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Options Market Making in Stressed Markets

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Enhancing Market-Making and Risk Management

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Abstract

In this thesis, we extend the high-frequency market-making literature by embedding discontinuous jumps into the mid-price dynamics via an exponential Lévy model and deriving a system of Hamilton–Jacobi–Bellman equations that jointly account for diffusion risk, jump risk, and volatility-spread effects. To render the resulting nonlinear integro-PDEs tractable, we introduce a Least-Squares Monte Carlo approximation that produces closed-form, quadratic-approximate quoting strategies. Implemented within the nanosecond-resolution ABIDES simulator, these jump-aware market makers—when calibrated with an enlarged order-arrival parameter outperform both uninformed and informed liquidity providers as well as the classical Avellaneda–Stoikov agent, generating substantial positive P&L while controlling inventory risk.

1 Introduction

High-frequency equity markets are characterized by rapid fluctuations and occasional extreme moves that continuous diffusion models fail to capture.[21, 1] The traditional geometric Brownian motion (GBM) model, central to the Black-Scholes-Merton framework, has long been a cornerstone in financial modeling. However, empirical inconsistencies—such as fat tails, skewness, volatility clustering, and most critically, the presence of jumps or discontinuities in asset prices—have motivated a shift towards more sophisticated stochastic processes. A compelling alternative arises from Lévy processes, a broad class of stochastic processes that generalize Brownian motion by allowing jumps, infinite activity, and non-Gaussian behavior. An increasing body of research supports their adoption for modeling stock prices.

One of the earliest and most influential contributions is from [13], who introduced hyperbolic distributions in finance, derived from Lévy processes. They showed that these distributions provide better fits for asset returns, capturing heavy tails and asymmetry that GBM fails to represent. Their work marked a turning point by showing that real market data exhibit much more kurtosis and skew than Gaussian models can handle, indicating the inadequacy of GBM. Building on this, [4] introduced normal inverse Gaussian (NIG) processes, a subclass of generalized hyperbolic Lévy processes. NIG processes were shown to model returns with both finite and infinite variance and possess analytical tractability for option pricing. These processes naturally accommodate the jump behavior and stochastic volatility observed in financial time series—features that are impossible to encode in the pure Brownian framework. Further endorsement comes from [27], who provided a systematic exposition of various Lévy-based models such as the Variance Gamma and CGMY models, all of which incorporate discontinuous paths and non-normal return distributions. Schoutens also emphasized that Lévy models support closed-form solutions for a wide class of financial derivatives, making them not only theoretically richer but also computationally feasible for practitioners. In his foundational chapter in Lévy Processes: Theory and Applications, [12] demonstrated that generalized hyperbolic Lévy motions outperform classical models in high-frequency finance. He emphasized that Lévy processes better reflect market microstructure noise, allowing for a semimartingale formulation that integrates seamlessly into the general framework of no-arbitrage pricing. The most comprehensive argument, however, comes from [10], whose seminal book makes a definitive case for modeling stock prices with jump processes. They show that market incompleteness, volatility smiles, and risk premia for jumps are all natural consequences of Lévy-based models. Their analytical framework shows that jump models, especially those driven by pure-jump Lévy processes, provide a more realistic depiction of market dynamics than continuous-path models like GBM.

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Early theoretical work on dealer behavior—most notably [20], who framed market-maker inventory management as an optimal-control problem, and [18], who distilled the essence of liquidity provision in a simple three-period model—laid the conceptual foundations for understanding trading incentives and equilibrium spreads. However, these models predate the era of ultra-electronic markets and offer little guidance for the design of real-time, algorithmic quoting strategies.

A revitalization came in 2008 with Avellaneda and Stoikov's seminal paper [2], which applied stochastic-control techniques to derive dynamic bid–ask quotes for a single asset in a limit-order-book setting. Since then, a rich body of work has extended this framework: [19] proved that, for exponential arrival intensities, the Hamilton–Jacobi–Bellman PDE reduces to a system of linear ODEs; [9] have incorporated alpha signals, ambiguity aversion, and alternative objective functions; and multi-asset extensions by Guéant and collaborators have shown how cross-asset inventory correlations can be managed via high-dimensional ODE systems. In the domain of option market making, [3] developed an algorithmic framework for a book of vanilla options on a single underlying that, by exploiting a constant-vega approximation, collapses the seemingly high-dimensional control problem into a low-dimensional functional equation solvable via simple Euler-interpolation schemes—yielding tractable optimal quotes even for large multi-strike and multi-maturity portfolios. Building on this,[22] recent Working Paper introduces a Cox process specification in which order-arrival intensities depend both on the posted spreads and on the trader's view of actual versus implied volatility. They derive closed-form, second-order approximations for bid and ask spreads that decompose neatly into (i) expected volatility-arbitrage profits, (ii) demand–elasticity terms, and (iii) inventory-risk penalties, and they show how the model extends naturally to general European derivatives and large option books under flexible risk controls . These advances enrich the core Avellaneda–Stoikov [2] paradigm by embedding volatility-arbitrage and multi-option dimensions directly into the dynamic quoting mechanism. Yet, despite its theoretical elegance, this literature remains largely confined to single-asset, diffusion-driven models, and is agnostic as to market microstructure beyond an intensity-based order arrival process.

In contrast, the modelling of options market making introduces new layers of complexity: the nonlinearity of option payoffs generates gamma and vega exposures even under continuous delta-hedging; and implied-volatility dynamics and stochastic volatility both shape order flow and risk. To date, only a handful of studies— [14], [29], and [3] —have addressed aspects of option quoting under stochastic-volatility assumptions or constant-vega approximations.

This thesis builds on the frameworks of [2], [3] and [22], by assuming the stock process in the market micro-structure follows an exponential levy process of finite variance, making it especially suitable for periods of high volatility/uncertainty. Furthermore, an algorithmic market-making model is developed for a portfolio of European options, which due to its formulation can be used for American options as well.

In this framework, the market maker's strategy unfolds along two intertwined dimensions. First, it continuously trades the underlying asset to mitigate inventory risk—but instead of classical delta-hedging (which leaves jump exposures unaddressed), it employs the Galtchouk–Kunita–Watanabe decomposition [10] to construct a jump-robust hedge, thereby stripping out all but the pure vega risk on the option book. Second, it dynamically posts bid and ask quotes whose executions arrive as a Poisson process, with an intensity function that depends on four key factors: the quoted spread, the market maker's view of actual versus implied volatility, the underlying's jump-risk profile, and the dealer's own risk-aversion parameter. By linking quote aggressiveness to both volatility-arbitrage opportunities and jump-risk considerations, this design ensures that the market maker adapts in real time to evolving market conditions while keeping residual risk to a minimum.

The problem of an option market maker boils down to solving a two-dimensional functional equation of the Hamilton-Jacobi-Bellman type. Under a second-order expansion of the value function, we derive closed-form approximations for optimal bid and ask quotes that decompose intuitively into (i) an expected volatility-arbitrage term capturing profit from actual versus implied volatility differentials, (ii) a demand-elasticity term reflecting order-flow sensitivity to spreads, and (iii) a risk-penalty term linked to running and terminal inventory. This can be tackled numerically using a simple Euler scheme and Least Squares Monte Carlo [17] along with interpolation techniques. Finally a market simulation is conducted where the P&L of the market maker quoting under the estimated rule is compared to one quoting under Avellaneda and Stoikov's (2008) estimated spread. [1]

[1] https://github.com/s3fushah33/Option-MarketMaking-in-stressed-Markets-.git

2 Description of the Problem

2.1 The Market

We model the mid-price S_t of the underlying as an exponential Lévy process.

$$S_t \;=\; S_0 \exp\Big(\int_0^t X_s \, ds\Big),$$

Define the log-return process by dX_t and suppose that[2]

$$dX_t = \alpha(t)\, dt \;+\; \beta(t)\, dW_t \;+\; \int_{\mathbb{R}} \gamma(t,z)\, \bar{N}(dt,dz),$$

where $(W_t)_{t\geq 0}$ is a Brownian motion and $\bar{N}$ is the compensated Poisson random measure with Lévy measure $\nu(dz)$. An application of Itô's Lemma then yields the price-process SDE in its general exponential-Lévy form

$$\frac{dS_t}{S_{t^-}} = \Big(\alpha(t) + \tfrac{1}{2}\beta(t)^2 + \int_{|z|<R} (e^{\gamma(t,z)} - 1 - \gamma(t,z))\, \nu(dz)\Big)\, dt + \beta(t)\, dW_t + \int_{\mathbb{R}} (e^{\gamma(t,z)} - 1)\, \bar{N}(dt,dz). \quad (1)$$

In particular, for the special case $\alpha(t) = \mu_t$, $\beta(t) = \sigma_t$, $\gamma(t,z) = z$, we obtain

$$\frac{dS_t}{S_{t^-}} = \Big(\mu_t + \tfrac{1}{2}\sigma_t^2 + \int_{|z|<R} (e^z - 1 - z)\, \nu(dz)\Big)\, dt + \sigma_t\, dW_t + \int_{\mathbb{R}} (e^z - 1)\, \bar{N}(dt,dz),$$

and under the risk-neutral measure $\mathbb{Q}$, with zero interest rate, the drift disappears and the log-price $Y_t = \ln S_t$ satisfies the compensated form

$$dY_t = \Big(-\tfrac{1}{2}\sigma_t^2 - \int_{|z|<R} (e^z - 1 - z)\, \nu(dz)\Big)\, dt + \sigma_t\, d\widetilde{W_t} + \int_{\mathbb{R}} z\, \widetilde{N}(dt,dz),$$

so that $(S_t)_{t\geq 0}$ is an exponential Lévy process[3].

