4.3 Transfer Function Concept
傳遞函數概念
The concept of transfer function for sampled systems can be defined similarly as it has been done for continuous-time one. To clarify this, let us refer to the Fig. 4.10 where the upstream sampler is a real one while the downstream one is a fictitious that we assume to be ideals and synchronized at the same sampling period. The second sampler is introduced for the purpose to define Y(z) and therefore define properly the pulse transfer function.
採樣系統的傳遞函數概念可以類似地定義已經做過連續的時間。為了澄清這一點，讓我們參考圖 4.10 上游採樣器是真實的，而下游採樣器是虛擬的，我們假設是理想狀態，並且在相同的採樣週期內保持同步。引入第二個採樣器是為了定義 Y（z），從而正確定義脈衝傳遞函數。
Based on the Fig. 4.10, we get:
根據圖 4.10，我們得到：

$$Y(s) = G(s)U^\star(s)$$

Since the output is sampled by the fictitious sampler, we can then have:
由於輸出是由虛擬採樣器採樣的，因此我們可以得到：

$$\begin{aligned} Y^\star(s) &= \left[G(s)U^\star(s)\right]^\star \\ &= G^\star(s)U^\star(s) \end{aligned}$$

圖 4.10 脈衝傳遞函數定義

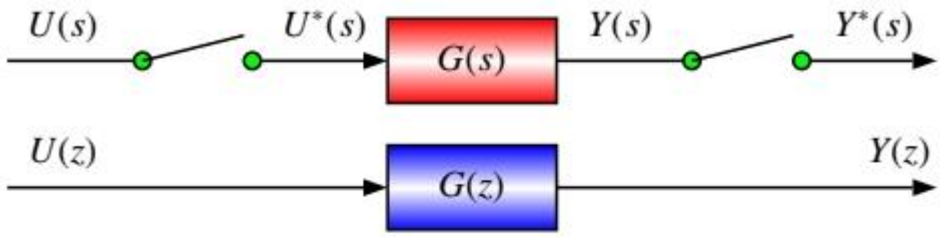

Fig. 4.10 Pulse transfer function definition

and if we apply the Z -transform, we obtain:
如果我們應用 Z 變換，則可以獲得：

$$Y(z) = G(z)U(z)$$

where $Y(z) = \mathscr{Z}\left[Y^\star(s)\right]$ and $U(z) = \mathscr{Z}\left[U^\star(s)\right]$.

This relation can be proved in an elegant way starting from the time domain. In fact, we have:

從時域開始可以很好地證明這種關係。 實際上，我們有：

$$y(t) = \mathscr{L}^{-1}\left[G(s)U^{\star}(s)\right]$$

Using now the convolution theorem, we get:
現在使用卷積定理，我們得到：

$$y(t) = \int_0^t g(t-\sigma)u^{\star}(\sigma)d\sigma$$

From the other side we know that u*(σ) can be written as follows:
另一方面，我們知道 u *（ σ ）可以寫成如下形式：

$$u^{\star}(\sigma) = \sum_{k=0}^{\infty} u(kT)\delta(t-kT)$$

Using this, the expression of y(t) becomes:
使用此，y（t）的表達式變為：

$$\begin{aligned} y(t) &= \int_0^t g(t-\sigma)\sum_{k=0}^{\infty} u(kT)\delta(t-kT)d\sigma \\ &= \sum_{k=0}^{\infty}\int_0^t g(t-\sigma)u(kT)\delta(t-kT)d\sigma \\ &= \sum_{k=0}^{\infty} g(t-kT)u(kT) \end{aligned}$$

Using now the definition of the Z -transform of the sampled signal y*(t) we have:
現在使用採樣信號 y *（t）的 Z 變換的定義，我們得到：

$$\begin{aligned} Y(z) &= \sum_{k=0}^{\infty} y(kT)z^{-k} \\ &= \sum_{k=0}^{\infty}\left[\sum_{l=0}^{\infty} g(kT-lT)u(lT)\right]z^{-k} \end{aligned}$$

Performing the change of variable, m = k − l, we get:
執行變量 m = k − l 的更改，我們得到：

$$Y(z) = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\left[g(mT)u(lT)\right]z^{-m-l}$$

that can be rewritten as follows:
可以重寫如下：

$$\begin{aligned} Y(z) &= \left[\sum_{m=0}^{\infty} g(mT)z^{-m}\right]\left[\sum_{l=0}^{\infty} u(lT)z^{-l}\right] \\ &= G(z)U(z) \end{aligned}$$

Finally, the transfer function is given by:
最後，傳遞函數由下式給出：

$$G(z) = \frac{Y(z)}{U(z)}$$

which is the ratio between the Z -transform of the output and Z -transform of the input .
When manipulating the block diagrams of sampled systems, care should be taken. The following relations will help for this purpose.
這是輸出的 Z 變換和輸入的 Z 變換之間的比率。
在操作採樣系統的框圖時，應格外小心。
以下關係將有助於此目的。

- If $Y(s) = G(s)U(s)$, then

$$Y(z) = \mathscr{Z}\left[Y^*(s)\right] = \mathscr{Z}\left[[G(s)U(s)]^*\right] \neq \mathscr{Z}\left[G^*(s)U^*(s)\right] = G(z)U(z).$$

- If $Y(s) = G(s)U^*(s)$, then

$$Y(z) = \mathscr{Z}\left[Y^*(s)\right] = \mathscr{Z}\left[[G(s)U^*(s)]^*\right] = \mathscr{Z}\left[G^*(s)U^*(s)\right] = G(z)U(z).$$

Example 4.3.1 In this example we consider the system of the Fig. 4.11 that represents two systems in serial with an ideal sampler between. The expression of the two transfer functions are:
例 4.3.1 在本例中，我們考慮圖 4.11 所示的系統，該系統表示兩個串行的系統，並且兩個系統之間有理想的採樣器。 這兩個傳遞函數的表達式是：

$$\begin{aligned} G_1(s) &= \frac{1}{s+a} \\ G_2(s) &= \frac{a}{s(s+a)} \end{aligned}$$

Our goal is to compute the equivalent transfer function for this system.
我們的目標是為該系統計算等效傳遞函數。

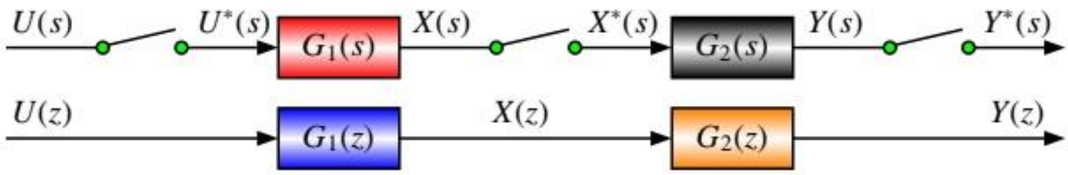

Fig. 4.11 Cascade transfer functions with sampler between

圖 4.11 級聯傳遞函數與採樣器之間

Based on this figure, we get:

根據此圖，我們得到：

$$\begin{aligned} Y^*(s) &= G_2^*(s)X^*(s) \\ X^*(s) &= G_1^*(s)U^*(s) \end{aligned}$$

which gives:

這使：

$$Y^*(s) = G_2^*(s)G_1^*(s)U^*(s)$$

that implies in turn:

這又意味著：

$$\frac{Y(z)}{U(z)} = G_1(z)G_2(z)$$

Using the table of Z -transform, we have:

使用 Z -transform 表，我們有：

$$\begin{aligned} \frac{Y(z)}{U(z)} &= G_1(z)G_2(z) = \frac{z}{(z-e^{-aT})}\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})} \\ &= \frac{z^2(1-e^{-aT})}{(z-1)(z-e^{-aT})^2} \end{aligned}$$

Example 4.3.2 In this example we consider the situation where the sample is removed between the two transfer function in serial. This situation is illustrated by the Fig. 4.12. The transfer function G1(s) and G2(s) are given by the following expression:

例 4.3.2 在本例中，我們考慮了在兩個串行傳遞函數之間移走樣品的情況。 圖 4.12 說明了這種情況。 傳遞函數 G1（s）和 G2（s）由下式給出
表達：

$$
\begin{aligned}
G_1(s) &= \frac{a}{s+a} \\
G_2(s) &= \frac{a}{s(s+a)}
\end{aligned}
$$

where a is a positive scalar.

Our goal is to compute the equivalent transfer function and compare it with the one obtained in the previous example.

其中 a 為正標量。

我們的目標是計算等效傳遞函數，並將其與前面示例中獲得的傳遞函數進行比較。

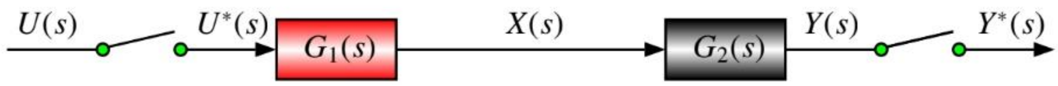

Fig. 4.12 Cascade transfer functions without sampler between

圖 4.12 之間沒有採樣器的級聯傳遞函數

In this case we have:

在這種情況下，我們有：

$$
\frac{Y^*(s)}{U^*(s)} = [G_1(s)G_2(s)]^*
$$

that gives in turn:

依次給出:

$$
\frac{Y^*(s)}{U^*(s)} = \mathscr{Z}[G_1(s)G_2(s)] = G_1G_2(z)
$$

It is important to notice that the equivalent transfer function we obtain for this case is different from the one we obtained for the system of the previous example.

值得要注意的事，在這種情況下，我們獲得的等效傳遞函數與在先前示例的系統中獲得的等效傳遞函數不同。

Using the expression of G1(s) and G2(s), we get:
使用 G1 和 G2 的表達式，我們得到：

$$G_1(s)G_2(s) = \frac{a^2}{s(s+a)^2}$$

Based on the table of Z -transform, we have:
根據 Z -transform 表，我們有：

$$\mathscr{Z}[G_1(s)G_2(s)] = G_1G_2(z) = \frac{z}{z-1} - \frac{z}{z-e^{-aT}} \frac{zaTe^{-aT}}{(z-e^{-aT})^2}$$

Example 4.3.3 In this example we consider the case where we have transfer functions in feedback and we search to compute the equivalent one as we did in the previous examples. The system is illustrated by the Fig. 4.13. The transfer functions are given by the following expression:
例 4.3.3 在本例中，我們考慮在反饋中具有傳遞函數，並且像在前面的例子中一樣搜索等效函數的情況。 該系統如圖 4.13 所示。 傳遞函數由以下表達式給出：

$$\begin{aligned} G(s) &= \frac{a}{s(s+a)} \\ H(s) &= \frac{1}{s} \end{aligned}$$

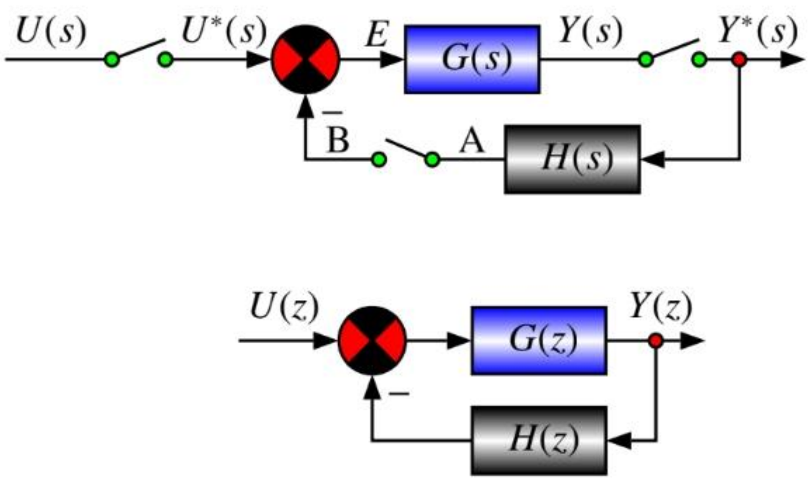

Fig. 4.13 Transfer functions in feedback

圖 4.13 反饋中的傳遞函數

Based on this figure we have:
根據此圖，我們可以：

$$
\begin{aligned}
A &= H(s)Y^*(s) \\
B &= [A]^* = [H(s)Y^*(s)]^* = H^*(s)Y^*(s) \\
E &= U^*(s) - B = U^*(s) - H^*(s)Y^*(s) \\
Y(s) &= G(s)\left[U^*(s) - H^*(s)Y^*(s)\right]
\end{aligned}
$$

that gives in turn:
依次給出：

$$Y^*(s) = [Y(s)]^* = [G(s)\left[U^*(s) - H^*(s)Y^*(s)\right]]^* = G^*(s)\left[U^*(s) - H^*(s)Y^*(s)\right]$$

