Financial Derivatives
(Binomial to Black Scholes)

Financial Engineering

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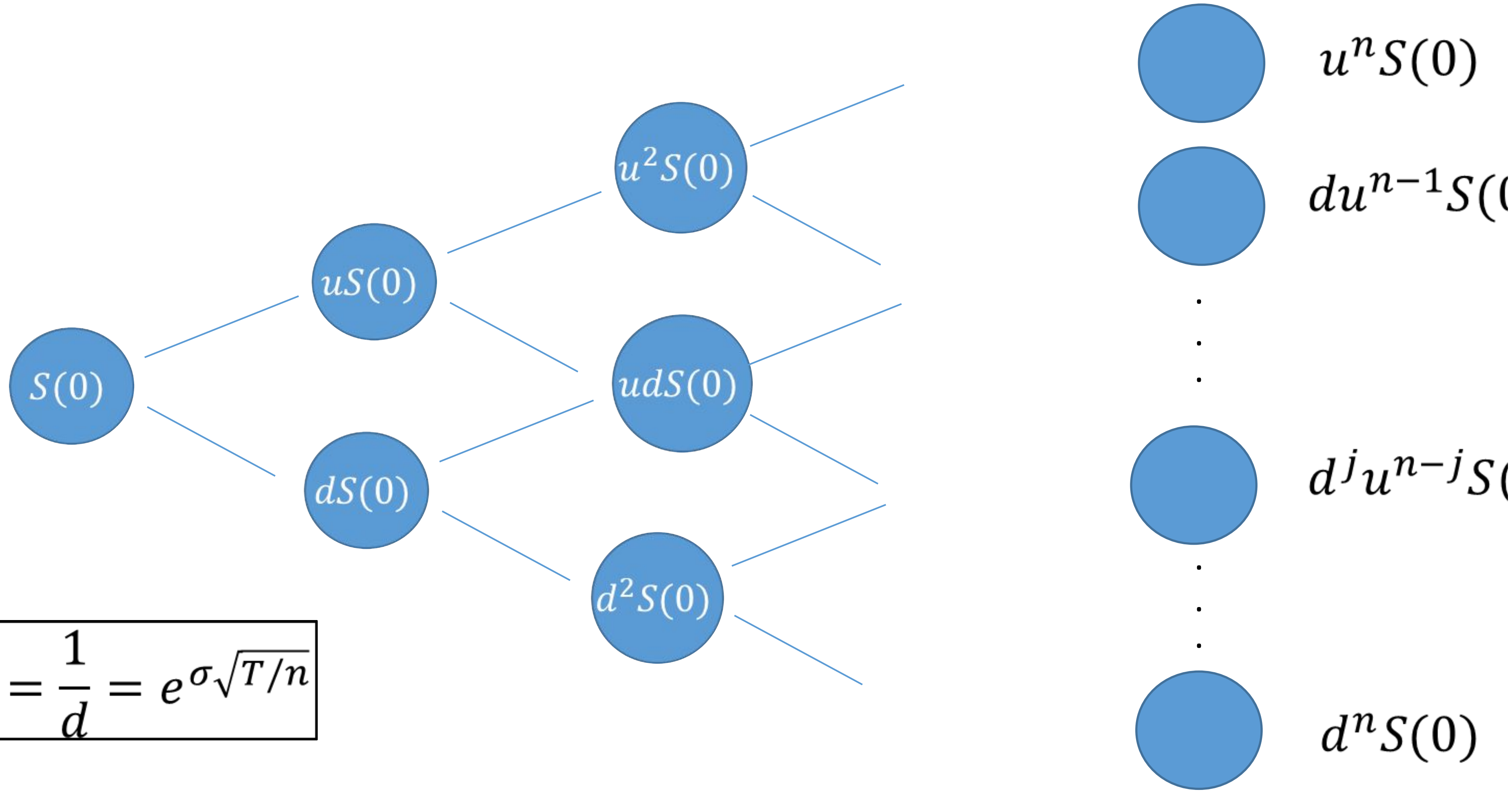
Option Pricing – n Period Binomial Model

