

# FROBENIUS PROBLEM AND DEAD ENDS IN INTEGERS

ZORAN ŠUNIĆ

**ABSTRACT.** Let  $a$  and  $b$  be positive and relatively prime integers. We show that the following are equivalent: (i)  $d$  is a dead end in the (symmetric) Cayley graph of  $\mathbb{Z}$  with respect to  $a$  and  $b$ , (ii)  $d$  is a Frobenius value with respect to  $a$  and  $b$  (it cannot be written as a non-negative or non-positive integer linear combination of  $a$  and  $b$ ), and  $d$  is maximal (in the Cayley graph) with respect to this property. In addition, for given integers  $a$  and  $b$ , we explicitly describe all such elements in  $\mathbb{Z}$ . Finally, we show that  $\mathbb{Z}$  has only finitely many dead ends with respect to any finite symmetric generating set. In the appendix we show that every finitely generated group has a generating set with respect to which dead ends exist.

## 1. INTRODUCTION

We first describe the variant of Frobenius Problem that is in our interest.

**Definition 1** (Frobenius values). Let  $S$  be a set of positive integers whose greatest common divisor is 1. An integer  $n$  is termed *positively generated* with respect to  $S$  if it is a non-negative integer linear combination of the elements in  $S$ , *negatively generated* if it is a non-positive integer linear combination of the elements in  $S$ , and is termed *Frobenius value* (with respect to  $S$ ) otherwise.

It is known that, for any set  $S$  of positive integers with greatest common divisor 1, there exist only finitely many Frobenius values. Frobenius Problem (also known as Linear Diophantine Problem of Frobenius) asks to find the largest Frobenius value for a given  $S$ . The largest Frobenius value is called the *Frobenius number* of  $S$ . It is known that, for  $S = \{a, b\}$ , where  $a$  and  $b$  are positive and relatively prime integers, the Frobenius number is  $ab - a - b$ . No explicit formula exists when  $S$  consists of at least three distinct numbers. On the positive side, upper bounds exist (see [BDR02] and [FR07] for some estimates and further references) as do polynomial time algorithms determining the Frobenius number for sets  $S$  of fixed size [Kan92].

While Frobenius Problem has a long history, the notion of a dead end is fairly recent. It appears explicitly in the work of Bogopol'skii [Bog97], who shows that, for a given non-elementary hyperbolic group with a given finite generating set, there exists a uniform bound on the depth of the dead ends in the group. Various results regarding dead ends in Thompson's group  $F$ , lamplighter groups, solvable groups, finitely presented groups, residually finite groups, etc., appear in the works of Cleary, Guba, Riley, Taback and Warshall [CT04] [CT05] [Gub05] [CR06] [RW06] [War06a] [War06b].

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**Definition 2** (Word length). Let  $G$  be a group generated by a finite set  $S$ . For an element  $g$  in  $G$ , define the *word length* (or simply *length*) of  $g$  with respect to  $S$ , denoted by  $\ell_S(g)$  (or simply by  $\ell(g)$  when  $S$  is assumed), to be the shortest length of a group word over  $S$  representing  $g$ , i.e.,

$$\ell_S(g) = \min\{k \mid g = s_1 s_2 \dots s_k, \text{ for some } s_1, \dots, s_k \in S \cup S^{-1}\}.$$

**Definition 3** (Cayley graph). Let  $G$  be a group generated by a finite set  $S$ . The (symmetric) *Cayley graph* of  $G$  with respect to  $S$  is the graph  $\Gamma(G, S)$  whose vertices are the elements of  $G$  and in which two vertices  $g$  and  $h$  are connected by an edge if and only if  $g = hs$ , for some  $s$  in  $S \cup S^{-1}$ .

**Example 1.** The Cayley graph of  $\mathbb{Z}$  with respect to  $S = \{3, 5\}$  is given in Figure 1.

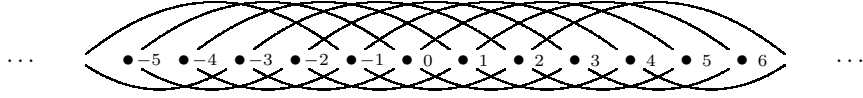


FIGURE 1. The Cayley graph of  $\mathbb{Z}$  with respect to  $S = \{3, 5\}$

It is clear that, for an element  $g$  in  $G$ , the combinatorial distance  $d_S(1, g)$  between 1 and  $g$  in the Cayley graph  $\Gamma(G, S)$  is precisely the word length  $\ell_S(g)$  of  $g$  with respect to  $S$ . The elements of length  $n$  in  $G$  with respect to  $S$  are precisely the elements on the *sphere*  $\Sigma(n)$  of radius  $n$  in  $\Gamma(G, S)$ .

