an interesting observation is that the polytope given by points $(1,0,2)$, $(1,1,2)$, $(2,1,2)$, $(-2,-1,-3)$ is not reflexive and yet has all positive integer coefficients up to a very high precision. Indeed, it has dual polytope given by points $(5,0,-3)$, $(-5,5,2)$, $(0,0,-1/2)$, $(0,-5,2))$, so that the dual is not even a lattice polytope, let alone one with single interior lattice point. However, applying the mirror map computation with precision 100 to this polytope outputs four polynomials, the first of which is given by the following.

$$
\begin{aligned}
1125410719756898661902098812967081289682681109609910474 7z_0^{20}z_1^{20}z_2^{20}z_3^{40}+\\
1554028517704684453426180328134729630518762982755686 7z_0^{19}z_1^{19}z_2^{19}z_3^{38}+\\
21545738647604568159094115041942904075725869886632z_0^{18}z_1^{18}z_2^{18}z_3^{36}+\\
300065266395167646260045916122007789063141205 90z_0^{17}z_1^{17}z_2^{17}z_3^{34}+\\
42000533743306900996258558183936452869240346z_0^{16}z_1^{16}z_2^{16}z_3^{32}+\\
591229023049651869212555694045669542896 68z_0^{15}z_1^{15}z_2^{15}z_3^{30}+\\
8376294882530921375819629698278900973 2z_0^{14}z_1^{14}z_2^{14}z_3^{28}+\\
1195499937923004716418398675030553 88z_0^{13}z_1^{13}z_2^{13}z_3^{2}6+\\
17208782056973315865051067758092 9z_0^{12}z_1^{12}z_2^{12}z_3^{2}4+\\
250197664435871540062715035215z_0^{11}z_1^{11}z_2^{11}z_3^{2}2+\\
3680900340226032710447661 08z_0^{10}z_1^{10}z_2^{10}z_3^{2}0+\\
5493072649279411757709 46z_0^{9}z_1^{9}z_2^{9}z_3^{1}8+\\
2068906461812132z_0^{6}z_1^{6}z_2^{6}z_3^{12}+\\
3434661338134z_0^{5}z_1^{5}z_2^{5}z_3^{10}+\\
6022714119z_0^{4}z_1^{4}z_2^{4}z_3^{8}+\\
11486973z_0^{3}z_1^{3}z_2^{3}z_3^{6}+\\
25352z_0^{2}z_1^{2}z_2^{2}z_3^{4}+\\
77z_0z_1z_2z_3^{2}+\\
1
\end{aligned}
$$

Clearly, the power series is positive and integral to a very high precision. Although we do not pursue this further in this project, the reader is encouraged to do so if they are intrigued by the result.

To conclude this chapter, we can see that the evidence suggests that Conjecture D holds for all reflexive polytopes of dimension 2 and 3. Furthermore, there may be evidence to suggest that the mirror map is positive integral also on lattice polytopes with only one interior point, as shown in the last example. This result is surprising and leaves space for future study.

