MATHS EXT 1 2022 Task 1

SOLUTIONS

① $\boxed{B}$ $y = \sin^{-1}(\cos x)$

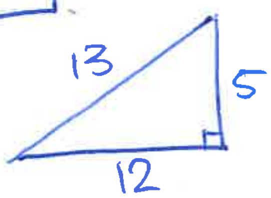
Check: when $x = 0$, $y = \sin^{-1}(\cos 0)$
$= \sin^{-1}(1)$
$= \frac{\pi}{2}$

② $\boxed{A}$ We know $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

So this means we have the following:

$\sin 3x \sin 4y = \frac{1}{2}(\cos(3x-4y) - \cos(3x+4y))$

③ $\boxed{D}$



Let $\alpha = \sin^{-1}\left(\frac{5}{13}\right)$

$\therefore \cot\alpha = \frac{\text{adjacent}}{\text{opposite}}$
$= \frac{12}{5}$

④ $\boxed{B}$ $y = \sin^{-1}x$ shifted up $\frac{\pi}{4}$ units becomes $y = \sin^{-1}x + \frac{\pi}{4}$

⑤ $\boxed{D}$ The range is $f(x) \geq 1$ is NOT true

⑥ $\boxed{A}$ We need $x^2 - 9 > 0$ here
$x^2 > 9$
$\therefore x < -3$ or $x > 3$
$\therefore x \in (-\infty, -3) \cup (3, \infty)$

7 (a) $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$

$$\therefore (\sin 75^\circ - \cos 75^\circ)^2 = 1 - \sin 150^\circ$$
$$= 1 - \left(\tfrac{1}{2}\right)$$
$$= \tfrac{1}{2}$$

$$\sin 75^\circ - \cos 75^\circ = \frac{1}{\sqrt{2}} \quad \text{since } \sin 75^\circ - \cos 75^\circ > 0$$
$$= \frac{\sqrt{2}}{2}$$

(b) Prove that for all integers $n \geq 1$:

$$(1\times 2) + (2\times 3) + (3\times 4) + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Prove true for $n=1$:

when $n=1$, LHS $= 1(1+1)$
$= 2$

$$\text{RHS} = \frac{1\times(1+1)\times(1+2)}{3}$$
$$= \frac{1\times 2\times 3}{3}$$
$$= 2$$
$$= \text{LHS}$$

$\therefore$ true for $n=1$

Assume true for $n=k$:

$$(1\times 2) + (2\times 3) + (3\times 4) + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

Prove true for $n=k+1$

$$(1\times 2) + (2\times 3) + (3\times 4) + k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)}{3}\left[k+3\right]$$

OR $$= \frac{(k+1)(k+2)(k+3)}{3}$$

$\therefore$ true for $n = k+1$

$\therefore$ By the principles of mathematical induction, the result is true for all integers $n \geq 1$.

(c) $y = 2\cos^{-1}(x-1)$

(i) We need $-1 \leq x-1 \leq 1$

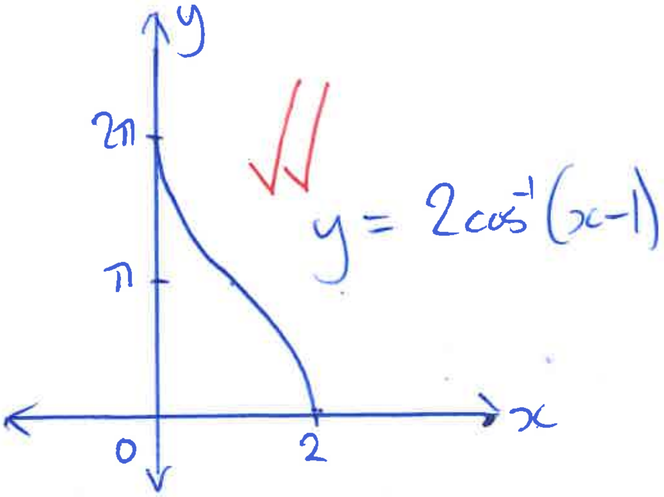
Domain: $\therefore$ $0 \leq x \leq 2$ (inequality notation)

OR $x \in [0,2]$ (interval notation)

(ii) Range: $0 \leq y \leq 2\pi$ (inequality notation)

OR $y \in [0, 2\pi]$ (interval notation)

(iii)



(d) Prove $4^n + 14$ is divisible by 6 for all integers $n \geqslant 1$:

Prove true for $n=1$:

when $n=1$, $4^n + 14 = 4^1 + 14$

$= 18$

$= 6(3)$ $\therefore$ true for $n=1$ ✓

Assume true for $n=k$:

$\therefore 4^k + 14 = 6P$ for $P \in \mathbb{Z}$

Prove true for $n=k+1$

$\therefore 4^{k+1} + 14 = 4 \times 4^k + 14$

$= 4 \times (6P - 14) + 14$ ✓

$= 24P - 56 + 14$

$= 24P - 42$

$= 6(4P - 7)$

$= 6Q$ for $Q \in \mathbb{Z}$ ✓

$\therefore$ true for $n=k+1$

$\therefore$ By the principles of mathematical induction, the result is true for all positive integers $n \geqslant 1$.

8 (a)

$$\tan 3\theta = \tan(2\theta + \theta)$$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$$

$$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\tan\theta}$$

$$= \frac{\frac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{1-\tan^2\theta}}{\frac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}}$$

$$= \frac{2\tan\theta + \tan\theta - \tan^2\theta}{\cancel{1-\tan^2\theta}} \times \frac{\cancel{1-\tan^2\theta}}{1-\tan^2\theta - 2\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^2\theta}{1 - 3\tan^2\theta}$$

as required

(b)(i) $y = 4x^2 - 2x^3$

$$\frac{dy}{dx} = 8x - 6x^2$$

$$= 2x(4 - 3x)$$

$$= 0 \text{ when } x = 0 \text{ or } \frac{4}{3}$$

$$\frac{d^2y}{dx^2} = 8 - 12x$$

P.T.O.

