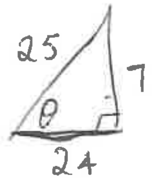
Ext 1 Yr 12 Ass #1

Suggested Solutions

Q1) Let $\cot^{-1}\frac{24}{7} = \theta$ $\therefore$



so $\sin\theta = \frac{7}{25} \longrightarrow$ A

Q2) $y = \frac{3^x - 1}{3^x + 1}$ as $x \longrightarrow +\infty$ $y \longrightarrow 1$

as $x \longrightarrow -\infty$ $y \longrightarrow -1$

hence A (e.g test large negative and positive values of x to get this result.

Q3) D All other cases are false

Q4) A All other cases are false

Q5) $\sin(3x + x) - \sin(3x - x)$

$\sin 3x \cos x + \cos 3x \sin x - (\sin 3x \cos x - \cos 3x \sin x)$

$= 2\cos 3x \sin x \longrightarrow$ B

Q6) D All others are false.

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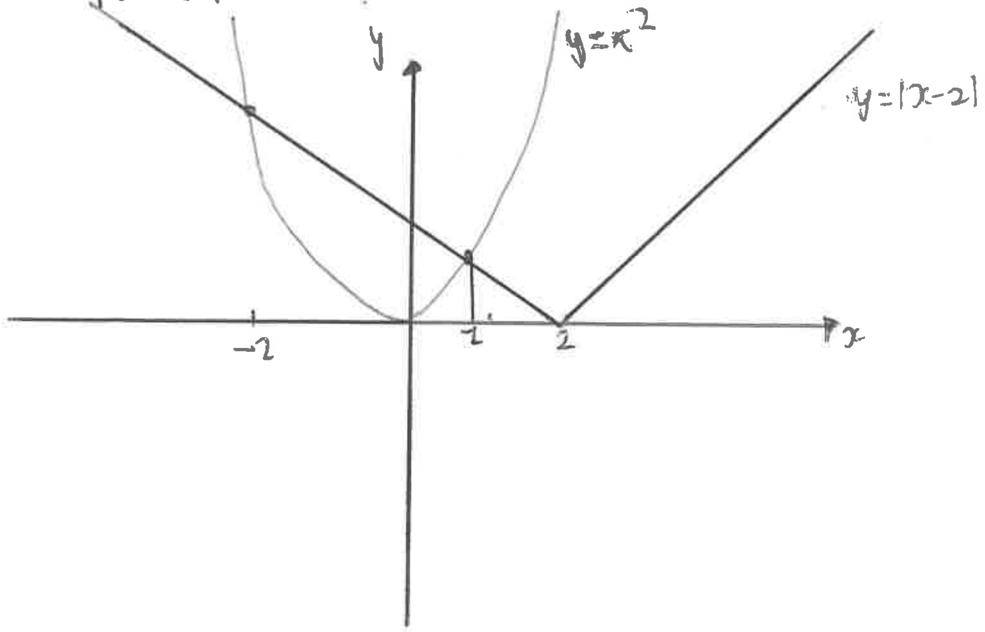
Question 7

a) $|x-2| \leq x^2$ solve.



when is x^2 value of y above $|x-2|$. or when is the parabola above $y=|x-2|$

The meet at $x^2 = -x+2$ (never meet at $x-2$)

$\therefore$ $x^2+x-2=0$, $(x+2)(x-1)=0$

meet at $x=-2$, $x=1$

$\therefore$ answer is $x \leq -2$ or $x \geq 1$.

b) Prove $2+4+ \dots +2n = n(n-1)$ $n \geq 1$

Test $n=1$

L.H.S $2(1)=2$ R.H.S $1(1-1)+2$
$= 2$

$\therefore$ L.H.S = R.H.S True for $n=1$

Q7b) Assume true for $n=k$ $k\geq 1$

$\therefore$ $2+4+ \ldots + 2k = k(k-1)+2$

Prove true for $n=k+1$

$\therefore$ $2+4+ \ldots + 2k + 2k+2 = (k+1)k + 2$

L.H.S $2+4+\ldots 2k + 2k+2$

$= k(k-1)+2 + 2k+2$ from assumption

$= k^2 - k + 2$

R.H.S $k(k+1)+2$

$k^2 + k + 2$

now L.H.S $\neq$ R.H.S hence cannot be proven by Mathematical induction

c) $\sin x - \cos x = 1$ $0 \leq x \leq 2\pi$

using t method let $\tan \frac{x}{2} = t$

$\therefore$ $\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1$

$2t + t^2 - 1 = 1+t^2$

$2t - 2 = 0$

$\therefore$ $t = 1$

so $\tan \frac{x}{2} = 1$

$\tan \frac{x}{2} = \tan \frac{\pi}{4}, \tan \frac{5\pi}{4}$

$\frac{x}{2} = \frac{\pi}{4}, \frac{5\pi}{4}$

$x = \frac{\pi}{2}, \frac{5\pi}{2}$ now $\frac{5\pi}{2} > 2\pi$

so $x = \frac{\pi}{2}$.

Q7c) however as $x = \frac{\pi}{2}$ is one of the $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ exact value solutions, we should test the other to see if the algebraic way has missed a solution.

