Sampling, reconstruction of signals and periodic signals demo

Outline today

- Introduction to sound and speech signals
- Analog and discrete signals
- Sampling, Nyquist and reconstruction
- Periodic basis functions
- Simple convolution filters
- Analysis of frequency content (power spectrum)

Next week: Acquire audio signals, analysis of pitch and understanding of the spectrogram

DTU Compute
Department of Applied Mathematics and Computer Science

Frequency vibrating string

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

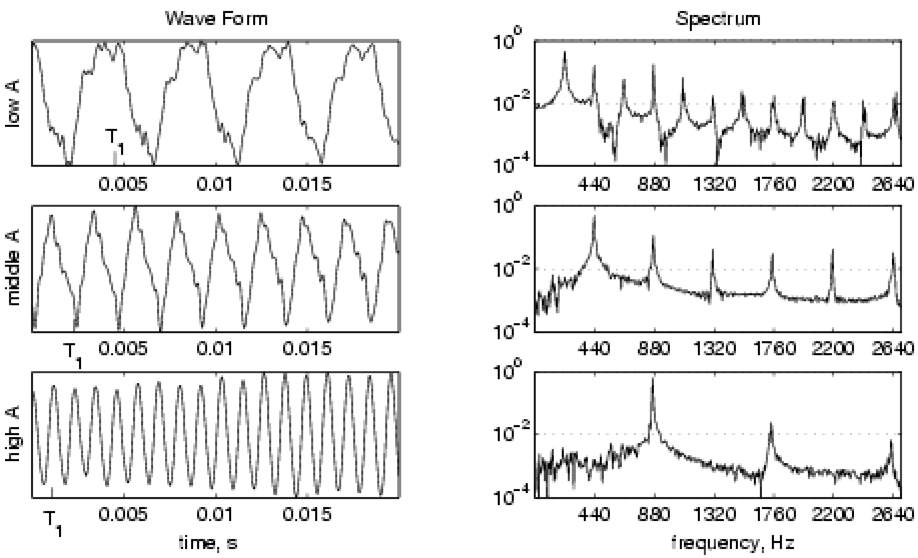
T tension,

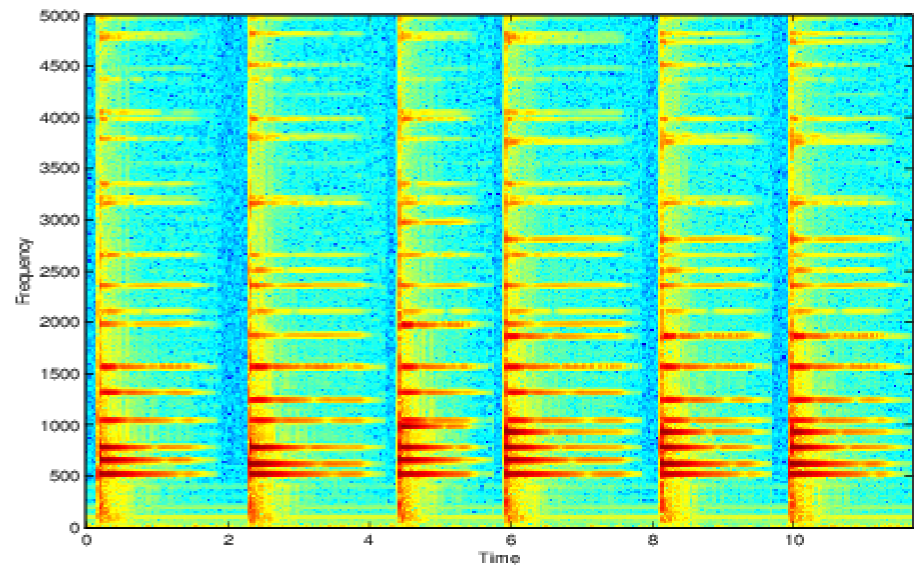
mu mass/length

L length

"Tones"