**Definition 4** (Dead end). Let  $G$  be a group generated by a finite set  $S$ . A *dead end* in  $G$  with respect to  $S$  is an element  $d$  in  $G$  such that

$$\ell(ds) \leq \ell(d)$$

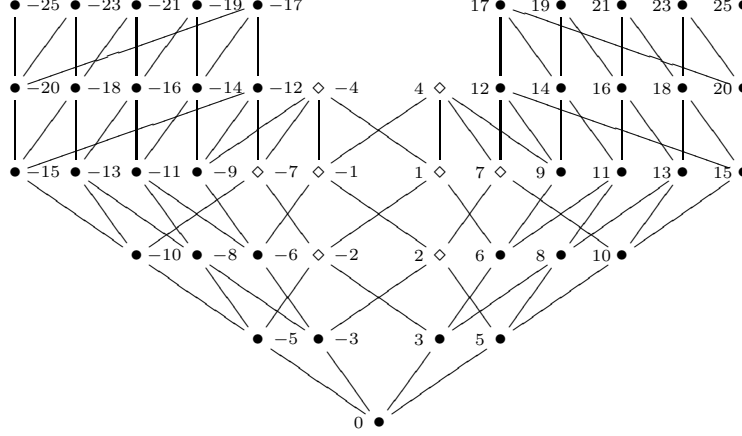
for every  $s$  in  $S \cup S^{-1}$ . A *strict dead end* is an element  $d$  such that  $\ell(ds) < \ell(d)$ , for every  $s$  in  $S \cup S^{-1}$ .

In the Cayley graph, a dead end is a vertex  $d$  for which a geodesic path from 1 to  $d$  cannot be further extended to a longer geodesic. In other words, a dead end of length  $n$  is a vertex in the sphere  $\Sigma(n)$  of radius  $n$  from which the sphere  $\Sigma(n+1)$  of radius  $n+1$  cannot be reached in one step.

**Example 2.** The Cayley graph  $\Gamma(\mathbb{Z}, S)$ , for  $S = \{3, 5\}$ , is presented in Figure 2 in such a way that the length of each element is apparent from the figure (it corresponds to the level at which it is drawn).

It is clear that  $\pm 4$  are the only dead end elements in  $\Gamma(\mathbb{Z}, S)$ . Note that the Frobenius values with respect to  $S$  are  $\pm 1, \pm 2, \pm 4, \pm 7$  (indicated by  $\diamond$  in the Cayley graph). While 7 is the largest Frobenius value, i.e., the Frobenius number for  $S$ , it is clear that  $\pm 4$  also play a special role among Frobenius values.

The Cayley graph  $\Gamma(G, S)$  induces a partial order on  $G$ . Namely,  $g \leq h$  if there exists a geodesic in  $\Gamma(G, S)$  from 1 to  $h$  that passes through  $g$ . Call this order on  $G$  the *Cayley order* (with respect to  $S$ ). Dead ends are precisely the maximal elements in  $G$  with respect to the Cayley order.

FIGURE 2. Spheres in the Cayley graph of  $\mathbb{Z}$  with respect to  $S = \{3, 5\}$ 

We are specifically interested in the relation between Frobenius values and dead ends in  $\mathbb{Z}$ . Of course, when  $S = \{1\}$  neither Frobenius values nor dead ends exist. In the case when  $S$  consists of 2 elements we completely describe the connection.

**Theorem 1.** *Let  $S = \{a, b\}$ , where  $a$  and  $b$  are positive and relatively prime integers. For an integer  $d$ , the following are equivalent:*

- (i)  $d$  is a dead end with respect to  $S$
- (ii)  $d$  is a maximal (in the Cayley order) Frobenius value with respect to  $S$ .

The question of existence of dead ends in  $\mathbb{Z}$  is touched upon in [RW06], where Riley and Warshall show that  $m$  is a dead end with respect to both  $\{2m, 2m+1\}$  and  $\{2m-1, 2m\}$  (with the exception of  $m = 1$  in the latter case). We explicitly describe all dead ends in  $\mathbb{Z}$ , for any generating set  $S$  consisting of two elements.

**Theorem 2.** *Let  $S = \{a, b\}$ , where  $a$  and  $b$  are relatively prime integers with  $a > b \geq 1$ . The dead ends in  $\mathbb{Z}$  (the maximal Frobenius values) with respect to  $S$  are given as follows.*

- (i) *If  $a + b$  is even, then there are exactly  $b - 1$  dead ends, they are all strict, they all have length  $(a + b)/2$ , and they are given by*

$$d = \frac{(a + b)(2\alpha - b)}{2},$$

for  $\alpha = 1, 2, \dots, b - 1$ .

- (ii) *If  $a + b$  is odd, then there are exactly  $2(b - 1)$  dead ends, none of them is strict, they all have length  $(a + b - 1)/2$ , and they are given by*

$$d = \frac{(a + b)(2\alpha - b) \pm b}{2},$$

for  $\alpha = 1, 2, \dots, b - 1$ .

Note that Theorem 2 implies that there are no dead ends when one of the generators is equal to 1 (it is immediately obvious that there are no Frobenius values in this case).

There are many examples of groups that have infinitely many dead ends with respect to some generating sets. Such examples can be found in [Bog97] (the triangle group  $T(3, 3, 3) = \langle a, b | a^3 = b^3 = (ab)^3 = 1 \rangle$ ), [CT04] (Thompson's group  $F$ ), [CT05] (lamplighter), [CR06, RW06] (some finitely presented examples), [War06b] (some lattices in  $Nil$  and  $Sol$ , namely the discrete Heisenberg group and any extension of  $\mathbb{Z}^2$  by a hyperbolic automorphism). We show that, on the contrary,  $\mathbb{Z}$  can only have finitely many dead ends.

