Lecture 11: Black-Scholes and the "Greeks"

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Stochastic Optimisation and Derivatives

Black-Scholes and the “Greeks”

Overview

Today's Lecture

We shall examine:

- the Black-Scholes formulæ for calls, puts and simple digitals
- the meaning and importance of the "greeks", delta, gamma, theta, vega and rho,
- formulæ for the "greeks" for calls, puts and simple digitals.

Today's Lecture

The Black-Scholes equation has solutions for calls, puts and some other contracts.

- We list the equations for calls, puts and binaries.
- The "delta", the first derivative of the option value with respect to the underlying, occurs as an important quantity in the derivation of the Black-Scholes equation.
- In this lecture we see the importance of other derivatives of the option price, with respect to the variables and with respect to some of the parameters.
- These derivatives are important in the hedging of an option position, playing key roles in risk management.

Overview

Call Option Value

$$\text{Call option value } = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and}$$

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Call Option Value

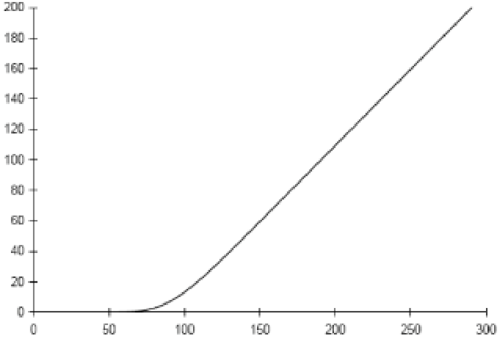

Figure: The value of a call option as a function of the underlying at a fixed time before expiry.

Call Option: At-the-money

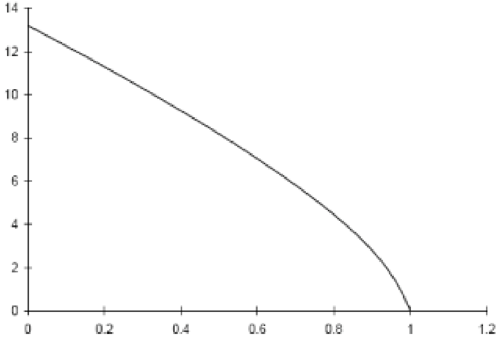

Figure: The value of a European call option at-the-money as a function of time.

Call Option: 3-D Graph

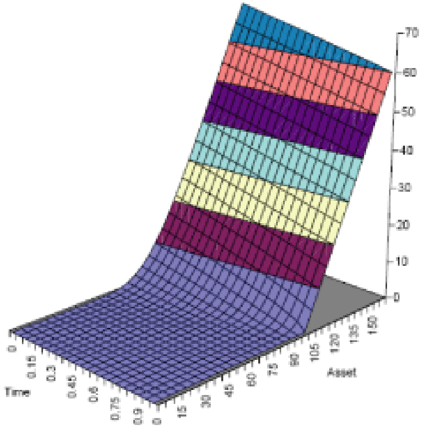

Figure: The value of a European call option as a function of the asset price and time.

Call Options on dividend-paying assets

When the option is for an asset with a continuous paying dividend or currency:

Call option value

$$Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\log(S/E) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/E) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

Figure: The value of a European call option as a function of the asset price and time.

Formula for a put

Given that the payoff for a put option is

$$Payoff(S) = max(E - S, 0),$$

the formula for a put option is:

$$\text{Put option value } = -SN(-d_1) + Ee^{-r(T-t)}N(-d_2)$$

NB: d_1 and d_2 are the same as for the call option previously.

Value of a Put Option

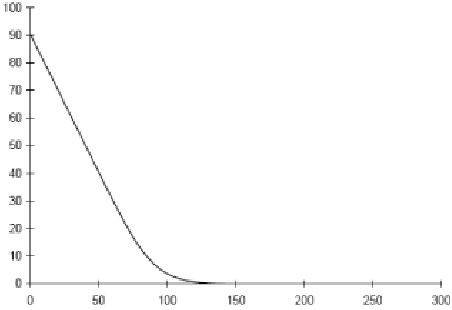

Figure: The value of a put option as a function of the underlying at a fixed time to expiry.

