THE PARADOX OF VOTING: PROBABILITY CALCULATIONS

by Mark B. Garman and Morton I. Kamien

Graduate School of Industrial Administration, Carnegie-Mellon University[1]

In this paper an expression for determining the frequency of the "voting paradox" as a function of the number of voters, the number of alternatives to be voted upon, and cultural characteristics of the voters, is derived. In the analysis a voter's ranking of the given alternatives is represented by a skewsymmetric matrix whose elements also satisfy a certain transitivity condition. The collective ranking of the alternatives is obtained under simple majority rule by summation of the actual individual rankings and consequently can likewise be represented by a skewsymmetric matrix. It is then shown that the existence of the paradox corresponds to the nonexistence of a row in the matrix denoting the collective ranking whose entries are all nonnegative. Cultural effects are incorporated by assigning to each possible individual ranking of alternatives a probability. It is then shown that when all possible rankings of three alternatives are equally likely and the number of voters becomes very large, the probability of the paradox approaches a definite limit, vig. .087. Exact values for the probability of the paradox for several cases in which the number of alternatives exceeds three are also presented.

In this paper we present some new results regarding the probability of occurrence of a phenomenon known in political theory as "the paradox of voting." The paradox can arise whenever a group of individuals seeks to choose from among several distinct alternatives a most preferred alternative by means of majority rule. The relevant group may be an entire constituency, an economic planning board, or the judges of a beauty contest, while the alternatives might be political candidates, states of the economy, or beauty contestants. The selection procedure involves voting upon alternatives in a pairwise fashion with the object of determining which of the pair receives a majority of the votes. The winner of the initial paired comparison is put up against a third alternative and so on until, by the process of elimination, a single alternative remains. *If the remaining alternative is to be the group's most preferred alternative then it should receive a majority of the votes in a pairwise comparison with every other alternative.* In those instances in which the remaining alternative does not possess this property or (equivalently, as shall be illustrated) is dependent on the order in which alternatives are compared, the voting paradox is said to occur. A brief example will help clarify this point: Suppose the relevant group consists of three judges J1, J2, and J3 who wish to select the most desirable alternative from among three distinct alternatives, a, b, c, in the manner described above. In this case three paired comparisons are possible, that is, a versus b, b versus c, and a versus c. Suppose further that if the outcomes of all comparisons were to be known, a defeats b, b defeats c, and c defeats a. Should the process of elimination prevail, only two comparisons will be made; for example, if a versus b is the initial comparison, a will win and subsequently be put against c. In this comparison c will win and the voting will terminate, since all single alternatives have come under consideration even though all pairs of alternatives have not. Likewise, if b versus c is the initial comparison then a will be the remaining alternative. Finally, if a versus c is the initial comparison then b will be the remaining alternative. Thus the remaining alternative in this case depends entirely upon the sequence of comparisons and does not possess the desired properties of a winner. On the other hand, were the outcome from a versus b reversed, alternative b would be the remaining alternative regardless of the initial comparison, as can be readily verified.

[1] We wish to thank David Klahr, University of Chicago, Richard Ruppert of University of Kansas, Nancy Schwartz of Carnegie-Mellon University and the referee for their helpful suggestions. We, of course, remain responsible for any errors or shortcomings of this paper.

For purposes of analysis, the actual voting need not take place if each judge can provide a ranking of alternatives reflecting his voting behavior on all paired comparisons. By consolidating these rankings it is possible to predict exactly the outcome of any vote that might actually take place. The problem of selecting the group's most preferred alternative then becomes one of aggregating the individual rankings into a group ranking. The aggregation procedure under consideration is majority rule.

Example 1: A case wherein majority rule does not yield a group ranking.

Suppose that the three judges J1, J2, and J3 have ranked the three alternatives a, b and c in order of individual preference, arriving at

J1: a b c

J2: c a b

J3: b c a,

which is shorthand for saying, "Judge 1 prefers a to b, b to c, and a to c. Judge 2 prefers c to a, a to b, and c to b. Judge 3 prefers b to c, c to a, and b to a." Next the group preferences are examined under the majority rule procedure. We discover that a is preferred to b by J1 and J2, a majority; hence a is preferred to b for the group. But likewise b is preferred to c (J1 and J3), and c is preferred to a (J2 and J3). Thus we are faced with the impossible task of finding a ranking in which a defeats b, b defeats c, and c defeats a.

The cyclical nature of this result prompted Lewis Carrol (Rev. C. L. Dodgson)[2] to call the phenomenon "cyclical majorities;" other names which have been attached to it are "l'effet Condorcet,"[3] "Arrow's Paradox," and herein "the paradox of voting." This restatement of the problem places it in the context of previous work in political theory and economics.

[2] Rev. C. L. Dodgson's work on the problem of cyclical majorities was discovered and reprinted by Duncan Black (1958).

[3] After the Frenchman who was reported the first to discover the paradox. For an excellent history and bibliography, see William H. Riker (1961).

