The Distortion of Cardinal Preferences in Voting

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Jeffrey S. Rosenschein
Hebrew University of Jerusalem

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Ariel D. Procaccia and Jeffrey S. Rosenschein

School of Engineering and Computer Science
Hebrew University, Jerusalem, Israel
{arielpro, jeff}@cs.huji.ac.il

Abstract. The theoretical guarantees provided by voting have distinguished it as a prominent method of preference aggregation among autonomous agents. However, unlike humans, agents usually assign each candidate an exact utility, whereas an election is resolved based solely on each voter's linear ordering of candidates. In essence, the agents' cardinal (utility-based) preferences are embedded into the space of ordinal preferences. This often gives rise to a *distortion* in the preferences, and hence in the social welfare of the outcome.

In this paper, we formally define and analyze the concept of distortion. We fully characterize the distortion under different restrictions imposed on agents' cardinal preferences; both possibility and strong impossibility results are established. We also tackle some computational aspects of calculating the distortion. Ultimately, we argue that, whenever voting is applied in a multiagent system, distortion must be a pivotal consideration.

1 Introduction

Social choice mechanisms have long been in the service of computer-scientists, a tool in the quest to reach consensus among agents. The problem is especially acute, as multiple heterogeneous, self-interested agents may (and often do) have conflicting preferences. Voting is a well-studied and well-understood method of preference aggregation, with numerous applications in multiagent systems. In practice, an election is held, and the winning candidate is declared to be the agreed choice; the candidates can be beliefs, joint plans [7], schedules [9], movies [8], or indeed entities of almost any conceivable sort.

A *social choice function*, also known as a *voting protocol*, is used to determine the winner of an election. The agents specify their preferences by reporting a linear order relation on the candidates. Such *ordinal* preferences are only natural when the voters are humans; a human might prefer, say, Ehud Olmert to Benjamin Netanyahu as the prime minister of Israel, but would probably find it impossible to evaluate each candidate precisely in terms of utility.

For computational agents, on the other hand, calculating utilities is a way of (artificial) life. In fact, even in settings where voting is used, it is usually assumed that agents compute the utility of each alternative. For instance, Ghosh et al. [8] describe a movie recommender system that relies on voting; with the guarantees provided by voting schemes, the system is able to generate convincing explanations for different recommendations, and is robust to small errors in the evaluation of the user's preferences. Aspects of these preferences are represented as dimensions, and every movie has

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a value (or utility) with respect to each dimension. The exact utilities are not taken into account: one movie is preferred over another with respect to a dimension if the former's utility is greater than the latter's.

So in some settings, designers of multiagent systems do away with exact *cardinal* (utility-based) preferences in order to exploit different properties of voting. Essentially, the cardinal preferences of agents are embedded into the space of ordinal preferences over candidates, in a way somewhat reminiscent of embeddings of metric spaces [10]. This embedding of preferences entails a degree of *distortion*, which depends on the properties of the social choice function used in the election.

Informally, we define the distortion of a social choice function to be the maximal ratio between the total utility of the candidate that maximizes social welfare, and the total utility of the candidate that is elected. The maximum is taken over all possible cardinal preference profiles, subject to certain restrictions.

We first explore distortion when the only restriction imposed on cardinal preferences is that all voters have the same sum of utilities for candidates. We establish some strong impossibility results regarding the degree of distortion in this model. Further, we show that these results also hold in an alternative model, where utilities are not constrained, but weighted voting is used. Another impossibility result is computational in nature: we prove that a decision problem associated with the computation of distortion is $\mathcal{NP}$-hard.[1]

The impossibility results mentioned above suggest that distortion is an obstacle that should be taken into account when applying voting in multiagent systems. Nevertheless, they motivate us to examine a model where the preferences of users are more restricted; in this context, we reformulate distortion as *misrepresentation*. We examine the misrepresentation of different well-known social choice functions. In addition, we analyze complexity issues related to calculating misrepresentation.

The paper proceeds as follows. In Section 2 we review some relevant issues in social choice theory. In Section 3, we put forward results concerning the distortion of social choice functions in models where preferences are little constrained. In Section 4, we examine the more specific setting of misrepresentation, especially with respect to important social choice functions. Finally, we give our conclusions in Section 5.

2 Preliminaries

In this section we give a brief introduction to classic social choice theory. Readers are urged to consult [3] for more information.

Let N be the set of voters, $|N| = n$, and let C be the set of candidates, $|C| = m$; we assume that $n \geq 2$ and $m \geq 3$, unless explicitly stated otherwise. We usually use the index i to refer to voters, and the index j to refer to candidates. When we discuss attributes of voters or candidates, the index of a voter usually appears in superscript, whereas the index of a candidate appears in subscript.

Let $\mathcal{L}$ be the set of all linear orders[2] on C. Each voter has ordinal preferences $\succ^i \in \mathcal{L}$. We refer to $\succ = \langle \succ^1, \ldots, \succ^n \rangle \in \mathcal{L}^N$ as an *ordinal preference profile*.

[1] Many recent articles have explored other computational aspects of voting; see for example [5,6,12,2].

[2] Binary relations that satisfy antisymmetry, transitivity, and totality.

Given $\succ^i$, let $j_1, \ldots, j_m$ be indices of candidates such that $j_1 \succ^i j_2 \succ^i \cdots \succ^i j_m$; we denote by p^i_l the candidate that voter i ranks in the l'th place, i.e., $p^i_l = j_l$. We denote by l^i_j the position in which candidate j is ranked by voter i; it holds that $p^i_{l^i_j} = j$.

