

# PHONONIC BRAGG REFLECTORS FOR THERMAL ISOLATION OF SEMICONDUCTOR QUBITS

SEBASTIAN KOCK

**Bachelor's Thesis in Physics**

presented to

**The Faculty of Mathematics, Computer Science and Natural Sciences at  
RWTH Aachen University**  
Department of Physics, Institute II C

*September 2021*

supervised by

Prof. Hendrik Bluhm, Ph.D.  
Tobias Hangleiter, M.Sc.



## ABSTRACT

---

In recent years, the experimental capability in realizing and exercising full control over multiple coupled quantum dots in semiconductor heterostructures has advanced to the point where tuning the dots to the few-electron regime required for quantum computation applications will become impractical to perform manually, establishing the demand for automated procedures in this area. Moreover, charge stability diagrams, which represent a crucial tool in the tuning of quantum dot systems, become increasingly complex to understand and analyze due to the growing number of parameters influencing their structure. The goal of this thesis is to take a first step towards a fully computer-automated implementation of the tuning process by focusing on the coarse tuning of the system to the zero-electron regime.

I simulate charge diagrams of multiple coupled quantum dots in a linear array using a classical capacitive model, reproducing the honeycomb structure of double quantum dot charge diagrams. In order to automatically recognize the zero-electron regime in double-dot charge diagrams by identifying the charge transitions, I introduce a line detection algorithm using the Radon transform of the charge diagram able to detect lines even in noisy images by exploiting the fact that, in a suitable charge diagram detail, several transitions of the same kind and hence the same slope will show. The strong noise performance of the algorithm promises fast scan times during the tuning process. Finally, I present and test an algorithm that attempts to automatically tune a simulated double-dot system to the zero-electron regime using the line detection algorithm developed.

This is still  
the wrong  
abstract



## CONTENTS

---

1	INTRODUCTION	3
2	THEORY	5
2.1	Elastic Properties of Materials	5
2.1.1	Stress Tensor	5
2.1.2	Strain Tensor	6
2.1.3	Stress-Strain Relations	6
2.2	Elastic Waves	7
2.2.1	Solution for Isotropic Materials	8
2.2.2	Elastic Wave Scattering at an Interface	9
2.2.3	Solving Methods for Stratified Media	13
3	IMPLEMENTATION	15
4	ANALYSIS	17
5	CONCLUSION	19
	Bibliography	21

## TODO LIST

---

<input type="checkbox"/> This is still the wrong abstract . . . . .	iii
<input type="checkbox"/> Introduction to notation . . . . .	3
<input type="checkbox"/> justification for modelling with elastic waves . . . . .	5
<input type="checkbox"/> check if description still applies, citation? . . . . .	5
<input type="checkbox"/> picture of stress definition? . . . . .	6
<input type="checkbox"/> reference to lecture 2? . . . . .	6
<input type="checkbox"/> mention of einstein convention? . . . . .	7
<input type="checkbox"/> consistent mode description . . . . .	10



## INTRODUCTION

---

### Introduction to notation

- notation of vectors and tensors





## THEORY

In order to understand the complexity of the problem and the utilised solving method, it is reasonable to revisit some fundamental concepts in the following chapter. At the same time, a consistent notation is introduced. At first, the tensor formalism for elastic properties of rigid bodies is introduced. It should be noted in advance, that vectors are denoted by a single underline and tensors or tensor fields of higher order by a number of underlines corresponding to their order. After that, elastic waves are derived and analysed further. This will be mostly based on [1], if not otherwise stated.

justification  
for modelling  
with elastic  
waves

check if de-  
scription still  
applies, cita-  
tion?

## 2.1 ELASTIC PROPERTIES OF MATERIALS

In the following, the formalism for bulk elastic properties of materials is introduced. As only small deformations are relevant for the considerations in this thesis, they are assumed to be in the regime of linear elastic behaviour, neglecting plastic or nonlinear behaviour.

In one dimension this can be described simply by Hooke's law, which states that a small displacement  $\Delta l$  caused by a force  $F$  is proportional to the displacement. Considering an object of length  $l$  and cross-sectional area  $A$ , **stress** and **strain** can be defined as

$$\text{Stress} \quad \sigma := \frac{F}{A} \quad \text{Strain} \quad \epsilon := \frac{\Delta l}{l} \quad (2.1)$$

Hooke's law can now be reformulated as

$$\sigma = C \epsilon \quad (2.2)$$

with **Young's Modulus**  $C$  as proportional constant.