**Theorem 3.** *Let  $S$  be a finite generating set of  $\mathbb{Z}$ . There exists only finitely many dead ends in  $\mathbb{Z}$  with respect to  $S$ .*

Recall that, for a dead end  $d$  of length  $n$ , the *depth* of  $d$  is its distance to the sphere  $\Sigma(n+1)$  decreased by 1 (some authors prefer not to subtract 1 here, but this hardly matters). Note that, in general, the condition that a group has only finitely many dead ends is stronger than the condition that the depth of the dead ends is uniformly bounded. For instance, the triangle group  $T(3, 3, 3)$  and Thompson's group  $F$  have infinitely many dead ends with respect to their standard 2-generator sets even though the depth is uniformly bounded (see [Bog97] and [CT04]).

In the appendix we address the existence of dead ends in arbitrary groups and show that every finitely generated group has a generating set with respect to which dead ends exist (see Theorem 4 and Proposition 1).

## 2. PROOFS

Let  $a$  and  $b$  be positive and relatively prime integers. Every integer  $c$  can be written as an integer linear combination  $c = \alpha a + \beta b$  of  $a$  and  $b$  in infinitely many ways. More precisely, if  $c = \alpha a + \beta b$  is one such representation then all other integer solutions to the equation  $c = xa + yb$  are given by

$$(1) \quad x = \alpha + bt, \quad y = \beta - at,$$

for integer values of  $t$ . There is a unique solution with  $0 \leq x < b$ , which we call the  $a$ -normal form of  $c$ , and a unique solution with  $0 \leq y < a$ , which we call the  $b$ -normal form of  $c$ .

Observe that, for any finite generating set  $S$  of  $\mathbb{Z}$ , the map  $c \mapsto -c$  is an automorphism of the Cayley graph of  $\mathbb{Z}$  with respect to  $S$  and this automorphism fixes 0. Therefore  $c$  and  $-c$  have the same length with respect to  $S$ , and  $c$  is a dead end if and only if so is  $-c$ . Similarly,  $c$  is a Frobenius value with respect to  $S$  if and only if so is  $-c$ , and  $c$  is a maximal Frobenius value if and only if so is  $-c$ . We will freely use this symmetry in the proofs that follow.

**2.1. Recognizing Frobenius values.** We begin by recalling a well known condition characterizing the positively generated integers with respect to  $S = \{a, b\}$  (it appears, for instance, in [NW72]).

**Lemma 1.** *Let  $S = \{a, b\}$ , where  $a$  and  $b$  are positive and relatively prime integers, and let  $c$  be a positive integer. The following conditions are equivalent:*

- (i)  $c$  is positively generated with respect to  $S$ .
- (ii) the  $a$ -normal form  $c = \alpha a + \beta b$  satisfies the condition  $\beta \geq 0$ .

The characterization in Lemma 1 will not be directly useful to us (note that imposing the condition  $\beta < 0$  in the  $a$ -normal form lumps together all positive Frobenius values and all negative integers), but the following slight modification will.

**Lemma 2.** *Let  $S = \{a, b\}$ , where  $a$  and  $b$  are positive and relatively prime integers, and let  $c$  be an integer (not necessarily positive). The following conditions are equivalent:*

- (i)  *$c$  is a Frobenius value with respect to  $S$ .*
- (ii) *the  $a$ -normal form  $c = \alpha a + \beta b$  satisfies the condition  $-a < \beta < 0 < \alpha < b$ .*
- (iii) *the  $b$ -normal form  $c = \alpha a + \beta b$  satisfies the condition  $-b < \alpha < 0 < \beta < a$ .*

*Proof.* (i) implies (ii). Let  $c$  be a Frobenius value with respect to  $S$  and let  $c = \alpha a + \beta b$  be the  $a$ -normal form of  $c$ . Since  $c$  is a Frobenius value neither  $\alpha$  nor  $\beta$  can be equal to 0. Thus  $0 < \alpha < b$  and  $\beta$  must be negative. If  $\beta \leq -a$  then  $c = (\alpha - b)a + (\beta + a)b$  and since  $\alpha - b < 0$  and  $\beta + a \leq 0$  we obtain that  $c$  is negatively generated, a contradiction. Thus  $-a < \beta < 0$ .

(ii) implies (i). Let the  $a$ -normal form  $c = \alpha a + \beta b$  satisfy the condition  $-a < \beta < 0 < \alpha < b$ . All other solutions to the equation  $c = xa + yb$  are given by [1]. For  $t \geq 0$ ,  $x = \alpha + bt \geq \alpha > 0$  and  $y = \beta - at \leq \beta < 0$ , while for  $t \leq -1$ ,  $x = \alpha + bt \leq \alpha - b < 0$  and  $y = \beta - at \geq \beta + a > 0$ . Thus, in any representation of  $c$  in the form  $c = xa + yb$ , one of the integers  $x$  and  $y$  is positive while the other is negative, implying that  $c$  is a Frobenius value with respect to  $S$ .

