Duality for normed spaces

4.1. Recall from the lectures that if $1 < p, q < +\infty$ and $1/p + 1/q = 1$, then there exists an isometric isomorphism $\ell^q \xrightarrow{\sim} (\ell^p)^*$. By using a similar argument, construct isometric isomorphisms **(a)** $\ell^\infty \xrightarrow{\sim} (\ell^1)^*$; **(b)** $\ell^1 \xrightarrow{\sim} (c_0)^*$. Does this approach give an isometric isomorphism $\ell^1 \cong (\ell^\infty)^*$?

4.2. Describe explicitly the duals of the following operators:
(a) the diagonal operator on ℓ^p (where $1 \leqslant p < \infty$) or on c_0;
(b) the right shift operator on ℓ^p (where $1 \leqslant p < \infty$) or on c_0;
(c) the operator of "taking the primitive" on $L^2[0,1]$ (see Exercise 2.6);
(d) the Hilbert-Schmidt integral operator on $L^2(X,\mu)$ (see Exercise 2.8).

4.3-B. Prove that c_0 is not isometrically isomorphic to the dual of a normed space[1].

4.4. Let X be a normed space.
(a) Prove that if X^* is separable, then so is X.
(b) Is the converse true?
(c) Prove that there is no topological isomorphism between $(\ell^\infty)^*$ and ℓ^1.

4.5. Let X be a normed space, and let $i_X \colon X \to X^{**}$ be the canonical embedding. Prove that for each operator $T \in \mathscr{B}(X,Y)$ the following diagram commutes.

$$\begin{tikzcd} X^{**} \arrow[r, "T^{**}"] & Y^{**} \\ X \arrow[u, "i_X"] \arrow[r, "T"] & Y \arrow[u, "i_Y"'] \end{tikzcd}$$

4.6. Prove that the composition of the canonical embedding $c_0 \to (c_0)^{**}$ and the standard isomorphism $(c_0)^{**} \cong \ell^\infty$ is the inclusion of c_0 into ℓ^∞. Deduce that c_0 is not reflexive.

4.7. Prove that **(a)** a Hilbert space is reflexive; **(b)** ℓ^1 is not reflexive; **(c)** $L^1[a,b]$ is not reflexive; **(d)** $C[a,b]$ is not reflexive.

4.8. Let X be a normed space, and let $i_X \colon X \to X^{**}$ be the canonical embedding. Find a relation between the operators $i_{X^*} \colon X^* \to X^{***}$ and $i_X^* \colon X^{***} \to X^*$.

4.9. **(a)** Prove that a Banach space X is reflexive $\iff$ X^* is reflexive.
(b) Deduce that ℓ^1, ℓ^∞, $L^\infty[a,b]$ are not reflexive.

4.10. Let X and Y be Banach spaces, and let $S \in \mathscr{B}(Y^*, X^*)$. Do we always have $S = T^*$ for some $T \in \mathscr{B}(X,Y)$?

4.11. Identify $(\ell^1)^*$ with ℓ^∞ (see Exercise 4.1), and consider c_0 as a subspace of $(\ell^1)^*$. Find ${}^\perp c_0$ and $({}^\perp c_0)^\perp$.

4.12. Let X be a nonreflexive Banach space. Prove that there exists a closed vector subspace $N \subsetneq X^*$ such that $N \neq ({}^\perp N)^\perp$.

4.13. Give an example of an injective operator $T \in \mathscr{B}(X,Y)$ between Banach spaces X and Y such that $\operatorname{Im} T^*$ is not dense in X^*. (*Hint:* X must be nonreflexive, see the lectures.) As a corollary, the equality $\overline{\operatorname{Im}(T^*)} = (\operatorname{Ker} T)^\perp$ can fail in the nonreflexive case.

4.14. Let X be a normed space, and let $X_0 \subset X$ be a closed vector subspace. Construct isometric isomorphisms $(X/X_0)^* \cong X_0^\perp$ and $X_0^* \cong X^*/X_0^\perp$. (*Hint:* use the universal property of quotients.)

[1] In fact, c_0 is not *topologically* isomorphic to the dual of a normed space. This seems to be much harder, and this will be discussed in a forthcoming course "Functional Analysis 2" (Spring 2022).

The three basic principles of Functional Analysis
(Hahn-Banach, Banach-Steinhaus, Open Mapping Theorem)

4.15. Let $X = \mathbb{R}^2$ equipped with the norm $\|\cdot\|_p$, and let $X_0 = \{(x,0) : x \in \mathbb{R}\} \subset X$. Define a linear functional $f_0 \colon X_0 \to \mathbb{R}$ by $f_0(x,0) = x$. We clearly have $\|f_0\| = 1$. Describe all "Hahn-Banach extensions" of f_0, i.e., all linear functionals $f \colon X \to \mathbb{R}$ such that $f|_{X_0} = f_0$ and $\|f\| = 1$. (Consider all possible $p \in [1, +\infty]$.)

4.16. Give an example of a normed space X and a pointwise bounded sequence (f_n) in X^* such that (f_n) is not norm bounded.

4.17. Let X, Y, Z be normed spaces.
(a) Prove that a bilinear operator $T \colon X \times Y \to Z$ is continuous if and only if there exists $C \geqslant 0$ such that $\|T(x,y)\| \leqslant C\|x\|\|y\|$ for all $x \in X,\ y \in Y$.
(b) Assume that either X or Y is complete. Prove that each separately continuous bilinear operator $X \times Y \to Z$ is continuous. (The separate continuity means that for each $x_0 \in X$, $y_0 \in Y$ the maps $Y \to Z$, $y \mapsto T(x_0, y)$, and $X \to Z$, $x \mapsto T(x, y_0)$, are continuous.) *Hint:* use the Uniform Boundedness Principle.
(c) Does (b) hold without the completeness assumption?

4.18-B. Let G be a compact topological group, and let π be a representation of G on a Banach space X. Suppose that π is continuous in the sense that the map $G \times X \to X$, $(g,x) \mapsto \pi(g)x$, is continuous. Prove that there exists an equivalent norm $\|\cdot\|_\pi$ on X such that all the operators $\pi(g)$ are isometric with respect to $\|\cdot\|_\pi$. (*Warning:* this has nothing to do with the Haar measure!).

4.19. **(a)** Deduce the Open Mapping Theorem from the Inverse Mapping Theorem.
(b) Deduce the Inverse Mapping Theorem from the Closed Graph Theorem.
(c)-B Deduce the Uniform Boundedness Principle from the Closed Graph Theorem.

4.20. **(a)** Give an example of a Banach space X, a normed space Y, and a bijective operator $T \in \mathscr{B}(X,Y)$ such that T^{-1} is unbounded.
(b)-B Give an example of a normed space X, a Banach space Y, and a bijective operator $T \in \mathscr{B}(X,Y)$ such that T^{-1} is unbounded.

4.21. Let $\|\cdot\|$ be a norm on $L^1(\mathbb{R})$ such that $(L^1(\mathbb{R}), \|\cdot\|)$ is complete and such that the convergence $f_n \to f$ with respect to $\|\cdot\|$ implies that $\int_{-\infty}^t f_n(s)\,ds \to \int_{-\infty}^t f(s)\,ds$ for all $t \in \mathbb{R}$. Prove that $\|\cdot\|$ is equivalent to the usual L^1-norm.