Convention. All vector spaces are over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Normed spaces

1.1. Let X be a normed space. Show that the operations $X \times X \to X$, $(x, y) \mapsto x + y$, and $\mathbb{K} \times X \to X$, $(\lambda, x) \mapsto \lambda x$, are continuous.

1.2. Let X be a normed space. Show that the closure $\overline{X_0}$ of a vector subspace $X_0 \subset X$ is a vector subspace as well.

1.3. Let $p, q \in (1, +\infty)$, and let $\frac{1}{p} + \frac{1}{q} = 1$.
(a) Prove *Young's inequality*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q} \qquad (a, b \geqslant 0).$$

(b) Given $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$, let $\|x\|_p = \bigl(\sum_{i=1}^n |x_i|^p\bigr)^{1/p}$. Show that Young's inequality implies *Hölder's inequality*

$$\sum_{i=1}^n |x_i y_i| \leqslant \|x\|_p \|y\|_q \qquad (x, y \in \mathbb{K}^n).$$

(c) Show that Hölder's inequality implies *Minkowski's inequality*

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p \qquad (x, y \in \mathbb{K}^n).$$

Thus $\| \cdot \|_p$ is a norm on $\mathbb{K}^n$. Let also $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|$. Clearly, $\| \cdot \|_1$ and $\| \cdot \|_\infty$ are norms as well.

1.4. Draw the unit ball on the plane $(\mathbb{R}^2, \| \cdot \|_p)$ for various $p \in [1, +\infty]$. Pay attention to the cases $p = 1$, $p = 2$, $p = \infty$. What happens with the ball when p grows?

1.5. Let $\| \cdot \|$ and $\| \cdot \|'$ be norms on a vector space X, and let B and B' denote the respective closed unit balls. Prove that $B \subseteq B'$ iff $\|x\|' \leqslant \|x\|$ for all $x \in X$ (in this case, we write $\| \cdot \|' \leqslant \| \cdot \|$).

1.6. Let $1 \leqslant p \leqslant q \leqslant +\infty$.
(a) Prove that $\| \cdot \|_q \leqslant \| \cdot \|_p$ on $\mathbb{K}^n$.
(b) Show that there exists a constant $C = C_{n,p,q} > 0$ such that $\| \cdot \|_p \leqslant C \| \cdot \|_q$ on $\mathbb{K}^n$.
(c) Can the above constant be chosen in such a way that it does not depend on n?
(d) Find the smallest possible $C_{n,p,q}$ with the above property.

1.7. Let c_{00} denote the space of all *finite* sequences (i.e., sequences $x = (x_n)$, $x_n \in \mathbb{K}$, such that $x_n = 0$ for all but finitely many n). Are the norms $\| \cdot \|_p$ and $\| \cdot \|_q$ equivalent on c_{00} for $p \neq q$?

1.8. Let X be a seminormed space, and let $N = \{x \in X : \|x\| = 0\}$. Show that the rule $\|x + N\|^\wedge = \|x\|$ determines a norm on X/N. In particular, show that $\| \cdot \|^\wedge$ is well defined (i.e., that $\|x\|$ depends only on the class $x + N \in X/N$ of $x \in X$).

Given a measure space (X, μ) and $p \in [1, +\infty)$, let $\mathscr{L}^p(X, \mu)$ denote the set of all measurable functions $f \colon X \to \mathbb{K}$ such that $|f|^p$ is μ-integrable. For each $f \in \mathscr{L}^p(X, \mu)$ we let

$$\|f\|_p = \Bigl(\int_X |f|^p d\mu\Bigr)^{1/p}.$$

1.9. Let (X, μ) be a measure space, and let $p, q \in (1, +\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.
(a) Show that for each $f \in \mathscr{L}^p(X, \mu)$ and $g \in \mathscr{L}^q(X, \mu)$ the product fg is integrable, and that *Hölder's inequality* holds:

$$\int_X |fg| \, d\mu \leqslant \|f\|_p \|g\|_q.$$

(b) Using Hölder's inequality, show that $\mathscr{L}^p(X,\mu)$ is a vector space, and that *Minkowski's inequality* holds:

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p \qquad (f,g \in \mathscr{L}^p(X,\mu)).$$

Thus $\|\cdot\|_p$ is a seminorm on $\mathscr{L}^p(X,\mu)$. Clearly, this result holds for $p=1$ as well.