- Single instruments
- Vibrating strings.. piano

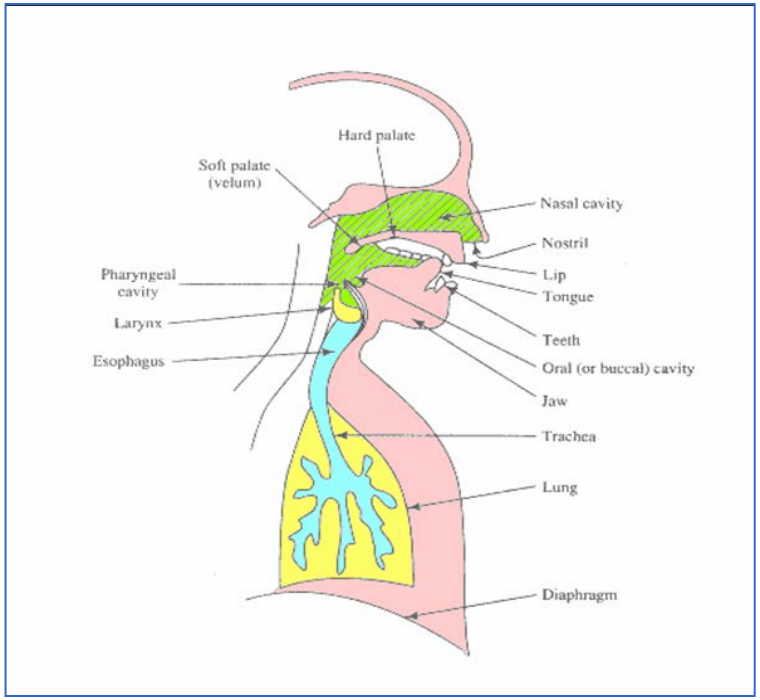




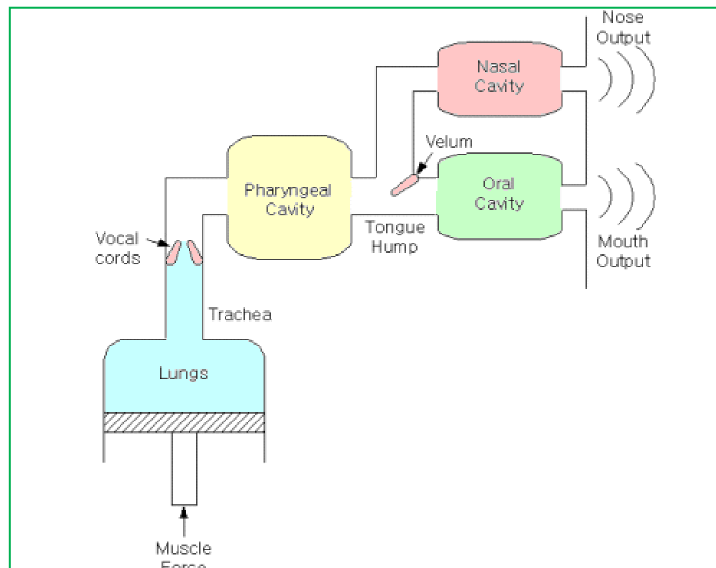
Sound: Speech signals

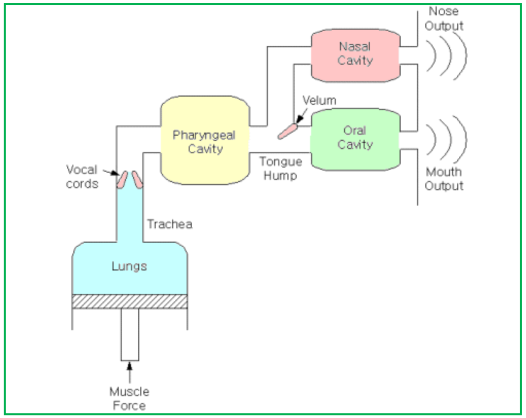
- Information in speech
- Phonetics: Voiced and unvoiced speech
- Periodic components – pitch and harmonics
- Unvoiced – the ‘s’ sound

