

## THE GREEKS

### BLACK AND SCHOLES (BS) FORMULA

The equilibrium price of the call option ( $C$ ; European on a non-dividend paying stock) is shown by Black and Scholes to be:

$$C_t = S_t N(d_1) - X e^{-r(T-t)} N(d_2),$$

Moreover  $d_1$  and  $d_2$  are given by

$$d_1 = \frac{\ln(\frac{S_t}{X}) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$
$$d_2 = \frac{\ln(\frac{S_t}{X}) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

Note that  $d_2 = d_1 - \sigma\sqrt{T}$

## Delta of a (European; non-dividend paying stock) call option:

The delta of a derivative security,  $\Delta$ , is defined as the rate of change of its price with respect to the price of the underlying asset.

For a European (on a non-dividend paying stock) call option is given by

$$\Delta = \frac{\vartheta C_t}{\vartheta S_t} = N(d_1) + S_t \frac{\vartheta N(d_1)}{\vartheta S_t} + \frac{-X e^{-r(T-t)} \vartheta N(d_2)}{\vartheta S_t} \quad (1)$$

where we have applied the product rule:

$$\begin{aligned} \frac{\vartheta[S_t N(d_1)]}{\vartheta S_t} &= \frac{\vartheta S_t}{\vartheta S_t} N(d_1) + S_t \frac{\vartheta N(d_1)}{\vartheta S_t} = \\ &= N(d_1) + S_t \frac{\vartheta N(d_1)}{\vartheta S_t} \end{aligned}$$

Next we apply the chain rule

$$\frac{\vartheta N(d_1)}{\vartheta S_t} = \frac{\vartheta N(d_1)}{\vartheta d_1} \frac{\vartheta d_1}{\vartheta S_t} \quad (2)$$

Since

$$N(d_1) = \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

it follows that

$$\frac{\vartheta N(d_1)}{\vartheta d_1} = N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \quad (3)$$

By using

$$d_{1,2} = \frac{\ln(\frac{S_t}{X}) + (r \pm \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

we have

$$\frac{\vartheta d_1}{\vartheta S_t} = \frac{\vartheta d_2}{\vartheta S_t} = \frac{1}{\sigma S_t \sqrt{T - t}} \quad (4)$$

Using equations (2)-(4) it can be shown that

$$S_t \frac{\vartheta N(d_1)}{\vartheta S_t} = X e^{-r(T-t)} \frac{\vartheta N(d_2)}{\vartheta S_t}$$

or

$$S_t N'(d_1) = X e^{-r(T-t)} N'(d_2)$$

Thus equation (1) reduces to

$$\Delta = \frac{\vartheta C_t}{\vartheta S_t} = N(d_1) > 0 \quad (5)$$

A delta neutral position:

Consider the following portfolio:

A short position in one call and a long position in  $\frac{\partial C}{\partial S} = \Delta$  stocks:

$$\begin{aligned}\Pi &= C - \frac{\partial C}{\partial S} S \Rightarrow \\ \frac{\partial \Pi}{\partial S} &= \frac{\partial C}{\partial S} - \frac{\partial C}{\partial S} = 0.\end{aligned}$$

The delta of the investor's hedge position is therefore zero.

The delta of the asset position offsets the delta of the option position.

A position with a delta of zero is referred to as being delta neutral.

It is important to realize that the investor's position only remains delta hedged (or delta neutral) for a relatively short period of time. This is because delta changes with both changes in the stock and the passage of time.

### The Gamma of a call option:

The second derivative of the call option with respect to the price of the stock is called the Gamma of the option and is given by

$$\frac{\vartheta^2 C_t}{\vartheta S_t^2} = \frac{\vartheta \Delta}{\vartheta S_t} = \frac{\vartheta N(d_1)}{\vartheta S_t} \quad (6)$$

Recall that from equation (2) we have

$$\frac{\vartheta N(d_1)}{\vartheta S_t} = \frac{\vartheta N(d_1) \vartheta d_1}{\vartheta d_1 \vartheta S_t}$$

where

$$\frac{\vartheta N(d_1)}{\vartheta d_1} = N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$$

and

$$\frac{\vartheta d_1}{\vartheta S_t} = \frac{\vartheta d_2}{\vartheta S_t} = \frac{1}{\sigma S_t \sqrt{T - t}}$$

$\Delta$  and the time to maturity for an at the money call option:

Recall that

$$\Delta = \frac{\vartheta C_t}{\vartheta S_t} = N(d_1)$$

Thus applying the chain rule gives

$$\frac{\vartheta \Delta}{\vartheta \tau} = N'(d_1) \frac{\vartheta d_1}{\vartheta \tau},$$

where

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} > 0$$

For an at the money call option ( $S_t = X$ ), since  $\ln(\frac{S_t}{X}) = 0$ , we have

$$d_1 = \frac{(r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = \frac{(r + \frac{1}{2}\sigma^2)\sqrt{\tau}}{\sigma}$$

Thus

$$\frac{\vartheta d_1}{\vartheta \tau} = \frac{(r + \frac{1}{2}\sigma^2)}{2\sigma\sqrt{\tau}} > 0$$

