Algorithmic Foundations 2

Section 3 – Sets & Functions

Dr. Gethin Norman

School of Computing Science
University of Glasgow

Sets – Notation

A set **A** is an **unordered** collection of **elements** or **members**

- that means order and number of occurrences do not matter

Common sets, using enumeration and curly brackets:

- natural numbers $\mathbb{N}=\{0,1,2,3,\ldots\}$
- integers $\mathbb{Z}=\{\ldots -3,-2,-1,0,1,2,3,\ldots\}$
- positive integers $\mathbb{Z}^{+}=\{1,2,3,4,5,\ldots\}$
- rational numbers $\mathbb{Q}=\{p/q \mid p,q$ integers and q not equal to $0\}$
- the real numbers $\mathbb{R}$

Also in this course we will use...

- the **floor** of a real x, denoted $\lfloor x \rfloor$ or **floor**(x), is the largest integer smaller than x, e.g. $\lfloor 2.4 \rfloor = 2$
- the **ceiling** of a real x, denoted $\lceil x \rceil$ or **ceil**(x), is the smallest integer greater than x, e.g. $\lceil 2.4 \rceil = 3$

Sets – CS1F material (covered in more depth there)

Set builder notation $S = \{x \mid x \in U \wedge P(x)\}$

- the set of all elements of the domain U that make the predicate $P(x)$ **true**

Intersection of two sets A and B is defined by the set $A \cap B$

Union of two sets A and B is defined by the set $A \cup B$

Set inclusion, A is included in the set B is defined by $A \subseteq B$

The empty set: $\varnothing$ (or $\{\}$) – the set that contains no elements

The universal set: U – the set that contains all elements

For set S, $|S|$ denotes the cardinality S (the number of elements in S)

For set S, $P(S)$ denotes the power set of S (the set of all subsets of S)

Collections of sets – Notation

Notation for unions and intersections over sets $A_1, A_2, \ldots, A_n$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

Sets – Power sets

If $|S|=n$ what is the size of $P(S)$?

Consider when $S=\{a,b,c\}$, we can represent each subset with a bit string of length $|S|=3$

- 1 if element is a member
- 0 if not a member

a	b	c	
0	0	0	∅
0	0	1	{c}
0	1	0	{b}
0	1	1	{b,c}
1	0	0	{a}
1	0	1	{a,c}
1	1	0	{a,b}
1	1	1	{a,b,c}

$P(S)$ is therefore of size 2^n

- number of bits string of length n (see section on Counting for why)

Detour – Russell's Paradox

Bertrand Russell (1872 – 1970)

A set can be a member of a set (it is a member of its power set), so…

Let $S = \{ x \mid x \notin x \}$

- the set of sets which are not members of themselves

Question: is S a member of S, i.e. is $S \in S$

- if S a member of S, then by definition of S, we have $S \notin S$ which is a contradiction
- if S is not a member of S, then by definition of S, S is a member of S which again is a contradiction

So S cannot exist, i.e. is not well defined

Detour – Russell's Paradox – In logic

Suppose there is a town with just one male barber

In this town, every man keeps himself clean-shaven by either:

- shaving himself
- going to the barber

Another way to state this is:

- the barber shaves only those men in town who do not shave themselves

In logic: $\exists x \in M.(\mathrm{barber}(x) \wedge \forall y \in M.(\mathrm{shaves}(x,y) \leftrightarrow \neg \mathrm{shaves}(y,y)))$

Who shaves the barber?

- if the barber does shave himself, then the barber must not shave himself
- if the barber does not shave himself, then the barber must shave himself

Contradiction since when $x=y$ we have $\mathrm{shaves}(x,x) \leftrightarrow \neg \mathrm{shaves}(x,x)$

Sets – Cartesian product (also covered in CS1Q)

A set of ordered tuples: $A \times B = \{ (a,b) \mid a \in A \land b \in B \}$

Example: $A=\{1,2,3,4\}$ and $B=\{a,b,c\}$

$A \times B=\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),$
$(3,a),(3,b),(3,c),(4,a),(4,b),(4,c)\}$

Ordered tuples so $A \times B$ and $B \times A$ are not the same set

In the example above…

- $(1,a) \in A \times B$ and $(1,a) \notin B \times A$
- $(a,1) \notin A \times B$ and $(a,1) \in B \times A$

Sets – Subsets

Improper subset: $A \subseteq B \equiv \forall x.(x \in A \rightarrow x \in B)$

- notice a set is a (improper) subset of itself

Is the empty set $\emptyset$ an improper subset of anything?

- the empty set is a subset of every set
  - $x \in \emptyset$ is always **false** making the implication always **true**

Is anything an improper subset of the emptyset?

- only the emptyset
  - for any other set A there exists x such that $x \in A$ is **true** while is $x \in \emptyset$ **false**

Proper (strict) subset: $A \subset B \equiv A \subseteq B \wedge \exists y.(y \in B \wedge y \notin A)$

If $A \subset B$, then A is strictly smaller than B, i.e. $|A|<|B|$

Equal sets

$$
\begin{aligned}
A=B \quad &\equiv \forall x.(x\in A \leftrightarrow x\in B) \\
&\equiv \forall x.((x\in A \rightarrow x\in B) \wedge (x\in B \rightarrow x\in A)) \\
&\equiv \forall x.(x\in A \rightarrow x\in B) \wedge \forall x.(x\in B \rightarrow x\in A) \\
&\equiv A\subseteq B \wedge B\subseteq A
\end{aligned}
$$

Therefore to prove two sets are equal it is equivalent to show that both **A** is a subset of **B** and **B** is a subset of **A**

Disjoint sets

Two sets are disjoint if they contain no common elements

- i.e. their intersection is empty

Formally: $(A \cap B = \emptyset) \rightarrow \texttt{disjoint}(A,B)$ alternatively $\forall x.(x \in A \rightarrow x \notin B)$

Confused as you expect the formula to be symmetric?

Recall an implication is logically equivalent to its contrapositive

- implication $p \rightarrow q$
- contrapositive $\neg q \rightarrow \neg p$

Hence... $\forall x.(x \in A \rightarrow x \notin B) \equiv \forall x.(\neg(x \notin B) \rightarrow \neg(x \in A))$

$\equiv \forall x.(x \in B \rightarrow x \notin A)$

Binary representation of sets

U = {0,1,2,3,4,5,6,7}

A = {0,1,3,5,7} binary representation: **10101011**

B = {0,2,4,7} binary representation: **10010101**

How do we compute the following?