(ii) is equivalent to (iii). If  $c = \alpha a + \beta b$  and  $-a < \beta < 0 < \alpha < b$ , then  $c = (\alpha - b)a + (\beta + a)b$  and  $-b < \alpha - b < 0 < \beta + a < a$ . Thus (ii) implies (iii) and, by symmetry, (iii) implies (ii).  $\square$

It is interesting to observe that Lemma 2 provides a rather simple proof of the following classical result of Sylvester [Syl82].

**Corollary 1.** *Let  $S = \{a, b\}$ , where  $a$  and  $b$  are positive and relatively prime integers. The number of positive Frobenius values with respect to  $S$  is equal to  $(a - 1)(b - 1)/2$ .*

*Proof.* Since Frobenius values are exactly the numbers  $\alpha a + \beta b$ , for  $-a < \beta < 0 < \alpha < b$ , and no two such numbers are equal (every integer has a unique  $a$ -normal form) the number of Frobenius values is  $(a - 1)(b - 1)$ . By symmetry, exactly half of them are positive.  $\square$

Three proofs of Sylvesters's result are offered in [RA05], but none of them uses the above argument for the simple reason that they all only consider and concentrate on positive Frobenius values. For instance, one of the proofs offered in [RA05] follows the argument of Nijenhuis and Wilf [NW72] (and is based on Lemma 1) and shows that an integer  $c$  in the closed interval  $[0, ab - a - b]$  is a Frobenius value if and only if  $ab - a - b - c$  is not.

**2.2. Recognizing maximal Frobenius values.** We now concentrate on description of all maximal Frobenius values.

**Lemma 3.** *Let  $S = \{a, b\}$ , where  $a$  and  $b$  are positive and relatively prime integers. Let  $c = \alpha a + \beta b$  be the  $a$ -normal form of the Frobenius value  $c$ . The length of  $c$  with respect to  $S$  is achieved either at the  $a$ -normal form or at the  $b$ -normal form of  $c$ , i.e.,*

$$\ell(c) = \min\{|\alpha| + |\beta|, a + b - (|\alpha| + |\beta|)\}.$$

*Proof.* Consider the solutions [1] to the equation  $c = xa + yb$ . We have

$$|x| + |y| = |\alpha + bt| + |\beta - at| = \begin{cases} (a+b)t + |\alpha| + |\beta|, & t \geq 0 \\ (a+b)|t| - (|\alpha| + |\beta|), & t \leq -1 \end{cases}.$$

Thus the length of  $c$  is achieved at  $t = 0$ , which is the  $a$ -normal form of  $c$ , or at  $t = -1$ , which is the  $b$ -normal form of  $c$ , and  $\ell(c) = \min\{|\alpha| + |\beta|, a+b - (|\alpha| + |\beta|)\}$ .  $\square$

Note that Lemma [3] implies that there are no Frobenius values of length greater than  $\lfloor \frac{a+b}{2} \rfloor$ . We describe explicitly the Frobenius values of length exactly  $\lfloor \frac{a+b}{2} \rfloor$ .

**Lemma 4.** *Let  $S = \{a, b\}$ , where  $a$  and  $b$  are relatively prime integers with  $a > b \geq 1$ . The Frobenius values of length  $\lfloor \frac{a+b}{2} \rfloor$  with respect to  $S$  are given as follows.*

(i) *If  $a+b$  is even, then there are exactly  $b-1$  Frobenius values of length  $(a+b)/2$  and they are given by*

$$c = \alpha a + \left( \alpha - \frac{(a+b)}{2} \right) b,$$

for  $\alpha = 1, 2, \dots, b-1$ .

(ii) *If  $a+b$  is odd, then there are exactly  $2(b-1)$  Frobenius values of length  $(a+b-1)/2$  and they are given by*

$$c = \alpha a + \left( \alpha - \frac{(a+b \mp 1)}{2} \right) b,$$

for  $\alpha = 1, 2, \dots, b-1$ .

*Proof.* The claim easily follows from Lemma [2] and Lemma [3].

(i) Note that, when  $0 < \alpha < b$ ,

$$-a < -\frac{a+b}{2} < \alpha - \frac{a+b}{2} < b - \frac{a+b}{2} = \frac{b-a}{2} < 0.$$

Thus  $\beta = \alpha - (a+b)/2$  satisfies the condition  $-a < \beta < 0$  and therefore  $c = \alpha a + \beta b$  is a Frobenius value. The length of  $c$  is  $|\alpha| + |\beta| = \alpha - \beta = (a+b)/2$ .

Conversely, if  $c = \alpha a + \beta b$  is a Frobenius value of length  $(a+b)/2$  and  $0 < \alpha$  then  $\beta$  must be negative and we must have  $(a+b)/2 = |\alpha| + |\beta| = \alpha - \beta$ , which implies  $\beta = \alpha - (a+b)/2$ .

(ii) The proof is analogous to the one given for (i). The difference in the number of solutions comes from the fact that if  $|\alpha| + |\beta|$  is equal to either  $(a+b-1)/2$  or  $(a+b+1)/2 = a+b - (a+b-1)/2$ , then the length of the corresponding Frobenius value  $c = \alpha a + \beta b$  is  $(a+b-1)/2$ .  $\square$

The next result shows that Lemma [4] describes exactly the maximal Frobenius values.