Let $O(t,S)$ denote the (mid)-price of a European option written on S_t. Under $\mathbb{Q}$, $O(t,S)$ solves the PIDE

$$\partial_t O(t,S) + \tfrac{1}{2}\sigma_t^2 S^2 \partial_{SS} O + \int_{|z|<R} [\, O(t,Se^z) - O(t,S) - \partial_S O(t,S)\, S\, z\,]\, \nu(dz) - r\, O(t,S) \;=\; 0,$$

In line with [3], the option price is specified by its terminal payoff. Because the horizon is very short, any intermediate boundary conditions can be safely ignored. The risk-free rate is also set to zero ($r = 0$), a perfectly reasonable simplification for a short-term optimization. Under these assumptions, the full pricing Partial integro-differential equation (PIDE) collapses to[4]:

$$\partial_t O + \tfrac{1}{2}\sigma_t^2 S^2 \partial_{SS} O + \int_{|z|<R} [\, O(t,Se^z) - O(t,S) - \partial_S O(t,S)\, S\, z\,]\, \nu(dz) = \; 0, \quad (2)$$

2.2 The Market-Making Control Problem

We study a market maker who continuously provides liquidity on a book of N European options $O_i(t, S_t)$, $i = 1, \ldots, N$, by posting symmetric bid–ask spreads around each mid-price . At time t, her inventory process in option i, which similar to [2] is denoted as

$$dq_t^i = dN_t^{i,b} - dN_t^{i,a} \quad (3)$$

where $N_t^{i,b}$ is the amount of Options of Asset i, bought by the Market maker at time t and $N_t^{i,a}$ is the amount of Options of Asset i sold at time t. $N_t^{i,a}$and $N_t^{i,b}$at are Poisson processes with intensities λ^b and λ^a. From a market-microstructure perspective, [2] assume that both bid and ask order arrival rates decay exponentially with the quoted spread. In particular, they propose the functional form

[2]Notation is taken from [24] Ch.1

[3]Detailed derivation is provided in the Appendix

[4]Derivation in the Appendix

$$\lambda^a(\delta) \;=\; \lambda^b(\delta) \;=\; A\, e^{-k\,\delta}, \tag{4}$$

where A denotes the baseline arrival rate at zero spread, and k measures the sensitivity of the order arrival intensity to the distance δ from the mid-market price.

At time t, the Market Makers Portfolio is denoted as

$$V_t \;=\; X_t \;+\; \sum_{i=1}^{N} q_t^i\, O_i(t, S_t) \;-\; \Delta_t S_t.$$

where X_t is the Cash account , $\sum_{i=1}^{N} q_t^i\, O_i(t, S_t)$ the option Inventory and $\Delta_t S_t$ the Hedge under the Galtchouk–Kunita–Watanabe decomposition [10] at time t .

The Cash process evolves as follows [3]

$$dX_t = \sum_{i=1}^{N}\left(\int_{\mathbb{R}_+^*} \big((O(t,S_t) + \delta_t^{i,A})\, dN_t^{i,a} - (O(t,S_t) - \delta_t^{i,B})\, dN_t^{i,b}\big) - O_t^i\, dq_t^i\right)$$
$$+ S_t\, d\Delta_t + d\langle \Delta, S\rangle_t.$$

with the help of Ito's Product rule one can write the inventory process by

$$dI_t = \sum_{i=1}^{N} O_t^i dq_t^i + \sum_{i=1}^{N} q_t^i dO_t^i$$

from that and the application of Ito's Product rule to the hedging component, the Portfolio process looks as follows

$$dV_t = dX_t + dI_t - \Delta_t dS_t - S_t d\Delta_t$$

plugging in the the respective processes,using that the quadratic variation of the stock process and the hedge term is zero and replacing $dO(t, S_t)$ by the pricing PIDE, the portfolio process is represented by

$$dV_t = \sum_{i=1}^{N}\int_{\mathbb{R}_+^*} \big((O(t,S_t) + \delta_t^{i,A})\, dN_t^{i,A} - (O(t,S_t) - \delta_t^{i,B})\, dN_t^{i,B}\big) + \sum_{i=1}^{N} q_t^i \big(\partial_t O(t,S_t)$$
$$+\frac{1}{2}\partial_{SS}^2 O(t,S_t) S(t^-)^2 \sigma^2 + \int_{|z|<R} \Big[O(t, S(t^-)e^z) - O(t, S(t^-))\Big]\nu(dz)$$

Furthermore, the well known Black and Scholes PDE is given by

$$\partial_t O(t,S_t) \; + \partial_S O(t,S_t) r S(t^-) + \frac{1}{2}\partial_{SS}^2 O(t,S_t) S(t^-)^2 \sigma_{imp}^2 - rO(t,S_t) = 0$$

since it was previously assumed $r = 0$ for the short term horizon the terms involving r vanish. As [5][5] and [22] show, the remaining equation must hold

$$\partial_t O(t,S_t) \;= -\frac{1}{2}\partial_{SS}^2 O(t,S_t) S(t^-)^2 \sigma_{imp}^2$$

Using this relation our Portfolio process can be expressed as

$$dV_t = \sum_{i=1}^{N}\int_{\mathbb{R}_+^*} \Big((O(t,S_t) + \delta_t^{i,A})\, dN_t^{i,A} - (O(t,S_t) - \delta_t^{i,B})\, dN_t^{i,B}\Big)$$
$$+ \sum_{i=1}^{N} q_t^i \Big(\frac{1}{2}\Big(\sigma^2 - \sigma_{imp}^2\Big) S(t^-)^2 \partial_{SS}^2 O(t,S_t)$$
$$+ \int_{|z|<R} \Big[O(t, S(t^-)e^z) - O(t, S(t^-))\Big]\nu(dz)\Big) - \Delta dS$$

which now involves a volatility trading part, where the market maker can profit from mis-priced volatility.

[5] page 4

Using a continuous time delta hedge in the presence of jumps would not accurately reduce or eliminate the inventory risk. In the classical Black–Scholes framework, where asset prices follow continuous paths, a pure delta-hedge—i.e. dynamically trading the underlying to match the partial derivative $\partial V/\partial S$—eliminates all risk. However, as soon as one introduces jumps into the asset price dynamics, perfect replication breaks down: an unexpected jump ΔS instantaneously invalidates the hedge, leaving a non-zero residual payoff. In fact, in any jump–diffusion market there is unhedgeable "jump risk," because jumps occur at discrete times and cannot be offset instantaneously by trading the underlying asset alone; moreover, the instantaneous variance contributed by jumps is unbounded over any short interval, and the market becomes incomplete, with more sources of uncertainty (jump times and sizes) than traded assets.

In the search of alternatives,[10] address several alternative hedging approaches in jump–diffusion models. Merton's extension [23] modifies the Black–Scholes pricing PDE to account for jumps but still relies on a pure delta-hedge, leaving jumps unhedged. Super-hedging computes the minimal initial capital required to dominate the payoff in all jump scenarios by solving an integro-PDE with a supremum over model parameters; this guarantees no shortfall but is often prohibitively expensive. Utility-based or indifference pricing chooses an investor's utility function U and finds the price at which the investor is indifferent between selling the derivative and trading only the underlying. the so called 'Certainty equivalent; the resulting indifference hedge maximizes expected utility and typically requires solving a nonlinear pricing equation.However, the indifference-pricing approach—which maximizes expected utility by solving a generally nonlinear pricing equation—has a notable drawback: to recover a linear pricing rule one must assume a quadratic utility function [10]. This assumption departs from the standard market-microstructure literature, where market makers almost invariably adopt a CARA (exponential) utility.

Quadratic hedging minimizes the mean-squared hedging error $E\big[(V_T - H)^2\big]$, and admits two main variants: global mean-variance hedging, which finds a self-financing strategy minimizing the L^2-norm of the terminal error, and local risk-minimization, which minimizes the instantaneous variance of the hedging error at each time. Both lead to a Galtchouk–Kunita–Watanabe (GKW) decomposition of the claim's martingale payoff $H = H_0 + \int_0^T \xi_t\, dS_t + L_T$, where the finite-variation remainder process L is orthogonal to the martingale part of S. The hedge ratio ξ_t is interpreted as the projection of the payoff's martingale onto the space spanned by the asset's martingale, and thus generalizes delta-hedging to incomplete markets by hedging only the "hedgeable" component of H.

Another class of methods selects an optimal equivalent martingale measure—such as the minimal-entropy or minimal-variance measure—prices under that measure, and then uses the corresponding delta-hedge; this embeds a choice of how jump scenarios are weighted. Finally, semi-static hedging exploits liquid vanilla options by forming static positions in calls and puts to cover jump risk, while dynamically delta-hedging the remaining diffusive component. This two-step approach can markedly reduce residual jump risk when sufficient vanilla liquidity exists.

In practice, mean-variance (or local risk-minimization) hedging via the GKW decomposition is often preferred because it explicitly accounts for jumps by projecting onto the hedgable component of S, yields explicit formulas in exponential-Lévy models, and minimizes a tractable quadratic risk metric. By contrast, pure delta-hedging ignores jump risk; super-hedging is overly conservative; utility-based methods can be model- and utility-specific; and semi-static hedges require liquid vanilla markets and careful calibration of static positions. In the Case of the Market makers Portfolio, the Hedging Problem under the mean-variance framework is

$$\text{Hedge Minimization problem: } \min_{\Delta} \mathbb{E}\left[\left| I_t - \int_0^t \Delta dS_t \right|^2\right]$$

Becasue r=0 the problem simplifies to the Option process at time t, namely the PIDE. From that, the exact soltution for the optimal Hedge is[6]

$$\text{GKW HEDGE :}\Delta_t^* = \sum_{i=1}^{N} \frac{\langle dO_i(t,S_t), dS_{t,i}\rangle}{\langle dS_{t,i}\rangle} = \frac{\sigma^2 \frac{\partial O}{\partial S}(t,S) + \frac{1}{S}\int_{\mathbb{R}} (e^z - 1)\big[O\big(t, Se^z\big) - O(t,S)\big]\,\nu(dz)}{\sigma^2 + \int_{\mathbb{R}} (e^z-1)^2\,\nu(dz)}$$

Under this optimal hedge, the instantaneous change in her total wealth decomposes into cash flows

[6]prove is given in ([10], chapter 10.4.3) for a Call Option and can be generalized to a Portfolio of Options as in the Market makers case, since it is assumed that the stock process has finite variance

from executed quotes and residual P&L from carrying option positions:

$$
\begin{aligned}
dV_t = &\sum_{i=1}^{N}\Big[(O_i(t,S_t)+\delta^a_{i,t})\,dN^{i,a}_t-(O_i(t,S_t)-\delta^b_{i,t})\,dN^{i,b}_t\Big]\\
&+\sum_{i=1}^{N}q^i_t\Big[\tfrac{1}{2}\big(\sigma^2-\sigma^2_{\text{imp}}\big)S_t^2\,\partial^2_{SS}O_i(t,S_t)+\int_{|z|<R}\big(O_i(t,S_te^z)-O_i(t,S_t)-\Delta^*_t dS_t\big)\,\nu(dz)\Big]\,dt.
\end{aligned}
\tag{5}
$$

From this final expression it is visible that each time step the market makers P&L is driven by the quoting, volatility trading and a non-hedgable residual risk of the hedging strategy. The market maker's objective is to choose quoting strategies $\{\delta^b_{i,t},\delta^a_{i,t}\}_{i=1}^N$ to maximize the expected utility of her terminal wealth

$$
\sup_{\delta\in\mathcal{A}}\ \mathbb{E}\big[\Phi(V_T)\big],
$$

where $\Phi(t,s,x,q)$=$-e^{-\gamma(V_T)}$ is the value function; a CARA utility with risk-aversion parameter $\gamma>0$. To simplify analysis and due to computational restrictions for the that follows in section 4-5, the problem will be reduced to a single Option written on a stock options so that the market makers selection of the optimal quoting strategy simplifies to choosing $\{\delta^b_t,\delta^a_t\}$.

