EE 102
Probability and Statistics in Electrical Engineering
MIDTERM 2

NAME: SOLUTION MANUAL

Problem 1	35	
Problem 2	35	
Problem 3	30	
Total	100	

Notes:

- Show your work for full/partial credit
- In the exam, $P[A]$ denotes the probability of event A happening.

Problem 1. For a random variable X, the cumulative distribution function (CDF) is given as

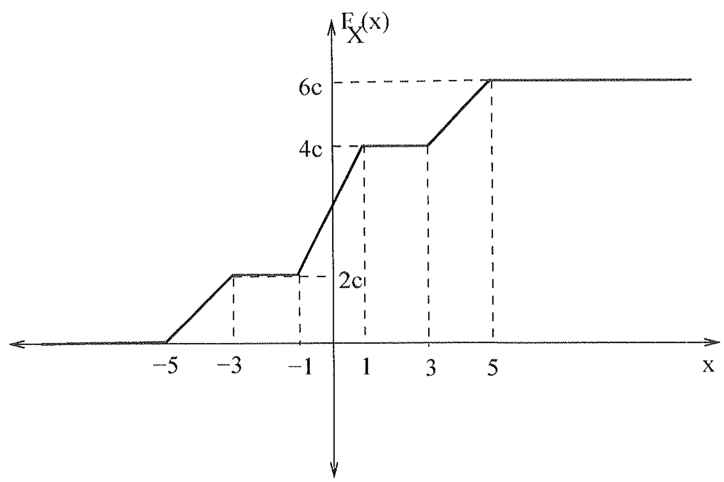


Figure 1: Problem 1

a) Find c such that $F_X(x)$ is a valid CDF.

b) Is X a continuous random variable? Explain your reasoning.

c) Find and sketch the PDF of X.

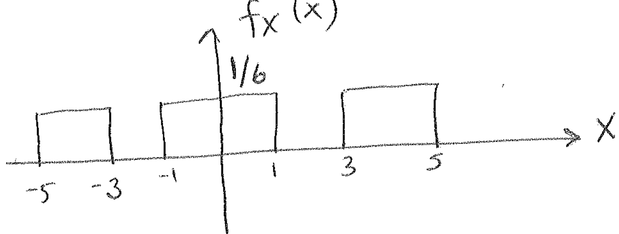
d) Find the sample space of X.

e) Find the mean $E[X]$ and $Var[X]$.

a) $F_X(\infty) = 1 \Rightarrow 6c = 1 \Rightarrow c = \frac{1}{6}$

b) YES, CDF has no jump points.

c) $f_X(x) = \dfrac{d\,F_X(x)}{dx}$



d) $S_X = \{ (-5,-3) \cup (-1,1) \cup (3,5) \}$

e) $E[X] = 0$ (symmetry)

$Var[X] = E[X^2] = \int_{-5}^{-3} x^2 \frac{1}{6}\,dx + \int_{-1}^{1} \frac{1}{6} x^2\,dx + \int_{3}^{5} x^2 \frac{1}{6}\,dx = 11$

Problem 2. The pdf of a continuous random variable X is given as

$$f_X(x) = \begin{cases} \beta e^{-x} & 0 < x < 2 \\ 0 & o.w. \end{cases}$$

a) What is the sample space of X.

b) Find β so that $f_X(x)$ is a valid pdf.

c) Find the mean of X, that is $E[X]$.

d) Find and sketch the CDF of X, $F_X(x)$.

e) Find the probability $P[-2 < X < 4 | X < 3]$

Hint: $\int xe^{-x}dx = -(x+1)e^{-x}$

a) $S_X = \{ (0,2) \}$

b) $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^2 \beta e^{-x}\,dx = 1$

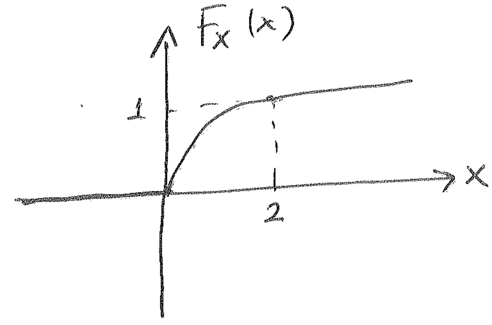
$\Rightarrow \beta \frac{e^{-x}}{-1}\Big|_0^2 = -\beta(e^{-2}-1) = 1 \Rightarrow \beta = \frac{1}{1-e^{-2}}$

c) $E[X] = \int_0^2 \beta x e^{-x}\,dx$

$= \beta[-(x+1)]e^{-x}\Big|_0^2$

$= \beta[1-3e^{-2}] = (1-3e^{-2})/(1-e^{-2})$

d) $F_X(x) = \begin{cases} \int_0^x \beta e^{-u}\,du = \beta[1-e^{-x}] & 0 < X < 2 \\ 0 & X < 0 \\ 1 & X > 2 \end{cases}$



e) $P[-2 < X < 4 | X < 3]$

$= \frac{P[-2 < X < 3]}{P[X < 3]} = \frac{1}{1} = 1$

Problem 3. *<u>Theorem</u>: Assume X is a Gaussian random variable with mean μ and variance σ^2. Let a and b be two constants (with $b \neq 0$). Then, the random variable $Y = aX + b$ is also Gaussian random variable.*

Use the above Theorem in solving the following problem.

A signal $S = 3$ is transmitted over a noisy channel and received as

$$Y = S + N$$

where N is a zero-mean Gaussian random variable with variance $\sigma_n^2 = 4$.

a) Find the mean and variance of the received signal Y.

b) Sketch the PDF of the received signal Y.

c) At the receiver, a detector finds an estimate of the transmitted signal based on its observation Y. The detector guesses that the transmitted signal is equal to 3 if the received signal is above a threshold τ:

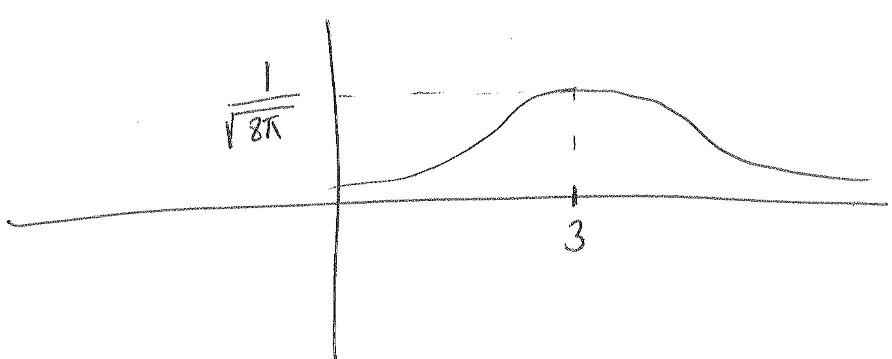
$$\hat{S} = 3, \quad \text{if } Y > \tau.$$

Find and sketch the probability that the detector makes the correct decision, that is $P[\hat{S} = S]$ in terms of τ.

a) $E[Y] = E[3+N] = 3$

$Var[Y] = Var[3+N] = Var[N] = 4$

b) $Y \sim \text{Gaussian}(3, 4)$



c) $P[\hat{S} = S] = P[Y > \tau]$

$= P[3 + N > \tau] = P[N > \tau - 3]$

$= 1 - \Phi\left(\frac{\tau - 3}{\underbrace{2}_{\sigma}}\right)$