It can be shown that  $\frac{\vartheta^2 \Delta}{\vartheta \tau^2} < 0$

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where  $d_1$  and  $d_2$  are given by

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$$d_2 = \frac{\ln(\frac{S_t}{X}) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

or

$$d_2 = d_1 - \sigma\sqrt{\tau}$$



Theta:

Theta is the rate of change of the price of the call with respect to time to maturity-with all else remaining the same:

$$\frac{\vartheta C_t}{\vartheta \tau} = S_t N'(d_1) \frac{\vartheta d_1}{\vartheta \tau} + r X e^{-r\tau} N(d_2) - X e^{-r(T-t)} N'(d_2) \frac{\vartheta d_2}{\vartheta \tau}$$

Utilizing the fact that

$$\frac{\vartheta d_2}{\vartheta \tau} = \frac{\vartheta d_1}{\vartheta \tau} - \frac{\sigma}{2\sqrt{\tau}}$$

we write

$$\frac{\vartheta C_t}{\vartheta \tau} = S_t N'(d_1) \frac{\vartheta d_1}{\vartheta \tau} + \underbrace{r X e^{-r\tau} N(d_2)}_{S_t N'(d_1)} - \underbrace{X e^{-r(T-t)} N'(d_2) \frac{\vartheta d_1}{\vartheta \tau}}_{S_t N'(d_1)} + \underbrace{X e^{-r(T-t)} N'(d_2) \frac{\sigma}{2\sqrt{\tau}}}_{S_t N'(d_1)}$$

Since

$$S_t N'(d_1) = X e^{-r(T-t)} N'(d_2)$$

$\frac{\partial C}{\partial \tau}$  simplifies to

$$\frac{\partial C_t}{\partial \tau} = +rXe^{-r\tau}N(d_2) + Xe^{-r(T-t)}N'(d_2)\frac{\sigma}{2\sqrt{\tau}} > 0$$

or

$$\Theta = \frac{\partial C_t}{\partial t} = -\frac{\partial C_t}{\partial \tau} < 0$$

which is negative.

This is because as the time to maturity decrease the option tends to become less valuable.

Theta is not the same type of hedge parameter as delta and gamma.

This is because although there is some uncertainty about the future stock price there is no uncertainty about the passage of time.

It does not make sense to hedge against the effect of the passage of time on an option portfolio.

Vega:

The vega of a derivative security,  $\Lambda$ , is defined as the rate of change of its price with respect to the volatility of the underlying asset.

For a European (on a non-dividend paying stock) call option is given by

$$\frac{\partial C_t}{\partial \sigma} = S_t N'(d_1) \frac{\partial d_1}{\partial \sigma} - X e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial \sigma}.$$

Utilizing the fact that

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{\tau}$$

we write

$$\begin{aligned} \frac{\partial C_t}{\partial \sigma} = & \underbrace{S_t N'(d_1)}_{X e^{-r(T-t)} N'(d_2)} \frac{\partial d_1}{\partial \sigma} - X e^{-r(T-t)} N'(d_2) \frac{\partial d_1}{\partial \sigma} \\ & + \underline{X e^{-r(T-t)} N'(d_2) \sqrt{\tau}}. \end{aligned}$$

$$\begin{aligned} \frac{\vartheta C_t}{\vartheta \sigma} = & \underbrace{S_t N'(d_1)}_{X e^{-r(T-t)} N'(d_2)} \frac{\vartheta d_1}{\vartheta \sigma} - X e^{-r(T-t)} N'(d_2) \frac{\vartheta d_1}{\vartheta \sigma} \\ & + \underline{X e^{-r(T-t)} N'(d_2) \sqrt{\tau}}. \end{aligned}$$

Since

$$S_t N'(d_1) = X e^{-r(T-t)} N'(d_2)$$

$\frac{\vartheta C}{\vartheta \tau}$  simplifies to

$$\begin{aligned} \frac{\vartheta C_t}{\vartheta \sigma} &= X e^{-r(T-t)} N'(d_2) \sqrt{\tau} = \\ &= S_t N'(d_1) \sqrt{\tau} > 0. \end{aligned}$$

Rho:

The rho of a derivative security is defined as the rate of change of its price with respect to the interest rate.

For a European (on a non-dividend paying stock) call option is given by

$$\begin{aligned} \frac{\partial C_t}{\partial r} = & \underbrace{S_t N'(d_1)}_{X e^{-r(T-t)} N'(d_2)} \underbrace{\frac{\partial d_1}{\partial r}}_{\frac{\partial d_2}{\partial r}} - X e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial r} \\ & + \underline{(T - t) X e^{-r(T-t)} N(d_2)}. \end{aligned}$$

Utilizing the fact that

$$\begin{aligned} \frac{\partial d_2}{\partial r} &= \frac{\partial d_1}{\partial r}, \text{ and} \\ S_t N'(d_1) &= X e^{-r(T-t)} N'(d_2) \end{aligned}$$