- membership of an element in a set
- union of two sets
- intersection of two sets
- complement of a set
- set difference

Set operations and logic – Union

The set of elements **x**, where **x** is in **A** or **x** is in **B**

$A \cup B = \{x \mid x \in A \lor x \in B\}$

Enumerate every possible combination of **x** in **A** and **x** in **B**, then explore consequences

- notice correspondence with truth table for disjunction

membership table

A	B	A∪B
0	0	0
0	1	1
1	0	1
1	1	1

Set operations and logic – Intersection

The set of elements x, where x is in A and x is in B

$A \cap B = \{x \mid x \in A \wedge x \in B\}$

Enumerate every possible combination of x in A and x in B, then explore consequences

– notice correspondence with truth table for conjunction

membership table

A	B	A∩B
0	0	0
0	1	0
1	0	0
1	1	1

Complement of a set

The set of elements **x** such that **x** is not in **A**: $\overline{A} = \{x \mid x \notin A\}$

Example:

- if $U=\{1,2,3,4,5,6,7,8\}$ and $A=\{1,2,3,4\}$, then $\overline{A}=\{5,6,7,8\}$

Enumerate every possible combination of **x** in **A**

then explore consequences

- notice correspondence with truth table for negation

membership table

A	$\overline{A}$
0	1
1	0

Set difference

The set of elements **x** where **x** is in **A** and **x** is not in **B**:

$A \setminus B = \{x \mid x \in A \land x \notin B\} = A \cap \overline{B}$

Example: if $A=\{1,2,3,4,5\}$ and $B=\{4,5,6,7,8\}$, then $A \setminus B=\{1,2,3\}$

Enumerate every possible combination of **x** in **A** and **x** in **B**, then explore consequences

- notice correspondence with truth table for $A \land \neg B$

membership table

A	B	A\B
0	0	0
0	1	0
1	0	1
1	1	0

Symmetric difference

The set of elements x where either x is in A and x is not in B or x is in B and x is not in A: $A \oplus B = (A \setminus B) \cup (B \setminus A)$

Example:

- if $A=\{1,2,3,4,5\}$ and $B=\{4,5,6,7,8\}$, then $A \oplus B=\{1,2,3,6,7,8\}$

Enumerate every possible combination of x in A and x in B, then explore consequences

- notice correspondence with truth table for exclusive or

membership table

A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

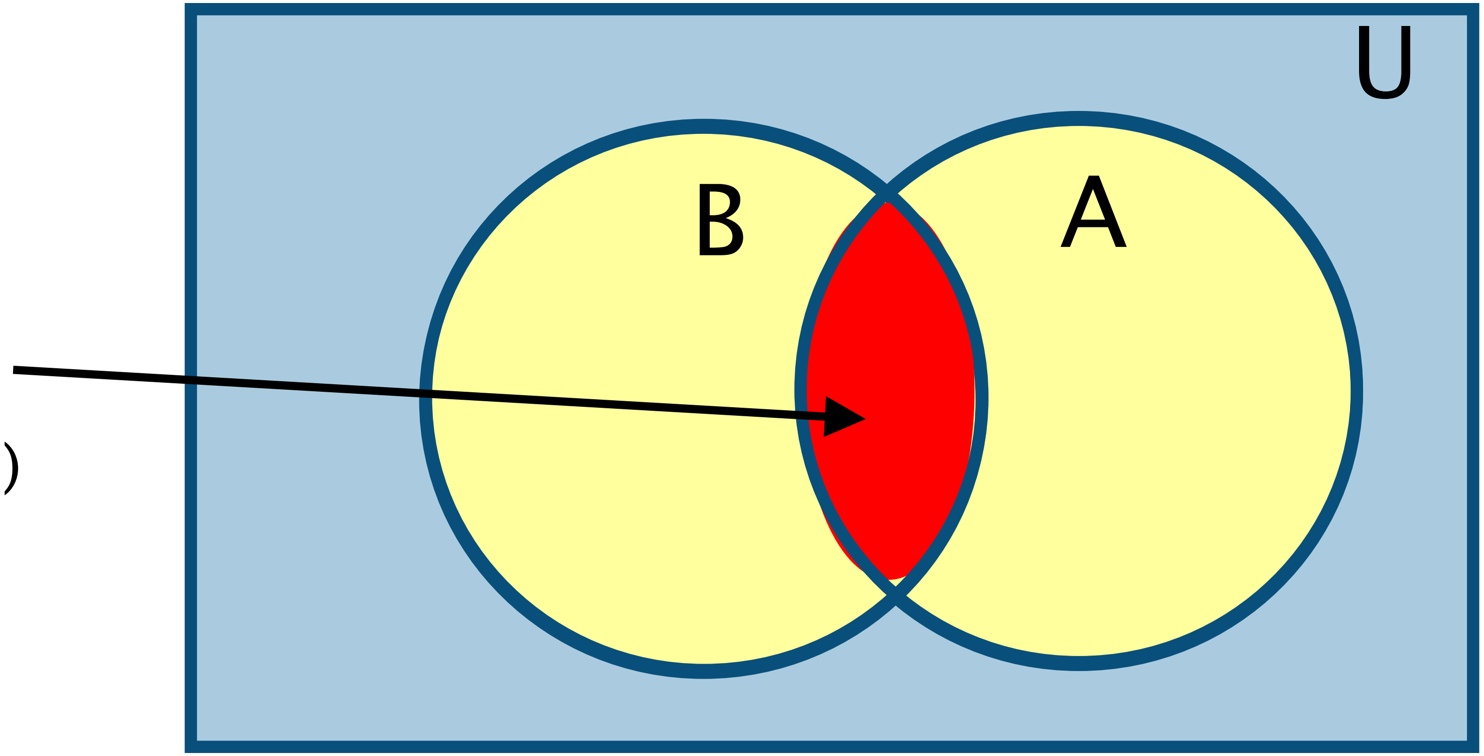
Cardinality of a set

Recall cardinality is the number of elements in a set

$$|A \cup B| = |A| + |B| - |A \cap B|$$

– the principle of inclusion-exclusion (see counting section)

potentially counted twice (“over counted”)