The paradox is of interest to political theorists because of its implications about the use of majority rule as a means of ascertaining community preference. It is not unreasonable to expect that the assumptions of the method will be met in numerous naturally occurring instances of voting, that, in fact, voters do rank alternatives[4], and that the majority rule procedure in fact experiences wide use. Black (1958) demonstrated that the paradox—together with the fact that not all pairs of alternatives are normally compared in many decision-making groups—can lead to distortions in the likelihood of acceptance for certain alternatives: namely, the later an alternative is introduced into the voting, the more likely it is to be the remaining one[5]. Such an effect must be termed undesirable, since it in no way depends on the perceived merit of that alternative. Were this only a theoretical difficulty it might not warrant major concern. However, Riker (1965) has described several instances in which the paradox may actually have occurred in Congress.

In economics, the voting paradox is related to the broader problem of constructing a social welfare function. Arrow showed (1951) that under a seemingly reasonable set of requirements a social welfare function could not be constructed. This result has come to be known as Arrow's Possibility Theorem. Majority rule is a method, considered by some an extremely appealing one (see Luce and Raiffa, 1957, pp. 357–363) of aggregating individual preferences into a collective ranking of alternatives. The existence of the voting paradox is a manifestation of the fact that because this procedure satisfies the requirements set forth by Arrow, it does not yield a social welfare function.

The relevance of the voting paradox to political theory and welfare economics has not gone unchallenged. Bergson (1966, pp. 27–49) summarizes the arguments of others and expresses some of his own doubts on the importance of Arrow's Possibility

[4] This is not totally uncontested. See for instance Kenneth O. May (1954).

[5] A fuller discussion is found in Black, pp. 39–45.

Theorem (and thereby of the voting paradox) to either of these areas. Attacks on the theorem usually consist of objections to one or several of the requirements for a social welfare function (see Luce and Raiffa, pp. 340–357). Seldom has it been argued that the theorem, though valid, is irrelevant since the impossibility it predicts can frequently be avoided. We think that a demonstration of the extreme unlikelihood of the paradox, even in theory, would substantially diminish the importance of the theorem to actual practice. In other words, much of the controversy regarding Arrow's Theorem could be ended if it were found that the paradox occurs very infrequently.

We are not the first in attempting to discover this probability. Black (1958) published a table consisting almost solely of blank entries with a challenge to combinatorial experts:

> It would be of interest to know in what fraction of all possible cases one motion [alternative] is able to get a simple majority against each of the others as n and m [the number of alternatives and judges, respectively] vary, and in what fraction the majorities are cyclical. Had we been able to do so, we would have calculated the figures . . . ; but we have been unable to derive the general series which would enable us to make the calculations for the table and have entered only a few figures in the cells (p. 51).

As yet, this table remains substantially unfilled though some progress has been made. Guilbaud (1952, p. 519) casually revealed some correct figures for the case of three alternatives and various odd numbers of judges, calculated "par les moyens usuels en analyse combinatoire." He does not describe how he arrived at these figures or how his method might be extended to more than three alternatives. Kendall (1947, pp. 421–428) calculated the maximum number of cyclical triads (three alternatives taken at a time) possible among any number of alternatives. While this result is interesting, it bears only indirectly on the problem at hand since, as will be demonstrated below, the presence of a cyclical triad does not preclude the existence of a most preferred alternative except when there are only three alternatives. Moreover, Kendall's results pertain to the comparisons of alternatives by an individual rather than a group.

More recently, two Monte Carlo (that is, randomly sampled by computer) investigations have appeared. Campbell and Tullock (1965), using this technique, estimated about 75 entries of Black's table. Unfortunately, they do not provide any measure of the accuracy of their results, and some anomalies interpreted by them as significant are in fact sampling errors. Klahr (1966) estimated fewer cells but with a corresponding gain in accuracy.

THE FORMAL FRAMEWORK

In this section we set forth precisely the concepts discussed above and introduce some new ones. The first notion is that of a set of alternatives $X = \{x_1, x_2, \cdots x_n\}$.[6]

The second concept is that of an individual preference relation, denoted by P_k, where the subscript distinguishes the k^{th} individual.

> *Definition 1:* For any pair of distinct elements x_i, x_j in X, $x_iP_kx_j$ denotes the propositional statement, "the k^{th} individual prefers alternative x_i to alternative x_j."

It is required that the relation P_k satisfy the following conditions:

> *Condition 1* (completeness). For every pair of alternatives x_i, x_j in X, $x_iP_kx_j \vee x_jP_kx_i$, for all k.[7]

This condition assures comparability among all the alternatives.

> *Condition 2* (antisymmetry). For each individual k and any pair of distinct alternatives x_i, x_j in X, $x_iP_kx_j \Rightarrow \sim x_jP_kx_i$.

This condition stipulates that each individual prefers either x_i to x_j or x_j to x_i; not

[6] There are no restrictions on the choice of alternatives in X. They may have single or multidimensional attributes, and we specifically allow the alternative "that no alternative be adopted."

