

Mathematics Extension 1 Task 2 2020

SOLUTIONS

Section 1:

1	2	3	4
B	C	D	C

$$\textcircled{1} \quad f(x) = \tan^{-1}\left(\frac{1}{x}\right)$$

$$f'(x) = -\frac{1}{x^2} \times \frac{1}{1 + \left(\frac{1}{x}\right)^2}$$

$$= -\frac{1}{x^2} \times \frac{1}{1 + \frac{1}{x^2}}$$

$$= -\frac{1}{x^2} \times \frac{x^2}{x^2 + 1}$$

$$= -\frac{1}{1+x^2}$$

B



$$\textcircled{2} \quad \sqrt{3}\cos\theta - \sin\theta = 2\cos(\theta + \alpha)$$

$$\sqrt{3}\cos\theta - \sin\theta = 2[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$$

Equate coefficients:

$$\cos\theta : \sqrt{3} = 2\cos\alpha \quad (1)$$

$$\sin\theta : -1 = -2\sin\alpha \quad (2)$$

$$(2) \div (1) \quad -\frac{1}{\sqrt{3}} = -\tan\alpha$$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6} \quad \therefore 2\cos\left(\theta + \frac{\pi}{6}\right)$$

C



$$\textcircled{3} \quad \cos 6x = \cos^2 3x - \sin^2 3x \\ = 1 - 2 \sin^2 3x$$

$$2 \sin^2 3x = 1 - \cos 6x$$

$$\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$$

$$\int \sin^2 3x \, dx = \frac{1}{2} \int 1 - \cos 6x \, dx \\ = \frac{1}{2} \left( x - \frac{1}{6} \sin 6x \right) + C$$

D

$$\textcircled{4} \quad y = \cos \left( 2t - \frac{\pi}{3} \right)$$

$$0 = \cos \left( 2t - \frac{\pi}{3} \right)$$

$$2t - \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$2t = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \dots$$

$$t = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \dots$$

$P$  is the second positive horizontal intercept here

$$\text{so } t = \frac{11\pi}{12}$$

$$\therefore P \text{ is } \left( \frac{11\pi}{12}, 0 \right)$$

C

## Section II:

$$\begin{aligned}
 (5)(a) \sec\theta + 2\tan\theta &= \frac{1}{\cos\theta} + 2\tan\theta \\
 &= \frac{1+t^2}{1-t^2} + 2 \times \frac{2t}{1-t^2} \\
 &= \frac{1+4t+t^2}{1-t^2}
 \end{aligned}$$

✓

$$(b) \int \frac{1}{\sqrt{9-x^2}} dx = \sin^{-1}\left(\frac{x}{3}\right) + C$$

✓

$$(c) (i) y = 30t - 5t^2$$

$\dot{y} = 0$  at maximum height

$$\dot{y} = 30 - 10t$$

$$0 = 30 - 10t$$

$$t = 3$$

$$\begin{aligned}
 \text{When } t=3, \quad y &= 30(3) - 5(3)^2 \\
 &= 90 - 45
 \end{aligned}$$

$$= 45 \text{ metres}$$

∴ Maximum height is 45 metres

$$(ii) \quad x = 30\sqrt{3}t$$

$$\dot{x} = 30\sqrt{3}$$

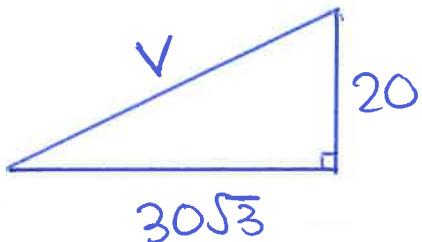
$$\text{When } t=1, \quad \dot{x} = 30\sqrt{3}$$

$$y = 30t - 5t^2$$

$$\dot{y} = 30 - 10t$$

$$\begin{aligned}
 \dot{y} &= 30 - 10(1) \\
 &= 20
 \end{aligned}$$

✓



$$\begin{aligned}v^2 &= (30\sqrt{3})^2 + 20^2 \\&= 2700 + 400 \\&= 3100\end{aligned}$$

$$\begin{aligned}v &= \sqrt{3100} \\&= 10\sqrt{31} \text{ m/s} \\&\doteq 55.7 \text{ m/s}\end{aligned}$$