Cardinality of a set

Recall cardinality is the number of elements in a set

$$|A \cup B| = |A|+|B|-|A \cap B|$$

- the principle of inclusion-exclusion

Example

- $A = \{1,3,5,7,8\}$, and hence $|A| = 5$
- $B = \{1,2,4,6,7\}$, and hence $|B| = 5$
- $A \cup B = \{1,2,3,4,5,6,7,8\}$, and hence $|A \cup B| = 8$
- $A \cap B = \{1,7\}$, and hence $|A \cap B| = 2$
- $|A \cup B| = |A| + |B| - |A \cap B| = 5 + 5 - 2 = 8$

Set equivalences

Identity laws:

- $A \cup \emptyset = A$
- $A \cap U = A$

Domination laws:

- $A \cup U = U$
- $A \cap \emptyset = \emptyset$

Idempotent laws:

- $A \cup A = A$
- $A \cap A = A$

Similarities with logic equivalences:

- U as true (universal)
- ∅ as false (empty)
- union as disjunction
- intersection as conjunction
- complement as negation

Set equivalences

Similarities with logic equivalences:

- U as true (universal)
- ∅ as false (empty)
- union as disjunction
- intersection as conjunction
- complement as negation

Commutative laws:

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Associative laws:

- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cap (B \cap C) = (A \cap B) \cap C$

Distributive laws:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan laws:

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proving sets are equal

Four ways to prove two sets A and B equal

- a membership table
- a containment proof
  - show that A is a subset of B
  - show that B is a subset of A
- set comprehension notation and logical equivalences
- Venn diagrams

Proving sets are equal – Example

Show $\overline{A \cup B} = \overline{A} \cap \overline{B}$ using a membership table

membership table

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
0	0	0	**1**	1	1	**1**
0	1	1	**0**	1	0	**0**
1	0	1	**0**	0	1	**0**
1	1	1	**0**	0	0	**0**

Proving sets are equal – Example

Show $\overline{A \cup B} = \overline{A} \cap \overline{B}$ using set comprehension and logical equivalence

$\overline{A \cup B}$	$= \{x \mid x \notin A \cup B\}$	definition of set difference
	$= \{x \mid \neg(x \in A \cup B)\}$	definition of negation
	$= \{x \mid \neg(x \in A \lor x \in B)\}$	definition of union
	$= \{x \mid \neg(x \in A) \land \neg(x \in B)\}$	de Morgan's law
	$= \{x \mid x \notin A \land x \notin B\}$	definition of negation
	$= \{x \mid x \in \overline{A} \land x \in \overline{B}\}$	definition of set difference
	$= \overline{A} \cap \overline{B}$	definition of intersection

Proving sets are equal

Show A=B using containment proof

Argue that an arbitrary element in A is in B

- i.e. that A is an improper subset of B

Argue that an arbitrary element in B is in A

- i.e. that B is an improper subset of A

Conclude by saying that since A is a subset of B, and vice versa the two sets are equal

Proving sets are equal – Example

Show $\overline{A \cup B} = \overline{A} \cap \overline{B}$ using containment proof

Argue that an arbitrary element in $\overline{A \cup B}$ is in $\overline{A} \cap \overline{B}$

- consider any $x \in \overline{A \cup B}$
- by definition of complement we have $x \notin A \cup B$
- by definition of union it follows that $x \notin A$ and $x \notin B$
- by definition of complement we have $x \in \overline{A}$ and $x \in \overline{B}$
- finally by definition of intersection it follows that $x \in \overline{A} \cap \overline{B}$

Proving sets are equal – Example

Show $\overline{A \cup B} = \overline{A} \cap \overline{B}$ using containment proof

Argue that an arbitrary element in $\overline{A \cup B}$ is in $\overline{A} \cap \overline{B}$ ✔

Argue that an arbitrary element in $\overline{A} \cap \overline{B}$ is in $\overline{A \cup B}$

- consider any $x \in \overline{A} \cap \overline{B}$
- by definition of intersection we have $x \in \overline{A}$ and $x \in \overline{B}$
- by definition of complement it follows that $x \notin A$ and $x \notin B$
- by definition of union we have $x \notin A \cup B$
- finally by definition of complement it follows that $x \in \overline{A \cup B}$

Proving sets are equal – Example

Show $\overline{A \cup B} = \overline{A} \cap \overline{B}$ using containment proof

We have...

Argued that an arbitrary element in $\overline{A} \cup \overline{B}$ is in $\overline{A \cap B}$

and hence $\overline{A} \cup \overline{B}$ is an improper subset of $\overline{A \cap B}$

Argued that an arbitrary element in $\overline{A \cap B}$ is in $\overline{A} \cup \overline{B}$

and hence $\overline{A \cap B}$ is an improper subset of $\overline{A} \cup \overline{B}$

Therefore the two sets are equal

Another example proof

We will show $A \cap (B \setminus A) = \emptyset$

- using set comprehension and logical equivalences

$A \cap (B - A) = \{x \mid (x \in A) \wedge (x \in B \setminus A)\}$	definition of intersection
$= \{x \mid (x \in A) \wedge ((x \in B) \wedge (x \notin A))\}$	definition of set difference
$= \{x \mid (x \in A) \wedge ((x \notin A) \wedge (x \in B))\}$	commutative law
$= \{x \mid ((x \in A) \wedge (x \notin A)) \wedge (x \in B)\}$	associative law
$= \{x \mid ((x \in A) \wedge \neg(x \in A)) \wedge (x \in B)\}$	definition of negation
$= \{x \mid \mathtt{false} \wedge (x \in B)\}$	contradiction law
$= \{x \mid \mathtt{false}\}$	domination law
$= \emptyset$	definition of the emptyset

Functions – Introduction

From mathematics at school you should be familiar with the concept of a real-valued function $f:\mathbb{R}\rightarrow\mathbb{R}$

- which assigns to each real number $x\in\mathbb{R}$ another real value $f(x)\in\mathbb{R}$

Examples

- $f(x) = 2\cdot x + 4$
- $f(x) = x^2$

However, the notion of a function generalizes to the concept of assigning to each element of a set to an element of another set

