EE 102
Probability and Statistics in Electrical Engineering
MIDTERM 1

NAME: ______________________________

Problem 1	44	
Problem 2	44	
Problem 3	12+ 20 (extra)	
Total	100 + 20	

Notes:

- Show your work for full/partial credit
- In the exam, $P[A]$ denotes the probability of event A happening.

Problem 1. For a random variable X, the probability mass function (PMF) is given as in Fig. 1.

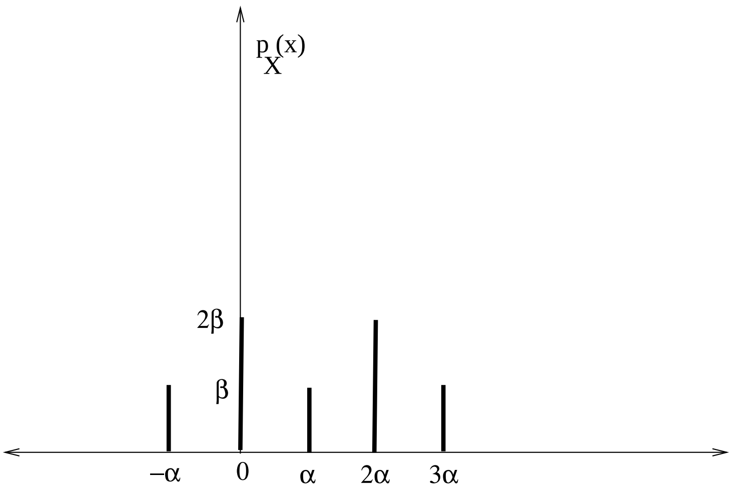

Figure 1: Problem 1

a) *[6 points + 2 points]* Find β such that $p_X(x)$ is a valid PMF.
Solution: Since $\sum_x P_X(x) = 1 \Rightarrow \beta = 1/7$

b) *[6 points + 2 points]* Write the sample space of X.
Solution:

$$S_X = \{-\alpha, 0, \alpha, 2\alpha, 3\alpha\}$$

c) *[6 points + 2 points]* Is X a discrete random variable? Explain your reasoning.
Solution: Yes, X is a discrete random variable, since the sample space is countable.

d) *[8 points + 2 points]* Sketch the cumulative distribution function (CDF) for X.
Solution: The CDF $F_X(x) = P(X \leq x)$ is shown in Fig, 2.

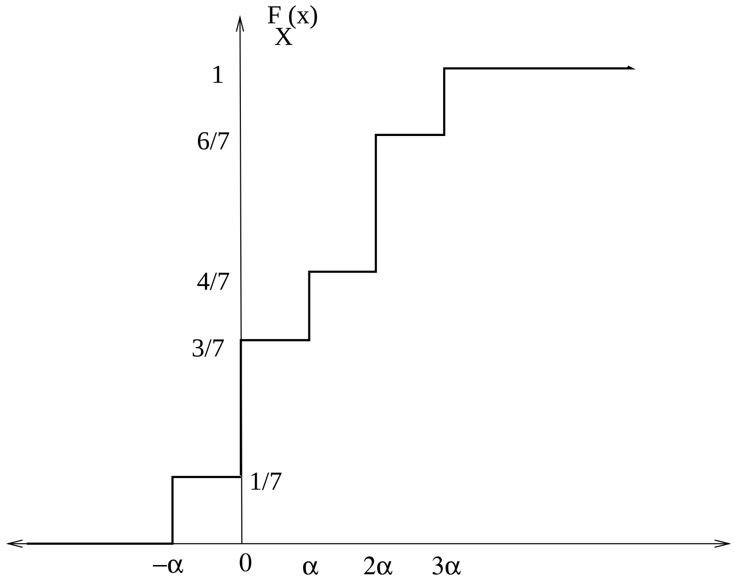

Figure 2: Problem 1

e) *[8 points + 2 points]* Let A be the event that X lies between α and 3α (that is $A = \{\alpha \leq X \leq 3\alpha\}$). Find the probability of A given that X is less than and equal to 2α.