$10\sqrt{31}$  or  $55.7$  are acceptable here

$\therefore$  the speed is  $10\sqrt{31}$  m/s after 1 second of motion

$$(d) \quad u = x+1$$

$$du = dx$$

$$x = u-1$$

$$\text{When } x=1, u=2$$

$$\text{When } x=0, u=1$$

$$\therefore \int_0^1 \frac{x}{\sqrt{x+1}} dx = \int_1^2 \frac{u-1}{\sqrt{u}} du$$

$$= \int_1^2 \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}} du$$

$$= \int_1^2 u^{1/2} - u^{-1/2} du$$

$$= \left[ \frac{2}{3}u^{3/2} - 2u^{1/2} \right]_1^2$$

$$= \left( \frac{2}{3} \times 2\sqrt{2} - 2\sqrt{2} \right) - \left( \frac{2}{3} - 2 \right)$$

$$= -\frac{2}{3}\sqrt{2} + \frac{4}{3}$$

$$= \frac{2}{3}(2-\sqrt{2})$$

Students can give an approximation here as their answer BUT the exact value is preferred

$$\frac{2}{3}(2-\sqrt{2}) \doteq 0.39$$

$$(e) (i) \quad y = 3\sin x + 2\cos x$$

$$3\sin x + 2\cos x = R \sin(x + \beta)$$

$$3\sin x + 2\cos x = R[\sin x \cos \beta + \cos x \sin \beta]$$

Equate coefficients:

$$\sin x: \quad 3 = R \cos \beta \quad (1)$$

$$\cos x: \quad 2 = R \sin \beta \quad (2)$$

$$\begin{aligned} (1)^2 + (2)^2: \quad 13 &= R^2 \sin^2 \beta + R^2 \cos^2 \beta \\ &= R^2 (\sin^2 \beta + \cos^2 \beta) \\ &= R^2 \end{aligned}$$

✓

$$\therefore R = \sqrt{13} \quad \text{since } R > 0$$

$$(2) \div (1) \quad \frac{2}{3} = \tan \beta$$

$$\beta \doteq 34^\circ \quad (\text{nearest degree})$$

$$\therefore y = \sqrt{13} \sin(x + 34^\circ)$$

$$(ii) \quad 3\sin x + 2\cos x = 1 \quad \text{for } 0^\circ \leq x \leq 360^\circ$$

$$\sqrt{13} \sin(x + 34^\circ) = 1 \quad \text{for } 34^\circ \leq x + 34^\circ \leq 394^\circ$$

$$\sin(x + 34^\circ) = \frac{1}{\sqrt{13}}$$

$$x + 34^\circ = 164^\circ \text{ or } 376^\circ$$

$$x = 130^\circ \text{ or } 342^\circ$$

(6)

(a) (i)

$$\cos^2 x = \sin^2 x$$

$$1 = \tan^2 x$$

$$x = -\frac{\pi}{4} \text{ or } \frac{\pi}{4} \quad \text{on the graph}$$

$$\therefore \alpha = \frac{\pi}{4}$$

$$\text{(ii) Area} = \int_a^b f(x) - g(x) \, dx \quad \text{"top curve - bottom curve"}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \, dx$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \, dx$$

$$= 2 \int_0^{\frac{\pi}{4}} \cos 2x \, dx \quad \text{because } \cos 2x \text{ is EVEN}$$