$\therefore$ When $x=0$, $y=0$ so $(0,0)$ is a stationary point

When $x=\frac{4}{3}$, $y = 4\left(\frac{4}{3}\right)^2 - 2\left(\frac{4}{3}\right)^3$

$= 4\left(\frac{16}{9}\right) - 2\left(\frac{64}{27}\right)$

$= \frac{64}{9} - \frac{128}{27}$

$= \frac{64}{27}$ so $\left(\frac{4}{3}, \frac{64}{27}\right)$ is a stationary point

Also: When $x=0$, $\frac{d^2y}{dx^2} = 8-12(0)$

$= 8$

> 0 so $(0,0)$ is a MINIMUM turning point

When $x=\frac{4}{3}$, $\frac{d^2y}{dx^2} = 8-12\left(\frac{4}{3}\right)$

$= -8$

< 0 so $\left(\frac{4}{3}, \frac{64}{27}\right)$ is a MAXIMUM turning point

(ii) $\frac{d^2y}{dx^2} = 8-12x$

$= 0$ when $x = \frac{2}{3}$

When $x=\frac{2}{3}$, $y = 4\left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)^3$

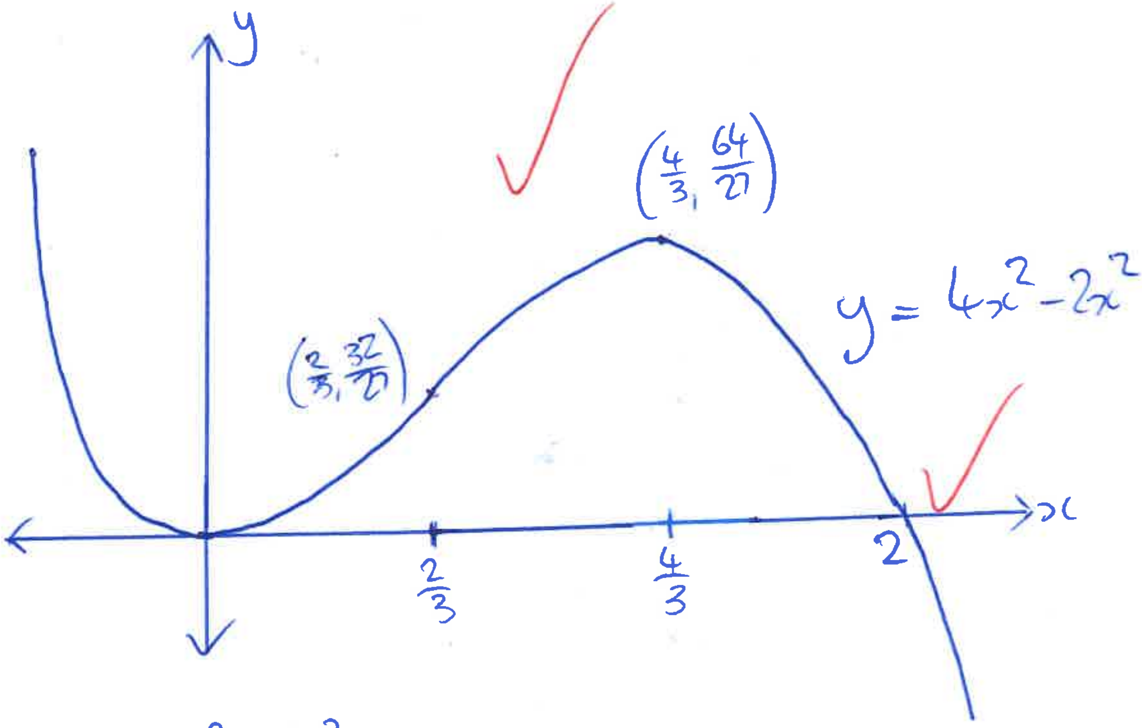
$= 4\left(\frac{4}{9}\right) - 2\left(\frac{8}{27}\right)$

$= \frac{16}{9} - \frac{16}{27}$

$= \frac{32}{27}$

$\therefore \left(\frac{2}{3}, \frac{32}{27}\right)$ is the point of inflection

(iii) $y = 4x^2 - 2x^3$



when $y = 0$, $0 = 4x^2 - 2x^3$

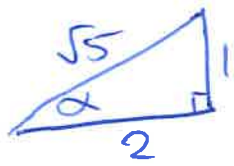
$0 = 2x^2(2 - x)$

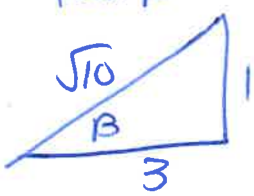
$\therefore x = 0$ or 2

(c) $\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right) = \sin^{-1} x$

let $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$ and let $\beta = \tan^{-1}\left(\frac{1}{3}\right)$

$\tan\alpha = \frac{1}{2}$ $\tan\beta = \frac{1}{3}$





$\therefore x = \sin\left(\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right)\right)$

$= \sin(\alpha - \beta)$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{1}{\sqrt{5}} \times \frac{3}{\sqrt{10}} - \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{10}}$$

$$= \frac{3}{\sqrt{50}} - \frac{2}{\sqrt{50}}$$

$$= \frac{1}{\sqrt{50}}$$

$$= \frac{1}{5\sqrt{2}}$$

$$= \frac{\sqrt{2}}{10}$$

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