[7] For compactness, we make use of the customary logical symbols. Read "not" for "$\sim$," "or" for "$\vee$," "and" for "$\wedge$"," "implies" for "$\Rightarrow$," and "is equivalent to" for "$\Leftrightarrow$."

both and not neither. The individual cannot be indifferent between two alternatives. By virtue of this condition the relation P_k is antisymmetric.

Condition 3 (transitivity). For each individual k and any three distinct alternatives x_h, x_i, x_j in X, $x_h P_k x_i \wedge x_i P_k x_j \Rightarrow x_h P_k x_j$.

This condition asserts that if an individual prefers x_h to x_i and x_i to x_j, then he must prefer x_h to x_j. Conditions 1, 2, and 3 together insure that all the alternatives in X can be ranked by every individual.

We next introduce the notion of a relevant group of individuals seeking to choose from amongst the set of alternatives X the most preferred alternative. We call such a group a committee and represent it by the set C. The elements of C are called judges, of which there are m in number; m is $\geqq 3$ and odd. Each judge, independently of each and all other judges, ranks the elements of X by means of the individual preference relation described above. In other words, we disallow negotiation amongst judges in the formation of individual preferences and require simultaneous presentation of individual rankings. We also assume that a judge does not deliberately misrepresent his preferences. Each judge represents only one complete ranking.

The committee arrives at a most desired alternative by simple majority rule. This procedure calls for a pairwise comparison of the alternatives in X. We call the most desirable alternative the majority winner and define it as follows:

Definition 2. An alternative x_*, an element of X, is said to be the majority winner if in a pairwise comparison with any other alternative x_i in X, x_* receives more than half the ballots (that is, at least $(m + 1)/2$ of the ballots).

The simple majority procedure gives rise to a group preference relation which we shall call P and define as follows:

Definition 3. For any pair of alternatives x_i, x_j in X, $x_i P x_j$ means that μ of the judges prefer x_i to x_j and $\mu \geqq (m + 1)/2$.

Accordingly, alternative x_* is the majority winner if and only if $x_* P x_i$ for every element x_i in the set $X - \{x_*\}$. The relation P gives us a committee ordering, that is, the set of $x_i P x_j$ for all x_i, x_j in X, $i \neq j$. It would be desirable for the collective preference relation P to have the same properties as the individual preference relation P_k. Consequently, we stipulate:

Condition 4 (group completeness). For every pair of alternatives x_i, x_j in X, $x_i P x_j \vee x_j P x_i$.

Condition 5 (group antisymmetry). For any pair of distinct alternatives x_i and x_j in X, $x_i P x_j \Rightarrow \sim x_j P x_i$.

This condition will be met since the number of judges in C is odd.

Condition 6 (group transitivity). For any three distinct alternatives x_h, x_i, and x_j in X, $x_h P x_i \wedge x_i P x_j \Rightarrow x_h P x_j$.

The existence of the voting paradox means that this condition will occasionally be violated. That is, if Condition 6 were never contradicted regardless of the judges' individual preference orderings, then the paradox would not occur and Arrow's theorem would be refuted. We must quickly add, however, that violation of this condition does not necessarily lead to the paradox except in the case of only three alternatives. Example 2 illustrates a situation wherein a majority winner exists, alternative a, despite the presence of intransitivity among a subset of alternatives, b, c, and d.

Example 2:

J1: a b c d
J2: a d b c
J3: a c d b

Following Klahr (1966, p. 385), we shall, therefore, distinguish between two cases.

Definition 4. A committee ordering of alternatives for which Condition 6 is violated and a majority winner exists is said to be a Type 1 committee ordering. A Type 2 committee ordering is one for which there does not exist a majority winner and Condition 6 is violated.

In this paper we are concerned only with the likelihood calculations for a Type 2 committee ordering since we believe this to be the more interesting case. So as to make Type 2 occurrence probabilities more amenable to computation and make possible the introduction of different societal assumptions, we need to develop some additional apparatus.

Since the set X consists of n alternatives, each judge can choose from $n!$ possible individual rankings. We shall call each such ranking a vote and represent it by a matrix $V = [v_{ij}]$, where

$$v_{ij} = \begin{cases} 1 \text{ if } x_i \text{ is preferred to } x_j\,,\ i \neq j \\ 0 \text{ if } i = j \\ -1 \text{ if } x_j \text{ is preferred to } x_i\,,\ i \neq j \end{cases}$$

for all x_i and x_j in X. If X consists of three elements, $X = \{x_1, x_2, x_3\}$, labeling the columns and rows of the matrix V by the x_i in their natural order yields, corresponding to the ranking "x_1 preferred to x_2 preferred to x_3," the vote

$$x_1\, x_2\, x_3 : V^1 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix},$$

where the superscript refers to a particular permutation of the elements of X. Likewise, each of five other possible rankings of these alternatives can be represented by the following votes respectively,

$$x_1\, x_3\, x_2 : V^2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$x_2\, x_1\, x_3 : V^3 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$x_2\, x_3\, x_1 : V^4 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$x_3\, x_1\, x_2 : V^5 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$x_3\, x_2\, x_1 : V^6 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

The superscripts on the votes are the lexographical indices of the permutations of x_1, x_2, and x_3.[8] Conditions 2 and 3, together with the definition of v_{ij}, cause the matrix V to be skewsymmetric,[9] and what we shall designate as transitive ($v_{hi} > 0$ and $v_{ij} > 0 \Rightarrow v_{hj} > 0$).

