M19MA014

Assignment

Financial Engineering

Q1) Given

$S(0) = 50$

$K = 52$

$T = 1$ y.

$u = 1.2$

$d = 0.8$

$r = 0.05$

$\Delta t = 1/2$

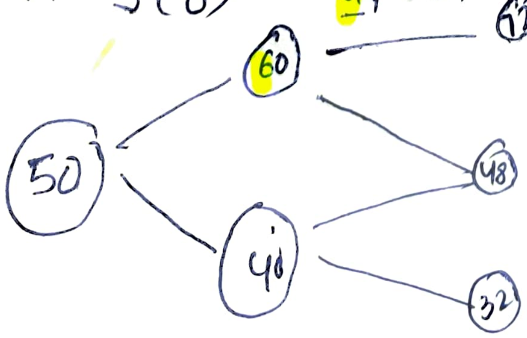
$$\tilde{p} = \frac{e^{r\Delta t - t}}{u-d} = \frac{e^{0.05 \times \frac{1}{2}} - 0.8}{\frac{1}{2} - 0.8}$$

$$= 0.56328$$

$$\simeq 0.56$$

where $\tilde{p}$: risk neutral probability

$\therefore$ S(0) after 6 mt (1/2) 1 year Payoff $(S(T)-K)^+$



20

$\boxed{0}$

$\boxed{0}$

$(S(T)-K)^+ = 72-52$

$= 20$

20

$48-52$

$= \boxed{0}$

$= \boxed{0}$

$\therefore$ European call price

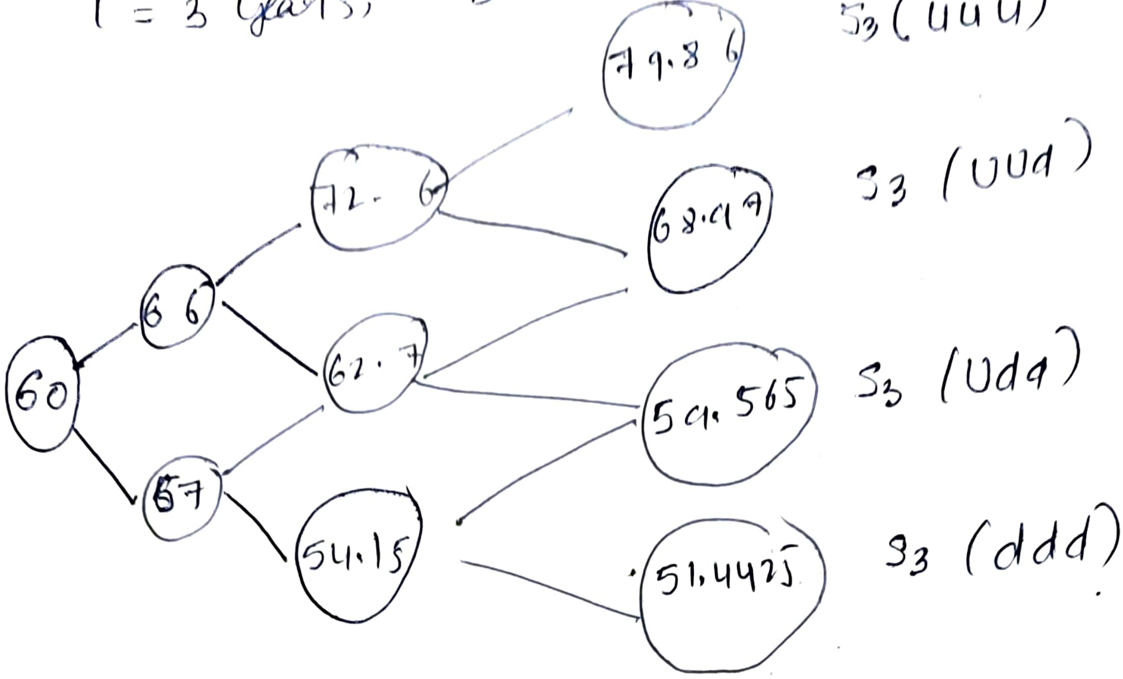
$$c(0) = e^{-2(0.05\times 1/2)}\left[\tilde{p}^2 \times 20 + 2\tilde{p}(1-\tilde{p})\times 0 + (1-\tilde{p})^2 \times 0\right]$$

