Student Number

2023 | **HSC ASSESSMENT TASK 3**

Mathematics Advanced

Year 12

General Instructions

- Working time – 45 minutes
- Weighting 25%
- Write using black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the end of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks:

35

Section I – 5 marks

- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II – 30 marks

- Attempt Questions 6 – 7
- Allow about 37 minutes for this section
- Write your solutions in the space provided

Section	Marks
Section I	/5
Section II	/30
Total marks	/35

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Section I

5 marks

Attempt Questions 1 to 5.

Allow about 8 minutes to complete this section.

Use the multiple-choice answer sheet for Questions 1-5.

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1   A particle that moves in a straight line with a displacement of x metres at time t seconds. For which of the following conditions could the particle be speeding up?

  A. $\frac{dx}{dt} < 0$ and $\frac{d^2x}{dt^2} > 0$

  B. $\frac{dx}{dt} > 0$ and $\frac{d^2x}{dt^2} < 0$

  C. $\frac{dx}{dt} > 0$ and $\frac{d^2x}{dt^2} > 0$

  D. $\frac{dx}{dt} < 0$ and $\frac{d^2x}{dt^2} = 0$

2   At time t a particle has displacement and acceleration functions $x(t)$ and $a(t)$. For which of the following functions does $x(t) = a(t)$?

  A. $x(t) = 3\sin(t) - e^t$

  B. $x(t) = 3\cos(t) - e^{-t}$

  C. $x(t) = e^t - e^{-t}$

  D. $x(t) = 3\sin(t) - 3\cos(t)$

3   The correct expression for the integral $\int \sin\frac{x}{5}\,dx$ is:

  A. $-\cos\frac{x}{5} + C$

  B. $5\cos\frac{x}{5} + C$

  C. $-5\cos\frac{x}{5} + C$

  D. $\frac{1}{5}\cos\frac{x}{5} + C$

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4 If $\log_e 3x = \log_e y - 2\log_e z$ where $x, y, z > 0$, which of the following is true?

A. $x = \frac{y - z^2}{3}$

B. $x = \frac{y}{3z^2}$

C. $\log_e 3x = \frac{y}{z^2}$

D. $\log_e 3x = \frac{\log_e y}{\log_e z^2}$
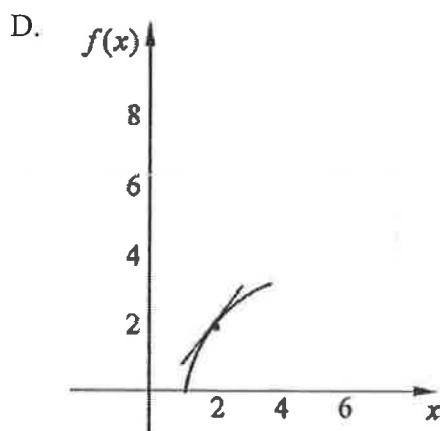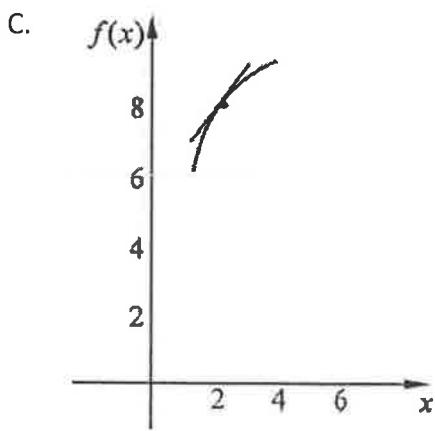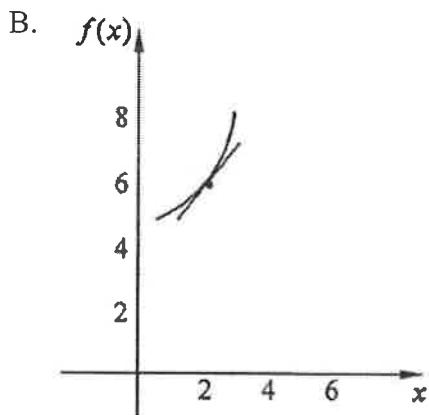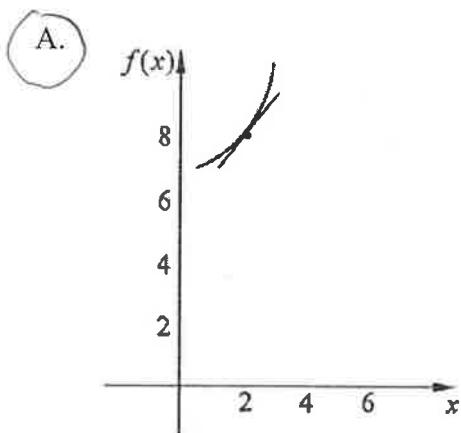
5 A function has the following properties:

$$f(2) = 8 \qquad\qquad f'(2) = 6 \qquad\qquad f''(2) = 2$$

Which sketch best matches the graph of the function near $x = 2$?

A.

B.

C.