In the next step, the dynamic programming principle is used in line with [24] ch.3 to show that the function $\Phi(t,s,x,q)$ solves the following Hamilton Jacobi Bellmann equation

$$
\begin{aligned}
\Phi_t(t,s,x,q)\ +\ \tfrac{1}{2}\,\sigma^2\,\Phi_{ss}(t,s,x,q)\ &+\big[\Phi\big(t,se^z,x,q\big)-\Phi\big(t,s,x,q\big)\big]\ \lambda(z)\\
+q\Bigg[\tfrac{1}{2}\big(\sigma^2-\sigma^2_{\text{imp}}\big)\partial_{ss}O(t,s)S(t)^2\ &+\ \Big[\,O(t,S(t^-)e^z)-O(t,S(t^-))\Big]\ \lambda(z)\Bigg]\\
+\ \max_{\delta^b}\Big[\,\Phi\big(t,s,x-(O(t,s)-\delta^b),q+1\big)-\ &\Phi\big(t,s,x,q\big)\Big]\lambda(\delta^b)\\
+\ \max_{\delta^a}\Big[\,\Phi\big(t,s,x+(O(t,s)+\delta^a),q-1\big)-\ &\Phi\big(t,s,x,q\big)\Big]\lambda(\delta^a)\\
&=\ 0\\
\text{With Terminal Condition:}&\\
\Phi(t,s,x,q)&=-aq-aq^2
\end{aligned}
\tag{6}
$$

This Hamilton–Jacobi–Bellman equation is analogous to that in the Avellaneda–Stoikov framework [2] with a linear and quadratic Inventory penalty. However in our case a jump and a volatility trading part are present as well.

The Hamilton–Jacobi–Bellman equation embodies the dynamic-programming principle by insisting that, under an optimal quoting strategy, the instantaneous expected change in the dealer's utility must vanish. the HJB balances five distinct contributions to the infinitesimal evolution of the value function: First, the mere passage of time carries an "opportunity-cost" drift: by waiting a vanishingly small interval without trading, the dealer forgoes potential profit opportunities, and her continuation-value must incorporate this deterministic time decay. Second, continuous Gaussian fluctuations in the underlying price impose a risk cost proportional to the curvature of her utility in the price dimension—intuitively, holding inventory exposes her to volatility, and the term involving the second derivative of the value function quantifies how that exposure detracts from expected utility.

Third, the possibility of sudden jumps in the asset price requires an integral term over all jump sizes: for each size and intensity, the HJB compares the value immediately before the jump to the value immediately after, thereby capturing the expected utility gain or loss from rare but potentially large price discontinuities. Fourth, because the dealer also carries an option book, any discrepancy between realized and implied variance generates a running profit or loss; this "volatility-trading" term enters the HJB as an additional drift, reflecting gains that accrue continuously whenever the actual variance deviates from the model's implied level.

Finally, and most crucially for market making, the HJB includes two optimization operators—one for the bid offset and one for the ask offset. Each operator evaluates the expected infinitesimal improvement in utility that would result from acquiring or shedding a single share at the chosen quote, multiplied by the probability of execution at that price. By maximizing these terms, the dealer determines, at each state and instant, the bid and ask spreads that best trade off profit per trade against execution risk.

3 Solution to the Market Makers Problem

To solve the integro–differential HJB in practice, we follow a sequence of systematic reductions very much in the spirit of [2].

First the Ansatz

$$\Phi(t,s,x,q) = -exp(-\gamma x)exp(-\gamma\Phi(t,s,q))$$

is made to separate the inventory dependent part from the inventory independent part. Plugging in this Ansatz into 6, the Inventory independent part cancels and the result after some algebraic reformulations is

$$\begin{aligned}
0 &= \Phi_t(t,s,q) + \tfrac{1}{2}\,\sigma^2\big(\Phi_{ss}(t,s,q) - \gamma\,\Phi_s(t,s,q)^2\big)\\
&+ \frac{\lambda(z)}{\gamma}\Big(1 - \exp\!\big(-\gamma\big[\Phi(t,se^z,q) - \Phi(t,s,q)\big]\big)\Big)\\
&+ q\Big(\tfrac{1}{2}\big(\sigma^2 - \sigma^2_{\text{imp}}\big)\,S(t)^2\partial^2_{ss}O(t,s) + \big[O(t,se^z) - O(t,s)\big]\Big)\\
&+ \max_{\delta^a}\Big\{\frac{\lambda^a(\delta^a)}{\gamma}\Big(1 - \exp\!\Big(-\gamma\,\Big(O(t,s) + \delta^a - \Big[\,\Phi\big(t,s,q\big) - \,\Phi\big(t,s,q-1\big)\Big]\,\Big)\Big)\Big)\Big\}\\
&+ \max_{\delta^b}\Big\{\frac{\lambda^b(\delta^b)}{\gamma}\Big(1 - \exp\!\Big(\gamma\Big(O(t,s) - \delta^b - \Big[\,\Phi\big(t,s,q+1\big) - \,\Phi\big(t,s,q\big)\Big]\Big)\Big)\Big)\Big\}
\end{aligned}$$

Note that the quoting part is identical to equation (3.6) in [2], except that we deal with mid-market Option prices rather than stock prices. They define the change in the value function with respect to the inventory as the reservation ask and bid price for each case separately.

$$\begin{aligned}
r_a &= \Big[\,\Phi\big(t,s,q\big) - \,\Phi\big(t,s,q-1\big)\Big]\\
r_b &= \Big[\,\Phi\big(t,s,q+1\big) - \,\Phi\big(t,s,q\big)\Big]
\end{aligned}$$

Which are the relative prices to which the market maker is indifferent to selling/ buying an additional unit of the stock/ option in this case.

Finally they derive the FOC and plug in for the mid market stock price and reservation bid/ask price, obtaining the expression[7]:

$$\frac{A}{k+\gamma}\big(e^{-k\delta_a} + e^{-k\delta_b}\big) \tag{7}$$

which I adopt.

Furthermore, (7) simplifies by making a Taylor expansion of first order so that the quoting expression becomes

$$\frac{A}{k+\gamma}\big(2 - k(\delta^a + \delta^b)\big) \tag{8}$$

Similarly, a Taylor expansion of second order is made of the stock jump part in the HJB, where the second degree is chosen to better capture large sudden movements.

To make further progress an asymptotic expansion must be made, inline with [2] a second Ansatz; an asyptotic expansion of the inventory variable is conducted.

$$\Phi(t,s,q) = \Phi^0(t,s) + q\Phi^1(t,s) + \frac{1}{2}q^2\Phi^2(t,s)$$

Using this expansion [2] add the reservation bid and ask prices to obtain

$$\delta_a + \delta_b \;=\; 2\,\Phi^2(t,s) \;+\; \frac{2}{\gamma}\,\ln\!\Big(1 + \frac{\gamma}{k}\Big) \tag{9}$$

From this expression it is visible that the Quotes are dependent on the risk aversion Parameter , sensitivity of the order arrival intensity and the second order value function. To obtain the optimal quotes it is

[7] FOC and solution method not shown here explicitly, for detailed solution check [2]

necessary to solve for this function.[2]interpret the second order value function as the sensitivity of the market maker's quotes to changes in inventory. Plugging this into (8) and expanding the HJB the following is obtained.

$$
\begin{aligned}
0 &= \Phi^0_t(t,s) + q\,\Phi^1_t(t,s) + \tfrac{1}{2}\,q^2\,\Phi^2_t(t,s)\\
&\quad + \tfrac{1}{2}\,\sigma^2\big(\Phi^0_{ss}(t,s) + q\,\Phi^1_{ss}(t,s) + \tfrac{1}{2}\,q^2\,\Phi^2_{ss}(t,s)\big)\\
&\quad - \gamma\big(\Phi^0_s(t,s) + q\,\Phi^1_s(t,s) + \tfrac{1}{2}\,q^2\,\Phi^2_s(t,s)\big)^2\\
&\quad + 2 - k\Big(2\,\Phi^2(t,s) + \frac{2}{\gamma}\,\ln\Big(1+\frac{\gamma}{k}\Big)\,\Big)\\
&\quad + \frac{\Lambda}{\gamma}\Big(1 - \,\big(\Phi^0(t,se^z) - \Phi^0(t,s)\big) + q\,[\Phi^1(t,se^z) - \Phi^1(t,s)] + \tfrac{1}{2}\,q^2\,[\Phi^2(t,se^z) - \Phi^2(t,s)]\Big)\\
&\quad + \Big(\big(\Phi^0(t,se^z) + q\,\Phi^1(t,se^z) + \tfrac{1}{2}\,q^2\,\Phi^2(t,se^z)\big) - \big(\Phi^0(t,s) + q\,\Phi^1(t,s) + \tfrac{1}{2}\,q^2\,\Phi^2(t,s)\big)\Big)^2\ \Big]\\
&\quad + q\,\Big[\tfrac{1}{2}(\sigma^2 - \sigma^2_{\text{imp}})\,\partial_{ss}S(t,s)\,O(t,s) + \big(O(t,se^z) - O(t,s)\big)\Big]
\end{aligned}
\tag{10}
$$