## 2.1.1 Stress Tensor

For finite size objects stress can be defined locally on infinitesimal, cubic volume elements. These volume elements get deformed by forces that are applied to

the object. If we consider a general force  $\Delta \underline{F}$  acting on a surface element  $\Delta A$  we can always divide this force into a normal component  $\Delta \underline{F}_n$  and two mutually perpendicular tangential components  $\Delta \underline{F}_{t1}$  and  $\Delta \underline{F}_{t2}$ . This implies the general definition of the stress tensor as

$$\sigma_{ij} = \frac{\text{force in direction } i}{\text{surface with normal in direction } j} \quad (2.3)$$

picture of stress definition?

whereas indices  $i$  and  $j$  denote one of the spatial directions  $x, y$  or  $z$ . If we claim the volume element to be static, it follows, that the normal forces on opposite sites and tangential forces on neighbouring sides equal each other. The latter can be expressed over  $\sigma_{ij} = \sigma_{ji}$ . This leaves 6 independent components, the three normal stresses  $\sigma_{ii}$  and the three shear stresses  $\sigma_{ij}$ .

### 2.1.2 Strain Tensor

Deformation of a three dimensional object can be described using the displacement field  $\underline{u}(\underline{r}, t) := \underline{r}' - \underline{r}$ , which defines a displacement vector for each point  $\underline{r}$  in space in comparison to the deformed position  $\underline{r}'$ . Local stress only relates to change in displacement relative to neighbouring positions which allows to consider the Taylor expansion in first order

$$\underline{u}(\underline{r} + \Delta \underline{r}, t) = \underline{u}(\underline{r}, t) + \underline{\underline{\nabla}} \underline{u}(\underline{r}, t) \Delta \underline{r} \quad (2.4)$$

with the Jacobian matrix  $(\underline{\underline{\nabla}} \underline{u})_{ij} = \frac{\partial u_i}{\partial r_j}$ . This in turn can be decomposed into a symmetric part and an antisymmetric part. The symmetric part is defined as strain tensor:

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} \right) \quad (2.5)$$

It is a dimensionless measure for local deformation in contrast to the antisymmetric part, which represents local rotation.

reference to lecture 2?

### 2.1.3 Stress-Strain Relations

With stress and strain tensor introduced, equation 2.2 can be generalised to three dimensions taking anisotropies of the material into account:

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} \quad \text{or} \quad \underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\epsilon}} \quad (2.6)$$

This defines the **Elasticity tensor** as order 4 tensor with unit force / area. In general this tensor contains 81 components. However, stress and strain tensor are

symmetric, so that  $C_{ijkl} = C_{jikl} = C_{ijlk}$ . This reduces the number of independent components to 36 and makes it possible to describe the Elasticity tensor as  $6 \times 6$  matrix. For this representation, the so called Voigt notation maps pairs of coefficients of elasticity, strain and stress tensor to a single index as in table 2.1 This simplifies the notation of elasticity significantly, but it should be treated

$$xx \rightarrow 1 \quad yy \rightarrow 2 \quad zz \rightarrow 3 \quad yz = zy \rightarrow 4 \quad xz = zx \rightarrow 5 \quad xy = yx \rightarrow 6$$

Table 2.1: Voigt notation

carefully. Tensors written in Voigt notation do not transform like vectors in each index.

One can also proof, that  $C_{ijkl} = C_{klij}$  considering elastic energy (see section 4.3.1 of [1]) which leads to 21 independent components. Further simplifications derive from crystal symmetry or limitations in anisotropy. In the following treatment, the material is assumed to be isotropic, which reduces the number of independent components to 2. These are typically introduced as Lamé constants  $\lambda$  and  $\mu$  defined by the relation

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \quad (2.7)$$

which derives from equation 2.6 by regarding isotropy of the material [2]. From this equation the elasticity tensor can be expressed by a simplified  $6 \times 6$  matrix as in:

$$\underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\epsilon}} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{pmatrix} \cdot \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} \quad (2.8)$$

mention of  
einstein con-  
vention?