This representation of the individual rankings in terms of votes leads naturally to an analogous representation for the committee ordering of alternatives by means of the arithmetic process of addition. Since each judge has an equal voice under the aggregation procedure employed, it does not matter which judges choose a particular vote, but only how many select it. We let the number of judges choosing vote t be represented by r_t, where $t = 1, 2, \cdots, n!$. Clearly,

$$\sum_{t=1}^{n!} r_t = m.$$

Definition 5. The matrix

$$Q = [q_{ij}] = \sum_{t=1}^{n!} r_t V^t \qquad (1)$$

is called the aggregate vote matrix.

Since q_{ij} denotes the difference between the number of judges favoring x_i over x_j and those favoring x_j over x_i, it follows that

$$q_{ij} \begin{cases} > 0 \text{ if and only if } x_i P x_j\,,\ i \neq j \\ = 0 \text{ if and only if } i = j \\ < 0 \text{ if and only if } x_j P x_i\,,\ i \neq j. \end{cases}$$

Q is also a skewsymmetric matrix, being the sum of skewsymmetric matrices. Transitivity of Q, however, is not preserved for every possible configuration of votes. It will prove convenient to represent a configuration of votes by a vector $\bar{r} = (r_1, r_2, \cdots r_{n!})$, to be called the vote profile. Thus, $\bar{r}$

[8] A lexographical ordering is one in which vectors are ordered in accordance with the values of their first elements. If the first elements are equal, then a comparison of the second elements is made, and so on. Thus, the ordered triple (1, 2, 3) precedes the ordered triple (1, 3, 2), while (2, 1, 3) follows (1, 3, 2) and precedes (2, 3, 1).

[9] A matrix $A = [a_{ij}]$ is said to be skewsymmetric if $-a_{ij} = a_{ji}$ for all i and j. A skewsymmetric matrix is necessarily a square matrix and has only zeros along the main diagonal.

completely characterizes the casting of votes for the set of alternatives X.

The committee's ordering of alternatives in terms of this notation can be applied to example 1 by letting $\bar{r} = (1, 0, 0, 1, 1, 0)$. The resulting aggregate vote matrix is

$$Q = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \qquad (2)$$

Translating back yields the committee ordering $\{x_1Px_2, \; x_2Px_3, \; x_3Px_1\}$, which indicates the lack of a majority winner. On the other hand, in example 3 a majority winner does exist.

Example **3:** Suppose the number of judges $m = 5$, and the number of alternatives $n = 3$. Let the vote profile be $r = (2, 1, 1, 0, 0, 1)$. Then

$$Q = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}. \qquad (3)$$

Transforming back we get the ordering $\{x_1Px_2, x_2Px_3, x_1Px_3\}$ which yields x_1 as the majority winner.

Comparison of the aggregate vote matrices (2) and (3) reveals that the former matrix does not contain a single row composed entirely of nonnegative elements while the latter does, namely the first. Referring back to the representation of individual rankings of three alternatives in terms of the matrices $V^1, V^2, \cdots, V^6$ it can be seen that each of these matrices also contains one row consisting only of nonnegative entries. The aggregate vote matrix (3) and the matrices V^t also share the property of transitivity. It can be shown that transitivity of Q implies the existence of a row composed entirely of nonnegative entries.[10] The converse is not true except in the case of three alternatives. It should also be noted that by virtue of skewsymmetry, Q cannot contain more than one nonnegative row. That is, the presence of a nonnegative row requires the presence of a nonpositive column. Consequently, each of the remaining rows must contain at least one negative element.

The above discussion, particularly the description of the composition of the aggregate vote matrix Q, leads to the following result.

Theorem: A necessary and sufficient condition for the existence of a majority winner, x_*, among the alternatives in X is that there exist a row of the aggregate vote matrix Q consisting only of nonnegative entries.

Proof: Suppose alternative x_* is the majority winner, then x_*Px_j, for all x_j is $X - \{x_*\}$. Consequently, $q_{*j} > 0$ for all $j \neq *$ and the $*^{\text{th}}$ row contains only nonnegative elements. Conversely, suppose the i^{th} row consists of nonnegative entries only, $q_{ij} > 0$ for all $j \neq i$. Then from the definition of q_{ij}, alternative x_i defeats every other alternative in a pairwise comparison and consequently is the majority winner, that is, $x_i = x_*$.

We conclude this section with the introduction of the concept of a culture.