# Notation

| Symbol/expression | Meaning |
|---|---|
| $\mathbb{m}$ | a vector $(m_1, \ldots, m_d) \in \mathbb{R}^d$ for some $d$ |
| $\mathbb{m} + \mathbb{n}$ | the pointwise sum $(m_1 + n_1, \dot{;} m_d + n_d)$ |
| $\lambda\mathbb{m}$ | the scalar multiple $(\lambda m_1, \ldots, \lambda m_d)$ |
| $\mathbb{m}^{(k)}$ | the $k$th entry of $\mathbb{m}$ |
| $\mathbb{m} \geq \mathbb{n}$ | inequality $m_i \geq n_i$ in every entry $1 \leq i \leq d$ |
| $\mathbb{m} \cdot \mathbb{n}$ | the scalar product $(m_1 n_1, \ldots, m_d n_d)$ |
| $\lvert\mathbb{m}\rvert$ | the sum of entries $m_1 + \cdots + m_d$ |
| $\mathbf{e}$ | a sequence of vectors $(\mathbb{e}^{(1)}, \mathbb{e}^{(2)}, \ldots, \mathbb{e}^{(q_1)})$ |
| $Q_{\mathbf{e},\mathbf{f}}(\mathbb{n})$ | the combinations function $\frac{(\mathbb{e}_1 \cdot \mathbb{n})! \ldots (\mathbb{e}_{q_1} \cdot \mathbb{n})!}{(\mathbb{f}_1 \cdot \mathbb{n})! \ldots (\mathbb{f}_{q_1} \cdot \mathbb{n})!}$ |
| $F_{\mathbf{e},\mathbf{f}}(\mathbb{z})$ | $\sum_{\mathbb{n} \geq 0} \frac{(\mathbb{e}_1 \cdot \mathbb{n})! \ldots (\mathbb{e}_{q_1} \cdot \mathbb{n})!}{(\mathbb{f}_1 \cdot \mathbb{n})! \ldots (\mathbb{f}_{q_1} \cdot \mathbb{n})!} \mathbb{z}^{\mathbb{n}}$ |
| $G_{e,f,k}(\mathbb{z})$ | $\sum_{\mathbb{n} \geq 0} \frac{(\mathbb{e}_1 \cdot \mathbb{n})! \ldots (\mathbb{e}_{q_1} \cdot \mathbb{n})!}{(\mathbb{f}_1 \cdot \mathbb{n})! \ldots (\mathbb{f}_{q_1} \cdot \mathbb{n})!} \left( \sum_{i=1}^{q_1} \mathbb{e}_i^{(k)} H_{\mathbb{e}_i \cdot \mathbb{n}} - \sum_{i=1}^{q_2} \mathbb{f}_i^{(k)} H_{\mathbb{f}_i \cdot \mathbb{n}} \right) \mathbb{z}^{\mathbb{n}}$ |
| $q_{\mathbf{e},\mathbf{f},k}(\mathbb{z})$ | the mirror map coordinates $z_k \exp(G_{\mathbf{e},\mathbf{f},k}(\mathbb{z}) / F_{\mathbf{e},\mathbf{f}}(\mathbb{z}))$ |
| $\Delta_{\mathbf{e},\mathbf{f}}(\mathbb{z})$ | Landau's function, $\sum_{i=1}^{q_1} \lfloor \mathbb{e}_i \cdot \mathbb{z} \rfloor - \sum_{i=1}^{q_2} \lfloor \mathbb{f}_i \cdot \mathbb{z} \rfloor$ |
| $D_{\mathbf{e},\mathbf{f}}$ | the semi-algebraic set $\mathbf{x} \in [0,1)^d$ s.t. $\exists \mathbb{b} \in \mathbf{e} \cup \mathbf{f}$ where $\mathbb{b} \cdot \mathbf{x} \geq 1$ |
| $\phi_0^{A(\Delta)}(\mathbb{z})$ | $\sum_{\mathbb{u} \in K_{\geq 0}} \mathrm{comb}(\mathbb{u}) \mathbb{z}^{\mathbb{u}}$ |
| $\phi_i^{A(\Delta)}(\mathbb{z})$ | $\sum_{\mathbb{u} \in K_{\geq 0}} \mathrm{comb}(\mathbb{u}) \cdot (H(u_1 + \cdots + u_e) - H(u_i)) \mathbb{z}^{\mathbb{u}}$ |
| $\psi_i^{A(\Delta)}(\mathbb{z})$ | the mirror map component $\exp(\phi_i / \phi_0)$ |
| $\mathrm{lcm}(\alpha_j)$ | lowest common multiple over $\alpha_j$ for $j \in J$ some indexing set |
| $\Psi_{\mathbb{k}}^{A(\Delta)}(\mathbb{z})$ | the mirror map $\prod_{i=1}^{e} \psi_i^{\mathbb{k}^{(i)}}$ |
| $\Omega(z^n)$ | sum of terms of $z$ with power greater than $n$ |

# Bibliography


[1] Paul S. Aspinwall, Brian R. Greene, and David R. Morrison. "Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory". In: *Nuclear Physics B* 416.2 (Mar. 1994), pp. 414–480. ISSN: 0550-3213. DOI: 10.1016/0550-3213(94)90321-2. URL: http://dx.doi.org/10.1016/0550-3213(94)90321-2.

[2] Victor V Batyrev and Duco Van Straten. "Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties". In: *Communications in mathematical physics* 168 (1995), pp. 493–533.

[3] P. Candelas et al. "A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory". In: *Nuclear Physics B* 359 (1 1991).

[4] David A Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*. Vol. 68. American Mathematical Society Providence, RI, 1999.

[5] Eric Delaygue. "A criterion for the integrality of the Taylor coefficients of mirror maps in several variables". In: *Advances in Mathematics* 234 (2013), pp. 414–452.

[6] Adams Douglas. *The Hitchhiker's Guide to the Galaxy*. Harmony Books, 1980.