$\frac{\partial C_t}{\partial r}$  simplifies to

$$\frac{\partial C_t}{\partial r} = (T - t) X e^{-r(T-t)} N(d_2) > 0$$

## Put-Call Parity:

Recall the put-call parity:

$$P_t = C_t + Xe^{-r(T-t)} - S_t$$

Utilizing the Black and Scholes formula for the call we write

$$\begin{aligned} P_t &= \underbrace{S_t N(d_1) - Xe^{-r(T-t)} N(d_2)}_{C_t} + Xe^{-r(T-t)} - S_t \\ &= S_t \underbrace{[N(d_1) - 1]}_{-N(-d_1)} + Xe^{-r(T-t)} \underbrace{[1 - N(d_2)]}_{N(-d_2)} \\ &= -N(-d_1)S_t + Xe^{-r(T-t)} N(-d_2). \end{aligned}$$

## Delta of a (European; non-dividend paying stock) put option

Rewrite the put call parity

$$P_t = C_t + Xe^{-r(T-t)} - S_t$$

It follows that the Delta of the put option is given by

$$\begin{aligned}\Delta_p &= \frac{\vartheta P_t}{\vartheta S_t} = \frac{\vartheta C_t}{\vartheta S_t} - 1 = \\ &= N(d_1) - 1 = -N(-d_1) < 0\end{aligned}$$

## Gamma of a (Europ.; non-dividend paying stock) put option

The Gamma of the put option is

$$\begin{aligned}\frac{\vartheta P_t}{\vartheta S_t} &= \frac{\vartheta C_t}{\vartheta S_t} - 1 \Rightarrow \\ \Gamma_p &= \frac{\vartheta^2 P_t}{\vartheta S_t^2} = \frac{\vartheta^2 C_t}{\vartheta S_t^2} = N'(d_1) \frac{\vartheta d_1}{\vartheta S_t} \\ &= \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}}_{N'(d_1)} \underbrace{\frac{1}{\sigma S_t \sqrt{T-t}}}_{\frac{\vartheta d_1}{\vartheta S_t}} > 0\end{aligned}$$

Since the Gamma of the call option is

$$\Gamma_c = \frac{\vartheta^2 C_t}{\vartheta S_t^2} = N'(d_1) \frac{\vartheta d_1}{\vartheta S_t}$$



## Theta of a (Europ.; non-dividend paying stock) put option

Rewrite the put call parity

$$P_t = C_t + Xe^{-r(T-t)} - S_t$$

It follows that the theta of the put option is given by

$$\begin{aligned}\frac{\vartheta P_t}{\vartheta \tau} &= \frac{\vartheta C_t}{\vartheta \tau} - rXe^{-r\tau} = \\ &\quad + \underbrace{\boxed{rXe^{-r\tau}} N(d_2) + Xe^{-r(T-t)} N'(d_2) \frac{\sigma}{2\sqrt{\tau}}}_{\frac{\vartheta C_t}{\vartheta \tau}} \\ &\quad - \boxed{rXe^{-r\tau}} \\ &= rXe^{-r\tau} \underbrace{[N(d_2) - 1]}_{-N(-d_2)} + \underbrace{Xe^{-r(T-t)} N'(d_2)}_{S_t N'(d_1)} \frac{\sigma}{2\sqrt{\tau}} \\ &= -rXe^{-r\tau} N(-d_2) + S_t N'(d_1) \frac{\sigma}{2\sqrt{\tau}}\end{aligned}$$

Thus

$$\begin{aligned}\Theta_p &= \frac{\vartheta P_t}{\vartheta t} = -\frac{\vartheta P_t}{\vartheta \tau} \\ &= rXe^{-r\tau} N(-d_2) - S_t N'(d_1) \frac{\sigma}{2\sqrt{\tau}}\end{aligned}$$

Vega:

For a European (on a non-dividend paying stock) put option the vega is given by

$$\Lambda_p = \frac{\vartheta P_t}{\vartheta \sigma} = \Lambda_c = \frac{\vartheta C_t}{\vartheta \sigma} = S_t N'(d_1) \sqrt{\tau} > 0$$

since according to the put-call parity

$$P_t = C_t + X e^{-r(T-t)} - S_t$$

Rho:

Recall the put-call parity

$$P_t = C_t + Xe^{-r(T-t)} - S_t$$

For a European (on a non-dividend paying stock) put option the rho is given by

$$\begin{aligned}\frac{\partial P_t}{\partial r} &= \frac{\partial C_t}{\partial r} - (T-t)Xe^{-r(T-t)} \\ &= \underbrace{(T-t)Xe^{-r(T-t)}}_{\frac{\partial C_t}{\partial r}} N(d_2) - (T-t)Xe^{-r(T-t)} \\ &= (T-t)Xe^{-r(T-t)} \underbrace{[N(d_2) - 1]}_{-N(-d_2)} \\ &= -(T-t)Xe^{-r(T-t)} N(-d_2) < 0\end{aligned}$$

since

$$\frac{\partial C_t}{\partial r} = (T-t)Xe^{-r(T-t)} N(d_2)$$