**ST PIUS X COLLEGE
CHATSWOOD**

HSC 2021 Stage 6
Year 12

Assessment Task #3

25% of School Based Assessment

MATHEMATICS ADVANCED

General Instructions

- Working time – 45 minutes
- Write using black or blue pen
  Black pen is preferred
- Draw diagrams using pencil
- NESA approved calculators may be used
- Marks may be deducted for careless or poorly arranged work
- Show all relevant mathematical reasoning and/or calculations
- Write your Student Number at the top of this cover page

Total Marks – 35

Section I – Multiple Choice 5 marks

- Attempt Questions 1 – 5
- Enter responses on the multiple choice answer sheet
- Allow 5 minutes for this section

Section II – 30 marks

- Attempt Questions 6 – 8
- Answer in the writing spaces provided
- Show all necessary working
- Allow 40 minutes for this section

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Mathematics Advanced – Multiple Choice Answer Sheet

Attempt all questions:

		A	B	C	D
Question	**1**	A ◯	B ◯	C ⬤	D ◯
	2	A ◯	B ◯	C ◯	D ⬤
	3	A ⬤	B ◯	C ◯	D ◯
	4	A ◯	B ⬤	C ◯	D ◯
	5	A ◯	B ◯	C ◯	D ⬤

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Using $y' = \dfrac{VU' - UV'}{V^2}$

4. What is the derivative of $\dfrac{x}{\cos x}$?

$= \dfrac{\cos x - -x\sin x}{\cos^2 x}$

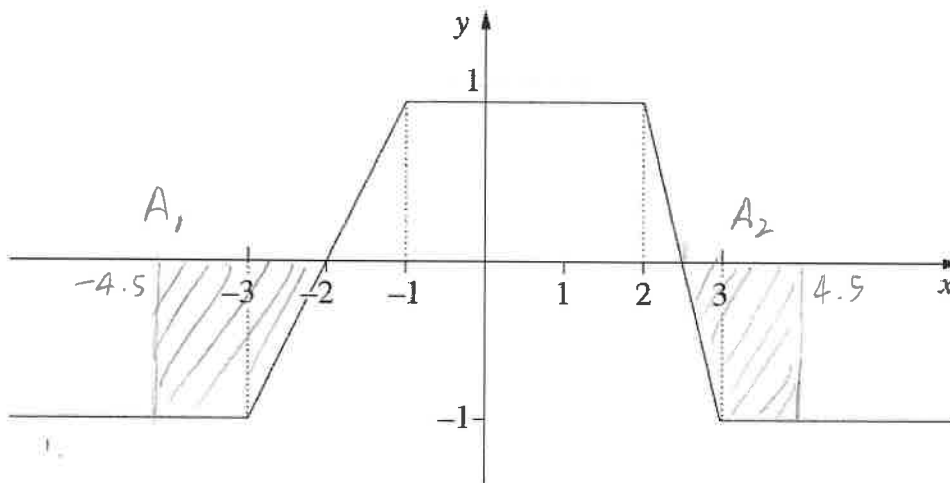$= \dfrac{\cos x + x\sin x}{\cos^2 x}$

(A) $\dfrac{\cos x - x \sin x}{\cos^2 x}$

(B) $\dfrac{\cos x + x \sin x}{\cos^2 x}$

(C) $\dfrac{x \sin x - \cos x}{\cos^2 x}$

(D) $\dfrac{-x \sin x - \cos x}{\cos^2 x}$

5. The diagram shows the graph $y = f(x)$.

What value of k, where $k > 0$, would make $\int_{-k}^{k} f(x)\,dx = 0$?