From this we get:
由此我們得到：

$$\frac{Y^*(s)}{U^*(s)} = \frac{G^*(s)}{1 + G^*(s)H^*(s)}$$

That gives the following pulse transfer function:
這提供了以下脈衝傳遞函數：

$$\frac{Y(z)}{U(z)} = \frac{G(z)}{1 + G(z)H(z)}$$

From the table of Z -transform we get:
從 Z -transform 表中，我們得到：

$$
\begin{aligned}
G(z) &= \frac{z(1 - e^{-at})}{(z-1)(z - e^{-aT})} \\
H(z) &= \frac{z}{z-1}
\end{aligned}
$$

Using this we obtain:
使用此我們可以獲得：

$$\frac{Y(z)}{U(z)} = \frac{G(z)}{1 + G(z)H(z)} = \frac{\frac{z(1-e^{-at})}{(z-1)(z-e^{-aT})}}{1 + \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}\frac{z}{(z-1)}}$$

$$= \frac{(1 - e^{-aT})z(z-1)(z - e^{-aT})}{(z-1)^2(z - e^{-aT}) + z^2(1 - e^{-aT})}$$

Example 4.3.4 As a second example of the previous case let us consider the system of the Fig. 4.14. The question is how to compute the pulse transfer function F(z) = Y(z)/G(z) of this system.
例 4.3.4 作為前面情況的第二個例子，讓我們考慮圖 4.14 的系統。 問題是如何計算該系統的脈衝傳遞函數 F（z）= Y（z）/ G（z）。

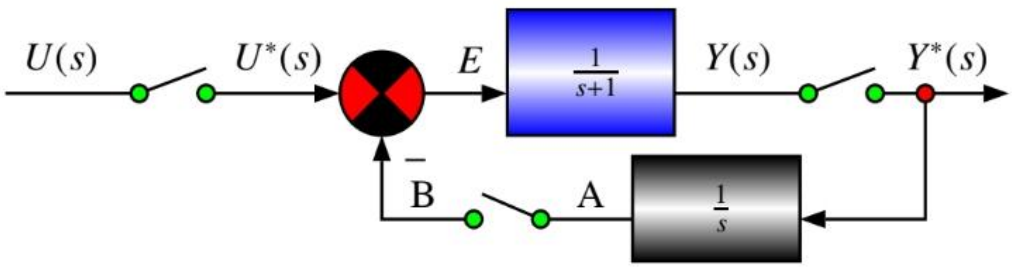

Fig. 4.14 Transfer functions in feedback

圖 4.14 反饋中的傳遞函數

Since (see the table for Z -transform)
由於（請參閱表 Z -transform）

$$G(z) = \mathscr{Z}\left[\frac{1}{s+1}\right] = \frac{z}{z - e^{-T}} \quad and\ H(z) = \mathscr{Z}\left[\frac{1}{s}\right] = \frac{z}{z-1}$$

we get the following expression for the closed-loop pulse transfer function:
對於閉環脈衝傳遞函數，我們得到以下表達式：

$$\frac{Y(z)}{U(z)} = \frac{G(z)}{1 + G(z)H(z)} = \frac{z(z-1)}{(z-e^{-T})(z-1)+z^2}$$

Example 4.3.5 In this example the system represented by the Fig. 4.15 where a zero order hold (ZOH) is used.

1. Find the open loop and closed loop pulse transfer functions Y(z)/U(z)
2. Find the unit-step response if K = 1 for T = 0.1

示例 4.3.5 在此示例中，圖 4.15 所示的系統使用零階保持（ZOH）。

1.找到開環和閉環脈衝傳遞函數 Y（z）/ U（z）

2.如果 K = 1 且 T = 0.1，則找到單位階躍響應

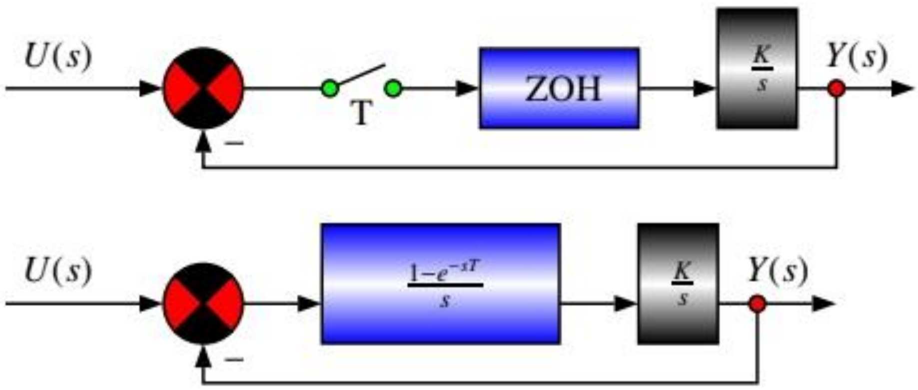

Fig. 4.15 Transfer functions in feedback

圖 4.15 反饋中的傳遞函數

The solution of this example can be obtained easily. In fact we have:

- Open loop:

此示例的解決方案可以輕鬆獲得。實際上，我們有：

- 開環：

$$\frac{Y(s)}{U(s)} = \frac{K}{s^2}(1-e^{-sT})$$

From which we have:

從中我們有：

$$\frac{Y(z)}{U(z)} = \frac{KTz}{(z-1)^2}\frac{z-1}{z} = \frac{KT}{z-1}$$

Finally we obtain:

最後我們得到：

$$Y(z) = \frac{KT}{z-1}U(z)$$

Closed loop:
閉環：

$$Y(z) = \frac{KT/(z-1)}{1+\frac{KT}{z-1}}U(z) = \frac{KT}{z-(1-KT)}\frac{z}{z-1}$$

Using the method of residues for z1 = 1 and z2 = 1 − KT, and the fact that K = 1, we find:
使用 z1 = 1 和 z2 = 1-KT 的殘差法，以及 K = 1，我們發現：

$$y(kT) \;= 1-(1-T)^k \quad pour \;\; k = 0,1,2,3,\cdots$$

If we use T = 0.1s, we get:
如果使用 T = 0.1s，則得到：

$$y(k) = 1-0.9^k$$

Example 4.3.6 Let us consider the system of the Fig. 4.16 and compute the transfer function.
示例 4.3.6 讓我們考慮圖 4.16 的系統併計算轉移功能。
Using this figure, we have:
使用此圖，我們可以：

$$\begin{aligned} E(s) \;&= R(s) - H(s)Y(s) \\ Y(s) \;&= G(s)E^\star(s) \end{aligned}$$

which gives in turn:
依次給出：

$$\begin{aligned} E^\star(s) \;&= R^\star(s) - [H(s)Y(s)]^\star \\ Y^\star(s) \;&= G^\star(s)E^\star(s) \end{aligned}$$

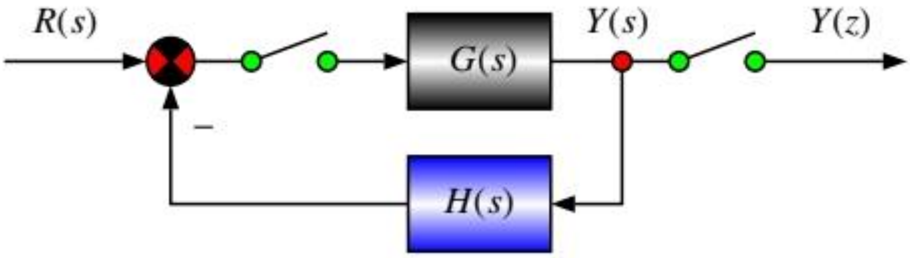

Fig. 4.16 Transfer functions in feedback

圖 4.16 反饋中的傳遞函數

Using the Z -transform, we obtain:

使用 Z 變換，我們獲得：

$$Y(z) = \frac{G(z)R(z)}{1 + GH(z)}$$

Example 4.3.7 Let us consider the system of the Fig. 4.17 and compute the transfer function.

例 4.3.7 讓我們考慮圖 4.17 的系統併計算轉移功能。

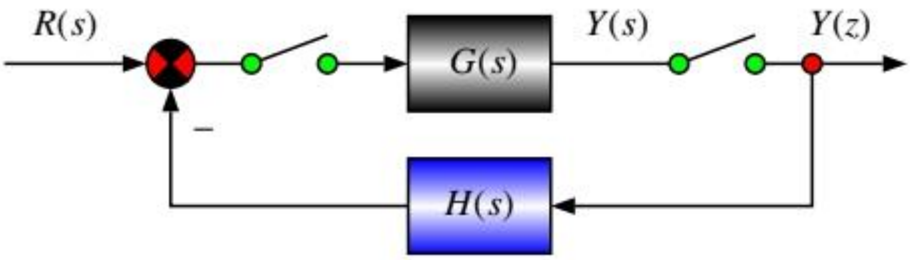

Fig. 4.17 Transfer functions in feedback

圖 4.17 反饋中的傳遞函數

Using this figure, we have:

使用此圖，我們可以：

$$\begin{aligned} E(s) &= R(s) - H(s)Y^{\star}(s) \\ Y(s) &= G(s)E^{\star}(s) \end{aligned}$$

which gives in turn:

依次給出：

$$
\begin{aligned}
E^{\star}(s) &= R^{\star}(s) - H^{\star}(s)Y^{\star}(s) \\
Y^{\star}(s) &= G^{\star}(s)E^{\star}(s)
\end{aligned}
$$

Using now the Z -transform, we obtain:
現在使用 Z 變換，我們得到：

$$Y(z) = \frac{G(z)R(z)}{1 + G(z)H(z)}$$

Example 4.3.8 Let us consider the dynamical system of the block diagram illustrated by Fig. 4.18
示例 4.3.8 讓我們考慮框圖的動態系統，如圖 4.18 所示

$$Y(z) = \frac{RG(z)}{1 + HG(z)}$$

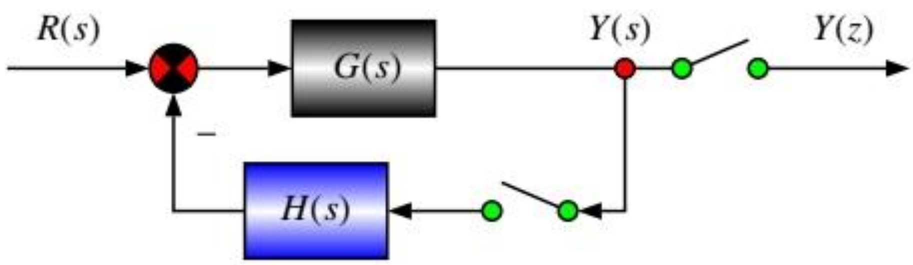

Fig. 4.18 Transfer functions in feedback

圖 4.18 反饋中的傳遞函數

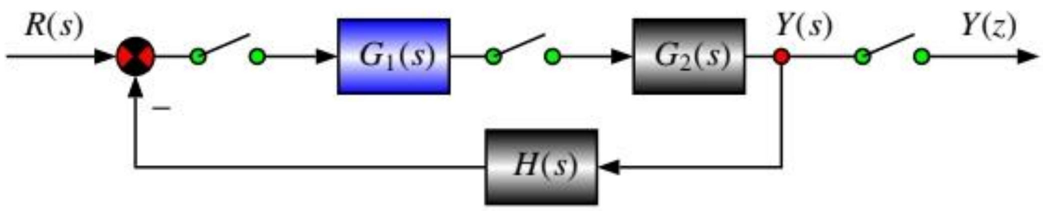

Fig. 4.19 Transfer functions in feedback

圖 4.19 反饋中的傳遞函數

Example 4.3.9 Let us consider the system of the block diagram of the figure 4.19 and compute the transfer function.
例 4.3.9 讓我們考慮圖 4.19 的框圖系統併計算傳遞函數

$$Y(z) = \frac{G_2(z)RG_1(z)}{1 + G_1G_2H(z)}$$

Example 4.3.10 Let us consider the system of the block diagram of the figure 4.20 and compute the transfer function.