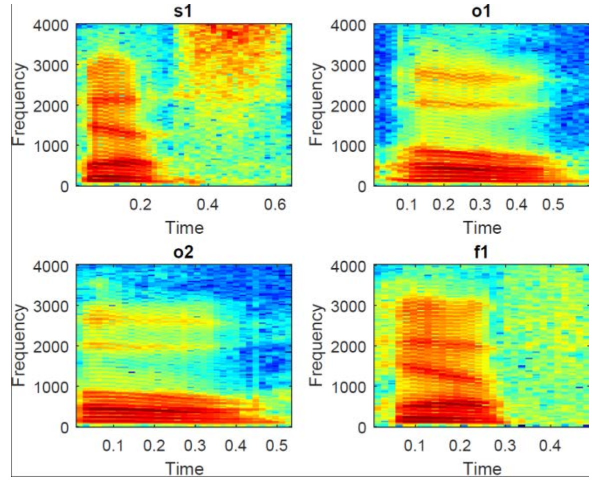
Speech production

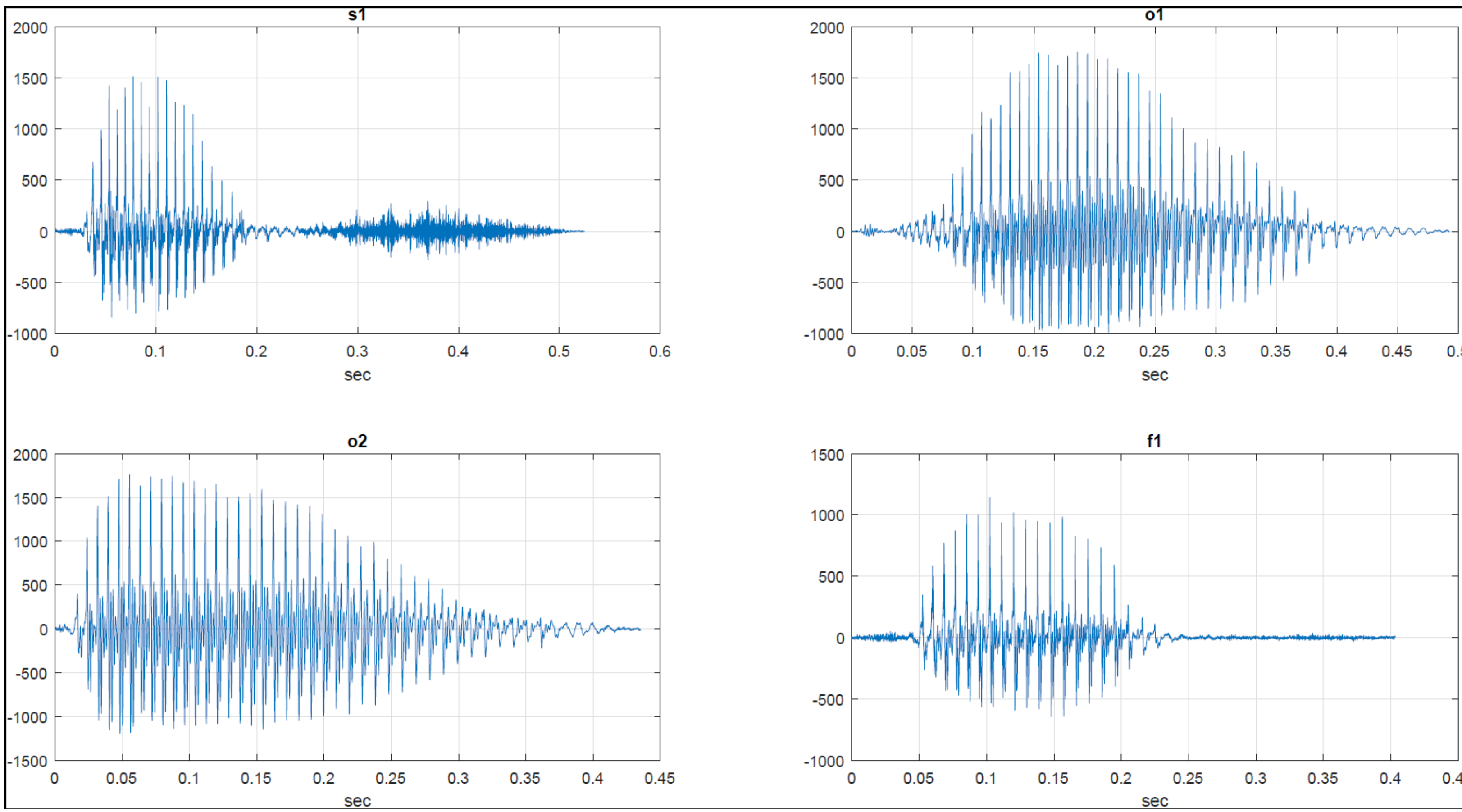


https://www.youtube.com/watch?v=uTOhDqhCKQs

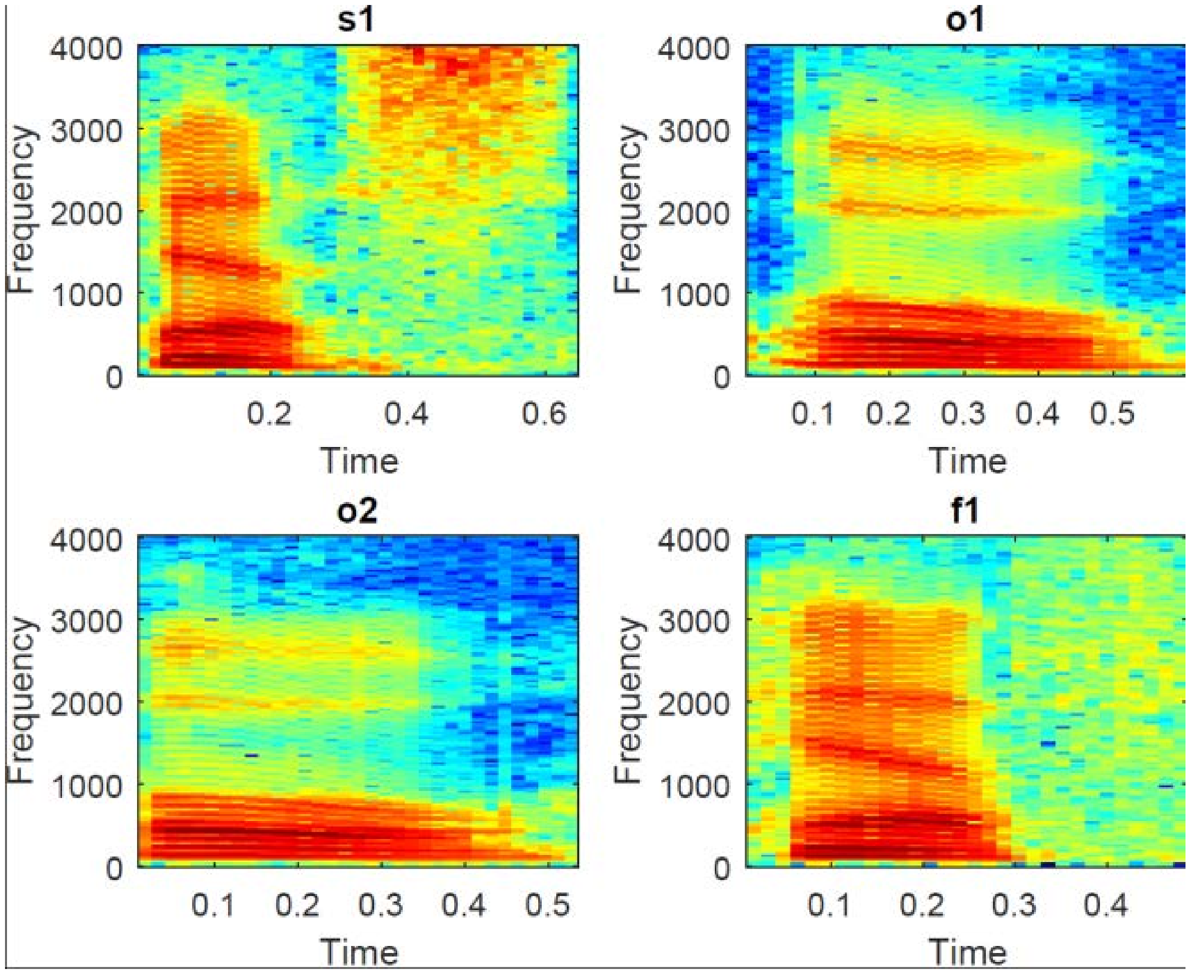



Nose Output
Nasal Cavity
Velum
Pharyngeal Cavity
Oral Cavity
Tongue Hump
Mouth Output
Vocal cords
Trachea
Lungs
Muscle Force


s1
o1
o2
f1
Frequency
Time


s1
o1
o2
f1
sec

Power distribution over time and frequency



Session 1

- Analog (continuous time) and discrete (sampled time) signals
- Maximum sampled frequency - Nyquist
- Reconstruction by interpolation

