Student Number

SOLUTIONS

ST PIUS X COLLEGE
CHATSWOOD

HSC 2020 Stage 6
Year 12

ASSESSMENT TASK #1

20% of School Based Assessment

MATHEMATICS EXTENSION 1

General Instructions

- Working time – 45 minutes
- Write using black or blue pen
  Black pen is preferred
- Draw diagrams using pencil
- NESA approved calculators may be used
- Marks may be deducted for careless or poorly arranged work
- Show all relevant mathematical reasoning and/or calculations
- Write your Student Number at the top of all pages

Total Marks – 30

Section I – 15 marks

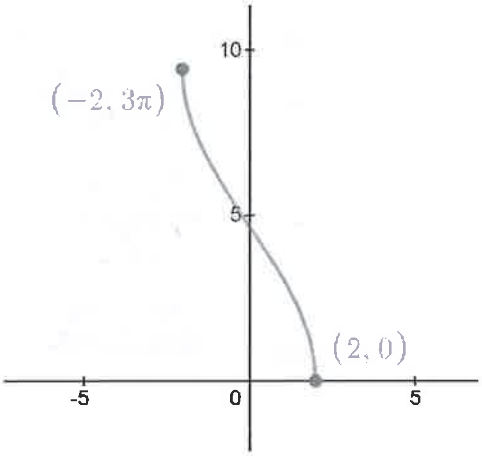
- Attempt all questions
- Show all necessary working
- ***Write solutions in space provided***
- Allow $22\frac{1}{2}$ minutes for this section

Section II – 15 marks

- Attempt all questions
- Show all necessary working
- ***Write solutions in space provided***
- Allow $22\frac{1}{2}$ minutes for this section

Section I 15 marks

1. Consider the graph of the inverse function below written in the format $y = a\cos^{-1}(bx)$. **2**



Find the values of a and b.

When $x=2$, $y=0$ $\therefore$ $0 = \cos^{-1}(2b)$

$1 = 2b$

$b = \frac{1}{2}$

When $x=-2$, $y=3\pi$ $\therefore$ $3\pi = a\cos^{-1}\left(\frac{1}{2}(-2)\right)$

$3\pi = a\cos^{-1}(-1)$

$3\pi = a\pi$

$a = 3$

$\therefore$ $a = 3$ and $b = \frac{1}{2}$

2. By considering the expansion of $\sin(A + B)$, find the exact value of $\sin 75^\circ$. Leave you solution in simplest surd form. **3**

$$\sin 75^\circ = \sin(30^\circ + 45^\circ)$$

$$= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$

3. Prove the trigonometric identity $\frac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$. **3**

$$LHS = \frac{2\cos A}{\operatorname{cosec} A - 2\sin A}$$

$$= \frac{2\cos A}{\frac{1}{\sin A} - 2\sin A}$$

$$= \frac{2\cos A}{\frac{1 - 2\sin^2 A}{\sin A}}$$

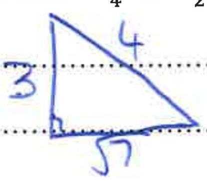
$$= \frac{2\sin A\cos A}{1 - 2\sin^2 A}$$

$$= \frac{\sin 2A}{\cos 2A}$$

$$= \tan 2A$$

$$= RHS \quad \therefore \frac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$$

4. If $\sin\theta = \frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$, evaluate in simplest surd form $\sec\theta$. **2**



$\sin\theta = \frac{3}{4}$ (2nd quadrant)