Functions – Definition

Let X and Y be sets

A function $f:X\rightarrow Y$ is a mapping from elements of X to elements of Y

- f can be consider as subset of $X\times Y$ i.e. by a set of tuples satisfying
  $\forall x\in X.\exists y\in Y.((x,y)\in f)$ every x gets mapped to some value
  $\forall x.(\forall y_1.\forall y_2.(((x,y_1)\in f \wedge (x,y_2)\in f) \rightarrow (y_1=y_2))$ & only one value

notice ordered tuples not sets

Example

- let X be the set of lecturers {David,Gethin,Michele,Nikos}
- let Y be the set of level 2 courses {ADS2,AF2,NOSE2,JP2,OOSE2,WAD2}
- $f:X\rightarrow Y$ where $f(x)$ returns the course that the lecturer x teaches
- f={(Mary Ellen,JP2),(Gethin,AF2),(Michele,ADS2),(Nikos,NOSE2)}

Functions – Introduction

For function $f:X \rightarrow Y$

- X is the domain and Y is codomain
- if $f(x)=y$, then y is the image of x and x is the preimage of y
- there may be more than one preimage of y
  - i.e. there can exist distinct x_1 and x_2 such that $f(x_1)=y$ and $f(x_2)=y$
- there is only one image of x (otherwise not a function)

There may be an element in the codomain with no preimage

- in the example neither WAD2 nor OOSE2 is taught by one of the lecturers

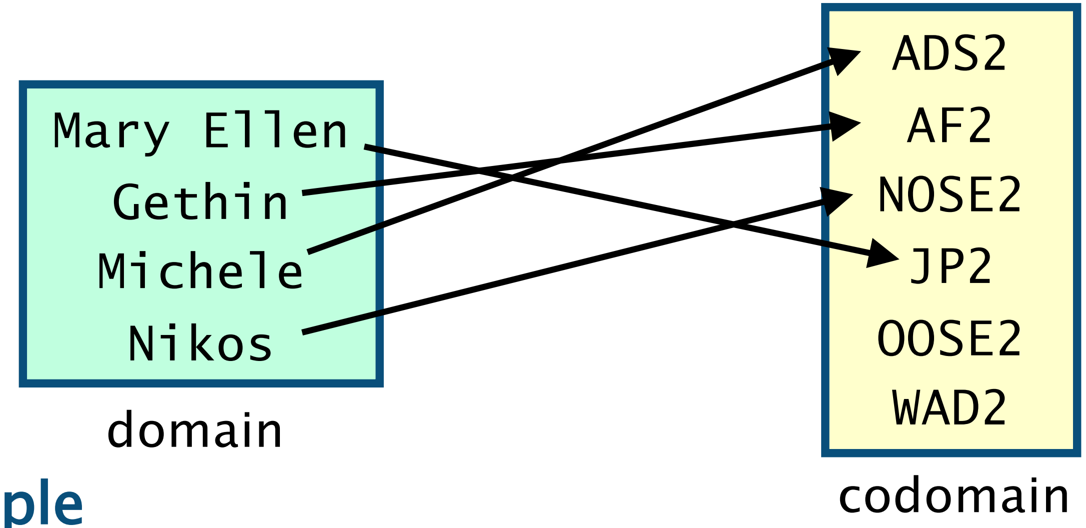
Range of f is the set of all images of X (i.e. the set of all results)

- e.g. in the previous example the rang is {ADS2, AF2, JP2, NOSE2}

Functions – Introduction

A value in the **domain** maps to only one value in **codomain**

otherwise it is not a function



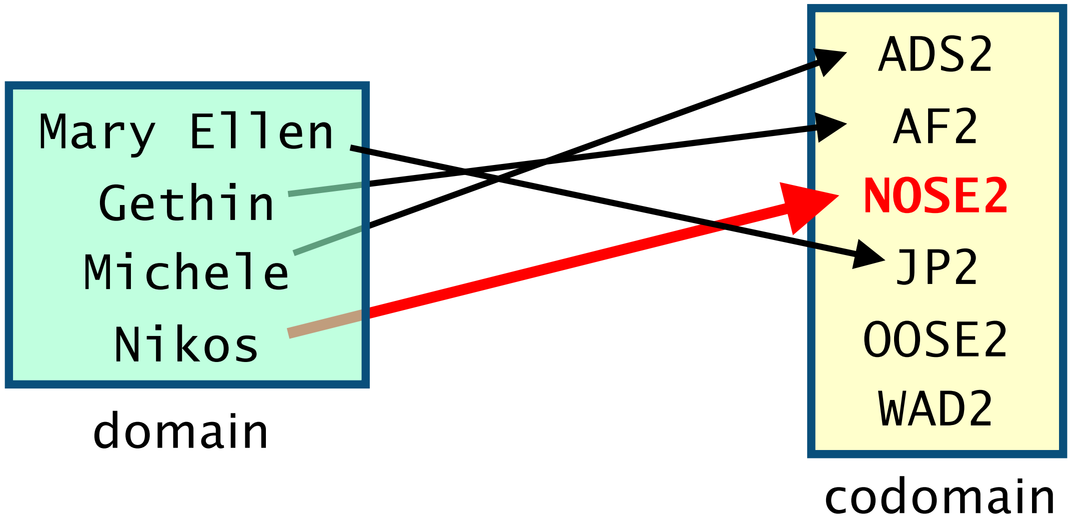
Example

- let X be the set of lecturers {David,Gethin, Michele ,Nikos}
- let Y be the set of level 2 courses {ADS2,AF2,NOSE2,JP2,OOSE2,WAD2}
- $f:X\rightarrow Y$ where f(x) returns the course that the lecturer x teaches
- f = {(Mary Ellen,JP2),(Gethin,AF2),(Michele,ADS2)},(Nikos,NOSE2)}

