Derivation of Portfolio Variance

Step 1: Unexpected Portfolio Return

Define the unexpected portfolio return as:

$$\tilde{r}_p - \bar{r}_p = w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2)$$

Where:

- $\tilde{r}_p$: Actual portfolio return.
- $\bar{r}_p$: Mean portfolio return.
- w_1, w_2: Weights of assets in the portfolio.
- $\tilde{r}_1, \tilde{r}_2$: Actual returns of the assets.
- $\bar{r}_1, \bar{r}_2$: Mean returns of the assets.

Step 2: Variance of Portfolio Return

The portfolio variance σ_p^2 is defined as:

$$\sigma_p^2 = \mathrm{Var}[\tilde{r}_p] = \mathbb{E}[(\tilde{r}_p - \bar{r}_p)^2]$$

Substitute $\tilde{r}_p - \bar{r}_p$ from Step 1:

$$\sigma_p^2 = \mathbb{E}\left[(w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2))^2\right]$$

Step 3: Expand the Square

Expanding the square:

$$\sigma_p^2 = \mathbb{E}\left[w_1^2(\tilde{r}_1 - \bar{r}_1)^2 + w_2^2(\tilde{r}_2 - \bar{r}_2)^2 + 2w_1w_2(\tilde{r}_1 - \bar{r}_1)(\tilde{r}_2 - \bar{r}_2)\right]$$

Step 4: Use Definitions of Variance and Covariance

Using the definitions:

- $\mathbb{E}[(\tilde{r}_1 - \bar{r}_1)^2] = \sigma_1^2$ (variance of asset 1).
- $\mathbb{E}[(\tilde{r}_2 - \bar{r}_r)^2] = \sigma_2^2$ (variance of asset 2).
- $\mathbb{E}[(\tilde{r}_1 - \bar{r}_1)(\tilde{r}_2 - \bar{r}_2)] = \text{Cov}(\tilde{r}_1, \tilde{r}_2) = \rho_{12}\sigma_1\sigma_2$ (covariance between asset 1 and asset 2).

Substituting these:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

Step 5: Generalize for n Assets (Matrix Form)

For n assets, the variance can be generalized as:

$$\sigma_p^2 = \mathbf{w}^\top \mathbf{\Sigma} \mathbf{w}$$

Where:

- $\mathbf{w} = [w_1, w_2, \ldots, w_n]^\top$: Vector of weights.
- $\mathbf{\Sigma}$: Covariance matrix, with $\Sigma_{ij} = \text{Cov}(\tilde{r}_i, \tilde{r}_j)$.