University of Bayreuth

Bachelor Seminar Tree Automata

Introduction to Ranked Tree Automata

Author:
Martin Braun

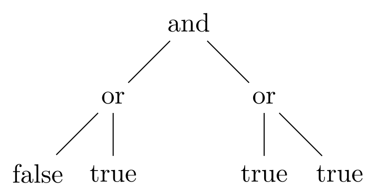
Supervisor:
Prof. Dr. Wim Martens

Introduction to Tree Languages

A good example for a tree language is the one consisting of all binary boolean expressions evaluating to true, for which an instance - if formatted in the right way - could look like this:

$$and(or(false, true), or(true, true))$$

In order to ease understanding, the elements of the language are often represented as a tree in a graphical way:



Just like for "normal" regular languages, it is of interest to know whether a given word (in this case a tree) is part of the (tree-)language. In order to describe an automaton that recognizes tree-languages we have to define what **Σ-trees** and (regular) **tree-languages** are, first.

Definition 1. *Σ-tree [1]*
*The set of Σ-trees T_Σ over the **alphabet** Σ is inductively defined as follows:*

$$\text{1. every } \sigma \in \Sigma \text{ is a } \Sigma\text{-tree}$$
$$\text{2. } \sigma \in T_\Sigma \text{ and } t_1, ..., t_n \in T_\Sigma, n \geq 1 \iff \sigma(t_1, ..., t_n) \in T_\Sigma$$

*Note: In general, there is no bound for the number of children in a tree (these trees are called unranked), but in this draft we will only take a look at **ranked trees**, which have such a bound.*

Definition 2. *tree-language [1]*
A tree language $L_{t\Sigma}$ over the alphabet Σ is defined as a subset of T_Σ:

$$L_{t\Sigma} \subseteq T_\Sigma$$

$\Rightarrow T_\Sigma$ is already a tree-language.

Next, we have to declare some special words in the context of $\Sigma - trees$.

Definition 3. *variables, terms, linear terms, ground-terms [2][5]*
*Let $v \in V, v \notin \Sigma = \emptyset$ be a constant (symbol with no child). We call v a **variable** (V is a set of Variables) in a **term** $t \in T_{\Sigma \cup V}$, if it is a placeholder for any given $\sigma \in \Sigma$ or yet another variable that is not necessarily part of V. Terms containing every v at most once are **linear terms**. All terms which don't contain any variables, are called **ground-terms** over Σ.*

We can now define (Non-Deterministic) Finite Tree Automata for tree languages.

Definition 4. *NFTA [2]*
A (Non-Deterministic) Finite Tree Automaton (NFTA) over the alphabet Σ is a tuple $A = (Q, \Sigma, Q_f, \Delta)$ where Q is a ***finite set of states****, $Q_f \subseteq Q$ is a* ***finite set of final states****, and Δ is a* ***finite set of transition rules*** *of the type:*

$$f(q_1, ..., q_n) \rightarrow q_x$$
$$\textit{where } n \geq 0, f \in \Sigma, q_x, q_1, ..., q_n \in Q$$

For $n = 0$, we write:

$$a \rightarrow q(a)$$
$$\textit{where } a \in \Sigma, q \in Q$$

Tree automata over Σ run on ground terms over Σ. An automaton starts at the leaves and moves upward, associating along a ***run*** *a state with each subterm inductively while reducing the tree via the transition rules.*

For a tree $t' \in T_{\Sigma \cup Q}$ that is the result of applying a transition rule on a tree $t \in T_{\Sigma \cup Q}$ we write:

$$t \rightarrow_A t'$$

If more or equal than one transition rules are applied we denote it like this:

$$t \rightarrow_A^* t'$$

$\rightarrow_A^*$ is the reflexive and transitive closure of $\rightarrow_A$.