(A) 3

(B) 3.5

(C) 4

(D) 4.5

Area of trapezium above x-axis is
$A = \frac{1}{2} \times 1 \times (3 + 4.5)$
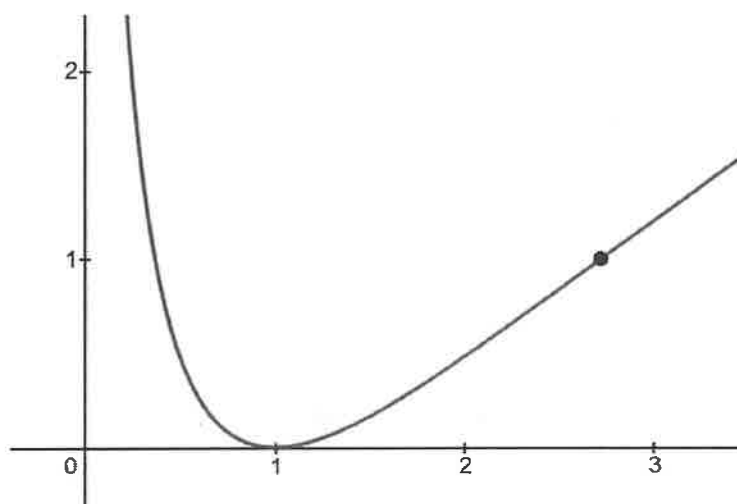$= 3.75$ square units.
Area below x-axis
$A_1 = \frac{1}{2} \times 1 \times (2.5 + 1.5) = 2$
$A_2 = \frac{1}{2} \times 1 \times (2 + 1.5) = 1.75$
Total $= 2 + 1.75 = 3.75$ square units

End of Multiple-Choice Section 1.

5

2. The diagram below shows the curve $y = (\log_e |x|)^2$.

What would be the gradient, m, of a tangent drawn to this curve at the point $(e, 1)$?

(A) $m = e$

(B) $m = \dfrac{1}{e}$

(C) $m = 2$

(D) $m = \dfrac{2}{e}$

Handwritten working:

$\dfrac{dy}{dx} = 2 \times \dfrac{1}{x} \times \log_e x$

When $x = e$

$\dfrac{dy}{dx} = \dfrac{2}{e} \times \log_e e$

$= \dfrac{2}{e}$

3. What is the derivative of $e^{3\ln x}$?

(A) $3x^2$

(B) $3e^{3\ln x}$

(C) $(3\ln x)e^{3\ln x - 1}$

(D) $(3\ln x)e^{3\ln x} \times \dfrac{1}{x}$

Handwritten working:

$x^3 = e^{3\ln x}$

∴ derivative of x^3 is

$3x^2$

Note If $y = e^{\ln x}$

$\ln y = \ln e^{\ln x}$

$\ln y = \ln x \times \ln e$ $(\ln e = 1)$

∴ $\ln y = \ln x$

and $y = x$

∴ $x = e^{\ln x}$

4

Use the multiple-choice answer sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A)   2   (B)   6   (C)   8   (D)   9
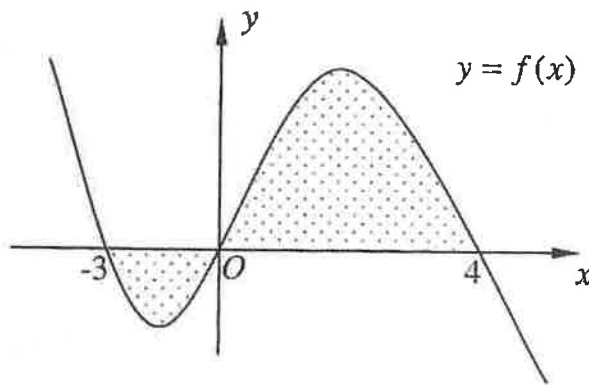
                                             A ⬭   B ⬤   C ⬭   D ⬭

If you think that you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

                              ✖   ⬤   ⬭   ⬭

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

                                   correct
                              ✖   ✖   ⬭   ⬭

1.   Consider the diagram below.

       Which of the following represents the shaded area?

(A) $\int_{-3}^{4} f(x)\,dx$ (B) $2\int_{0}^{4} f(x)\,dx$

(C) $\int_{0}^{4} f(x)\,dx - \int_{-3}^{0} f(x)\,dx$ (D) $\int_{-3}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx$

Integral from -3 to 0 is negative. We want to add its absolute value to integral from 0 to 4, Therefore the negative of a negative will make it a positive.