## 2.2 ELASTIC WAVES

This formalism can now be used to introduce elastic waves. Their existence and behaviour depends on the equations of motion of the regarded medium. To obtain these, we consider at first a small volume  $\Delta V = \Delta x \Delta y \Delta z$ , that is subject to a stress  $\sigma_{xx}(x)$  on the one side and to  $\sigma_{xx}(x + \Delta x)$  on the other side. The resulting net force becomes thus

$$\Delta F_x = [\sigma_{xx}(x + \Delta x) - \sigma_{xx}(x)] \Delta x \Delta y \Delta z = \frac{\partial \sigma_{xx}}{\partial x} \Delta x \Delta y \Delta z \quad (2.9)$$

by approximating  $\sigma_{xx}(x + \Delta x)$  in first order and setting the frame of reference to the center of mass of the volume. The force leads to a displacement  $u_x$  of the volume in  $x$  direction and equals the product of the mass  $\rho \Delta x \Delta y \Delta z$  and acceleration in  $x$  direction  $\frac{\partial^2 u_x}{\partial t^2}$ . This yields the one dimensional partial differential equation

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} \quad (2.10)$$

Assuming an isotropic material and using relation 2.6 we get

$$\rho \frac{\partial^2 u_x}{\partial t^2} = C_{11} \frac{\partial \epsilon_{xx}}{\partial x} = C_{11} \frac{\partial^2 u_x}{\partial x^2} \quad (2.11)$$

In an anisotropic medium the other stress components need to be taken into account leading to the general wave equation

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} = C_{ijkl} \frac{\partial^2 u_l}{\partial x_j \partial x_k} \quad (2.12)$$

### 2.2.1 Solution for Isotropic Materials

In the following analysis, only isotropic media will be considered. In this case the isotropic stress strain relation, equation 2.8, can be inserted to the wave equation which yields

$$\rho \frac{\partial^2 \underline{u}}{\partial t^2} = (\lambda + \mu) \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) + \mu \underline{\nabla}^2 \underline{u} \quad (2.13)$$

By using the relation  $\underline{\nabla}^2 \underline{u} = \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{u})$  for the vector laplace operator one can express it as

$$\rho \frac{\partial^2 \underline{u}}{\partial t^2} = (\lambda + 2\mu) \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) - \mu \underline{\nabla} \times (\underline{\nabla} \times \underline{u}) \quad (2.14)$$

If we now consider the Helmholtz decomposition of the displacement fields

$$\underline{u} = \underline{\nabla} \Phi + \underline{\nabla} \times \underline{\Psi} \quad (2.15)$$

with elastic potentials  $\Phi$  and  $\Psi$  and insert it into 2.14, it decouples to the two equations (see [3])

$$\frac{\partial^2 \Phi}{\partial t^2} = c_L^2 \underline{\nabla}^2 \Phi \quad (2.16)$$

$$\frac{\partial^2 \underline{\Psi}}{\partial t^2} = c_T^2 \underline{\nabla}^2 \underline{\Psi} \quad (2.17)$$

$$\text{with } c_L = \left( \frac{\lambda + 2\mu}{\rho} \right)^{1/2} \quad \text{and} \quad c_T = \left( \frac{\mu}{\rho} \right)^{1/2} \quad (2.18)$$

Those equations represent four wave equations for the elastic potentials with phase velocities  $c_L$  and  $c_T$ .

In analogy to electromagnetic waves we can now make the harmonic plane wave ansatz  $\Phi(\underline{x}, t) = u_L e^{i(\underline{k}_L \underline{x} - \omega t)}$  and  $\Psi(\underline{x}, t) = \underline{u}_T e^{i(\underline{k}_T \underline{x} - \omega t)}$  with arbitrary amplitudes  $u_L \in \mathbb{C}$  and  $\underline{u}_T \in \mathbb{C}^3$ . At this, the vector  $\underline{k} = \frac{2\pi}{\lambda}$  denotes the wave vector defined over the wavelength  $\lambda$  and  $\omega$  is the circular frequency defined over the oscillation frequency  $f$  with  $\omega = 2\pi f$ . The differentiation between  $\underline{k}_L$  and  $\underline{k}_T$  is necessary because of the different phase velocities for the elastic potentials resulting in different wave numbers according to  $c = \frac{\omega}{|\underline{k}|}$ .