例 4.3.10 讓我們考慮圖 4.20 的框圖系統併計算傳遞函數

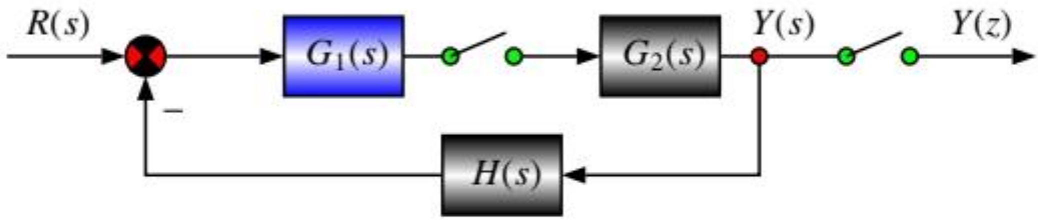

Fig. 4.20 Transfer functions in feedback

圖 4.20 反饋中的傳遞函數

Using this figure, we have:

使用此圖，我們可以：

$$\begin{aligned} E(s) &= R(s) - H(s)Y(s) \\ Y(s) &= [G(s)E(s)]^{\star} \end{aligned}$$

which gives in turn:

依次給出：

$$Y^{\star}(s) = \left[[(R(s) - H(s)Y(s))\, G_1(s)]^{\star} \right]^{\star} G_2(s)$$

Using now the Z -transform, we obtain:

現在使用 Z 變換，我們得到：

$$Y(z) = \frac{G_1(z)G_2(z)R(z)}{1 + G_1(z)G_2(z)H(z)}$$

Based on these examples, we are always able to compute the transfer function of the system and its expression is given by:

根據這些示例，我們始終能夠計算出該系統及其表達方式如下：

$$G(z) = \frac{Y(z)}{U(z)}$$

where Y(z) and U(z) are respectively the Z -transform of the output Y(s) and the input U(s).
其中 Y（z）和 U（z）分別是輸出 Y（s）和 Z 的 Z 變換，輸入 U（s）。
This transfer function is always in the following form:
此傳遞函數始終採用以下形式：

$$
\begin{aligned}
G(z) &= \frac{N(z)}{D(z)} \\
&= \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}
\end{aligned}
$$

where ai and bi are real scalars and n is an integer representing the degree of the system.
其中 ai 和 bi 是實數標量，n 是一個整數，表示整數的階數系統。
The roots of the polynomials N(z) and D(z), i.e.: the solutions of the following equations:
多項式 N（z）和 D（z）的根，即以下項的解等式：

$$
\begin{aligned}
N(z) &= 0 \\
D(z) &= 0
\end{aligned}
$$

are called respectively zeros and poles of the system.
分別稱為系統的零點和極點。
The poles play an important role in the system response. Their location is very important and it related to the system performances like the stability, the transient regime, etc. as it will be shown later on.
極點在系統響應中起重要作用。他們的位置非常很重要，並且與系統性能（例如穩定性，瞬態狀態等）有關。如稍後所示。
Example 4.3.11 Let us consider a dynamical system with the following transfer function:
例 4.3.11 讓我們考慮一個具有以下轉移的動力系統功能：

$$
\begin{aligned}
G(z) &= \frac{N(z)}{D(z)} \\
&= \frac{z^2 - z + 0.02}{z^3 - 2.4z^2 + z - 0.4}
\end{aligned}
$$

Compute the poles and zeros of the system and plot them in the z-domain.
From the expression of the transfer function we have:
計算系統的極點和零點並將它們繪製在 z 域中。

根據傳遞函數的表達式，我們有：

$$N(z) = z^2 - z + 0.02$$
$$D(z) = z^3 - 2.4z^2 + z - 0.4 = (z-2)\left(z^2 - 0.4z + 0.2\right)$$

The roots of this polynomials are 0.1± 0.1 j for the zeros and 2 and 0.2 ± 0.4 j for the poles. The zeros are all inside the unit circle. The complex poles are also inside the unit circle while the real one is outside this circle.
多項式的根為零時為 0.1±0.1 j，極點為 2 時為 0.2±0.4 j。零都在單位圓內。
複雜極點也位於單位圓內，而真實極點在此圓外。
We have introduced the concept of transfer function and we have learnt how to manipulate the block diagrams. It is now time to compute the time response of the system for given signal inputs. This is the subject of the next section.
我們介紹了傳遞函數的概念，並且學習瞭如何操作框圖。現在是時候計算傳感器的時間響應了。給定信號輸入的系統。這是下一部分的主題。

4.4 Time Response and Its Computation

4.4 時間響應及其計算

More often, the control system has to guarantee certain performances such as:
通常，控制系統必須保證某些性能，例如：

- the settling time at a given percentage　　穩定時間為給定百分比
- the overshoot　　過衝
- the damping ratio　　阻尼比

etc 等

For time definitions we ask the reader to look to the Fig. 4.21. To have an idea on the concept of the settling time, the overshoot, etc., let us consider a linear time invariant system with an input r(t) and an output y(t). If we apply a step function at the input, the output of this system will be as shown in Fig. 4.21. From this figure, it can be seen that the settling time is defined as the time for the system response, y(t) to reach the error band (that is defined with a certain percentage, 2 %, 5 %, etc.) and stay for the rest of the time. The lower the percentage is, the longer the settling time will be.
對於時間定義，我們要求讀者看一下圖 4.21。 為了對穩定時間，過衝等概念有所了解，讓我們考慮具有輸入 r（t）和輸出 y（t）的線性時不變系統。 如果我們在輸入上應用階躍函數，該系統的輸出將如圖 4.21 所示。 從這個數字來看，可以看到建立時間定義為系統響應時間，
y（t）達到誤差帶（定義為一定百分比，2％，5％等），並在其餘時間停留。 百分比越低，建立時間越長。

The overshoot is another characteristic of the time response of a given system.
If we refer to the previous figure, the overshoot is defined as the maximum exceed of the steady state value of the system output. More often, we use the percentage overshoot, which is defined as the maximum value of the output minus the step value divided by the step value.
The error is also another characteristic of the output behavior. It is defined as the difference between the steady value taken by the output and the desired value. For a closed-loop system with a unity feedback, the error, E(z), is defined mathematically as:
過衝是給定係統時間響應的另一個特徵。
如果我們參考上圖，則過衝定義為系統輸出的穩態值的最大超出值。
通常，我們使用百分比超調，它被定義為輸出的最大值減去步長值除以步長值。
錯誤也是輸出行為的另一個特徵。
它定義為輸出所獲得的穩定值與期望值之間的差。
對於具有統一反饋的閉環系統，誤差 E（z）在數學上定義為：

$$E(z) = R(z) - Y(z)$$

where R(z) is the reference input and Y(z) is the output.
Previously we developed tools that can be used to compute the expression in time of a given signal. Here we will use this to compute the time response of a given system to a chosen input that may be one or a combination of the following signals:
其中 R（z）是參考輸入，Y（z）是輸出。
以前，我們開發了可用於在給定信號時間內計算表達式的工具。
在這裡，我們將使用它來計算給定係統對選定輸入的時間響應，該輸入可能是以下信號之一或組合：

- Dirac impulse　狄拉克脈衝
- step　步
- ramp　坡道

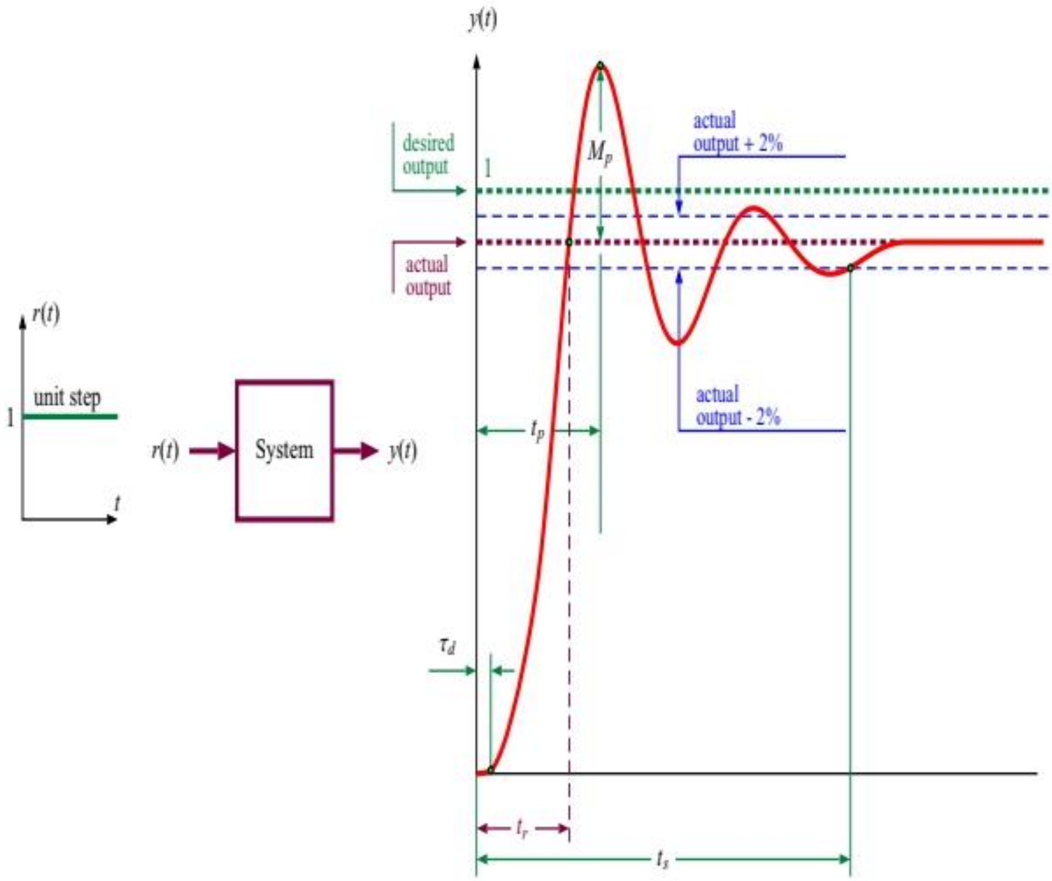

Fig. 4.21 Behavior of the time response for a step input

圖 4.21 步進輸入的時間響應行為

To compute the time response let us consider a system which has a pulse transfer function G(z) with a given input signal, U(z), and consider the computation of the expression of y(kT). The system is represented in Fig. 4.22. This figure may represent either an open loop pulse transfer function or its equivalent closed-loop pulse transfer function that we get after simplifying the system block diagram.

From this figure, we get:

為了計算時間響應，讓我們考慮一個具有給定輸入信號 U（z）的脈衝傳遞函數 G（z）的系統，並考慮 y（kT）表達式的計算。

該系統如圖 4.22 所示。 該圖可能代表開環脈衝傳遞函數或其等效的閉環脈衝傳遞函數，這是我們在簡化系統框圖後得到的。

從這個數字，我們得到：

$$Y(z) = G(z)U(z)$$

The computation of time response, y(kT), is brought to the computation of the inverse Z -transform that be can be determined using one of the following methods:
將時間響應 y（kT）的計算帶入 Z 逆變換的計算，可以使用以下方法之一確定該逆 Z 變換：

Fig. 4.22 Block diagram (BD)

- expansion into partial fraction　　擴展為部分分數
- polynomial division　　多項式除法
- residues method　　殘留法

To illustrate how the time response, let us consider the following examples.
為了說明時間響應如何，讓我們考慮以下示例。
Example 4.4.1 In this example we consider the speed control of a dc motor driving via a gear a given mechanical load. We assume that the system is controlled using a microcontroller. The transfer function of the system is given by:
例 4.4.1 在本例中，我們考慮了在給定機械負載下通過齒輪驅動的直流電動機的速度控制。 我們假設系統是使用微控制器控制的。 系統的傳遞函數由下式給出：

$$G(s) = \frac{K}{\tau s + 1}$$

with K = 2 and τ = 2.
其中 K = 2 和 τ = 2
The system is considered in open-loop. In this case since we have the presence of a ZOH, we obtain:
該系統被認為是開環的。 在這種情況下，由於我們存在
ZOH，我們獲得：

$$G(z) = \left(1 - z^{-1}\right) \mathscr{Z}\left[\frac{2}{s\,(2s + 1)}\right]$$

Using the Z -transform table, we get:
使用 Z -transform 表，我們得到：

$$G(z) = \left(1 - z^{-1}\right)\left[\frac{z\left(1 - e^{-\frac{T}{2}}\right)}{(z-1)\left(z - e^{-\frac{T}{2}}\right)}\right]$$
$$= \frac{\left(1 - e^{-\frac{T}{2}}\right)}{\left(z - e^{-\frac{T}{2}}\right)}$$

where T is the sampling period.For our system, since the time constant is equal to 2 sec, a proper choice for the sampling period is T = 0.2sec. Using this, we get:
對於我們的系統，由於時間常數等於 2 秒，因此採樣時間的適當選擇是 T = 0.2 秒。 使用這個，我們得到：

$$G(z) = \frac{0.0952}{z - 0.9048}$$

If now we consider that the signal input is unit step, we get
如果現在我們認為信號輸入是單位步長，我們得到

$$Y(z) = \frac{0.0952z}{(z-1)(z-0.9048)}$$

To compute the time response either we can use the table or proceed with the expansion into partial fraction.
要計算時間響應，我們可以使用表格或將其擴展為部分分數。
Using the Z -transform table, we have:
使用 Z -transform 表，我們得到：

$$y(kT) = 1 - e^{-0.1k}$$

With the expansion into partial fraction we obtain:
通過擴展為部分分數，我們得到：

$$\frac{Y(z)}{z} = \frac{0.0952}{(z-1)(z-0.9048)}$$
$$= \frac{K_1}{z-1} + \frac{K_2}{z-0.9048}$$
$$= \frac{1}{z-1} + \frac{-1}{z-0.9048}$$