3

Attempt Questions 6 to 8.
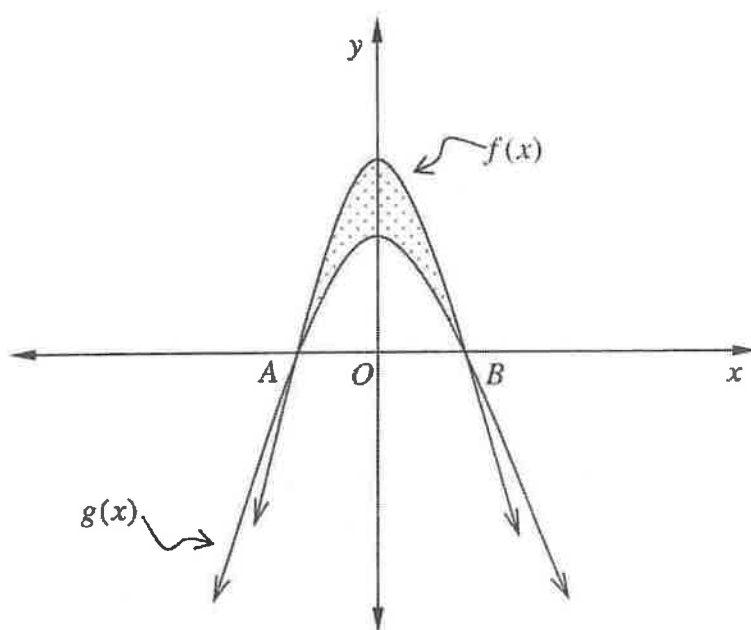**Allow about 40 minutes for this section.**

In Questions 6 to 8 your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (10 marks)

Write your solutions in the spaces provided   **Marks**

(a)   The graphs of $f(x) = (5+x)(5-x)$ and $g(x) = \dfrac{2}{5}(5+x)(5-x)$ intersect at points A and B, as shown in the diagram below.

(i)   Show that the area of the shaded region is given by $A = \dfrac{6}{5}\displaystyle\int_0^5 (25-x^2)\ dx$.   1

$f(x) = 25-x^2$ x-Intercepts at $x=-5$ and $x=5$

$g(x) = \dfrac{2}{5}(25-x^2)$

$\therefore A = 2\displaystyle\int_0^5 f(x) - g(x)\ dx$

1 mark
either of
these two
lines

$\left\{ \begin{array}{l} = 2\displaystyle\int_0^5 (25-x^2) - \dfrac{2}{5}(25-x^2)\ dx \\[4mm] = 2\displaystyle\int_0^5 \dfrac{3}{5}(25-x^2)\ dx \qquad = \dfrac{6}{5}\displaystyle\int_0^5 (25-x^2)\ dx \end{array} \right.$

(ii) Hence evaluate the area of the shaded region. **2**

$$A = \frac{6}{5}\int_0^5 (25 - x^2)\,dx = \frac{6}{5}\left[25x - \frac{x^3}{3}\right]_0^5$$

