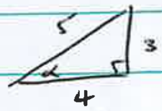
2022 YEAR 12 MATHEMATICS EXTENSION 1

ASSESSMENT #3

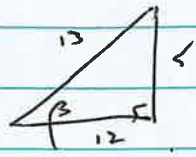
SAMPLE SOLUTIONS ONLY

SECTION 1.

1. $\sin\alpha = \frac{3}{5}$



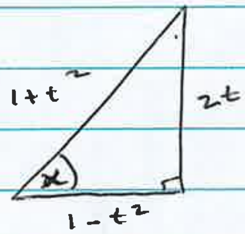
$\sin\beta = \frac{5}{13}$



$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\tfrac{4}{5}\right)\left(\tfrac{12}{13}\right) - \left(\tfrac{3}{5}\right)\left(\tfrac{5}{13}\right) \\ &= \frac{48}{65} - \frac{15}{65} \\ &= \frac{33}{65}\end{aligned}$$

C.

2. $\sec x - \tan x$



$$\begin{aligned}&= \frac{1}{\cos x} - \tan x \\ &= \frac{1+t^2}{(1-t^2)} - \frac{2t}{(1-t^2)} \\ &= \frac{1-2t+t^2}{(1-t^2)} \\ &= \frac{(t-1)(t-1)}{(1-t)(1+t)} \\ &= \frac{-1\cancel{(1-t)}(t-1)}{\cancel{(1-t)}(1+t)} \\ &= \frac{(1-t)}{(1+t)}\end{aligned}$$

B.

3. D.

4. $$\begin{aligned}&\frac{d}{dx}\left[\cos(\ln x)\right] \\ &= -\sin(\ln x)\cdot\frac{1}{x} \\ &= -\frac{\sin(\ln x)}{x}\end{aligned}$$

D.

5. $$\begin{aligned}&\frac{d}{dx}\sin^{-1}(x\sqrt{x}) \\ &= \frac{d}{dx}\sin^{-1}\left(x^{\frac{3}{2}}\right) \\ &= \frac{1}{\sqrt{1-x^{\frac{3}{2}}}}\cdot\frac{3}{2}x^{\frac{1}{2}} \\ &= \frac{3\sqrt{x}}{2\sqrt{1-x^{\frac{3}{2}}}}\end{aligned}$$

A

SECTION II

Q6a. $4\cos A = \sec A, \quad 0^\circ \le A \le 360^\circ$

$4\cos A = \frac{1}{\cos A}$

$4\cos^2 A = 1$

$\cos^2 A = \frac{1}{4}$

$\cos A = \frac{\pm 1}{2}$ (1)

$\therefore A = 60^\circ, 120^\circ, 240^\circ$ or 360° (1)

b. $y = \tan^{-1}\left(\frac{1}{2}x + 1\right)$

$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{1}{2}x + 1\right)^2} \cdot \frac{1}{2}$ (1)

$= \frac{1}{2\left(1 + \frac{1}{4}x^2 + x + 1\right)}$

$= \frac{1}{2\left(\frac{1}{4}x^2 + x + 2\right)}$

$= \frac{1}{\left(\frac{1}{2}x^2 + 2x + 4\right)}$ (1)

c.(i) $y = 3\sin x - 2\cos x$

Let $3\sin x - 2\cos x = R\sin(x - \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$

$3\sin x - 2\cos x = R\cos\alpha\sin x - R\sin\alpha\cos x$

$R = \sqrt{3^2 + (-2)^2}$

Now $\sqrt{13}\sin\alpha = 2$ $\quad\therefore R = \sqrt{13}$ (1)

$\sin\alpha = \frac{2}{\sqrt{13}}$

$\alpha = 0.588$ radians (1)

ii $3\sin x - 2\cos x = \sqrt{13}\sin(x - 0.588)$

6c (ii) $\sqrt{13}\sin(x - 0.588) = 1$

$\sin(x - 0.588) = \frac{1}{\sqrt{13}}$

$x - 0.588 = 0.281$ or $\cancel{2.861}$ (1)

$x = 0.281 + 0.588$

$\therefore x = 0.869$ (3SF) $0 < \alpha < \frac{\pi}{2}$ (1)

d. $\int \cos^2\left(\frac{x}{3}\right) dx = \int \frac{1}{2} + \frac{1}{2}\cos\left(2 \cdot \frac{x}{3}\right) dx$

$= \int \frac{1}{2}\, dx + \frac{1}{2}\int \frac{3}{2} \cdot \frac{2}{3}\cos\left(\frac{2x}{3}\right) dx$ (1)

$= \frac{x}{2} + \frac{3}{4}\sin\left(\frac{2x}{3}\right) + c$ (1)