From this we get:
由此我們得到：

$$Y(z) = \frac{z}{z-1} + \frac{-z}{z-0.9048}$$

Using now the Z -transform table, we get:
使用 Z -transform 表，我們得到：

$$y(kT) = 1 - e^{-0.1k}$$

$since\ e^{-0.1} = 0.9048.$

Example 4.4.2 In this example we consider the position control of a dc motor driving via a gear a given mechanical load. We assume that the system is controlled using a microcontroller. The transfer function of the system is given by:
例 4.4.2 在本例中，我們考慮了在給定機械負載下通過齒輪驅動的直流電動機的位置控制。 我們假設系統是使用微控制器控制的。 系統的傳遞函數由下式給出：

$$G(s) = \frac{K}{s(\tau s + 1)}$$

with K = 2 and τ = 2. 其中 K = 2 和 τ = 2
The system is considered in open-loop. In this case since we have the presence of a ZOH, we obtain:
該系統被認為是開環的。 在這種情況下，由於我們存在 ZOH，因此我們獲得：

$$G(z) = \left(1 - z^{-1}\right) \mathscr{Z}\left[\frac{2}{s^2(2s+1)}\right]$$

Using the Z -transform table with T = 0.2second, we get:
使用 T = 0.2 秒的 Z 轉換表，我們得到：

$$\begin{aligned} G(z) &= \left(1 - z^{-1}\right)\left[\frac{Tz}{(z-1)^2} - \frac{z\left(1 - e^{-\frac{T}{2}}\right)}{0.5(z-1)\left(z - e^{-\frac{T}{2}}\right)}\right] \\ &= \frac{(0.4048z - 0.5476)}{0.5(z-1)(z-0.9048)} \end{aligned}$$

If now we consider that the signal input is unit step, we get
如果現在我們認為信號輸入是單位步長，我們得到

$$Y(z) = 0.8096\frac{z(z-1.3528)}{(z-1)^2(z-0.9048)}$$

To compute the time response either we can use the Z -transform table or proceed with the method of expansion into partial fraction or with the method of residues.
為了計算時間響應，我們可以使用 Z 變換錶，也可以採用擴展為部分分數的方法或採用殘差的方法。

Using the Z -transform table, we get:
使用 Z -transform 表，我們得到：

$$y(kT) = kT - \frac{1}{a}\left[1 - e^{akT}\right]$$

with a = 0.5 and T = 0.2 a = 0.5 和 T = 0.2

With the method expansion into partial fraction we have:
通過將方法擴展為部分分數，我們可以：

$$Y(z) = \frac{K_1}{(z-1)^2} + \frac{K_2}{(z-1)} + \frac{K_3}{(z-0.9048)}$$

With the method of residues, we obtain:
用殘基的方法，我們得到：

$$y(kT) = \sum residues\ of\ 0.8096\frac{z(z-1.3528)z^{k-1}}{(z-1)^2(z-0.9048)}$$

at the poles z = 1 and z = 0.9048.
These residues are computed as follows:
在極點 z = 1 和 z = 0.9048 處。
這些殘基計算如下：
residue at pole z = 1 z 極處的殘差= 1

$$\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-1)^2 0.8096\frac{z(z-1.3528)z^{k-1}}{(z-1)^2(z-0.9048)}\right]_{|z=1}$$
$$= \frac{d}{dz}\left[0.8096\frac{(z-1.3528)z^k}{(z-0.9048)}\right]_{|z=1}$$
$$= 119.0464 - 2.9568k$$

residue at pole z = 0.9048 z 極處的殘留量= 0.9048

$$\left[0.8096\frac{(z-1.3528)\,z^k}{(z-1)^2}\right]_{|_{z_r=0.9048}}$$
$$= -40.0198\,(0.9048)^k$$

Using now the table we get:
現在使用該表，我們得到：

$$y(kT) = 1 - e^{-0.1k}$$

since e−0.1 = 0.9048.
因為 e-0.1 = 0.9048。
From the time response we computed in the previous section, it can be seen that for a given system the output can take either finite or infinite value for a given signal signal input. The question is why this happen. The answer of this question is given by the stability analysis and this will be covered in the next section.
從上一節中計算的時間響應可以看出，對於給定的系統，對於給定的信號輸入，輸出可以取有限值或無限值。 問題是為什麼會這樣。 這個問題的答案由穩定性分析給出，這將在下一部分中介紹。
4.5 Stability and Steady-State Error
4.5 穩定性和穩態誤差
For systems in the continuous-time domain, the stability implies that all the poles must have negative real parts. With the transform z = eT s, with T is the sampling period, we saw that the left half plane of the s-domain corresponds to the inside unit circle and therefore, in the z-domain, the system will be stable if all the poles are inside this unit circle.
對於連續時間域中的系統，穩定性意味著所有極點都必須具有負實部。 在變換 z = eT s 的情況下，以 T 為採樣週期，我們看到 s 域的左半平面對應於內部單位圓，因此，在 z 域中，如果所有 極點在此單位圓內。
To analyze the stability of discrete-time systems, let us consider the system of the Fig. 4.23. The closed loop transfer function of this system is given by:
為了分析離散時間系統的穩定性，讓我們考慮圖 4.23 的系統。 該系統的閉環傳遞函數由下式給出：

$$F(z) = \frac{Y(z)}{R(z)} = \frac{C(z)G(z)}{1 + C(z)G(z)}$$

where R(z) and Y(z) are respectively the input and the output.
The poles of the system are the solution of the following characteristic equation:
其中 R（z）和 Y（z）分別是輸入和輸出。

系統的極點是以下特徵方程的解：

$$1 + C(z)G(z) = 0$$

The study of stability requires the computation of these roots. For small order system we can always solve the characteristic equation by hand and then obtain the poles and the conclusion on stability will be done based on the fact where the poles are located. For high order this approach is not recommended and an alternate is needed. Some criterions have been developed to study the stability. Among these criterions we quote the one of Jury and the one of Raible.

穩定性的研究需要計算這些根。 對於小階系統，我們總是可以手動求解特徵方程，然後求出極點，並根據極點所在的事實得出穩定性結論。 對於高階，不建議使用此方法，而需要替代方法。 已經開發了一些標準來研究穩定性。 在這些標準中，我們引用了一個陪審團和一個 Raible。

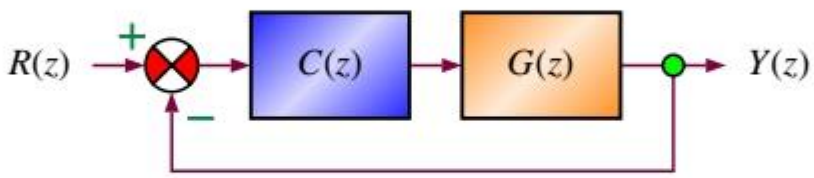

Fig. 4.23 Block diagram of the closed-loop

圖 4.23 閉環框圖

Let $z = e^{sT}$ with $s = \sigma \pm j\omega$. Therefore,

令 z = esT，其中 s = σ ±jω。 因此，

if σ < 0 then |z| < 1 and the system is stable

if σ > 0 then |z| > 1 and the system is unstable

if σ = 0 then |z| = 1 and the system is at the limit of stability

如果σ <0，則| z | <1 並且系統穩定

如果σ > 0，則| z | > 1，系統不穩定

如果σ = 0，則| z | = 1，系統處於穩定性極限

Example 4.5.1 Let us consider a dynamical system with the following characteristic equation:

例 4.5.1 讓我們考慮一個具有以下特徵方程的動力系統：

$$1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = 0$$

The roots of the characteristic equation are: z = 1/2 and z = 1/4 . These roots are located inside the unit circle and therefore the system is stable.
特徵方程的根為：z = 1/2 和 z = 1/4。 這些根位於單位圓內，因此系統穩定。
Example 4.5.2 Let us consider a dynamical system with the following characteristic equation:
例 4.5.2 讓我們考慮一個具有以下特徵方程的動力系統：

$$1 - 2z^{-1} + \frac{5}{4}z^{-2} = 0$$

or equivalently:
或等效地：

$$z^2 - 2z + \frac{5}{4} = 0$$

The roots of the system are z1,2 = 1 ± j1/2 and are both outside the unit circle which implies that the system is unstable.
系統的根是 z1,2 = 1±j1 / 2，並且都在單位圓之外，這表示系統不穩定。
A direct approach to study the stability of discrete-time system is to convert it to an equivalent continuous-time one, and then use the Routh-Hurwitz's Criterion.
The idea is to find an adequate application that maps the inside of the unit circle onto the left-hand half plane. Then, we can apply the Routh-Hurwitz criterion. The transformation we're looking for is:
研究離散時間系統穩定性的直接方法是將其轉換為等效的連續時間系統，然後使用 Routh-Hurwitz 的準則。
這個想法是找到一個合適的應用程序，將單位圓的內部映射到左側的半平面上。然後，我們可以應用 Routh-Hurwitz 準則。 我們正在尋找的轉變是：

$$z = \frac{1 + w}{1 - w} \text{ with } w \neq 1$$

Replacing z by this expression in the characteristic equation will give a new one in w and we can apply the Routh-Hurwitz's Criterion.
在特性方程式中用該表達式替換 z 將得到 w 的一個新值，我們可以應用 Routh-Hurwitz 的準則。
Example 4.5.3 To show how we use the Routh-Hurwitz's Criterion, let us consider the dynamical system with the following characteristic equation:
例 4.5.3 為了展示我們如何使用勞斯·赫維茲準則，讓我們考慮具有以下特徵方程的動力學系統：

$$z^3 - 2.4z^2 + z - 0.4 = 0$$

It can be shown that the poles are 2 and 0.2 ± 0.4 j. Therefore the system is unstable.

Let us now replace z by 1+w/1−w in the characteristic equation. This gives:

可以看出極點為 2 和 0.2±0.4 j。 因此系統不穩定。

現在讓我們在特徵方程中將 z 替換為 1 + w / 1-w。 這給出：

$$\left[\frac{1+w}{1-w}\right]^3 - 2.4\left[\frac{1+w}{1-w}\right]^2 + \left[\frac{1+w}{1-w}\right] - 0.4 = 0$$

which can be put in the following form:

可以採用以下形式：

$$4.8w^3 + 3.2w^2 + 0.8w - 0.8 = 0$$

The Routh-Hurwitz's Criterion consists then of filling the following table:

勞斯·赫維茲準則的條件包括填寫下表：

$$\begin{array}{c|lll} w^3 & 4.8 & 0.8 & 0 \\ w^2 & 3.2 & -0.8 & 0 \\ \hline w^1 & 2 & 0 & \\ w^0 & -0.8 & & \end{array}$$

Based on the first column, we can see that there one change in the sign and therefore the system is unstable. This confirm the results we has already remarked earlier.

在第一列的基礎上，我們可以看到符號發生了變化，因此系統不穩定。 這證實了我們前面已經提到的結果。

It is also important to notice that the roots of the characteristic equation in w are given by:

還需要注意的是，w 中特徵方程的根由下式給出：

$$\begin{aligned} w_1 &= 0.3333 \\ w_{2,3} &= -0.5000 \pm 0.5000j \end{aligned}$$

These roots can also be obtained from the ones in z-domain using w = z−1/z+1

這些根也可以使用 w = z−1 / z + 1 從 z 域中的根獲得。

Example 4.5.4 Consider the characteristic equation:

例 4.5.4 考慮特徵方程：

$$z^2 + z(6.32K - 1.368) + 0.368 = 0$$

Applying the bilinear transform yields:
應用雙線性變換可得出：

$$\left(\frac{1+w}{1-w}\right)^2 + \left(\frac{1+w}{1-w}\right)(6.32K - 1.368) + 0.368 \ = 0$$

that gives in turn:
依次給出：

$$w^2[2.736 - 6.32K] + 1.264w + (6.32K - 1) \ = 0$$

Applying Routh-Hurwitz gives:
應用勞斯·赫維茲準則可以得出：

w^2	$2.736 - 6.32K$	$6.32K - 1$
w^1	1.264	0
w^0	$6.32K - 1$	

To guarantee the stability we should determine the range of the parameter K such that we don’t have sign change in the first column. For the row w0, we should have 6.32K − 1 > 0, i.e. K > 1/6.32 = 0.158. For the row w2, we should also have2.736 − 6.32K > 0, i.e. K < 2.736/6.32 = 0.4329. If we look to these two conditions, we conclude that the system is stable for 0.158 < K < 0.4349.
為了保證穩定性，我們應該確定參數 K 的範圍，以使第一列中的符號沒有變化。對於 w0 行，我們應該有 6.32K − 1> 0，即 K> 1 / 6.32 = 0.158。 對於 w2 行，我們還應該有 2.736 − 6.32K> 0，即 K <2.736 / 6.32 = 0.4329。 如果我們看這兩個條件，可以得出結論，系統對於 0.158 <K <0.4349 是穩定的。
To check this, let us consider K = 0.2, which is inside the interval. Using this value, we obtain the following characteristic equation:
為了檢查這一點，讓我們考慮在間隔內的 K = 0.2。 使用該值，我們獲得以下特徵方程式：

$$z^2 - 0.104z + 0.368 = 0$$

that has as roots z1 = 0.052 + j0.6044 and z2 = 0.052 − j0.6044. The roots are located inside the unit circle and therefore, the system is then stable. For K = 1, we obtain:
其根為 z1 = 0.052 + j0.6044 和 z2 = 0.052-j0.6044。 根位於單位圓內，因此系統穩定。 對於 K = 1，我們獲得：

$$z^2 + 4.952z + 0.368 = 0$$

The roots are z1 = −0.076 and z2 = −4.876. The system is then unstable because|z2|> 1.