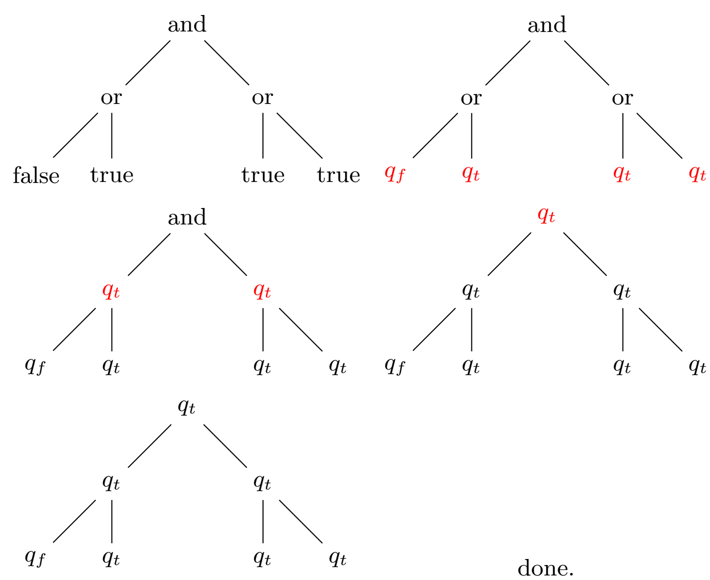
Note: There is no initial state in an NFTA but the ground-terms (which can be considered to be the "initial rules" of the NFTA) act alike by transitioning constant symbols into a state.

Our binary-boolean-expression NFTA can now be written as:

Example 1. binary-boolean-statement NFTA
$A = (Q, \Sigma, Q_f, \Delta)$
$\Sigma = \{or, and, not, true, false\}$
$Q = \{q_f, q_t\}$
$Q_f = \{q_t\}$
$\Delta = \{false \to q_f, true \to q_t,$
$\quad and(q_t, q_t) \to q_t, and(q_t, q_f) \to q_f, and(q_f, q_t) \to q_f, and(q_f, q_f) \to q_f,$
$\quad or(q_t, q_t) \to q_t, or(q_t, q_f) \to q_t, or(q_f, q_t) \to q_t, or(q_f, q_f) \to q_f,$
$\quad not(q_f) \to q_t, not(q_t) \to q_f\}$

A run with the example input from the beginning of this chapter looks like this:

Example 2. running a NFTA



Or in written form:
$and(or(false, true), or(true, true)) \to_A^* and(or(q_f, q_t), or(q_t, q_t))$
$\to_A^* and(q_t, q_t) \to_A^* q_t$

$q_t \in Q_f \Rightarrow A$ accepts $w \Rightarrow w \in L_A$ with L_A being the language recognized by the automaton.

Determinization

Non Deterministic Finite Tree Automata (NFTA) can be determinized just like Non Determinitistic Automata (NFA) in the word case. By knowing that there exists a DFTA for every NFTA, definitions, proofs and algorithms become much easier, since we don't have to take special care of the properties of NFTAs. We will now take a look at how this is done. But first we have to define formally, what being deterministic means in the context of FTAs.

Definition 5. *Deterministic Finite Tree Automaton*
A tree automaton with no two rule of the type:

$$f(q_1, ..., q_n) \to q_x$$
$$f(q_1, ..., q_n) \to q_y$$
(this includes the ground-terms)

or

$$\epsilon(q_1, ..., q_n) \to q_x$$
(state changes, even though no actual symbol is read)

with $n \geq 0, q_x, q_y, q_1, ...q_n \in Q, q_x \neq q_y, f \in \Sigma$ *is called a* ***Deterministic Finite Tree Automaton*** *(DFTA).*

Similar to the algorithm for Determinization in the word case, there exists a power set construction algorithm for determizing Tree Automata.

Definition 6. *Algorithm DET for Tree Automata [2]*
Note: statesOf(x) returns the set of states that contributed to the creation of the state x, while state(X) returns a state representing all states in the set X.

Data: *NFTA* $A = (Q, \Sigma, Q_f, \Delta)$
$Q_d := \emptyset$
$\Delta_d := \emptyset$
while Δ_d *grew last cycle* **do**
  $f(q_1, ..., q_n) \in \Delta$
  $s_1, ..., s_n \in Q_d$

  / meta-state representing the set of reachable states */*
  $s := state(\{q \in Q \mid q_1 \in statesOf(s_1), ..., q_n \in statesOf(s_n), f(q_1, ...q_n) \to q \in \Delta\})$

  $Q_d := Q_d \cup \{s\}$
  $\Delta_d := \Delta_d \cup f(s_1, ..., s_n) \to s$
end
$Q_{f_d} := \{s \in Q_d \mid \{s\} \cap Q_d \neq \emptyset\}$
Result: *DFTA* $A_d = (Q_d, \Sigma, Q_{f_d}, \Delta_d)$