Functions – Introduction

A value in the domain maps to only one value in codomain

otherwise it is not a function



f(Nikos) = NOSE2

- NOSE2 is the image of Nikos
- Nikos is the preimage of NOSE2

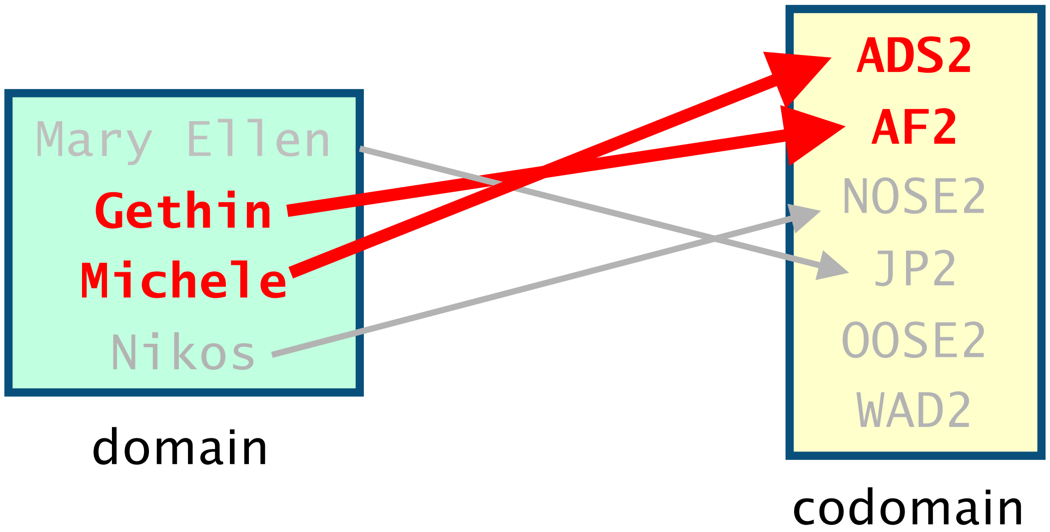
Functions – Image

The image of a set S where S is a subset of the domain

- f(S) = {f(x) | x∈S}
- i.e. the range of f when the domain of f is restricted to S

Example

- f({Gethin, Michele})={AF2, ADS2}



Composition of functions

We can add and multiply functions

- as long as the domains and codomains match and addition and multiplication makes sense
  - recall domain on the left hand side
  - codomain on the right hand side (set of results)
- addition: $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- multiplication: $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$

Examples:

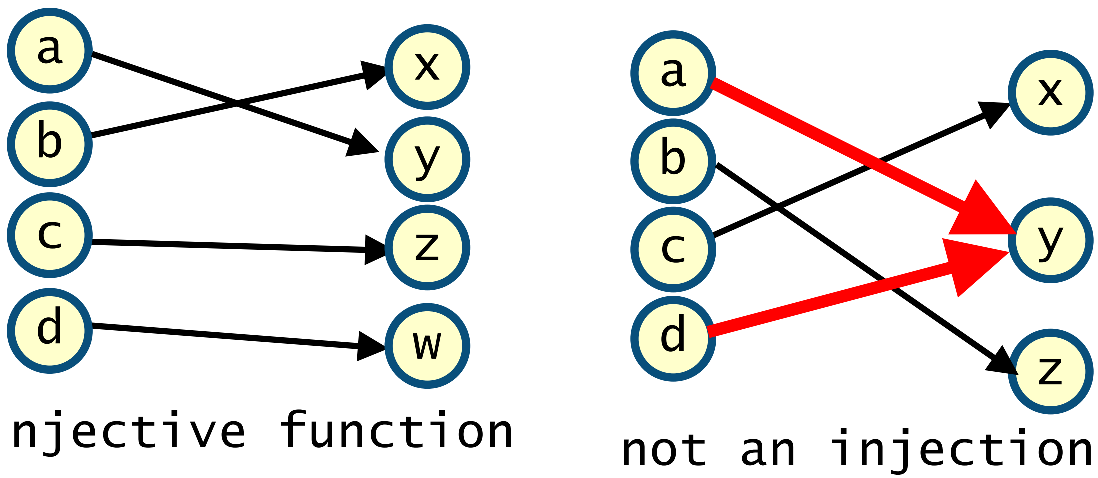
- if $f_1(x)=x^2$ and $f_2(x)=2x$, then
- $(f_1 + f_2)(x) = x^2 + 2x$
- $(f_1 \cdot f_2)(x) = 2x^3$

Functions – Injective

Injective or one-to-one-function $f{:}X \rightarrow Y$

- informally f maps different elements of X to different elements of Y
- formally $\forall x{\in}X.\forall y{\in}X.(x{\neq}y \rightarrow f(x){\neq}f(y))$
- or equivalently $\forall x{\in}X.\forall y{\in}X.(f(x){=}f(y) \rightarrow x{=}y)$

  (equivalence due to contrapositive $p{\rightarrow}q \equiv \neg q{\rightarrow}\neg p$)



Functions – Injective

Injective or one-to-one-function $f:X \to Y$

- informally f maps different elements of X to different elements of Y
- formally $\forall x \in X. \forall y \in X. (x \neq y \to f(x) \neq f(y))$
- or equivalently $\forall x \in X. \forall y \in X. (f(x) = f(y) \to x = y)$

  (equivalence due to contrapositive $p \to q \equiv \neg q \to \neg p$)

For injective functions pre-image is unique

- follows by the definition of injective

If $f:X \to Y$ is injective, then $|X| \leq |Y|$

- i.e. the size of the domain is smaller than the codomain
- since each element of X yields a different value of Y under f

Functions – Strictly increasing/decreasing

Consider a function $f{:}X \rightarrow Y$

- where X and Y are subsets of the real numbers
- the function f is strictly increasing if $x<y$, then $f(x)<f(y)$
- the function f is strictly decreasing if $x>y$, then $f(x)<f(y)$

A strictly increasing/decreasing function must be injective

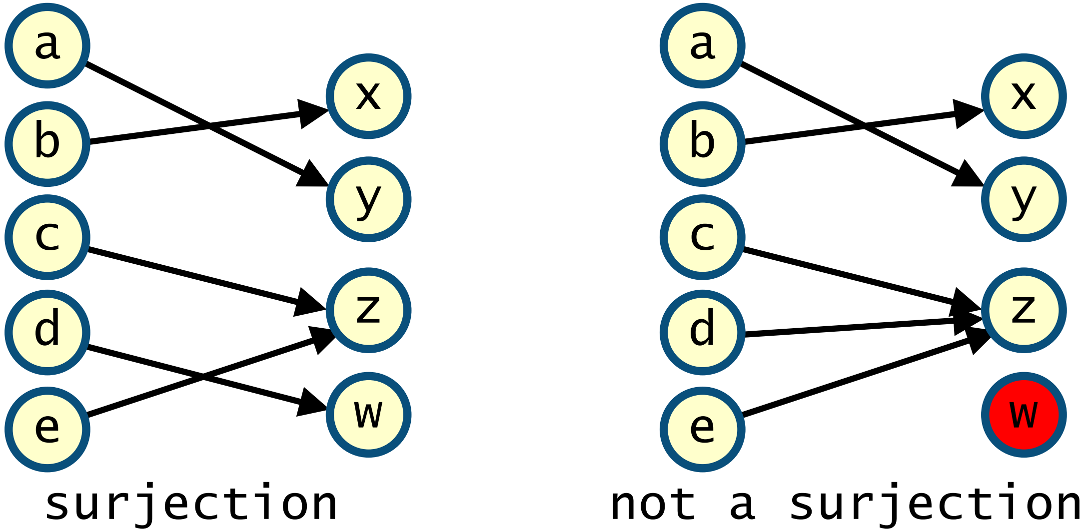
- if $f(x)=f(y)$, then $x=y$

  since if $x \neq y$ we have either $x<y$ or $y<x$, and hence $f(x) \neq f(y)$

Functions – Surjective

Surjective or onto function $f{:}X \to Y$

- informally each value in the codomain has a preimage
- formally $\forall y \in Y . \exists x \in X . (f(x)=y)$