2.1 Social Choice Functions

A *social choice function*, also known as a *voting protocol*,[3] is a function $F : \mathcal{L}^N \to C$, i.e., a mapping from preferences of voters to candidates. We shall consider the following voting protocols:

- *Scoring protocols* are defined by a vector $\boldsymbol{\alpha} = \langle \alpha_1, \ldots, \alpha_m \rangle$.[4] Given $\succ \in \mathcal{L}^N$, the score of candidate j is $s_j = \sum_i \alpha_{l^i_j}$. The candidate who wins the election is $F(\succ) = \text{argmax}_j s_j$. Some of the well-known scoring protocols are:
  - *Borda*: $\boldsymbol{\alpha} = \langle m-1, m-2, \ldots, 0 \rangle$.
  - *Plurality*: $\boldsymbol{\alpha} = \langle 1, 0, \ldots, 0 \rangle$.
  - *Veto*: $\boldsymbol{\alpha} = \langle 1, \ldots, 1, 0 \rangle$.

  Some of our results (in particular regarding complexity) concentrate on scoring protocols, as these voting protocols can be concisely represented by the vector $\boldsymbol{\alpha}$.
- *Copeland*: we say that candidate j beats j' in a pairwise election if $|\{i \in N : l^i_j < l^i_{j'}\}| > n/2$. The score s_j of candidate j is the number of candidates that j beats in pairwise elections, and Copeland$(\succ) = \text{argmax}_j s_j$.
- *Maximin*: the maximin score of candidate j is the candidate's worst performance in a pairwise election: $s_j = \min_{j'} |\{i \in N : l^i_j < l^i_{j'}\}|$, and Maximin$(\succ) = \text{argmax}_j s_j$.
- *Single Transferable Vote* (STV): the election proceeds in rounds (a total of $m-1$ rounds); in each round, the candidate with the fewest votes ranking him first among the remaining candidates is eliminated.
- *Plurality with Runoff*: similar to STV, but there are only two rounds. After the first round, only the two candidates that maximize $|\{i \in N : l^i_j = 1\}|$ survive. In the second round, a pairwise election is held between these two candidates.
- *Bucklin*: for any candidate j and $l \in \{1, \ldots, m\}$, let $B_{j,l} = \{i \in N : l^i_j \leq l\}$. It holds that Bucklin$(\succ) = \text{argmin}_j(\min\{l : |B_{j,l}| > n/2\})$.

It is also to possible to consider weighted voting. A voter i with weight K and preferences $\succ^i$ is taken into account as K voters, each with preferences $\succ^i$.

2.2 Properties of Social Choice Functions

In this subsection we formulate several criteria that are commonly used to compare social choice functions.

- Majority criterion: $[\exists j \in C \; s.t. \; |\{i \in N : l^i_j = 1\}| > n/2] \Rightarrow F(\succ) = j$.
- Participation: if $F(\succ) = j$ and one adds a ballot that ranks j above j', then the winner is not j' (it is better to vote honestly than not to vote at all).

[3] We use the two terms interchangeably.

[4] More formally, a scoring protocol is defined by a sequence of such vectors, one for each value of m, but we abandon this formulation for clarity's sake.

- Monotonicity: If $F(\succ) = j$, and $\succ'$ is an ordinal preference profile where some of the voters rank j higher compared to $\succ$ (but none rank j lower), then $F(\succ') = j$.
- Consistency: if the electorate is partitioned in two and a candidate wins in both parts, then he wins overall.

3 Distortion of General Cardinal Preferences

Let $\mathcal{U} = (\mathbb{N} \cup \{0\})^C$ be the set of all possible cardinal preferences on C. Each voter is associated with preferences in $\mathcal{U}$, $\boldsymbol{u}^i = \langle u_1^i, \ldots, u_j^i \rangle$, where $u_j^i \in \mathbb{N} \cup \{0\}$ is voter i's utility for candidate j; denote $u_j = \sum_i u_j^i$, and denote the *cardinal preference profile* by $\boldsymbol{u} = \langle \boldsymbol{u}^1, \ldots, \boldsymbol{u}^n \rangle \in \mathcal{U}^N$.

When voting is used to aggregate preferences, agents' cardinal preferences are translated into ordinal preferences in the natural way.

Definition 1. *Let $\boldsymbol{u} \in \mathcal{U}^N$ and $\succ \in \mathcal{L}^N$. $\succ$ is* derived *from u iff both of the following conditions hold:*

1. $\forall i \in N,\ j_1, j_2 \in C:\ u_{j_1}^i > u_{j_2}^i \Rightarrow j_1 \succ^i j_2$.
2. $\forall i \in N,\ j_1, j_2 \in C:\ u_{j_1}^i = u_{j_2}^i \Rightarrow j_1 \succ^i j_2 \vee j_2 \succ^i j_1$, *but not both.*

Definition 2. *Let F be a social choice function. The* distortion *of F with n voters and m candidates, denoted $\Delta_m^n(F)$, is $\max \frac{\max_j u_j}{u_{F(\succ)}}$, where the first maximum is taken over all $\boldsymbol{u} \in \mathcal{U}^N$ and $\succ \in \mathcal{L}^N$, under the restrictions that there exists $K \in \mathbb{N}$ such that for all voters i, $\sum_j u_j^i = K \geq 1$, and $\succ$ is derived from u. It is additionally assumed that in case several candidates are tied in the election, the one that minimizes social welfare is elected.*[5] *If the denominator is 0 but the numerator is not 0, we write $\Delta_m^n(F) = \infty$, where $\infty > k$ for all $k \in \mathbb{N}$.*

Less formally, the distortion of F with n voters and m candidates is the worst-case ratio between the utility of the candidate that maximizes social welfare and the winner according to F, when one considers all possible cardinal preference profiles u with fixed utility-sum for each voter, and derived ordinal preference profiles $\succ$.