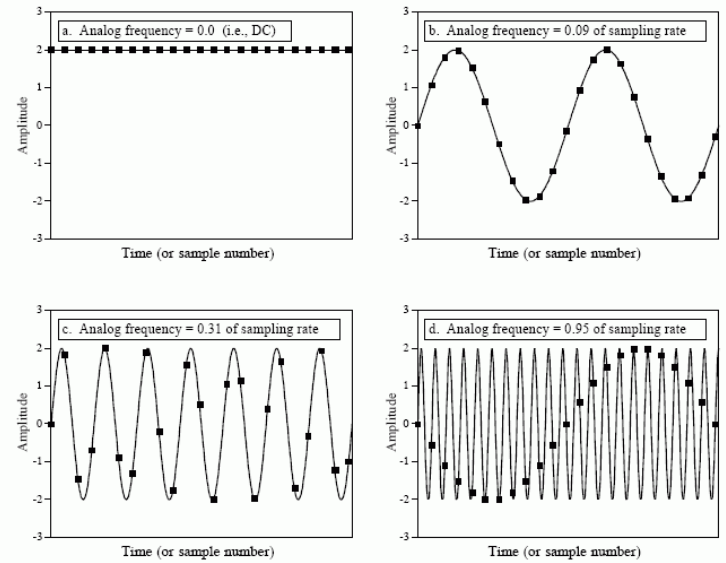


FIGURE 3-3
Illustration of proper and improper sampling. A continuous signal is sampled *properly* if the samples contain all the information needed to recreate the original waveform. Figures (a), (b), and (c) illustrate *proper sampling* of three sinusoidal waves. This is certainly not obvious, since the samples in (c) do not even appear to capture the shape of the waveform. Nevertheless, each of these continuous signals forms a unique one-to-one pair with its pattern of samples. This guarantees that reconstruction can take place. In (d), the frequency of the analog sine wave is greater than the Nyquist frequency (one-half of the sampling rate). This results in *aliasing*, where the frequency of the sampled data is different from the frequency of the continuous signal. Since aliasing has corrupted the information, the original signal cannot be reconstructed from the samples.

Sampling and reconstruction

- Let an $f(t)$ be an 'analog' signal - ie. a real value function of time
- Let $x_j = f(t_0 + jT)$ be a discrete signal sampled at time points separated by T and indexed by $j \in \mathbb{Z}$
- 'Sampling theorem':

If $f(t)$ is a band-limited signal with no energy above the half sampling frequency $\frac{F_s}{2} = \frac{1}{2T}$ then the function can be exactly reconstructed from the interpolation $\hat{f}(t) = \sum_{j=-\infty}^{\infty} x_j \mathrm{sinc}(\frac{t-jT}{T}) = f(t)$.