根是 z1 = -0.076 和 z2 = -4.876。 由於| z2 |> 1，因此系統不穩定。

For discrete-time Jury has developed a criterion that gives an idea on stability of any system without solving the characteristic equation.

To show how this approach works, let us consider the following characteristic polynomial with real coefficients:

對於離散時間，陪審團已經制定了一個準則，該準則給出了任何系統穩定性的思想，而無需求解特徵方程。

為了說明這種方法是如何工作的，讓我們考慮以下具有實係數的特徵多項式：

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

where an > 0 and ai is a real scalar.

其中 an> 0 且 ai 是實標量。

Jury's stability criterion consists of building the following array of coefficients:

評審團的穩定性標準包括建立以下一系列係數：

$$\begin{array}{l|l|l|l|l|l|l|l|l|}
\text{row } 1 & a_0 & a_1 & a_2 & \cdots & a_{n-k} & \cdots & a_{n-1} & a_n \\
\text{row } 2 & a_n & a_{n-1} & a_{n-2} & \cdots & a_k & \cdots & a_1 & a_0 \\
\text{row } 3 & b_0 & b_1 & b_2 & \cdots & b_{n-k} & \cdots & b_{n-1} & \\
\text{row } 4 & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_k & \cdots & b_0 & \\
\text{row } 5 & c_0 & c_1 & c_2 & \cdots & & c_{n-2} & & \\
\text{row } 6 & c_{n-2} & c_{n-3} & c_{n-4} & \cdots & & c_0 & & \\
\vdots & \vdots & \vdots & \vdots & \cdots & & & & \\
\text{row 2n-5} & p_0 & p_1 & p_2 & p_3 & & & & \\
\text{row 2n-4} & p_3 & p_2 & p_1 & p_0 & & & & \\
\text{row 2n-3} & q_0 & q_1 & q_2 & & & & &
\end{array}$$

The Jury's array coefficients are computed as follows:

評審團的陣列係數計算如下：

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix},$$

$$d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \qquad \cdots$$

$$q_0 = \begin{vmatrix} p_0 & p_3 \\ p_3 & p_0 \end{vmatrix}, \qquad q_2 = \begin{vmatrix} p_0 & p_1 \\ p_3 & p_2 \end{vmatrix}$$

The necessary and sufficient conditions that the system described by P(z) is stable are:

P（z）描述的系統穩定的必要和充分條件是：

$$P(1) > 0$$
$$P(-1) \quad \begin{cases} > 0 & \text{if n is even} \\ < 0 & \text{if n is odd} \end{cases}$$

$$\text{with (n - 1) constraints} \quad \begin{bmatrix} |a_0| < a_n & |b_0| > |b_{n-1}| \\ |c_0| > |c_{n-2}| & |d_0| > |d_{n-3}| \\ \cdots & \cdots \\ |q_0| > |q_2| & \end{bmatrix}$$

Example 4.5.5 Examine the stability of the system described by the following polynomial:

示例 4.5.5 檢查以下多項式描述的系統的穩定性：

$$1 + K\frac{z}{(z-1)(z-0.4)} = 0$$

where K is a parameter to determine such that the system is stable.

This characteristic equation can be rewritten as follows:

其中 K 是確定參數以使系統穩定的參數。

此特徵方程式可重寫如下：

$$z^2 + (K - 1.4)z + 0.4 = 0$$

Applying Jury criterion gives:

- P(1) > 0, which gives K > 0
- P(−1) > 0 which gives K < 2.8

・ |a0| < an, i.e: 0.4 < 1 which is true
應用評審標準可得出：
・P（1）> 0，使 K> 0
・P（−1）> 0，這使得 K <2.8
・| a0 | <an，即：0.4 <1，這是正確的
Therefore, our system will be stable if K ∈]0, 2.8[. For instance, if we fix K to 2, which gives the following characteristic equation:
因此，如果 K∈] 0，2.8 [，我們的系統將是穩定的。 例如，如果我們將 K 固定為 2，
給出以下特徵方程式：

$$z^2 + 0.6z + 0.4 = 0$$

the roots are z1,2 = −0.3000±0.5568 j which are inside the unit circle since |z1,2| < 1.
根是 z1,2 = -0.3000±0.5568 j，由於| z1,2 |而在單位圓內 <1。
Another criterion to study the stability bas been developed by Raible. This sta-bility Criterion consists also as for the Jury criterion to fill an array and then conclude on stability. To show how this criterion works, let us consider the following characteristic equation:
Raible 提出了另一個研究穩定性基礎的標準。 此穩定性標準還包括作為陪審團標準來填充數組，然後得出穩定性的結論。 為了說明此標準的工作原理，讓我們考慮以下特徵方程式：

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

where ai is a real scalar.
其中 ai 是真正的標量。

row 1	a_0	a_1	$\cdots$	a_{n-1}	a_n	multiplier
row 2	a_n	a_{n-1}	$\cdots$	a_1	a_0	$\alpha_n = \frac{a_n}{a_0}$
row 3	$a_0^{(n-1)}$	$a_1^{(n-1)}$	$\cdots$	$a_{n-1}^{(n-1)}$	0	multiplier
row 4	$a_{n-1}^{(n-1)}$	$a_{n-2}^{(n-1)}$	$\cdots$	$a_0^{(n-1)}$	0	$\alpha_{n-1} = \frac{a_{n-1}^{(n-1)}}{a_0^{(n-1)}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
row 2n-1	$a_0^{(1)}$	$a_1^{(1)}$				multiplier
row 2n	$a_1^{(1)}$	$a_0^{(1)}$				$\alpha_1 = \frac{a_1^{(1)}}{a_0^{(1)}}$
row 2n+1	$a_0^{(0)}$					

・ The 1st row is formed by the polynomial coefficients
・ The 2nd row is formed by the same coefficients but in the opposite order
・ The 3rd row is obtained by multiplying the 2nd row by αn = an/a0, then by subtracting the result of the 1st row
・ The 4th row is formed by coefficients of the 3rd row placed in the opposite order.
・第一行由多項式係數組成
・第二行由相同的係數組成，但順序相反
・通過將第二行乘以α n= an / a0，然後減去第一行的結果來獲得第三行
・第 4 行由第 3 行的係數以相反的順序排列而成。
These procedures are repeated until the array gets 2n + 1 rows. The last row contains only one number.
重複這些過程，直到數組獲得 2n +1 行。 最後一行僅包含一個數字。
Raible’s Stability Criterion
Raible 的穩定性標準
When a0 > 0, the roots of the polynomial are all inside the unit circle if and only if a(i)0 > 0, i = 0, 1, ⋯ , n – 1
當 a0> 0 時，且僅當 a（i）0> 0，i = 0，1，⋯，n − 1 時，多項式的根都在單位圓內
The coefficients a(i)0 > 0, i = 0, 1, ⋯ , n − 1 appear in the Raible’s array .
係數 a（i）0> 0，i = 0，1，⋯，n − 1 出現在 Raible 數組中。

Remark 4.5.1 The assumption a0 > 0 is not restrictive. In fact, when a0 < 0, it is enough to change the signs of all coefficients of the polynomial P(z) to obtain −P(z), which in turn is used for Raible’s criterion.
備註 4.5.1 假設 a0> 0 不是限制性的。 實際上，當 a0 <0 時，足以改變多項式 P（z）的所有係數的符號以獲得-P（z），
依次用於 Raible 的標準。
This procedure is correct since the roots of P(z) and of −P(z) are identical.
該過程是正確的，因為 P（z）和-P（z）的根相同。
Example 4.5.7 To show how the Raible’s criterion works, let us consider the following characteristic equation:
例 4.5.7 為了說明 Raible 準則是如何工作的，讓我們考慮以下特徵方程式：

$$P(z) = -z^3 - 0.7z^2 - 0.5z + 0.3$$

The coefficient a0 must be positive, then we form the coefficient array of the polynomial −P(z) = z^3 + 0.7z^2 + 0.5z − 0.3

係數 a0 必須為正，然後形成多項式的係數數組-P（z）= z ^ 3 + 0.7z ^ 2 + 0.5z-0.3

1	0.7	0.5	−0.3	
−0.3	0.5	0.7	1	$\alpha_3 = \frac{0.3}{-1} = -0.3$
0.91	0.85	0.71		
0.71	0.85	0.91		$\alpha_2 = \frac{0.71}{0.91} = 0.78$
0.36	0.19			
0.19	0.36			$\alpha_1 = \frac{0.19}{0.36} = 0.53$
0.26				

The system is stable because a(i)0 > 0, i = 0, 1, ··· , n – 1
系統穩定是因為 a（i）0> 0，i = 0，1，···，n – 1
We have presented some techniques to study the stability of discrete-time sys-tems. It is also important to notice that we can also apply the criterions in the frequency domain.
我們提出了一些技術來研究離散時間系統的穩定性。 還必須注意，我們也可以在頻域中應用這些準則。

4.6 Root Locus Technique

4.6 根軌跡技術

The root locus technique is a powerful approach that is usually used for continuous-time or discrete-time systems either for analysis or design. The technique gives an idea on how the poles of the closed-loop dynamics behave when a gain or more (aparameter or more) are changed. The direct conclusion is that we know immedi-ately how the stability and the other performances of the system are affected by the parameters changes.
根軌跡技術是一種強大的方法，通常用於連續時間或離散時間系統以進行分析或設計。 該技術給出了當增益或更大（參數或更大）改變時閉環動力學的極點表現方式的想法。 直接的結論是，我們立即知道參數的變化如何影響系統的穩定性和其他性能。
Nowadays there exist many tools to plot the root loci of any dynamical system some of them are available free for use. In the rest of this section, we will use Matlab for our plotting but we will develop rules of how obtain a sketch of the root locus in case we don’t have a computer at hand.
如今，有許多工具可以繪製任何動態系統的根基因座，其中一些是免費提供的。 在本節的其餘部分中，我們將使用 Matlab 進行繪圖，但是我們將製定規則，以防萬一沒有計算機時如何獲取根軌跡草圖。
As for the continuous case, the root locus for the discrete system is described by the characteristic equation that we write in the following form:
對於連續情況，離散系統的根軌跡由我們用以下形式編寫的特徵方程式描述：

$$1 + KG(z) = 0$$

where K is the parameter that varies and
其中 K 是變化的參數，

$$G(z) = \frac{(z-n_1)(z-n_2)\cdots(z-n_m)}{(z-z_1)(z-z_1)\cdots(z-z_n)}$$

with z1,z2, ⋯ ,zn are the poles and n1, n2, ⋯ , nm are the zeros of the open loop transfer function.
其中 z1，z2，... n 是極點，n1，n2，... nm 是開環傳遞函數的零點。
When the parameter K varies from 0 to infinity (∞). The same rules as we use for the plotting of the root locus of the continuous-time systems in the s-plane apply to the plotting of the one of discrete-time systems in the z-plane, except that the interpretation of the results is different mainly in regard of stability.
當參數 K 從 0 變為無窮大（∞）時。 與在 s 平面中繪製連續時間系統的根軌跡所用的規則相同，適用於在 z 平面中繪製離散時間系統之一的圖，除了對結果的解釋是 主要在穩定性方面有所不同。
From the characteristic equation, we get the following conditions:
從特徵方程式，我們得到以下條件：

$$\frac{1}{K} = \frac{\Pi_{i=1}^{m}|z-n_i|}{\Pi_{i=1}^{n}|z-z_i|} \tag{4.4}$$

$$\sum_{i=1}^{m}\arg(z-n_i) - \sum_{i=1}^{n}\arg(z-z_i) = (2k+1)\pi, k = 0, 1, 2, \cdots, \tag{4.5}$$