Remark 1. Clearly, when one eschews the assumption that $\sum_j u_j^i = K$ for all i, it is not possible to bound the distortion even when a small number of voters and candidates is considered. For example, assume $n = 3$ and $m = 2$, and F is the plurality protocol. Let $u_1^1 = c$ for some $c > 2$, $u_2^1 = 0$, $u_1^2 = u_1^3 = 0$, $u_2^2 = u_2^3 = 1$. The derived ordinal preference profile is $1 \succ^1 2$, $2 \succ^2 1$, $2 \succ^3 1$, therefore candidate 2 is chosen by the plurality protocol. The distortion is $c/2$.

The following proposition is a strong impossibility result; it implies that no voting protocol is optimal in terms of distortion, even for very small values of n and m.

Proposition 1. *Let F be a social choice function. Then $\Delta_2^3(F) > 1$.*

[5] This assumption is justified as we engage here in a *worst-case* analysis.

Proof. Consider the cardinal utility profile $u_1^1 = 3$, $u_2^1 = 2$, $u_1^2 = 0$, $u_2^2 = 5$, $u_1^3 = 3$, $u_2^3 = 2$. The only derived ordinal preference profile is $1 \succ^1 2$, $2 \succ^2 1$, $1 \succ^3 2$. Since $u_1 < u_2$, if $F(\succ) = 1$ then we are done. Otherwise, suppose $F(\succ) = 2$, and consider the cardinal preference profile $u_1^1 = 5$, $u_2^1 = 0$, $u_1^2 = 0$, $u_2^2 = 5$, $u_1^3 = 5$, $u_2^3 = 0$. Again, the only derived preference profile is $\succ$, but now $u_1 > u_2$. □

Definition 3. *Let F be a social choice function. We say that F* has unbounded distortion *if there exists $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $\Delta_m^n(F) > k$ for infinitely many values of n.*

Proposition 2. *Let F be a scoring protocol with*

$$\alpha_2 \geq \frac{1}{m-1} \sum_{l \neq 2} \alpha_l \tag{1}$$

for some m. Then F has unbounded distortion.

Proof. Let n such that $m-1$ divides n. Consider the profile $\boldsymbol{u} \in \mathcal{U}^N$ where for every candidate $j \neq 1$, exactly $n/(m-1)$ voters i have utility $u_j^i = 1$ and $u_{j'}^i = 0$ for every $j' \neq j$. Let $\succ$ be a derived ordinal preference profile; define for all

$$P_{j,l} = \{i \in N : \, p_l^i = j\}.$$

It must hold that for all $j \neq 1$, $|P_{j,1}| = n/(m-1)$. Moreover, it is possible to derive an ordinal preference profile such that for all $j \neq 1$ and $l \neq 2$, $|P_{j,l}| = n/(m-1)$, and with respect to candidate 1, $|P_{1,2}| = n$. Without loss of generality, let $\succ$ be such a profile. The score of candidate 1 in this election is $n\alpha_2$, and the score of every other candidate is $\frac{n}{m-1} \sum_{l \neq 2} \alpha_l$. Further, it holds that $u_1 = 0$, and $u_j = n/(m-1)$ for all $j \neq 1$. By Equation (1) and the assumption that in case of a tie the candidate that minimizes utility wins, it follows that candidate 1 wins the election, but for any other candidate, say candidate 2, $\frac{u_2}{u_1} = \frac{n/(m-1)}{0}$. Thus, the distortion of F is unbounded. □

It follows from Proposition 1 that in many reasonable scoring protocols, the distortion is unbounded. In particular:

Corollary 1. *The Borda and Veto Protocols have unbounded distortion.*

3.1 An Alternative Model

So far, we have analyzed the distortion with respect to cardinal preference profiles that satisfy, for all voters, i: $\sum_j u_j^i = K$. If one allows for weighted voting, it is possible to obtain a generalization of this model. Indeed, let $K^i = \sum_j u_j^i$, possibly $K^i \neq K^{i'}$ for $i \neq i'$. However, when an election is held based on a derived ordinal preference profile $\succ$, voter i has weight K^i. The definition of distortion can be reformulated in the obvious way to apply to this model; we denote the worst-case ratio between the candidate that maximizes utility and the one that wins the weighted election governed by F, when different K^i are allowed, by $\widetilde{\Delta}_m^n(F)$.

The next proposition shows that the two models are equivalent with respect to distortion.

Proposition 3. *For all social choice functions F, n_1 and m, $\Delta_m^{n_1}(F) \leq \widetilde{\Delta}_m^{n_1}(F)$, and there exists $n_2 \geq n_1$ such that $\widetilde{\Delta}_m^{n_1}(F) \leq \Delta_m^{n_2}(F)$.*

Proof. For the first inequality, let $n_1, m \in \mathbb{N}$. Let $\boldsymbol{u} \in \mathcal{U}^N$ and $\succ \in \mathcal{L}^N$ that maximize, in the first model, the ratio $\frac{\max_j u_j}{u_{F(\succ)}}$, subject to: for all i, $\sum_j u_j^i = K$, and $\succ$ is derived from u. In the second model, $F(\succ)$ is as before, since all voters have identical weights in the election (K). Therefore $\frac{\max_j u_j}{u_{F(\succ)}}$ in the second model is at least as large as in the first.

