Algorithmic Foundations 2

Section 2 – Predicate logic

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Predicate logic – Introduction

Predicate logic is the foundation of mathematical logic and culminated in Gödel's incompleteness theorem, which reveals the ultimate limits of mathematical thought:

given any finitely describable, consistent proof procedure, there will still be some true statements that can never be proven by that procedure (Kurt Gödel 1906–1978)

This means we can not discover all mathematical truths, unless we sometimes resort to making guesses

Predicate logic – Introduction

We often want to specify statements which involve variables

- e.g. $x > 3$, $x = y + 3$ or $x + y = z$
- these statements are neither **true** nor **false** when the values of the variables (i.e. x, y and z) are not specified

Predicates allow us to construct propositions which include such statements

Example statement: $x>3$

- this states "x is greater than 3"
- the variable x is the subject of the statement
- the predicate $x>3$ refers to a property the subject can have
- can be expressed by $P(x)$ where P is the predicate "is greater than 3"

Predicates – Definition

A predicate P is propositional (or Boolean) function

- (more on functions later in the course)
- a mapping from some domain (or universe of discourse) U to truth values
- $P : U \rightarrow \{\mathbf{true}, \mathbf{false}\}$
- for any element x of U, we have P(x) is either **true** or **false**

Example: let U equals the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and let the predicate P(x) be given by x>0

- P(-2) is **false**
- P(42) is **true**
- P(0) is **false**

Predicates – Definition

A predicate **P** is propositional (or Boolean) function

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Predicates can have more that one argument

Example: let the predicate **Q(x,y)** be given by **x>y**

- Q(1,2) is **false**
- Q(2,1) is **true**

Example: let the predicate **R(x,y,z)** be given by **x+y+z=4**

- R(-2,2,0) and R(8,4,4) are **false**
- R(-2,6,0) and R(1,1,2) are **true**

Predicates – Examples

A predicate is a boolean function

- i.e. delivers as a result **`true`** or **`false`**

Examples:

- `isOdd(x), isEven(x),`
- `isMarried(x), isWoman(x) …`
- `isGreaterThan(x,y)`
- `sumsToOneHundred(a,b,c,d,e)`

Predicates – Free and bounded variables

Predicates become propositions (`true` or `false`) if

- variables are assigned values or
- variables are bound with values from its domain U through quantifiers
  - quantifiers are coming soon

In the predicate P(y) the variable y is free or unbounded

- i.e. the value of y is no yet specified
- hence P(y) could be either **`true`** or **`false`** depending on the value of y

For example: $P(y) \wedge \neg P(1)$ is not a proposition

- since the variable y is free

Predicates – Another example

Let U equals the set of integers $\mathbb{Z}$ = {…,-2,-1,0,1,2,…}

– put another way: "let the domain of discourse be the set of all integers"

Let R(x,y,z) denote the statement x + y = z

What is the truth value of

– R(2,-1,3) is **false** since 2 + (-1) = 1 ≠ 3

– R(x,3,z) is unknown since both x and y are free (unbounded)

– R(3,6,9) is **true** since 3 + 6 = 9

Quantifiers – Universal

The universal quantifier asserts that a property holds for all values of a variable in a given domain of discourse

$\forall x.P(x)$ "for all values of x the predicate $P(x)$ holds (is `true`)"

However, we should really state the domain (universe of discourse)

$\forall x \in U.P(x)$ "for all values of x in domain U the predicate $P(x)$ holds"

Example: $\forall x \in \{1,2,3\}.P(x)$ is the same as $P(1) \land P(2) \land P(3)$

- correspondence between universal quantification and conjunction

Quantifiers – Existential

The existential quantifier asserts that a property holds for some values of a variable in a given domain of discourse

$\exists x.P(x)$ "for some values of x the predicate $P(x)$ holds (is `true`)"

However, again we should really state the domain

$\exists x \in U.P(x)$ "for some values of x in domain U predicate $P(x)$ holds"

Example: $\exists x \in \{1,2,3\}.P(x)$ is the same as $P(1) \lor P(2) \lor P(3)$