Put Option: At-the-money

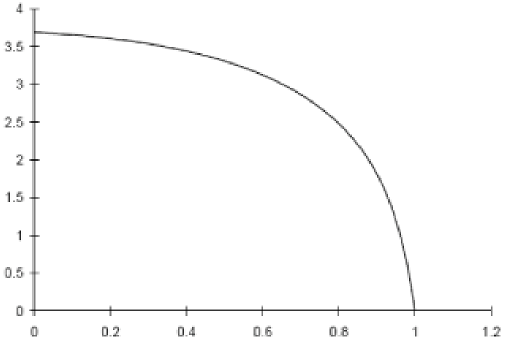

Figure: The value of a European put option at-the-money as a function of time.

Put Option: 3-D Graph

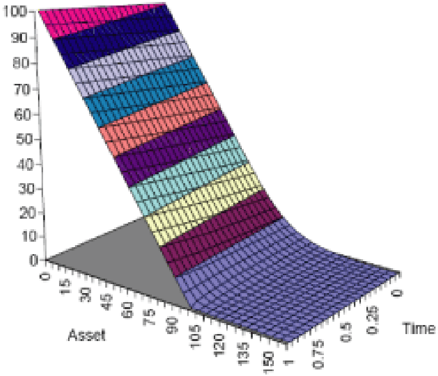

Figure: The value of a European put option as a function of the asset price and time.

Put Options on dividend-paying assets

When the put option is for an asset with a continuous paying dividend or currency:

Put option value

$$-Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$$

Figure: The value of a European put option as a function of the asset price and time.

Binary Call

The binary call has payoff:

$$Payoff(S) = \mathcal{H}(S = E),$$

where $\mathcal{H}$ is the Heaviside function (taking the value 1 if its argument is positive and zero otherwise).

Binary call option value

$$e^{-r(T-t)}N(d_2)$$

Binary Call Value

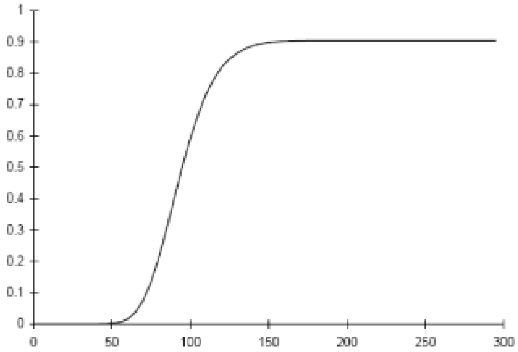

Figure: The value of a binary call option.

Binary Put Value

The binary put has a payoff of one if $S < E$ at expiry. It has a value of:

Binary put option value

$$e^{-r(T-t)}(1 - N(d_2))$$

A binary call and a binary put must add up to the present value of \$1 received at time T.

Binary Put Value

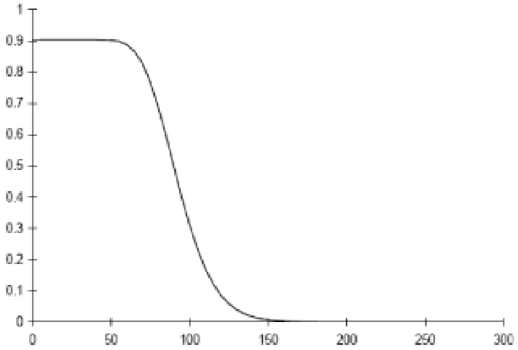

Figure: The value of a binary put option.

Overview

Delta

The delta of an option or a portfolio of options is the **sensitivity** of the option or portfolio to the underlying. It is the *rate of change of value with respect to the asset*:

$$\Delta = \frac{\delta V}{\delta S}$$

Here V can be the value of a single contract or of a whole portfolio of contracts. The delta of a portfolio of options is just the sum of the deltas of all the individual positions.

Delta Hedging

The theoretical device of delta hedging for eliminating risk is far more than that, it is a very important practical technique.

- Delta hedging means holding one of the option and short a quantity Δ of the underlying.
- Delta can be expressed as a function of S and t.
- This function varies as S and t vary.
- This means that the number of assets held must be continuously changed to maintain a *delta neutral position*, this procedure is called **dynamic hedging**.
- Changing the number of assets held requires the continual purchase and/or sale of the stock. This is called **rehedging** or **rebalancing** the portfolio.