$$= e^{-0.05} \times \left[(0.56)^2 \times 20 + 0 + 0\right]$$

$$= 5.966$$

Q2) $S(0) = 60$, $K = 62$, $u = 1.1$, $d = 0.95$, $r = 0.03$

$T = 3$ years, $\Delta t = 1$



European put option

pay off $(K - S(T))^+$

$$\tilde{p} = \frac{e^r - d}{u - d}$$

$f(S_3(uuu)) = 0$

$f(S_3(uud)) = 0$

$f(S_3(udd)) = 2.435$

$f(S_3(ddd)) = 10.55075$

$$= \frac{e^{0.03} - 0.95}{1.1 - 0.95}$$

$$= 0.5364$$

Now $\tilde{E}f(S(T)) = {}^3C_0\, \tilde{p}^3\, f(S_3(uuu)) + {}^3C_1\, \tilde{p}^2(1-\tilde{p})\, f(S_3(uud)) + {}^3C_2\, \tilde{p}(1-\tilde{p})^2 f(S_3(udd)) + {}^3C_3 (1-\tilde{p})^3 f(S_3(ddd))$

$= (0.5364)^3 \times 0 + 3 \times 0.4636 \times (0.5364)^2 \times 0$

$+ 3 \times (0.5364)(0.4636)^2 \times 2.435$

$+ (0.4636)^3 \times 10.5575$

$= 0 + 0 + 0.8422 + 1.0519$

$= 1.8941$

Now $P(0) = e^{-3r} \tilde{E}(f(S(T)))$

$= e^{-0.03 \times 3} \times 1.8941$

$= 1.731$

European Call option

Pay off $(S(T) - K)^+$

$f(S_3(uuu)) = 17.86$

$f(S_3(uud)) = 6.97$

$f(S_3(udd)) = 0$

$f(S_3(ddd)) = 0$

$\tilde{E} f(S(T)) = {}^3C_0 \tilde{p}^3 f(S_3(uuu)) + {}^3C_1 \tilde{p}^2 (1-\tilde{p}) + (S_3(uud)) + {}^3C_2 \tilde{p} (1-\tilde{p})^2 + (S_3(udd)) + {}^3C_3 (1-\tilde{p})^3 f(S_3(ddd))$

$= (0.5364)^3 \times 17.86 + 3 \times (0.5364)^2 \times 0.4636 \times 6.97 + 3(0.5364)(0.4636)^2 \times 0 + (0.4636)^3 \times 0$

