YEAR 12 2022 MATHEMATICS ADVANCED TASK # 3

SAMPLE SOLUTIONS.

SECTION I.

1. $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx$

$= \tan x - x + c$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

B.

2. $N = N_0 e^{0.04t}$

$2 = 1 e^{0.04t}$

$\log_e 2 = 0.04t$

$t = \dfrac{\log_e 2}{0.04}$

$= \dfrac{100 \log_e 2}{4}$

$\therefore t = 25 \log_e 2$

D.

3. $t = 6$

C

4. $\dfrac{d^2x}{dt^2} < 0$, $\dfrac{dx}{dt} > 0$

C

5. $\dfrac{d}{dx} e^{x \sin 3x}$

$= e^{x \sin 3x}(x \cdot \cos 3x \cdot 3 + \sin 3x \cdot 1)$

$= e^{x \sin 3x}(3x \cos 3x + \sin 3x)$

B.

1. B
2. D
3. C
4. C
5. B.

SECTION II

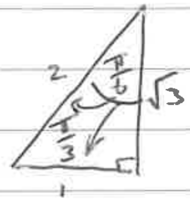
6a. $\frac{d}{dx}(\sin^2 x) = \frac{d}{dx}(\sin x)^2$

$= 2(\sin x)^1 \cdot \cos x$

$= 2\sin x \cos x$

b. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x \cos x \, dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\sin x \cos x \, dx$

$= \frac{1}{2}\left[\sin^2 x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$



$= \frac{1}{2}\left[\sin^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{6}\right)\right]$

$= \frac{1}{2}\left[\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{2}\right)^2\right]$

$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right]$

$= \frac{1}{2}\left[\frac{1}{4}\right]$

$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x \cos x \, dx = \frac{1}{8}$.

c. $\int \cos \frac{x}{5} x \, dx = \frac{1}{5}\int 5 \cos \frac{x}{5} \, dx$

$= 5 \sin \frac{x}{5} x + c$

d. $\int \frac{2x+2}{4x^2+8x+2} \, dx$

$\frac{d}{dx}(4x^2+8x+2)$

$= 8x+8$

$= \frac{1}{4}\int \frac{4(2x+2)}{(4x^2+8x+2} \, dx$

$= \frac{1}{4}\log_e(4x^2+8x+2) + c$

6a (i) $\displaystyle\int_1^e \frac{8}{x}\,dx = 8\int_1^e \frac{1}{x}\,dx$

$= 8\left[\log_e x\right]_1^e$

$= 8\left[\log_e e - \log_e 1\right]$

$= 8\left[1 - 0\right]$

$\therefore \displaystyle\int_1^e \frac{8}{x}\,dx = 8$

(ii) $\displaystyle\int_a^e \frac{8}{x}\,dx = 16$

$8\left[\log_e x\right]_a^e = 16$

$8\left[\log_e e - \log_e a\right] = 16$

$8\cancel{\log_e e} - 8\log_e a = 16$

$8 - 8\log_e a = 16$

$-8\log_e a = 8$

$\log_e a = \frac{8}{-8}$

$\log_e a = -1$

$\therefore e^{-1} = a$

ie $a = \frac{1}{e}$.

6f. (i) When $y = \sin x$ and $y = \sqrt{3}\cos x$.

$$\sin x = \sqrt{3}\cos x$$

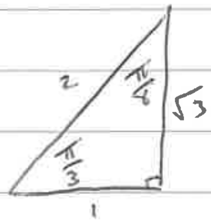
$(\div \cos x)$

$$\tan x = \sqrt{3}$$

$$x = \tan^{-1}(\sqrt{3})$$

$$\therefore x = \frac{\pi}{3} \text{ or } \left(\pi + \frac{\pi}{3}\right)$$

$$\text{i.e. } x = \frac{\pi}{3} \text{ or } \frac{4\pi}{3}.$$



(ii) Area, $A = \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} (\sin x - \sqrt{3}\cos x)\,dx$

$$= \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \sin x\,dx - \sqrt{3}\int \cos x\,dx$$