D.

End of Section I

Section II

Question 6 (15 marks)

a. Differentiate $e^{x\cos x}$. **2**

$$\frac{d}{dx} e^{x\cos x} = (\cos x - x\sin x)e^{x\cos x}$$

b. Find the exact value of $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sec^2(2x)\, dx$. **2**

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 2\sec^2(2x)\, dx$$

$$= \frac{1}{2}\left[\tan(2x)\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$$

$$= \frac{1}{2}\left[\tan\left(\frac{4\pi}{3}\right) - \tan(\pi)\right]$$

$$= \frac{1}{2}\left[\sqrt{3} - 0\right]$$

$$= \frac{\sqrt{3}}{2}$$

c. Find $\int \frac{x-1}{3x^2-6x}\,dx.$ **2**

$$= \frac{1}{6} \int \frac{(6x-6)}{(3x^2-6x)}\,dx$$

$$= \frac{1}{6} \log_e \left| 3x^2 - 6x \right| + c$$

d. (i) Differentiate $\log_e(\sin x).$ **1**

$$= \frac{1}{\sin x} \cdot \cos x$$

$$= \frac{\cos x}{\sin x}$$

$$= \cot x$$

(ii) Hence find $\int \cot x\,dx.$ **1**

$$= \log_e(\sin x) + c$$

e. Find the equation of the normal to the curve $y = 2e^{2x} - e^x$ at the point where $x = 0$, in general form. **3**

$\frac{dy}{dx} = 4e^{2x} - e^x$

when $x = 0$

$\frac{dy}{dx} = 4e^0 - e^0$

$= 3$

$\therefore$ normal has gradient $m = \frac{-1}{3}$

when $x = 0$

$y = 2e^0 - e^0$

$= 1$

$\therefore (0, 1)$ and $m = \frac{-1}{3}$

$y - 1 = \frac{-1}{3}(x - 0)$

$y - 1 = \frac{-x}{3}$

$3y - 3 = -x$

$x + 3y - 3 = 0$

$\therefore$ normal has equation $x + 3y - 3 = 0$.

f. Consider the function $y = 2\sin x + \sin^2 x$.

(i) Show that $\frac{dy}{dx} = 2\cos x (1 + \sin x)$. **1**

$$\frac{dy}{dx} = 2\cos x + 2\sin x \cos x$$

$$\therefore \frac{dy}{dx} = 2\cos x (1 + \sin x)$$

(ii) Find the co-ordinates of the minimum turning point in the domain $[0, 2\pi]$. **3**

When $\frac{dy}{dx} = 0$

$2\cos x (1 + \sin x) = 0$
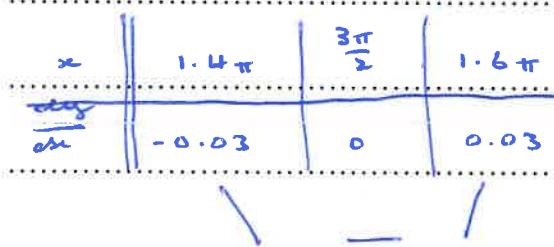
$2\cos x = 0$	$1 + \sin x = 0$
$\cos x = 0$	$\sin x = -1$
$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$	$x = \frac{3\pi}{2}$

$\therefore$ Turning points at $\left(\frac{\pi}{2}, 3\right)$ and $\left(\frac{3\pi}{2}, -1\right)$.

x	1.4π	$\frac{3\pi}{2}$	1.6π
$\frac{dy}{dx}$	-0.03	0	0.03
	$\searrow$	$-$	$\nearrow$

$\therefore$ MINIMUM turning point at $\left(\frac{3\pi}{2}, -1\right)$.

Question 7 (15 marks)

a. A radioactive substance decomposes from 100 grams to 90 grams in 2 hours according to the law $\frac{dM}{dt} = -kM$.

(i) Show $M = M_o e^{-kt}$ is a solution and find show $k = -\frac{1}{2}\log_e\left(\frac{9}{10}\right)$.   **2**

$$\frac{dM}{dt} = -k \times M_o e^{-kt}$$

$$= -kM \quad \text{as required.}$$

$$M = 100 e^{-kt} \qquad\qquad \therefore k = \frac{-1}{2}\log_e\left(\frac{9}{10}\right)$$

$$90 = 100 e^{-2k}$$

$$0.9 = e^{-2k}$$

$$-2k = \log_e(0.9)$$

(ii) Calculate the half-life of this radioactive substance, correct to the nearest minute.   **2**

$$\frac{1}{2} = e^{-kt}$$

$$\log_e\left(\frac{1}{2}\right) = -kt$$

$$t = \frac{\log_e\left(\frac{1}{2}\right)}{-\left(-\frac{1}{2}\log_e\left(\frac{9}{10}\right)\right)}$$

$$\therefore t = 13 \text{ hours } 9 \text{ minutes}$$

(iii) What percentage of the original amount has decayed after 6 hours?   **1**

$$M = 100 e^{-6k}$$

$$= 100 e^{3\log_e\left(\frac{9}{10}\right)}$$

$$\therefore M = 72.9\%$$

$$100 - 72.9 = 27.1\%$$

$$\therefore 21.7\% \text{ has decayed}$$

b. A particle moving in a straight line has velocity $v = 3e^t - 12e^{-2t}$.

