Tutorial Group	**LB12**
Name	**Sidhant Bhavnani**
Student ID	**2482327B**

Algorithmic Foundations 2

Assessed Exercise 1

Notes for guidance

1. This is the first of two assessed exercises. Each is worth 10% of your final grade for this module. Your answers must be the result of your own individual efforts.
2. Please use the latex template and submit your the generated pdf via moodle (do not submit the latex source file).
3. Please ensure you have filled out your tutorial group, name and student id.
4. **Failure to follow the submission instructions will lead to a penalty for non-adherence to submission instructions of 2 bands**.
5. As stated on the cover sheet deadline for completing this assessed exercise is **16:30 Monday October 28, 2019**.
6. The exercise is marked out of 30 using the included marking scheme. Credit will be given for partial answers.

1. Using the laws of logical equivalence and show (please ensure you give the rules used in each step):

   (a) $p \to (p \lor q)$ is a tautology [2]

Solution:
Given: $p \to (p \lor q)$
To Prove: That the given logical equivalence is a Tautology, or in other words, is True.
By logical equivalence laws we have:

$$\begin{array}{ll} p \to (p \lor q) & \\ \equiv \neg p \lor (p \lor q) & \text{Implication Law} \\ \equiv (\neg p \lor p) \lor q & \text{Associative Law} \\ \equiv (True) \lor q & \text{Tautology Law} \\ \equiv True & \text{Domination Law} \end{array}$$

Thus, the given logical equivalence is a Tautology.

   (b) $((p \to q) \lor (\neg p \to r)) \to (q \lor r) \;\equiv\; q \lor r$ [4]

Solution:
To Prove: $((p \to q) \lor (\neg p \to r)) \to (q \lor r) \;\equiv\; q \lor r$
Proof:
Starting from LHS and by using logical equivalence laws we have:

$$\begin{array}{ll} ((p \to q) \lor (\neg p \to r)) \to (q \lor r) & \\ \equiv \neg((\neg p \lor q) \lor (p \lor r)) \lor (q \lor r) & \text{Implication Law} \\ \equiv (\neg(\neg p \lor q) \land \neg(p \lor r)) \lor (q \lor r) & \text{De Morgan's Law} \\ \equiv (p \land \neg q \land \neg p \land \neg r) \lor (q \lor r) & \text{De Morgan's Law} \\ \equiv ((p \land \neg p) \land (\neg q \land \neg r)) \lor (q \lor r) & \text{Associative Law} \\ \equiv ((False) \land (\neg q \land \neg r)) \lor (q \lor r) & \text{Negation Law} \\ \equiv (False) \lor (q \lor r) & \text{Domination Law} \\ \equiv (q \lor r) & \text{Identity Law} \\ \equiv RHS. & \end{array}$$

Since LHS=RHS, the proof is complete.

(c) $\neg\forall x \in U.\,(P(x) \lor \neg Q(x)) \;\equiv\; \exists x \in U.\,(\neg P(x) \land Q(x))$ [2]

Solution:
To Prove: $\neg\forall x \in U.\,(P(x) \lor \neg Q(x)) \;\equiv\; \exists x \in U.\,(\neg P(x) \land Q(x))$
Proof:
Starting from LHS and by using logical equivalence laws we have:

$$\begin{aligned} &\neg\forall x \in U.\,(P(x) \lor \neg Q(x)) && \\ &\equiv \exists x \in U.\,\neg(P(x) \lor \neg Q(x)) && \text{Negation of Quantifier} \\ &\equiv \exists x \in U.\,(\neg P(x) \land Q(x)) && \text{De Morgan's Law} \\ &\equiv RHS. && \end{aligned}$$

Since LHS=RHS, the proof is complete.

2. Assuming the following predicates:

- $E(x)$: x is even
- $O(x)$: x is odd
- $P(x)$: x is prime
- $L(x, y)$: x is less than or equal to y
- $Eq(x, y)$: x is equal to y

determine which of the following formulae are `true` (in your answer include an expression of the formula in concise (good) English and without variables).

(a) $\forall x \in \mathbb{Z}^+.\,(\neg O(x) \to E(x))$ [2]

Solution:
"Every positive integer that is not odd is even."
The given formula is **True**.

(b) $\exists y \in \mathbb{Z}.\,\forall x \in \mathbb{Z}.\,L(y, x)$ [2]

Solution:
"There is an integer less than or equal to every integer."
The given formula is **False**.

(c) $\forall x \in \mathbb{N}.(P(x) \to (\exists y \in \mathbb{N}.\,(L(x, y) \land \neg Eq(x, y)))$ [2]

Solution:
"Every natural number that is prime, there is a natural number less than it."
The given formula is **True**.

Next, using the above predicates and quantifiers were necessary, express the following English statements in logic.