Regarding the second inequality, let $n_1, m \in \mathbb{N}$, and let $\widetilde{\boldsymbol{u}} \in \mathcal{U}^N$ and a derived $\widetilde{\succ} \in \mathcal{L}^N$ that maximize the ratio $\frac{\max_j \widetilde{u}_j}{u_{F(\widetilde{\succ})}}$ (with weighted voting). $\widetilde{\boldsymbol{u}}$ may not be a valid cardinal preference profile in the first model, but we construct a profile that is. Let $n_2 = \sum_i K^i$; for each one of the original voters $\widetilde{i} = 1, \ldots, n_1$, consider $K^{\widetilde{i}}$ voters i whose utility is $u_j^i = u_j^{\widetilde{i}} \cdot \prod_{\widetilde{i}' \neq \widetilde{i}} K^{\widetilde{i}'}$. Let $K = \prod_{\widetilde{i}} K^{\widetilde{i}}$; it holds that for all i, $\sum_j u_j^i = K$, hence $\boldsymbol{u}$ is valid in the first model. Further, for every candidate j it holds that $u_j = K \cdot \widetilde{u}_j$. Notice that $\widetilde{\succ}^{\widetilde{i}}$ can be derived from $\boldsymbol{u}^i$ for every voter i that corresponds to $\widetilde{i}$; denote the ordinal preference profile that is obtained by replicating $\succ^{\widetilde{i}}$ $K^{\widetilde{i}}$ times, once for each voter that corresponds to $\widetilde{i}$, by $\succ$. In the new election, we have $K^{\widetilde{i}}$ voters casting identical ballots to the one cast by voter $\widetilde{i}$, and this voter had weight $K^{\widetilde{i}}$ in the original election. Therefore, $F(\widetilde{\succ})$ with weighted voting is identical to $F(\succ)$ without. To conclude, we have obtained that:

$$\frac{\max_j u_j}{u_{F(\succ)}} = \frac{K \max_j \widetilde{u}_j}{K \widetilde{u}_F(\widetilde{\succ})} = \frac{\max_j \widetilde{u}_j}{\widetilde{u}_F(\widetilde{\succ})}. \qquad \square$$

Corollary 2. *Let F be a social choice function. Then $\widetilde{\Delta}_2^3(F) > 1$.*

Corollary 3. *Let F be a social choice function. F has unbounded distortion in the first model iff F has unbounded distortion in the second model.*

3.2 Complexity Issues

The existence of an algorithm that efficiently computes (or approximates) the distortion of a given voting protocol is, clearly, a basic prerequisite for comparing voting protocols in terms of distortion. As we shall see in Subsection 4.1, one of the building blocks of such an algorithm is a procedure that efficiently decides the following problem:

Definition 4. *In the* MIN-SCORE-MAX-UTIL *(MSMU) problem, we are given the number of voters n, the number of candidates m, a scoring protocol F defined by parameters $\alpha_1, \ldots, \alpha_m$, for each voter i, a sequence of nonnegative integers $b^i = \langle b_1^i, \ldots, b_m^i \rangle$, and $y, z \in \mathbb{N}$. We are asked whether there are n permutations on C, $\pi^1, \ldots, \pi^n$, such that for the cardinal preference profile u defined by $u_j^i = b_{\pi^i(j)}^i$ and a derived ordinal preference profile $\succ$, it holds that $u_1 \geq y$ but $s_1 \leq z$.*

To put it less formally, we are given a scoring protocol, and for each voter, a sequence of m numbers. We know what the utilities of each voter are in general, but it is still left to determine how each voter assigns these utilities to candidates. Essentially, this is equivalent to choosing an ordinal preference relation for each voter, and then assigning the maximal element in b^i to p_1^i, the second largest element to p_2^i, etc. — and this is the approach that will later become relevant.

Remark 2. It is not assumed here that $\sum_l b_l^i = K$ for all i.

Proposition 4. *MSMU is $\mathcal{NP}$-complete.*

Proof. Reduction from KNAPSACK; omitted due to space constraints.

4 Misrepresentation

Impossibility results regarding the general model, manifested above as Propositions 1, 2, and (to a lesser degree) 4, motivate us to impose restrictions on agents' possible cardinal preference profiles. In this section, we examine a slight variation on the concept of distortion that allows for possibility results.

Monroe [11] defines a measure of *misrepresentation*; using our notations, voter i's misrepresentation with respect to candidate j is $\mu_j^i = l_j^i - 1$. To put it differently, if voter i ranks candidate j first, then i's misrepresentation w.r.t. to j is 0, the misrepresentation w.r.t. the second highest-ranked candidate is 1, and so forth. The misrepresentation of candidate j is $\mu_j = \sum_i \mu_j^i$.

Definition 5. *Let F be a social choice function. The* misrepresentation *of F with n voters and m candidates, denoted $\mu_m^n(F)$, is* $\max \frac{\mu_{F(\succ)}}{\min_j \mu_j}$*, where the maximum is taken over all ordinal preference profiles $\succ^i$ and their associated misrepresentation values. If several candidates are tied in an election, the one that maximizes misrepresentation is elected.*

Misrepresentation values can, of course, be interpreted as cardinal preferences (e.g., $u_j^i = m - \mu_j^i - 1$), albeit restricted ones: a voter's ordinal preference relation $\succ^i$ fixes a (perfect) matching between candidates and the utilities $0, 1, \ldots, m-1$.[6] Consequently, the misrepresentation of a social choice function F can be easily reformulated as distortion. In fact, similar results can be obtained, but the latter formulation favors candidates that are ranked last by few voters, whereas the former formulation rewards candidates that are placed first by many voters.

When is misrepresentation an issue? The following scenario provides a compelling, albeit somewhat artificial, example. Consider the meeting scheduling problem discussed in [9]: scheduling agents schedule meetings on behalf of their associated users, based on given user preferences; a winning schedule is decided in an election. Say three possible schedules are being voted on. These schedules, being fair, conflict with at most two of the requirements specified by any user. In other words, a user's misrepresentation with respect to a certain schedule is 0 if there are no conflicts, 1 if there is a single

[6] Unlike the general model, in the current setting there is a unique derivation of misrepresentation values from ordinal preferences, and vice versa.

conflict, and 2 if there are two conflicts.[7] In this case, having no conflicts at all is vastly superior to having at least one conflict, as even one conflict may prevent a user from attending a meeting. As noted above, this issue is taken into account in the calculation of misrepresentation — emphasis is placed on candidates that were often ranked first.