[7] Sheel Ganatra et al. *Integrality of mirror maps and arithmetic homological mirror symmetry for Greene–Plesser mirrors*. 2024. arXiv: 2312.01949 [math.SG].

[8] Christian Haase and Ilarion V Melnikov. "The reflexive dimension of a lattice polytope". In: *Annals of Combinatorics* 10 (2006), pp. 211–217.

[9] Christian Haase, Benjamin Nill, and Andreas Paffenholz. "Lecture notes on lattice polytopes". In: *Fall School on Polyhedral Combinatorics* (2012).

[10] Martin Henk, Jürgen Richter-Gebert, and Günter M Ziegler. "Basic properties of convex polytopes". In: *Handbook of discrete and computational geometry*. Chapman and Hall/CRC, 2017, pp. 383–413.

[11] Th Kaluza. "Über die koeffizienten reziproker potenzreihen". In: *Mathematische Zeitschrift* 28.1 (1928), pp. 161–170.

[12] Christian Krattenthaler and Tanguy Rivoal. *Analytic properties of mirror maps*. 2011. arXiv: 1102.5375 [math.CA].

[13] Christian Krattenthaler and Tanguy Rivoal. “Multivariate $p$-adic formal congruences and integrality of Taylor coefficients of mirror maps”. In: *arXiv preprint arXiv:0804.3049* (2008).

[14] Maximilian Kreuzer and Harald Skarke. “Classification of reflexive polyhedra in three dimensions”. In: *arXiv preprint hep-th/9805190* (1998).

[15] D. R. Morrison. “Mirror Symmetry and Rational Curves on Quintic Threefolds: A Guide for Mathematicians”. In: *Journal of the American Mathematical Society* 6 (1993), pp. 223–247.

[16] W. A. Stein et al. *Sage Mathematics Software*. `http://www.sagemath.org`. The Sage Development Team. 2024.

[17] Shing-Tung Yau. *Mirror Symmetry I*. 1998.

# Appendix A

# Sagemath code

To check that all reflexive polytopes of two dimensions have positive integer coefficients, we must calculate 16 mirror maps. The prospect of doing this by hand is undesirable, yet at least somewhat attainable. To calculate mirror maps of all 3-dimensional reflexive polytopes, we must find 4319 of these. The decision was made to use computational assistance.

The first cell of code defines the harmonic function

$$\mathrm{Harm}(n) = \sum_{i=1}^{n} \frac{1}{i},$$

the combinations function

$$\mathrm{comb}(u_1, u_2, \ldots, u_n) = \frac{(u_1 + u_2 + \cdots + u_n)!}{u_1! u_2! \ldots u_n!},$$

the power function

$$\mathbf{z}^{\mathbf{u}} = z_1^{u_1} z_2^{u_2} \ldots z_n^{u_n},$$

and the $i$th harmonic function

$$H(u_1 + u_2 + \cdots + u_n) - H(i) = \sum_{j=1}^{u_1+\cdots+u_n} \frac{1}{j} - \sum_{j=1}^{i} \frac{1}{j}.$$

```
def Harm(naturalNumber):

    '''
    Takes a natural number.

    Outputs the harmonic function on that number.
    '''

    harm = sum([1/n for n in range(1,naturalNumber+1)])

    return harm


def comb(listOfU):

```

```
    '''
    Takes a vector.

    Outputs the combinations function on that vector.
    '''

    sumOfU = sum([i for i in listOfU])

    prodOfUFact = prod([factorial(i) for i in listOfU])

    comb = factorial(sumOfU) / prodOfUFact

    return ZZ(comb)


def power(listZ, listU):

    '''
    Takes two lists of the same length, where listZ is a list of variables and listU is a list of numbers.

    Outputs the product of each entry of listZ to the power of the corresponding entry of listU.
    '''

    return prod([listZ[i]**listU[i] for i in range(0,len(listU))])


def harm_i(listU, i):

    '''
    Takes a list of numbers which sum to H and a natural number less H.

    Outputs the subtraction of the ith harmonic function from the Hth harmonic function.
    '''

    H=0

    for u in listU:

        H+=u

    return Harm(H)-Harm(listU[i])
```

Listing A.1: The basic functions

The second cell of code formulates versions of multiplication and inversion up to the given precision, so as not to expend any unnecessary energy on the calculation of higher precision terms. The truncate function removes all terms higher than a given precision. Using this, the polynomial multiplication function multiplies two polynomials up to a certain precision and forgets all higher-precision terms. Similarly, the polynomial power function multiplies the polynomial by itself some number of times, forgetting higher-precision terms with each iteration.