Further collecting the terms, the expression looks as follows

$$
\begin{cases}
\underbrace{\Big[\frac{\Lambda}{\gamma}\Big(1 - k\big(\Phi^0(t,se^z) - \Phi^0(t,s)\big) + \big(\Phi^0(t,se^z) - \Phi^0(t,s)\big)^2\Big) + \frac{A}{\kappa+\gamma}\Big(2 - k\Big(2\,\Phi_2(t,s) + \tfrac{2}{\gamma}\ln\big(1+\tfrac{\gamma}{k}\big)\Big)\Big) \\ \Phi^0_t + \tfrac{1}{2}\,\sigma^2\,\Phi^0_{ss} - \gamma\,(\Phi^0_s)^2\Big]}_{\text{coefficient of } q^0}\\
+\,q\,\underbrace{\Big[\,\Phi^1_t + \tfrac{1}{2}\,\sigma^2\,\Phi^1_{ss} - 2\gamma\,\Phi^0_s\,\Phi^1_s + \frac{\Lambda}{\gamma}\big(-k\,(\Phi^1(t,se^z) - \Phi^1(t,s)) + 2\,(\Phi^0(t,se^z) - \Phi^0(t,s))\ (\Phi^1(t,se^z) - \Phi^1(t,s))\big) \\ +\ \tfrac{1}{2}(\sigma^2 - \sigma^2_{\text{imp}})\,S^2\partial_{ss}\,O(t,s) + \big(O(t,\,se^z) - O(t,s)\Big]}_{\text{coefficient of } q^1}\\
+\,q^2\,\underbrace{\Big[\,\tfrac{1}{2}\,\Phi^2_t + \tfrac{1}{4}\,\sigma^2\,\Phi^2_{ss} - \gamma\big[(\Phi^1_s)^2 + \Phi^0_s\,\Phi^2_s\big] + \frac{\Lambda}{\gamma}\big(-\tfrac{k}{2}\,(\Phi^2(t,se^z) - \Phi^2(t,s))\ . \\ +\,((\Phi^1(t,se^z) - \Phi^1(t,s)))^2 + (\Phi^0(t,se^z) - \Phi^0(t,s))\,(\Phi^2(t,se^z) - \Phi^2(t,s))\big)\Big]}_{\text{coefficient of } q^2}
\end{cases}
$$

Equating each bracket to zero gives the system

$$
\begin{cases}
\Phi^0_t + \tfrac{1}{2}\,\sigma^2\,\Phi^0_{ss} - \gamma\,(\Phi^0_s)^2 + \frac{\Lambda}{\gamma}\big(1 - k\,(\Phi^0(t,se^z) - \Phi^0(t,s)) + ((\Phi^0(t,se^z) - \Phi^0(t,s))^2\big) + \\
\frac{A}{\kappa+\gamma}\Big(2 - k\Big(2\,\Phi_2(t,s) + \tfrac{2}{\gamma}\ln\big(1+\tfrac{\gamma}{k}\big)\Big)\Big) = 0\\
\Phi^1_t + \tfrac{1}{2}\,\sigma^2\,\Phi^1_{ss} - 2\gamma\,\Phi^0_s\,\Phi^1_s + \frac{\Lambda}{\gamma}\big(-k\,(\Phi^1(t,se^z) - \Phi^1(t,s)) + 2\,(\Phi^0(t,se^z) - \Phi^0(t,s))(\Phi^1(t,se^z) - \Phi^1(t,s))\big)\\
+\tfrac{1}{2}(\sigma^2 - \sigma^2_{\text{imp}})\,\partial_{ss}S^2\,O(t,s) + \big(O(t,se^z) - O(t,s)\big) = 0,\\
\tfrac{1}{2}\,\Phi^2_t + \tfrac{1}{4}\,\sigma^2\,\Phi^2_{ss} - \gamma\big[(\Phi^1_s)^2 + \Phi^0_s\,\Phi^2_s\big] + \frac{\Lambda}{\gamma}\big(-\tfrac{k}{2}\,(\Phi^2(t,se^z) - \Phi^2(t,s)) + ((\Phi^1(t,se^z) - \Phi^1(t,s)))^2\\
+(\Phi^0(t,se^z) - \Phi^0(t,s))\,(\Phi^2(t,se^z) - \Phi^2(t,s))\big) = 0.
\end{cases}
$$

with terminal conditions.

$$
\Phi^0(T,s) = 0, \qquad \Phi^1(T,s) = -a, \qquad \Phi^2(T,s) = -\,2a.
$$

Finally one can solve the non-linear differential equations, obtain the solution and replace right hand sight of equation (9) with the left hand sight and reformulate the terms for the spread yielding the following:

$$\delta_a+\delta_b \;=\; \frac{1}{k}\Bigg[2+\frac{\kappa+\gamma}{A}\Big(\Phi^0_t+\tfrac{1}{2}\,\sigma^2\,\Phi^0_{ss}-\gamma\,(\Phi^0_s)^2+\frac{\Lambda}{\gamma}\big(1-k\,((\Phi^0(t,se^z)-\Phi^0(t,s))+((\Phi^0(t,se^z)-\Phi^0(t,s))^2)\big)\Big)\Bigg] \tag{11}$$

This term can be read in one seamless narrative as follows.

First, even if the dealer ignores all dynamic risks, she still needs to charge a non-zero "baseline" spread, simply to cover the two-sided cost of immediacy plus any fixed overhead or running-inventory penalty built into κ. On top of this static component sits a correction term—scaled by her risk aversion γ and inversely by the overall order-arrival rate A—that precisely encodes how anticipated diffusion, jumps, and time-decay alter her willingness to narrow the quotes.

Within that correction, $\Phi_{0,t}$ captures the deterministic "time cost" of waiting, while $\frac{1}{2}\sigma^2\,\Phi_{0,ss}$ measures her exposure to continuous-path volatility through the curvature of her zero-inventory value function. The subtractive $-\gamma\,\Phi_{0,s}^2$ term arises from exponential utility, tempering the spread when her value function is highly sensitive to small price moves. Finally, the jump-risk adjustment $\frac{\Lambda}{\gamma}\big(1-k\,\Delta\Phi_0+(\Delta\Phi_0)^2\big)$ bundles both the expected and variance effects of rare discontinuities—by comparing pre- and post-jump valuations of the book—into a single penalty, with larger estimated jump intensities leading to larger spreads and vice verca

All of these economic forces are then translated into price increments by the factor $1/k$, which reflects market liquidity: when execution probability declines steeply with price (large k), the dealer can quote tighter spreads for the same economic comfort, whereas in a thin market (small k) she must leave a wider cushion. In this way, the compact expression above unifies static immediacy costs, risk-aversion, diffusion and jump penalties, and liquidity considerations into one transparent rule for setting optimal bid–ask quotes.

3.1 Estimation of the Value Functions via Least–Squares Monte Carlo

Since further analytical progress is difficult to make we proceed by doing a numerical approximation of the value function, using Least–Squares Monte Carlo.[17]

The least–squares Monte Carlo (LSMC) approach is intrinsically aligned with the backward-induction structure of dynamic programming. By simulating forward the underlying state variables and then performing a regression-based approximation of the conditional expectation at each time step, the method naturally implements the Bellman recursion without ever requiring an explicit grid in the high-dimensional state space. [17] demonstrates how LSMC efficiently values American-style options: at each potential exercise date one simply regresses the realized payoffs (or continuation values) of the simulated paths onto chosen basis functions to estimate the continuation value, and then compares it to the immediate exercise payoff. This yields an approximate optimal stopping rule that converges to the true dynamic programming solution as the number of paths and the richness of the basis increase.

LSCM can be used beyond American-option pricing [17] ,its flexibility extends to stochastic control problems such as the Market Makers problem we deal with.

To tackle our problem, we begin by discretizing the time interval $[0,T]$ into a uniform grid $0=t_0<t_1<\cdots<t_M=T$ with time step $\Delta t=T/M$. On each of N Monte Carlo sample paths we simulate the underlying asset price under the jump–diffusion model. Specifically, we initialize $S_0^{(i)}=S_0$ for path i, and for $n=0,\ldots,M-1$,draw independent increments $\Delta W_n^{(i)}\sim\mathcal{N}(0,\Delta t)$ and $N_n^{(i)}\sim\text{Poisson}(\lambda\Delta t)$, then update

$$S_{n+1}^{(i)}=S_n^{(i)}\exp\!\big((r-\tfrac{1}{2}\sigma^2-\lambda(\mathbb{E}[e^z]-1))\,\Delta t+\sigma\,\Delta W_n^{(i)}\big)\,e^{z\,N_n^{(i)}}$$

under the risk neutral measure.

At the terminal time $t_M=T$, the boundary conditions $\Phi_0=0$, $\Phi_1=-a$, and $\Phi_2=-2a$ are imposed on each path.

Then a backward induction from $n=M-1$ down to $n=0$ is performed. At each time step, for each path i, we evaluate the right–hand side of the PDE at $(t_{n+1},S_{n+1}^{(i)})$, including the diffusion term $\frac{1}{2}\sigma^2\partial_{ss}\Phi_j$, the nonlinear term $-\gamma(\partial_s\Phi_j)^2$, the jump corrections $\frac{\Lambda}{\gamma}(1-k\,\Delta_j+\Delta_j^2)$ where $\Delta_j=\Phi_j(t_{n+1},S_{n+1}^{(i)}e^z)-\Phi_j(t_{n+1},S_{n+1}^{(i)})$, and, in the case $j=0$, the spread term $\frac{A}{\kappa+\gamma}(2-k(\delta_a+\delta_b))$. Option-gamma contributions are computed via a Black–Scholes formula with implied volatility σ_{imp}. An explicit Euler step then yields provisional values

$$Y_{j,n}^{(i)}=\Phi_j\big(t_{n+1},S_{n+1}^{(i)}\big)+f_j\big(t_{n+1},S_{n+1}^{(i)}\big)\,\Delta t,\quad j=0,1,2.$$

To recover the functional form of $\Phi_j(t_n, s)$, we assume a polynomial basis of degree d and regress the provisional values $\{Y_{j,n}^{(i)}\}_{i=1}^{N}$ against the monomials $\{1, s, \ldots, s^d\}$ evaluated at $S_n^{(i)}$. Ordinary least-squares fitting yields coefficients $\{\beta_{j,r}\}$ such that

$$\Phi_j(t_n, s) \approx \sum_{r=0}^{d} \beta_{j,r}\, s^r,$$

and we then set $\Phi_j^{(i)}(t_n) = \sum_{r=0}^{d} \beta_{j,r} (S_n^{(i)})^r$ to carry the values forward.