The first condition is referred to as the magnitude condition while the secon is referred to as angle condition. Any point in the z-plane that satisfies these two conditions belongs to the root locus of the system. To this point corresponds a gain Kz0 . If this point is z0, then we have:
第一個條件稱為幅度條件，而第二個條件稱為角度條件。 Z 平面中滿足這兩個條件的任何點都屬於系統的根軌跡。 在這一點上對應於增益 Kz0。 如果這點是 z0，那麼我們有：

$$\frac{1}{K_{z_0}} = \frac{\Pi_{i=1}^{m}|z_0-n_i|}{\Pi_{i=1}^{n}|z_0-z_i|}$$

$$\sum_{i=1}^{m}\arg(z_0-n_i) - \sum_{i=1}^{n}\arg(z_0-z_i) = \theta_0$$

where θ0 is the corresponding angle of this point.
其中θ0 是該點的對應角度。
A point of the z-plane will belong to the root locus, if it satisfies these two condi-tions. In general plotting the exact root locus for a given system is a hard task unless we have the appropriate tools for that. More often a sketch of this root locus can be easily obtained using some simple rules. Some of these rules are:
如果滿足上述兩個條件，則 z 平面的點將屬於根軌跡。
通常，除非給我們提供合適的工具，否則為給定係統繪製確切的根軌跡是一項艱鉅的任務。
通常，可以使用一些簡單的規則輕鬆獲得此根軌蹟的草圖。
其中一些規則是：

1. the number of branches is equal to the order of the system, i.e.: n;
1. 分支的數量等於系統的階數，即 i.e.:n;
2. the root locus is symmetric with respect to the real axis. This is due to the fact that the roots of the characteristic equation are either real or complex. And if there is a complex root, we have automatically its conjugate.
2. 根軌跡相對於實軸對稱。 這是由於特徵方程的根是實數還是複數的事實。 如果存在復雜的根，我們將自動獲得其共軛。
3. The loci originate from the poles of the open loop transfer function and termi-nate on the zeros of the this transfer function. To explain why the loci originate from the poles, we can make K equal to zero, while why the loci terminate on the zeros can be explained by letting K goes to ∞ in Eq. (4.4).
3. 軌跡起源於開環傳遞函數的極點，並終止於該傳遞函數的零點。 為了解釋為什麼基因座起源於極點，我們可以使 K 等於零，而為什麼基因座終止於零點可以通過讓 K 在等式中變為∞來解釋。 （4.4）。
4. the number of asymptotes is equal to the difference between the number of poles, n, and the number of zeros, m, of the open loop transfer function. These asymptotes are characterized by:
4. 漸近線的數量等於開環傳遞函數的極數 n 和零數 m 之差。 這些漸近線的特徵是：

$$\delta = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$$
$$\beta_k = (2k+1)\frac{\pi}{n-m}, k = 0, 1, 2, \cdots,$$

The parameter, δ, gives the intersections of the asymptotes with the real axis,while βk gives the angle that make each asymptote with the real axis.
參數δ給出漸近線與實軸的交點，而βk 給出使每個漸近線與實軸的夾角。

5. for the breakpoints of the root locus, firstly we determine the expression of the varying parameters K, i.e.:
5. 對於根軌蹟的斷點，首先我們確定各種參數 K 的表達式，即：

$$K = \frac{\Pi_{i=1}^{n}|z - z_i|}{\Pi_{i=1}^{m}|z - n_i|}$$

The breakpoints are solution of the following equation:
斷點是以下方程式的解：

$$\frac{dK}{dz} = 0$$

It is important to select from the roots of this equation those are feasible solution for the breakpoints.
從方程式的根中選擇對於斷點而言可行的解決方案非常重要。

6. the intersection of the imaginary axis in the z-plane can be determined by replacing z by jv in the characteristic equation and write it as follows:
6. 通過在特性方程中用 jν 替換 z 並確定，可以確定 z 平面中虛軸的交點，並將其寫為：

$$\Re(K, \nu) + j\Im(K, \nu) = 0$$

that gives in turn two equations:
依次給出兩個方程式：

$$\begin{aligned} \Re(K, \nu) &= 0 \\ \Im(K, \nu) &= 0 \end{aligned}$$

The solution gives the frequency at which the intersection occurs and the corresponding gain.
該解給出了相交發生的頻率和相應的增益。

7. the angle of departure from a complex pole or the angle of arrival to a complex zero is computed using the angle condition. If the point at which we want to calculate the angle is z0, the condition angle becomes:
7. 使用角度條件計算從復數極點出發的角度或到零點的到達角度。 如果要計算角度的點為 z0，則條件角度變為：

$$\sum_{i=1}^{m} \arg(z_0 - n_i) - \sum_{i=1}^{n} \arg(z_0 - z_i) - \theta_0 = 180$$

where θ0 is the corresponding angle of this point.
其中 θ 0 是該點的對應角度。

Example 4.6.1 To show how the technique of root locus works, let us consider the system of the Fig. 4.24 where the plant is the double integrator and the controller is a proportional action with a gain K, that we will assume to change between zero and infinity for some physical reasons like heating, aging, etc.
為了說明根源技術是如何工作的，讓我們考慮圖 4.24 中的系統，其中設備是雙積分器，而控制器是具有增益 K 的比例動作，由於某些物理原因（例如加熱，老化等），我們將假定在零和無窮大之間進行切換。
Using the Z -transform table and the expression of the closed-loop transfer function we get the following characteristic equation of this system:
使用 Z 變換錶和閉環傳遞函數的表達式，我們得到了該系統的以下特徵方程：

$$1 + K\frac{(z+1)}{(z-1)^2} = 0, with\ \ K = \frac{k}{2}$$

- Number of branches: n = 2
- Finite number of branches: m = 1
- Infinite number of branches: n − m = 2 − 1 = 1
- Angle of asymptotes: β = π(2k+1)/n−m = π(2k+1)/2−1 = π, k = 0

・分支數量：n = 2
・有限的分支數量：m = 1
・無限數量的分支：n − m = 2 − 1 = 1
・漸近角：β =π（2k + 1）/ n−m =π（2k + 1）/ 2-1 =π，k = 0

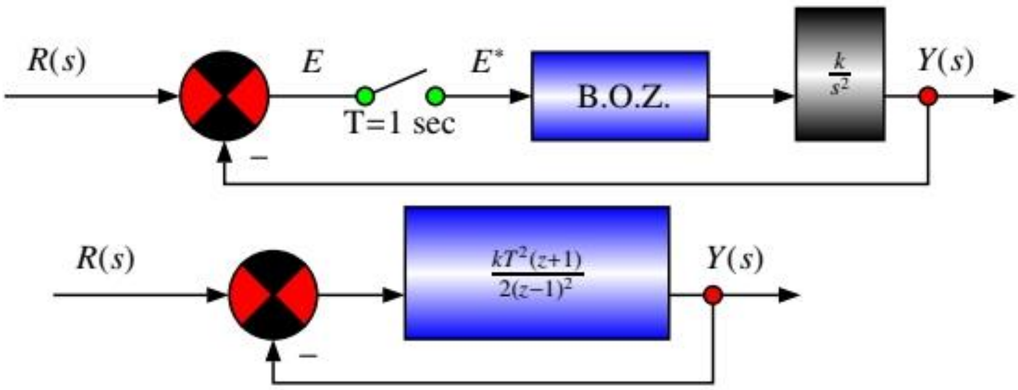

Fig. 4.24 BD of the system with characteristic eqn: 1 + K (z+1)/(z−1)2 = 0
具有特徵方程的系統的 BD: 1 + K (z+1)/(z−1)2 = 0

・ Intersection of the asymptote with the real axis:

δ = (1)+(1)−(−1)/2−1 = 3

・ Intersection of the locus with the real axis: dK/dz = 2z2 + 4z − 6 = 0, which gives z1 = −1 et z2 = −3.

・漸近線與實軸的交點：

δ =（1）+（1）-（-1）/ 2-1 = 3

・軌跡與實軸的交點：dK / dz = 2z2 + 4z-6 = 0，得出 z1 = -1 和 z2 = -3。

The root locus is illustrated in Fig. 4.25. All the roots are outside the unit circle in blue. The system is unstable. THis means that a proportional controller is not able to stabilize a double integrator.

根軌跡如圖 4.25 所示。 所有的根都在藍色的單位圓之外。 系統不穩定。 這意味著比例控制器無法穩定雙積分器。

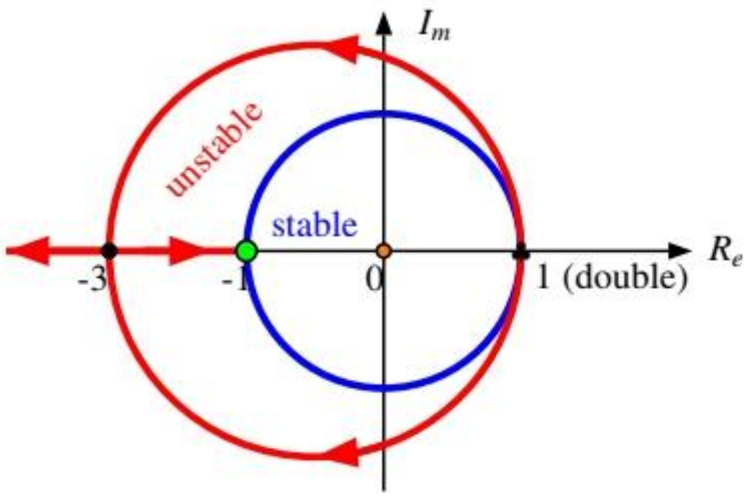

Fig. 4.25 RL of the system with characteristic eqn: $1 + K\frac{(z+1)}{(z-1)^2} = 0$

圖 4.25 具有特徵 eqn 的系統的 RL：1 + K（z + 1）/（z-1）2 = 0

Example 4.6.2 As a second example for the root locus technique let us consider the system of the Fig. 4.26.

示例 4.6.2 作為根軌跡技術的第二個示例，讓我們考慮圖 4.26 的系統。

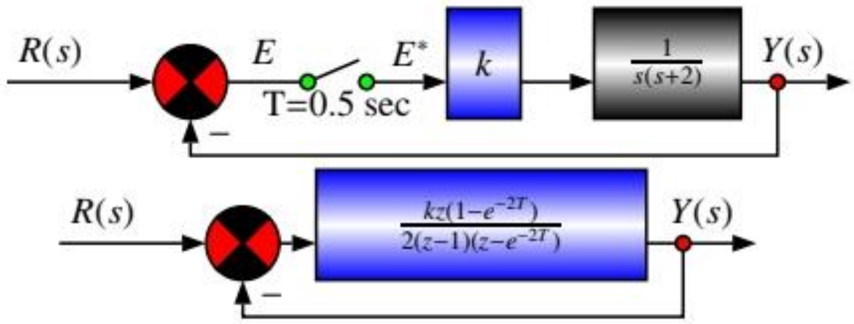

Fig. 4.26 BD of the system with characteristic eqn: $1 + K\frac{z}{(z-1)(z-0.368)} = 0$

圖 4.26 特徵方程為 eqn 的系統的 BD：1 + K * Z /（z-1）（z-0.368）= 0

The characteristic equation of this system is given by:
該系統的特徵方程由下式給出：

$$1 + k\frac{z(1-e^{-2T})}{2(z-1)(z-e^{-2T})} = 1 + K\frac{z}{(z-1)(z-0.368)} = 0$$
$$\text{with } K = 0.316k$$

- Number of branches: n = 2.
- Finite Number of branches: m = 1.
- Infinite Number of branches n − m = 2 − 1 = 1.
- Angle of asymptotes: β = π(2k+1)/n−m = π(2k+1)/2−1 = π.
- Intersection of the locus with the real axis: dK/dz = −z2+0.368 = 0. The resolution

of this equations gives: z1 = −0.606 et z2 = +0.606.