Proposition 1 stated that there is no social choice function with distortion 1. Clearly this is not the case here:

Proposition 5. *Let $m \in \mathbb{N}$, and let F be a scoring protocol with parameters $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m$. Then $\mu_m^n(F) = 1$ for all n iff there exist a and b such that $\alpha_l = -a \cdot l + b$ for all $l = 1, \ldots, m$.*

Proof. Assume first that there exist a and b such that $\alpha_l = -a \cdot l + b$ for all $l = 1, \ldots, m$, and let $n \in \mathbb{N}$, $\succ \in \mathcal{L}^N$. Candidate j's score is:

$$\sum_i \alpha_{l_j^i} = \sum_i [-a \cdot l_j^i + b] = \sum_i [-a(\mu_j^i + 1) + b] = n[b - a] - a \sum_i \mu_j^i,$$

so the candidate that maximizes the score is the one that minimizes misrepresentation.

In the other direction, assume there do not exist a and b such that $\alpha_l = -a \cdot l + b$ for all $l = 1, \ldots, m$. It follows that there exist l_0, a and a' such that $a \neq a'$, and $\alpha_1 - \alpha_2 = a$ but $\alpha_1 - \alpha_{l_0} = a'(l_0 - 1)$, and $a, a' > 0$.[8] Assume w.l.o.g. that $a > a'$. Consider the following ballot: n' voters vote $2 \succ^i 1 \succ^i \ldots$, $n' - x$ voters rank $1 \succ^i 2 \succ^i \ldots$, and y voters cast their ballots in a way that $p_1^i = 1$, $p_{l_0}^i = 2$, for some $x, y \in \mathbb{N}$ (we have that $n = 2n' - x + y$). When comparing the scores of candidates 2 and 1, we have:

$$s_2 - s_1 = xa - ya'(l_0 - 1). \tag{2}$$

Further, it holds that:

$$\mu_2 - \mu_1 = -x + y(l_0 - 1). \tag{3}$$

It is sufficient to show that it is possible to make candidate 2 win the election, and in particular guarantee that candidate 2's score be higher than 1's, but simultaneously ensure that candidate 2's misrepresentation be higher than 1's. Indeed, by Equations (2) and (3) both conditions are satisfied whenever

$$\frac{x}{l_0 - 1} < y < \frac{a}{a'} \cdot \frac{x}{l_0 - 1}. \tag{4}$$

Choosing $x > 3(l_0 - 1)\frac{a'}{a - a'}$, it is possible to choose y that satisfies Equation (4). Moreover, it is clearly now possible to choose n' large enough so as to guarantee that candidate 2 wins the election, since for all candidates $j \neq 1, 2$, there are at most y voters such that $p_2^i = j$, and $\alpha_1 > \alpha_l$ for all $l \neq 1$. □

Corollary 4. *For all n, m, μ_m^n(Borda) $= 1$.*

[7] We implicitly assume that for each user there is one schedule with no conflicts, one with a single conflict, and one with two conflicts.

[8] It is safe to assume that $a > 0$ (and therefore $a' > 0$), because if $\alpha_1 = \alpha_2$ then the result is obvious.

Corollary 4 establishes the optimality of the Borda protocol in terms of misrepresentation. Unfortunately, this protocol is notoriously easy to manipulate, and is plagued by other disadvantages. Therefore, it is worthwhile to explore the misrepresentation of other protocols.

The concept of unbounded misrepresentation can be defined analogously to Definition 3. In the framework of misrepresentation, we have the following proposition.

Proposition 6. *Let F be a scoring protocol with parameters $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m$. Then F has unbounded misrepresentation iff $\alpha_1 > \alpha_2$.*

Proof. Suppose first that $\alpha_1 > \alpha_2$. Let $n, m \in \mathbb{N}$, $\succ \in \mathcal{L}^N$, and assume w.l.o.g. that $\text{argmin}_j \mu_j = 1$ and $F(\succ) = 2$. Let $k = |\{i \in N : l_1^i = 1\}|$ be the number of voters that ranked candidate 1 first. The number of points candidate 2 received is at most $s_2 \leq (n-k)\alpha_1 + k\alpha_2$, and the number of points candidate 1 received is at least $s_1 \geq k\alpha_1$. We have:

$$(n-k)\alpha_1 + k\alpha_2 \geq s_2 \geq s_1 \geq k\alpha_1.$$

Therefore, $k \leq n\frac{\alpha_1}{2\alpha_1 - \alpha_2}$; this implies that $\mu_1 \geq n\frac{\alpha_1 - \alpha_2}{2\alpha_1 - \alpha_2}$. As $\mu_2 \leq n(m-1)$, we have that

$$\frac{\mu_2}{\mu_1} \leq \frac{n(m-1)}{n\frac{\alpha_1-\alpha_2}{2\alpha_1-\alpha_2}} = \frac{(m-1)(2\alpha_1 - \alpha_2)}{\alpha_1 - \alpha_2}.$$

For a fixed m, this expression is a constant, even as n grows.

In the other direction, suppose $\alpha_1 = \alpha_2$, and consider $\succ \in \mathcal{L}^N$ where for all voters i, $1 \succ^i 2 \succ^i \cdots$. It holds that $\mu_1 = 0$, $\mu_2 = n$. We can assume w.l.o.g. that $F(\succ) = 2$, since in case of a tie a candidate that maximizes misrepresentation is elected, hence the winner must have misrepresentation at least as high as μ_2. The proposition follows from the fact that $\frac{\mu_2}{\mu_1} = \infty$. □

Corollary 5. *The Veto protocol has unbounded misrepresentation.*

Remark 3. Corollary 5 implies that the Participation, Monotonicity, and Consistency properties (even together) do not guarantee that a voting protocol has bounded misrepresentation, as the Veto protocol satisfies all three properties.