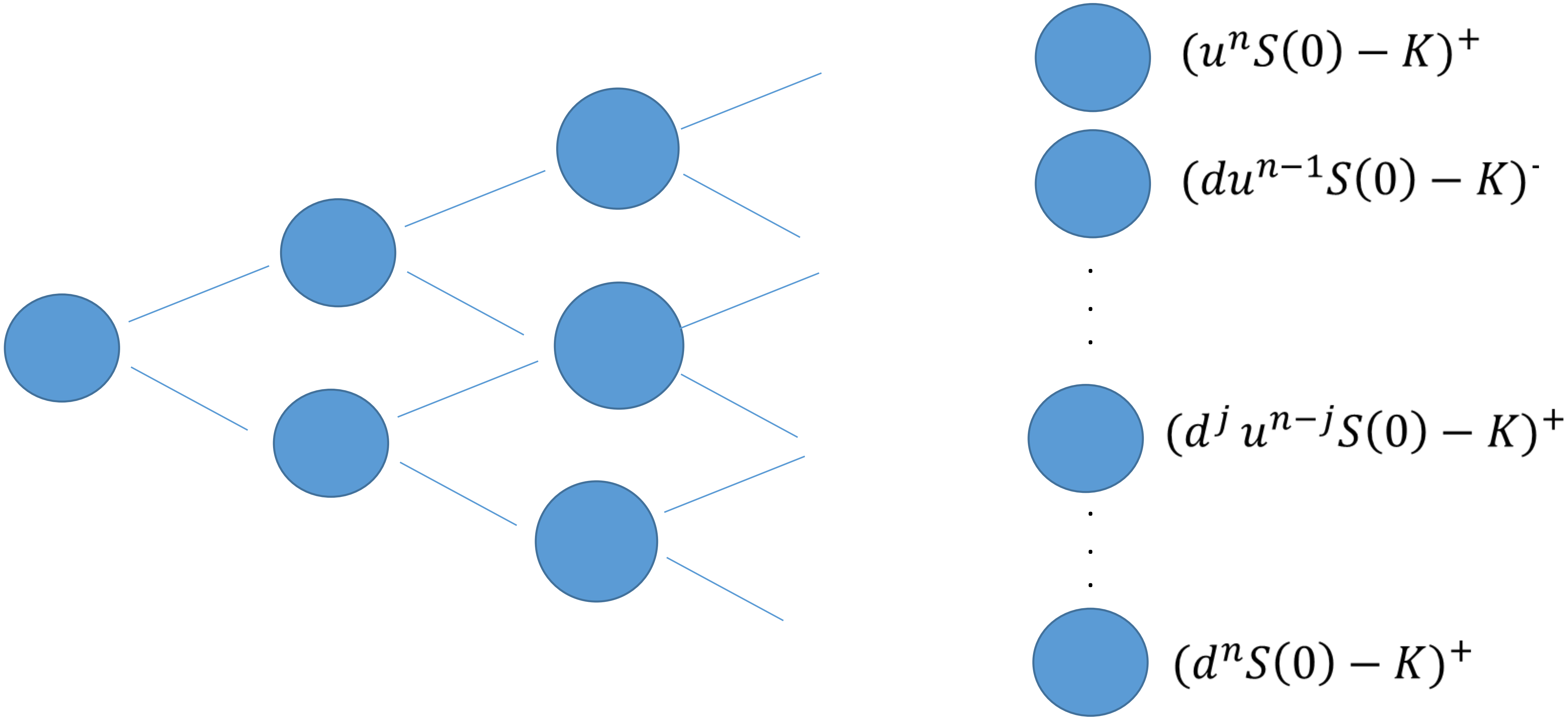
Stock Price Diagram



$$\boxed{u = \frac{1}{d} = e^{\sigma