・分支數量：n = 2。
・有限分支數：m = 1。
・無限數量的分支 n – m = 2 – 1 = 1。
・漸近角：β =π（2k + 1）/ n–m =π（2k + 1）/ 2-1 =π。
・軌跡與實軸的交點：dK / dz = -z2 + 0.368 =0。此等式的分辨率為：z1 = -0.606 et z2 = +0.606。

If we replace z by −1 in the characteristic equation, we find:
如果在特性方程式中用-1 代替 z，我們會發現：

$$1 + K\frac{z}{(z-1)(z-0.368)} = 1 + K\frac{(-1)}{(-1-1)(-1-0.368)} = 0$$

which implies in turn:
這又意味著：

$$K = 2.738$$

$$K = 0.316k$$

which gives:
這使：

$$k = \frac{K}{0.316} = \frac{2.738}{0.316} = 8.65$$

The root locus is drawn in Fig. 4.27. All the the roots are inside the unit circle in blue. Therefore, the system is stable for all gains k < 8.65.
根軌跡如圖 4.27 所示。 所有的根都在藍色的單位圓內。
因此，系統對於所有增益 k <8.65 都是穩定的。

4.7 Bode Plot Technique
4.7 波特圖技術

The frequency response plays an important role in the analysis and design of continuous-time and discrete-time systems. As for the time response, the frequency
頻率響應在連續時間和離散時間系統的分析和設計中起著重要作用。 時間響應，頻率

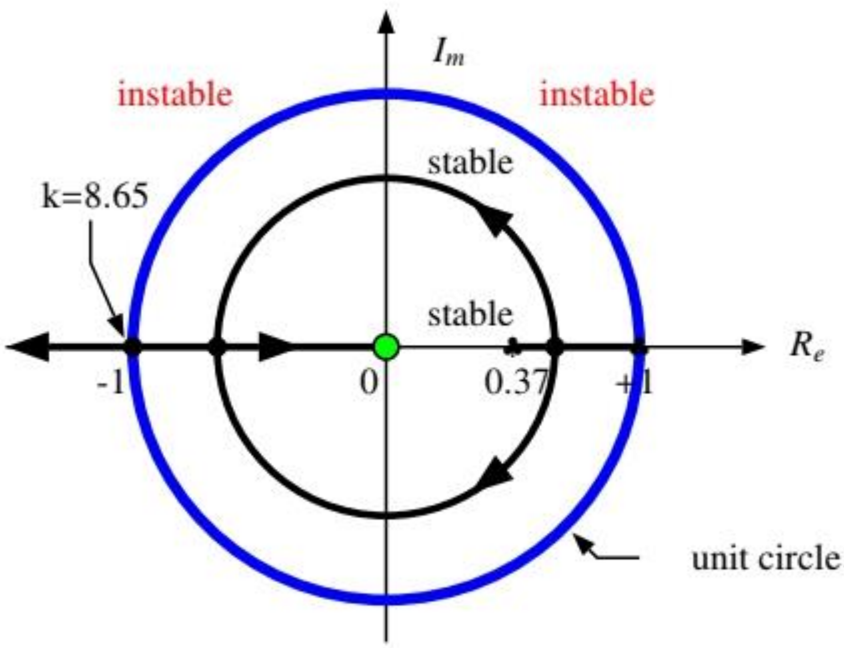

Fig. 4.27 RL of the system with characteristic eqn: $1 + K\frac{z}{(z-1)(z-0.368)} = 0$

圖 4.27 具有特徵方程式的系統的 RL：1 + K * z /（z-1）（z-0.368）= 0

response consists of exciting the system by a sinusoidal input. In the continuous-time system, it was proven that for a sinusoidal input, the output of the a stable linear system is sinusoidal with same frequency of the input, and the magnitude and the phase of the output are function of this frequency. For discrete-time system, the output is also sinusoidal with the same frequency as the input signal and the phase and the magnitude are still function of this frequency. To show this, let us consider a stable linear system with the following transfer function:

響應包括通過正弦輸入激勵系統。 在連續時間系統中，已證明對於正弦輸入，穩定線性系統的輸出是正弦的，並且輸入頻率相同，並且輸出的幅度和相位是該頻率的函數。 對於離散時間系統，輸出也是與輸入信號相同頻率的正弦波，並且相位和幅度仍是該頻率的函數。 為了說明這一點，讓我們考慮具有以下傳遞函數的穩定線性系統：

$$\begin{aligned} G(z) &= \frac{Y(z)}{R(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} \\ &= K \frac{\Pi_{i=1}^{m} (z - n_i)}{\Pi_{i=1}^{n} (z - z_i)} \end{aligned}$$

Let the input r(t) has the following expression:

令輸入 r（t）具有以下表達式：

$$r(t) = \sin(wt)$$

where w is the frequency of the input. The magnitude is taken here equal to one. The Z -transform of this signal is given by (see Z -transform table):

其中 w 是輸入頻率。 此處的幅度等於 1。

該信號的 Z 變換由下式給出（請參見 Z 變換錶）：

$$R(z) = \frac{z \sin(wt)}{z^2 - 2zcost(wt) + 1} = \frac{zsin(wT)}{(z - e^{-jwT})(z - e^{jwT})}$$

Now if we consider that the system is excited by R(z) the corresponding output,Y(z) is given by:

現在，如果我們認為系統由 R（z）激勵，則相應的輸出 Y（z）由下式給出：

$$\begin{aligned} Y(z) &= G(z)R(z) \\ &= K \frac{\Pi_{i=1}^{m} (z - n_i)}{\Pi_{i=1}^{n} (z - z_i)} \frac{zsin(wT)}{(z - e^{-jwT})(z - e^{jwT})} \end{aligned}$$

To get the expression of the output, let us proceed with a partial fraction of Y(z).This gives:
為了得到輸出的表達式，讓我們繼續處理 Y（z）的一部分，得到：

$$Y(z) = \frac{cz}{z - e^{-jwT}} + \frac{\bar{c}z}{z - e^{jwT}} + \text{terms due to } G(z)$$

Let s now multiply both sides this equality by (z−e−jwT)/ z to get the following:
現在讓 s 將等式的兩邊乘以（z−e−jwT）/ z 得到以下結果：

$$G(z)\frac{sinwT}{\left(z - e^{jwT}\right)} = c + \frac{\bar{c}\left(z - e^{-jwT}\right)}{z - e^{jwT}} + \left[\frac{\left(z - e^{-jwT}\right)}{z}\right]\text{terms due to } G(z)$$

Where 而

$$\begin{aligned} c &= \left[G(z)\frac{sin(wT)}{\left(z - e^{jwT}\right)}\right]_{|_{z=e^{-jwT}}} \\ \bar{c} &= \text{conjugate of } c \end{aligned}$$

Notice that e−jwT = cos−wT + jsin−wT = coswT − jsinwT, which implies that
請注意，e-jwT = cos-wT + jsin-wT = coswT-jsinwT，這意味著

$$\left(z - e^{jwT}\right)_{|_{z=e^{-jwT}}} = -2jsinwT$$

Using this we get:
使用這個我們得到：

$$\begin{aligned} c &= \frac{G(e^{-jwT})}{-2j} \\ \bar{c} &= \frac{G(e^{jwT})}{2j} \end{aligned}$$

Using now the fact that for any complex number we have:
現在利用以下事實：對於任何複數，我們都有：

$$G(e^{jwT}) = M(w)e^{j\theta(w)}$$

where M and θ represent respectively the magnitude and the phase at the frequency w.The steady state, the terms due to G(z) vanish and we have:

其中 M 和θ分別代表頻率 w 的幅度和相位。
穩態，歸因於 G（z）的項消失，我們有：

$$\begin{aligned} Y(z) &= \frac{G(e^{-jwT})}{-2j}\frac{z}{z-e^{-jwT}} + \frac{G(e^{jwT})}{2j}\frac{z}{z-e^{jwT}} \\ &= \frac{M(w)}{2j}\left[-\frac{e^{-\theta(w)}z}{z-e^{-jwT}} + \frac{e^{\theta(w)}z}{z-e^{jwT}}\right] \end{aligned}$$

The Z -transform inverse of Y(z) at the steady state is given by:
穩態下 Y（z）的 Z 變換逆由下式給出：

$$\begin{aligned} y(kT) &= \frac{M(w)}{2j}\left[e^{j\theta(w)}e^{jwT} - e^{-j\theta(w)}e^{-jwT}\right] \\ &= \frac{M(w)}{2j}\left[e^{j(\theta(w)+wT)} - e^{-j(\theta(w)+wT)}\right] \\ &= M(w)sin\,(wT + \theta(w)) \end{aligned}$$

Remark 4.7.1 It is important to mention that the magnitude and the phase of the output for a sinusoid input are both functions of its frequency.
Therefore, their values will change when the frequency changes.
備註 4.7.1 重要的是要提到正弦輸入的輸出幅度和相位都是其頻率的函數。
因此，它們的值將隨頻率變化而變化。
A certain parallel can be made with frequency response of continuous time. In fact, for these system, the frequency response can be obtained from the transfer function, G(s) that describes the system by doing the following:
可以通過連續時間的頻率響應進行一定的並聯。 實際上，對於這些系統，可以通過執行以下操作從描述系統的傳遞函數 G（s）中獲得頻率響應：

- the magnitude M(w) is given by:

・大小 M（w）由下式給出：

$$M(w) = |G(jw)|$$

- the phase θ(w) is given by:

・相位θ（w）由下式給出：

$$\theta(w) = \arg\,(G(jw))$$

This means that the magnitude and the phase of the output at frequency w are obtained from the transfer function of the system by replacing firstly s by jw and then compute the magnitude and the phase using the previous formulas.
這意味著在頻率為 w 時輸出的幅度和相位是通過首先用 sw 替換 s 然後從系統的傳遞函數獲得的，然後使用先前的公式來計算幅度和相位。

For the discrete time, the same reasoning applies except that we have to replace z by ejwT and use the following formulas:
對於離散時間，除了我們必須用 ejwT 替換 z 並使用以下公式之外，採用相同的推理：

・ the magnitude M(w) is given by:
・大小 M（w）由下式給出：

$$M(w) = \left|G(e^{jwT})\right|$$

・ the phase θ(w) is given by:
・相位 θ（w）由下式給出：

$$\theta(w) = \arg\left(G(e^{jwT})\right)$$

Some precautions have to be taken for the frequency response of discrete time system. In fact, the Z -transform is obtained by replacing z by esT . Therefore, the primary and the complementary strips of the left hand side of the s-domain are mapped to the interior of the unit circle in the z-domain. If we replace in turn z by ejwT to get the frequency response of the discrete time system, the result we will get has no sense since it deals with the entire z-plane. To avoid this the following transformation is usually used:
對於離散時間系統的頻率響應，必須採取一些預防措施。 實際上，Z 變換是通過用 esT 替換 z 獲得的。 因此，s 域左側的主要條帶和互補條映射到 z 域中單位圓的內部。 如果我們用 ejwT 依次替換 z 以獲得離散時間系統的頻率響應，則由於涉及整個 z 平面，因此得到的結果沒有意義。 為了避免這種情況，通常使用以下轉換：

$$z = \frac{1 + \frac{T}{2}\omega}{1 - \frac{T}{2}\omega}$$

which implies:
這意味著：

$$\omega = \frac{2}{T}\frac{z-1}{z+1}$$

Using the Z -transform and and the w-transform respectively, the primary trip of the left half of the s-plane is then transformed into the unit circle which in turn transformed to the entire left half of the w-plane. More specifically, the range of frequencies in the s-plane − ωs/2 ≤ w ≤ ω2/2 is firstly transformed into the unit circle in the z-plane, which in turn transformed into the entire left half of the w-plane.

然後分別使用 Z 變換和 w 變換，將 s 平面左半部分的一次行程轉換為單位圓，然後將其轉換為 w 平面的整個左半部分。 更具體地，首先將 s 平面-ωs/2≤w≤ω2/ 2 中的頻率範圍變換為 z 平面中的單位圓，然後將其變換為 w 平面的整個左半部。

Finally, it is important to notice the relationship between the frequencies ω and ν . In fact, ω is defined by:

最後，重要的是要注意頻率ω和ν之間的關係。 實際上，ω定義為：

$$\omega_{|_{\omega=iv}} = j\nu = \left[\frac{2}{T}\frac{z-1}{z+1}\right]_{|_{z=e^{jwT}}}$$
$$= \frac{2}{T}\frac{e^{jwT}-1}{e^{jwT}+1}$$

Multiplying the numerator and the denominator by e−jwT , we get:

分子和分母乘以 e-jwT，得到：

$$w_{|_{w=iv}} = j\nu$$
$$= j\frac{2}{T}\tan\left(\frac{wT}{2}\right)$$

which gives the following relationship between w and v:

給出 w 和ν之間的以下關係：

$$w = \frac{2}{T}\tan\left(\frac{wT}{2}\right)$$

At low frequencies, we have equality between these frequencies. In fact, when w is low, we have tan(wT/2) = wT/2 , which gives w = v.

在低頻下，這些頻率之間是相等的。 實際上，當 w 低時，我們的 tan（wT / 2） = wT / 2，得出 w = ν 。

Based on this remark, the frequency response of the discrete time consists thenof replacing w by jv, with v is a fictitious frequency, in the new expression of thetransfer function obtained after replacing z by z = (1+ T/2 w) / (1− T/2 w) . To have an idea on how the frequency response can be plotted, let us consider the following example.