**Lemma 5.** *Let  $S = \{a, b\}$ , where  $a$  and  $b$  are positive and relatively prime integers. A Frobenius value  $c$  with respect to  $S$  is maximal Frobenius value if and only if its length is  $\lfloor \frac{a+b}{2} \rfloor$ .*

*Proof.* If a Frobenius value  $c$  has length  $\lfloor \frac{a+b}{2} \rfloor$  then it is certainly maximal Frobenius value, since the length of a maximal Frobenius value cannot be greater than  $\lfloor \frac{a+b}{2} \rfloor$  (by Lemma [3]).

Conversely, let  $c$  be a Frobenius value of length strictly smaller than  $\lfloor \frac{a+b}{2} \rfloor$  and let  $c = \alpha a + \beta b$  be its  $a$ -normal form.

Let the length of  $c$  be achieved at the  $a$ -normal form, i.e., let  $\ell(c) = |\alpha| + |\beta|$ . We cannot have  $\alpha = b - 1$  and  $\beta = 1 - a$  (otherwise  $\ell(c) = a + b - 2 \geq (a + b)/2 + 3/2 - 2 = (a + b - 1)/2$ , a contradiction). Thus either  $c' = c + a$  (if  $\alpha \neq b - 1$ ) or  $c' = c - b$  (if  $\beta \neq 1 - a$ ) is a Frobenius value. Since  $|\alpha| + |\beta| + 1 \leq \lfloor \frac{a+b}{2} \rfloor$  we in each case obtain that  $\ell(c') = |\alpha| + |\beta| + 1 = \ell(c) + 1$ . Thus  $c$  is not a maximal Frobenius value.

If the length of  $c$  is achieved at the  $b$ -normal form then the length of  $-c$  is achieved at its  $a$ -normal form (multiplying the  $b$ -normal form of  $c$  by  $-1$  throughout provides the  $a$ -normal form of  $-c$ ). By the previous argument,  $-c$  is not a maximal Frobenius value and, by symmetry,  $c$  is not a maximal Frobenius value either.  $\square$

**2.3. Connection to dead ends.** We now provide the proofs of the statements relating the maximal Frobenius values and the dead ends. In fact, we provide two proofs that maximal Frobenius values are dead ends. One is based on the fact that we already know explicitly all maximal Frobenius values and can relatively easily and directly check that the length of any of their neighbors in the Cayley graph does not exceed  $\lfloor \frac{a+b}{2} \rfloor$ , which is the length of the maximal Frobenius values. The other proof is more conceptual and relies solely on the definition of a maximal Frobenius value (it does not use Lemma 2, Lemma 4 or Lemma 5). While the first proof is slightly shorter and contributes to the proof of Theorem 2 the second proof is more likely to be amenable to generalizations to larger generating sets (since it is unlikely that explicit descriptions of maximal Frobenius values for such generating sets can be found).

*First proof of Theorem 1 (ii) implies (i).* Let  $c$  be a maximal Frobenius value. Without loss of generality, assume that the length of  $c$  is achieved at the  $a$ -normal form  $c = \alpha a + \beta b$  (otherwise we may consider  $-c$ ). Thus  $-a < \beta < 0 < \alpha < a$  and  $\ell(c) = |\alpha| + |\beta| = \lfloor \frac{a+b}{2} \rfloor$ .

Since

$$c - a = (\alpha - 1)a + \beta b \quad \text{and} \quad c + b = \alpha a + (\beta + 1)b$$

and  $|\alpha - 1| + |\beta| = |\alpha| + |\beta| - 1 = |\alpha| + |\beta + 1|$ , we see that  $\ell(c - a), \ell(c + b) \leq \ell(c) - 1$  (and therefore  $\ell(c - a) = \ell(c + b) = \ell(c) - 1$ ).

Further,

$$\begin{aligned} c + a &= (\alpha + 1)a + \beta b = (\alpha + 1 - b)a + (\beta + a)b, \\ c - b &= \alpha a + (\beta - 1)b = (\alpha - b)a + (\beta - 1 + a)b, \end{aligned}$$

and  $|\alpha + 1 - b| + |\beta + a| = a + b - (|\alpha| + |\beta|) - 1 = |\alpha - b| + |\beta - 1 + a|$ . Therefore, if  $a + b$  is even,  $\ell(c + a), \ell(c - b) \leq a + b - (|\alpha| + |\beta|) - 1 = \frac{a+b}{2} - 1 = \ell(c) - 1$  (implying that  $\ell(c + a), \ell(c - b) = \ell(c) - 1$ ) and, if  $a + b$  is odd,  $\ell(c + a), \ell(c - b) \leq a + b - (|\alpha| + |\beta|) - 1 = \frac{a+b+1}{2} - 1 = \ell(c)$ .  $\square$

*Second proof of Theorem 1 (ii) implies (i).* Assume that  $c$  is a maximal Frobenius value. Without loss of generality, assume that  $c = \alpha a + \beta b$ , with  $\beta < 0 < \alpha$ , and  $\ell(c) = |\alpha| + |\beta|$  (note that neither  $\alpha$  nor  $\beta$  can be 0, since  $m$  is a Frobenius value).

