EE 102
Probability and Statistics in Electrical Engineering
MIDTERM 2

NAME: ______________________________

PICK THREE PROBLEMS !

Problem 1	33	
Problem 2	33	
Problem 3	33	
Problem 4	33	
Attendance	1	
Total	100	

Notes:

- Show your work for full/partial credit

Problem 1. For a continuous random variable X, the probability density function (PDF) is given as in Fig. 1.

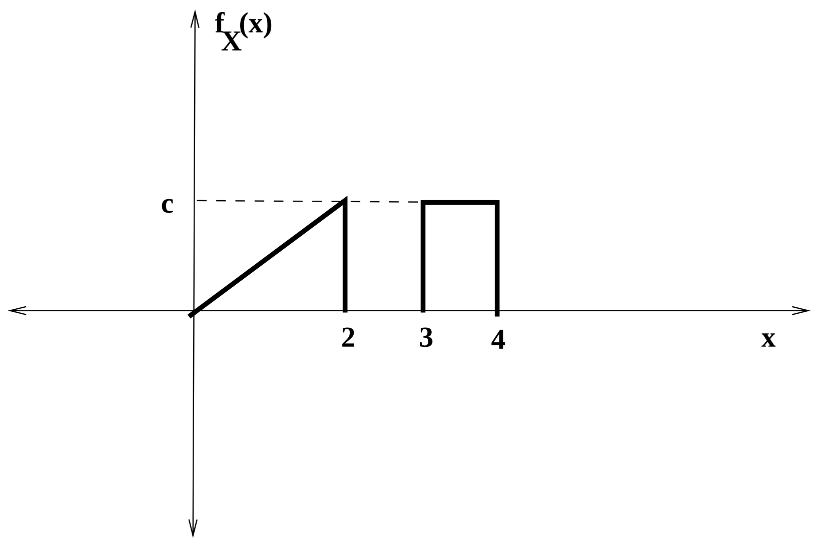

Figure 1: Problem 1

a) Find the sample space for X.
Solution:

$$S_X = \{x \mid (0,2) \cup (3,4)\}$$

b) Find the value of c so that the given PDF is valid.
Solution:

$$\int f_X(x)dx = 1 \Rightarrow \frac{2c}{2} + 1c = 2c = 1 \Rightarrow c = 0.5$$

c) Find the mean $E[X]$ and variance $Var[X]$ of X.
Solution:

$$E[X] = \int x f_X(x)dx = \int_0^2 x(x/4)dx + \int_3^4 x0.5dx = 2/3 + 7/4 = 29/12$$

$$E[X^2] = \int x^2 f_X(x)dx = \int_0^2 x^2(x/4)dx + \int_3^4 x^2 0.5dx = 46/3$$

$$Var[X] = E[X^2] - E[X]^2 == 9.4931$$

d) Sketch the cumulative distribution function (CDF) for X. **Solution:**

$$F_X(x) = P[X \leq x] = \int_0^x f_X(x)dx$$

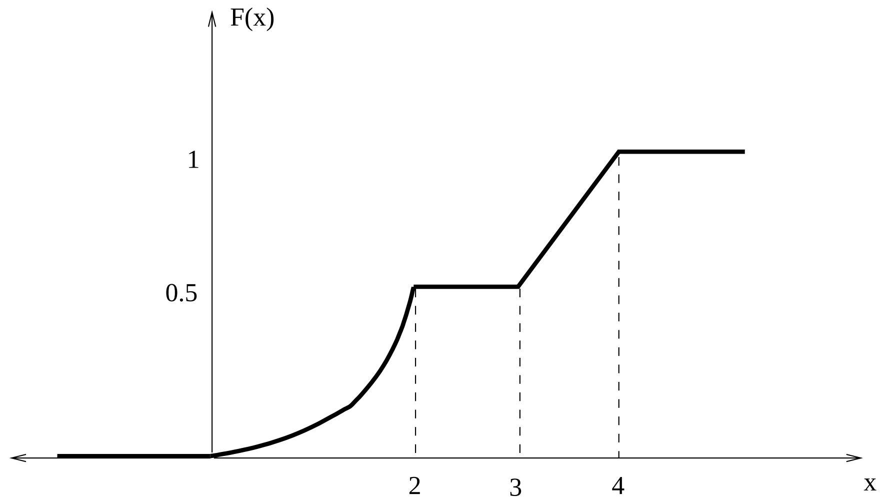

Figure 2: Problem 1-d

Problem 2. The cumulative distribution function (CDF) for a continuous random variable is given as follows

$$F_X(x) = \beta - e^{-2x}, \quad x \geq 0.$$

a) Find the value of β so that $F_X(x)$ is a valid CDF.
Solution: Since
$$\lim_{x\to\infty} F_X(x) = 1 \rightarrow \beta = 1.$$

b) Find and sketch the probability density function (PDF) for X.
Solution:
$$f_X(x) = \frac{dF_X(x)}{dx} = 2e^{-2x}, \quad x \geq 0.$$

c) What type of random variable is X? Find the mean, $E[X]$, and the variance, $Var[X]$.
Solution: X is an exponential random variable with parameter 2:
$$X \sim Exponential(2) \Rightarrow E[X] = \frac{1}{2}, Var[X] = 1/4.$$

d) Find the probability $P[-2 < X \leq 1 | X \leq 2]$.
Solution:

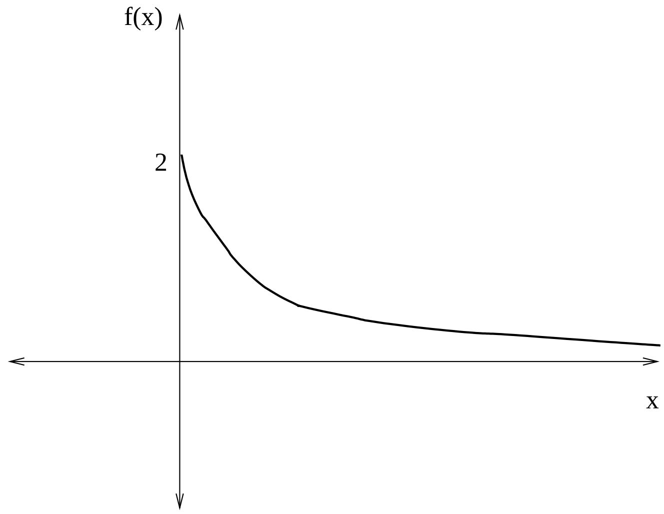

Figure 3: Problem 2-c

$$
\begin{aligned}
P[-2 < X \leq 1 | X \leq 2] &= \frac{P[\{-2 < X \leq 1\} \cap \{X \leq 2\}]}{P[X \leq 2]} \\
&= \frac{P[\{-2 < X \leq 1\}]}{P[X \leq 2]} = \frac{F_X(1) - F_X(-2)}{F_X(2)} \\
&= \frac{1 - e^{-2}}{1 - e^{-4}}
\end{aligned}
$$

Problem 3. A student studies for the midterm depending on the weather conditions. If the weather is good, the student prefers to play basketball and spends less time for the midterm. The time spent by the student for the exam when the weather is GOOD has probability mass function (PMF)

$$P_T(t \mid weather\ is\ GOOD) = \begin{cases} 0.7 & t = 0 \\ 0.2 & t = 1 \\ 0.1 & t = 2 \end{cases}$$

The time spent by the student for the exam when the weather is BAD has probability mass function (PMF)

$$P_T(t \mid weather\ is\ BAD) = \begin{cases} 0.3 & t = 0 \\ 0.3 & t = 1 \\ 0.3 & t = 2 \\ 0.1 & t = 3 \end{cases}$$

The unit for the time is hour.

a) On a good day (perfect for playing basketball), find the probability that the student spends at least one hour for the midterm.
Solution:

$$P[T \geq 1|GOOD] = P[T = 1|GOOD] + P[T = 2|GOOD] = 0.3$$

a) Assume the students lives in California which has a chance of 80% good weather. Find the probability mass function for the time T spent by the student for the midterm.
Solution: It is given that $P[GOOD] = 0.8, P[BAD] = 0.2$.

$$P_T(t) = P_T(t|GOOD)P[GOOD] + P_T(t|BAD)P(BAD).$$

$$P_T(t) = \begin{cases} 0.7 \times 0.8 + 0.3 \times 0.2 = 0.62 & t = 0 \\ 0.2 \times 0.8 + 0.3 \times 0.2 = 0.22 & t = 1 \\ 0.1 \times 0.8 + 0.3 \times 0.2 = 0.14 & t = 2 \\ 0 \times 0.8 + 0.1 \times 0.2 = 0.02 & t = 3 \end{cases}$$

b) Assume the students lives in California which has a chance of 80% good weather. Find the average time spent by the student for the midterm.
Solution:

$$E[T] = \sum_{t=0}^{3} tP_T(t) = 0.62 \times 0 + 0.22 \times 1 + 0.14 \times 2 + 0.02 \times 3 = 0.56$$

c) Comment on how would your answer change in parts b) and c) if the student lives in NewYork.

Solution: Assuming weather is worse in Newyork ($P[GOOD] < 0.8$ and $P[BAD] > 0.2$), the average time spent will increase ($E[T] > 0.56$); in addition the pmf $P_T(t)$ will be larger for large t values ($P_T(3) > 0.02$).

Problem 4. TRUE or FALSE. Explain your reasoning. If you have no explanation, you do not get any points.

1) The distribution of a Gaussian random variable is uniquely determined by its mean and variance.
   Solution: TRUE:
   $$X \sim Gaussian(\mu, \sigma) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}.$$
2) Let $f_X(x)$ be a valid pdf; then $0 \leq f_X(x) \leq 1$
   Solution: FALSE. $f_X(x) \geq 0$ always; however $f_X(x)$ could be larger than 1. For example, uniform distribution in the interval $(0, 0.5)$, $f_X(x) = 2, \quad 0 < x < 0.5$.
3) If the variance of a random variable is small (close to zero), then its pdf is concentrated around its mean.
   Solution: TRUE. Since variance shows the deviation from the mean.
4) The cdf of a continuous random variable has jump points.
   Solution: FALSE. The cdf of a DISCRETE random variable has jump points.
5) Both the mean and variance of an exponential random variable are always greater than zero.
   Solution: TRUE. Exponential random variable always take nonnegative values, hence the mean is always greater than zero. The variance of any random variable is always greater than zero (could be equal to zero).
6) Assume that $X \sim Gaussian(3, 4)$. Then, the probability
   $$P[-2 < X < 2] = \Phi(0.5) - \Phi(-0.5)$$
   where $\Phi(z)$ denotes the standard normal CDF.
   Solution: FALSE.
   $$P[-2 < X < 2] = \Phi\left(\frac{2-3}{4}\right) - \Phi\left(\frac{-2-3}{4}\right) = \Phi(0.25) - \Phi(-1.25)$$
7) The mean of a discrete random variables is always zero.
   Solution: FALSE. For example, $X \sim Bernoulli(p)$ has mean p.
8) Let $Y \sim Uniform(-5, 5)$, then $E[Y^2] = 50$ (*Hint: Use variance and mean formula of continuous uniform random variable*).
   Solution: FALSE.
   $$E[Y^2] = Var[Y] + (E[Y])^2 = (5 - (-5))^2/12 + 0 = 100/12.$$