[10] We deduce that transitivity of Q, along with skewsymmetry, implies the existence of a nonnegative row by assuming the opposite and showing that it leads to a contradiction. Thus, suppose Q is transitive and that each row contains at least one negative entry. Let $q_{1h} < 0$, $h \neq 1$, and $q_{hi} < 0$, $h \neq i$, then by transitivity $q_{1i} < 0$, $i \neq 1$. (Note that the transitivity condition (6) also obtains when the sense of the inequalities is reversed.) There must, in other words, be at least two negative entries in the first row. Reapplying this argument for $q_{1i} < 0$ and $q_{ij} < 0$, $i \neq j$ we get $q_{1j} < 0$, $j \neq 1$. Also, by skewsymmetry, $j \neq h$, and it follows that the first row must contain three negative entries. Repeating the argument once more for $q_{jk} < 0$ it follows that $q_{1k} < 0$ and that $k \neq j, k \neq i$, and $k \neq h$. The last result is obtained from the consideration that presence of at least three negative entries has been established for the j^{th} row as well as for the first. Consequently, we can select from among these three that one whose index $k \neq h$. (It follows from skewsymmetry that $k \neq i$ and $k \neq j$.) Continuing in this fashion we establish that the first row must contain $n - 1$ negative entries. The second row must contain $n - 2$ negative entries and so on leading to the conclusion that the last row cannot contain any negative entries. This fact contradicts the assumption that each row contains at least one negative entry and consequently justifies our initial assertion.

In the case of only three alternatives, the presence of a nonnegative row, together with skewsymmetry of Q, implies the transitivity property as can be readily verified by simply considering any three off-diagonal entries.

Definition 6. To each vote V^t a number s_t is assigned denoting the probability that a judge will select that vote. The resulting vector $\bar{s} = (s_1, s_2, \cdots, s_{n!})$ is called a culture, where $\sum_{t=1}^{n!} s_t = 1$.

The concept of a culture permits us to investigate the occurrence probability of the voting paradox under alternate assumptions regarding communal effects on individual preference orderings.[11]

The assignment of probabilities s_t to the votes V^t gives the r_t, $t = 1, 2, \cdots, n!$ the character of random variables. The imposition of a culture in effect amounts to sampling with replacement from the population of votes, taking as many samples as there are judges. The numbers of each type of vote chosen, $r_1, r_2, \cdots r_{n!}$, therefore, follow a multinomial distribution. The probability of obtaining any specific vote profile is

$$P(\bar{r}) = \binom{m}{r_1, r_2, \cdots, r_{n!}} \prod_{t=1}^{n!} s_t^{r_t}. \quad (4)$$

Example 4: Suppose the number of judges $m = 3$, number of alternatives $n = 3$, and the vote profile $\bar{r} = (1, 2, 0, 0, 0, 0)$. The number of ways in which this vote profile can arise is $\binom{3}{1, 2, 0, 0, 0, 0} = 3$. These are:

(i) $J1$ picks V^1, $J2$ and $J3$ pick V^2, with probability $s_1 s_2^2$

(ii) $J2$ picks V^1, $J1$ and $J2$ pick V^2, with probability $s_1 s_2^2$

(iii) $J3$ picks V^1, $J1$ and $J2$ pick V^2, with probability $s_1 s_2^2$.

Since our objective is the occurrence probability of a Type 2 intransitive group ordering, we cumulate the probabilities (4) over all profiles $\bar{r}$ for which each row of the matrix Q has at least one negative entry. We introduce the set $R = \{\bar{r}$: there does not exist a row i for which $q_{ij} \geqq 0, j = 1, 2, \cdots, n\}$, for notational convenience. We then have the function of m, n, and $\bar{s}$,

$$P(m, n, \bar{s}) = \sum_{\bar{r} \in R} \binom{m}{r_1, r_2, \cdots, r_{n!}} \cdot \prod_{t=1}^{n!} s_t^{r_t}, \quad (5)$$

the keystone of all that follows. The function $P(m, n, \bar{s})$ is the probability that a committee of m judges under the culture $\bar{s}$ will not select a majority winner from among the n alternatives.

THEORETICAL RESULTS: GUILBAUD'S NUMBER, PARTIAL AND IMPARTIAL CULTURES

We shall now present several analytic results for the case of three alternatives and an arbitrary odd number of judges with the apparatus developed above. Recalling the definition of Q, (1), we have in terms of the rs

$$Q = \begin{bmatrix} 0 & q_{12} & q_{13} \\ q_{21} & 0 & q_{23} \\ q_{31} & q_{32} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & r_1 + r_2 - r_3 - r_4 + r_5 - r_6 & r_1 + r_2 + r_3 - r_4 - r_5 - r_6 \\ -r_1 - r_2 + r_3 + r_4 - r_5 + r_6 & 0 & r_1 - r_2 + r_3 + r_4 - r_5 - r_6 \\ -r_1 - r_2 - r_3 + r_4 + r_5 + r_6 & -r_1 + r_2 - r_3 - r_4 + r_5 + r_6 & 0 \end{bmatrix} \quad (6)$$

and

$$P(m, 3, \bar{s}) = \sum_{\bar{r} \in R} \binom{m}{r_1, r_2, \cdots, r_6} \prod_{t=1}^{6} s_t^{r_t}. \quad (7)$$

