University of Bayreuth

Bachelor Seminar Tree Automata

Introduction to Ranked Tree Automata

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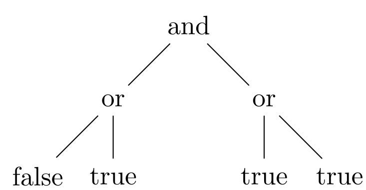
Introduction to Tree Languages

Regular Tree Languages are a powerful tool when it comes to parsing data given in a textual form. However, they lack in the context of parsing hierarchical data. Using Tree Languages to define your data structure can help with this shortcoming. A good example for a tree language is the one consisting of all binary boolean expressions evaluating to true, for which an instance - if formatted in the right way - could look like this:

$$and(or(false, true), or(true, true))$$

To simplify, the elements of the language are often represented as a tree in a graphical way:



Just like for regular word languages, it is of interest to know whether a given word (in this case a tree) is part of the (tree-)language. In order to describe an automaton that recognizes tree-languages we have to define what **ranked tree alphabets**, ***Σ*-trees** and (regular) **tree-languages** are, first.

Definition 1. *ranked tree alphabet [2][5]*
A ***ranked tree alphabet*** *Σ of* ***arity*** *n is a refinement of an ordinary alphabet such that each symbol $\sigma \in \Sigma$ either has zero or exactly n arguments. Symbols with zero arguments are called* ***constants****.*

Note: There also exists a definition for tree alphabets that dont have the restriction of arity. These are called ***unranked tree alphabets****, but in this paper we will only take a look at the ranked case.*

Definition 2. *Σ-tree [1]*
The set of Σ-trees T_Σ over the ranked tree alphabet Σ of arity n is inductively defined as follows:

1. every $\sigma \in \Sigma$ is a Σ-tree
2. $\sigma \in \Sigma$ and $t_1, ..., t_n \in T_\Sigma \iff \sigma(t_1, ..., t_n) \in T_\Sigma$

Definition 3. *tree-language [1]*
A tree language $L_{t\Sigma}$ over the alphabet Σ is defined as a subset of T_Σ:

$$L_{t\Sigma} \subseteq T_\Sigma$$

From that definition, we can see that T_Σ is already a tree-language. Next, we have to declare some terminology in the context of $\Sigma - trees$.

We can now define (Non-Deterministic) Finite Tree Automata (NFTA) for tree languages. One can get a good grasp of how they work if you consider them to be NFAs with the possibility to have multiple states in their transition rules.

Definition 4. *NFTA [2]*
A (Non-Deterministic) Finite Tree Automaton (NFTA) over the alphabet Σ of arity n is a tuple $A = (Q, \Sigma, Q_f, \Delta)$ where Q is a ***finite set of states****, $Q_f \subseteq Q$ is a* ***finite set of final states****, and Δ is a* ***finite set of transition rules*** *of the type:*

$$f(q_1, ..., q_n) \rightarrow q_x$$
$$\textit{where } f \in \Sigma, q_x, q_1, ..., q_n \in Q$$

For constants, we write:

$$a \rightarrow q$$
$$\textit{where } a \in \Sigma, q \in Q$$

Note: These rules transition into the initial states of a NFTA
(that's why we call them initial rules rather informally)

These transition rules are applied from the bottom up to a given input tree. By doing so, the tree is reduced until no transition rule can be found. A tree $t \in T_{\Sigma \cup Q}$ can be reduced to another tree $t' \in T_{\Sigma \cup Q}$ iff they only differ in one sub-tree such that t contains $t_{sub} = \sigma(q_1, ..., q_n), \sigma \in \Sigma$ and t' contains $t'_{sub} = q, q \in Q$ (both being at the exact same spot respectively) and a transition rule $\delta \in \Delta, \delta = \sigma(q_1, ..., q_n) \rightarrow q$ exists.

Note: This definition also applies for reductions on constants. The only difference being that σ doesn't have any arguments in that case.

