Linear operators

2.1. Let X, Y be normed spaces. Suppose that X is finite-dimensional. Prove that each linear operator $T\colon X \to Y$ is bounded.

2.2. Choose $t_0 \in [a,b]$, and consider the linear functional

$$F\colon (C[a,b], \|\cdot\|_p) \to \mathbb{K}, \quad F(f) = f(t_0).$$

(a) Find all $p \in [1, +\infty]$ such that F is bounded. **(b)** Find $\|F\|$.

2.3. Define a linear functional F on $(C[0,1], \|\cdot\|_\infty)$ by

$$F(f) = 2f(0) - 3f(1) + \int_0^1 f(t)\,dt.$$

(a) Prove that F is bounded. **(b)** Find $\|F\|$.

2.4 (*the multiplication operator on* $C(I)$)**.** Let $I = [a,b]$, and let $f \in C(I)$. For each $p \in [1, +\infty]$, define $M_f\colon (C(I), \|\cdot\|_p) \to (C(I), \|\cdot\|_p)$ by

$$M_f(g) = fg \qquad (g \in C(I)).$$

(a) Prove that M_f is bounded. **(b)** Find $\|M_f\|$.
Hint: consider separately the cases $p = \infty$ and $p < \infty$.

2.5 (*the multiplication operator on* L^p)**.** Let (X, μ) be a σ-finite measure space, and let $f\colon X \to \mathbb{K}$ be an essentially bounded measurable function. For each $p \in [1, +\infty]$, define $M_f\colon L^p(X,\mu) \to L^p(X,\mu)$ by

$$M_f(g) = fg \qquad (g \in L^p(X,\mu)).$$

(a) Prove that M_f is bounded. **(b)** Find $\|M_f\|$.

2.6. Let X be either $L^p[0,1]$ $(1 \leqslant p < +\infty)$ or $C[0,1]$. Define $T\colon X \to X$ by

$$(Tf)(x) = \int_0^x f(t)\,dt \qquad (f \in X).$$

(a) Prove that T is bounded. Find $\|T\|$ in the cases where **(b)** $X = C[0,1]$ and **(c)** $X = L^1[0,1]$.

Remark. If the above operator T acts on $L^2[0,1]$, then $\|T\| = 2/\pi$. We will be able to prove this in due course.

2.7 (*the integral operator on* $C(I)$)**.** Let $I = [a,b]$, and let $K \in C(I \times I)$. Define $T\colon C(I) \to C(I)$ by

$$(Tf)(x) = \int_a^b K(x,y) f(y)\,dy.$$

Prove that T takes $C(I)$ to $C(I)$, that T is bounded, and that $\|T\| \leqslant \|K\|_\infty (b-a)$.

2.8 (*the Hilbert-Schmidt integral operator*)**.** Let (X,μ) be a σ-finite measure space, and let $K \in L^2(X \times X, \mu \times \mu)$. Define $T\colon L^2(X,\mu) \to L^2(X,\mu)$ by

$$(Tf)(x) = \int_X K(x,y) f(y)\,d\mu(y).$$

Prove that the above integral exists for almost all $x \in X$, that T takes $L^2(X,\mu)$ to $L^2(X,\mu)$, that T is bounded, and that $\|T\| \leqslant \|K\|_2$.

Banach spaces

2.9. Show that c_0 is a closed vector subspace of ℓ^∞. As a consequence, c_0 is a Banach space.

2.10. Show that $(c_{00}, \|\cdot\|_p)$ is not complete for each $p \in [1, +\infty]$, Show that $(\ell^p, \|\cdot\|_q)$ is not complete if $q > p$. Describe the completions of these spaces.

2.11. Show that the dimension of an infinite-dimensional Banach space is uncountable.

2.12. For each $p < \infty$, construct a Cauchy sequence in $(C[a,b], \|\cdot\|_p)$ which does not converge. Describe the completion of $(C[a,b], \|\cdot\|_p)$.

2.13. **(a)** Prove that $C^n[a,b]$ is a Banach space with respect to the norm $\|f\| = \max\limits_{0 \leqslant k \leqslant n} \|f^{(k)}\|_\infty$.
(b) Is $C^n[a,b]$ complete with respect to the sup-norm? Describe the completion of this space.

2.14. Let (X, μ) be a measure space. Prove that $L^\infty(X, \mu)$ is a Banach space.