It is easy to see that the algorithm produces a deterministic automaton A_d as we are automatically constructing meta-states for all reachable states and therefore eliminating all possible non-deterministic behaviour. However, we still have to prove $L(A) = L(A_d)$. For this, we have to show that the meta-states $s \in Q_d$ are "built correctly", or in formal terms:

$$For\ any\ tree\ t : t \rightarrow^*_{A_d} s \iff s = state(\{q \in Q \mid t \rightarrow^*_A q\})$$

Proof. $L(A) = L(A_d)$ (Correctness of DET) [2]
This proof is done via an induction over the structure of the symbols in Σ.

- **Base case:** For any tree $t = a \in \Sigma$ we take a look at the corresponding ground-term $a \rightarrow q(a)$. Because of the way we defined s as the meta-state representing the set of all reachable states in a given situation this is inherently correct.

- **induction step:** $t = f(q_1, ..., q_n)$

  - 1.: $t \rightarrow^*_{A_d} s \Rightarrow (s = state(\{q \in Q \mid t \rightarrow^*_A q\})$

    Supposing $t \rightarrow^*_{A_d} f(s_1, ..., s_n) \rightarrow_{A_d} s$, by induction hypothesis, for each $i \in 1, ..., n$, we can see $s_i = state(\{q \in Q \mid q_i \rightarrow^*_A q\}$.

    Because states $s_i \in Q_d$, rules $f(s_1, ..., s_n) \rightarrow s \in \Delta_d$ are added by the determinization algorithm and $s := state(\{q \in Q \mid q_1 \in statesOf(s_1), ..., q_n \in statesOf(s_n), f(q_1, ...q_n) \rightarrow q \in \Delta\})$, we learn $s = state(\{q \in Q \mid t \rightarrow^*_A q\})$.

  - 2.: $s = state(\{q \in Q \mid t \rightarrow^*_A q\}) \Rightarrow t \rightarrow^*_{A_d} s$

    Considering $s = state(\{q \in Q \mid f(q_1, ..., q_n) \rightarrow^*_A q\})$ with state sets S_i defined as $S_i := \{q \in Q \mid q_i \rightarrow^*_A q\}$, by induction hypothesis for each $i \in \{1, ..., n\}$ we know $q_i \rightarrow^*_{A_d} s_i, s_i = state(S_i)$. Thus $s = state(\{q \in Q \mid q_1 \in S_1, ..., q_n \in S_n, f(q_1, ...q_n) \rightarrow q \in \Delta\})$.

    By the definition of Δ_d in the determinization algorithm, $f(s_1, ..., s_n) \in \Delta_d$ and thus $t \rightarrow^*_{A_d} s$.

$\square$

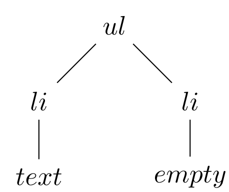
Following is an example of how a NFTA can be determinized with this algorithm.

Example 3. Running the DET algorithm
consider a non deterministic FTA given like this:
$A = (Q, \Sigma, Q_f, \Delta)$
$\Sigma = \{ul, li, text, empty\}$
$Q = \{q_{ul}, q_{li1}, q_{li2}, q_{text}, q_{empty}\}$
$Q_f = \{q_{ul}\}$
$\Delta = \{ul(q_{li1}, q_{li2}) \rightarrow q_{ul}, ul(q_{li2}, q_{li1}) \rightarrow q_{ul},$
$\mathbf{li}(\mathbf{q_{text}}) \rightarrow \mathbf{q_{li1}}, \mathbf{li}(\mathbf{q_{text}}) \rightarrow \mathbf{q_{li2}},$
$text \rightarrow q_{text}, empty \rightarrow q_{empty},$
$\epsilon(\mathbf{q_{empty}}) \rightarrow \mathbf{q_{text}}\}$

This recognizes all trees that represent unordered lists (ul) in HTML notation, which contain 2 list items (li):

```
<ul>
      <li>text</li>
      <li>empty</li>
</ul>
```

Or as a tree input:



If we start determinizing with the rules containing no state and then go "up in the hierarchy" and generate all the states on-the-fly, we get these new rules:

$text \rightarrow state(\{q_{text}\})$
$empty \rightarrow state(\{q_{text}, q_{empty}\})$
$li(state(\{q_{text}\}))) \rightarrow state(\{q_{li1}, q_{li2}\})$
$li(state(\{q_{text}, q_{empty}\})) \rightarrow state(\{q_{li1}, q_{li2}\})$
$ul(state(\{q_{li1}, q_{li2}\}), state(\{q_{li1}, q_{li2}\})) \rightarrow state(\{q_{ul}\})$

And the set of final states is $Q_{f_d} = \{state(\{q_{ul}\})\}$.