The particle is initially at the origin, with t is measured in seconds and v in metres per second.

(i) Find the initial velocity of the particle. **1**

When $t = 0$. $\qquad$ $V = 3 - 12$

$$\therefore V_0 = -9 \text{ m/s}.$$

(ii) Determine by calculation if the particle is ever at rest. **2**

When $V = 0$.

$$3e^t - 12e^{-2t} = 0$$

$$12e^{-2t} = 3e^t$$

$$\frac{4}{e^{2t}} = e^t$$

$$4 = e^{3t}$$

$$\therefore 3t = \log_e 4$$

$$t = \frac{1}{3}\log_e 4$$

$$\therefore t = \frac{2}{3}\log_e 2 \quad \text{or} \quad 0.46 \text{ seconds}.$$

(iii) Find the displacement of the particle when $t = 4$ seconds, correct to 1 decimal place. **2**

$x = 3e^t + 6e^{-2t} + C$ $\qquad$ When $t = 4$

When $t = 0$, $x = 0$ $\qquad$ $x = 3e^4 + 6e^{-8} - 9$

$0 = 3 + 6 + C$ $\qquad$ $\therefore x = 154.8 \text{ m} \ (1DP)$.

$\therefore C = -9$
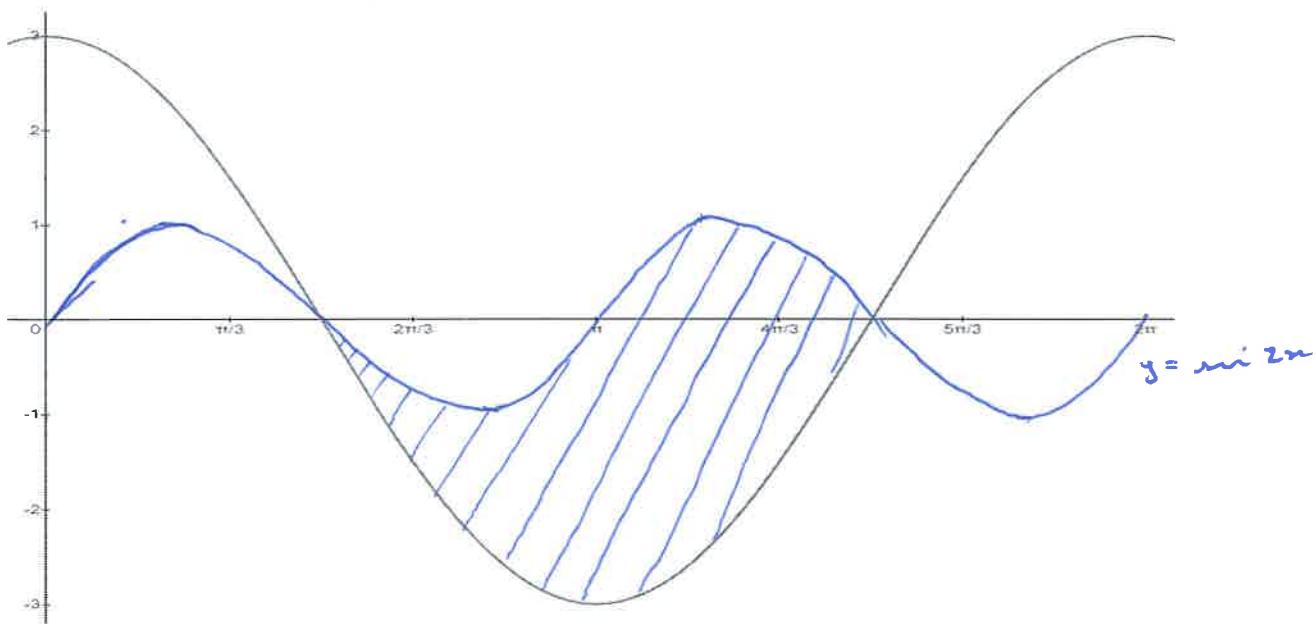
$x = 3e^t + 6e^{-2t} - 9$

c. The graph below shows the curve $y = 3 \cos x$.

$y = \sin 2x$

(i)   On the same graph draw $y = \sin 2x$ with domain $[0, 2\pi]$.   **2**

(ii)   Hence find the exact area bounded by the curves.   **3**

$$\text{Area}, A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\sin 2x - 3 \cos x \right) dx$$

$$= \left[-\frac{1}{2} \cos 2x - 3 \sin x \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left[-\frac{1}{2}(-1) - 3(-1) \right] - \left[-\frac{1}{2}(-1) - 3(1) \right]$$

$$= \frac{7}{2} - \left(\frac{-5}{2} \right)$$

$$\therefore A = 6 \text{ units}^2$$

End of Task

EXTRA WRITING SPACE

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