Introduction

One of the most used standards in the world-wide-web (www) is the XML standard or variations of it (HTML, XHTML, etc.).

XML

June 2014

Martin Braun
Bachelor Seminar Tree Automata
Supervisor: Prof. Dr. Wim Martens

Organization

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Table of Contents

Hamiltonian Mechanics unter besonderer Berücksichtigung der höhreren Lehranstalten

Ivar Ekeland[1], Roger Temam[2] Jeffrey Dean, David Grove, Craig Chambers, Kim B. Bruce, and Elsa Bertino

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Keywords: computational geometry, graph theory, Hamilton cycles

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$$\dot{x} = JH'(t,x)$$
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with $H(t,\cdot)$ a convex function of x, going to $+\infty$ when $\|x\| \to \infty$.

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In this section, we will consider the case when the Hamiltonian $H(x)$ is autonomous. For the sake of simplicity, we shall also assume that it is C^1.

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Now if $\widetilde{x}$ has a lower period, T/k say, we would have, by Corollary 31:

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To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family x_T, $T \in \left(2\pi\omega^{-1}, 2\pi b_\infty^{-1}\right)$ of periodic solutions, $x_T(0) = x_T(T)$, with x_T going away to infinity when $T \to 2\pi\omega^{-1}$, which is the period of the linearized system at 0.

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$$f = \frac{d}{dt} F + f_o \qquad \text{with} \ \ F \in L^2 \left(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n} \right) \ , \tag{30}$$

where $f_o := T^{-1} \int_o^T f(t) dt$. For instance,

$$f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi \ , \tag{31}$$

where δ_k is the Dirac mass at $t = k$ and $\xi \in \mathbb{R}^{2n}$ is a constant, fits the prescription. This means that the system $\dot{x} = JH'(x)$ is being excited by a series of identical shocks at interval T.

Definition 1. *Let $A_\infty(t)$ and $B_\infty(t)$ be symmetric operators in $\mathbb{R}^{2n}$, depending continuously on $t \in [0,T]$, such that $A_\infty(t) \leq B_\infty(t)$ for all t.*

A Borelian function $H : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is called (A_∞, B_∞)-subquadratic at infinity if there exists a function $N(t,x)$ such that:

$$H(t,x) = \frac{1}{2} \left(A_\infty(t) x, x \right) + N(t,x) \tag{32}$$

$$\forall t \ , \quad N(t,x) \qquad \text{is convex with respect to} \ \ x \tag{33}$$

$$N(t,x) \geq n \left(\left\| x \right\| \right) \qquad \text{with} \ \ n(s) s^{-1} \to +\infty \ \ \text{as} \ \ s \to +\infty \tag{34}$$

$$\exists c \in \mathbb{R} \ : \quad H(t,x) \leq \frac{1}{2} \left(B_\infty(t) x, x \right) + c \quad \forall x \ . \tag{35}$$

If $A_\infty(t) = a_\infty I$ and $B_\infty(t) = b_\infty I$, with $a_\infty \leq b_\infty \in \mathbb{R}$, we shall say that H is (a_∞, b_∞)-subquadratic at infinity. As an example, the function $\left\| x \right\|^\alpha$, with $1 \leq \alpha < 2$, is $(0, \varepsilon)$-subquadratic at infinity for every $\varepsilon > 0$. Similarly, the Hamiltonian

$$H(t,x) = \frac{1}{2} k \left\| k \right\|^2 + \left\| x \right\|^\alpha \tag{36}$$

is $(k, k + \varepsilon)$-subquadratic for every $\varepsilon > 0$. Note that, if $k < 0$, it is not convex.

Notes and Comments. The first results on subharmonics were obtained by Rabinowitz in [5], who showed the existence of infinitely many subharmonics both in the subquadratic and superquadratic case, with suitable growth conditions on H'. Again the duality approach enabled Clarke and Ekeland in [2] to treat the same problem in the convex-subquadratic case, with growth conditions on H only.

Recently, Michalek and Tarantello (see [3] and [4]) have obtained lower bound on the number of subharmonics of period kT, based on symmetry considerations and on pinching estimates, as in Sect. 5.2 of this article.

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Hamiltonian Mechanics2

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where $f_o := T^{-1} \int_o^T f(t) dt$. For instance,

$$f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi \ , \tag{31}$$

where δ_k is the Dirac mass at $t = k$ and $\xi \in \mathbb{R}^{2n}$ is a constant, fits the prescription. This means that the system $\dot{x} = JH'(x)$ is being excited by a series of identical shocks at interval T.

Definition 1. *Let $A_\infty(t)$ and $B_\infty(t)$ be symmetric operators in $\mathbb{R}^{2n}$, depending continuously on $t \in [0,T]$, such that $A_\infty(t) \le B_\infty(t)$ for all t.*

A Borelian function $H : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is called (A_∞, B_∞)-subquadratic at infinity if there exists a function $N(t,x)$ such that:

$$H(t,x) = \frac{1}{2} \left(A_\infty(t)x, x \right) + N(t,x) \tag{32}$$

$$\forall t \ , \quad N(t,x) \qquad \text{is convex with respect to} \ \ x \tag{33}$$

$$N(t,x) \ge n \left(\left\| x \right\| \right) \qquad \text{with} \ \ n(s)s^{-1} \to +\infty \ \ \text{as} \ \ s \to +\infty \tag{34}$$

$$\exists c \in \mathbb{R} \ : \quad H(t,x) \le \frac{1}{2} \left(B_\infty(t)x, x \right) + c \quad \forall x \ . \tag{35}$$

If $A_\infty(t) = a_\infty I$ and $B_\infty(t) = b_\infty I$, with $a_\infty \le b_\infty \in \mathbb{R}$, we shall say that H is (a_∞, b_∞)-subquadratic at infinity. As an example, the function $\left\| x \right\|^\alpha$, with $1 \le \alpha < 2$, is $(0, \varepsilon)$-subquadratic at infinity for every $\varepsilon > 0$. Similarly, the Hamiltonian

$$H(t,x) = \frac{1}{2} k \left\| k \right\|^2 + \left\| x \right\|^\alpha \tag{36}$$

is $(k, k + \varepsilon)$-subquadratic for every $\varepsilon > 0$. Note that, if $k < 0$, it is not convex.

Notes and Comments. The first results on subharmonics were obtained by Rabinowitz in 1985, who showed the existence of infinitely many subharmonics both in the subquadratic and superquadratic case, with suitable growth conditions on H'. Again the duality approach enabled Clarke and Ekeland in 1981 to treat the same problem in the convex-subquadratic case, with growth conditions on H only.

Recently, Michalek and Tarantello (see Michalek, R., Tarantello, G. 1982 and Tarantello, G. 1983) have obtained lower bound on the number of subharmonics of period kT, based on symmetry considerations and on pinching estimates, as in Sect. 5.2 of this article.

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