As we can see, there is no ϵ-rule left and we don't have to choose which rule to apply when reading

Minimization

Now that we can obtain a DFTA for each NFTA, we can take a look at how we can minimize these newly determinized automata.
Just like in the word case there exists a Myhill-Nerode theorem for Finite Tree Automata. But before we can use it, we have to define **Contexts**, **Congruence** and $\equiv_L$.

Definition 7. *Context [2][3]*
*Let V_n be a set of n variables. Then, a linear term $C \in T_{\Sigma \cup V_n}$ is called a **context**. Furthermore, $C[t_1, ..., t_n], t_1, ..., t_n \in T_\Omega$ is known as a **context application**, meaning that variables $v_i \in V_n$ are replaced by (sub-)trees $t_i \in T_\Omega$ with $T_\Omega \supseteq T_{\Sigma \cup V_n}$*

*Note: T_Ω **can** contain new variables.*

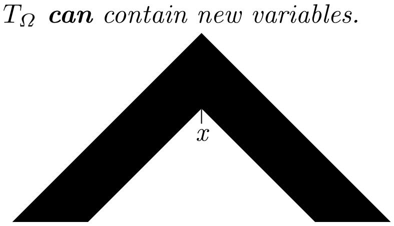


Fig. 1: Context with one variable (x) [3]

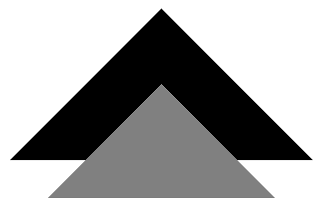


Fig. 2: Context application [3]

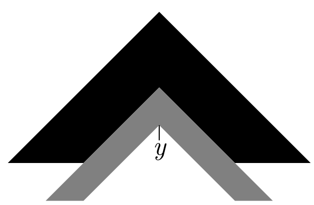


Fig. 3: Context application with a new context [3]

Definition 8. *Congruence [2]*
*An equivalence relation $\equiv$ on T_Σ is a **congruence** on T_Σ if for every $f \in \Sigma$ with n arguments applies:*

$$T_\Sigma \ni u_i \equiv w_i \in T_\Sigma, 1 \leq i \leq n \Rightarrow f(u_1, ..., u_n) \equiv f(w_1, ..., w_n)$$
$$\#\ of \equiv -classes\ is\ finite \Rightarrow \equiv\ is\ of\ \textbf{finite index}.$$

Additionally a congruence is an equivalence relation closed under context. This means that for any $C \in T_{\Sigma \cup V}$, if $u \equiv w \Rightarrow C[u] \equiv C[w]$.

Definition 9. $\equiv_L$ *[2]*
For any given tree language $L \in T_\Sigma$, we define the congruence $\equiv_L$ on T_Σ by: $T_\Sigma \ni u \equiv_L w \in T_\Sigma$, if for all Contexts $C \in T_{\Sigma \cup V}$ applies:

$$C[u] \in L \iff C[v] \in L$$

For the sake of easier proofs, we consider all DFTAs as **complete** and **reduced**.

Definition 10. *Completeness and reduction [6]*
A FTA A is ***complete*** *if there is at least one transition rule available for every possible symbol-states combination. A state q is* ***accessible*** *if there exists a ground term t such that $t \rightarrow_A^* q$. A NFTA is* ***reduced*** *if all its states are accessible.*

Note: All examples for Finite Tree Automata given in this draft are supposed to be complete and reduced. We only do not add a capturing state for all impossible symbol-state combinations for the sake of simplicity. Once such a capturing state would be reached there'd no way to get to another state.

We can now give the Myhill-Nerode theorem.

Theorem 1. *Myhill-Nerode [2]*
These statements are equivalent:

(i) L is a regular tree language

(ii) L is the union of some congruence classes of finite index

(iii) the relation $\equiv_L$ is a congruence of finite index

Proof.