Functions – Surjective

Surjective or onto function $f{:}X \to Y$

- informally each value in the codomain has a preimage
- formally $\forall y \in Y. \exists x \in X. (f(x)=y)$

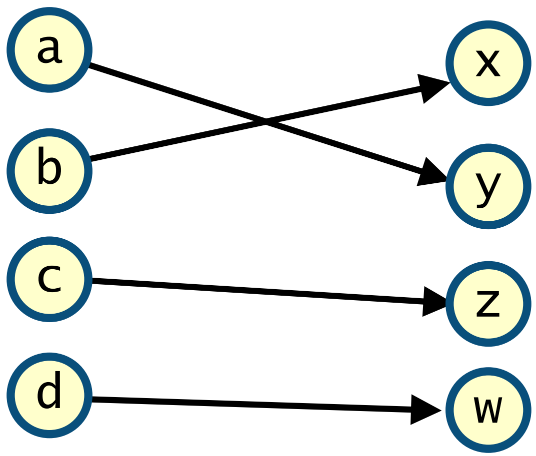
If $f{:}X \to Y$ is surjective, then $|Y| \leq |X|$

- i.e. the size of the codomain is smaller than the domain
- since each element of Y has a corresponding value (preimage) in X

Functions – Bijections

A function **f:X→Y** is **bijective** if it is both injective and surjective

- f(a)=f(b) if and only if a=b (injective)
- each element of codomain has a preimage (surjective)



bijection

Functions – Bijections

A function $f:X \to Y$ is **bijective** if it is both injective and surjective

- $f(a)=f(b)$ if and only if $a=b$ (injective)
- each element of codomain has a preimage (surjective)

If $f:X \to Y$ is bijection, then $|X| = |Y|$

- i.e the sets X and Y are of the same size (have the same cardinality)

Follows from the fact that we have shown

- if $f:X \to Y$ is injective, then $|X| \leq |Y|$
- if $f:X \to Y$ is surjective, then $|Y| \leq |X|$

(More to come later with regards to countability)

Examples

Are the following functions injections, surjections and/or bijections

- $f : \mathbb{Z} \rightarrow \mathbb{Z}$ where $\mathbb{Z}$ is the set of integers and $f(x)=x^2$
- $g : \mathbb{N} \rightarrow \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers and $f(x)=x^2$
- $h : \mathbb{Z} \rightarrow \mathbb{E}^{\mathbb{Z}}$ where $\mathbb{E}^{\mathbb{Z}}$ is the even integers and $f(x)=2 \cdot x$

Inverse of a function

For the inverse of a function to exist the function must be a bijection

For bijective function $f:X\rightarrow Y$, the inverse of f is the function $f^{-1}:Y\rightarrow X$ where $f^{-1}(y)=x$ if $f(x)=y$

- for such an x to always exist we need f to be surjective
- for x to be unique we need f to be injective
- the inverse is also a bijection

In follows that $f^{-1}(f(x))=x$ and $f(f^{-1}(y))=y$ for all $x\in X$ and $y\in Y$

Function composition

Can only compose functions **f** and **g** if the range of **f** is a subset of the domain of **g**

For functions $f{:}X \to Y$ and $g{:}Y \to Z$ the composition is denoted $g \circ f$

- "g composed with f" or "g after f" or "g following f" or "g of f"

and defined by $g \circ f(x) = g(f(x))$

- to be well defined we need f(x) to be in the domain of g

Example

- let f be the function from student numbers to students
- let g be the function from students to postal codes
- $(g \circ f)(a)=g(f(a))$ delivers postal code for a student number

Function composition – Another example

Order of composition matters…

Suppose $f(x) = 2x+3$ and $g(x)=x^2$, then

- $(g \circ f)(x) = g(f(x)) = g(2x+3) = (2x+3)^2 = 4x^2+6x+9$

- $(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2+3$

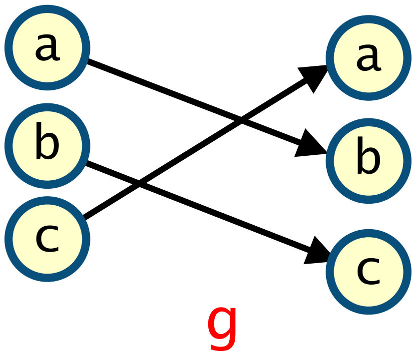
Function composition – Final Example

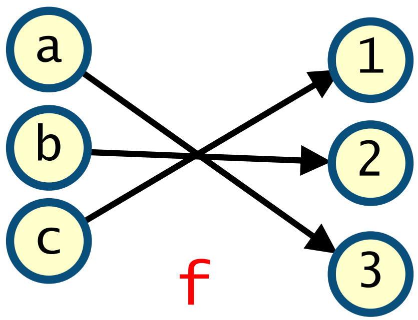
Let X={a,b,c} and Y={1,2,3}

– if $f:X\rightarrow Y$ where $f=\{(a,3),(b,2),(c,1)\}$
  and $g:X\rightarrow X$ where $g=\{(a,b),(b,c),(c,a)\}$