The inverse function finds the inverse of a polynomial up to the $p$th term of its geometric series. The exponential function finds the exponential of a polynomial using the first $p$ terms of its Taylor expansion.

```
def truncate(precision, P):

    '''
    Takes a natural number and a polynomial.

    Outputs this polynomial truncated to total degree p = precision.
    '''

    trunc = {dd: c for dd, c in P.dict().items() if sum(dd) <= precision}

    return P.parent()(trunc)


def polynomial_multiplication(precision, poly1, poly2):

    return truncate(precision, poly1 * poly2)

def polynomial_power(precision, polynomial, power):

    poly = 1

    for i in range(power):

        poly = polynomial_multiplication(precision, poly, polynomial)

    return poly


def inv(precision, polynomial):

    '''
    Takes a natural number and a polynomial.

    Outputs the inverse of the polynomial to the given precision.
    '''

    a_0 = polynomial.constant_coefficient()

    r = a_0 - polynomial

    inverse = 0

    for k in range(precision+1):

        inverse += a_0^(-k+1)*r^k

    return inverse

```

```
def exp_prec(precision, polynomial):

    '''
    Takes a natural number and a polynomial.

    Outputs the exponential of the polynomial up to the given precision.
    '''

    return sum(polynomial_power(precision, polynomial,k)/factorial(k) for k in range(0,precision+1))
```

Listing A.2: Functions to adjust to precision

The function `A list` takes a list of lattice points and finds whether the edges formed by taking the polyhedron on these points have any other lattice points intersecting them. This finds all lattice points on the polyhedron occurring on faces of codimension 2.

```
def A_list(polyhedron):

    '''
    Takes a list of lattice points as vertices of a polyhedron.

    Outputs all lattice points on codimension 2 faces of the polytope
    '''
    L=[]

    for face in polyhedron.faces(codim=2):

        L = L+list(face.points())

    seen = []

    for e in L:

        if e in seen:

            continue

        seen.append(e)

    return seen
```

Listing A.3: Function which finds all lattice points on codimension 2 faces of the polytope

The function `K` finds the kernel of the matrix of vertices, such that no vertex $\mathbb{u}$ has sum $|\mathbb{u}| \geq p$. This is to say it is bounded above by the hyperplane $x_1 + x_2 + \cdots + x_e = p$.

```
def K(precision, vertices_matrix):

    '''
    Takes a natural number and a list of lattice points in the form of a matrix.
```

```

    Outputs all lattice points in the kernel of that matrix with sum of coordinates less than the given precision.
    '''

    e_m = len(vertices_matrix)

    d_m = len(vertices_matrix[0])

    # Create a list L1 which has e vectors with e+1 entries, where the first one is 0 and subsequent ones are all 0 except for a single 1.

    L1=[zero_vector(QQ, e_m+1)]

    for i in range(e_m+1):

        vec = zero_vector(QQ, e_m+1)

        vec[i]=1

        L1.append(vec)

    # Append the vector (precision,-1,...,-1) to L1, corresponding to the inequality \sum x_i \le precision for terms of degree at most precision

    L1.append([precision]+[-1 for i in range(len(L1)-2)])

    # Create a list L2 corresponding to the equalities imposed by being in K.

    L2=[]

    for col in range(d_m):

        L2.append([0]+[vert[col] for vert in vertices_matrix])


    # The Polyhedron function generates the kernel from this information.

    P = Polyhedron(ieqs=L1, eqns=L2)

    return P.integral_points()
```

Listing A.4: Function to find the kernel of a given polytope

The function `mirror map of polytope` takes a polyhedron, finds the polynomials `phi 0` and `phi(j)`, representing the two power series

$$\phi_0^{A(\Delta)}(\mathbb{z}) := \sum_{\mathbb{u}\in K_{\geq 0}} \mathrm{comb}(\mathbb{u})\mathbb{z}^{\mathbb{u}},$$

and

$$\phi_i^{A(\Delta)}(\mathbb{z}) := \sum_{\mathbb{u}\in K_{\geq 0}} \mathrm{comb}(\mathbb{u}) \cdot (H(u_1 + \dots + u_e) - H(u_i))\mathbb{z}^{\mathbb{u}},$$

respectively, and uses them to output `psi list`, the list of coefficients of

$$\psi_i^{A(\Delta)}(\mathbf{z}) := \exp\left(\frac{\phi_i}{\phi_0}\right) \in \mathbb{Q}[[z]].$$