The normed space associated with $\mathscr{L}^p(X,\mu)$ (see Exercise 1.8) is denoted by $L^p(X,\mu)$. Thus we have $L^p(X,\mu) = \mathscr{L}^p(X,\mu)/\{f : f = 0 \text{ a.e.}\}$. Observe that, if $X=\mathbb{N}$ and μ is the counting measure, then $\mathscr{L}^p(X,\mu) = L^p(X,\mu)$, and that $L^p(X,\mu)$ is nothing but

$$\ell^p = \Big\{ x = (x_n) \in \mathbb{K}^{\mathbb{N}} : \|x\|_p = \Big(\sum_n |x_n|^p\Big)^{1/p} < \infty \Big\}.$$

1.10. Let $1 \leqslant p \leqslant q \leqslant +\infty$.
(a) Show that there exists a constant $C = C_{a,b,p,q} > 0$ such that $\|\cdot\|_p \leqslant C\|\cdot\|_q$ on $C[a,b]$.
(b) Find the smallest possible $C_{a,b,p,q}$ with the above property.
(c) Are the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ equivalent on $C[a,b]$ for $p \neq q$?

Let (X,μ) be a measure space. A measurable function $f\colon X \to \mathbb{K}$ is *essentially bounded* if there exists a measurable set $E \subset X$ such that $\mu(X \setminus E) = 0$ and that f is bounded on E. The *essential supremum* of $|f|$ is given by

$$\operatorname{ess\,sup}|f| = \inf\big\{ \sup_{x\in E} |f(x)| : E \subset X,\ \mu(X\setminus E) = 0 \big\}. \tag{1}$$

1.11. Show that inf in (1) is attained at some E. As a corollary, $\operatorname{ess\,sup}|f| = 0$ iff $f=0$ a.e.

1.12. Let $f \in C[a,b]$. Prove that $\operatorname{ess\,sup}|f| = \sup_{x\in[a,b]}|f(x)|$.

The set of all essentially bounded measurable functions on (X,μ) is denoted by $\mathscr{L}^\infty(X,\mu)$.

1.13. Show that $\mathscr{L}^\infty(X,\mu)$ is a vector space, and that the rule $\|f\| = \operatorname{ess\,sup}|f|$ determines a seminorm on $\mathscr{L}^\infty(X,\mu)$.

The normed space associated with $\mathscr{L}^\infty(X,\mu)$ (see Exercise 1.8) is denoted by $L^\infty(X,\mu)$. Thus we have $L^\infty(X,\mu) = \mathscr{L}^\infty(X,\mu)/\{f : f=0 \text{ a.e.}\}$. Observe that, if $X=\mathbb{N}$ and μ is the counting measure, then $\mathscr{L}^\infty(X,\mu) = L^\infty(X,\mu)$, and that $L^\infty(X,\mu)$ is nothing but the space ℓ^∞ of all bounded sequences equipped with the supremum norm.

1.14. Let $1 \leqslant p < q \leqslant \infty$. Show that
(a) $\ell^p \subset \ell^q$, but $\ell^p \neq \ell^q$;
(b) if $\mu(X) < \infty$, then $L^q(X,\mu) \subset L^p(X,\mu)$, and the inclusion is proper provided that X contains infinitely many disjoint measurable sets of positive measure;
(c) $L^p(\mathbb{R}) \not\subset L^q(\mathbb{R})$ and $L^q(\mathbb{R}) \not\subset L^p(\mathbb{R})$.

1.15. Show that a normed space X is separable iff there exists a dense vector subspace $X_0 \subset X$ of at most countable dimension.

1.16. Show that c_0, $C[a,b]$, ℓ^p, $L^p[a,b]$, $L^p(\mathbb{R})$ $(p<\infty)$ are separable, while ℓ^∞, $C_b(\mathbb{R})$, $L^\infty[a,b]$, $L^\infty(\mathbb{R})$ are not separable.