Proposition 7. *For all n, m, μ_m^n(Plurality) $= \mu_m^n$(Plurality with Runoff) $= m - 1$.*

Proof. Omitted due to space constraints.

Proposition 8. *For all n, m, μ_m^n(Copeland) $\leq m - 1$.*

Proof. Let $\succ \in \mathcal{L}^N$; w.l.o.g. suppose $\text{argmin}_j u_j = 1$ and Copeland$(\succ) = 2$. Additionally, denote by C' the set of candidates that candidate 2 beats in a pairwise election, $|C'| = k$. For each candidate $j \in C'$, at least $\lceil n/2 \rceil$ voters have $l_2^i < l_j^i$. Let $C^i = \{j \in C : l_2^i < l_j^i\}$; for all i, $l_2^i = m - |C^i|$. It holds that $\sum_i |C^i| \geq k\lceil n/2 \rceil$, but this implies that:

$$\mu_2 = \sum_i \mu_2^i = \sum_i (l_2^i - 1) = \sum_i [(m-1) - |C^i|] \leq n(m-1) - k\lceil n/2 \rceil.$$

We distinguish two cases:

Case 1: $k = m$. In this case, candidate 1 has not won the pairwise election against 2, and thus there are at least $\lceil n/2 \rceil$ voters i such that $l_2^i < l_1^i$. This implies that $\mu_1 \geq \lceil n/2 \rceil$, and hence $\frac{\mu_2}{\mu_1} \leq \frac{n(m-1)-m\lceil n/2 \rceil}{\lceil n/2 \rceil} \leq m-2$.

Case 2: $k \leq m-1$. Candidate 1 won at most k pairwise elections. In each pairwise election that 1 did not win against candidate j, at least $\lceil n/2 \rceil$ voters voted $l_j^i < l_1^i$. By the same reasoning as before, $\mu_1 \geq (m-k)\lceil n/2 \rceil$. Therefore,

$$\frac{\mu_2}{\mu_1} \leq \frac{n(m-1)-k\lceil n/2 \rceil}{(m-k)\lceil n/2 \rceil} \leq \frac{2m-k-2}{m-k}. \quad (5)$$

The ratio in Equation (5) on the right is monotonic increasing as a function of k when $1 \leq k \leq m-1$, and thus is bounded by $m-1$. □

Proposition 9. *For all n, m, μ_m^n(Bucklin) $\leq m$.*

Proof. Let $\succ \in \mathcal{L}^N$; assume w.l.o.g. that $\text{argmin}_j u_j = 1$, and Bucklin$(\succ) = 2$. Let $l_0 = \min\{l \in \{1, \ldots, m\} : \exists j \ s.t. \ B_{j,l} > n/2\}$. At least $\lceil n/2 \rceil$ voters i have $p_l^i = 2$ for $l \leq l_0$. Therefore, $\mu_2 \leq \lceil n/2 \rceil (l_0 - 1) + \lfloor n/2 \rfloor (m-1)$. We now examine two cases.

Case 1: $l_0 = 1$. It cannot be the case that $B_{2,1} > n/2$ and $B_{1,1} > n/2$ simultaneously. Therefore, it must be true that at least $\lceil n/2 \rceil$ voters i have $l_1^i \geq 2$, and hence $\mu_1 \geq \lceil n/2 \rceil$. We have that $\frac{\mu_2}{\mu_1} \leq m-1$.

Case 2: $l_0 \geq 2$. At most $\lfloor n/2 \rfloor$ voters i have $l_1^i \leq l_0 - 1$, therefore $\mu_1 \geq \lceil n/2 \rceil (l_0 - 1)$. It holds that

$$\frac{\mu_2}{\mu_1} \leq \frac{\lceil n/2 \rceil (l_0 - 1) + \lfloor n/2 \rfloor (m-1)}{\lceil n/2 \rceil (l_0 - 1)}.$$

The ratio is maximized when $l_0 = 2$; it follows that $\frac{\mu_2}{\mu_1} \leq m$. □

Proposition 10. *For all n, m, $\mu_m^n(Maximin) \leq \frac{2}{\sqrt{5}-1}(m-1) \approx 1.62(m-1)$.*

Proof. Let $\succ \in \mathcal{L}^N$. Assume w.l.o.g. that $\text{argmin}_j u_j = 1$, and Maximin$(\succ) = 2$. Additionally, suppose that candidate 2's Maximin score is k. With foresight, we denote $c = \frac{3-\sqrt{5}}{2}$. We distinguish two cases:

Case 1: $k > cn$. At least cn voters i have $l_2^i < l_1^i$. In the worst case, $(1-c)n$ voters i vote $l_1^i = 1, l_2^i = m$, and cn voters have $l_2^i = 1$ and $l_1^i = 2$. Therefore, in this case,

$$\frac{\mu_2}{\mu_1} \leq \frac{(1-c)}{c}(m-1) \approx 1.62(m-1).$$

Case 2: $k \leq cn$. There exists a candidate, w.l.o.g. candidate 3, s.t. at most k voters i have $l_1^i < l_3^i$, i.e., at least $(1-c)n$ voters do not rank 1 first. Since $\mu_2 \leq n(m-1)$, it holds that

$$\frac{\mu_2}{\mu_1} \leq \frac{n(m-1)}{(1-c)n} = \frac{1}{1-c}(m-1) \approx 1.62(m-1).$$

This concludes the proof.[9] □

[9] c was chosen such that $\frac{1-c}{c} = \frac{1}{1-c}$.