Since

$$c - a = (\alpha - 1)a + \beta b \quad \text{and} \quad c + b = \alpha a + (\beta + 1)b$$

and  $|\alpha - 1| + |\beta| = |\alpha| + |\beta| - 1 = |\alpha| + |\beta + 1|$ , we see that  $\ell(c - a), \ell(c + b) \leq \ell(c) - 1$  (and therefore  $\ell(c - a) = \ell(c + b) = \ell(c) - 1$ ).

Consider  $c - b$  and  $c + a$ . Let  $c - b = \alpha'a + \beta'b$ , with  $\ell(c - b) = |\alpha'| + |\beta'|$ . Then  $c = \alpha'a + (\beta' + 1)b$ ,  $c + a = (\alpha' + 1)a + (\beta' + 1)b$ , and since  $c$  is a Frobenius value, either  $\beta' + 1 < 0 < \alpha'$  or  $\alpha' < 0 < \beta' + 1$ . In the former case

$$\ell(c + a) \leq |\alpha' + 1| + |\beta' + 1| = (\alpha' + 1) - (\beta' + 1) = \alpha' - \beta' = |\alpha'| + |\beta'| = \ell(c - b)$$

and in the latter

$$\ell(c + a) \leq |\alpha' + 1| + |\beta' + 1| = -(\alpha' + 1) + (\beta' + 1) = -\alpha' + \beta' = |\alpha'| + |\beta'| = \ell(c - b).$$

Thus  $\ell(c + a) \leq \ell(c - b)$  and, by symmetry, it follows that  $\ell(c - b) = \ell(c + a)$ .

Assume that  $c$  is not a dead end. Then both  $c - b$  and  $c + a$  have length  $\ell(c) + 1$ . Since  $c$  is a maximal Frobenius value, none of  $c - b$  and  $c + a$  is a Frobenius value. However,  $c + a$  cannot be negatively generated since  $c = (c + a) - a$  would then also be negatively generated, which contradicts the assumption that  $c$  is a Frobenius value. If  $c + a = \alpha''a + \beta''b$ , where  $\alpha'', \beta'' \geq 0$ , then  $\alpha'' = 0$  (if  $\alpha'' > 0$  then  $c = (\alpha'' - 1)a + \beta''b$  would be positively generated). Thus  $c + a = \beta''b$ , for some positive  $\beta''$ . Moreover, since  $c$  is a Frobenius value, its length is at least 2, implying  $\ell(c + a) \geq 3$  and therefore  $\beta'' \geq 3$ . By a symmetric argument,  $c - b$  must be negatively generated, and  $c - b = -\alpha''a$ , for some  $\alpha'' \geq 3$ . But then

$$a + b = (c + a) - (c - b) = \beta''b + \alpha''a \geq 3b + 3a = 3(a + b),$$

a contradiction.  $\square$

*Proof of Theorem 1* (i) implies (ii). Without loss of generality, assume that  $a > b$ .

Let  $c$  be a positive integer and let  $c = \alpha a + \beta b$ , where  $\alpha$  and  $\beta$  are integers (we are not assuming any normal form here). All other solutions to the equation  $c = xa + yb$  are given by (1). The length of  $c$  with respect to  $S$  is the minimal value of the function

$$f_c(t) = |\alpha + bt| + |\beta - at|,$$

at an integer value of  $t$ . We have

$$f_c(t) = \begin{cases} -\alpha + \beta - (a + b)t, & t \leq -\alpha/b, \\ c/b, & t = -\alpha/b, \\ \alpha + \beta - (a - b)t, & -\alpha/b \leq t \leq \beta/a, \\ c/a, & t = \beta/a, \\ \alpha - \beta + (a + b)t, & \beta/a \leq t \end{cases}$$

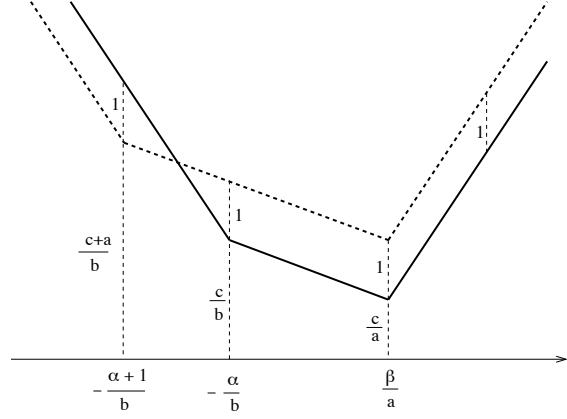
and the graph of  $f_c(t)$  is given in Figure 3 (full line).

Since the function  $f_c(t)$  is decreasing for values of  $t$  smaller than  $\beta/a$ , achieves its minimum at  $\beta/a$  and is increasing for values of  $t$  greater than  $\beta/a$ , the length of  $c$  is obtained either at the integer  $\ell = \lfloor \beta/a \rfloor$  that is closest to  $\beta/a$  to the left of  $\beta/a$  or the integer  $r = \lceil \beta/a \rceil$  that is closest to  $\beta/a$  to the right of  $\beta/a$ .