Formulæ

Here are some formulæ for the deltas of common contracts (assuming that the underlying pays dividends or is a currency):

Deltas of common contracts

Call $e^{-D(T-t)}N(d_1)$

Put $e^{-D(T-t)}(N(d_1)-1)$

Binary call $\dfrac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$

Binary put $-\dfrac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$

$$N'(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

Deltas

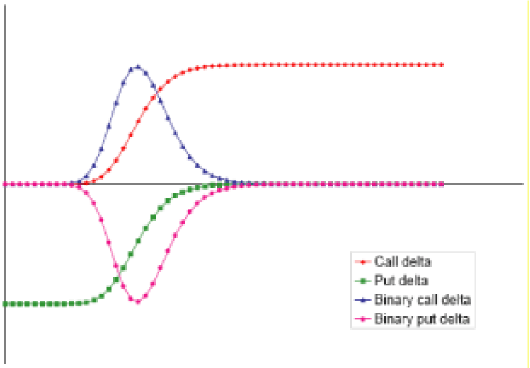

Figure: The deltas of call, put and binary options.

Gamma

The gamma, Γ, of an option or a portfolio of options is the second derivative of the position with respect to the underlying:

$$\Gamma = \frac{\delta^2 V}{\delta S^2}$$

Since gamma is the sensitivity of the delta to the underlying it is a measure of by how much or *how often a position must be rehedged* in order to maintain a delta-neutral position.

Gamma

Because costs can be large and because one wants to reduce exposure to model error it is natural to try to minimize the need to rebalance the portfolio too frequently.

- Since gamma is a measure of sensitivity of the hedge ratio Δ to the movement in the underlying, the hedging requirement can be decreased by a gamma-neutral strategy.
- This means buying or selling more *options*, not just the underlying.

Gamma contd.

- Because the gamma of the underlying (its second derivative) is zero, we cannot add gamma to our position just with the underlying.
- We can have as many options in our position as we want, we choose the quantities of each such that both delta and gamma are zero.

Gamma Equations

Gammas of common contracts

Call	$\dfrac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Put	$\dfrac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Binary call	$-\dfrac{e^{-r(T-t)}d_1 N'(d_2)}{\sigma^2 S^2 (T-t)}$
Binary put	$\dfrac{e^{-r(T-t)}d_1 N'(d_2)}{\sigma^2 S^2 (T-t)}$

Gammas

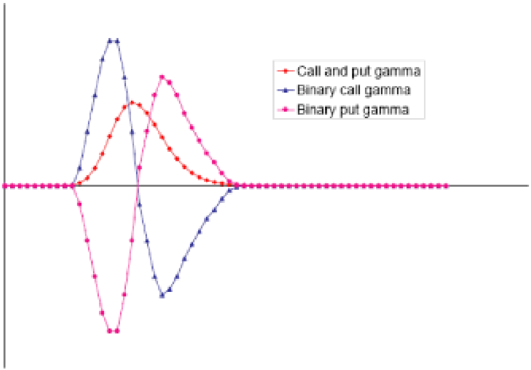

Figure: The gammas of a call, put and binary option.

Theta

Theta, Θ, is the rate of change of the option price with time.

$$\Theta = \frac{\delta V}{\delta t}$$

- The theta is related to the option value, the delta and the gamma by the Black-Scholes equation.
- In a delta-hedged portfolio the theta contributes to ensuring that the portfolio earns the risk-free rate.

Theta equations

Thetas of common contracts

Call	$-\frac{\sigma S e^{-D(T-t)} N'(d_1)}{2\sqrt{T-t}} + DSN(d_1)e^{-D(T-t)} - rEe^{-r(T-t)}N(d_2)$
Put	$-\frac{\sigma S e^{-D(T-t)} N'(-d_1)}{2\sqrt{T-t}} - DSN(-d_1)e^{-D(T-t)} + rEe^{-r(T-t)}N(-d_2)$
Binary call	$re^{-r(T-t)}N(d_2) + e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$
Binary put	$re^{-r(T-t)}(1 - N(d_2)) - e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$

Figure: Theta Equations

Theta contd.