\sqrt{T/n}}}$$

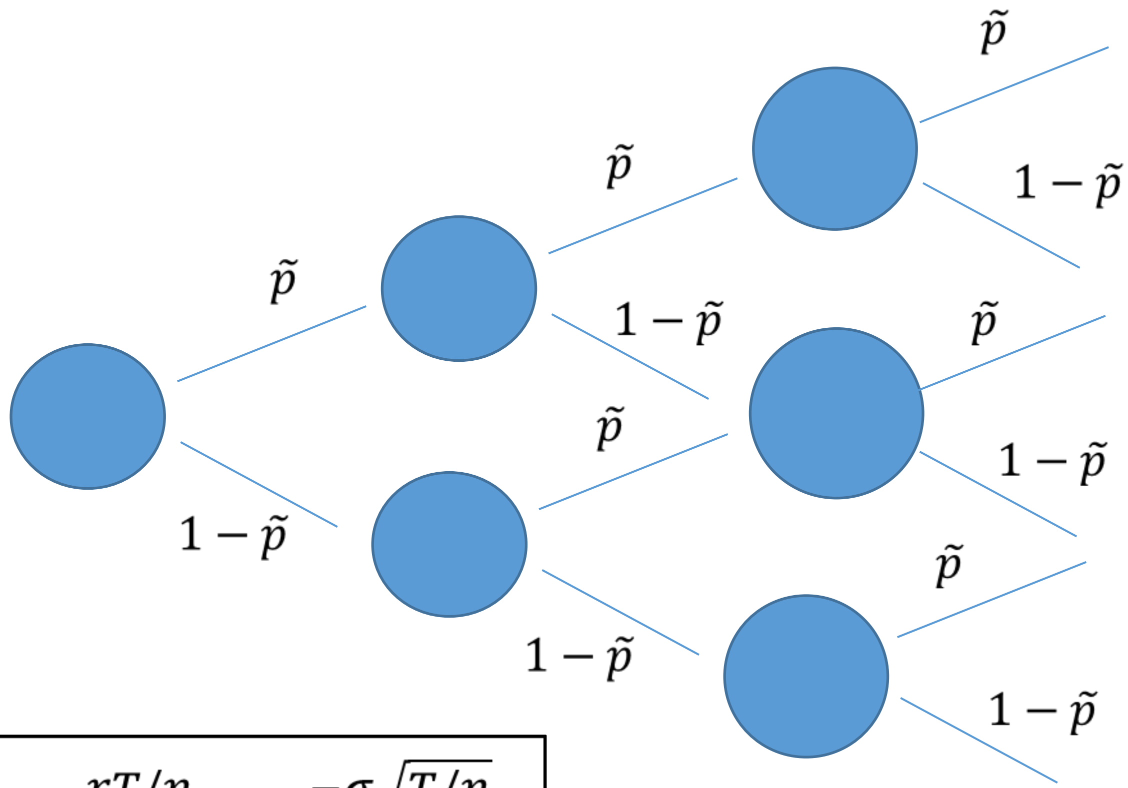
Option Pricing – n Period Binomial Model