- correspondence between existential quantification and disjunction

Quantifiers – binding and scope

Variables can be **bound** through **quantifiers**

– unbound variables are also called free variables

A variable x is bound to quantifier ∀x or ∃x if

– it appears free within the scope of the quantifier

Examples: $\forall x.(P(y) \wedge Q(x))$ $\exists x.\forall y.(R(y,x) \wedge Q(x))$

If a quantifier does not bind any variables it can be removed

Example: $\forall y.\exists x.P(x)$

– since y is not a free variable in $\exists x.P(x)$, the "$\forall y$" quantifier is not used to bind any variables and can be removed

Quantifiers – binding and scope

Variables can be **bound** through **quantifiers**

- unbound variables are also called free variables

A variable x is **bound** to quantifier $\forall x$ or $\exists x$ if

- it appears free within the scope of the quantifier

Examples: $\forall x.(P(y) \wedge Q(x))$ $\exists x.\forall y.(R(y,x) \wedge Q(x))$

If a quantifier does not bind any variables it can be removed

Example: $\forall x.\exists x.P(x)$

- since x is not a free variable in $\exists x.P(x)$, the "$\forall x$" quantifier is not used to bind any variables and can be removed

Quantifiers – binding and scope

More examples…

For the formula $(\forall x.P(x)) \wedge Q(x)$ the variable x appearing in $Q(x)$ is outside of the scope of the "$\forall x$" quantifier, and is therefore free

However in $\forall x.(P(x) \wedge Q(x))$ both x's are within the scope of "$\forall x$"

$(\forall x.P(x)) \wedge (\exists x.Q(x))$ is a valid formula and the x's are different

Take note: parentheses (brackets) are again important

Also omitted "$\in U$" to simplify the presentation

Quantifiers – binding and scope

$\forall x \in U. P(x) \quad \equiv \quad \forall y \in U. P(y) \quad \equiv \quad \forall z \in U. P(z)$

$\forall x \in U. \exists y \in U. R(x,y) \equiv \forall x \in U. \exists z \in U. R(x,z) \equiv \forall y \in U. \exists x \in U. R(y,x)$

$\forall x \in U. \exists y \in U. R(x,y)$ and $\forall x \in U. \exists x \in U. R(x,x)$ are not equivalent

- the second formula is actually the same as $\exists x \in U.\ R(x,x)$
- since both x's in R are bound to the inner existential quantifier

Also, if a variable does not appear free in a formula/predicate we can change the scope, to include the formula. For example:

- $\forall x \in U. P(x) \land \exists y \in U. Q(y) \equiv \forall x \in U. \exists y \in U. (P(x) \land Q(y))$
- $\forall x \in U. (P(x) \rightarrow \exists y \in U. R(x,y)) \equiv \forall x \in U. \exists y \in U. (P(x) \rightarrow R(x,y))$

Nesting of quantifiers – Ordering matters

$\forall x.\exists y.Q(x,y)$ for all x we can find a y such that $Q(x,y)$ holds

$\exists y.\forall x.Q(x,y)$ we can find a y such that $Q(x,y)$ holds for all x

If $\exists y.\forall x.Q(x,y)$ holds, then $\forall x.\exists y.Q(x,y)$ also holds

- if we can find an y such that $Q(x,y)$ holds for all x
  then clearly for any x we can find a y such that $Q(x,y)$ holds

If $\forall x.\exists y.Q(x,y)$ holds, then it does not follow $\exists y.\forall x.Q(x,y)$ holds

- if for any x we can find a y such that $Q(x,y)$ holds
  then the y's might be different for each x so it does **not** mean
  we can find a y such that $Q(x,y)$ holds for all x

Nesting of quantifiers – Ordering matters

Let **U** equals the set of integers $\mathbb{Z}=\{\dots,-2,-1,0,1,2,\dots\}$

- put another way: "let the domain of discourse be the set of all integers"

Let **P(x,y)** denote the statement **x>y**

- $\forall x.\forall y.P(x,y)$ for all integers x and y we have $x>y$
  this statement is **false** take $x=y=1$, then $1>1$ does not hold
- $\forall x.\exists y.P(x,y)$ for all integers x there exists an integer y such that $x>y$
  this statement is **true** take for example $y=x-1$
- $\exists x.\ \forall y.\ P(x,y)$ there exists an integer x such that $x>y$ for all integers y
  this statement is **false** take $y=x$ (or $y=x+1$), then $x\leq y$
- $\exists x.\exists y.P(x,y)$ there exists integers x and y such that $x>y$
  this statement is **true** take $x=2$ and $y=1$, then $x>y$