Algorithm 1

```
procedure MIN-MISREP(n, m, α, y)
    for l ← 0, y do                                   ▷ Initialization
        a_{0,l} ← 0
    end for
    for k ← 1, n do
        for l ← 0, y do
            q ← min{l, m − 1}
            a_{k,l} ← min_{p=0,...,q}(a_{k−1,l−p} + α_{p+1})   ▷ Induces rankings for candidate 1
        end for
    end for
    return a_{n,b}
end procedure
```

Proposition 11. *For all n, m, $\mu_m^n(STV) \leq \frac{3}{2}(m-1)$.*

Proof. Omitted due to space constraints.

Remark 4. It is easy to show that if F is a voting protocol that satisfies the majority criterion, then $\mu_m^n(F) \leq 2(m-1)$.

4.1 Complexity Issues

In this subsection we address complexity issues related to calculating misrepresentation. We begin by reformulating the MSMU problem, presented in Section 3, in the context of misrepresentation.

Definition 6. *In the* MIN-SCORE-MIN-MISREPRESENTATION *(MSMM) problem, we are given the number of voters n, the number of candidates m, a scoring protocol F defined by parameters $\boldsymbol{\alpha} = \langle \alpha_1, \ldots, \alpha_m \rangle$, and $y, z \in \mathbb{N}$. We are asked whether there exists $\succ \in \mathcal{L}^N$ such that it holds that $\mu_1 \leq y$ but $s_1 \leq z$.*

Unlike the general formulation of the problem, here we have:

Lemma 1. *MSMM can be decided in time polynomial in n and m.*

Proof. We describe a dynamic programming algorithm MIN-MISREP, given as Algorithm 1. The algorithm keeps a matrix $A = (a_{kl})_{k \in \{0,\ldots,n\}, l \in \{0,\ldots,y\}}$; entry a_{kl} is the minimal score candidate 1 may have under the constraints that k voters have cast their vote, and $\mu_1 \leq l$.

The correctness of the algorithm can be easily proven by induction on k. As $y = O(nm)$, the running time of the algorithm is $O(n^2m^2)$. Now, the given instance of MSMM is a "yes" instance iff the output of MIN-MISREP is at most z: $a_{n,y} \leq z$. □

We now consider the following problem:

Definition 7. *In the* LOSER-WITH-MIN-MISREPRESENTATION *(LWMM) problem, we are given the number of voters n, the number of candidates m, a scoring protocol F defined by parameters $\alpha_1, \ldots, \alpha_m$, and $y \in \mathbb{N}$. We are asked whether there exists $\succ \in \mathcal{L}^N$ such that it holds that $\mu_1 \leq y$ but candidate 1 loses the election.*

Lemma 2. *LWMM can be decided in time polynomial in n and m.*

Proof. Algorithm 1 can easily be adapted to return candidate 1's minimal score, under the former constraint that $\mu_1 \le y$, and the additional constraint that exactly n' voters, $0 \le n' \le n$, satisfy $l_1^i = 1$. This can be accomplished, for example, by running MIN-MISREP once for each value of n', and assuming that n' voters have already cast their vote (ranking candidate 1 first), whereas the remaining $n - n'$ cast their vote according to the algorithm.

For each value of n', it is possible to assume the remaining voters rank candidate 2 as high as possible, i.e., candidate 2 receives $s_2 = (n - n')\alpha_1 + n'\alpha_2$ points. Clearly, there exists a value of n' such that $s_2 \ge s_1$ iff the given instance of LWMM is a "yes" instance.[10] □

Given n, m, and $\boldsymbol{\alpha}$, we have shown so far that it is possible to find rankings ${l_1^i}^{ALG}$ for candidate 1, such that the associated misrepresentation of candidate 1 satisfies:

$$\mu_1^{ALG} = \min\{\mu : \exists \succ \in \mathcal{L}^N \ s.t. \ \mu_1 = \mu \wedge \text{candidate 1 loses the election}\}.$$

Ultimately, we would like to be able to compute $\mu(F) = \max \frac{\mu_{F(\succ)}}{\min_j \mu_j}$; let $\succ^* \in \mathcal{L}^N$ that maximizes this ratio, let μ_j^* be the associated total misrepresentation values of candidates, s_j^* be the associated scores, and ${l_j^i}^*$ be the associated rankings; assume w.l.o.g. that $\text{argmin}_j u_j^* = 1$, and $F(\succ^*) = 2$.

Definition 8. *Let F be a scoring protocol. F has the* popular loser property *iff the rankings ${l_1^i}^{ALG}$ are identical to the rankings ${l_1^i}^*$, up to the order of the voters.*

Definition 9. *Let F be a scoring protocol. F has the* even match property *iff, given s_1^*, the rankings ${l_2^i}^*$ are the ones that maximize μ_2^*, under the constraint $s_2^* \ge s_1^*$.*

In other words, a scoring protocol has the popular loser property if any ordinal preference profile such that candidate 1 has maximal misrepresentation, under the constraint that candidate 1 is not the winner, is optimal in the sense that candidate 1's ranking by voters is identical to candidate 1's ranking in the preference profile that maximizes misrepresentation. A scoring protocol has the even match property if, once the above rankings for candidate 1 are known, in order to find the misrepresentation of the protocol it is sufficient to find rankings for candidate 2 that maximize candidate 2's misrepresentation, while guaranteeing that 2 has a higher score than 1.

Certainly, if F has both properties, then Lemma 2 is a step forward towards calculating the misrepresentation of F. But are there protocols that possess both properties?

Example 1. The Plurality and Veto protocols have the popular loser property and the even match property.

If so, some important protocols possess the properties. Characterizing more fully the protocols that possess both properties remains an open question.

[10] It is enough to demand a weak inequality in $s_2 \ge s_1$, as candidate 1 is the candidate that minimizes the score while achieving misrepresentation μ_1; if $s_2 = s_1$ then it must hold that $\mu_2 \ge \mu_1$, hence candidate 2 still wins the election.