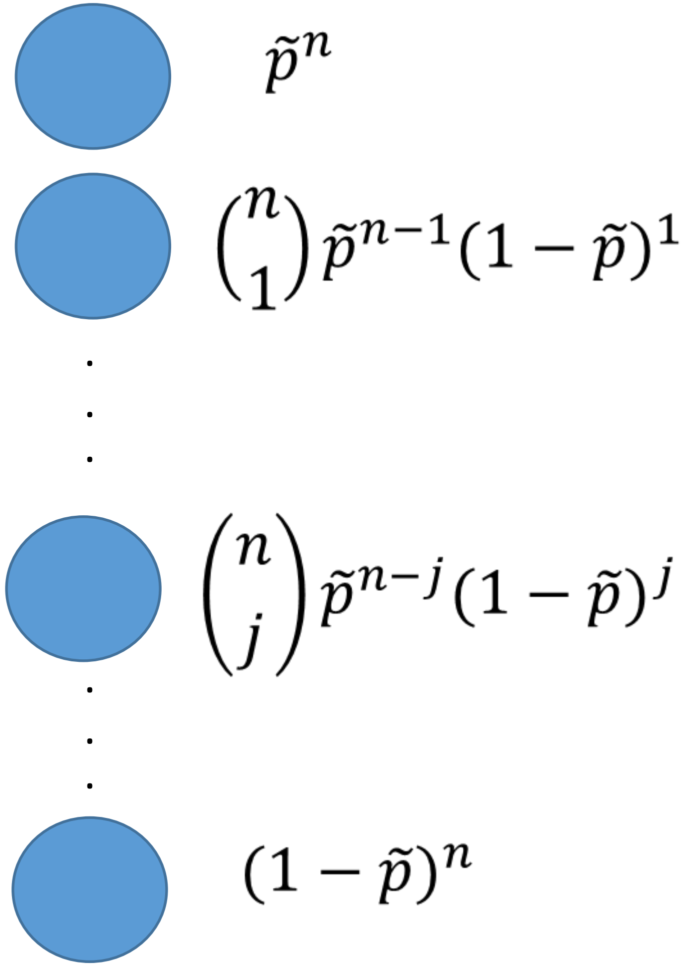
Call Option Price (Pay off) Diagram



Option Pricing – n Period Binomial Model

Probability Diagram





$$\tilde{p} = \frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}$$

Option Pricing – Multi Period Binomial Model

Assuming n period between time 0 and T, price of a European option is given by

$$C(0) = e^{-rT}\tilde{E}(f(S(T))), \qquad \Delta t = T/n$$

- $$C(0) = e^{-rT}\sum_{j=1}^{n}\binom{n}{j}\tilde{p}^{j}(1-\tilde{p})^{n-j}(d^{j}u^{n-j}S(0)-K,0)^{+},$$

$$(d^{j}u^{n-j}S(0)-K,0)^{+} = \begin{cases} d^{j}u^{n-j}S(0)-K, & if\ \ d^{j}u^{n-j}S(0)-K>0 \\ 0, & otherwise \end{cases}$$

Option Pricing – Multi Period Binomial Model

$$C(0) = e^{-rT} \sum_{for\ some\ j} \binom{n}{j} \tilde{p}^{j} (1-\tilde{p})^{n-j} (d^j u^{n-j} S(0) - K)$$

$$For\ j\ such\ that\ d^j u^{n-j} S(0) - K > 0$$

$$That\ is,\ j > \frac{n}{2} - \frac{\ln(\frac{S(0)}{K})}{\sigma\sqrt{T/n}} = \propto (say)$$

Thus,

$$C(0) = e^{-rT}(S(0)\ A - K\ B)$$

Option Pricing – Approximation

$$\textit{Where} \quad A = \sum_{j>\propto} \binom{n}{j} \tilde{p}^j (1-\tilde{p})^{n-j} d^j u^{n-j}$$

$$\textit{and} \quad B = \sum_{j>\propto} \binom{n}{j} \tilde{p}^j (1-\tilde{p})^{n-j}$$

Clearly $B = P(X > \propto)$, *where* $X \sim B(n, \tilde{p})$

By Central Limit Theorem, if X follows Binomial with parameters n and p, then for large n,

$$\frac{X - np}{\sqrt{np(1-p)}} = Z \sim N(0,1)$$

Option Pricing – Multi Period Binomial Model

$$\textit{Therefore,} \quad B = P\left(Z > \frac{\propto - n\tilde{p}}{\sqrt{n\tilde{p}(1-\tilde{p})}}\right) = N\left(\frac{n\tilde{p} - \propto}{\sqrt{n\tilde{p}(1-\tilde{p})}}\right)$$

$$\textit{Using} \propto = \frac{n}{2} - \frac{\ln\left(\frac{S(0)}{K}\right)}{\sigma\sqrt{T/n}} \ \textit{and} \ \tilde{p} = \frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}$$

$$\textit{We get} \ B = N(d_2),$$

$$\textit{Where} \ d_2 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Option Pricing – Approximation

$$\textit{Similarly,}\quad A = e^{rT} N(d_1),$$

$$\textit{Where}\ \ d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\textit{Hence,}\ C(0) = e^{-rT}\left(S(0)\ A - K\ B\right)\ \textit{implies}$$

$$\boxed{C(0) = S(0)\ N(d_1) - \ e^{-rT} K\, N(d_2)}$$

This is Black-Scholes-Merton formula

Option Pricing – Black-Scholes formula

$$\boxed{C(0) = S(0)\, N(d_1) - e^{-rT} K\, N(d_2)}$$

$$Where\ \ d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$and \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Here, r *– interest rate,* σ *– annual volatility*

$S(0)$ *– spot price,* K *– strike price,* T *– maturity time*

Example

- Current price of a stock is Rs 42 which has annual volatility of 20%. The risk free rate is 10%. Value a 6 months call option having strike price Rs 40. We have S(0)=42, K=40, r=0.1, T=1/2, σ=0.2

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = 0.77 \qquad d_2 = d_1 - \sigma\sqrt{T} = 0.63$$

$$\begin{aligned} C(0) &= S(0)\, N(d_1) - e^{-rT} K\, N(d_2) \\ &= 42\, N(0.77) - 38.05\, N(0.63) \\ &= 4.76 \end{aligned}$$