From (6) we deduce that

$$R = \{\bar{r}: (q_{12} < 0 \vee q_{13} < 0) \wedge (q_{21} < 0 \vee q_{23} < 0) \wedge (q_{31} < 0 \vee q_{32} < 0)\}. \quad (8)$$

[11] The implicit assumption, in previous work in this area, that a society does not create similarities in individual preference orderings appears to be somewhat heroic. Coombs' work appears to suggest the opposite. (See Luce and Raiffa, 1957, 353–357.)

Since Q is skewsymmetric, a further reduction of R is possible,

$$R = \{\bar{r}: (q_{12} > 0 \wedge q_{23} > 0 \wedge q_{31} > 0) \ \underline{\vee}\ (q_{13} > 0 \wedge q_{32} > 0 \wedge q_{21} > 0)\}, \quad (9)$$

where the "exclusive or" symbol $\underline{\vee}$, indicates that the two possibilities are disjoint. Interpreting (9), R consists of two disjoint subsets which express the only possibilities for intransitivity to occur among the three alternatives.

Definition 7. An impartial culture is one for which the selection of any vote is equally likely, that is, $s_t = 1/n!$, $t = 1, 2, \cdots, n!$. We denote an impartial culture by $\bar{s}'$.

Under the assumption of an impartial culture and taking account of the symmetry between the subsets of R we have

$$P(m,3,\bar{s}') = 2\sum_{\bar{r}\in R'} \binom{m}{r_1, r_2, \cdots, r_6} (1/6)^m \quad (10)$$

where now

$$R' = \{\bar{r}: (q_{13} > 0 \wedge q_{32} > 0 \wedge q_{21} > 0)\}.$$

As m gets very large, the multinomial distribution approaches a multinormal distribution. According to the statistics of this limiting case, the components of $\bar{r}$ all have means $= ms_t$, variances $ms_t(1 - s_t)$, and covariances $- ms_t s_{t'}$, where $t \neq t'$ (see, for instance, Cramer, 1946, p. 318). When $s_t = 1/6$ for all t these are $m/6$, $5m/36$, and $-m/36$, respectively. If we perform the linear transformation of variables from $r_1, r_2, r_3, r_4, r_5, r_6$ to q_{12}, q_{23}, and q_{31} according to the Q matrix entries given earlier in (6), then q_{13}, q_{32}, and q_{21} are also multinormally distributed. Elementary calculations for this linear transformation will make it clear that the means for these new variables are all $= 0$, variances all $= m$, and covariances all $= -m/3$.

We shall next derive Guilbaud's result, namely that

$$\lim_{m\to\infty} P(m, 3, \bar{s}') = 1 - \frac{3}{\pi}\cos^{-1}\frac{1}{\sqrt{3}}. \quad (11)$$

We do this by noting that in the limit, the sum (10) becomes an integral evaluated over the positive orthant in the transformed variables, namely $q_{13} > 0$, $q_{32} > 0$, and $q_{21} > 0$.[12] David (1953) gives the value of this integral as

$$\frac{1}{4\pi}[2\pi - \cos^{-1}\rho_1 - \cos^{-1}\rho_2 - \cos^{-1}\rho_3], \quad (12)$$

where ρ_1 is the correlation between q_{13} and q_{32}, ρ_2 is the correlation between q_{32} and q_{21}, and ρ_3 is the correlation between q_{13} and q_{21}. In the present case, these correlations are all $-1/3$.[13] Inserting this value into David's expression (and multiplying by 2 again for the two disjoint regions) yields

$$\begin{aligned}\lim_{m\to\infty} P(m, 3, \bar{s}') &= 2\cdot\frac{1}{4\pi}\left[2\pi - 3\cos^{-1}\left(-\frac{1}{3}\right)\right] \\ &= 1 - \frac{3}{\pi}\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= .08773983876\cdots,\end{aligned} \quad (13)$$

Guilbaud's number. It represents the probability that a very large committee voting under an impartial culture on three alternatives will be unable to decide on a majority winner.

Although Guilbaud's number is one of the most important results in this area of inquiry, its relevance to actual situations is

---

[12] An introduction to multinormal integral forms and linear transformations of multinormally distributed variables is given in Cramer, pp. 310–313.

