Efficient Frontier: Maximizing Return and Minimizing Risk

1. Maximizing Return for a Given Level of Risk (Variance)

Given:

- Portfolio return: $r_p = \sum_{i=1}^{n} w_i r_i$
- Portfolio variance: $\sigma_p^2 = \mathbf{w}^\top \Sigma \mathbf{w}$

Objective: Maximize r_p for a fixed risk $\sigma_p^2 = \sigma_0^2$.
Lagrangian:

$$\mathcal{L} = \sum_{i=1}^{n} w_i r_i - \lambda \left(\mathbf{w}^\top \Sigma \mathbf{w} - \sigma_0^2 \right)$$

Take the derivative with respect to w_i and set it to zero:

$$\frac{\partial \mathcal{L}}{\partial w_i} = r_i - 2\lambda \sum_{j=1}^{n} \Sigma_{ij} w_j = 0$$

This gives the system of equations for the optimal portfolio weights.
Solve for $\mathbf{w}$ (optimal weights):

$$\mathbf{w} = \frac{1}{2\lambda} \Sigma^{-1} \mathbf{r}$$

2. Minimizing Risk for a Given Level of Return

Given:

- Portfolio return: $r_p = \sum_{i=1}^{n} w_i r_i$
- Portfolio variance: $\sigma_p^2 = \mathbf{w}^\top \Sigma \mathbf{w}$

Objective: Minimize σ_p^2 for a fixed return $r_p = r_0$.
Lagrangian:

$$\mathcal{L} = \mathbf{w}^\top \Sigma \mathbf{w} - \lambda \left(\sum_{i=1}^{n} w_i r_i - r_0 \right)$$

Take the derivative with respect to w_i and set it to zero:

$$\frac{\partial \mathcal{L}}{\partial w_i} = 2 \sum_{j=1}^{n} \Sigma_{ij} w_j - \lambda r_i = 0$$

This gives the system of equations for the optimal portfolio weights.
Solve for $\mathbf{w}$:

$$\mathbf{w} = \frac{\lambda}{2} \Sigma^{-1} \mathbf{r}$$

Conclusion

By solving these optimization problems for different values of σ_0^2 or r_0, we can plot the efficient frontier, which represents the set of portfolios offering the best trade-off between risk and return.