$\cos\theta = -\frac{\sqrt{7}}{4}$

$\sec\theta = -\frac{4}{\sqrt{7}}$

$= -\frac{4\sqrt{7}}{7}$

5. Solve the equation $2\sin^2\theta = \sin 2\theta$ for $0 \le \theta \le 2\pi$. **3**

$2\sin^2\theta = 2\sin\theta\cos\theta$

$\sin^2\theta = \sin\theta\cos\theta$

$\sin^2\theta - \sin\theta\cos\theta = 0$

$\sin\theta(\sin\theta - \cos\theta) = 0$

$\therefore \sin\theta = 0$ OR $\sin\theta - \cos\theta = 0$

$\theta = 0, \pi, 2\pi$

$\sin\theta = \cos\theta$

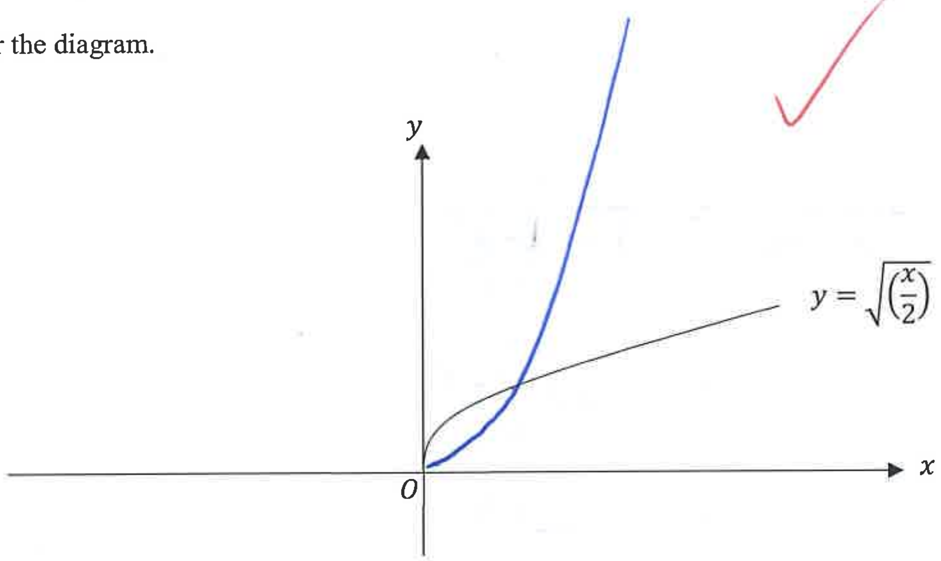
$\frac{\sin\theta}{\cos\theta} = 1$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\therefore \theta = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$ or 2π

6. Consider the diagram.



a. Find the inverse function of $y = \sqrt{\left(\frac{x}{2}\right)}$. **1**

$x = \sqrt{y/2}$

$x^2 = y/2$

$y = 2x^2$ for $x \geq 0$

b. Sketch, on the above diagram, the inverse function of $y = \sqrt{\left(\frac{x}{2}\right)}$. **1**

See above

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End of Section I

1. a. Use the substitution $t = \tan\frac{\theta}{2}$, to express $\frac{1}{1+\cos\theta}$ in a simplified form in terms of t. **2**

$$\frac{1}{1+\cos\theta} = \frac{1}{1+\frac{1-t^2}{1+t^2}}$$

$$= \frac{1}{\frac{1+t^2+1-t^2}{1+t^2}}$$

$$= \frac{1+t^2}{2}$$

$$= \frac{1}{2} + \frac{t^2}{2}$$

b. Hence, or otherwise, solve the equation $\frac{1}{1+\cos\theta} = 2$, for $-\pi \le \theta \le \pi$. **2**

$$\frac{1}{2} + \frac{t^2}{2} = 2$$

$$\boxed{\times 2} \quad 1 + t^2 = 4$$

$$t^2 = 3$$

$$t = \pm\sqrt{3}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{3} \quad \text{for } -\frac{\pi}{2} < \frac{\theta}{2} < \frac{\pi}{2}$$

$$\frac{\theta}{2} = \pm\frac{\pi}{3}$$

$$\theta = \pm\frac{2\pi}{3}$$

$$\therefore \theta = -\frac{2\pi}{3} \text{ or } \frac{2\pi}{3}$$

2. Prove that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n. **3**

(1) Prove the result is true for $n=1$

When $n=1$, $9^{1+2} - 4^1 = 9^3 - 4$

$= 729 - 4$

$= 725$

$= 5(145)$

$\therefore$ true for $n = 1$

(2) Assume true for $n = k$:

$9^{k+2} - 4^k = 5p$ for $p \in \mathbb{Z}$

(3) Prove true for $n = k+1$

$$9^{k+3} - 4^{k+1} = 9 \times 9^{k+2} - 4 \times 4^k$$
$$= 9(5p + 4^k) - 4 \times 4^k$$
$$= 45p + 9 \times 4^k - 4 \times 4^k$$
$$= 45p + 5 \times 4^k$$
$$= 5(9p + 4^k)$$
$$= 5Q \text{ for } Q \in \mathbb{Z}$$

$\therefore$ true for $n = k+1$

(4) By the principles of mathematical induction, the result is true for all positive integers n.

3. Prove $1 + 8 + 27 + \cdots + n^3 = \frac{n^2}{4}(n+1)^2$ for all positive integers n. **3**

(1) Prove true for $n = 1$

When $n = 1$,
$$\text{LHS} = 1^3 = 1$$
$$\text{RHS} = \frac{1^2}{4}(1+1)^2 = \frac{1}{4} \times 4 = 1 = \text{LHS}$$
$\therefore$ true for $n = 1$

(2) Assume true for $n = k$

$$1 + 8 + 27 + \cdots + k^3 = \frac{k^2}{4}(k+1)^2$$

(3) Prove true for $n = k+1$

$$1 + 8 + 27 + \cdots + k^3 + (k+1)^3 = \frac{k^2}{4}(k+1)^2 + (k+1)^3$$
$$= (k+1)^2\left[\frac{k^2}{4} + (k+1)\right]$$
$$= (k+1)^2\left[\frac{k^2 + 4k + 4}{4}\right]$$
$$= \frac{(k+1)^2}{4}(k+2)^2$$
$\therefore$ true for $n = k+1$

(4) By the principles of mathematical induction, the result is true for all positive integers

4. Consider the statement $3^n > n^2 + 20$ for $n \in \mathbb{Z}^+$.

a. Is the statement true for $n = 1$? Justify your answer. **1**

When $n=1$, $3^1 = 3$ and $1^2 + 20 = 21$

But $3 < 21$ so, NO, the statement is NOT true for $n=1$

b. Find the smallest positive integer n for which the statement is true. Show your working. **1**

When $n=2$, $3^2 = 9$ and $2^2 + 20 = 24$

But $9 < 24$ so $n=2$ is not allowed

When $n=3$, $3^3 = 27$ and $3^2 + 20 = 29$

But $27 < 29$ so $n=3$ is not allowed

When $n=4$, $3^4 = 81$ and $4^2 + 20 = 36$

$81 > 36$ so $n=4$ is the smallest positive integer n for which the statement is true

5. a. Use the compound angle results to show $\tan(\pi - \theta) = -\tan\theta$. **1**

$$\tan(\pi - \theta) = \frac{\tan\pi - \tan\theta}{1 + \tan\pi\tan\theta}$$

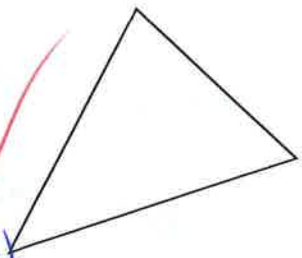
$$= \frac{0 - \tan\theta}{1 + 0}$$

$$= -\tan\theta$$

as required

b. Hence, or otherwise, prove that for any triangle with angles A, B and C that: **2**

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$



$A + B + C = \pi$ ($\angle$ sum of a Δ)

$A + B = \pi - C$

$\tan(A + B) = \tan(\pi - C)$

$= -\tan C$ from part (a)

$$\therefore -\tan C = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$-\tan C(1 - \tan A \tan B) = \tan A + \tan B$

$-\tan C + \tan A \tan B \tan C = \tan A + \tan B$

$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C$ as required

End of Task.

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Additional working space. Ensure you identify the question number.