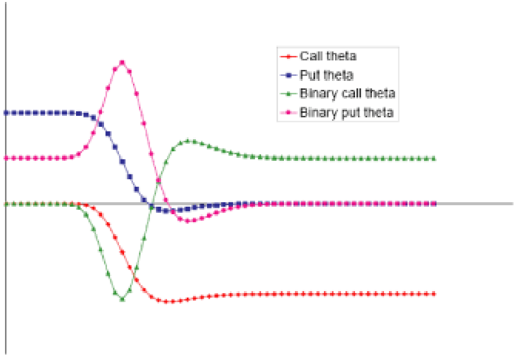

Figure: The thetas of a call, put and binary options.

Vega

Vega (also known as zeta and kappa) is a very important but confusing quantity.

- It is the sensitivity of the option price to volatility.

$$Vega = \frac{\delta V}{\delta \sigma}$$

- This is a completely different from the other greeks since it is a derivative with respect to a parameter and not a variable.
- As with gamma hedging, one can vega hedge to reduce sensitivity to the volatility.
- This is a major step towards eliminating some model risk, since it *reduces dependence on a quantity that is not known very accurately*.

Vega

Vegas of common contracts

Call $S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$

Put $S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$

Binary call $-e^{-r(T-t)}N'(d_2)\left(\sqrt{T-t}+\frac{d_2}{\sigma}\right)$

Binary put $e^{-r(T-t)}N'(d_2)\left(\sqrt{T-t}+\frac{d_2}{\sigma}\right)$

Vega

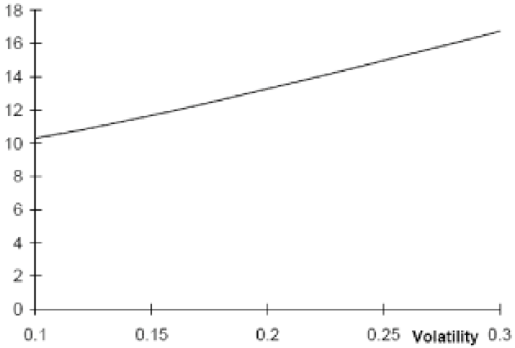

Figure: An at-the-money call option as a function of volatility.

Rho

Rho, ρ is the sensitivity of the option value to the interest rate used in Black-Scholes:

$$\rho = \frac{\delta V}{\delta r}$$

Rhos of common contracts

Call	$E(T-t)e^{-r(T-t)}N(d_2)$
Put	$-E(T-t)e^{-r(T-t)}N(-d_2)$
Binary call	$-(T-t)e^{-r(T-t)}N(d_2) + \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$
Binary put	$-(T-t)e^{-r(T-t)}(1-N(d_2)) - \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

Overview

Implied Volatility

- The Black-Scholes formula for a call option takes as input the expiry, the strike, the underlying and the interest rate together with the volatility to output the price.

All but the volatility are easily measured.

How do we know what volatility to put into the formulæ?

A trader can see on his screen that a certain call option with four months until expiry and a strike of 100 is trading at 6.51 with the underlying at 101.5 and a short-term interest rate of 8%. Can we use this information in some way?

Implied Volatility Continued

Turn the relationship between volatility and an option price on its head, if we can see the price at which the option is trading, we can ask

- "What volatility must I use to get the correct market price?"
- The implied volatility is the volatility of the underlying which when substituted into the Black-Scholes formula gives a theoretical price equal to the market price.

In a sense it is the market's view of volatility over the life of the option.

Solving Black-Scholes backwards

Because there is no simple formula for the implied volatility as a function of the option value we must solve the equation

$$v_{BS} = (S_0, t_0; \sigma, r; E, T) = \text{Known Market Value},$$

for σ. (V_{BS} is the Black-Scholes formula, todays asset price is S_0, date is t_0.) Everything is known in this equation except σ.

Volatility Smile

In practice if we calculate the implied volatility for many different strikes and expiries on the same underlying then we find that the volatility is not constant.

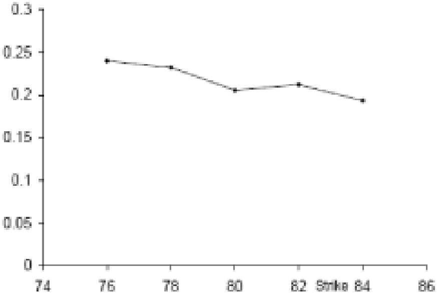

Figure: Implied volatilities for the DJIA.

Summary

- Black-Scholes Introduction
- Option Values for calls, puts
- "Greeks" (Delta, Gamma, Theta, Vega, Rho)
- Implied Volatility