We denote such a relation with:

$$t \rightarrow_A t'$$

If one or more transition rules are applied, we write:

$$t \rightarrow_A^* t'$$

If a given input tree t_{input} can be reduced to a tree $t_{final} = q, q \in Q_f$, then it is accepted by A. The set of all input trees accepted by A is called the ***Language of A****, which is denoted by L_A.*

Our binary-boolean-expression NFTA can now be written as:

Example 1. binary-boolean-statement NFTA
$A = (Q, \Sigma, Q_f, \Delta)$
$\Sigma = \{or, and, not, true, false\}$
$Q = \{q_f, q_t\}$
$Q_f = \{q_t\}$
$\Delta = \{false \to q_f, true \to q_t,$
$\quad and(q_t, q_t) \to q_t, and(q_t, q_f) \to q_f, and(q_f, q_t) \to q_f, and(q_f, q_f) \to q_f,$
$\quad or(q_t, q_t) \to q_t, or(q_t, q_f) \to q_t, or(q_f, q_t) \to q_t, or(q_f, q_f) \to q_f,$
$\quad not(q_f) \to q_t, not(q_t) \to q_f\}$

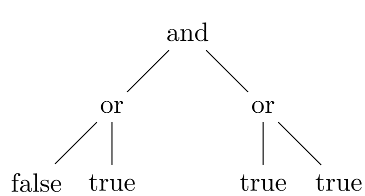
We will now show how the above automaton processes the tree from the beginning of this chapter.

Example 2. running a NFTA

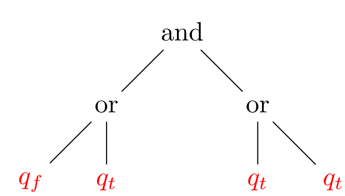
$and(or(false, true), or(true, true))$
$\to_A and(or(q_f, true), or(true, true))$
$\to_A and(or(q_f, q_t), or(true, true))$
$\to_A and(or(q_f, q_t), or(q_t, true))$
$\to_A and(or(q_f, q_t), or(q_t, q_t))$
$\to_A and(q_t, or(q_t, q_t))$
$\to_A and(q_t, q_t)$
$\to_A q_t$

We see that the tree can successfully be reduced to the accepting state $q_t \in Q_f$ and therefore that A accepts w and that w is in the language L_A recognized by the automaton.

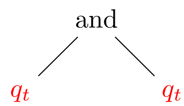
This process can be represented in a graphical way as well. In order to keep things shorter, we condensed some of the steps together, but the general point is still visible.



The start.



$false$ and $true$, according to our initial rules, are reduced to q_f and q_t.



$or(q_f, q_t)$ and $or(q_t, q_t)$ are reduced to q_t each.

q_t

$and(q_t, q_t)$ is reduced to q_t.

q_t

The final result.

Determinization

Non Deterministic Finite Tree Automata (NFTA) can be determinized just like Non Determinitistic Automata (NFA) in the word case. By knowing that there exists a DFTA for every NFTA, definitions, proofs and algorithms become much easier, since we don't have to take special care of the properties of NFTAs. We will now take a look at how this is done. But first we have to define formally, what being deterministic means in the context of FTAs.

Definition 5. *Deterministic Finite Tree Automaton*
A tree automaton with no two rules of the type:

$$\begin{gathered} f(q_1, ..., q_n) \to q_x \\ f(q_1, ..., q_n) \to q_y \\ \textit{with } q_x \neq q_y \end{gathered}$$

with $n \geq 0, q_x, q_y, q_1, ...q_n \in Q, q_x \neq q_y, f \in \Sigma$ *is called a* ***Deterministic Finite Tree Automaton*** *(DFTA).*

Similar to the algorithm for Determinization in the word case, there exists a power set construction algorithm for determizing Tree Automata.

powerset benutzen hier

Definition 6. *Algorithm DET for Tree Automata [2]*
Note: statesOf(x) returns the set of states that contributed to the creation of the state x, while state(X) returns a state representing all states in the set X.

Data: *NFTA* $A = (Q, \Sigma, Q_f, \Delta)$
$Q_d := \emptyset$
$\Delta_d := \emptyset$
while Δ_d *grew last cycle* **do**
  $f(q_1, ..., q_n) \in \Delta$
  $s_1, ..., s_n \in Q_d$

  / meta-state representing the set of reachable states */*
  $s := state(\{q \in Q \mid q_1 \in statesOf(s_1), ..., q_n \in statesOf(s_n), f(q_1, ...q_n) \to q \in \Delta\})$

  $Q_d := Q_d \cup \{s\}$
  $\Delta_d := \Delta_d \cup f(s_1, ..., s_n) \to s$
end
$Q_{f_d} := \{s \in Q_d \mid \{s\} \cap Q_f \neq \emptyset\}$
Result: *DFTA* $A_d = (Q_d, \Sigma, Q_{f_d}, \Delta_d)$

It is easy to see that the algorithm produces a deterministic automaton A_d as we are automatically constructing meta-states for all reachable states and therefore eliminating all possible non-deterministic behaviour. However, we still have to prove $L(A) = L(A_d)$. For this, we have to show that the meta-states $s \in Q_d$ are "built correctly", or in formal terms:

$$\textit{For any tree } t : t \rightarrow^*_{A_d} s \iff s = \mathit{state}(\{q \in Q \mid t \rightarrow^*_A q\})$$

Proof. $L(A) = L(A_d)$ (Correctness of DET) [2]
This proof is done via an induction over the structure of the symbols in Σ.