Q7 a

$$\int_{e}^{e^2} \frac{1}{x(\log_e x)^2}\, dx$$

Let $u = \log_e x$

$\frac{du}{dx} = \frac{1}{x}$ $\quad dx = x\,du$

$\frac{dx}{x} = du$

$$= \int_{e}^{e^2} \frac{1}{(\log_e x)^2} \cdot \frac{dx}{x}$$

When $x = e^2$

$u = \log_e e^2$

$= 2 \cancel{\log_e e}$

$u = 2$ $\uparrow$

$$= \int_{1}^{2} \frac{1}{u^2} \cdot du$$

$$= \cancel{u^{-2}]} \int_{1}^{2} u^{-2}\, du$$

When $x = e$

$u = \log_e e$

$u = 1$ $\downarrow$

$$= \left[\frac{u^{-1}}{-1}\right]_1^2$$

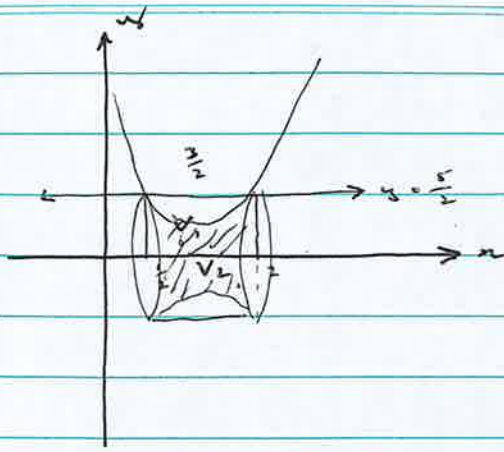
$$= \left[\frac{-1}{u}\right]_1^2$$

$$= \frac{-1}{2} - \frac{-1}{1}$$

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}.$$

7b.



$$V_1 = \pi\left(\frac{5}{2}\right)^2\left(\frac{3}{2}\right)$$

$$= \pi\left(\frac{25}{4}\right)\left(\frac{3}{2}\right)$$

$$V_1 = \frac{75\pi}{8}$$

1

$$V_2 = \pi\int_{\frac{1}{2}}^{2}\left(x + \frac{1}{x}\right)^2 dx$$

$$= \pi\int_{\frac{1}{2}}^{2}\left(x^2 + 2 + x^{-2}\right) dx$$

$$= \pi\left[\frac{x^3}{3} + 2x - \frac{1}{x}\right]_{\frac{1}{2}}^{2}$$

$$= \pi\left[\left(\frac{8}{3} + 4 - \frac{1}{2}\right) - \left(\frac{1}{24} + 1 - 2\right)\right]$$

$$= \pi\left[\frac{37}{6} + \frac{23}{24}\right]$$

1

$$= \frac{57\pi}{8}$$

$$\therefore\ V_3 = \frac{75\pi}{8} - \frac{57\pi}{8}$$

$$V = \frac{9\pi}{4}.$$

1

7c.

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{1\,dx}{\sqrt{4-x^2}} = \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{2^2-x^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}} \quad (1)$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} \quad (1)$$

$$\therefore \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{12} \quad (1)$$

8a (i)

$x = 35t\cos\theta$ $\qquad\qquad$ $y = -5t^2 + 35t\sin\theta + 20$

Sub $t = \dfrac{x}{35\cos\theta}$ into $y = -5t^2 + 35t\sin\theta + 20$

$$\therefore \quad y = -5\left(\frac{x}{35\cos\theta}\right)^2 + \cancel{35}\left(\frac{x}{\cancel{35}\cos\theta}\right)\sin\theta + 20 \qquad (1)$$

$$= \frac{-5x^2}{1225\cos^2\theta} + x\tan\theta + 20 \qquad (1)$$

$$\therefore y = \frac{-x^2}{245}\sec^2\theta + x\tan\theta + 20 \qquad (1)$$

(ii) When $x = 140$, $y = (-20)$ $\qquad$ $\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$$\frac{-(140)^2}{245}\sec^2\theta + 140\tan\theta + 20 = -20$$

$$-\frac{19600}{245}\sec^2\theta + 140\tan\theta + 40 = 0$$

$$-80\sec^2\theta + 140\tan\theta + 40 = 0 \qquad (1)$$

$(\div\ -20)$

$$4\sec^2\theta - 7\tan\theta - 2 = 0$$

$$4(\tan^2\theta + 1) - 7\tan\theta - 2 = 0$$

$$4\tan^2\theta + 4 - 7\tan\theta - 2 = 0 \qquad (1)$$

$$4\tan^2\theta - 7\tan\theta + 2 = 0$$

$$\tan\theta = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(4)(2)}}{2(4)}$$

$$\tan\theta = \frac{7 \pm \sqrt{49 - 32}}{8}$$

$$\tan\theta = \frac{7 \pm \sqrt{17}}{8}$$

$$\therefore \theta = \tan^{-1}\left(\frac{7 - \sqrt{17}}{8}\right) \text{ or } \tan^{-1}\left(\frac{7 + \sqrt{17}}{8}\right)$$

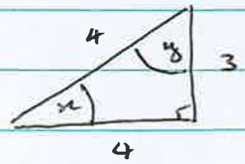
$$= 19.779\ldots \text{ or } 54.275\ldots$$

$$\therefore \theta = 20^\circ \text{ or } 54^\circ \qquad (1)$$

8b. Let $\tan^{-1}\left(\frac{3}{4}\right) = x$ and $\cos^{-1}\left(\frac{3}{5}\right) = y$

$\therefore \tan x = \frac{3}{4}$ $\therefore \cos y = \frac{3}{5}$ (1)

As both x and y are 1st quadrant angles then



$\therefore x + y = \frac{\pi}{2}$ (1)

c. $y = 2\sin\left(x - \frac{\pi}{6}\right) + 2$

(1) (1)