Using this, the function determines whether there are any negative or noninteger coefficients, and otherwise indicates that all coefficients are nonnegative integers.

```

def mirror_map_of_polytope(precision, polyhedron):

    '''
    Takes a precision level, a list of vertices (lists) as a
   matrix and a basis for its kernel.

    Finds the Taylor polynomial associated with the mirror map of
     that polytope up to the given precision, and outputs True if
   all coefficients are positive integers and False otherwise.
    '''

    # Find all lattice points in faces of the polyhedron of
   codimension 2

    vertices_matrix = A_list(polyhedron)

    # d is the dimension of the space the vertices live in

    d_m = len(vertices_matrix[1])

    # e is the number of vertices

    e_m = len(vertices_matrix)



    # Make the power series ring in e-many variables

    R2 = PolynomialRing(QQ,e_m,'z')

    # Give names to the variables

    z = R2.gens()

    # Find the kernel of the map

    ker = K(precision, vertices_matrix)


    # Define phi_0 with the use of the comb and delta function

    phi_0 = sum(comb(M)*prod(z[i]^M[i] for i in range(e_m)) for M
     in ker)


    # Define phi_j

```

```
    def phi(j):

        return sum(comb(M)*harm_i(M,j)*prod(z[i]^M[i] for i in range(e_m)) for M in ker)


    # Make a list of coefficients of psi

    psi_list = []

    for i in range(0,e_m):

        phi_i_by_0 = polynomial_multiplication(precision, phi(i), inv(precision, phi_0))

        psi_list.append(exp_prec(precision, phi_i_by_0).coefficients())


    # Check the coefficients are positive and integral

    for i in range(len(psi_list)):

        print(f"psi_{i} has coefficients", psi_list[i], 'up to precision', precision)

        l = []

        for i in range(len(psi_list)):

            for j in range(len(psi_list[i])):

                if not(psi_list[i][j] in ZZ):

                    print('Non-integer coefficient!')

                    return False

                if psi_list[i][j] < 0:

                    print('Negative coefficient!')

                    return False

    return True
```

Listing A.5: Mirror map function

We access the PALPreader library and call the function which retrieves all of the 2-dimensional reflexive polytopes. Our code then checks that the mirror map on each of these polytopes is positive and integral.

```
from sage.geometry.polyhedron.palp_database import PALPreader

# Download all 2-dim'l reflexive polytopes

polytopes_2 = PALPreader(2, output='list')
```

```


# Use all CPUs available using the parallel function

@parallel(96)

def check_conjecture(i):

    '''
    Takes a natural number.

    Outputs True if all coefficients up to the specified
   precision are positive integers, and False otherwise.
    '''

    # Give a range of precisions, so that the lower-rank
   polytopes are calculated to higher precision.

    precisions
   =[30,30,30,30,30,20,10,9,8,7,6,5,4,4,3,3,3,3,3,3,3,3,3]

    return mirror_map_of_polytope(precisions[len(A_list(
   LatticePolytope(polytopes_2[i])))],LatticePolytope(polytopes_2
   [i]))

output = check_conjecture([i for i in range(len(list(polytopes_2)
   ))])

# Check for all reflexive polytopes

output = check_conjecture([i for i in range(16)])

for o in list(output):

    if o[1]==False:

        print(o)

print('Done!')
```

Listing A.6: Imports all 2-dimensional reflexive polytopes and checks the conjecture holds to the specified precision

We use the PALPreader library to call the function which retrieves all of the reflexive polytopes of dimension 3. As before, our code then checks that the mirror map on each of these polytopes is positive and integral.

```
# Download all 3-dim'l reflexive polytopes

polytopes_3 = PALPreader(3,output='list')


# Use all CPUs available using the parallel function

@parallel(96)

def check_conjecture(i):
```

```
    '''
    Takes a natural number.

    Outputs True if all coefficients up to the specified precision are positive integers, and False otherwise.
    '''

    # Give a range of precisions, so that the lower-rank polytopes are calculated to higher precision.

    precisions =[30,30,30,30,30,20,10,9,8,7,6,5,4,4,3,3,3,3,3,3,3,3,3]

    return mirror_map_of_polytope(precisions[len(A_list(LatticePolytope(polytopes_3[i])))],LatticePolytope(polytopes_3[i]))

# Check for all reflexive polytopes

output = check_conjecture([i for i in range(4319)])

for o in list(output):

    if o[1]==False:

        print(o)

print('Done!')
```

Listing A.7: Imports all 3-dimensional reflexive polytopes and checks the conjecture holds to the specified precision