$$= 2 \left[ \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}}$$

$$= [\sin 2x]_0^{\frac{\pi}{4}}$$

$$= \sin \frac{\pi}{2} - \sin 0$$

$$= 1 - 0$$

$$= 1 \text{ unit}^2 \quad (1 \text{ square unit})$$

$$(b) \quad y = \frac{3}{\sqrt{1+4x^2}}$$

$$y^2 = \frac{9}{1+4x^2}$$

$$\text{Volume} = \pi \int_a^b y^2 dx$$

$$= \pi \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{9}{1+4x^2} dx$$

$$= 9\pi \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{4(1+x^2)} dx$$

$$= \frac{9\pi}{4} \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{(\frac{1}{2})^2 + x^2} dx$$

$$= \frac{9\pi}{4} \times \left[ \frac{1}{\frac{1}{2}} \times \tan^{-1}\left(\frac{x}{\frac{1}{2}}\right) \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{9\pi}{2} \left[ \tan^{-1}(2x) \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{9\pi}{2} \left( \tan^{-1}\sqrt{3} - \tan^{-1}1 \right)$$

$$= \frac{9\pi}{2} \left( \frac{\pi}{3} - \frac{\pi}{4} \right)$$

$$= \frac{9\pi}{2} \times \frac{\pi}{12}$$

$$= \frac{3\pi^2}{8} \text{ units}^3$$

Answer must be given as an EXACT value in its simplest form  
for full marks

(c) (i) When  $t = 2$ ,  $x = 8$  and  $y = -2$  ✓

$$\therefore 8 = 2V \cos \theta \quad \text{and} \quad -2 = 2V \sin \theta - 19.6$$

$$V \cos \theta = 4 \quad (1)$$

$$V \sin \theta = 8.8 \quad (2)$$

$$(2) \div (1) \quad \frac{V \sin \theta}{V \cos \theta} = \frac{8.8}{4}$$

$$\therefore \tan \theta = 2.2$$

mark here if  
you recognise  $y = -2$   
when  $t = 2$

(ii) When  $y = -6$  the projectile hits the ground

$$\therefore -6 = V t \sin \theta - 4.9 t^2$$

$$-6 = 8.8t - 4.9t^2 \quad \text{since } V \sin \theta = 8.8$$

$$4.9t^2 - 8.8t - 6 = 0$$

$$t = \frac{8.8 \pm \sqrt{(-8.8)^2 - 4(4.9)(-6)}}{2(4.9)}$$

$$= \frac{8.8 \pm \sqrt{195.04}}{9.8}$$

$$\therefore -0.527 \text{ or } 2.323$$

But  $t > 0$  here

So the projectile hits the ground approximately 2.3 seconds after it is launched and 0.3 seconds after it clears the vertical post ✓

$$(iii) \quad x = Vt \cos \theta$$

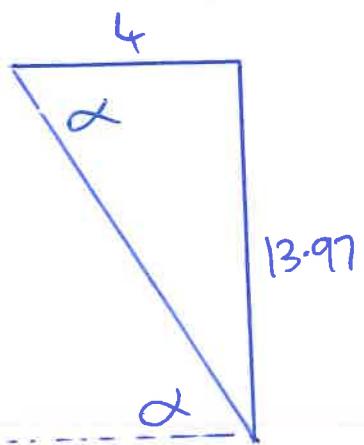
$$\dot{x} = V \cos \theta \\ = 4$$

$$y = Vt \sin \theta - 4.9t^2$$

$$\dot{y} = V \sin \theta - 9.8t \\ = 8.8 - 9.8t$$

using part (i)

$$\text{When } t = 2.323, \quad \dot{x} = 4 \quad \text{and} \quad \dot{y} = 8.8 - 9.8(2.323) \\ = -13.97 \quad \checkmark$$



let  $\alpha$  be the angle at which the projectile strikes the ground

$$\tan \alpha = \frac{13.97}{4}$$

$$\alpha = \tan^{-1} \left( \frac{13.97}{4} \right)$$

$$\approx 74^\circ \quad \checkmark$$

So the projectile strikes the ground at an angle of  $74^\circ$  with the horizontal