Substituting the wave ansätze into equation 2.15, we get

$$\underline{u} = u_L \underline{k} e^{i(\underline{k}_L \underline{x} - \omega t)} + \underline{k} \times \underline{u}_T e^{i(\underline{k}_T \underline{x} - \omega t)} \quad (2.19)$$

Here we can see now, that the elastic potential  $\Phi$  is responsible for waves with longitudinal polarisation and  $\Psi$  for waves with transversal propagation. Furthermore we can identify an orthonormal polarisation basis depending on the propagation direction  $\hat{\underline{k}} = \frac{\underline{k}}{|\underline{k}|}$  so that  $\underline{u}$  decomposes to

$$\underline{u} = a_L \underline{p}_L e^{i(\underline{k}_L \underline{x} - \omega t)} + (a_{TH} \underline{p}_{TH} + a_{TV} \underline{p}_{TV}) e^{i(\underline{k}_T \underline{x} - \omega t)} \quad (2.20)$$

In general, the polarisation vector for longitudinal polarisation  $\underline{p}_L$  is fixed to be  $\hat{\underline{k}}$ . If scattering at an interface is considered, a plane of incidence can be defined that is spanned by  $\hat{\underline{k}}$  and the normal vector of the interface. The transversal polarisation vectors are then defined as transversal horizontal polarisation  $\underline{p}_{TH}$  pointing out of the plane of incidence and transversal vertical polarisation lying in plane of incidence orthonormal to  $\hat{\underline{k}}$ .

The coefficients  $a_i$  may be complex to express an additional phase between the components. However, it should be noted, that this is merely convenient for calculation and that the physical wave behaves like the real part of the shown equations.

### 2.2.2 Elastic Wave Scattering at an Interface

#### Boundary Conditions

On the way to describe wave propagation through complex layer structures it is useful to consider a single interface first. In the upper half space  $S_1$  and lower half space  $S_2$  we assume homogeneous materials with different elastic constants and thus different sound velocities. At the interface, which is chosen to be the xy-plane the elastic properties are discontinuous. However, we can assume that

those layers connected strongly so that the displacement field and normal stresses are continuous at the boundary [4, pp. 182, 185]. This leads to the boundary conditions:

$$\underline{u}^{(1)}(\underline{r}, t)|_{\underline{r} \in \partial S_1} = \underline{u}^{(2)}(\underline{r}, t)|_{\underline{r} \in \partial S_2} \quad (2.21)$$

$$\sigma_{13}^{(1)}(\underline{r}, t)|_{\underline{r} \in \partial S_1} = \sigma_{13}^{(2)}(\underline{r}, t)|_{\underline{r} \in \partial S_2} \quad (2.22)$$

$$\sigma_{23}^{(1)}(\underline{r}, t)|_{\underline{r} \in \partial S_1} = \sigma_{23}^{(2)}(\underline{r}, t)|_{\underline{r} \in \partial S_2} \quad (2.23)$$

$$\sigma_{33}^{(1)}(\underline{r}, t)|_{\underline{r} \in \partial S_1} = \sigma_{33}^{(2)}(\underline{r}, t)|_{\underline{r} \in \partial S_2} \quad (2.24)$$

In these six equations displacement and stress from upper and lower medium are differentiated by the given superscript index.

### Law of Refraction

If we now consider a plane wave that is incident on that interface, we can define a plane of incidence as in section 2.2.1 and rotate the coordinate system so that it equals the yz-plane. This is only valid if we assume an isotropic material as stated before. In that setting, the wave vector  $\underline{k}_i$  for a particular mode  $i \in \{L, TV, TH\}$  can be parametrised by the angle of incidence  $\theta$  as

$$\underline{k}_i = \frac{\omega}{c_i} (0, \sin(\theta_i), -\cos(\theta_i))^T \quad (2.25)$$

The polarisation basis can then be defined as

$$\underline{p}_L = \hat{\underline{k}} = (0, \sin(\theta_i), -\cos(\theta_i))^T \quad (2.26)$$

$$\underline{p}_{TH} = (1, 0, 0)^T \quad (2.27)$$

$$\underline{p}_{TV} = \underline{p}_L \times \underline{p}_{TH} = (0, -\cos(\theta), -\sin(\theta))^T \quad (2.28)$$

It is now possible to extract a law of refraction from the boundary conditions in equation 2.21 (see [4, 168ff]). A simple approach is to consider scattering of the transversal horizontal mode (TH) so that the displacement is only in x-direction. An incoming plane wave with wave vector  $\underline{k}_{in}$  is partially reflected to a wave along  $\underline{k}_r$  and partially transmitted through the interface to a wave with  $\underline{k}_t$ . Also, different frequencies  $\omega_i$  are assumed for the scattered waves so that the wave in the upper medium is

$$u_1^{(1)} = a_{in} e^{i(k_{in} \sin \theta_{in} y - \omega_{in} t)} + a_r e^{i(k_r (\sin \theta_r y - \omega_r t))} \quad (2.29)$$