Nesting of quantifiers – Ordering matters

Let **P(x,y)** denote the statement **x>y**

$\forall x \in \{1,2\}. \forall y \in \{3,4\}. P(x,y) \equiv P(1,3) \land P(1,4) \land P(2,3) \land P(2,4)$

$\forall x \in \{1,2\}. \exists y \in \{3,4\}. P(x,y) \equiv (P(1,3) \lor P(1,4)) \land (P(2,3) \lor P(2,4))$

$\exists x \in \{1,2\}. \forall y \in \{3,4\}. P(x,y) \equiv (P(1,3) \land P(1,4)) \lor (P(2,3) \land P(2,4))$

$\exists x \in \{1,2\}. \exists y \in \{3,4\}. P(x,y) \equiv P(1,3) \lor P(1,4) \lor P(2,3) \lor P(2,4)$

Examples

For **P** and **Q** the **universe of discourse** (domain) is set of integers **U**

- $P(x)$ denote the statement $x>3$
- $Q(x,y)$ denote the statement $x+y = 0$

Consider the following:

- $\forall x.P(x)$ this is **false**, for example take $x=2$
- $\forall x.\exists y.Q(x,y)$ this is **true**, for any x take $y = -x$
- $\exists y.\forall x.Q(x,y)$ this is **false**, no single value of y for all values of x

Notice again the ordering of the quantifiers is important

Again have omitted “$\in$U” to simplify the presentation

Examples

Let U equals the set of integers $\mathbb{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$

- put another way: “let the domain of discourse be the set of all integers”

Let $R(x,y,z)$ denote the statement $x + y = z$

$\forall x \in U. \forall y \in U. \exists z \in U. R(x,y,z)$ **true** or false?

- for all x and for all y there exists z such that $z = x+y$
- given any x and y we can choose $z = x+y$ and $R(x,y,z)$ holds

$\exists z \in U. \forall x \in U. \forall y \in U. R(x,y,z)$ true or **false**?

- there exists z such that for all x and for all y we have $z = x+y$
- given any z, then fails for $x=z$ and $y=1$

Equivalences – Ordering

Above we said that the ordering of quantifiers was important

However, we can swap the ordering when they are of the same form:

$\forall x.\forall y.Q(x,y) \equiv \forall y.\forall x.Q(x,y)$

$\exists x.\exists y.Q(x,y) \equiv \exists y.\exists x.Q(x,y)$

Equivalences – Quantifier Negation Laws

$\neg(\exists x.\neg P(x)) \equiv \forall x.P(x)$

- there does not exist an x such that $P(x)$ does not hold
- $P(x)$ holds for all x

$\neg(\forall x.\neg P(x)) \equiv \exists x.P(x)$

- $\neg P(x)$ does not hold for all x
- there exists an x such that $P(x)$ holds

What if we have $\neg(\exists x.P(x))$ and we want to express using $\forall$

$\neg(\exists x.P(x))$	$\equiv \neg(\exists x.\neg\neg P(x))$	double negation law
	$\equiv \exists\forall.\neg P(x)$	quantifier negation law

What if we have $\neg(\forall x.P(x))$ and we want to express using $\exists$

$\neg(\forall x.P(x))$	$\equiv \neg(\forall x.\neg\neg P(x))$	double negation law
	$\equiv \exists x.\neg P(x)$	quantifier negation law

Equivalences – Quantifier Negation Laws

$\neg(\forall x.\neg P(x)) \equiv \exists x.P(x)$

Example derivation using de Morgan laws

- assume domain of discourse U is $\{1,2\}$
- therefore $\forall x.P(x)$ is the same as $P(1) \wedge P(2)$
- therefore $\exists x.P(x)$ is the same as $P(1) \vee P(2)$

$\neg(\forall x.\neg P(x))$	$\equiv \neg(\neg P(1) \wedge \neg P(2))$	
	$\equiv \neg\neg(P(1) \vee P(2))$	de Morgan ($\neg(P \vee Q) \equiv \neg P \wedge \neg Q$)
	$\equiv P(1) \vee P(2)$	double negation law
	$\equiv \exists x.P(x)$	