基於這一點，在將 z 替換為 z =（1+ T / 2 w）/（之後得到的傳遞函數的新表達式中，離散時間的頻率響應包括：用 jv 替換 w，其中ν是虛擬頻率。 1−T / 2 w）。為了了解如何繪製頻率響應，讓我們考慮以下示例。

Example 4.7.1 As a first example of the frequency response, let us consider the system of the Fig. 4.28. It represents the speed control of a load driven by a dc motor. The controller is a proportional. The transfer function of the system and the controller is given by:

例 4.7.1 作為頻率響應的第一個例子，讓我們考慮圖 4.28 的系統。 它表示由直流電動機驅動的負載的速度控制。 控制器是成比例的。 系統和控制器的傳遞函數由下式給出：

$$\bar{G}(s) = \frac{K_p k}{\tau s + 1} = \frac{K}{\tau s + 1}$$

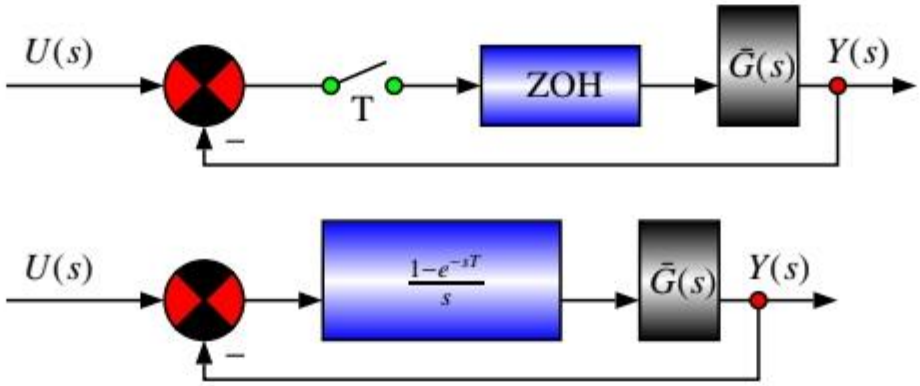

Fig. 4.28 Speed control of mechanical part driven by a dc motor

圖 4.28 由直流電動機驅動的機械零件的速度控制

Firstly, let us compute the open loop transfer function of the system in Fig. 4.26. Since we have a ZOH we get:

首先，讓我們計算圖 4.26 中系統的開環傳遞函數。

既然有了 ZOH，我們得到：

$$G(s) = \left(1 - e^{-sT}\right) \frac{K}{s(\tau s + 1)}$$

where K = Kpk = 2, τ = 1s and T is the sampling period used for our system and it is equal to 0.1s.

Using the Z -transform table we get:

其中 K = Kpk = 2，τ = 1s，T 是用於我們系統的採樣週期，它等於 0.1s。

使用 Z -transform 表，我們得到：

$$
\begin{aligned}
G(z) &= K\frac{(z-1)}{z}\frac{z\left(1-e^{-T}\right)}{(z-1)\left(z-e^{T}\right)} \\
&= K\frac{\left(1-e^{-T}\right)}{\left(z-e^{T}\right)} \\
&= \frac{0.1903}{z-0.9048}
\end{aligned}
$$

Replacing now z by (1+ T/2 w) /(1− T/2 w) = 1+0.05w/1−0.05w, we get:
現在將 z 替換為（1+ T / 2 w）/（1- T / 2 w）= 1 + 0.05w / 1-0.05w，我們得到：

$$
\begin{aligned}
G(z) &= \frac{0.1903}{\frac{1+0.05w}{1-0.05w}-0.9048} \\
&= \frac{0.1903\,(1-0.05w)}{0.0952+0.0952w} \\
&= \frac{1.9989\,(1-0.05w)}{1+w}
\end{aligned}
$$

Using Matlab, we can get the bode diagram of this transfer function as illustrated by Fig. 4.29.
使用 Matlab，我們可以獲得該傳遞函數的伯德圖，如圖 4.29 所示。

4.8 Conclusions
4.8 結論

This chapter covers the analysis tools based on the transfer function concept. Mainly, we developed the techniques of how to compute the time response and determine the system performances. We also presented the root locus and bode plot techniques.
本章介紹基於傳遞函數概念的分析工具。 主要是，我們開發瞭如何計算時間響應和確定係統性能的技術。 我們還介紹了根軌跡和波德圖技術。

4.9 Problems
4.9 問題

1. Compute the Z -transform of the following signals:
(a) the unit step
(b) the unit ramp
(c) the unit exponential
(d) r(t) = t + sinwt
(e) 1 – coswt

1.計算以下信號的 Z 變換：

(a ）單位步長

（b ）單位坡道

（c ）單位指數

（d ）r（t ）= t + sinwt

（e ）1-重量

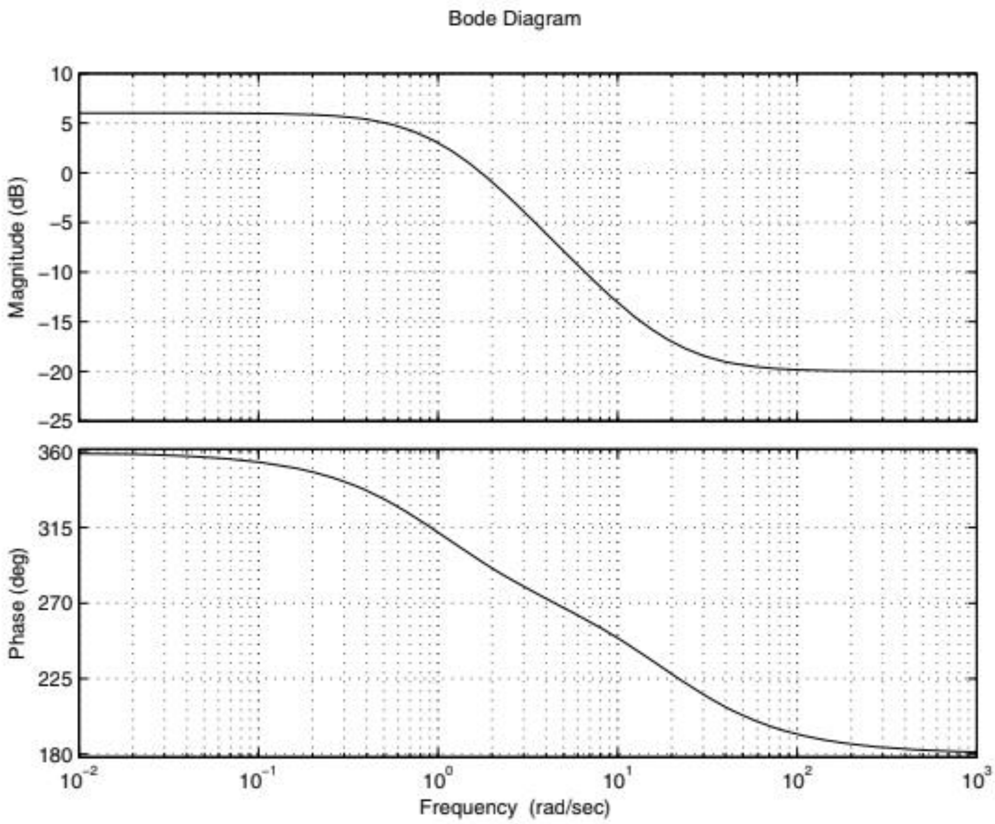

Fig. 4.29 Bode diagram of $\frac{1.9989(1-0.05w)}{1+w}$

2. Compute the expression of the signal in time of the following ones in z:

2. 按以下時間計算信號的表達式：

(a) $Y(z) = \frac{Tze^{aT}}{(z-e^{-aT})^2}, a > 0$

(b) $Y(z) = \frac{z(1-e^{aT})}{(z-1)(z-e^{-aT})}, a > 0$

(c) $Y(z) = \frac{1}{b-a}\left[\frac{z}{z-e^{aT}} - \frac{z}{z-e^{bT}}\right], a > 0, b > 0 \text{ and } a \neq b$

3. For the dynamical systems with the input u(t) and the output y(t) with the following dynamics:

3.對於動力系統，輸入為 u（t），輸出為 y（t），具有以下動態：

- $\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} = u(t)$
- $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 4u(t)$
- $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 8u(t)$
- $\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} = u(t)$

(a) determine the sampling period T

(b) using the approximation methods determine the relationship between the input U(z) and the output Y(z)

(c) determine the pulse transfer function for each dynamics

(d) using Matlab compute the step response of each dynamics

(e) using now the zero-order-hold, determine the corresponding transfer function and compute the step response. Compare this response to the one of the previous question

（a）確定採樣週期 T

（b）使用逼近方法確定輸入 U（z）與輸出 Y（z）之間的關係

（c）確定每種動力學的脈衝傳遞函數

（d）使用 Matlab 計算每個動力學的階躍響應

（e）現在使用零階保持，確定相應的傳遞函數併計算階躍響應。 將此響應與上一個問題進行比較

4. In this problem we consider the system of the Fig. 4.30 where the transfer function of the system is given by:

4.在這個問題中，我們考慮圖 4.30 的系統，其中系統的傳遞函數由下式給出：

$$G(s) = \frac{10}{(s+1)(s+10)}$$

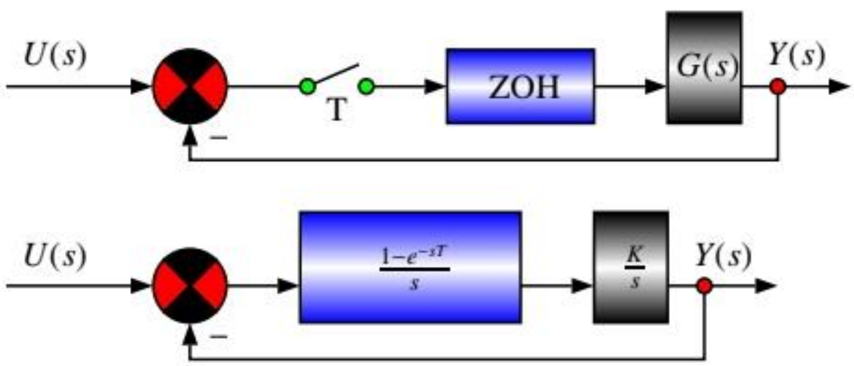

Fig. 4.30 Transfer functions in feedback

圖 4.30 反饋中的傳遞函數

(a) determine the sampling period that we can use for this system

(b) using this sampling period determine the open loop transfer function and the closed-loop one

(c) determine the step response of the system

(d) plot the behavior of the output with respect to time

（a）確定我們可以用於該系統的採樣週期

（b）使用該採樣週期確定開環傳遞函數和閉環傳遞函數

（c）確定係統的階躍響應

（d）繪製輸出相對於時間的行為

5. Study the stability of the dynamical systems with the following characteristic equation:

5. 用以下特徵方程式研究動力系統的穩定性：

(a) z3 + 0.8z2 + 0.17z + 0.01

(b) z4 + 1.4z3 + 0.65z2 + 0.112z + 0.006

(c) z5 + 2.39z4 + 2.036z3 + 0.7555z2 + 0.1169z + 0.0059

(d) z5 + 11.4z4 + 14.65z3 + 6.6120z2 + 1.126z + 0.06

6. In this problem we consider the dynamical system show in the block diagram illustrated by the Fig. 4.31. The transfer functions are given by:

6. 在這個問題中，我們考慮動力學系統，如圖 4.31 所示。 傳遞函數由下式給出：

$$
\begin{aligned}
G(z) &= \frac{z\left(1-e^{aT}\right)}{(z-1)\left(z-e^{-aT}\right)} \\
C(z) &= K
\end{aligned}
$$

with a = 0.1 and T = 0.01

跟 a = 0.1 和 T = 0.01

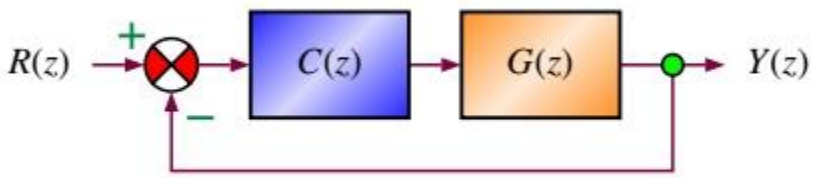

Fig. 4.31 Block diagram of the closed-loop

圖 4.31 閉環框圖

(a) study the stability in function of the gain K

(b) plot the root locus of the system and conclude on the stability

（a）研究增益 K 的函數穩定性
（b）繪製系統的根軌跡並得出穩定性結論
7. Consider the system of the Fig. 4.30 with the following expression for G(s):
7.考慮圖 4.30 的系統，其中 G（s）具有以下表達式：

$$G(s) = \frac{K}{s(\tau s + 1)}$$

with K is the gain and τ = 1s is the time constant of the system.
其中 K 是增益，τ = 1s 是系統的時間常數。

- determine the sampling period
- compute the transfer function G(z)
- plot the root locus the system when the gain K varies between 0 and ∞

・確定採樣週期
・計算傳遞函數 G（z）
・當增益 K 在 0 和∞之間變化時，繪製系統的根軌跡

8. Consider the system of the Fig. 4.30 with the following expression for G(s):
8. 考慮圖 4.30 的系統，其中 G（s）具有以下表達式：

$$G(s) = \frac{K}{s(\tau s + 1)}$$

with K = 10 is the gain and τ = 0.1s is the time constant of the system.
其中 K = 10 是增益，而 τ = 0.1s 是系統的時間常數。

- determine the sampling period
- compute the transfer function G(z)
- plot the Bode diagram of the system

・確定採樣週期
・計算傳遞函數 G（z）
・繪製系統的伯德圖