(d) "the only prime natural number that is even is two" [2]

Solution:
$\forall x \in \boldsymbol{N}.(P(x) \wedge E(x) \rightarrow Eq(x, 2))$

(e) "there is an integer less than or equal to all other integers greater than 0" [2]

Solution:
$\exists x \in \boldsymbol{Z}.\forall y \in \boldsymbol{Z}^{+}.L(x, y)$

3. (a) Prove that $A \cap (B \cup A) = A$ using a containment proof. Explain your steps. [3]

Solution:
Given: Let A and B be sets.
To Prove: $A \cap (B \cup A) = A$
Proof:
First we show that $A \cap (B \cup A) \subseteq A$.
Let $x \in A \cap (B \cup A)$ be an arbitrary element.

$x \in A \cap (B \cup A)$	
$\Rightarrow\ x \in (A \cap B) \cup (A \cap A)$	Distributive Law
$\Rightarrow\ x \in (A \cap B) \cup A$	Idempotent Law
$\Rightarrow\ x \in (A \cap B)$ or $x \in A$	By order of Union
We know that $(A \cap B) \subseteq A$	
Thus, $x \in A$	

Thus, since $x \in A \cap (B \cup A)$ is an arbitrary element, we have $A \cap (B \cup A) \subseteq A$.

To complete the proof we will now show $A \subseteq A \cap (B \cup A)$.

Let $x \in A$ be an arbitrary element.

$$
\begin{array}{lr}
\text{We know that } (A \cap B) \subseteq A. \text{ Thus,} & \\
\Rightarrow\ x \in (A \cap B) \text{ or } x \in A & \text{By order of Union} \\
\Rightarrow\ x \in (A \cap B) \cup A & \text{By order of Union} \\
\Rightarrow\ x \in (A \cap B) \cup (A \cap A) & \text{Idempotent Law} \\
\Rightarrow x \in A \cap (B \cup A) & \text{Distributive Law}
\end{array}
$$

Thus, since $x \in A$ is an arbitrary element, we have $A \subseteq A \cap (B \cup A)$ completing the proof. QED.

(b) Prove $(A - C) \cup (B - C) = (A \cup B) - C$ using set builder notation and logical equivalences. Explain your steps. [3]

Solution:
Given: Let A, B and C be sets.
To Prove: $(A - C) \cup (B - C) = (A \cup B) - C$
Starting from RHS, we can write the expression as:

$$
\begin{array}{lr}
(A \cup B) - C \ = \ \{x \mid x \in (A \cup B) - C\} & \\
= \ \{x \mid x \in (A \cup B) \cap \overline{C}\} & \text{By order of set subtraction} \\
= \ \{x \mid x \in (A \vee B) \wedge \overline{C}\} & \text{By order of Union and Intersection} \\
= \ \{x \mid x \in (A \wedge \overline{C}) \vee (B \wedge \overline{C})\} & \text{Distributive Law} \\
= \ \{x \mid x \in (A \cap \overline{C}) \cup (B \cap \overline{C})\} & \text{By order of Union and Intersection} \\
= \ \{x \mid x \in (A - C) \cup (B - C)\} & \text{By order of set subtraction} \\
= \text{LHS} &
\end{array}
$$

Since LHS = RHS, the proof is complete, QED.

4. For each of the following functions find the inverse or explain why no inverse exists.

(a) $f : \mathbb{N} \to \mathbb{N}$ where $f(x) = x + 1$ [2]

Solution:

For the inverse of f to exist, it must be injective and surjective.

Surjective

f is not surjective as 1, an element in the co-domain of f has no pre-image in Domain of f.

Thus, the inverse of f doesn't exist.

(b) $g : \mathbb{Z} \to \mathbb{Z}$ where $g(x) = 2 - x$ [2]

Solution:

For the inverse of g to exist, it must be injective and surjective.

Injective

If g is Injective, $g(x) = g(y) \to x = y$.

$g(x) = g(y) \ \equiv \ 2 - x = 2 - y$

$\Rightarrow \ -x = -y$

$\Rightarrow \ x = y$

Thus, g is Injective.

Surjective

Let $y \in codomain(g)$ be an arbitrary element.

$y = g(x)$, for some $x \in domain(g)$.

$\Rightarrow y = 2 - x$

$\Rightarrow x = 2 - y$

We see that the expression is linear in nature, and since the domain and co-domain of g are Z, g is Surjective.

Inverse

g is Injective and Surjective, thus, its inverse exists and is defined as the following:

$g^{-1}(x) = 2 - x$ where $g^{-1} : \mathbb{Z} \to \mathbb{Z}$

(c) $h : \mathbb{Z} \to \mathbb{Z}$ where $h(x) = x^2 + 2{\cdot}x - 8$ [2]

Solution:

For the inverse of h to exist, it must be injective and surjective.

Injective

If h is Injective, $h(x) = h(y) \to x = y$.

$h(x) = h(y) \ \equiv \ x^2 + 2{\cdot}x - 8 = y^2 + 2{\cdot}y - 8$

$\Rightarrow \ x^2 - y^2 = 2y - 2x$

$\Rightarrow \ (x + y)(x - y) = -2(x - y)$

$\Rightarrow\ (x + y) = -2$
Since, $x \neq y$, h is not Injective.

Therefore,
Since h is not Injective, it is not invertable.