Solution: The event $A = \{\alpha \leq X \leq 3\alpha\}$.

$$
\begin{aligned}
P(A|X \leq 2\alpha) &= \frac{P(\{\alpha \leq X \leq 3\alpha\} \cap \{X \leq 2\alpha\})}{P(X \leq 2\alpha)} \\
&= \frac{P(\{\alpha \leq X \leq 2\alpha\})}{P(X \leq 2\alpha)} \\
&= \frac{\beta + 2\beta}{\beta + 2\beta + \beta + 2\beta} \\
&= \frac{1}{2} \qquad (1)
\end{aligned}
$$

Problem 2. A single bit is transmitted over a noisy channel. The noise flips the bit with probability p (see Fig. 3). Assume that the binary messages sent by the transmitter that are "bit=1"s are double the number of messages that are "bit=0"s.

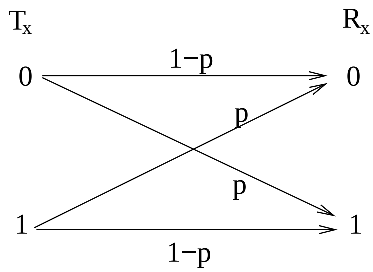

Figure 3: Binary Symmetric Channel. T_x = Transmitter, R_x: Receiver

a) *[6 points + 2 points]* Find the transmitter message probabilities $P(T_x = 0)$ or $P(T_x = 1)$.
Solution: $P(T_x = 0) = 1/3$ and $P(T_x = 1) = 2/3$.

b) *[6 points + 2 points]* Find the probability of error if a "bit=0" has been sent, i.e. $P(error|T_x = 0)$.
Solution:

$$P(error|Tx = 0) = P(R_x = 1|T_x = 0) = p.$$

c) *[8 points + 2 points]* Find the total probability of error, $P(error)$, at the receiver in terms of p. Does $P(error)$ depend on $P(T_x = 0)$ or $P(T_x = 1)$?
Solution: Using total probability law,

$$\begin{aligned} P(error) &= P(error|T_x = 0)P(T_x = 0) + P(error|T_x = 1)P(T_x = 1) \\ &= p\frac{1}{3} + p\frac{2}{3} = p. \end{aligned}$$

$P(error)$ does not depend on the value of $P(T_x = 0)$ and $P(T_x = 1)$ for a binary symmetric channel.

d) *[8 points + 2 points]* If an error has occurred, what is the probability that the transmitter has sent "bit=0".
Solution: Using Bayes thm,

$$P(T_x = 0|error) = \frac{P(error|T_x = 0)P(T_x = 0)}{P(error)} = \frac{p(1/3)}{p} = \frac{1}{3}.$$

e) *[6 points + 2 points]* Assume that transmitter uses the *repetition coding* by mapping each bit to three identical bits ($0 \to 000$ and $1 \to 111$). Similar to part-a, each bit is transmitted over the binary symmetric channel (see Fig. 3). Assume that the receiver uses *majority decoding*. That is, the receiver counts number of "bit=0"s and "bit=1"s for each message. If the number of "bit=0"s exceeds the number of "bit=1"s, then the message is decoded as "bit=0". Otherwise, the message is decoded as "bit=1" (For example, let's say transmitter wants to send the message "bit=0". Using repetition coding, the transmitter sends "000". Due to noise the receiver observes "011" and then decodes it as "bit=1".)

What is the probability of error in terms of p for the 3-bit repetition coded transmission/reception? (HINT: Error occurs when all three bits are flipped OR 2 out of 3 bits are flipped.)

Solution:

$$\begin{aligned} P(error) &= \\ &\quad P(error|T_x = 000)P(T_x = 000) + P(error|T_x = 111)P(T_x = 111) \end{aligned}$$

$$\begin{aligned} P(error|T_x = 000) &= P(R_x = 110|T_x = 000) + P(R_x = 101|T_x = 000) \\ &+ P(R_x = 011|T_x = 000) + P(R_x = 111|T_x = 000) \\ &= 3p^2(1-p) + p^3 \end{aligned}$$

$$\begin{aligned} P(error|T_x = 111) &= P(R_x = 001|T_x = 111) + P(R_x = 010|T_x = 111) \\ &+ P(R_x = 100|T_x = 111) + P(R_x = 000|T_x = 111) \\ &= 3p^2(1-p) + p^3 \end{aligned}$$

$P(error|T_x = 111) = 3p^2(1-p) + p^3$
$P(error) = 3p^2(1-p) + p^3 = 3p^2 - 2p^3$.

