EE 102
Probability and Statistics in Electrical Engineering
MIDTERM 2

NAME: ______________________________

Problem 1	33	
Problem 2	33	
Problem 3	33	
Attendance	1	
Total	100	

Notes:

- Show your work for full/partial credit

Problem 1. For a continuous random variable X, the probability density function (PDF) is given as in Fig. 1.

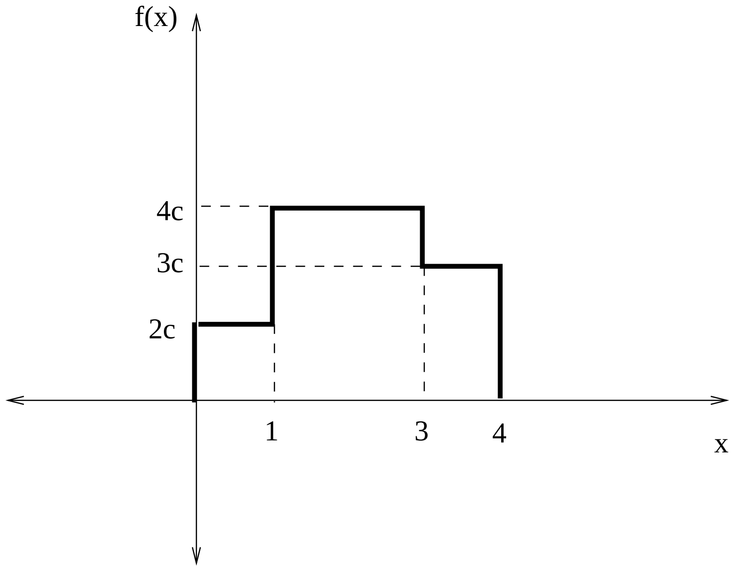

Figure 1: Problem 1

a) Find the sample space for X. **Solution:**

$$S_X = \{x | x \in (0, 4)\}$$

b) Find the value of c so that the given PDF is valid.
Solution: We know that $\int f_X(x)dx = 1 \Rightarrow 2c + 8c + 3c = 1 \Rightarrow c = 1/13$.

c) Find the mean $E[X]$ and variance $Var[X]$ of X.
Solution:

$$\begin{aligned} E[X] &= \int x f_X(x)dx \\ &= \int_0^1 2cxdx + \int_1^3 4cxdx + \int_3^4 3cxdx \\ &= c + 16c + (21/2)c = 55/26 \end{aligned} \quad (1)$$

$$\begin{aligned} E[X^2] &= \int x^2 f_X(x)dx \\ &= 217/39 \end{aligned} \quad (2)$$

$$Var[X] = E[X^2] - (E[X])^2 = 1.089$$

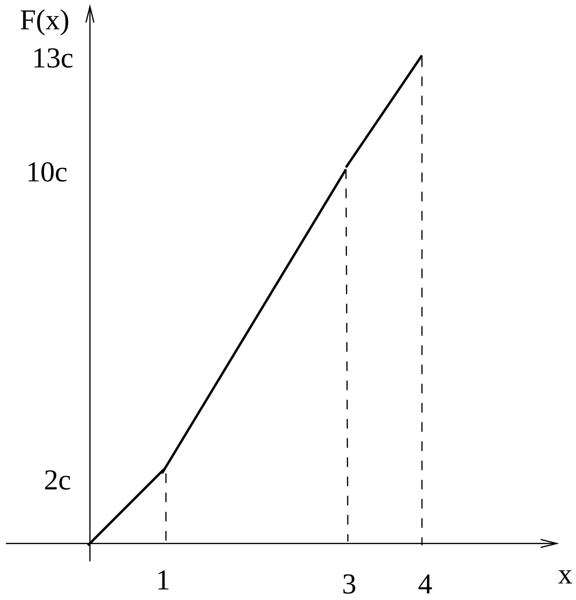

Figure 2: Problem 1- CDF

d) Sketch the cumulative distribution function (CDF) for X.
Solution:

$$F_X(x) = \begin{cases} \frac{2x}{13} & 0 \le x < 1 \\ \frac{(4x-2)}{13} & 1 \le x < 3 \\ \frac{(3x+1)}{13} & 3 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