Finally, once the surface $\Phi_0(t_0, s)$ is obtained, the optimal bid–ask spread at time zero follows by isolating the spread contribution in the PDE:[8]

$$\delta_a + \delta_b = \frac{2}{k} + \frac{\kappa + \gamma}{k\,A} \Big[\partial_t \Phi_0 + \tfrac{1}{2}\sigma^2\, \partial_{ss}\Phi_0 - \gamma(\partial_s \Phi_0)^2 + \tfrac{\Lambda}{\gamma}(1 - k\,\Delta_0 + \Delta_0^2) \Big]_{t=0}.$$

This approach relies on the assumptions that the model parameters remain constant, the explicit Euler scheme is stable for sufficiently small Δt, the value functions are well-approximated by polynomials of degree d, antithetic sampling is used to reduce variance, jump corrections are handled by evaluating polynomials at s and se^z, and option gamma is computed under Black–Scholes with implied volatility σ_{imp}.

4 Market Simulation

4.1 Introduction to the ABIDES Framework

The Agent-Based Interactive Discrete Event Simulation (ABIDES) framework is a modular, event-driven simulation engine designed to study financial markets at the micro-structural level. At its core, ABIDES employs a discrete-event kernel that advances simulated time only when events occur (order arrivals, wake-ups, cancellations, etc.), ensuring computational efficiency even when simulating large populations of interacting agents. Agents in ABIDES register with the kernel, specifying wake-up times or subscribing to event streams (e.g. updates to the limit order book), and respond by issuing new orders, cancellations, or simply recording market data.

The framework provides a built-in 'ExchangeAgent', who maintains a central limit order book (CLOB) matching limit and market orders according to price-time priority. Key exchange parameters such as tick size, book depth, and streaming history length can be configured to mimic venue rules found in real markets. Through an extensible agent interface, researchers can implement a wide variety of trading behaviors—from zero-intelligence noise traders to sophisticated market-making algorithms—while the kernel handles asynchronous message passing, agent latencies, and customizable computational delays.[7]

4.2 Simulation Setup and Configuration

In our experiments, we base the market-making simulation on the Reference Market Simulation Configuration (RMSC-4) but we extend and tailor it through a custom routine to match our purpose of testing and comparing the market makers quoting strategies. First, we fix a pseudorandom seed to guarantee that each run is reproducible. We then specify the trading date and hours—by default opening at 09:30:00 and closing at 20:00:00 with nanosecond precision—so that every simulated day follows the same temporal boundaries.

Within the configuration, a single 'ExchangeAgent' is instantiated to manage a central limit order book with full price-time priority. We enable book logging, capturing the top ten levels of both bid and ask, and set the stream history length to buffer recent trades for agents that require trade-by-trade information. Optionally, an order-by-order log may be turned on to facilitate detailed post-mortem analyses of order submission and execution dynamics.

The agent population mirrors the RMSC-4 assumptions in scale and diversity: one thousand "noise" agents generate both limit and market orders at Poisson-distributed wake-up times throughout the trading session, drawing order sizes from an empirical distribution defined by the 'OrderSizeModel'. A cohort of 102 "value" agents observe an exogenous latent fundamental process r_t that evolves according to an

[8] A practical implementation of this can be found in https://github.com/s3fushah33/Option-MarketMaking-in-stressed-Markets-.git, where an explicit spread solver class is created using LSMC .The parameters can be set to ones liking, including terminal condition and polynomial degree

Ornstein–Uhlenbeck stochastic differential equation and submit market orders whenever the discrepancy between the mid-price and their private estimate of r_t exceeds a predefined threshold. In addition, twelve "momentum" agents compute discrete-time returns over a fixed look-back window and place trades in the direction of recent price trends.

Finally, two sophisticated market-maker agents are included externally into the configuration— a classical 'AvellanedaStoikovMarketMaker' that quotes according to equation (3.18) in [2] and a 'JumpRiskMarketMakerAgent' that quotes according to our estimated spread in equation 11.Finally there is a built in Market Maker called bespoke 'AdaptiveMarketMakerAgent' it simply cancels any existing orders at each wake-up and then reposts symmetric bid and ask limit orders at a fixed number of ticks around the current mid-price, sizing each quote proportionally to recent trading volume via a participation-rate cap.[7]

To synchronize value agents' views of the fundamental process, we embed a shared "oracle", broadcasting updates of r_t at each kernel tick. We also model both network and computational latencies: every message between agents and the exchange incurs a draw from a common latency distribution, and each agent experiences a fixed computation delay before acting on received information.

Once the configuration dictionary is assembled, it is passed, the configuration initializes the discrete-event kernel, registers all agents, and advances simulated time by jumping from event to event until the designated close time. At the end of the run, ABIDES returns an `end_state` object containing comprehensive logs of executed trades, order-book snapshots, and the profit-and-loss trajectories of each agent, ready for downstream analysis.

4.3 Market Structure and Modeling Assumptions

To cleanly isolate the impact of our market-making strategies, we model the trading environment as a single continuous double-auction central limit order book (CLOB) in which all limit orders on the asset "ABM" are fully visible and matched strictly by price–time priority with a fixed tick size Δp. We deliberately omit hidden or iceberg orders and do not charge transaction fees or issue rebates, so that agents incur no explicit costs beyond adverse selection and inventory risk. By restricting trading to a single asset, we avoid cross-asset hedging effects or correlation structures, and we grant each agent an effectively infinite credit line—initialized with a large cash balance (e.g. \$100 000 in cents) and no margin constraints—so that the focus remains squarely on quoting behavior rather than funding limitations.

The only exogenous input to the system is a latent fundamental value process r_t, which evolves as an Ornstein–Uhlenbeck diffusion,

$$dr_t = -\kappa_{\text{oracle}} \, (r_t - \bar{r}) \, dt + \sigma_{\text{fund}} \, dW_t, \tag{12}$$

and is broadcast asynchronously to our value agents via an oracle. We impose no additional external data feeds, shocks, or jumps—apart from this mean-reverting signal, there are no exogenous noise components driving prices. Instead, every change in the displayed mid-price emerges endogenously from agents' order flow: market orders consume visible liquidity at successive price levels (up to $\lceil v/\bar{q} \rceil$ levels for an order of size v, where $\bar{q}$ is the average queue depth), producing a piecewise-linear temporary impact, and limit orders enter and exit the book to alter depth.

Order arrivals for noise and value agents follow independent Poisson processes, whereas market-maker agents wake either at fixed intervals or according to their own Poisson schedule, depending on the strategy. Realized volatility of the mid-price is thus a direct outcome of the mix of Poisson-driven order flow, value-based trading around r_t, momentum-based trend-following, and strategic quoting. In the absence of any submitted orders, the reference mid-price coincides exactly with the fundamental value, so any bid–ask spread at the top of the book reflects purely strategic placement by liquidity-providing agents and stochastic deviations in value-agent submissions.

Under these simplifying assumptions, our simulation constitutes a closed system in which fundamental-driven trades and strategic quoting within the ABIDES CLOB give rise to all observed market phenomena—spread dynamics, volatility clustering, inventory accumulation, and price-impact curves—thereby allowing us to attribute emergent behaviors unambiguously to the specified agent architectures and their parameterizations.

These simplifications strike a balance between realism and tractability, ensuring that emergent phenomena can be attributed to agent strategies rather than ancillary market features. Since no external market data or real-world time-series are injected into the simulation, the entire price dynamics emerge endogenously from the interactions of our agent populations and the exogenous fundamental process broadcast via the oracle. In particular, we assume:

In our closed-system simulation, all mid-price movements emerge directly from the mechanics of the central limit order book: every tick change reflects the matching of buy and sell orders placed by agents, with no external price-impact function imposed. When a market order arrives, it simply consumes available liquidity at the best price levels, and any subsequent limit-order arrivals or cancellations adjust the book's depth in real time. Beyond this discrete order-book matching, we introduce no additional exogenous noise . The only external signal is the mean-reverting fundamental value broadcast via the oracle , which serves as a proxy for all latent news; agents observe it through noisy, asynchronous updates, but there are no sudden shocks or discontinuities injected into the price process.

Volatility in our model is entirely agent-driven. Rather than specify a stochastic volatility model, we let realized mid-price variance arise from the interplay of Poisson-driven order arrivals, the empirical order-size distribution of noise traders, threshold-based value-agent trading around the OU process, and momentum-based trend-followers. In this way, clustering of volatility and occasional bursts of activity are natural consequences of the heterogeneous strategies and order-flow imbalances in the ABIDES environment.

Price impact is purely temporary and linear through the book: a market order of size v will walk the book across up to $\lceil v/\bar{q} \rceil$ price levels (where $\bar{q}$ denotes the average queue depth), producing a piecewise-linear relationship between executed volume and instantaneous price movement. There is no permanent impact component beyond the new best bid or ask that results from the execution. Finally, in the absence of any submitted orders, the reference mid-price coincides exactly with the latent fundamental value; any observed bid–ask spread is thus purely the strategic outcome of market-maker quotes and stochastic variation in value-agent submissions.

Under these assumptions—endogenous order-book matching, no external jumps, agent-generated volatility, linear temporary impact, and zero latent spread—our simulated market becomes a self-contained laboratory in which every emergent phenomenon, from spread dynamics to inventory accumulation, can be attributed unambiguously to the specified agent behaviors and their parametrizations.