- **Base case:** For any tree $t = a \in \Sigma$ we take a look at the rule $a \rightarrow q(a)$. Because of the way we defined s as the meta-state representing the set of all reachable states in a given situation this is inherently correct.

- **induction step:** $t = f(q_1, ..., q_n)$

  - 1.: $t \rightarrow^*_{A_d} s \Rightarrow (s = \mathit{state}(\{q \in Q \mid t \rightarrow^*_A q\})$

    Supposing $t \rightarrow^*_{A_d} f(s_1, ..., s_n) \rightarrow_{A_d} s$, by induction hypothesis, for each $i \in 1, ..., n$, we can see $s_i = \mathit{state}(\{q \in Q \mid q_i \rightarrow^*_A q\}$.

    Because states $s_i \in Q_d$, rules $f(s_1, ..., s_n) \rightarrow s \in \Delta_d$ are added by the determinization algorithm and $s := \mathit{state}(\{q \in Q \mid q_1 \in \mathit{statesOf}(s_1), ..., q_n \in \mathit{statesOf}(s_n), f(q_1, ...q_n) \rightarrow q \in \Delta\})$, we learn $s = \mathit{state}(\{q \in Q \mid t \rightarrow^*_A q\})$.

  - 2.: $s = \mathit{state}(\{q \in Q \mid t \rightarrow^*_A q\}) \Rightarrow t \rightarrow^*_{A_d} s$

    Considering $s = \mathit{state}(\{q \in Q \mid f(q_1, ..., q_n) \rightarrow^*_A q\})$ with state sets S_i defined as $S_i := \{q \in Q \mid q_i \rightarrow^*_A q\}$, by induction hypothesis for each $i \in \{1, ..., n\}$ we know $q_i \rightarrow^*_{A_d} s_i, s_i = \mathit{state}(S_i)$. Thus $s = \mathit{state}(\{q \in Q \mid q_1 \in S_1, ..., q_n \in S_n, f(q_1, ...q_n) \rightarrow q \in \Delta\})$.

    By the definition of Δ_d in the determinization algorithm, $f(s_1, ..., s_n) \in \Delta_d$ and thus $t \rightarrow^*_{A_d} s$.

$\square$

Following is an example of how a NFTA can be determinized with this algorithm.

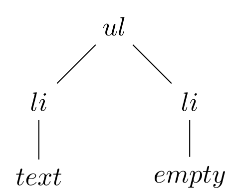
Beispiel ohne epsilon

Example 3. Running the DET algorithm
consider a non deterministic FTA given like this:
$A = (Q, \Sigma, Q_f, \Delta)$
$\Sigma = \{ul, li, text, empty\}$
$Q = \{q_{ul}, q_{li1}, q_{li2}, q_{text}, q_{empty}\}$
$Q_f = \{q_{ul}\}$
$\Delta = \{ul(q_{li1}, q_{li2}) \to q_{ul}, ul(q_{li2}, q_{li1}) \to q_{ul},$
$\mathbf{li}(\mathbf{q_{text}}) \to \mathbf{q_{li1}}, \mathbf{li}(\mathbf{q_{text}}) \to \mathbf{q_{li2}},$
$text \to q_{text}, empty \to q_{empty},$
$\epsilon(\mathbf{q_{empty}}) \to \mathbf{q_{text}}\}$

This recognizes all trees that represent unordered lists (ul) in HTML notation, which contain 2 list items (li):

```
<ul>
    <li>text</li>
    <li>empty</li>
</ul>
```

Or as a tree input:



If we start determinizing with the rules containing no state and then go "up in the hierarchy" and generate all the states on-the-fly, we get these new rules:

$text \to state(\{q_{text}\})$
$empty \to state(\{q_{text}, q_{empty}\})$
$li(state(\{q_{text}\}))) \to state(\{q_{li1}, q_{li2}\})$
$li(state(\{q_{text}, q_{empty}\})) \to state(\{q_{li1}, q_{li2}\})$
$ul(state(\{q_{li1}, q_{li2}\}), state(\{q_{li1}, q_{li2}\})) \to state(\{q_{ul}\})$

And the set of final states is $Q_{f_d} = \{state(\{q_{ul}\})\}$.