Problem 2. The cumulative distribution function (CDF) for a continuous random variable is given as follows

$$F_X(x) = \beta - Q(2x-3), \quad -\infty \leq x \leq \infty,$$

where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$ denotes the Q-function. Note that $F_X(x)$ can be rewritten as

$$F_X(x) = \beta - 1 + \Phi(2x-3), \quad -\infty \leq x \leq \infty,$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = 1 - Q(x).$$

Note that $\lim_{x\to\infty} Q(x) = 0$, and $\lim_{x\to\infty} \Phi(x) = 1$.

a) Find the value of β so that $F_X(x)$ is a valid CDF.
Solution:

$$\lim_{x\to\infty} F_X(x) = \beta = 1$$

b) What type of random variable is X? Find the mean, $E[X]$, and the variance, $Var[X]$.
Solution: X is a Gaussian random variable with mean 3/2 and variance $Var(X) = \sigma^2 = 1/4$.

c) Find and sketch the probability density function (PDF) for X. **Solution:**

$$f_X(x) = \frac{1}{\sqrt{\pi/2}} e^{-2(x-3/2)^2}, \quad -\infty < x < \infty.$$

d) Find the probability $P[-2 < X \leq 2 | X \leq 1]$.
Solution:

$$P[-2 < X \leq 2 | X \leq 1] = \frac{P[-2 < X \leq 1]}{P[X \leq 1]} = \frac{Q(-7) - Q(-1)}{Q(1)}$$

Problem 3. A desert shop makes different types of deserts depending on the weather conditions. If the weather is GOOD, all the deserts are served with ice-cream. When the weather is BAD, the number of desert types that come with ice-cream has probability mass function (PMF)

$$P_N(n \mid weather\ is\ BAD) = \begin{cases} 0.6, & n = 0 \\ 0.2, & n = 1 \\ 0.1, & n = 2 \\ 0.1, & n = 3 \end{cases}$$

Assume that the store makes 5 different types of deserts each day.

a) On a BAD day, find the probability that the store makes at most 2 different types of deserts that come with ice-cream.
Solution:

$$P(N \leq 2 \mid weather\ is\ BAD) = \sum_{n=0}^{2} P_N(n \mid weather\ is\ BAD) = 0.9$$

a) Assume the store is in California which has a chance of 80% good weather. Find the probability mass function for the "the number of desert types that comes with ice-cream" (N).
Solution:

$$P_N(n \mid weather\ is\ GOOD) = \begin{cases} 1, & n = 5 \\ 0, & otherwise \end{cases}$$

$$\begin{aligned} P_N(n) &= P_N(n \mid weather\ is\ BAD)P(\ weather\ is\ GOOD) \\ &\quad \ldots + P_N(n \mid weather\ is\ GOOD)P(\ weather\ is\ GOOD) \end{aligned}$$

$$P_N(n \mid weather\ is\ BAD) = \begin{cases} 0.12, & n = 0 \\ 0.04, & n = 1 \\ 0.02, & n = 2 \\ 0.02, & n = 3 \\ 0.8 & n = 5 \end{cases}$$

b) Assume the store is in California which has a chance of 80% good weather. Find the average number of desert types that comes with ice-cream.
Solution:

$$E[N] = 0.04 + 0.04 + 0.06 + 4 = 4.14$$

c) Comment on how would your answer change in parts b) and c) if the store is in NewYork.

Solution: Assuming weather is worse in Newyork ($P[GOOD] < 0.8$ and $P[BAD] > 0.2$), the average number of deserts with icecream will decrease ($E[N] < 4.14$); in addition the pmf $P_N(n)$ will be smaller for large n values ($P_N(n = 5) < 0.8$).