By these assumptions, our simulated market becomes a closed system: the fundamental value drives valuation-based trades, and all price movements are a direct reflection of agent interactions within the ABIDES limit order book. This design ensures that observed dynamics—spread clustering, volatility bursts, inventory accumulation, and price-impact curves—can be unambiguously attributed to the specified agent behaviors and their parameterization." Similar research designs have been used to study the financial markets e.g [8] study bubble formation and find that participants split into "value" traders—who bought or sold based on deviations from known fundamental value—and "momentum" traders—who based their decisions on recent price trends. The experiments showed that even though value traders worked to correct mispricings, momentum traders amplified initial price deviations, causing prices to consistently overshoot fundamentals and form bubbles. These findings among the diverse literature regarding this topic, encourage us t use Abides as the testing plattform, as external data in that frequency is highly expensive and very scarce.

5 Simulation and Results

The modeling of optimal bid and ask quotes in high-frequency trading frameworks requires careful empirical calibration of key parameters governing market order execution. In particular, the parameters A and k in the execution intensity function of equation 4,

$$\lambda(\delta) = Ae^{-k\delta}$$

must reflect realistic properties of market order flow and microstructure dynamics.

The constant A represents the baseline arrival rate of market orders. Empirical studies of intraday trading activity [6][28] suggest that market order arrivals follow approximately Poissonian statistics with stable rates across short intervals. Thus, A is set to a level consistent with observed trading frequencies on electronic exchanges, such as $A = 140$ in simulations of liquid stocks.

The decay rate k captures how sharply the probability of execution decreases with the distance δ from the mid-price. This decay is linked to the distribution of market order sizes and their associated price impacts, as mentioned in section 2. Empirical studies [16][30] find that order sizes obey a power-law distribution and that price impact scales either logarithmically or as a power of order size. Combining these results, the decay constant can be written as $k = \alpha K$, where $\alpha \approx 1.5$ is the tail exponent of the order size distribution and K is derived from the market impact function. Thus, $k = 1.5$ represents an empirically grounded estimate.

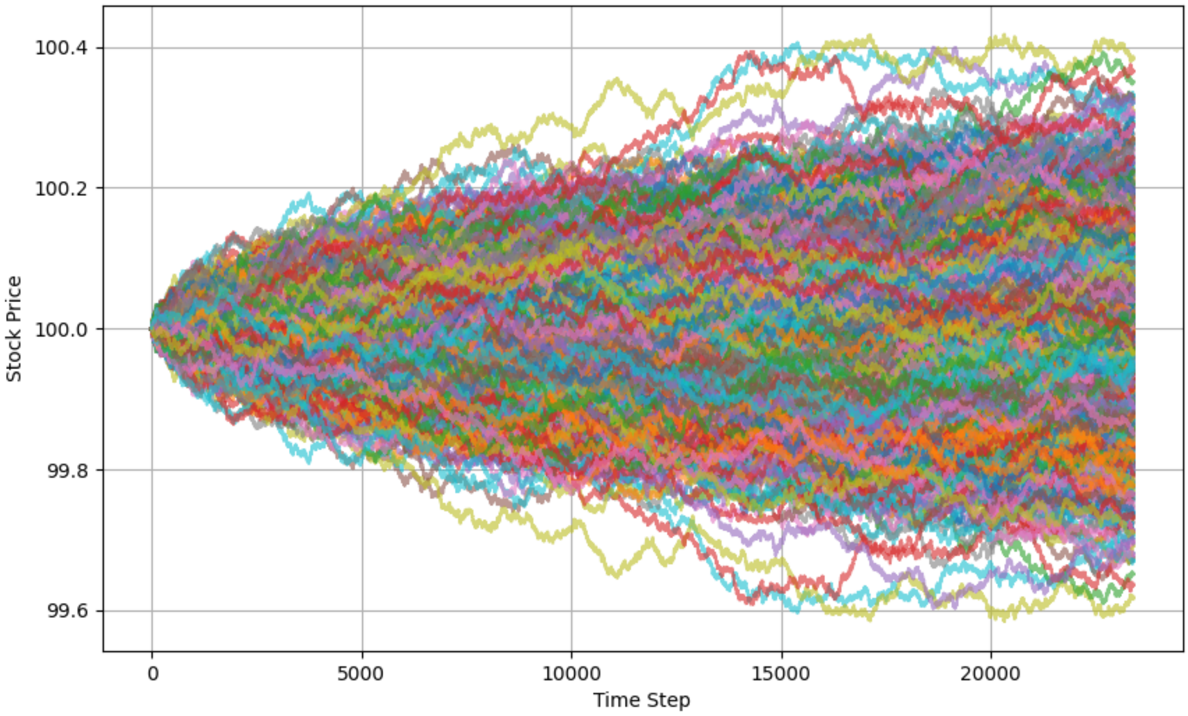

Figure 1: Simulation Stock Paths with Jumps

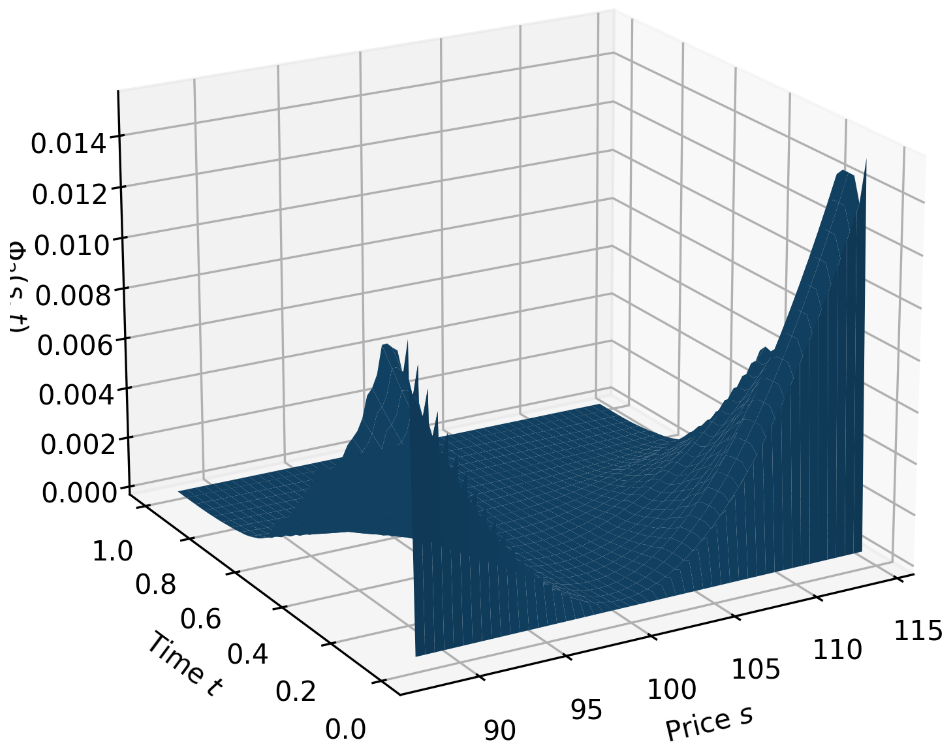

Figure 2: Estimated Φ_2 Value Function surface for K=100

In addition to modeling execution intensity through an exponential decay, modern high-frequency models and we as well, incorporate jump dynamics to capture abrupt movements in asset prices. When stock returns are modeled as exponential Lévy processes, the jump intensity function $\lambda(z)$, which characterizes the arrival rate of price jumps of size z, becomes critical.

In high-frequency financial models where asset prices follow exponential Lévy processes, the jump intensity λ — representing the average number of price jumps per unit time — varies considerably depending on the asset class. For liquid equities such as AAPL or MSFT, empirical studies suggest typical jump intensities in the range of 2 to 3 jumps per day, which translates approximately to 0.002 to 0.005 jumps per minute and 0.02/60 to 0.05/60 in our high-frequency case. For more volatile instruments, such as cryptocurrencies (e.g., Bitcoin futures), the estimated jump intensity is significantly higher, often between 10 and 20 jumps per day, or around 0.02 to 0.05 jumps per minute. Regarding the size of jumps, the literature indicates that the expected magnitude $\mathbb{E}[Z]$ of log-price jumps lies between 0.5% and 2%, reflecting the impact of rare but substantial price changes. [11], [1], [15] Potential Stock paths for over $2 * 10^4$ time steps under such calibration can be seen in Figure 1.

These findings are crucial for accurate modeling of short-term price dynamics and market risk, particularly in limit order books where liquidity crises and sudden revaluations are common.

This empirically informed calibration ensures that the trading intensity model used in our dealer framework reflects actual microstructure features of high-frequency markets.

Furthermore, the parameter choices play an important role in the estimated value function, which is dependent on such parameters as seen in the previous sections and therefore the trading behavior of the Market making can be heavily influenced by such parameter choices. An example of a value function estimate using LSMC Φ_2 under is given in figure2 for an Option with Strike = 100. Recall from 9, that the spread of the quotes is dependent on this value function. In figure2, we see that as an option nears maturity, its value function grows exponentially with the absolute distance from the strike, causing the quoted spread to widen when the underlying price moves further away. Conversely, at longer times to maturity, both the value function and the spread decline. This pattern reflects inventory risk management: because the market maker can offload positions before expiration and does not carry the option through to payoff, the potential payoff-related risk diminishes with time, resulting in tighter spreads when there is more remaining time to maturity. Once the configuration is built, a couple of aditional assumptions are necessary to make the results applicable to the options market. In our jump-PDE market-making framework, we do not incorporate a dynamic volatility surface; rather, we assume that the implied volatility σ_{imp} remains constant over the entire trading session so that all time-varying effects in option quotes arise solely from realized variance generated endogenously by order-flow and the jump–diffusion fundamental process. This is admittedly a semi-strong assumption, since in real markets the volatility surface is typically recalibrated several times per day to reflect new information and shifts in supply–demand balance. By holding σ_{imp} fixed from market open to close, we isolate the influence of jump risk and realized-variance fluctuations on the optimal bid–ask spread without confounding from evolving market expectations of future volatility.