1 mark correct integration

$$= \frac{6}{5}\left[\left(125 - \frac{125}{3}\right) - 0\right]$$

$$= \frac{6}{5} \times \frac{280}{3}^{50}$$

$$= 100$$

1 mark correct answer

(b) Find a primitive for $\dfrac{5}{x^2} - 8x$. **2**

$$\int \frac{5}{x^2} - 8x\,dx = \int 5x^{-2} - 8x\,dx$$

$$= -5x^{-1} - 4x^2 + c$$

$$= -\frac{5}{x} - 4x^2 + c \quad \text{or} \quad \frac{-4x^3 + 5}{x} + c$$

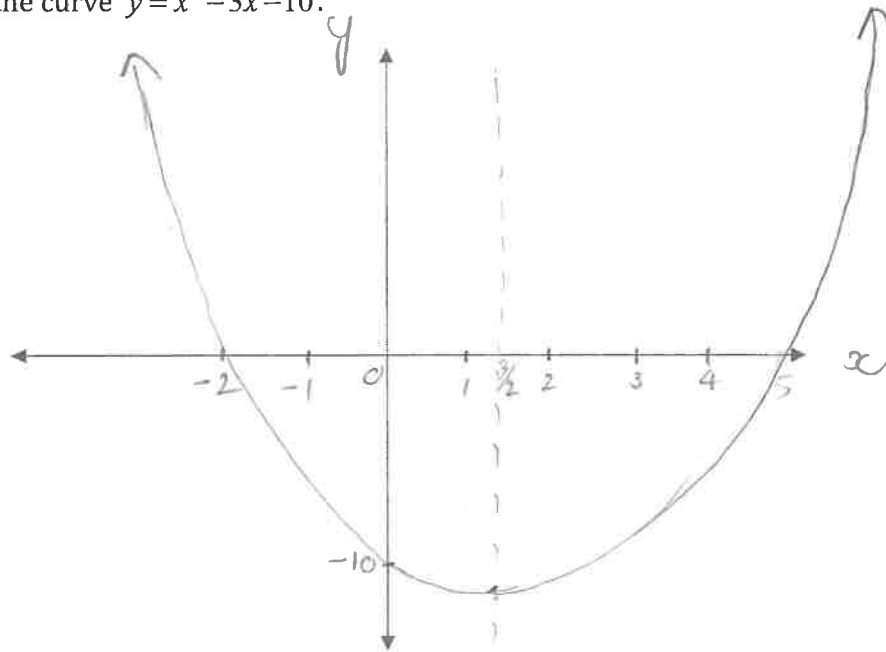**(c)** Find $\displaystyle\int \frac{2}{\sqrt{3x-1}}\,dx = 2\int \frac{1}{\sqrt{3x-1}}\,dx$ **2**

$$= 2\int (3x-1)^{-\frac{1}{2}}\,dx$$

$$= 2 \times \frac{2}{3}(3x-1)^{\frac{1}{2}} + c$$

$$= \frac{4}{3}(3x-1)^{\frac{1}{2}} + c \quad \text{or} \quad \frac{4}{3}\sqrt{3x-1} + c$$

(d) (i) Sketch the curve $y = x^2 - 3x - 10$. **1**

When $x = 0$, $y = -10$

Axis of symmetry at $x = \dfrac{3}{2}$; min at $-12\dfrac{1}{4}$

When $y = 0$

$x = \dfrac{3 \pm \sqrt{9 - (4 \times -10)}}{2} = \dfrac{3 \pm 7}{2}$

$= -2 \text{ or } 5$

(ii) Hence find the area between the curve and the x-axis. **2**

$A = \left| \displaystyle\int_{-2}^{5} x^2 - 3x - 10 \, dx \right| = \left[\dfrac{x^3}{3} - \dfrac{3x^2}{2} - 10x \right]_{-2}^{5}$

$= \left| \left(\dfrac{125}{3} - \dfrac{75}{2} - 50 \right) - \left(\dfrac{-8}{3} - 6 + 20 \right) \right|$

$= \left| -\dfrac{275}{6} - \dfrac{34}{3} \right|$

$= \dfrac{343}{6}$

$= 57\dfrac{1}{6} \text{ square units}$

Question 7 on next page.

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Question 7 (10 marks)

(a) Differentiate with respect to x:

    (i) $x \log_e 2x$ $y' = vu' + uv'$ **2**

$$y' = \log_e 2x + \frac{2x}{2x}$$
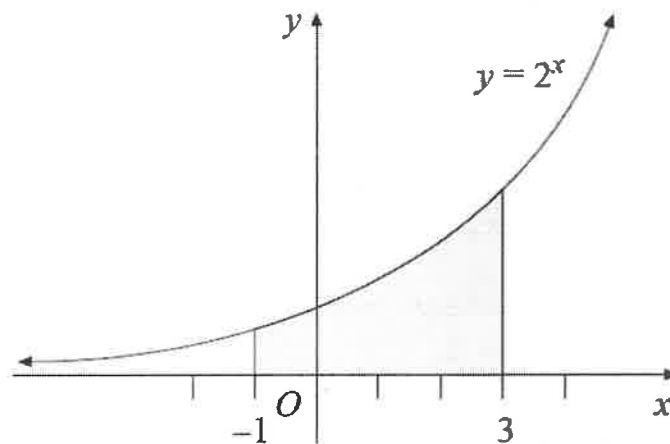
$$= \log_e 2x + 1$$

    (ii) $4e^{2x}$ **1**

$$y' = 8e^{2x}$$

(b) Find $\int \frac{x}{x^2 - 7} dx$ **1**

$$\frac{1}{2} \ln (x^2 - 7) + C$$

(c) Consider the function $y = 2^x$ shown below.

