This exercise sheet is meant to be discussed at the blackboard during a “traditional” seminar, and is also intended for self-study (that is, is plays no role in the grading). Nevertheless, it is strongly recommended to look at it before the exam.

Spectra

5.1. Show that for each subset $S \subset \mathbb{C}$ there exist a unital algebra A and $a \in A$ such that $\sigma_A(a) = S$. (Do not forget about $S = \varnothing$.)

5.2. Show that for each nonempty compact subset $K \subset \mathbb{C}$ there exists a bounded linear operator T on a Banach space such that $\sigma(T) = K$.

5.3. Prove that the spectrum of a bijective isometry on a Banach space is contained in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

5.4. Find the point spectrum, the continuous spectrum, and the residual spectrum of the diagonal operator on ℓ^∞.

5.5. Let (X, μ) be a measure space, and let $f \colon X \to \mathbb{C}$ be a measurable function. Recall (see the lectures) that $\lambda \in \mathbb{C}$ is an *essential value* of f if for each neighborhood $U \ni \lambda$ we have $\mu(f^{-1}(U)) > 0$. The set of all essential values of f is called the *essential range* of f. Also recall (see the lectures) that, if f is essentially bounded, then the spectrum $\sigma_{L^\infty(X,\mu)}(f)$ is equal to the essential range of f.
(a) Show that $f(X)$ is not necessarily contained in the essential range of f.
(b) Show that the essential range of f is not necessarily contained in $f(X)$.
(c) Show that, if $X = [a, b]$ or $X = \mathbb{T}$ with the Lebesgue measure, and if f is continuous, then the essential range of f is equal to $f(X)$.

5.6. Let (X, μ) be a σ-finite measure space, f be an essentially bounded measurable function on X, and M_f be the multiplication operator on $L^p(X, \mu)$ acting by the rule $g \mapsto fg$ (where $1 \leqslant p \leqslant \infty$). Find the point spectrum, the continuous spectrum, and the residual spectrum of M_f. Pay special attention to the case of $M_t \colon L^2[0, 1] \to L^2[0, 1]$, $(M_t g)(t) = tg(t)$.

5.7. Find the point spectrum, the continuous spectrum, and the residual spectrum of the operator $T \colon L^2[-\pi, \pi] \to L^2[-\pi, \pi]$ acting by the rule

$$(Tf)(t) = \int_{-\pi}^{\pi} \sin^2(t - s) f(s)\, ds.$$

(*Hint*: replace T by a unitary equivalent operator on $\ell^2(\mathbb{Z})$.)

5.8. Find the point spectrum, the continuous spectrum, and the residual spectrum of the left and right shift operators on **(a)** c_0; **(b)** ℓ^1; **(c)-B** ℓ^∞.

5.9. Given $\zeta \in \mathbb{T}$, define the shift operator $T_\zeta \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by $(T_\zeta f)(z) = f(\zeta^{-1} z)$. Find the point spectrum, the continuous spectrum, and the residual spectrum of T_ζ.

5.10 (*the Volterra operator*)**.** Let $I = [a, b]$, let $H = L^2(I)$, and let $K \in L^2(I \times I)$. The *Volterra operator* $V_K \colon L^2(I) \to L^2(I)$ is given by

$$(V_K f)(x) = \int_a^x K(x, y) f(y)\, dy$$

(a) Prove that V_K is quasinilpotent whenever K is bounded.
(b)-B Prove that V_K is quasinilpotent for each $K \in L^2(I \times I)$.

Compact operators

5.11. Let X be a normed space, let $f \in X^* \setminus \{0\}$, and let $X_0 = \operatorname{Ker} f$. Show that there exists a 0-perpendicular to X_0 in X iff f is norm-attaining (which means that there exists $x \in X$, $\|x\| = 1$, such that $|f(x)| = \|f\|$). Give an example showing that this is not always the case.

5.12. **(a)** Prove that a subset $S \subset c_0$ is relatively compact iff there exists $y \in c_0$ such that $|x_n| \leqslant |y_n|$ for all $x \in S$ and all $n \in \mathbb{N}$. **(b)** Does a similar result hold for ℓ^p?

5.13. Are the left and right shift operators on ℓ^p and on c_0 compact?

5.14. Can the image of a compact operator between Banach spaces contain an infinite-dimensional closed vector subspace?

5.15. Prove that the inclusion $C^1[a, b] \to C[a, b]$ is a compact operator.

5.16. **(a)** Let $f \in C[a, b]$, and let M_f denote the respective multiplication operator on $C[a, b]$. Find a condition on f that is necessary and sufficient for M_f to be compact.
(b) Let $I \subset \mathbb{R}$ be an interval (not necessarily open or closed, not necessarily bounded), let $f\colon I \to \mathbb{C}$ be an essentially bounded measurable function, and let M_f denote the respective multiplication operator on $L^p(I)$ ($1 \leqslant p \leqslant \infty$). Find a condition on f that is necessary and sufficient for M_f to be compact.

5.17. Given an integrable function f on $[0, 1]$, define a function Tf on $[0, 1]$ by

$$(Tf)(x) = \int_0^x f(t)\, dt.$$

Is T a compact operator **(a)** from $C[0, 1]$ to $C[0, 1]$? **(b)** from $L^p[0, 1]$ to $C[0, 1]$ (where $1 < p \leqslant \infty$)? **(c)** from $L^p[0, 1]$ to $L^p[0, 1]$ (where $1 < p \leqslant \infty$)? **(d)** from $L^1[0, 1]$ to $C[0, 1]$? **(e)** from $L^1[0, 1]$ to $L^1[0, 1]$?

5.18. Let $I = [a, b]$, and let $K \in C(I \times I)$. Prove that the integral operator $T\colon C(I) \to C(I)$,

$$(Tf)(x) = \int_a^b K(x, y) f(y)\, dy,$$

is compact.

5.19. Let (X, μ) be a measure space, and let $K \in L^2(X \times X, \mu \times \mu)$. Prove that the *Hilbert–Schmidt integral operator* $T_K\colon L^2(X, \mu) \to L^2(X, \mu)$,

$$(T_K f)(x) = \int_X K(x, y) f(y)\, d\mu(y),$$

is compact.

Hint: show that functions of the form $K(x, y) = f(x)g(y)$, where $f, g \in L^2(X, \mu)$, span a dense subspace of $L^2(X \times X, \mu \times \mu)$, and use the fact that $\|T_K\| \leqslant \|K\|_2$ (see Exercise 2.8).

5.20. Calculate the norm of the operator $T\colon L^2[0, 1] \to L^2[0, 1]$ defined in Exercise 5.17.
Hint: T^*T is compact and selfadjoint.
Answer: see Exercise 2.6.