As we can see, there is no ϵ-rule left and we don't have to choose which rule to apply when reading

Minimization

Now that we can obtain a DFTA for each NFTA, we can take a look at how we can minimize these newly determinized automata.
Just like in the word case there exists a Myhill-Nerode theorem for Finite Tree Automata. But before we can use it, we have to define **Contexts**, **Congruence** and $\equiv_L$.

For the definition of a **Context** it is convenient to define a **Slot** first.

Definition 7. *Slot (is this definition sufficient?)*
*A **Slot** $s \in S, S \cap \Sigma = \emptyset$ is a special token, that, if found in a tree $t_1 \in T_{\Sigma \cup S}$, can be replaced by any tree $t_1 \in T_{\Sigma \cup S}$ (t_2 can contains **slots** as well).*

As an abstract representation, a tree with a slot is often drawn as a triangle with a marker for every slot:

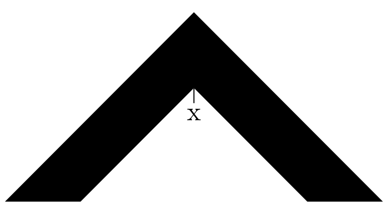


Fig. 1: Tree with the slot x [3]

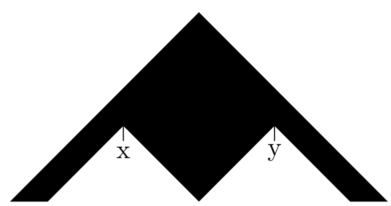


Fig. 2: Tree with the slots x and y [3]

Defining a Context is straightforward now.

Definition 8. *Context [2][3]*
*A tree with slots is called a **Context**. Furthermore, if C is a context with slots $s_1, ..., s_n \in S$, then $C[t_1, ..., t_n], t_1, ..., t_n \in T_\Omega$ is known as a **context application**, with the slots s_i being replaced by (sub-)trees $t_i \in T_\Omega, T_\Omega \supseteq T_{\Sigma \cup S}$.*

*Note: T_Ω **can** contain new slots.*

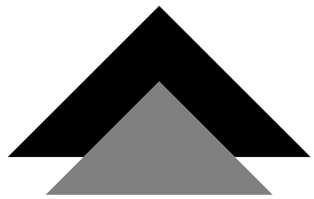


Fig. 3: Context application [3]

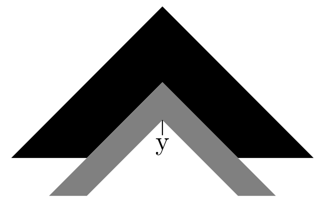


Fig. 4: Context application with a new context [3]

Definition 9. *Congruence [2]*
An equivalence relation $\equiv$ *on* T_Σ *is a* ***congruence*** *on* T_Σ *if for every* $f \in \Sigma$ *with* n *arguments applies:*

$$T_\Sigma \ni u_i \equiv w_i \in T_\Sigma, 1 \leq i \leq n \Rightarrow f(u_1, ..., u_n) \equiv f(w_1, ..., w_n)$$
$$\# \ of \equiv -classes \ is \ finite \Rightarrow \equiv \ is \ of \ \boldsymbol{finite \ index}.$$

Additionally a congruence is an equivalence relation closed under context. This means that for any $C \in T_{\Sigma \cup V}$*, if* $u \equiv w \Rightarrow C[u] \equiv C[w]$.

Definition 10. $\equiv_L$ *[2]*
For any given tree language $L \in T_\Sigma$*, we define the congruence* $\equiv_L$ *on* T_Σ *by:* $T_\Sigma \ni u \equiv_L w \in T_\Sigma$*, if for all Contexts* $C \in T_{\Sigma \cup V}$ *applies:*

$$C[u] \in L \iff C[v] \in L$$

For the sake of easier proofs, we consider all following DFTAs to be **complete** and **reduced**.

Definition 11. *Completeness and reduction [5]*
A FTA A is ***complete*** *if there is at least one transition rule available for every possible symbol-states combination. A state* q *is* ***accessible*** *if there exists a tree* t *such that* $t \rightarrow_A^* q$*. A NFTA is* ***reduced*** *if all its states are accessible.*

Note: All examples for Finite Tree Automata given in this paper are supposed to be complete and reduced. However, we do not add a capturing state for all impossible symbol-state combinations for the sake of simplicity. Let $q_c \in Q$ *be the capturing state, then every transition rule that contained it state on the left side look like* $f(..., q_c, ...) \rightarrow q_c$*. This means, that once the capturing state is reached, there is no way of getting to a different state anymore.*

We can now give the Myhill-Nerode theorem.