(i) Complete the following tables of values for $y = 2^x$: **1**

x	−1	0	1	2	3
2^x	½	1	2	4	8

(ii) Use the Trapezoidal rule with these five function values to find an estimate **2**
for the area of the shaded region in the diagram.

$h = \dfrac{b-a}{n}$ 5 function values means 4 sub-intervals

$= \dfrac{3--1}{4}$

$= 1$

$\therefore \displaystyle\int_{-1}^{3} 2^x \, dx \doteqdot \dfrac{1}{2}\left(\dfrac{1}{2} + 2\times1 + 2\times2 + 2\times4 + 8 \right)$

$= 11.25 \text{ square units}$

(iii)   Find the EXACT area by evaluating the integral $\int_{-1}^{3} 2^x \, dx$.   **2**

Using $\int f'(x) a^{f(x)} \, dx = \dfrac{a^{f(x)}}{\ln a} + C$

$\int_{-1}^{3} 2^x \, dx = \left[\dfrac{2^x}{\ln 2} \right]_{-1}^{3}$

$= \dfrac{8}{\ln 2} - \dfrac{\frac{1}{2}}{\ln 2}$

$= \dfrac{8}{\ln 2} - \dfrac{1}{2\ln 2} \qquad \left(\doteq 11.19498674 \right)$

(iv)   By what percentage does the Trapezoidal rule in part (ii) overestimate the   **1**
true area bounded by the curve and the x-axis between $x = -1$ and $x = 3$?

$\dfrac{11.25 - \left(\dfrac{8}{\ln 2} - \dfrac{1}{2\ln 2} \right)}{\left(\dfrac{8}{\ln 2} - \dfrac{1}{2\ln 2} \right)} \times 100 \doteq 0.49\%$

$\dfrac{11.25 - 11.19}{11.19} \times 100 \doteq 0.54\%$

Question 8 on next page.

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Question 8 (10 marks)

(a) Differentiate the following with respect to x:

(i) $5x + \sin 5x$

1

$$\frac{dy}{dx} = 5 + 5\cos 5x$$

(ii) $\cos(x^2 - 3)$

1

$$\frac{dy}{dx} = -2x \sin(x^2 - 3)$$

(b) Find $\int_0^{\frac{\pi}{4}} \sec^2 3x \; dx$.

2

$$\int_0^{\frac{\pi}{4}} \sec^2 3x \; dx = \left[\frac{\tan 3x}{3} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\tan \frac{3\pi}{4}}{3} - \frac{\tan 0}{3}$$

$$= -\frac{1}{3} \quad \text{(2 marks)}$$
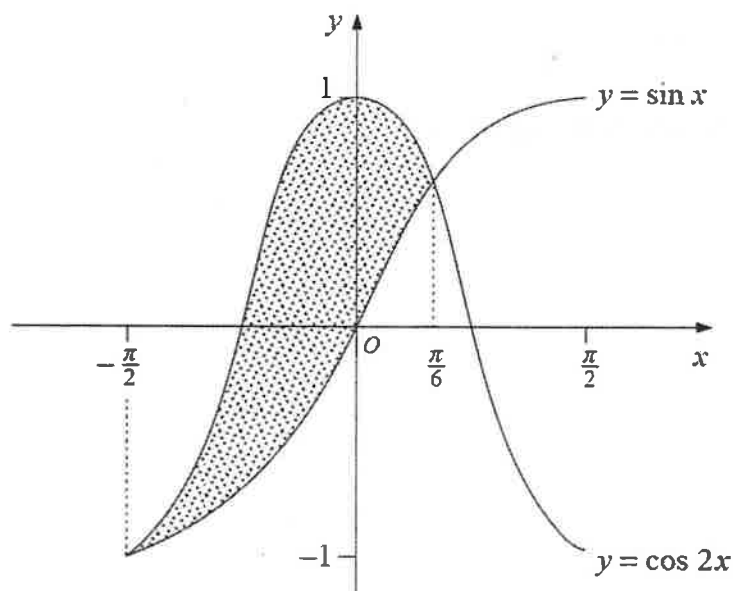
OR