Consider the function  $f_{c+a}(t)$  that determines the length of  $c + a$ . We have  $c + a = (\alpha + 1)a + \beta b$  and the graph of the function  $f_{c+a}(t)$  is given in Figure 3 (dotted line). If there exists an integer in the closed interval  $[-\alpha/b, \beta/a]$ , then  $c$  cannot be a dead end (to the right of  $-\alpha/b$  the function  $f_{c+a}(t)$  is 1 unit above  $f_c(t)$ , implying that the length of  $c + a$  is larger than the length of  $c$ ).

Assume that  $d$  is a positive dead end,  $d = \alpha a + \beta b$ , and  $\alpha$  and  $\beta$  are chosen so that  $\ell(d) = |\alpha| + |\beta|$ . This means that the minimum of  $f_d(t)$  at an integer point is achieved at  $t = 0$  and there is no integer in the interval  $[-\alpha/b, \beta/a]$ . Thus, either  $-1 < -\alpha/b < \beta/a < 0$  or  $0 < -\alpha/b < \beta/a < 1$ . In the former case



FIGURE 3. Graphs of the functions  $f_c(t)$  and  $f_{c+a}(t)$ 

$-a < \beta < 0 < \alpha < b$ , while in the latter  $-b < \alpha < 0 < \beta < a$ . In each case we conclude that  $d$  is a Frobenius value by Lemma 2.

If  $d$  is a negative dead end, then  $d$  is just negative of some positive dead end, so  $d$  is a Frobenius value in this case as well.

Thus we proved that all dead ends are Frobenius values. Since dead ends are maximal in the Cayley order they must be maximal Frobenius values as well.  $\square$

*Proof of Theorem 2.* By Lemma 5, the maximal Frobenius values are precisely the Frobenius values of length  $\lfloor \frac{a+b}{2} \rfloor$ , and these values are explicitly described in Lemma 4. By Theorem 1 these values are precisely the dead ends in  $\mathbb{Z}$  with respect to  $S$ .

Since

$$\alpha a + \left( \alpha - \frac{(a+b)}{2} \right) b = \frac{(a+b)(2\alpha - b)}{2},$$

and

$$\alpha a + \left( \alpha - \frac{(a+b \mp 1)}{2} \right) b = \frac{(a+b)(2\alpha - b) \pm b}{2},$$

for  $\alpha = 1, 2, \dots, b-1$ , the lists of numbers given in the claims of Lemma 4 and Theorem 2 coincide.

The claim on the lengths of dead ends follows from the corresponding claim on the lengths of maximal Frobenius values (Lemma 5).

In the course of the first proof of the direction (ii) implies (i) of Theorem 1 we already proved that, if  $a+b$  is even then, for any dead end  $c$ ,  $\ell(c \pm a) = \ell(c \pm b) = \ell(c) - 1$ . Thus, in this case all dead ends are strict.

The equality

$$\frac{(a+b)(2\alpha - b) + b}{2} - b = \frac{(a+b)(2\alpha - b) - b}{2},$$

for  $\alpha = 1, \dots, b-1$ , shows that, when  $a+b$  is odd, each dead end has another dead end as a neighbor in the Cayley graph, which then means that none of them is a strict dead end (since they all have the same length).  $\square$

*Proof of Theorem 3* Without loss of generality, let  $S = \{b_1, b_2, \dots, b_k\}$  be a generating set for  $\mathbb{Z}$  such that  $k \geq 1$  and  $b_1 > b_2 > \dots > b_k > 0$ . Denote  $b_1 = a$ .

Let

$$c = x_1 a + x_2 b_2 + \dots + x_k b_k,$$

and let the length  $\ell(c)$  of  $c$  with respect to  $S$  be given by

$$\ell(c) = |x_1| + |x_2| + \dots + |x_k|.$$

We claim that  $|x_i| < a$ , for  $i = 2, \dots, k$ . Indeed, assume  $x_i \geq a$ , for some  $i = 2, \dots, k$ . Then we have

$$c = (x_1 + b_i)a + x_2 b_2 + \dots + (x_i - a)b_i + \dots + x_k b_k,$$

which implies that

$$\ell(c) \leq |x_1 + b_i| + |x_2| + \dots + |x_i - a| + \dots + |x_k| \leq |x_1| + \dots + |x_k| + b_i - a < \ell(c),$$

a contradiction. Thus we must have  $x_i < a$ , for  $i = 2, \dots, k$ . A symmetric argument shows that  $-a < x_i$ , for  $i = 2, \dots, k$ .

We show that if  $c > (a - 1)b$ , where  $b = b_2 + \dots + b_k$ , then  $x_1$  must be positive. Indeed, if  $x_1$  is not positive then

$$c = x_1 a + x_2 b_2 + \dots + x_k b_k \leq x_2 b_2 + \dots + x_k b_k \leq (a - 1)(b_2 + \dots + b_k) = (a - 1)b.$$

Thus, by contraposition, if  $c$  is large enough  $x_1$  must be positive.