and in the lower medium

$$u_1^{(2)} = a_t e^{i(k_t \sin \theta_t y - \omega_t t)} \quad (2.30)$$

consistent  
mode descrip-  
tion

Here it was used, that the interface is at  $z = 0$  which simplifies the following derivation. According to the boundary conditions, these displacements must equal each other at any time and for all points on the interface. This leads to the immediate conclusion, that

$$\omega = \omega_{in} = \omega_r = \omega_t \quad (2.31)$$

After removing the common factor  $e^{-i\omega t}$  from the equation and expressing the wave numbers by frequency and sound velocity over  $k = \frac{\omega}{c}$ , we get

$$a_{in} e^{i \frac{\omega}{c_{T,1}} \sin \theta_{in} y} + a_r e^{i \frac{\omega}{c_{T,1}} \sin \theta_r y} = a_t e^{i \frac{\omega}{c_{T,2}} \sin \theta_t y} \quad (2.32)$$

This however can be only fulfilled for all  $y \in \mathbb{R}$ , if the relations

$$a_{in} + a_r = a_t \quad (2.33)$$

$$\frac{\sin \theta_{in}}{c_{T,1}} = \frac{\sin \theta_r}{c_{T,1}} = \frac{\sin \theta_t}{c_{T,2}} \quad (2.34)$$

are met. Equation 2.34 can be also interpreted as the equality of the wave vector component parallel to the interface  $k_y = \sin \theta_i \frac{\omega}{c_i}$ .

The same procedure can be applied to the y and z component of the displacement field, where transversal vertical (TV) and longitudinal (L) modes need to be considered [4, p. 185]. However, one significant difference is that even for a single incident mode, both L and TV mode are possible after being reflected or transmitted. This will only add a term of the form  $a_i e^{i \frac{\omega}{c} \sin \theta_i y}$  to each side of equation 2.32, but will result in analogous relations to 2.31 and 2.34, namely the invariance of frequency  $\omega$  and  $k_2$  of each mode under scattering at the interface.

To conclude, the angle  $\theta_{out}$  of any outgoing polarisation mode can be determined from the incoming wave's angle  $\theta_{in}$  and the sound velocities for incoming and outgoing wave  $c_{in}, c_{out}$  by

$$\frac{\sin \theta_{in}}{c_{in}} = \frac{\sin \theta_{out}}{c_{out}} \quad (2.35)$$

This can be transferred to the optical law of snellius by introducing an effective refraction index  $n_i = \frac{1}{c_i}$ . Consequently, TH-polarised waves scatter like electromagnetic waves and for example the reflected angle is always the incident angle. The main difference to optics becomes clear for L and TV polarised waves. Scattering within the same polarisation mode is the same as for the TH mode, but now those modes can scatter into each other. Equation 2.35 still applies, but energy is also emitted to additional modes as shown in figure 2.1.

It is also possible to get the analytical solutions for refraction of a single mode at an interface by evaluating the remaining boundary conditions and the results



from continuity of displacement. The results are usually expressed by defining a reflection coefficient  $r_i$  and a transmission coefficient  $t_i$  for participating mode  $i \in \{L, TH, TV\}$  as

$$r_i := \frac{a_{r,i}}{a_{in,i}} \quad (2.36)$$

$$t_i := \frac{a_{t,i}}{a_{in,i}} \quad (2.37)$$

Usually those coefficients have a functional dependency on the incident angle.

### *Total Internal Reflection*

#### *Conservation of Energy at a Single Interface*

According to [4, p. 166] the transmitted time averaged power per unit area of a transversal elastic wave in a homogeneous medium is