To embed this asset-market simulation in an options-market context, we further assume that all derivative contracts are European-style calls or puts exercisable only at the single maturity T and strike K , and that discounting occurs at a constant risk-free rate r=0 with zero dividend yield. Between jumps, the underlying price follows a geometric Brownian motion with volatility σ, punctuated by Poisson-driven jumps of log-size z at intensity λ. We calibrate the constant implied volatility σ_{imp} at market open to match observed option mid-prices at the reference strike, and thereafter treat it as immutable in our PDE driver for Φ_1. In particular, we use Black–Scholes formulas for $O(S, K, r, \sigma_{\text{imp}}, \tau)$ and its gamma Γ_O to compute the continuation-value and jump-induced revaluation terms. We ignore additional transaction costs or liquidity constraints in the option itself, so that the quoted option spread reflects solely the strategic behavior of our market-making agents under fixed implied vol. Under these assumptions, every change in our option quotes over the session is driven by realized variance and jump exposure rather than shifts in the volatility surface, thereby allowing a clear analysis of liquidity provision under jump risk in an options LOB.

The ABIDES simulation[9] was executed on February 5, 2024 from market open (09:30:00) to close, processing over 1.1 million events for the trading day. At the end of the run, we compute the mean ending (mark-to-market) value of each agent type, as summarized in Table 1. Finally the Simulation for the trading day is conducted and the spread is of the market maker is estimated once before the trading day using only 1000 simulations because of computational constraints. [10] The Noise Agents, acting as zero-intelligence

[9]https://github.com/s3fushah33/Option-MarketMaking-in-stressed-Markets-.git

[10]Each path contains approx. 24000 steps in the simulation, at each step three value functions,its derivatives and the

Table 1: Mean Ending Value by Agent Type

Agent Type	Mean Ending Values in cents
NoiseAgent	324
ValueAgent	90 686
AdaptivePOVMarketMakerAgent	-4 551 342
JumpRiskMM	-318 548
AvellanedaStoikovMarketMaker	-24 781
MomentumAgent	17 949

traders, submit limit and market orders at Poisson-distributed times with random sizes. Their aggregate performance yields a small positive mean ending value of approximately \$3.24, suggesting that while they contribute liquidity at favorable prices, they do not systematically exploit any market signals. In contrast, the Value Agents—mean-reverting fundamental traders who observe an exogenous Ornstein–Uhlenbeck process and trade whenever the mid-price deviates beyond a predetermined threshold—realize a substantial profit of around \$906.86, indicating effective capture of mispricings relative to the latent fundamental. Among the liquidity providers, the AdaptivePOVMarketMaker Agents, which adapt their quoted sizes to recent volume across multiple price levels, incur a large mean loss of approximately \$45513.42, implying that adverse selection and inventory-holding costs outweigh their spread earnings under this parameterization. The JumpRiskMM Agents, whose quoting is driven by the solution of the jump-PDE system to account for jump risk, experience a more moderate negative P&L of about \$3185.48, reflecting that the cost of managing inventory slightly exceeds compensation from spreads. Classical Avellaneda–Stoikov Market Makers, which deploy analytic optimal-control formulas to set symmetric bid–ask quotes, fare somewhat better, with a modest mean loss of \$247.81, balancing spread capture against inventory-risk aversion more effectively than the adaptive POV strategy yet remaining vulnerable to order-flow noise. Finally, Momentum Agents—trend-followers who place trades in the direction of recent price movements—achieve a positive mean ending value of approximately \$179.49, indicating exploitable short-term trends, albeit on a smaller scale than those harvested by value traders. Taken together, these results demonstrate that market making strategies require careful calibration: in our nanosecond-resolution simulation, every market-maker agent (adaptive-POV, JumpRiskMM, Avellaneda–Stoikov) ended the day with negative P&L, whereas buyer-oriented traders (value and momentum agents) generated net profits. This stark contrast highlights that without precise tuning of participation rates/Order Arrival rates, spread parameters, and risk-aversion coefficients, liquidity providers can easily incur losses even in the absence of significant market stress.

5.1 Effect of Increased Order Arrival Parameter A

In a follow-up experiment, we increased the participation-of-volume parameter A from its empirically estimated value of 140 [26] to 1 400 , thus tightening the quoted spreads substantially. Because our simulation runs at nanosecond resolution, this more aggressive (i.e. tighter) quoting rule allowed both the jump-PDE and the Avellaneda–Stoikov market makers to earn sufficient compensation for diffusion and jump risk over extremely short intervals. As shown in Table 2, the JumpRiskMM agent achieved a mean profit of approximately \$11 760.64, while the Avellaneda–Stoikov market maker also generated a modest positive P&L (\$11.12) despite the adverse selection pressures of ultra-high-frequency quoting. By contrast, all other agent classes—the uninformed noise traders, mean-reverting value traders, adaptive-POV liquidity providers, and momentum agents—suffered significant losses, ranging from tens to hundreds of dollars in mark-to-market terms. These results underscore that, in a true nanosecond trading environment, sufficiently tight spreads can not only mitigate inventory and jump risks but even convert them into net gains for strategically calibrated market makers.

Overall, ABIDES provides a flexible yet high-performance environment to evaluate market-making strategies under controlled yet richly interactive agent dynamics. The discrete-event kernel ensures that even with thousands of agents and high-frequency interactions, simulation runs remain tractable on standard computing hardware, enabling extensive parameter sweeps and statistical analysis critical for robust thesis results.

implied volatility are estimated (Computationally expensive)

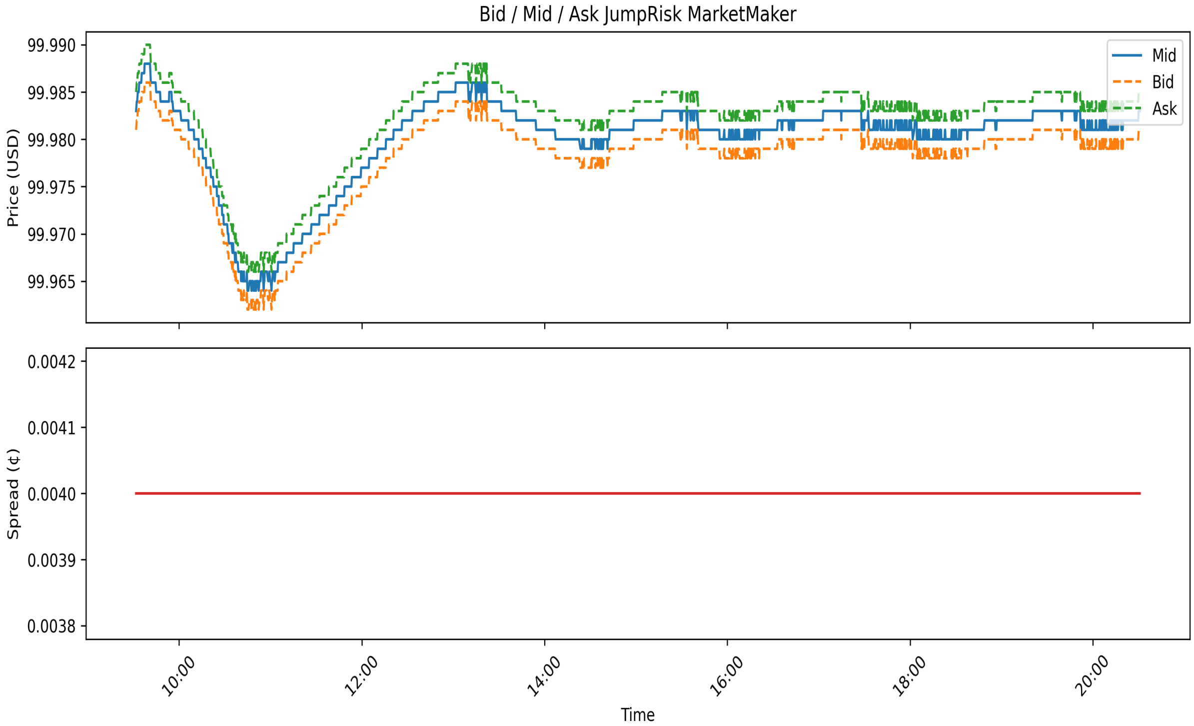

Figure 3: Jump Risk Market Maker Spread throughout the trading day

Table 2: Mean Ending Value by Agent Type with $A = 1400$ (Spreads tightened)

Agent Type	**Mean Ending Value (CENTS)**
NoiseAgent	−49
ValueAgent	−10 265
AdaptivePOVMarketMakerAgent	−59 132
JumpRiskMM	1 176 064
AvellanedaStoikovMarketMaker	1 112
MomentumAgent	−95 012

6 Conclusion

In this thesis, we have extended the classical high-frequency market-making paradigm by embedding discontinuous jumps into the underlying asset dynamics and deriving a tractable, jump-aware quoting rule within a Hamilton–Jacobi–Bellman framework. Starting from an exponential Lévy model for the mid-price, we obtained a coupled system of nonlinear integro-PDEs characterizing the zero-, first-, and second-order components of the market maker's value function. To sidestep the analytical intractability of this system, we introduced a Least-Squares Monte Carlo (LSMC) scheme that approximates each value-function surface via backward induction on simulated jump-diffusion paths. The resulting estimates of Φ_0, Φ_1, and Φ_2, together with their derivatives, yield a closed-form approximate spread rule that simultaneously accounts for diffusion risk, jump risk, time decay, and inventory aversion.

We then implemented both the jump-PDE–based and the classical Avellaneda–Stoikov market-making strategies in the ABIDES discrete-event simulator, alongside benchmark agents including zero-intelligence noise traders, mean-reverting value traders, momentum-based trend followers, and adaptive participation-of-volume liquidity providers. Calibrated to identical model parameters, these agents were evaluated on end-of-day profitability and inventory control. In our baseline experiment, informed strategies (value and momentum) generated positive P&L, uninformed liquidity providers incurred losses, and the jump-PDE market maker delivered intermediate performance.

To explore the sensitivity of quoting aggressiveness, we conducted a follow-up run in which we increased the participation parameter A tenfold (from 140 to 1 400), thereby tightening quoted spreads in our nanosecond–resolution market. Under this more aggressive quoting rule, the jump-PDE agent realized a substantial mean profit of approximately \$11 760.64, and the Avellaneda–Stoikov market maker also turned a positive P&L of \$11.12, while all other agents suffered significant losses. These findings demonstrate that, with sufficiently tight spreads calibrated to the ultra–high-frequency environment, strategically informed market makers can not only offset diffusion and jump risks but actually profit on average, whereas uninformed or adaptive-only strategies remain unprofitable.