We show that if  $c > (a - 1)b$  then  $c$  cannot be a dead end. Indeed, in that case  $c + a$  is also large enough, so that if  $c + a = x'_1 a + x'_2 b_2 + \dots + x'_k b_k$ , with  $\ell(c) = |x'_1| + |x'_2| + \dots + |x'_k|$ , then  $x'_1$  is positive. But then  $c = (x'_1 - 1)a + x'_2 b_2 + \dots + x'_k b_k$  and therefore

$$\ell(c) \leq |x'_1 - 1| + |x'_2| + \dots + |x'_k| = |x'_1| + |x'_2| + \dots + |x'_k| - 1 = \ell(c + a) - 1,$$

which shows that  $c$  is not a dead end.

Thus there are only finitely many positive dead ends and, by symmetry, there are only finitely many dead ends in  $\mathbb{Z}$ .  $\square$

#### APPENDIX: ALL GROUPS HAVE DEAD ENDS

Warshall showed [War06a] that if  $G$  is a finitely generated group with infinitely many finite homomorphic images then, for every  $n$ ,  $G$  has a finite generating set with respect to which  $G$  has dead ends of depth at least  $n$ .

We show that, surprisingly, if one is only interested in existence of dead ends, no conditions on the group are needed (for applications involving just the existence of dead ends see [Bar06]).

**Theorem 4.** *Every infinite, finitely generated group  $G$  has a finite generating set  $S$  with respect to which  $G$  has dead ends.*

*If  $G$  is generated by  $m$  elements, the size of  $S$  can be chosen to be no greater than  $4m + 2$ .*

*Proof.* Let  $G$  have an element  $a$  of order at least 5 (including the possibility of infinite order). Let  $S''$  be any generating set for  $G$  and let  $S'$  be the set obtained from  $S''$  by removing any generators that happen to be in  $\langle a \rangle$ . For every  $b$  in  $S'$  define

$$S_b = \{b, ab, ab^{-1}, aba^{-1}\},$$

and let

$$S = \left( \bigcup_{b \in S'} S_b \right) \cup \{a^2, a^3\}.$$

It is clear that  $S$  is a generating set for  $G$  (of size at most  $4|S'| + 2$ ). We claim that  $a$  is a dead end of length 2 with respect to  $S$ .

Since  $a = a^3 a^{-2}$  the length of  $a$  is no greater than 2. On the other hand,  $a$  is not equal to any of the generators in  $S$  or their inverses (this is because the order of  $a$  is at least 5 and  $b \notin \langle a \rangle$ ). Thus  $a$  has length 2.

We have

$$\begin{array}{ll} a \cdot a^2 = a^3 & a \cdot a^{-2} = a^2 \cdot a^{-3} \\ a \cdot a^3 = a^2 \cdot a^2 & a \cdot a^{-3} = a^{-2} \\ a \cdot b = ab & a \cdot b^{-1} = ab^{-1} \\ a \cdot ab = a^2 \cdot b & a \cdot (ab)^{-1} = (aba^{-1})^{-1} \\ a \cdot ab^{-1} = a^2 \cdot b^{-1} & a \cdot (ab^{-1})^{-1} = aba^{-1} \\ a \cdot aba^{-1} = a^2 \cdot (ab^{-1})^{-1} & a \cdot (aba^{-1})^{-1} = a^2 \cdot (ab)^{-1}, \end{array}$$

which shows that no neighbor of  $a$  in the Cayley graph of  $G$  with respect to  $S$  has length higher than 2. Therefore  $a$  is a dead end of length 2.

Let  $G$  have an element  $a$  of order 4. Define  $S'$  as before. A similar argument to the one given above then shows that  $a^2$  is a dead end of length 2 with respect to the generating set  $S$  defined by

$$S = \left( \bigcup_{b \in S'} S_b \right) \cup \{a\}.$$

Finally, if  $G$  has no elements of order 4 or higher, then  $G$  is a finitely generated group in which  $g^6 = 1$ , for all  $g \in G$ . Therefore  $G$  is a homomorphic image of the free Burnside group  $B(m, 6)$ . Since  $B(m, 6)$  is finite [Hal58], this means that  $G$  is finite, contradicting the assumption that  $G$  is infinite.  $\square$

The statement in Theorem 4 is concerned only with infinite groups since finite groups always have dead ends. However, for completeness, we observe that dead ends of length 2 can be achieved in any group that is sufficiently large to allow elements of length 2 to exist (thus all groups but the trivial group and the cyclic groups of order 2 and 3).

**Proposition 1.** *Every finitely generated group  $G$  that has at least 4 elements has a finite generating set  $S$  with respect to which  $G$  has a dead end of length 2.*

*Proof.* The proof of Theorem 4 applies unless  $G$  is a finite group in which every element has order 2 or 3. In that case, let  $a$  be any nontrivial element of  $G$ . Since  $a$  has order 2 or 3 and  $|G| \geq 4$  the set  $\{1, a, a^{-1}\}$  is a proper subgroup of  $G$ . The set  $S = G - \{1, a, a^{-1}\}$  is then a finite generating set for  $G$  (since the complement of a proper subgroup always generates the group). The element  $a$  is then a dead end of length 2 with respect to  $S$ .  $\square$

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368, USA