Theorem 1. *Let F be a scoring protocol with the popular loser property and the even match properties. Then the problem of calculating $\mu_m^n(F)$ has a Fully Polynomial Time Approximation Scheme (FPTAS).*

Proof (Sketch). Observe the rankings ${l_1^i}^{ALG}$ fixed by the algorithm from the proof of Lemma 2, on the given input. By the assumptions, it is sufficient to find rankings l_2^i for candidate 2 in a way that μ_2 is maximal, under the constraint $s_2 \geq s_1^{ALG}$.

The above problem reduces to the exact KNAPSACK problem with cardinality constraints (E-kKP). In this problem, we are given n items, each with a weight w_i and a value v_i, and a weight limit K; the goal is to find a subset S of items of size k, that maximizes $\sum_{i \in S} v_i$, subject to $\sum_{i \in S} w_i \leq K$.

In our setting, l_2^i can take any value in $\{1, \ldots, m\} \setminus \{{l_1^i}^{ALG}\}$; let there be an item associated with each possible value of l_2^i, $i = 1, \ldots, n$ (there are $n(m-1)$ items). The value of the item associated with $l_2^i = l$ is $\mu_2^i = l - 1$, and its weight is $\alpha_1 - \alpha_l$. Exactly n items are to be chosen; the weight limit is $n\alpha_1 - s_1^{ALG}$.

This is a polynomial time reduction. Indeed, given rankings l_2^i, $i = 1, \ldots, n$ such that $s_2 \geq s_1^{ALG}$, choose n corresponding items in the knapsack instance. The items' total value is exactly μ_2. Moreover, the total weight associated with the items is at most $n\alpha_1$ minus the total score of the associated rankings, which is at least s_1^{ALG}. The other direction is similar.

Caprara et al. [4] present an FPTAS for E-kKP. Therefore, for any $\epsilon > 0$, it is possible to find (in polynomial time) μ_2 such that $\mu_2 \leq \mu_2^* \leq (1 + \epsilon)\mu_2$. In addition, recall that $\mu_1^{ALG} = \mu_1^*$. Therefore:

$$\frac{\left(\frac{\mu_2^*}{\mu_1^*}\right)}{\left(\frac{\mu_2}{\mu_1^{ALG}}\right)} = \frac{\mu_1^{ALG}\mu_2^*}{\mu_1^*\mu_2} = \frac{\mu_2^*}{\mu_2} \leq 1 + \epsilon. \qquad \square$$

5 Conclusions

We have defined the distortion of a social choice function as the worst-case ratio between the total utility of the candidate that maximizes social welfare, and the elected candidate. At first, we have focused on a model where, for all voters, the sum of utilities is identical. We have shown that every social choice function is distorted, even when the number of voters and the number of candidates are small. Moreover, we have established a sufficient condition for unbounded distortion — a result which implies that several well-known scoring protocols have unbounded distortion in the general model. We have shown our model to be equivalent, in terms of distortion, to another model where the voters' cardinal preferences are unconstrained, but each voter's weight is the sum of its utilities. Finally, we have proven that a problem associated with calculating distortion is $\mathcal{NP}$-complete when utilities are unconstrained.

Motivated by the impossibility results mentioned above and the work of Monroe [11], we have reformulated the concept of distortion as misrepresentation. The main difference between the two settings is, essentially, that in the misrepresentation setting voters' cardinal preferences are quite restricted. We have established a necessary and

Table 1. The misrepresentation of common voting protocols

Voting Protocol	Misrepresentation
Borda	1
Veto	Unbounded
Plurality	$= m - 1$
Plurality with Runoff	$= m - 1$
Copeland	$\leq m - 1$
Bucklin	$\leq m$
Maximin	$\leq 1.62(m - 1)$
STV	$\leq 1.5(m - 1)$

sufficient condition for a social choice function to be optimal in terms of misrepresentation, and have also characterized the scoring protocols with unbounded misrepresentation. More importantly, we have given bounds — in some cases tight — for the misrepresentation of specific voting protocols; these bounds are summarized in Table 1.

Last, we have tackled the problem of calculating misrepresentation. Moving through several sub-problems, we have ultimately demonstrated that there is a fully polynomial time approximation scheme (FPTAS) for this problem, when the voting protocol is a scoring protocol that possesses the popular loser and even match properties. It remains an open issue to characterize the scoring protocols that have these properties.

The results presented in Section 3 suggest that distortion may be a major obstacle for designers of multiagent systems who wish to apply voting. This is true, however, only if the agents' cardinal preferences are almost unconstrained. On the other hand, we have seen in Section 4 that restricting the preferences overturns some of the impossibility results.

In the context of restricted preferences, the results imply that the distortion of a voting protocol should be a major criterion in the comparison of different protocols — alongside other classical criteria like manipulability. For instance, whenever misrepresentation is a concern (as in the meeting scheduling example discussed in Section 3), one might prefer to employ the Borda protocol, which is optimal in terms of misrepresentation but highly manipulable, rather than the STV protocol, which is difficult to manipulate [1] but has high misrepresentation. In a three candidate example, STV's misrepresentation might be 3 times higher than Borda's; in the scheduling domain, this might imply three times as many conflicts with user preferences — certainly a steep price to pay for preventing strategic behavior.

We briefly mention two directions for future research. Our computational complexity analysis of distortion is rather rough. It seems true that calculating distortion (or even misrepresentation) in scoring protocols is $\mathcal{NP}$-complete, but currently there is no proof. Second, our approximation scheme relies on the popular loser and even match properties; it remains an open issue to characterize the scoring protocols that have these properties. Further, can these assumptions be abandoned?

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