$$= \left[-\cos x\right]_{\frac{\pi}{3}}^{\frac{4\pi}{3}} - \sqrt{3}\left[\sin x\right]_{\frac{\pi}{3}}^{\frac{4\pi}{3}}$$

$$= -1\left[\cos\frac{4\pi}{3} - \cos\frac{\pi}{3}\right] - \sqrt{3}\left[\sin\frac{4\pi}{3} - \sin\frac{\pi}{3}\right]$$

$$= -1\left[-\frac{1}{2} - \frac{1}{2}\right] - \sqrt{3}\left[\frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right]$$

$$= -1[-1] - \sqrt{3}\left[-\frac{\cancel{2}\sqrt{3}}{\cancel{2}}\right]$$

$$= 1 - \sqrt{3}(-\sqrt{3})$$

$$= 1 + 3$$

$$\therefore A = 4 \text{ units}^2.$$

Q7a. $\dot{x} = 8 - 16\sin t$

(i) $\ddot{x} = -16\cos(t) \cdot 1$

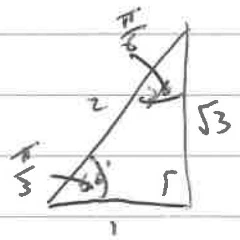
$\ddot{x} = -16\cos(t)$

When $t = 0$

$\ddot{x} = -16(1)$

$\ddot{x} = -16$

(ii) When $\dot{x} = 0$



$8 - 16\sin t = 0$

$16\sin t = 8$

$\sin t = \frac{1}{2}$

$t = \sin^{-1}\left(\frac{1}{2}\right)$

$\therefore t = \frac{\pi}{6}$.

(iii) When $t=0$, $x=0$

$x = \int 8 - 16\sin t \, dt$

$= \int 8\,dt - 16\int \sin t \, dt$

$= 8t - 16 \cdot -\cos t + c$

$\therefore x = 8t + 16\cos t + c$

Now, $0 = \cancel{8(0)} + 16(\cos 0) + c$

$0 = 16(1) + c$

$\therefore c = (-16)$

$\therefore x = 8t + 16\cos t - 16$.

7b (i) $P = 1000e^{kt}$

When $t = 0$

$P = 1000e^{k(0)}$

$\therefore P = 1000.$

(ii) When $t = 5$, $P = 15000$

$\therefore 15000 = 1000e^{5k}$

$15 = e^{5k}$

$5k = \log_e 15$

$\therefore k = \frac{1}{5}\log_e 15$

(iii) When $P = 2\,500\,000$

$2\,500\,000 = 1000e^{kt}$

$2500 = e^{kt}$

$kt = \log_e 2500$

$t = \dfrac{\log_e 2500}{k}$

$= \dfrac{\log_e 2500}{\frac{\log_e 15}{5}}$

$= \dfrac{\log_e 2500}{1} \times \dfrac{5}{\log_e 15}$

$\therefore t = \dfrac{5\log_e 2500}{\log_e 15}$

7b) (i) $\dot{x} = 1 - 2\cos t$ at $t = 0$

$= 1 - 2\cos 0$

$= 1 - 2$

$= -1$ m/s.

(ii) Max velocity when $\ddot{x} = 0$

$\ddot{x} = 2\sin t$

$0 = 2\sin t$ $\therefore t = 0, \pi, 2\pi$

$\dot{x} = 1 - 2\cos t$

$= 1 - 2\cos\pi$

$= 1 - 2(-1)$

$= 3$

(iii) $\dot{x} = 1 - 2\cos t$

$x = t - 2\sin t + c$ when $t = 0$ $x = 3$

$3 = 0 - 2\sin 0 + c$

$3 = c$

$\therefore x = t - 2\sin t + 3$

(iv) $\dot{x} = 1 - 2\cos t$ $\dot{x} = 0$

$1 - 2\cos t = 0$

$-2\cos t = -1$

$\cos t = \frac{1}{2}$

$t = \frac{\pi}{3}$

$x = \frac{\pi}{3} - 2\sin\frac{\pi}{3} + 3$

$= \frac{\pi}{3} - 2\left(\frac{\sqrt{3}}{2}\right) + 3$ $= \frac{\pi}{3} - \sqrt{3} + 3$