[13] In fact, since alternative x_i beats both alternatives x_j and x_k (i, j, k distinct) in exactly one third of all possible votes, the correlation between q_{ij} and q_{ik}—two elements of a single row—is always $= 1/3$, independent of n. If we let $\theta(n)$ be the probability that a large number of judges will choose V^i as majority winner from among n alternatives, then $\theta(n)$ can be calculated by a multinormal integral over $q_{i1} \geq 0 \wedge q_{i2} \geq 0 \wedge \cdots \wedge q_{in} \geq 0$ in which the correlations are all $1/3$. Unfortunately, a closed expression for $\theta(n)$ for larger n does not exist, so far as we know; however, Ruben (1954) has tabulated $\theta(n)$ by means of a recursive integral function. The relation of $\theta(n)$ to the paradox probability is $\lim_{m\to\infty} P(m, n, \bar{s}') = 1 - n\theta(n)$. (We are grateful to the referees of *Behavioral Science* for bringing Ruben's work to our attention.)

TABLE 1
IMPARTIAL CULTURE

m	n — $P(m, n, \bar{s}')$: 3	4	5	6	7	8	…	∞
3	12*/216 (.05556)	1,536/13,824 (.11111)	276,480/1,728,000 (.16000)	75,479,040/373,248,000 (.20222)	30,571,914,240/128,024,064,000 (.23880)	(.27075)	…	?**
5	540/7,776 (.06944)	1,105,920/7,962,624 (.13889)	4,964,820,480/24,883,200,000 (.19953)				…	?
7	21,000/279,936 (.07502)	688,128,000/4,586,471,424 (.15003)	(.21533)				…	?
9	785,820/10,077,696 (.07798)	(.15595)					…	?
11	28,956,312/362,797,056 (.07981)	(.15963)					…	?
13	(.08107)						…	?
15	(.08198)						…	?
17	(.08267)						…	?
	⋮	⋮	⋮	⋮	⋮	⋮	…	⋮
***∞	$1 - \frac{3}{\pi} \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (.08774)	$2\left[1 - \frac{3}{\pi} \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$ (.17548)	(.25131)	(.31524)	(.36918)	(.41509)	…	(1.00000)

* The fraction given has as denominator the number of ways in which m voters may cast $n!$ voters $= (n!)^m$. The numerator is the number of those ways which yield no majority winner. This form of $P(m, n, \bar{s}')$ is given where feasible to aid those seeking a more closed formula.

** Entries marked "?" are likely 1.000, but this is unproven.

*** $P(\infty, 3, \bar{s}')$ and $P(\infty, 4, \bar{s}')$ were computed analytically. The remainder of the entries in this row are taken from Ruben, p. 223. (See footnote 13.)

limited by the supposition of an impartial culture. It is not difficult to construct cultures for which the corresponding probability exceeds Guilbaud's number or is exceeded by Guilbaud's number. The former class of cultures might be termed "antagonistic," while the latter class might be thought of as "similar" cultures.

It is apparent that the culture $\bar{s}_1 = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0)$, for example, will invariably yield x_1 as the majority winner. Hence $P(m, 3, \bar{s}_1) = 0$ for all values of m. It is not so obvious, however, that for the culture $\bar{s}_2 = (\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0)$, a sufficiently large panel of judges will drive $P(m, 3, \bar{s}_2)$ arbitrarily close to 1.0. To see this, we substitute $r_2 = r_3 = r_6 = 0$ into expression (7) and obtain

$$P(m, 3, \bar{s}_2) = \sum_{\bar{r} \in R} \binom{m}{r_1, r_4, r_5} \frac{1}{3^m} \quad (14)$$

where

$$R = \{\bar{r}: r_2 = r_3 = r_6 = 0$$

and

$$[(r_1 + r_4 - r_5 > 0) \wedge (r_1 - r_4 + r_5 > 0) \wedge (-r_1 + r_4 + r_5 > 0)]\}.$$

Again, r_1, r_4, r_5 will be distributed multinormally in the limit for large m, with means $= m/3$, variances $= 2m/9$, covariances $= -m/9$. Transforming to new variables u, v, w, by

$$\begin{aligned} u &= (r_1 + r_4 - r_5)/m \\ v &= (r_1 - r_4 + r_5)/m \qquad (15) \\ w &= (-r_1 + r_4 + r_5)/m, \end{aligned}$$

we have u, v, w possessing means equal to $\frac{1}{3}$. We may now appeal to the law of large numbers which tells us that as m increases, the values of u, v, and w approach their means ($\frac{1}{3}$).[14] Since these means are all positive, the probability that $u > 0$, $v > 0$, $w > 0$, that is, that the voting paradox occurs, approaches 1.0 for this culture.

The point of the foregoing result is that though Guilbaud's figure—about nine percent—might be considered to be an acceptable failure rate for the majority rule procedure, it is by no means an upper bound on this failure rate. Societies may exist in which the paradox is extremely likely.

NUMERICAL RESULTS

In this section we shall evaluate expression (5) for selected values of m, n, and $\bar{s}$. Since a large number of multinomial coefficients must be calculated and many points $\bar{r}$ tested for membership in the set R, it is

[14] The law of large numbers will be found in most elementary statistics texts, for example, Mood and Graybill (1963).