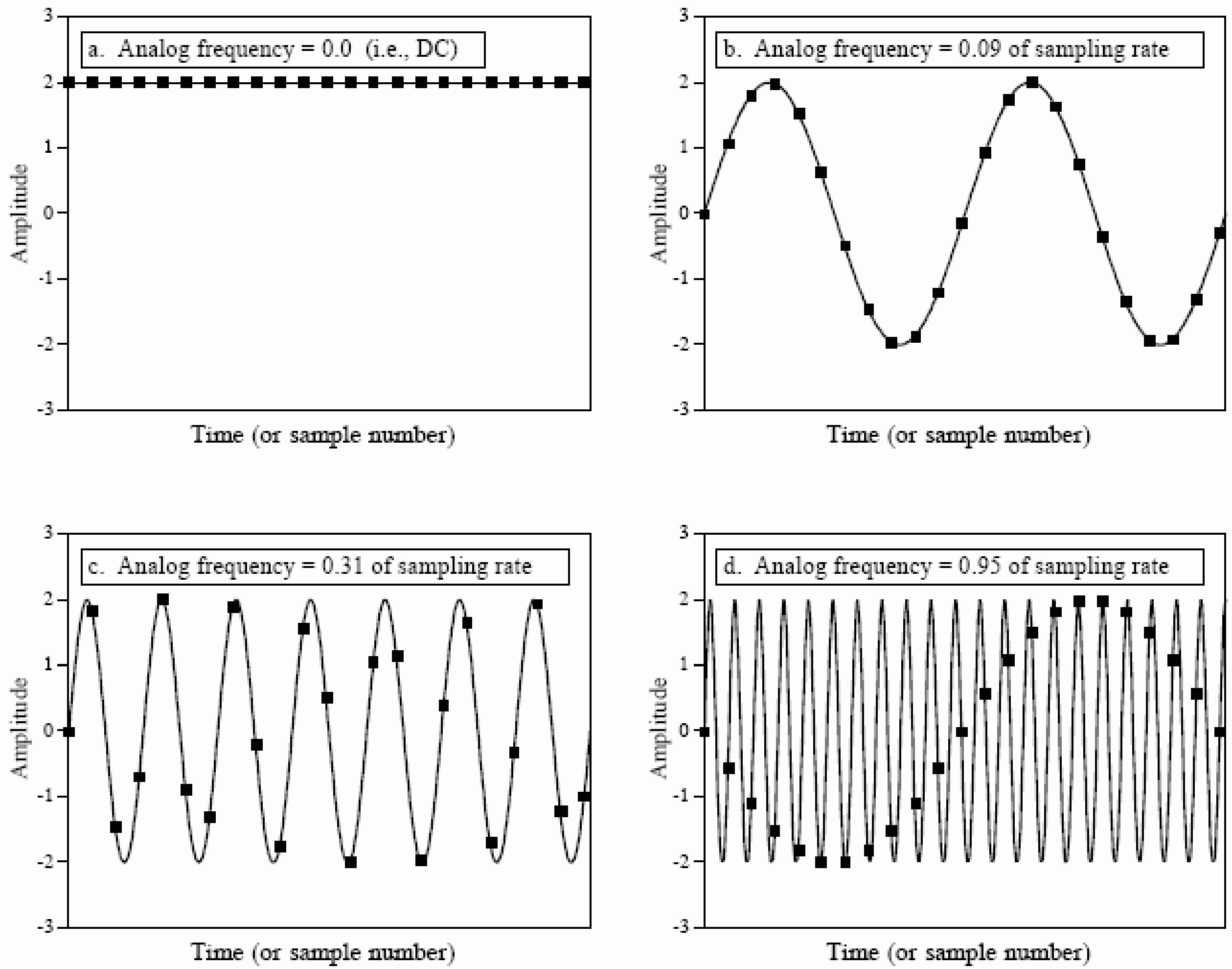


FIGURE 3-3
Illustration of proper and improper sampling. A continuous signal is sampled *properly* if the samples contain all the information needed to recreate the original waveform. Figures (a), (b), and (c) illustrate *proper sampling* of three sinusoidal waves. This is certainly not obvious, since the samples in (c) do not even appear to capture the shape of the waveform. Nevertheless, each of these continuous signals forms a unique one-to-one pair with its pattern of samples. This guarantees that reconstruction can take place. In (d), the frequency of the analog sine wave is greater than the Nyquist frequency (one-half of the sampling rate). This results in *aliasing*, where the frequency of the sampled data is different from the frequency of the continuous signal. Since aliasing has corrupted the information, the original signal cannot be reconstructed from the samples.

http://www.dspguide.com/ch3/2.htm

Sampling – The Nyquist rate

A simple analog signal is the harmonic oscillation:

$$x_a(t) = A\sin(2\pi F\, t + \theta)$$

The frequency F is measured in cycles per second (hertz).
The discrete signal obtained with a sampling rate $F_s = \frac{1}{T}$ is:

$$x_d(n) = A\sin\left(2\pi\frac{F}{F_s}n + \theta\right)$$

Note, that if the frequency increases $F' = F + kF_s$ for some integer k, the discrete signal becomes:

$$x_d'(n) = A\sin\left(2\pi\frac{F + kF_s}{F_s}n + \theta\right) = A\sin\left(2\pi\frac{F}{F_s}n + \theta\right)$$

Such higher frequencies, F', are called aliases of the frequence F as $x_d'(n) = x_d(n)$.
The Shannon-Nyquist sampling theorem states that the continuous signal can only be properly sampled when the sampling frequency is at least twice the frequency of the signal, ie. $F_s > 2F$ (The Nyquist rate).

Session 2

Periodic basis functions for sampled signals

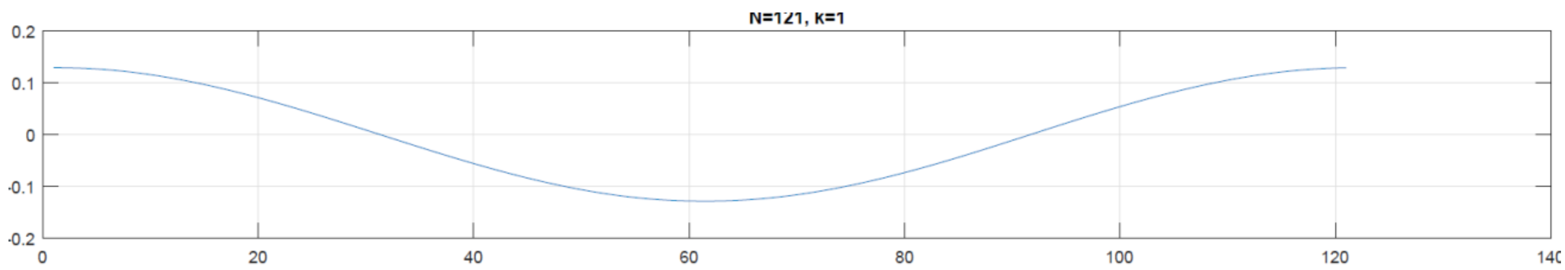
Periodic functions as basis vectors

Let us consider signals sampled on the interval $n \in [0, N-1]$.
The fundamental frequency functions

