EE 102
Probability and Statistics in Electrical Engineering
MIDTERM
FALL 2014

NAME: ____

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Notes:

- Show your work for full/partial credit
- In the exam, $P[A]$ denotes the probability of event A happening.
- Show your work explicitly.

Problem 1. For a discrete random variable X, the probability mass function (PMF) is given as

$$P_X(k) = \begin{cases} \frac{\beta}{3^k k!} & k = 0, 1, 2, 3, \ldots \\ 0 & otherwise \end{cases}$$

a) (10 points) Find β so that this is a valid PMF.
(Hint: You can use the formula $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$.)
Solution: $\sum_{k=0}^{\infty} P_X(k) = 1 \rightarrow \beta \sum_{k=0}^{\infty} \frac{(1/3)^k}{k!} = \beta e^{1/3} = 1 \rightarrow \beta = e^{-1/3}$

b) (10 points) What kind of random variable is X?
Solution: (10 points) $X \sim Poisson(1/3)$

c) (5 points) Find $E[X]$.
Solution: $E[X] = 1/3$

Problem 2. A player has two decks of cards: RED and BLUE. The RED deck is missing the King of Spades, the Queen of Spades, the Jack of Spades, and the Ace of Spades (**K♠, Q♠, J♠, A♠**), and the BLUE deck is missing all the Aces (**A♣**, **A**♡, **A**♢, **A♠**).

*Hint: Remember that a standard deck of playing cards is 52-cards. There are 4 suits in a deck: clubs (♣), hearts (♡), diamonds (♢) and spades (♠). Each suit has 13 cards (**A,2, 3, . . . , 10, J, Q, K**).*

A FAIR die is thrown, and a deck is selected according to the number showing on the die: RED deck is selected if the die shows a number less than 5. BLUE deck is selected otherwise.

From the selected deck, we pick TWO cards.

a) (15 points) Find the probability of picking exactly one King (**K**), and one 2.

   Solution:

$$P[RED] = P[Die < 5] = \frac{4}{6} = \frac{2}{3}$$
$$P[BLUE] = P[Die \geq 5] = \frac{2}{6} = \frac{1}{3}$$

   Using total probability theorem,

$$P[Getting\ a\ King\ and\ a\ 2] = P[\mathbf{K2}] = P[\mathbf{K2}|RED]P[RED] + P[\mathbf{K2}|BLUE]P[BLUE].$$

   Since RED deck has 48 cards and 3 Kings and 4 twos,

$$P[\mathbf{K2}|RED] = \frac{C_3^1 C_4^1}{C_{48}^2}$$

   Since BLUE deck has 48 cards and 4 Kings and 4 twos,

$$P[\mathbf{K2}|BLUE] = \frac{C_4^1 C_4^1}{C_{48}^2}.$$

   Hence,

$$P[\mathbf{K2}] = \frac{C_3^1 C_4^1}{C_{48}^2}\frac{2}{3} + \frac{C_4^1 C_4^1}{C_{48}^2}\frac{1}{3}.$$

b) (5 points) If the hand we picked has one King (**K**), and one **2**, what is the probability that the BLUE deck was selected?

Solution: Using Bayes Theorem,

$$P[BLUE|\mathbf{K2}] = \frac{P[\mathbf{K2}|BLUE]P[BLUE]}{P[\mathbf{K2}]} = \frac{2}{5}$$

c) (5 points) Discuss what kind of an UNFAIR die (instead of the FAIR die above) you would use if you want to increase your chance of getting exactly one King (**K**), and one **2** in the above random experiment.

Solution: Since $P[\mathbf{K2}|BLUE] > P[\mathbf{K2}|RED]$, it is better to choose the BLUE deck more often. So using a die that favors 5 & 6 will increase the chance of getting a King and a two.

Problem 3. A student is studying for a test which is on next day. He has a set of 40 sample problems. He does not know which problems are more relevant to the exam. So he flips a FAIR coin. If the coin is heads, he attempts to solve the problem. If the coin is tails, he skips to the next problem.

a) (11 points) Is **X = "the number of problems he attempts to solve"** a random variable? If so, what type of random variable is it? Provide the necessary parameters.

   Solution:

$$X \sim Binomial(n, p)$$

   where $n = 40$ questions $p = \frac{1}{2}$ which is the probability of getting a head.

b) (11 points) He passes the test if he attempts to solve at least 95% of the problems. What is the probability that he will pass the test?

   Solution:

   95% of the problems = 95% * 40 = 38 problems.

$$P[Passing\ the\ Test] = P[X \geq 38] = P[X = 38] + P[X = 39] + p[X = 40]$$

$$P[Passing\ the\ Test] = C_{38}^{40}\frac{1}{2^{40}} + C_{39}^{40}\frac{1}{2^{40}} + C_{40}^{40}\frac{1}{2^{40}}$$

c) (3 points) Answer part-a if he had an endless list of sample problems instead of 40 problems.

   Solution: If the time is not limited, and there is an endless list of sample problems, then X is no longer a random variable because the student will solve ∞ problems almost surely.

Problem 4. Each question has one correct answer. Each question is 5 points.

1. Which of the following is FALSE for a discrete random variable **X**?
   ✓a) the sample space of **X** can only contain integers as outcomes.
   b) the sample space of **X** may contain finite number of outcomes.
   c) the sample space of **X** may contain infinite number of outcomes.
   d) the sample space of **X** is countable
2. Which of the followings is TRUE definition for the Probability Mass Function (PMF) of a discrete random variable **X**?
   a) $P_{\mathbf{X}}(x) = P[X \leq x]$
   b) $P_{\mathbf{X}}(x) = P[X > x]$
   ✓c) $P_{\mathbf{X}}(x) = P[X = x]$
   d) $P_{\mathbf{X}}(x) = P[X < x]$
3. Which of the followings is FALSE for the Probability Mass Function (PMF), $P_{\mathbf{X}}(x)$, of a discrete random variable **X**?
   a) For any x, $P_{\mathbf{X}}(x) \geq 0$.
   b) $\sum_{x \in S_{\mathbf{X}}} P_{\mathbf{X}}(x) = 1$ where $S_{\mathbf{X}}$ denotes the sample space of **X**.
   c) For any event B, $P[B] = \sum_{x \in B} P_{\mathbf{X}}(x)$
   ✓d) $P_{\mathbf{X}}(x)$ is right-continuous and has jump points at the outcomes in the sample space.
4. Which of the followings is FALSE for the Cumulative Distribution Function (CDF), $F_{\mathbf{X}}(x)$, of a discrete random variable **X**?
   a) For any x, $0 \leq F_{\mathbf{X}}(x) \leq 1$.
   b) For any x, $F_{\mathbf{X}}(x) = P[X \leq x]$.
   ✓c) For any event B, $P[B] = \sum_{x \in B} F_{\mathbf{X}}(x)$
   d) $F_{\mathbf{X}}(x)$ is right-continuous and has jump points at the outcomes in the sample space.
5. Which of the followings is FALSE for the MEAN of a discrete random variable **X**?
   a) The mean is equal to $E[X] = \sum_{x_i \in S_{\mathbf{X}}} x_i P[\mathbf{X} = x_i]$ where $S_{\mathbf{X}}$ denotes the sample space.
   ✓b) The mean of a discrete random variable is always an integer.
   c) The mean of a discrete random variable can be negative.
   d) The mean for $\mathbf{X} \sim Bernoulli(p)$ is equal to $E[X] = p$.