tan x has asymptote at $x = 90°$ or $\frac{\pi}{2}$

∴ tan $3x$ has asymptote at $x = 30°$ or $\frac{\pi}{6}$ $\frac{\pi}{6} < \frac{\pi}{4}$

So cannot integrate $\sec^2 3x$ over $\frac{\pi}{6}$

(2 marks)

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(c)

The diagram above shows the graphs of the functions $y = \cos 2x$ and $y = \sin x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. The two graphs intersect at $x = \frac{\pi}{6}$ and $x = -\frac{\pi}{2}$.

Calculate the EXACT area of the shaded region.

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \cos 2x - \sin x \, dx = \left[\frac{\sin 2x}{2} + \cos x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$$
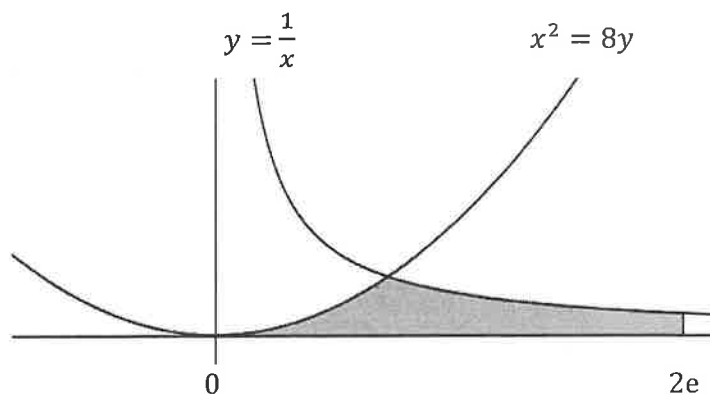
$$= \left(\frac{\sin \frac{\pi}{3}}{2} + \cos \frac{\pi}{6} \right) - \left(\frac{\sin(-\pi)}{2} + \cos\left(-\frac{\pi}{2}\right) \right)$$

$$= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} \right) - \left(0 + 0 \right)$$

$$= \frac{\sqrt{3} + 2\sqrt{3}}{4}$$

$$= \frac{3\sqrt{3}}{4}$$

(d)

$$y = \frac{1}{x} \qquad\qquad x^2 = 8y$$

0   2e

The graph above shows the parabola $y = \frac{x^2}{8}$ and the hyperbola $y = \frac{1}{x}$.

The curves intersect at a point in the 1st quadrant. The region between the x-axis and the curves from $x = 0$ to $x = 2e$ has been shaded.

(i) By solving simultaneously, show that the curves meet at the point $(2, \frac{1}{2})$.   **1**

$$\frac{1}{x} = \frac{x^2}{8}$$
$$x^3 = 8$$
$$\therefore x = 2$$

When $x = 2$, $y = \frac{1}{2}$ or $y = \frac{2^2}{8} = \frac{1}{2}$

$\therefore$ Point $(2, \frac{1}{2})$
of intersection

(ii) Show that the area of the shaded region is $1\frac{1}{3}$ square units.   **2**

$$A = \int_{2}^{2e} \frac{1}{x} \, dx + \int_{0}^{2} \frac{x^2}{8} \, dx$$

$$= \left[\ln x \right]_{2}^{2e} + \left[\frac{x^3}{24} \right]_{0}^{2}$$

$$= \left(\ln 2e - \ln 2 \right) + \left(\frac{8}{24} - 0 \right)$$

$$= 1 + \frac{1}{3}$$

$$= 1\frac{1}{3} \text{ square units}$$

End of Task

Section II extra writing space

If you use this space, clearly indicate which question you are answering.