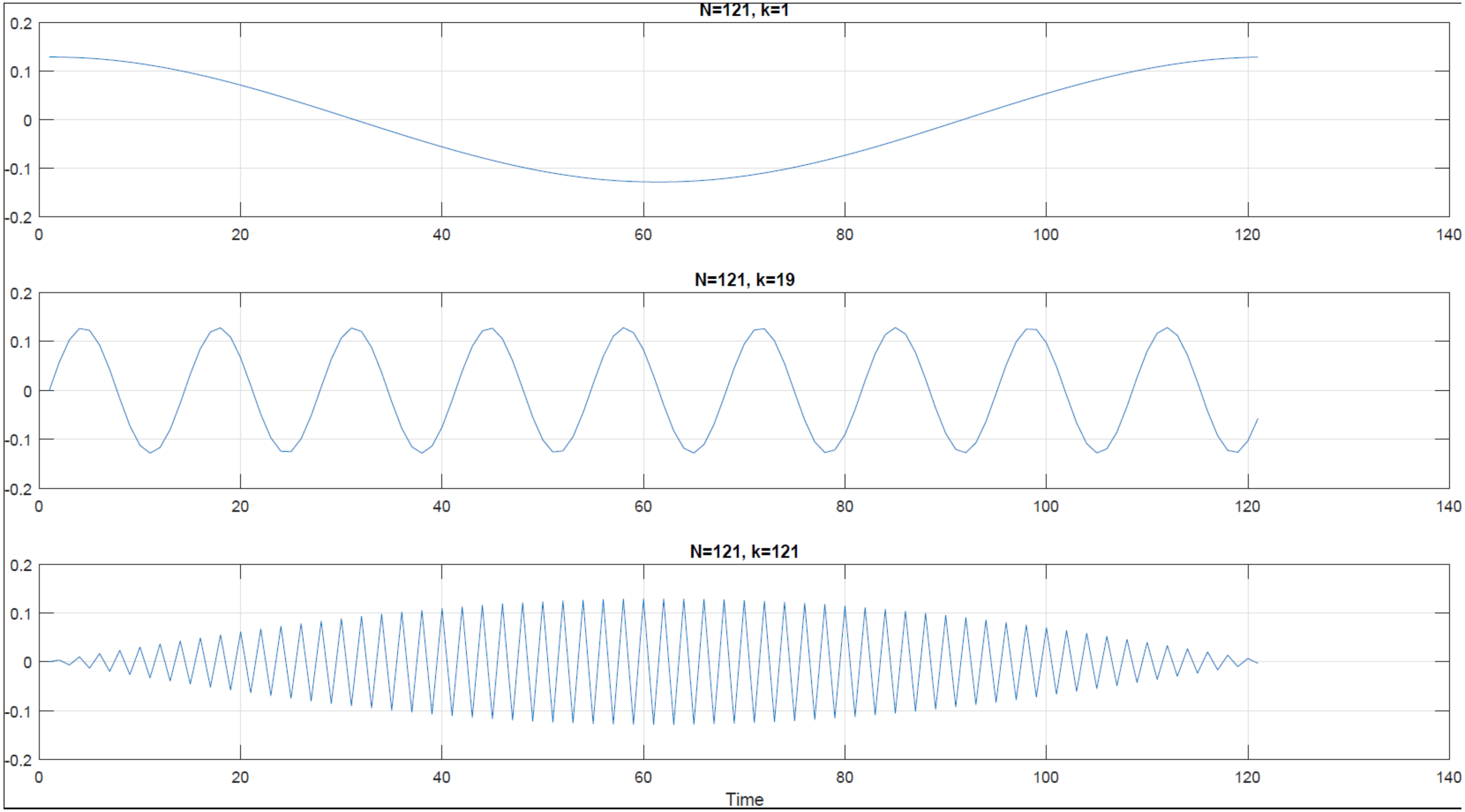
$$x_1(n) = A \sin\left(\frac{2\pi}{N} n\right)$$

$$y_1(n) = B \cos\left(\frac{2\pi}{N} n\right)$$

are periodic with $x(n+N) = x(n),\ y(n+N) = y(n)$



Periodic functions as basis vectors



Periodic functions as basis vectors

The 'higher' harmonics are given by

$$x_k(n) = A \sin\left(\frac{2\pi k}{N} n\right)$$

$$y_k(n) = B \cos\left(\frac{2\pi k}{N} n\right)$$

are periodic with $x(n + \frac{N}{k}) = x(n)$, $y(n + \frac{N}{k}) = y(n)$, i.e. their frequencies are k times the higher that the fundamental frequency.

Periodic functions as basis vectors

What is the highest frequency harmonic, i.e., the largest k?
For N even

$$y_{N/2}(n) = B\cos\left(\frac{2\pi\frac{N}{2}}{N}n\right) = B\cos(\frac{2\pi\frac{N}{2}}{N}n = (-1)^n B$$

For N odd

$$x_{(N-1)/2}(n) = A\sin\left(\frac{2\pi\frac{N-1}{2}}{N}n\right) = A\sin(\frac{2\pi\frac{N-1}{2}}{N}n = (-1)^n A\sin\left(\frac{\pi n}{N}\right)$$

$$y_{(N-1)/2}(n) = B\cos\left(\frac{2\pi\frac{N-1}{2}}{N}n\right) = B\cos(\frac{2\pi\frac{N-1}{2}}{N}n = (-1)^n B\cos\left(\frac{\pi n}{N}\right)$$

Periodic functions as basis vectors

The range of k's is $k = [0, N/2]$ for even N and $k = [0, (N-1)/2]$ for odd. With $k = 0$ being the constant function

$$x_{(N-1)/2}(n) = B\cos\left(\frac{2\pi 0}{N}n\right) = B$$

So in total we have precisely N discrete sampled harmonic functions or vectors if they are visualized as points in $\mathbb{R}^N$.

$$(\mathbf{x}_k)_n = x_k(n)$$

Periodic functions as basis vectors

In the exercise we will show that harmonics are mutually orthogonal, so when properly normalized and numbered from 0 to $N-1$

$$\mathbf{u}_{2k} = \mathbf{x}_k / ||\mathbf{x}_k||$$

$$\mathbf{u}_{2k+1} = \mathbf{y}_k / ||\mathbf{y}_k||$$

we have a complete orthonormal basis set!
Hence, we can form a *basis matrix* $\mathbf{U} = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \quad \mathbf{u}_N)$ where $||\mathbf{u}|| = 1$

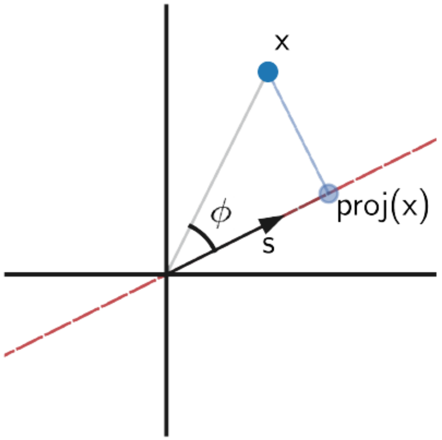
Periodic functions as basis vectors

- Given the periodic function basis we can decompose the "energy" in the sample vector $(\mathbf{x})_j = x_j$, with individual contributions given by the projections

$$\mathbf{z}_m = \left(\mathbf{U}^\top \mathbf{x}\right)_m = \mathbf{u}_m^\top \mathbf{x} = \sum_{j=0}^{N-1} \cos\left(\frac{\pi m j}{N}\right) x(j)$$

$$\sum_k \mathbf{z}_k^2 = \sum_k \left(\mathbf{U}^\top \mathbf{x}\right)_k^2 = \left(\mathbf{U}^\top \mathbf{x}\right)^\top \mathbf{U}^\top \mathbf{x} = \mathbf{x}^\top \mathbf{U}\mathbf{U}^\top \mathbf{x} = \|\mathbf{x}\|^2$$