In doing so. $\sin\pi - \cos\pi = 1$ is the only solution that works.

hence full solutions are $x = \frac{\pi}{2}, \pi$.

d) i) $\frac{1}{(k+1)!} - \frac{k+1}{(k+2)!} = \frac{1}{(k+2)!}$

L.H.S

$$\frac{1}{(k+1)!} - \frac{k+1}{(k+2)(k+1)!}$$

$$\frac{k+2}{(k+2)(k+1)!} - \frac{k+1}{(k+2)(k+1)!}$$

$$\frac{k+2-k-1}{(k+2)!}$$

$$= \frac{1}{(k+2)!} = \text{R.H.S.}$$

ii) Prove $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Test for $n = 1$

L.H.S $\frac{1}{2!} = \frac{1}{2}$ R.H.S $1 - \frac{1}{2!} = \frac{1}{2}$

$\therefore$ true for $n = 1$

7d ii) Assume true for $n=k$ $k \geq 1$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$$

Prove true for $n = k+1$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

L.H.S

$$1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$$ from assumption.

$$\frac{(k+1)! - 1}{(k+1)!} + \frac{(k+1)}{(k+2)!}$$

$$\frac{(k+2)[(k+1)! - 1]}{(k+2)\,(k+1)!} + \frac{(k+1)}{(k+2)(k+1)!}$$

$$\frac{(k+2)! - (k+2) + (k+1)}{(k+2)!}$$

$$= \frac{(k+2)! - k - 2 + k + 1}{(k+2)!}$$

$$= \frac{(k+2)! - 1}{(k+2)!}$$

$$= 1 - \frac{1}{(k+2)!} = R.H.S$$

By the principles of Mathematical induction, the result is true for integers $n \geq 1$.

Q8 a) $\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$ Letting $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$

$$\text{L.H.S} = \frac{1-\left(\frac{1-t^2}{1+t^2}\right)}{\frac{2t}{1+t^2}}$$

$$= \frac{1+t^2-1+t^2}{1+t^2} \div \frac{2t}{1+t^2}$$

$$= \frac{2t^2}{2t}$$

$$= t.$$

$$\text{R.H.S} = \frac{\frac{2t}{1+t^2}}{1+\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \frac{2t}{1+t^2} \div \frac{1+t^2+1-t^2}{1+t^2}$$

$$= \frac{2t}{2}$$

$$= t$$

$\therefore$ L.H.S = R.H.S

b) $f(x) = \frac{x^2-9}{x^2-4}.$

i) $x^2-4 \neq 0$ $\therefore x \neq 2, -2$ and these are vertical asymptotes

ii) $$f'(x) = \frac{2x(x^2-4)-2x(x^2-9)}{(x^2-4)^2}$$

$$= \frac{2x(5)}{(x^2-4)^2}$$

$$f'(x) = \frac{10x}{(x^2-4)^2}$$

$f'(x)=0$ for turning points

$\therefore x = 0.$

iii) Test $x=-1$ left of $x=0$ $f'(-1) = \frac{-10}{9}$ so \ negative

Test $x=1$ right of $x=0$ $f'(1) = \frac{10}{9}$ so / positive

This meets the condition of a minimum turning point.

Q8 c) i) $\dfrac{\cos 3x - \cos 11x}{\sin 9x + \sin 5x} = 2\sin 2x$

L.H.S $\cos 3x = \cos(7x - 4x) = \cos 7x \cos 4x + \sin 7x \sin 4x$

$\cos 11x = \cos(7x + 4x) = \cos 7x \cos 4x - \sin 7x \sin 4x$

$\sin 5x = \sin(7x + 2x) = \sin 7x \cos 2x + \cos 7x \sin 2x$

$\sin 9x = \sin(7x + 2x) = \sin 7x \cos 2x + \cos 7x \sin 2x$

$$\therefore \frac{\overset{(\cos 3x)}{(\cos 7x \cos 4x + \sin 7x \sin 4x)} - \overset{\cos 11x}{(\cos 7x \cos 4x - \sin 7x \sin 4x)}}{\underset{(\sin 9x)}{(\sin 7x \cos 2x + \cos 7x \sin 2x)} + \underset{\sin 5x}{(\sin 7x \cos 2x - \cos 7x \sin 2x)}}$$

$$= \frac{2\sin 7x \sin 4x}{2 \sin 7x \cos 2x}$$

$$= \frac{\sin 4x}{\cos 2x}$$

$$= \frac{2\sin 2x \cdot \cos 2x}{\cos 2x}$$

$\sin 4x = 2\sin 2x \cdot \cos 2x$ from $\sin 2x = 2 \sin x \cdot \cos x$

$= 2 \sin 2x$

$=$ R.H.S

ii) This turns into $2\sin 2x = 2$

$\therefore \sin 2x = 1$

$\sin 2x = \sin \frac{\pi}{2}, \sin \frac{5\pi}{2}$, but in the negative direction too will be $\sin -\frac{3\pi}{2}, \sin -\frac{7\pi}{2}$

$\therefore x = \frac{\pi}{4}, -\frac{3\pi}{4}, \frac{5\pi}{4}, -\frac{7\pi}{4}$. all closest answers to the origin.