- (i) $\Rightarrow$ (ii): Assume that the tree language L is recognized by some complete DFTA $A = (Q, \Sigma, Q_f, \delta)$ with δ being a transition function (i). Let us consider the relation $\equiv_A$ defined on T_Σ by: $T_\Sigma \ni u \equiv v \in T_\Sigma$, if $\delta(u) = \delta(v)$. Since we know that Q only has a finite amount of states in it and the number of equivalence classes may at most be equal to the size of Q, we can deduce that $\equiv_A$ is a congruence of finite index (ii).

- (ii) $\Rightarrow$ (iii): By denoting the congruence of finite index as $\cong$ and assuming that $T_\Sigma \ni u \cong v \in T_\Sigma$, it can be proven that $C[u] \cong C[v]$ for all contexts $C \in T_{\Sigma \cup V}$ by an easy induction on the structure of terms. Since L is the union of some equivalence classes of the congruence of finite index $\cong$ (ii), we have $C[u] \in L \iff C[v] \in L$. Therefore we know that $u \equiv_L v$ and that $\equiv_L$ contains the equivalence class of u in $\cong$. Furthermore we now know that $index(\equiv_L) \leq index(\cong) \Rightarrow index(\equiv_L)\ is\ finite$ (iii).

- (iii) $\Rightarrow$ (i): By representing the set of equivalence classes of $\equiv_L$ (iii) as the finite set of states Q_{min} with $|Q_{min}| = |\equiv_L|$, we know that every equivalence class has its own state. By denoting the equivalence class of a term $u \in T_\Sigma$ as $[u]$ we define the transition function δ_{min} for every $f \in \Sigma$ with n arguments as:

$$\delta_{min}(f, [u_1], ..., [u_n]) = [f(u_1, ..., u_n)]$$

  The definition of δ is consistent because $\equiv_L$ is a congruence. With $Q_{min_f} := \{[u] | u \in L\}$ the resulting DFTA $A_{min} := (Q_{min}, \Sigma, Q_{min_f}, \delta_{min}$ recognizes the tree language L (i). $\square$

As a consequence of this theorem we can deduce the following:

Corollary 1. *[2]*
The minimum DFTA recognizing a tree language L is unique up to renaming the states and is given by A_{min} in the proof of the Myhill-Nerode Theorem.

This means that we can minimize a tree automaton by computing the congruence classes of the language it recognizes. But before we can put this to use, we have to prove the corollary first.

Proof. [2]
Assume that L is recognized by some DFTA $A = (Q, \Sigma, Q_f, \delta)$. Then the relation $\equiv_A$ is a refinement of $\equiv_L$ with $index(\equiv_A) \geq index(\equiv_L)$, thus $|Q| \geq |Q_{min}|$. We know that A is reduced (all states are accesible), because otherwise a state could be removed contradicting to the definition of $\equiv_A$. Let $q \in Q$ and $u \in T_\Sigma$, such that $\delta(u) = q$. Then the state q can be consistently identified with the state $\delta_{min}(u)$, since δ is a refinement of δ_{min} and we can see that every state $q \in Q$ has a corresponding state $q_{min} \in Q_{min}$. $\square$

By using this Corollary and the construction given in the Myhill-Nerode theorem we can deduce an algorithm to minimize Deterministic Finite Tree Automata:

Definition 11. *Algorithm MIN for Tree Automata [4]*

Data: *complete and reduced DFTA* $A = (Q, \Sigma, Q_f, \Delta)$
Set $P = \{(q_f, q) \mid q_f \in Q_f, q \in Q \setminus Q_f\}$
Set $P' = P$
while $P' \neq P$ **do**
  $P = P'$
  $\forall p_1, p_2, p_3 \in Q, p_1 \neq p_2, p_1 \neq p_3, p_2 \neq p_3,$
  define $\neg(p_1 P' p_2) \iff$
    /* *coudln't distinguish to anything in the last cycle* */
  ist das hier p_3? Weil sonst bricht das ja sofort ab...
    1.$\neg(p_1 P p_3)$ *or*
    /* *can't distinguish p_1 from p_2, yet* */
    2.$\exists f \in \Sigma$ *with n arguments,* $\exists q_1, ..., q_{i-1}, q_{i+1}, ..., q_n \in Q$,
     $\neg(r_1 P r_2), r_1, r_2 \in Q$, *where:*
      $f(q_1, ..., q_{i-1}, p_1, q_{i+1}, ..., q_n) \rightarrow r_1$ *and*
      $f(q_1, ..., q_{i-1}, p_2, q_{i+1}, ..., q_n) \rightarrow r_2$
      (Note: this works for multiple occurences of p_1 and p_2 as well, see the example on the next page)
end
Q_{min} = *set of equivalence classes of* P
$\Delta_{min} = \{f([q_1], ..., [q_n]) \rightarrow [q] \mid f(q_1, ..., q_n) \rightarrow q \in \Delta\}$
$Q_{f_{min}} = \{[q] \mid q \in Q_f\}$
Result: *complete, reduced and minimal DFTA*
  $A_{min} = (Q_{min}, \Sigma, Q_{f_{min}}, \Delta_{min})$

While we are not giving a complete proof for this algorithm, we can at least go over why it is correct for a given language $L = L(A)$ and the automaton $A = (Q, \Sigma, Q_f, \Delta)$ [2]:

- In the while loop we are marking all tuples p_1, p_2 as distinguishable when they are added to the relation
- In order to get the equivalence classes for Q_{min}, we are merging all pairs of indistinguishable states to a new one representing both. After having done this incrementally for all not marked combinations, these "artificial" states have to be distinguishable to all other states and Q_{min} must therefore be minimal. This can easily be proven correct by contradiction.
- It isn't hard to see that the construction of $Q_{f_{min}}$ and Δ_{min} is correct.

Example 4. Running the MIN algorithm
After cleaning up the Automaton of our previous unordered list example, one might already see that it isn't minimal yet:

$A = (Q, \Sigma, Q_f, \Delta)$ $\Sigma = \{ul, li, text, empty\}$
$Q = \{q_{ul}, \mathbf{q_{text}}, \mathbf{q_{text2}}, q_{li}\}$
$Q_f = \{q_{ul}\}$
$\Delta = \{text \to q_{text},$
$empty \to q_{text2},$
$\mathbf{li(q_{text}) \to q_{li}},$
$\mathbf{li(q_{text2}) \to q_{li}},$
$ul(q_{li}, q_{li}) \to q_{ul}\}$

While we should be able to fix this by hand, we are now using the MIN algorithm to minimize this automaton. In the following table, we are marking all tuples p_1, p_2 that are distinguishable in that cycle by the index of that cycle, so we can see the process in action.

	q_{text}	q_{text2}	q_{li}	q_{ul}
q_{text}	-	-	-	-
q_{text2}	(merge)	-	-	-
q_{li}	1	1	-	-
q_{ul}	0	0	0	-

As predicted q_{text} and q_{text2} have to be merged in order to minimize A. The resulting automaton is:

$A = (Q, \Sigma, Q_f, \Delta)$ $\Sigma = \{ul, li, text, empty\}$
$Q = \{q_{ul}, \mathbf{q_{text_{1\&2}}}, q_{li}\}$
$Q_f = \{q_{ul}\}$
$\Delta = \{text \rightarrow \mathbf{q_{text_{1\&2}}},$
$empty \rightarrow \mathbf{q_{text_{1\&2}}},$
$\mathbf{li}(\mathbf{q_{text_{1\&2}}}) \rightarrow \mathbf{q_{li}},$
$ul(q_{li}, q_{li}) \rightarrow q_{ul}\}$

References

1. Automata theory for XML researchers, Frank Neven, University of Limburg, frank, neven luc, ac. be, http://homepages.inf.ed.ac.uk/libkin/dbtheory/frank.pdf, 03/11/2015
2. Tree Automata and Techniques, Hubert Comon et. al, Pages 19-39
3. Automata and Logic on Trees, Wim Martens, Stijn Vansummeren, http://lrb.cs.uni-dortmund.de/~martens/data/esslli07/lecture01.pdf, 03/17/2015
4. Automata and Logic on Trees, Wim Martens, Stijn Vansummeren, http://lrb.cs.uni-dortmund.de/~martens/data/esslli07/lecture02.pdf, 03/21/2015
5. http://en.wikipedia.org/wiki/Ground_expression, 03/16/2015
6. http://en.wikipedia.org/wiki/Tree_automaton, 03/20/2015