Give the function compositions f∘g and g∘f

– $f\circ g = \{(a,2),(b,1),(c,3)\}$





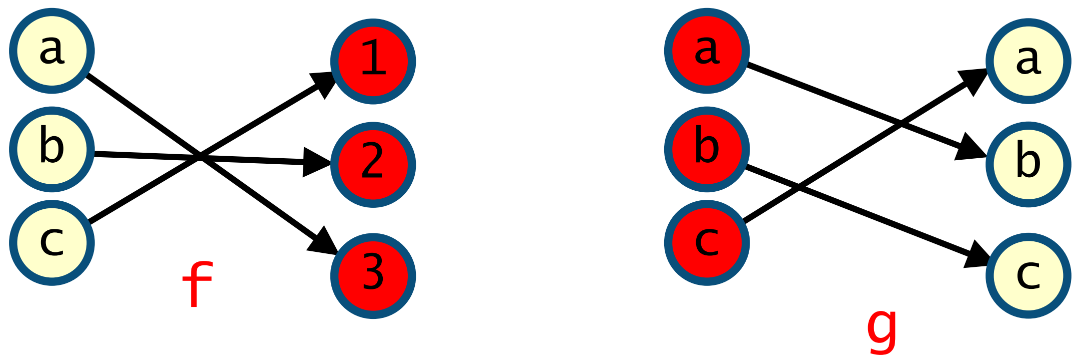
Function composition – Final Example

Let X={a,b,c} and Y={1,2,3}

– if f:X→Y where f={(a,3),(b,2),(c,1)}
  and g:X→X where g={(a,b),(b,c),(c,a)}

Give the function compositions f∘g and g∘f

– g∘f = bad type since the range of f is not a subset of the domain of g



Countability

Recall the cardinality of a set equals the number of elements in a set

- i.e. the size of the set

The cardinality of a set A is equal to the cardinality of a set B if and only if there exists a bijection from A to B

- equivalently if there is a bijection from B to A
  (just use the inverse which is also a bijection)

Countability – Countable & uncountable sets

The cardinality of a set **A** is equal to the cardinality of a set **B** if there exists a bijection from **A** to **B**

If a set has the same cardinality as a subset of the natural numbers $\mathbb{N}$, then we say is is countable

- being countable implies that one can index/list the elements of the set i.e. first, second, third, …, 100th, …
- hence we can count the elements

Conversely if the set does not have the same cardinality as any subset of the natural numbers, then we say it is uncountable

Countability – Countably infinite

If a set has the same cardinality as ℕ (the natural numbers), then the set is called countably infinite

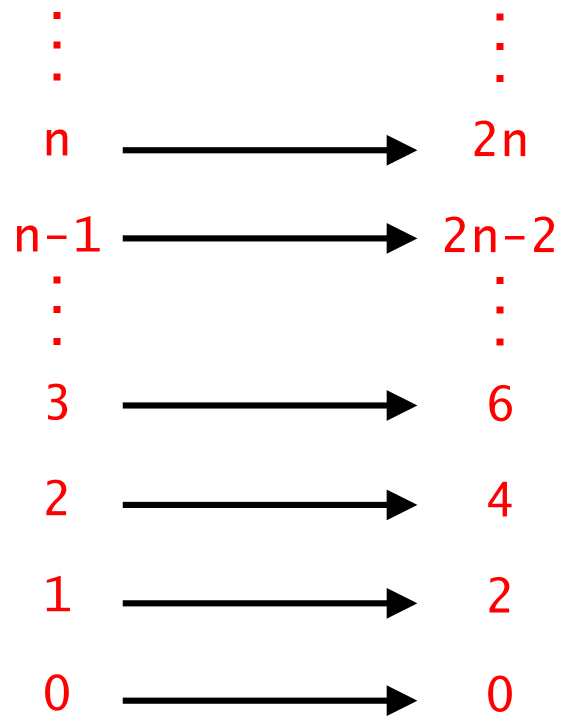
- countable as same cardinality as a subset of the natural numbers ℕ
- infinite as same cardinality as ℕ (which is infinite)

ℕ={0,1,2,3,4,…} the natural numbers

𝔼={0,2,4,6,8,…} the even natural numbers

Consider $f:\mathbb{N}\rightarrow\mathbb{E}$ where $f(x)=2\cdot x$

- the function is a bijection (we have seen this)
- with inverse $f^{-1}:\mathbb{E}\rightarrow\mathbb{N}$ where $f^{-1}(y)=y/2$



Therefore the even numbers is countably infinite

- the same cardinality as the natural numbers

Countability – Countably infinite

Similarly holds for odd numbers...

$\mathbb{N}$={0,1,2,3,4,...} the natural numbers

$\mathbb{O}$={1,3,5,7,9,...} the odd natural numbers

Consider **g:$\mathbb{N}\rightarrow\mathbb{O}$** where **g(x)=2x+1**

- the function is a bijection
- with inverse $g^{-1}:\mathbb{O}\rightarrow\mathbb{N}$ where $g^{-1}(y)=(y-1)/2$

$$
\begin{array}{ccc}
\vdots & & \vdots \\
n & \longrightarrow & 2n+1 \\
n-1 & \longrightarrow & 2n-1 \\
\vdots & & \vdots \\
3 & \longrightarrow & 4 \\
2 & \longrightarrow & 5 \\
1 & \longrightarrow & 3 \\
0 & \longrightarrow & 1
\end{array}
$$

Countability – Countably infinite

Can extend approach to the integers...

$\mathbb{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$ the integers

$\mathbb{E}^{\mathbb{Z}}=\{\ldots,-4,-2,0,2,4,\ldots\}$ the even integers

Consider $f:\mathbb{Z}\rightarrow\mathbb{E}^{\mathbb{Z}}$ where $f(x)=2x$

- the function is a bijection
- with inverse $f^{-1}:\mathbb{E}^{\mathbb{Z}}\rightarrow\mathbb{Z}$ where $f^{-1}(y)=y/2$

$$
\begin{array}{ccc}
\vdots & & \vdots \\
n & \longrightarrow & 2n \\
n-1 & \longrightarrow & 2n-2 \\
\vdots & & \vdots \\
3 & \longrightarrow & 6 \\
2 & \longrightarrow & 4 \\
1 & \longrightarrow & 2 \\
0 & \longrightarrow & 0 \\
-1 & \longrightarrow & -2 \\
-2 & \longrightarrow & -4 \\
\vdots & & \vdots \\
-n+1 & \longrightarrow & -2n+2 \\
-n & \longrightarrow & -2n \\
\vdots & & \vdots
\end{array}
$$

Countability – Countably infinite

If a set has the same cardinality as $\mathbb{N}$ (the natural numbers), then the set is called **countably infinite**

$\mathbb{N}=\{0,1,2,3,4,\ldots\}$ the natural numbers

$\mathbb{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$ the integers

Can construct a function $f:\mathbb{Z}\to\mathbb{N}$ which is a **bijection**

- see tutorial sheet 3

Therefore the **integers** are countably infinite

- the same cardinality as the natural numbers

Countability – Countably infinite

Is $\mathbb{Q}^{+}=\{p/q \mid p,q\in\mathbb{N}\}$ (set of positive rationals) countable infinite?