$$I_L = \frac{1}{2}(\lambda + 2\mu)\frac{\omega^2}{c_L}|a_L|^2 \quad (2.38)$$

and similarly for a transversal elastic wave of a single polarisation mode

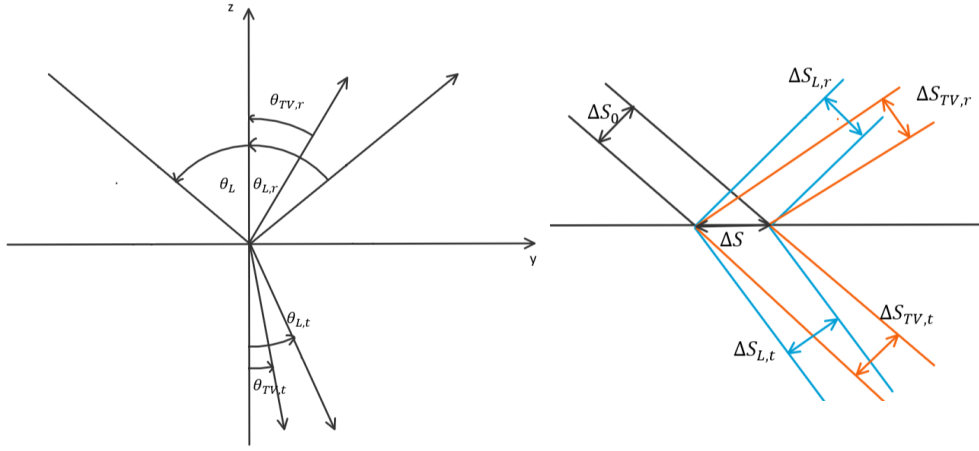
$$I_T = \frac{1}{2}\mu\frac{\omega^2}{c_T}|a_T|^2 \quad (2.39)$$

where  $a_T$  is either  $a_{TV}$  or  $a_{TH}$  depending on the regarded mode. This quantity is also known as time averaged intensity. By using the definition of  $c_L$  and  $c_T$  from equation 2.18 we get the more general relation

$$I_p = \frac{\rho\omega^2}{2}|a_p|^2c_p \quad (2.40)$$

with  $p \in \{L, TH, TV\}$  denoting the polarisation mode. It should be noted that the referred unit area is perpendicular to the direction of propagation.

In the setting of an arbitrary polarised wave scattered at an interface, there are six possible outgoing modes as discussed previously. To get an energy relation between the intensities of each participating mode we consider the energy balance of a surface element  $\Delta S$  of the interface, on which an arbitrary polarised beam with cross-sectional area  $\Delta S_0$  and intensity  $I_{in}$  is incident. The cross-sectional areas of the outgoing beams are named according to figure 2.1b which can be also transferred to the case of an incident TH polarised wave.



(a) Refraction of a Longitudinal Elastic Wave (b) Conservation of Energy at Surface Element  $\Delta S$  for an TV polarized beam

Figure 2.1

The power each beam transmits or emits is given by the product  $P_p = I_p \Delta S_p$ .  $\Delta S_p$  is related to  $\Delta S$  by  $\Delta S_p = \cos \theta_p \Delta S$ ,  $p$  denoting any of the depicted beams and  $\theta_p$  being the according angle.

In conclusion, the energy balance results in

$$I_{in} \cos \theta_{in} = \sum_{i=L,TH,TV} I_{i,r} \cos \theta_{i,r} + I_{i,t} \cos \theta_{i,t} \quad (2.41)$$

From that we can define reflectivity  $\mathcal{R}$  and transmittivity  $\mathcal{T}$  of the interface as fraction of reflected and transmitted power by

$$\mathcal{R} := \frac{I_{L,r} \cos \theta_{L,r} + I_{TH,r} \cos \theta_{T,r} + I_{TV,r} \cos \theta_{T,r}}{I_{in} \cos \theta_{in}} \quad (2.42)$$

$$\mathcal{T} := \frac{I_{L,t} \cos \theta_{L,t} + I_{TH,t} \cos \theta_{T,t} + I_{TV,t} \cos \theta_{T,t}}{I_{in} \cos \theta_{in}} \quad (2.43)$$

$$(2.44)$$

so that  $\mathcal{R} + \mathcal{T} = 1$  is fulfilled. This can be later used to verify the obtained results.

### 2.2.3 Solving Methods for Stratified Media

#### Transfer Matrix Method

#### Linear System of Equations



## IMPLEMENTATION

---



ANALYSIS

---



## CONCLUSION

---





## BIBLIOGRAPHY

---

- <sup>1</sup>R. Gross and A. Marx, *Festkörperphysik* (De Gruyter Oldenbourg, 2014), pp. 143–155.
- <sup>2</sup>T. Kundu, *Ultrasonic and electromagnetic nde for structure and material characterization: engineering and biomedical applications* (Taylor & Francis, 2012), pp. 27–30.
- <sup>3</sup>A. Bedford and D. Drumheller, *Introduction to elastic wave propagation* (Jan. 1994), p. 43.
- <sup>4</sup>J. Achenbach, N.-H. P. Company, and A. E. P. Company, *Wave propagation in elastic solids*, Applied Mathematics and Mechanics Series (North-Holland Publishing Company, 1973).