Coursework 1

Description Q1

The task was to find numerical solutions to the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = (4g(2t) - s)\ w(t) \tag{1}$$

with periodic boundary conditions $w(0) = w(2\pi)$. $g(t)$ is a triangular periodic signal approximating $\cos(t)$ as shown in the assignment text.

To apply the method of finite differences the time interval $[0, 2\pi]$ is discretised to a finite number N of evenly spaced values t_i, $i \in I_N := \{i \in \mathbb{N}_0 \mid i \leq N\}$. The boundary values are defined as $t_0 = 0$ and $t_N = 2\pi$. Then, the second derivative of $w(t)$ can be approximated by the *three-point-difference* formula

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = \frac{w(t_{i+1}) - 2w(t_i) + w(t_{i-1}))}{\Delta t^2}, \quad \Delta t := t_{i+1} - t_i\ i \in I_N \tag{2}$$

known from the weekly material. Inserting this into equation 1 yields a system of N linear equations

$$\frac{w(t_{i+1}) - 2w(t_i) + w(t_{i-1})}{\Delta t^2} = (4g(2t_i) - s)\ w(t_i) \quad \forall i \in I_N \setminus \{N\}\ . \tag{3}$$

The boundary conditions can be expressed as $w(t_0) = w(t_N)$ so that $w(t_{-1}) = w(t_{N-1})$ which is important for the equations with $i = 0$ and $i = N-1$ and enables to leave out $i = N$ in the calculation.

In order to solve these equations they need to be expressed in matrix vector form. Multiplying by Δt^2 and adding $4\Delta t^2 g(2t_i)$ to each equation yields the expression

$$\left(2 + 4\Delta t^2 g(2t_i)\right) w(t_i) - w(t_{i+1}) - w(t_{i-1}) = \Delta t^2 s\ w(t_i) \forall i \in I_N \setminus \{N\}\ . \tag{4}$$

Because of the boundary conditions, $w(t_N)$ does not need to be calculated so that the equations with mentioned $i = 0$ and $i = N-1$ are

$$\left(2 + 4\Delta t^2 g(2t_0)\right) w(t_0) - w(t_1) - w(t_{N-1}) = \Delta t^2 s\ w(t_0) \tag{5}$$

$$\left(2 + 4\Delta t^2 g(2t_{N-1})\right) w(t_{N-1}) - w(t_0) - w(t_{N-2}) = \Delta t^2 s\ w(t_{N-1}) \tag{6}$$

This can now be written in vector notation by constructing a N dimensional vector

$$\vec{w} := (w(t_0), w(t_1), ..., w(t_{N-1}))^T \tag{7}$$

so that the equations become an eigenvalue problem

$$\mathbf{A}\vec{w} = \lambda\vec{w} = \Delta t^2 s\vec{w} \tag{8}$$

with $N \times N$ matrix[1]

$$\mathbf{A} = \begin{pmatrix} d_0 & -1 & 0 & \cdots & 0 & -1 \\ -1 & d_1 & -1 & 0 & \cdots & 0 \\ 0 & -1 & d_2 & -1 & \cdots & 0 \\ \vdots & 0 & -1 & \ddots & 0 & 0 \\ 0 & \vdots & \vdots & 0 & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & d_{N-1} \end{pmatrix}, \quad d_i = 2 + 4\Delta t^2 g(2t_i)\ . \tag{9}$$

[1] see below for an example with $N = 5$

This problem can be easily implemented and solved by eigenvalue solvers. The resulting eigenvalues λ need to be divided by Δt^2 in order to obtain the s of a solution. The calculated eigenvectors represent periodic solutions and can be plotted over t.

Example with N=5

For more clarity the case with $N = 5$ is shown here

$$\mathbf{A} = \begin{pmatrix} d_0 & -1 & 0 & 0 & -1 \\ -1 & d_1 & -1 & 0 & 0 \\ 0 & -1 & d_2 & -1 & 0 \\ 0 & 0 & -1 & d_3 & -1 \\ -1 & 0 & 0 & -1 & d_4 \end{pmatrix} , \quad d_i = 2 + 4\Delta t^2 g(2t_i) \ . \tag{10}$$