- Where we remembered $\mathbf{U}\mathbf{U}^\top = \mathbf{U}^\top \mathbf{U} = \mathbf{I}$



Analysis of random signals and filters

- Simplest random signal is the identical, independent distributed (iid.) signal $x_j \sim \mathcal{N}(0, \sigma^2)$

$$\mathrm{cov}(x(j), x(j')) = \mathbb{E}\{x_j x_{j'}\} = \sigma^2 \quad \mathbf{\Sigma} = \mathbb{E}\{\mathbf{x}\mathbf{x}^\top\} = \sigma^2 \mathbf{I}$$

- We can project the iid. signal on the basis $\mathbf{z} = \mathbf{U}^\top \mathbf{x}$

$$\mathrm{cov}(z(k), z(k')) = \mathbb{E}\{z_j z_{j'}\}$$

$$\mathbb{E}\{\mathbf{U}^\top \mathbf{x} \left(\mathbf{U}^\top \mathbf{x}\right)^\top\} = \mathbf{U}^\top \mathbb{E}\{\mathbf{x}\mathbf{x}^\top\}\mathbf{U} = \sigma^2 \mathbf{U}^\top \mathbf{I}\mathbf{U} = \sigma^2 \mathbf{I}$$

- Conclusion: the iid. random signal has evenly distributed energy over the set of harmonics

Analysis of random signals and filters

- We can construct a simple random periodic signal using the basis

$$x(n) = a\cos\left(\frac{2\pi mn}{N}\right) + b\sin\left(\frac{2\pi mn}{N}\right) = c\cos\left(\frac{2\pi mn}{N} + \phi\right)$$

Where the amplitude is $c = \sqrt{a^2 + b^2}$ and the phase is given by $\tan\phi = \frac{a}{b}$

- Let $a_m, \_m \sim \mathcal{N}(0, \sigma^2)$ to create a signal of frequency $F = \frac{m}{N}$, and with random amplitude and phase

The signal can be written in matrix notation

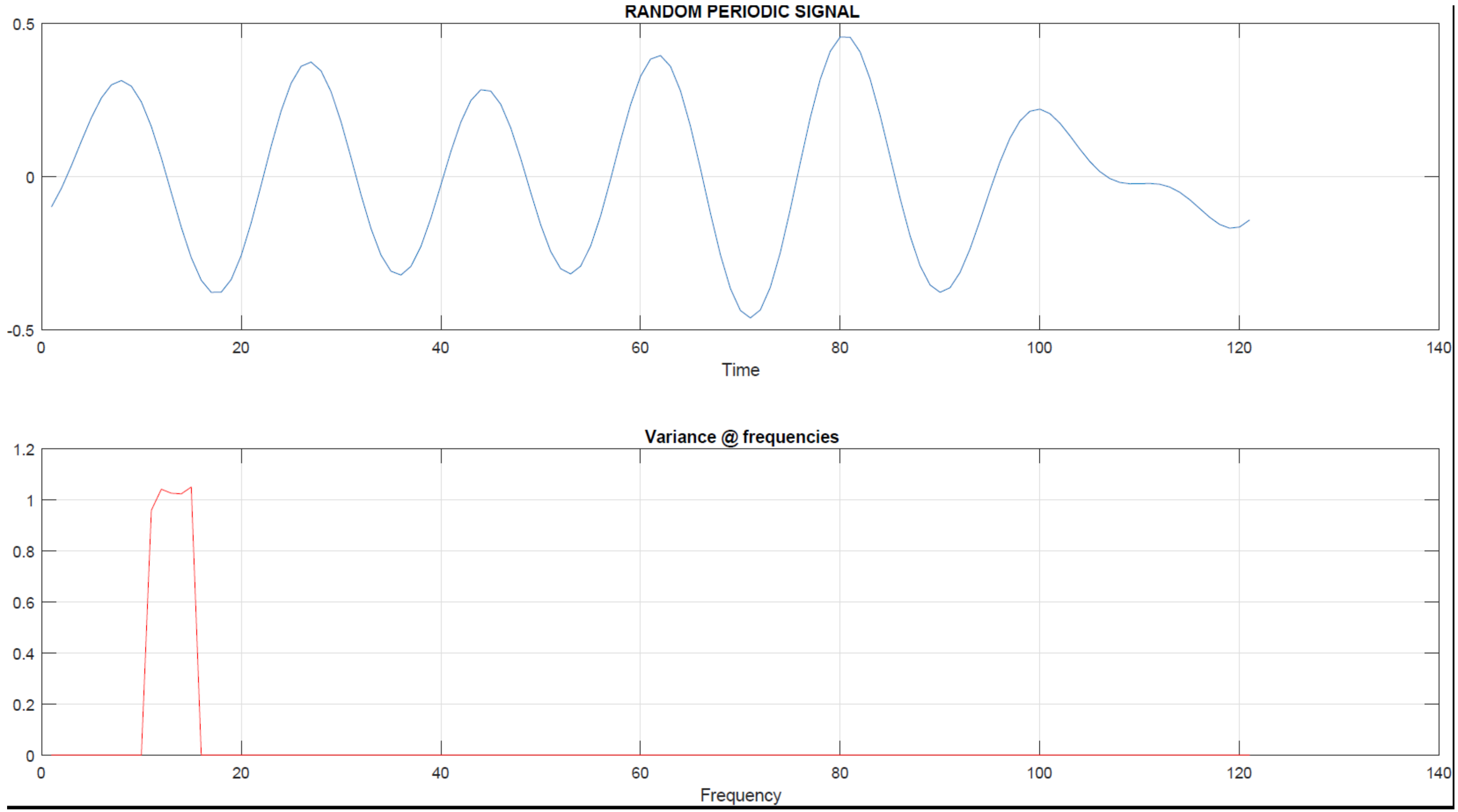
$$\mathbf{x} = a\mathbf{u}_m + b\mathbf{u}_{m+1}$$

Hence if we project on the basis

$$\mathbf{z} = \mathbf{U}^\top(a\mathbf{u}_m + b\mathbf{u}_{m+1}) = (0, 0, 0, 0, a, b, 0, 0 \ldots 0)^\top$$

The energy of the signal is confined - as expected - to the two basis functions at the given frequency.

