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Definition

Definition

An integer n > 1 is called a *Carmichael number* if n is composite and $(a, n) = 1 \rightarrow a^{n-1} \equiv 1 \mod n$ for all $a \in \mathbb{Z}$.

Korselt's theorem

Theorem (Korselt's Criterion)

A composite integer n > 1 is a Carmichael number if and only if

- 1 n is squarefree.
- 2 for every prime p dividing n, (p-1)|(n-1).

Carmichael is Squarefree.

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800

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- $(1+p)^{n-1} \equiv 1 \pmod{p^2}$.
- $1 + (n-1)p \equiv 1 \pmod{p^2}$.
- 6 $p \equiv 0 \pmod{p^2}$.



Proof

Divisibility rule

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Korselt's criterion 800

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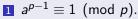
- $\exists b$: $b \mod p$ has order p-1.
- $\exists a: a \equiv b \pmod{p}$ and $a \equiv 1 \pmod{n/p}$.
- $a^{n-1} \equiv b^{n-1} \equiv 1 \pmod{p}.$



Sufficient conditions

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For each prime p, for which p|n, if $a \in \mathbb{Z}$ satisfies gcd(a, n) = 1



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- $2 a^{n-1} \equiv 1 \pmod{p}.$

Then if *n* is squarefree, $a^{n-1} \equiv 1 \pmod{n}$.



Theorem statement

Theorem

A composite integer n is a Carmichael number if and only if $a^n \equiv a \mod n$ for all $a \in \mathbb{Z}$.

Lemma

for all
$$a \in Z$$
, $a^n \equiv a \mod n \Rightarrow (a, n) = 1 \rightarrow a^{n-1} \equiv 1 \mod n$



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Another characterization

- **1** Let $a \in \mathbb{Z}$ s.t. (a, n) = 1.
- $a^{-1}a^n \equiv a^{-1}a \mod n$.

Lemma

for all $a \in Z$, $a^n \equiv a \mod n \Rightarrow (a, n) = 1 \rightarrow a^{n-1} \equiv 1 \mod n$

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- $a^{p-1} \equiv 1 \mod p.$
- $(p-1)|(n-1) \Rightarrow a^{n-1} \equiv 1 \mod p$.
- 6 $a^n \equiv a \mod p$.



561

$$561 = 3 * 11 * 17$$
$$2|560$$
$$10|560$$
$$16|560$$

561 is the lowest Carmichael number

Lemma (Decision criteria)

For every number n, so that n < 561, try:

- **1** *If* p|n *then* p-1|n-1
- 2 n is squarefree

Corollary (Jack Chernick, 1939)

If k is a positive integer such that 6k + 1, 12k + 1, and 18k + 1 are all prime, then the product n = (6k + 1)(12k + 1)(18k + 1) is a Carmichael number.

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■ Is odd.

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Every Carmichael number

- Is odd.
- Has at least three different prime factors.

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Lemma

Every Carmichael number

- Is odd.
- Has at least three different prime factors.
- Satisfies that every prime factor of n is less than \sqrt{n} .



Chernick's construction

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Proof.

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- n composite greater than 1
- n is squarefree

Carmichael numbers

Corollary (Jack Chernick, 1939)

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- n composite greater than 1
- n is squarefree
- for every prime p dividing n, also (p-1)|(n-1).

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If k is a positive integer such that 6k + 1, 12k + 1, and 18k + 1 are all prime, then the product n = (6k + 1)(12k + 1)(18k + 1) is a Carmichael number.

Proof.

Korselt

- n composite greater than 1
- n is squarefree
- for every prime p dividing n, also (p-1)|(n-1).
 - $n \equiv (0+1)(0+1)(0+1) \equiv 1 \mod 6k$.
 - $n \equiv (6k+1)(0+1)(6k+1) \equiv 1 \mod 12k$.
 - $n \equiv (6k+1)(12k+1)(0+1) \equiv 1 \mod 18k$.

Proof.

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Every Carmichael number is odd.

Proof.

- 1 n-1 relatively prime with n.
- $(n-1)^{(n-1)} \equiv (-1)^{(n-1)} \equiv 1 \mod n.$
- $3 n > 2 \Rightarrow -1 \not\equiv 1 \mod n.$
- alg n-1 even.



Let n be a Carmichael number. Every prime factor of n is less than \sqrt{n} .

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$$\frac{1}{p-1} = \frac{p(n/p)-1}{p-1} = \frac{(p-1)(n/p)+n/p-1}{p-1} = \frac{n}{p} + \frac{n/p-1}{p-1}.$$

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$$p-1|n/p-1$$
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$$p \le n/p \Rightarrow p^2 \le n.$$

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4
$$n \neq p^2$$
 (squarefree).



Every Carmichael number has at least three different prime factors.

Proof.

Contradiction

$$n = pq \Rightarrow p > \sqrt{n} \text{ or } q > \sqrt{n}$$



Future work

Conjecture (Dickson's conjecture)

There are infinitely many numbers generated by Chernick's construction.

Theorem (W. R. Alford, A. Granville, C. Pomerance, 1994)

There are infinitely many Carmichael numbers.