Problem 3. A student is studying for the exam by solving practice problems one by one. He is playing a game with M&Ms while studying: he draws M&Ms from a bag, and solves problems based on the color of the M&M. The M&M bag contains 1 brown, 2 yellow, and 1 red M&Ms. For each problem, he picks one M&M from the bag. He loves the brown M&Ms. If the M&M is brown, he eats the M&M and stops solving the rest of the problems. If the selected M&M is red, he solves an easy problem and puts the M&M back in the bag. If the selected M&M is yellow, he solves a hard problem and puts the M&M back in the bag. Assume there are enough (infinite) number of problems (hard or easy) to be solved.

Let X denote the number of problems that the student attempts to solve. Let X_e denote the number of EASY problems that the student attempts to solve. Let X_h denote the number of HARD problems that the student attempts to solve.

a) *[12 points]* What type of random variables X, X_e, and X_h are? Please specify with the parameters.

Solution:

Sol1) Define Y as the number of M&M draws from the bag. Note that $Y \sim Geometric(p)$ with sample space $S_Y = \{1, 2, \ldots\}$. This implies $P(Y = k) = (1-p)^{k-1}p$.
Here $p = \frac{1}{4}$ is the probability of drawing the brown M&M. The number of problems that the student attempts to solve is $X = Y - 1$.

Sol2) $X \sim Geometric(p)$ with sample space $S_X = \{0, 1, 2, \ldots\}$. This implies, $P(X = k) = (1-p)^k p$. Here $p = \frac{1}{4}$ is the probability of drawing the brown M&M.

Given the value of $X = x$, $X_e \sim Binomial(x, \frac{1}{3})$ and $X_h \sim Binomial(x, \frac{2}{3})$. Hence,

$$P(X_e = k|X = n) = \binom{n}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

and

$$P(X_h = k|X = n) = \binom{n}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{n-k}.$$

The sample space for $X_e = \{0, 1, 2, \ldots\}$. Using the total probability thm,

$$\begin{aligned} P(X_e = k) &= \sum_n P(X_e = k|X = n)P(X = n) \\ &= \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k} (1-p)^n p. \end{aligned}$$

The sample space for $X_h = \{0, 1, 2, \ldots\}$. Using the total probability thm,

$$\begin{aligned} P(X_h = k) &= \sum_n P(X_h = k|X = n)P(X = n) \\ &= \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{n-k} (1-p)^n p. \end{aligned}$$

b) *[12 points]* EXTRA CREDIT Consider the following events

$$\begin{aligned} A &= \text{"The student solves at most 2 easy problems"} \\ B &= \text{"The student solves 4 problems"} \end{aligned}$$

Are A, B mutually exclusive? Are A, B independent? Use the hint if needed.

Solution: As student can solve 2 easy problems, and 2 hard problems ending up 4 problems in total. Hence, $A \cap B \neq \emptyset$ which implies A, B are not mutually exclusive.

$$P(A|B) = P(X_e \leq 2|X = 4) = \sum_{k=0}^{2} \binom{4}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

and

$$P(A) = P(X_e \leq 2) = \sum_{k=0}^{2} P(X_e = k)$$

where $P(X_e = k)$ is provided in part-a.

Since $P(A|B) \neq P(A)$, A and B are NOT independent.

c) *[8 points]* EXTRA CREDIT The student should attempt to solve at least 2 of the hard problems in order to be successful in the exam. Find the probability that the student will be successful if he attempts to solve 4 problems. Use the hint if needed.

Solution:

$$P(success|X = 4) = P(X_h \geq 2|X = 4) = \sum_{k=2}^{4} \binom{4}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{4-k}$$