Equivalences – Conjunction and disjunction

$\forall x.(P(x) \wedge Q(x)) \equiv (\forall x.P(x)) \wedge (\forall x.Q(x))$

- for all x we have both $P(x)$ and $Q(x)$
- for all x we have $P(x)$ and for all x we have $Q(x)$

$\exists x.(P(x) \vee Q(x)) \equiv (\exists x.P(x)) \vee (\exists x.Q(x))$

- there exists an x such that $P(x)$ or $Q(x)$
- there exists an x such that $P(x)$ or there exists an x such that $Q(x)$

Universal quantification and disjunction

$\forall x.(P(x) \vee Q(x))$ is not the same as $(\forall x.P(x)) \vee (\forall x.Q(x))$

- for all x we have either $P(x)$ or $Q(x)$
- for all x we have $P(x)$ or for all x we have $Q(x)$

$(\forall x.P(x)) \vee (\forall x.Q(x))$ implies $\forall x.(P(x) \vee Q(x))$ but not vice versa

- $\forall x.(Odd(x) \vee Even(x))$ is **true**
- $(\forall x.Odd(x)) \vee (\forall x.Even(x))$ is **false**

Existential quantification and conjunction

$\exists x.(P(x) \land Q(x))$ is not the same as $(\exists x.P(x)) \land (\exists x.Q(x))$

- there exists an x such that P(x) and Q(x)
- there exists an x such that P(x) and there exists an x such that Q(x)
- in the first x must be the same while in the second it does not

$\exists x.(P(x) \land Q(x))$ implies $(\exists x.P(x)) \land (\exists x.Q(x))$ but not vice versa

- $(\exists x.\mathrm{Equals4}(x)) \land (\exists x.\mathrm{Equals6}(x))$ is **true**
- $\exists x.(\mathrm{Equals4}(x) \land \mathrm{Equals6}(x))$ is **false**

More examples

Universe of discourse U: people

Predicates:

- $F(x)$ denotes the statement x is a female
- $P(x)$ denotes the statement x is a parent
- $M(x,y)$ denotes the statement x is the mother of y

If somebody is female and a parent then she is someone's mother:

$$\forall x \in U. ((F(x) \wedge P(x)) \rightarrow \exists y \in U. M(x,y))$$

or equivalently

$$\forall x \in U. \exists y \in U. ((F(x) \wedge P(x)) \rightarrow M(x,y))$$

Quantifiers as games

Thinking in terms of a two player game can help you tell whether a proposition with nested quantifiers is holds

The game has two players:

- `verifier`: wants to demonstrate that the proposition is **`true`**
- `falsifier`: wants to demonstrate that the proposition is **`false`**

The rules of the game:

- read the quantifiers from left to right, choosing values of variables
- if you see "$\exists$`x.`", then the `verifier` gets to select the value for `x`
- if you see "$\forall$`x.`", then the `falsifier` gets to select the value for `x`
- if the `verifier` can always win, then the proposition is **`true`**
- if the `falsifier` can win, then it is **`false`**

Quantifiers as games

The rules of the game:

- read the quantifiers from left to right, choosing values of variables
- if you see "$\exists x.$", then the **verifier** gets to select the value for x
- if you see "$\forall x.$", then the **falsifier** gets to select the value for x
- if the **verifier** can always win, then the proposition is **true**
- if the **falsifier** can win, then it is **false**

Let U equals the set of integers {…,-2,-1,0,1,2,…}

You can be the verifier and I will be the falsifier

- $\forall x.\ \exists y.\ (x=y+1)$
  - I get to choose x and then you can choose y
- $\exists y.\ \forall x.\ (x=y+1)$
  - now you have to choose y first

Summary

Predicates

- a Boolean function i.e. returns either **`true`** or **`false`**

Quantifiers

- universal quantifier asserts a property holds for all values of a variable
- existential quantifier asserts a property holds for some value of a variable

Nesting quantifiers and binding

- need to be careful
- order of quantifiers matters

Logical equivalences

- using negation can define one type of quantifier with the other