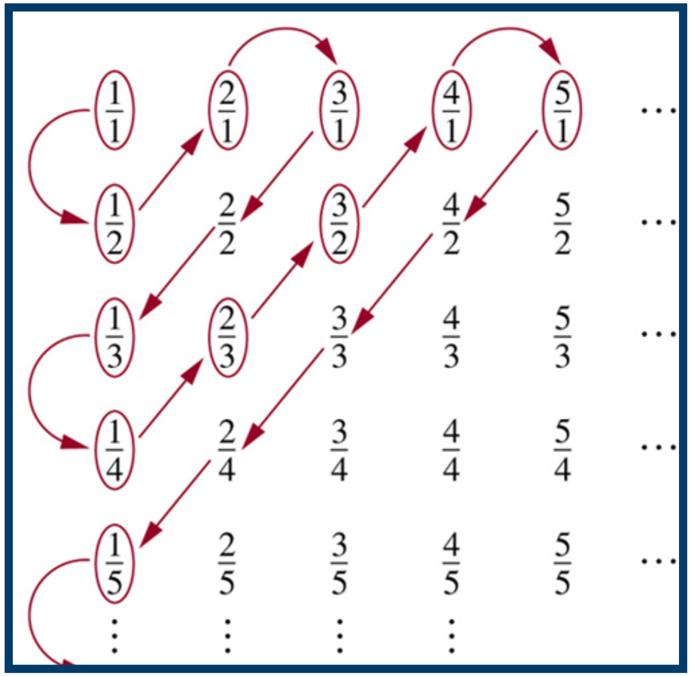
- i.e. can we arrange them in a sequence $r_0, r_1, \ldots$ containing every rational

We can do this by forming a two dimensional (infinite) grid

- the ith row list all fractions with numerator p (in an increasing order)
- the jth column list all fractions with denominator q (in increasing order)

The sequence if formed by traversing the grid in the manner shown

- skipping the duplicates (those not circled)
- corresponds to
  - first list all p/q such that $p+q=2$
  - second list all p/q such that $p+q=3$
  - third list all p/q such that $p+q=4$
  - ...



Countability – Countably infinite

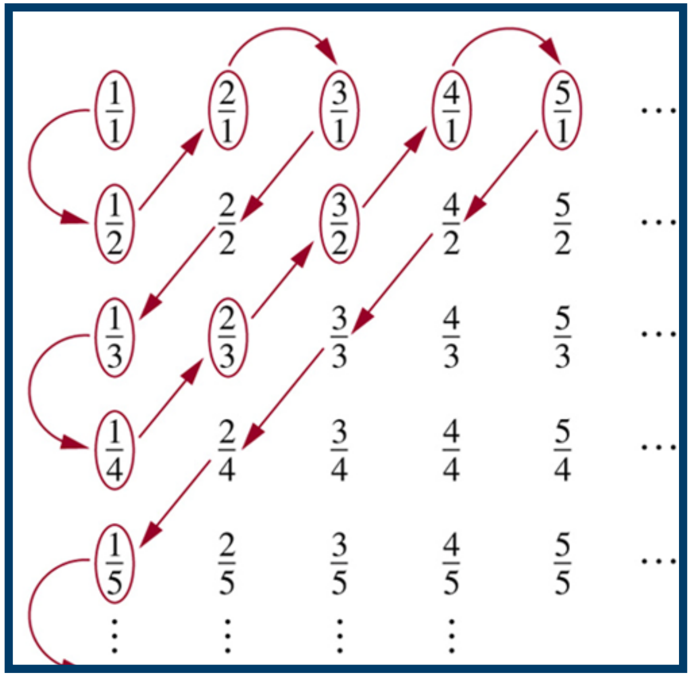
Notice we cannot just go along each row one at a time

- we would never get to the end of the first row

Using a similar argument one can show $\mathbb{N} \times \mathbb{N}$ is countably infinite

- this time there is no need to skip entries
- again we cannot go along each row

Can also show the set of all rationals $\mathbb{Q}$ is countably infinite



Countability – Countably infinite

If a set has the same cardinality as $\mathbb{N}$ (the natural numbers), then the set is called countably infinite

The set of Java programs is countably infinite

Proof sketch…

- a Java program is a (finite) string of characters over a given (finite) alphabet
- we can order these strings lexicographically (see later in the course)
- if a program fails to compile delete it
- we now have an ordered listing of all Java programs
- this implies a bijection from $\mathbb{N}$ to the list of Java programs

Only works because strings are finite (see next slide)

Countability – Uncountable sets

The set of real numbers between 0 and 1 is uncountable

Proof sketch…

$$
\begin{aligned}
r_0 &= 0.d_{0,0}d_{0,1}d_{0,2}\ldots d_{0,i}\ldots \\
r_1 &= 0.d_{1,0}d_{1,1}d_{1,2}\ldots d_{1,i}\ldots \\
r_2 &= 0.d_{2,0}d_{2,1}d_{2,2}\ldots d_{2,i}\ldots \\
&\vdots \\
r_i &= 0.d_{i,0}d_{i,1}d_{i,2}\ldots d_{i,i}\ldots \\
&\vdots
\end{aligned}
$$

- will assume that it is countable and show that yields a contradiction
- if it is countable, we can list all the real numbers between 0 and 1 as shown
  - i.e. we have $\mathbb{R}=\{r_i \mid i\in\mathbb{N}\}$
- using the list we can create a real number $x=0.x_0x_1x_2\ldots\in\mathbb{R}$ not in the list by choosing for each $i\in\mathbb{N}$ the value x_i such that $x_i \neq d_{i,i}$
  - using this construction we have x is not equal to r_i for any $i\in\mathbb{N}$
- this contradicts the fact that the list contained all the real numbers in $[0,1]$
- hence we cannot construct such a list and the set must be uncountable

Countability

A number between 0 and 1 is computable if there is a program which when given the input `i` produces the `i`th digit of the decimal expansion of that number

There are numbers between 0 and 1 that are not computable

Proof very rough sketch…

- the real numbers between 0 and 1 are uncountable
- the set of programs are countable
- hence there are more numbers than there are programs to compute them