Several limitations temper these conclusions. Due to computational constraints, the jump-PDE quoting rule was calibrated on a one-second grid rather than true nanosecond ticks, and the calibration was performed only at market open, rendering the strategy static thereafter. Agent parameters were selected from empirical observations at coarser time scales and were not optimized dynamically during the trading session. Moreover, our study was confined to a single trading day, and we did not incorporate hedging strategies such as delta- or gamma-weighted portfolios.[11]

Looking forward, this work suggests several promising extensions. Incorporating a dynamic implied-volatility surface or stochastic volatility process would allow quoting rules to adapt to evolving market expectations. Extending the LSMC solver to multi-asset or multi-strike option books could enable comprehensive, jump-robust market making across derivative portfolios. Integrating reinforcement-learning agents within ABIDES may uncover novel quote-execution feedback effects under realistic latency and order-flow dynamics. Finally, embedding a real-time volatility predictor—whether a rule-based estimator like Yang–Zhang or a deep-learning model—could further enhance market-maker profitability in volatile and uncertain regimes.

In summary, by unifying jump-diffusion modeling, optimal-control theory, numerical Monte Carlo, and agent-based simulation, this thesis delivers a comprehensive framework for designing and evaluating jump-aware market-making strategies. The analytical developments and empirical results not only advance the theoretical literature on liquidity provision under discontinuities but also provide practical guidance for algorithmic traders seeking to manage tail risk in fast-moving electronic markets.[12]

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A Derivation of Stock Process under $\mathbb{R}$

[13]

$$S_t = S_0 \exp\left\{\int_0^t X_s \, ds\right\}$$

$$dX_t = \alpha(t)dt + \beta(t)W_t + \int_{\mathbb{R}} \gamma(t,z)\,\bar{N}(dt,dz)$$

$$g(t, X_t) = S_0 \exp\left\{\int_0^t X_s \, ds\right\}$$

$$\frac{\partial g}{\partial x} = S_0 \exp\left\{\int_0^t X_s \, ds\right\} \quad \text{and} \quad \frac{\partial^2 g}{\partial x^2} = S_0 \exp\left\{\int_0^t X_s \, ds\right\}$$

[13] Derivation with the help of [25] and [24]

$$dg(t,X_t) = dS(t) = S_0 \exp\left\{\int_0^t X_s\,ds\right\} dX_t + \frac{1}{2} S_0 \exp\left\{\int_0^t X_s\,ds\right\} \langle dX_t\rangle$$
$$+ \int_{|z|<R}\Big[S_0 \exp\left\{\int_0^{t-} X_s\,ds\right\} \left(e^{\gamma(t,z)} - 1 - \gamma(t,z)\right)\Big]\nu(dz)dt$$
$$+ \int_{\mathbb{R}}\Big[S_0 \exp\left\{\int_0^{t-} X_s\,ds\right\} \left(e^{\gamma(t,z)} - 1)\right)\Big]\bar{N}(dt,dz)$$

$$dS(t)/S(t-) = dX_t + \frac{1}{2}\langle dX_t\rangle + \int_{|z|<R}\Big[e^{\gamma(t,z)} - 1 - \gamma(t,z))\Big]\nu(dz)dt \ + \int_{\mathbb{R}}\Big[e^{\gamma(t,z)} - 1\Big]\bar{N}(dt,dz)$$

$$dS(t)/S(t-) = \alpha(t)dt + \beta(t)W_t + \int_{\mathbb{R}} \gamma(t,z)\,\bar{N}(dt,dz) + \frac{1}{2}\beta(t)^2 dt$$
$$+ \int_{|z|<R}\Big[e^{\gamma(t,z)} - 1 - \gamma(t,z))\Big]\nu(dz)dt \ + \int_{\mathbb{R}}\Big[e^{\gamma(t,z)} - 1\Big]\bar{N}(dt,dz)$$
$$dS(t)/S(t-) = \alpha(t)dt + \beta(t)W_t + \int_{\mathbb{R}} \gamma(t,z)\,\bar{N}(dt,dz) + \frac{1}{2}\beta(t)^2 dt$$
$$+ \int_{|z|<R}\Big[e^{\gamma(t,z)} - 1 - \gamma(t,z)\Big]\nu(dz)dt \ + \int_{\mathbb{R}}\Big[e^{\gamma(t,z)} - 1\Big]\bar{N}(dt,dz)$$

$$\alpha(t) = \mu_t, \quad \beta(t) = \sigma_t, \quad \gamma(t,z) = z$$

$$dS(t)/S(t-) = \mu_t dt + \sigma_t W_t + \int_{\mathbb{R}} z\,\bar{N}(dt,dz) + \frac{1}{2}\ \sigma_t^2 dt$$
$$+ \int_{|z|<R}\Big[(e^z - 1 - z)\Big]\nu(dz)dt \ + \int_{\mathbb{R}}\Big[e^z - 1\Big]\bar{N}(dt,dz)$$
$$dS(t)/S(t-) = \big(\mu_t + \frac{1}{2}\ \sigma_t^2 + \int_{|z|<R}\Big[e^z - 1 - z\Big]\nu(dz)\big)dt + \sigma_t W_t + \int_{\mathbb{R}}\Big[e^z - 1\Big]\bar{N}(dt,dz)$$

B Derivation of Stock Process under $\mathbb{Q}$

[14]

$$dY_t = d\ln S_t = \frac{1}{S_{t^-}}\,dS_t - \frac{1}{2}\frac{1}{{S_{t^-}}^2}(dS_t)_c^2 + \int_{\mathbb{R}}\Big[\ln\big(S_{t^-} + \Delta S_t\big) - \ln S_{t^-} - \frac{\Delta S_t}{S_{t^-}}\Big]\,\bar{N}(dt,dz)$$
$$= \mu_t dt + \sigma_t\,dW_t + \int_{\mathbb{R}} z\,\bar{N}(dt,dz).$$
$$dR(t) = \frac{dS_t}{M(t)} - \frac{rS(t)}{M(t)}dt$$
$$\mathbb{E}\big[dR(t)\big] = 0$$
$$\mathbb{E}\Big[\frac{dS_t}{M(t)} - \frac{rS(t)}{M(t)}dt\Big] = 0$$

$$\mathbb{E}\Big[S(t^-)\Big(\big(\mu_t + \frac{1}{2}\ \sigma_t^2 + \int_{|z|<R}\Big[e^z - 1 - z\Big]\nu(dz)\big)dt + \sigma_t W_t + \int_{\mathbb{R}}\Big[e^z - 1\Big]\bar{N}(dt,dz)\Big) - rS(t^-)dt\Big] = 0$$

$$\mathbb{E}\Big[S(t^-)\big(\mu_t - r + \frac{1}{2}\ \sigma_t^2 + \int_{|z|<R}\Big[\Big(e^z - 1 - z\Big)\nu(dz)\big)dt\Big]$$
$$+\mathbb{E}\Big[S(t^-)\Big(\sigma_t W_t + \int_{\mathbb{R}}\Big[e^z - 1\Big]\bar{N}(dt,dz)\Big)\Big] = 0$$

[14]Derivation with the help of [25]

$$\tilde{u}_t = (r - \frac{1}{2}\ \sigma_t^2 - \int_{|z|<R} \Big[e^z - 1 - z \Big] \nu(dz)\big) dt$$

$$r = 0$$

$$d\tilde{S(t)}/S(t-) = \sigma_t \tilde{W}_t + \int_{\mathbb{R}} \Big[e^z - 1 \Big] \tilde{N}(dt, dz)$$

$$d\tilde{Y}_t = d \ln \tilde{S}_t = (-\frac{1}{2}\ \sigma_t^2 - \int_{|z|<R} \Big[e^z - 1 - z \Big] \nu(dz)\big) dt + \sigma_t \tilde{W}_t + \int_{\mathbb{R}} z \tilde{N}(dt, dz)$$

C Derivation of PIDE

PIDE: $O(t, S_t) = \mathbb{E}\left[\frac{O(T, S_T)}{M(T)} \middle| \mathcal{F}_t \right], \quad d\left(\frac{O(t, S_t)}{M(t)} \right) = dO(t, S_t), dO(t, S_t) = \frac{dO(t, S(t)}{M(t)} - \frac{rO(t, S(t))}{M(t)} dt$

$$dO(t, S_t) = \partial_t O(t, S_t)\, dt + \partial_S O(t, S_t)\, d\tilde{S}_t + \frac{1}{2} \partial_{SS}^2 O(t, S_t)\, \langle d\tilde{S}_t \rangle + \int_{|z|<R} \Big[O(t, S(t^-) e^z) - O(t, S(t^-)) -$$

$$\partial_S O(t, S(t^-))\, z \Big] \nu(dz) \big) dt + \int_{\mathbb{R}} \Big[O(t, S(t^-) e^z) - O(t, S(t^-)) \Big] \tilde{N}(dt, dz)$$

$$0 = \partial_t O(t, S_t)\, dt + \partial_S O(t, S_t) S(t^-) \Big(\sigma_t \tilde{W}_t + \int_{\mathbb{R}} \Big[e^z + z - 1 \Big] \tilde{N}(dt, dz) \Big) + \frac{1}{2} \partial_{SS}^2 O(t, S_t) S(t^-)^2 \sigma^2 dt$$

$$+ \int_{|z|<R} \Big[O(t, S(t^-) e^z) - O(t, S(t^-)) - \partial_S O(t, S(t^-))\, z \Big] \nu(dz) dt$$

$$+ \int_{\mathbb{R}} \Big[O(t, S(t^-) e^z) - O(t, S(t^-)) \Big] \tilde{N}(dt, dz)$$

$$0 = \partial_t O(t, S_t)\ + \frac{1}{2} \partial_{SS}^2 O(t, S_t) S(t^-)^2 \sigma^2 + \int_{|z|<R} \Big[O(t, S(t^-) e^z) - O(t, S(t^-)) -$$

$$\partial_S O(t, S(t^-))\, z \Big] \nu(dz) \big) - rO(t, S_t)$$