Random periodic signal



Filters - circular convolutions

- Define the filter or 'circular convolution'

Local weighted average or difference

$$w(m) = h_{-1}x\left(\mathrm{mod}(m-1,N)\right) + h_0 x\left(\mathrm{mod}(m,N)\right) + h_1 x\left(\mathrm{mod}(m+1,N)\right)$$

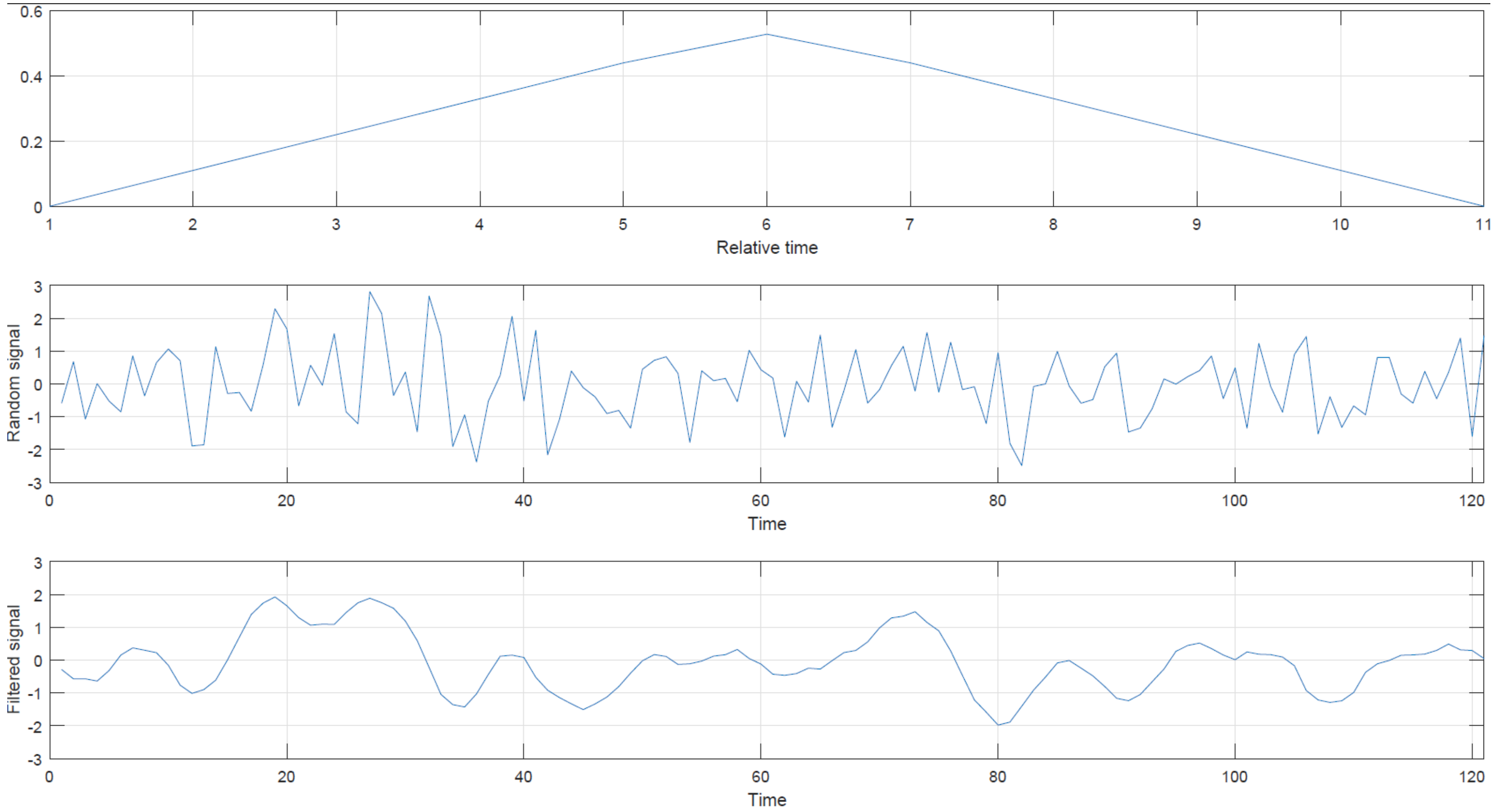
$$w(m) = \sum_{t=-L}^{L} h_t x\left(\mathrm{mod}(m+t,N)\right) \equiv \sum_{j=1}^{N} \tilde{h}(\mathrm{mod}(m-j,N))x(j) \equiv \sum_{j=1}^{N} H_{m,j}x(j)$$

Where the matrix H is banded

$$\mathbf{w} = \mathbf{H}\mathbf{x}$$

- With the 'convolution' we can enhance different parts of the spectrum. If the convolution is "local averaging" we damped high frequencies and vice versa if the convolution is "local differentiation", it will enhance high frequencies frequencies ... examples in the exercise

Local averaging convolution (filter)



Local averaging convolution (filter)