Theorem 1. *Myhill-Nerode [2]*
These statements are equivalent:

(i) L is a regular tree language

(ii) L is the union of some congruence classes of finite index

(iii) the relation $\equiv_L$ *is a congruence of finite index*

Proof.

- (i) $\Rightarrow$ (ii): Assume that the tree language L is recognized by some complete DFTA $A = (Q, \Sigma, Q_f, \delta)$ with δ being a transition function (i). Let us consider the relation $\equiv_A$ defined on T_Σ by: $T_\Sigma \ni u \equiv v \in T_\Sigma$, if $\delta(u) = \delta(v)$. Since we know that Q only has a finite amount of states in it and the number of equivalence classes may at most be equal to the size of Q, we can deduce that $\equiv_A$ is a congruence of finite index (ii).

- (ii) $\Rightarrow$ (iii): By denoting the congruence of finite index as $\cong$ and assuming that $T_\Sigma \ni u \cong v \in T_\Sigma$, it can be proven that $C[u] \cong C[v]$ for all contexts $C \in T_{\Sigma \cup V}$ by an easy induction on the structure of terms. Since L is the union of some equivalence classes of the congruence of finite index $\cong$ (ii), we have $C[u] \in L \iff C[v] \in L$. Therefore we know that $u \equiv_L v$ and that $\equiv_L$ contains the equivalence class of u in $\cong$. Furthermore we now know that $index(\equiv_L) \leq index(\cong) \Rightarrow index(\equiv_L)\ is\ finite$ (iii).

- (iii) $\Rightarrow$ (i): By representing the set of equivalence classes of $\equiv_L$ (iii) as the finite set of states Q_{min} with $|Q_{min}| = |\equiv_L|$, we know that every equivalence class has its own state. By denoting the equivalence class of a term $u \in T_\Sigma$ as $[u]$ we define the transition function δ_{min} for every $f \in \Sigma$ with n arguments as:

$$\delta_{min}(f, [u_1], ..., [u_n]) = [f(u_1, ..., u_n)]$$

The definition of δ is consistent because $\equiv_L$ is a congruence. With $Q_{min_f} := \{[u] | u \in L\}$ the resulting DFTA $A_{min} := (Q_{min}, \Sigma, Q_{min_f}, \delta_{min}$ recognizes the tree language L (i). $\square$

As a consequence of this theorem we can deduce the following:

Corollary 1. *[2]*
The minimum DFTA recognizing a tree language L is unique up to renaming the states and is given by A_{min} in the proof of the Myhill-Nerode Theorem.

This means that we can minimize a tree automaton by computing the congruence classes of the language it recognizes. But before we can put this to use, we have to prove the corollary first.

Proof. [2]
Assume that L is recognized by some DFTA $A = (Q, \Sigma, Q_f, \delta)$. Then the relation $\equiv_A$ is a refinement of $\equiv_L$ with $index(\equiv_A) \geq index(\equiv_L)$, thus $|Q| \geq |Q_{min}|$. We know that A is reduced (all states are accesible), because otherwise a state could be removed contradicting to the definition of $\equiv_A$. Let $q \in Q$ and $u \in T_\Sigma$, such that $\delta(u) = q$. Then the state q can be consistently identified with the state $\delta_{min}(u)$, since δ is a refinement of δ_{min} and we can see that every state $q \in Q$ has a corresponding state $q_{min} \in Q_{min}$. □

By using this Corollary and the construction given in the Myhill-Nerode theorem we can deduce an algorithm to minimize Deterministic Finite Tree Automata:

Definition 12. *Algorithm MIN for Tree Automata [4]*

Data: *complete and reduced DFTA* $A = (Q, \Sigma, Q_f, \Delta)$
Set $P = \{(q_f, q) \mid q_f \in Q_f, q \in Q \setminus Q_f\}$
Set $P' = P$
while $P' \neq P$ **do**
  $P = P'$
  $\forall p_1, p_2, p_3 \in Q, p_1 \neq p_2, p_1 \neq p_3, p_2 \neq p_3,$
  define $p_1 P' p_2 \iff$
    / could distinguish in the last cycle */*
    1.$p_1 P p_2$ *or*
    / can distinguish* p_1 *from* p_2*, with: */*
    2.$\exists f \in \Sigma$ *with* n *arguments,* $\exists q_1, ..., q_{i-1}, q_{i+1}, ..., q_n \in Q,$
     $r_1 P r_2, r_1, r_2 \in Q$*, where:*
      $f(q_1, ..., q_{i-1}, p_1, q_{i+1}, ..., q_n) \to r_1$ *and*
      $f(q_1, ..., q_{i-1}, p_2, q_{i+1}, ..., q_n) \to r_2$
      (Note: this works for multiple occurences of p_1 *and* p_2 *as well, see the example on the next page)*
end
Q_{min} = *set of equivalence classes of* P
$\Delta_{min} = \{f([q_1], ..., [q_n]) \to [q] \mid f(q_1, ..., q_n) \to q \in \Delta\}$
$Q_{f_{min}} = \{[q] \mid q \in Q_f\}$
Result: *complete, reduced and minimal DFTA*
   $A_{min} = (Q_{min}, \Sigma, Q_{f_{min}}, \Delta_{min})$

While we are not giving a complete proof for this algorithm, we can at least go over why it is correct for a given language $L = L(A)$ and the automaton $A = (Q, \Sigma, Q_f, \Delta)$ [2]:

– In the while loop we are marking all tuples p_1, p_2 as distinguishable when they are added to the relation

– To get the equivalence classes for Q_{min}, we are merging all pairs of indistinguishable states to a new one representing both. After having done this incrementally for all not marked combinations, these "artificial" states have to be distinguishable to all other states and Q_{min} must therefore be minimal. This can easily be proven correct by contradiction.

– It isn't hard to see that the construction of $Q_{f_{min}}$ and Δ_{min} is correct.

Example 4. Running the MIN algorithm
After cleaning up the Automaton of our previous unordered list example, one might already see that it isn't minimal yet:

$A = (Q, \Sigma, Q_f, \Delta)$ $\Sigma = \{ul, li, text, empty\}$
$Q = \{q_{ul}, \mathbf{q_{text}}, \mathbf{q_{text2}}, q_{li}\}$
$Q_f = \{q_{ul}\}$
$\Delta = \{text \rightarrow q_{text},$
$empty \rightarrow q_{text2},$
$\mathbf{li(q_{text}) \rightarrow q_{li}},$
$\mathbf{li(q_{text2}) \rightarrow q_{li}},$
$ul(q_{li}, q_{li}) \rightarrow q_{ul}\}$

While we should be able to fix this by hand, we are now using the MIN algorithm to minimize this automaton. In the following table, we are marking all tuples p_1, p_2 that are distinguishable in that cycle by the index of that cycle, so we can see the process in action.

	q_{text}	q_{text2}	q_{li}	q_{ul}
q_{text}	-	-	-	-
q_{text2}	(merge)	-	-	-
q_{li}	1	1	-	-
q_{ul}	0	0	0	-

As predicted q_{text} and q_{text2} have to be merged in order to minimize A. The resulting automaton is:

$A = (Q, \Sigma, Q_f, \Delta)$ $\Sigma = \{ul, li, text, empty\}$
$Q = \{q_{ul}, \mathbf{q_{text_{1\&2}}}, q_{li}\}$
$Q_f = \{q_{ul}\}$
$\Delta = \{text \rightarrow \mathbf{q_{text_{1\&2}}},$
$empty \rightarrow \mathbf{q_{text_{1\&2}}},$
$\mathbf{li}(\mathbf{q_{text_{1\&2}}}) \rightarrow \mathbf{q_{li}},$
$ul(q_{li}, q_{li}) \rightarrow q_{ul}\}$

References

1. Automata theory for XML researchers, Frank Neven, University of Limburg, frank, neven luc, ac. be, http://homepages.inf.ed.ac.uk/libkin/dbtheory/frank.pdf, 03/11/2015
2. Tree Automata and Techniques, Hubert Comon et. al, Pages 19-39
3. Automata and Logic on Trees, Wim Martens, Stijn Vansummeren, http://lrb.cs.uni-dortmund.de/~martens/data/esslli07/lecture01.pdf, 03/17/2015
4. Automata and Logic on Trees, Wim Martens, Stijn Vansummeren, http://lrb.cs.uni-dortmund.de/~martens/data/esslli07/lecture02.pdf, 03/21/2015
5. http://en.wikipedia.org/wiki/Tree_automaton, 03/20/2015