SELECTED NONIMPARTIAL CULTURES
($n = 3$)

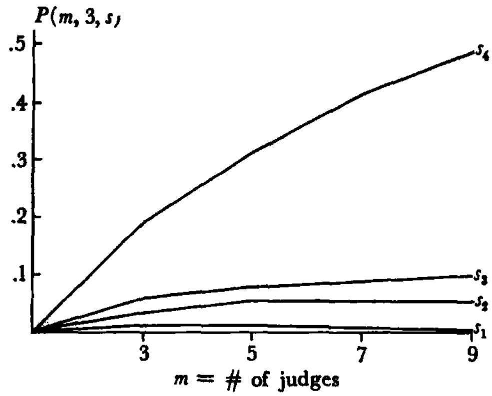

FIG. 1

Cultures shown:
$s_1 = (10/33, 1/33, 10/33, 1/33, 10/33, 1/33)$
$s_2 = (1/4, 1/4, 1/8, 1/8, 1/8, 1/8)$
$s_3 = (1/4, 1/8, 1/8, 1/4, 1/8, 1/8)$
$s_4 = (20/63, 1/63, 1/63, 20/63, 20/63, 1/63)$

natural to turn to a computer program to perform this service. An algorithm for the evaluation of expression (5) was written in the ALGOL-20 language, a variant of ALGOL-60.[15] Entries in Table 1 are values produced by that algorithm. It should be stressed that these are not Monte Carlo samples, but exact results.

CONCLUSION

From the primitive notion of an individual judge's preference between two alternatives we have developed a structure for producing group preferences by means of simple majority rule. The group preference may yield one alternative which is preferred to all others, or no such alternative. The latter case has been characterized by a probability, $P(m, n, \bar{s})$, a function of the number of judges m, the number of alternatives n, and the culture $\bar{s}$. The derived expression for $P(m, n, \bar{s})$ was seen to have a twofold significance: on the practical side, it allowed us to calculate hitherto undiscovered probabilities associated with the paradox of voting; on the theoretical side, its functional form permitted useful insights into limiting-case behavior for large numbers of judges.

[15] Copies available upon request.

While the numerical results reported herein have been restricted primarily to small values of m and n, this limitation is due to the availability of computer time rather than the representation of $P(m, n, \bar{s})$, expression (5). In other words, our formulation of $P(m, n, \bar{s})$ would yield the correct probability of the paradox for any number of alternatives, providing sufficient computer time were available. Since this proviso can seldom be met in practice however, further work aimed at finding a simpler expression is warranted.

Our framework has also permitted us to introduce societal assumptions into the problem in a natural way. We think that this step brings the entire question of the probability of the paradox closer to reality than has been the case heretofore. Another step in the same direction, though not explored here, would be consideration of the voting paradox in the context of aggregation procedures other than simple majority rule, for instance two-thirds majorities.

REFERENCES

Arrow, K. J. *Social choice and individual values.* Cowles Commission Monograph, No. 12. New York: John Wiley, 1951.

Bergson, A. *Essays in normative economics.* Cambridge, Mass.: Belknap, 1966.

Black, Duncan. *The theory of committees and elections.* Cambridge: Cambridge University Press, 1958.

Campbell, C. D., & Tullock, G. A measure of the importance of cyclical majorities. *Econ. J.*, 1965, 75, 853–857.

Cramer, H. *Mathematical methods of statistics.* Princeton: Princeton University Press, 1946.

David, F. N. A note on the evaluation of the multivariate normal integral. *Biometrika*, 1953, 60, 458–459.

Guilbaud, G. T. Les théories de l'interet general et les problèmes logique de l'agregation. *Economie Appliqué*, 1952, 5, 501-584.

Kendall, M. G. *The advanced theory of statistics.* Vol. 1. (3rd ed.) London: Griffin, 1947.

Klahr, D. A computer simulation of the paradox of voting. *Amer. pol. sci. Rev.*, 1966, 60, 384–390.

Luce, R. D., & Raiffa, H. *Games and decisions.* New York: John Wiley, 1957.

May, K. O. Intransitivity, utility, and the aggregation of preference patterns. *Econometrica*, 1954, 22, 1–13.

Mood, A. M., & Graybill, F. A. *Introduction to the theory of statistics.* New York: McGraw-Hill, 1963.

Riker, W. H. *Arrow's theorem and some examples of the paradox of voting.* Arnold Foundation Monograph. Dallas: Southern Methodist University, 1965.

Riker, W. H. Voting and the summation of preferences: An interpretive bibliographical review of selected developments during the last decade. *Amer. pol. sci. Rev.*, 1961, 5, 900–911.

Ruben, H. On the moments of order statistics in samples from normal populations. *Biometrika*, 1954, 41, 200–227.

(Manuscript received September 13, 1967)

I have no inclination to keep the domain of the psychological floating as it were in the air, without any organic foundation . . . Let the biologists go as far as they can and let us go as far as we can. Some day the two will meet.

SIGMUND FREUD