$= 2.7564 + 2.7892$

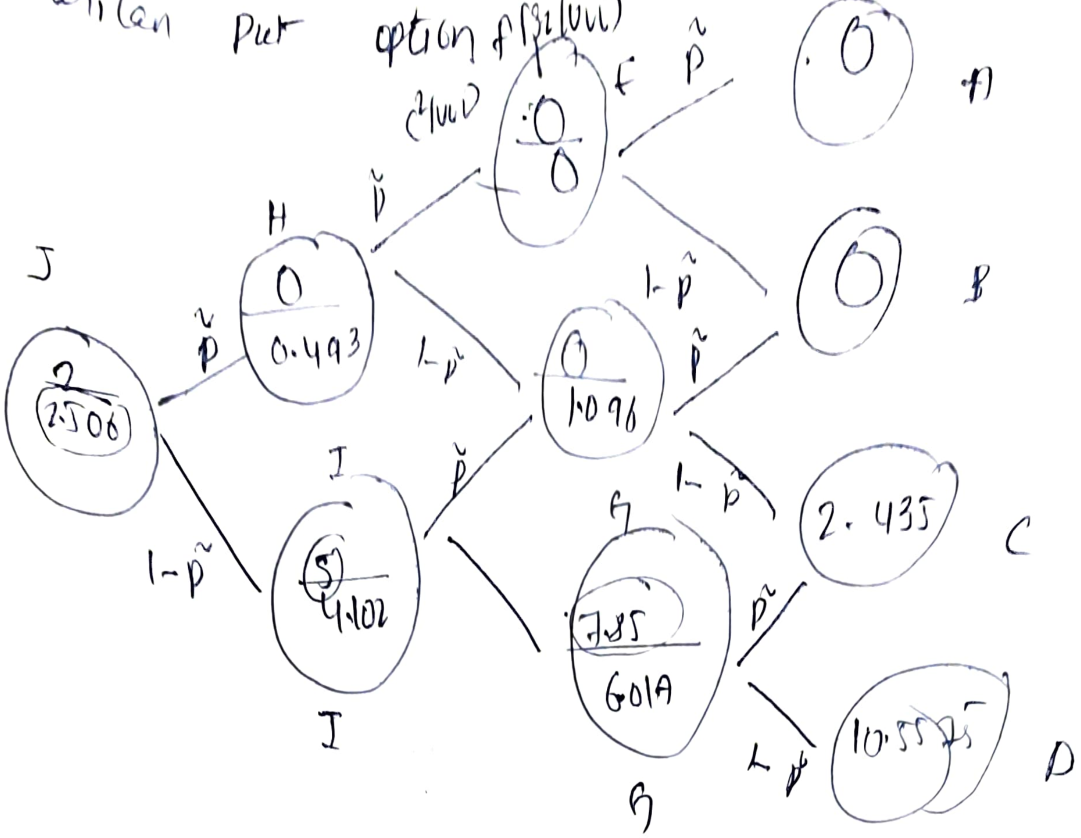
$= 5.5456$

$C(0) = e^{-3r} \tilde{E}(f(S(T)))$

$= e^{-0.09} \times 5.5456$

$= 5.0683$

American Put option



So $p^A(0) = 2.506$

Put option can be exercised at node G & I since at these nodes pay off is greater than c_i.

American Call option

$c^A(0) = 5.068$

$\therefore$ c^A should be exercised at time of expiry i.e. T = 3 years

Q3)

$S(0) = ₹\ 100$

$K = ₹\ 80$

$\sigma = 20\% = 0.2$

$n = 2, \quad T = 4$ yrs

$\therefore \quad \Delta t = \frac{T}{n} = 2$ yrs

$\therefore$ By C-R-R model

$u = e^{\sigma\sqrt{\Delta t}}$

$= e^{0.2 \times \sqrt{2}}$

$= 1.326$

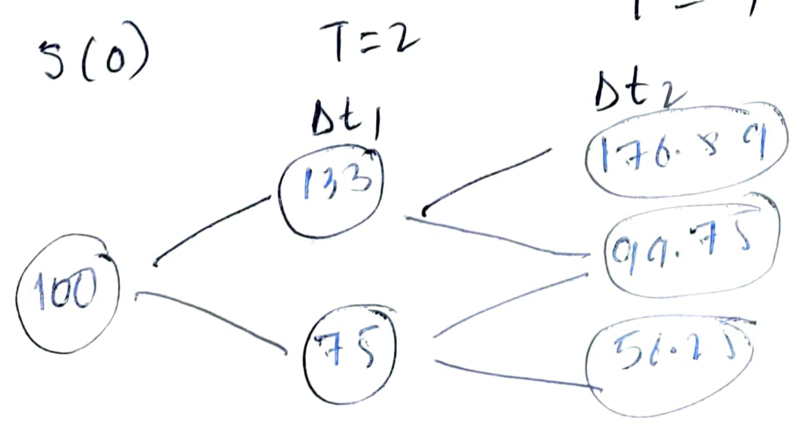
≈ 1.33

$d = e^{-\sigma\sqrt{\Delta t}}$

$= \frac{1}{1.33}$

$= 0.75$

Hence it will go up by 33 % and down by 25 %.



pay off

$(S(T) - K)^+$

96.86

19.75

0

Now,

$\tilde{p}$ [illegible]

$= 0.61$

$\therefore 1-\tilde{p} = 0.39$

$\therefore$ By 2 period C.R.R binomial option pricing model

E.C.O

$$c^E(0) = e^{-0.05\times 4}\left[\tilde{p}^2 \times 96.86 + 2\tilde{p}(1-\tilde{p}) \times 19.75 + (1-\tilde{p})^2 \times 0\right]$$

$$= e^{-0.2}\left[0.61^2 \times 96.86 + 2\times 0.61\times 0.39\times 19.75\right]$$

$$\simeq 37.2$$

Q4). Given

X : Continuous Random variable

$f(x)$: pdf

k : constant

Expected value
$$E(X-K)^+ = \int_{-\infty}^{\infty} (x-K)^+ f(x)\,dx.$$

$$= \lim_{K\to\infty} \int_{k}^{\infty} (x-K) f(x)\,dx.$$

Let $x - K = u$ & $v = \int_{-\infty}^{\infty} f(t)\,dt$

$du = dx,$ $dv = f(x)\,dx.$

$$\therefore E\left[(x-K)^+\right] = \lim_{M\to\infty}\left(\left[\int_{-\infty}^{x} f(t)\,dt\right](x-K)\Big|_{K}^{n} - \int_{K}^{M}\int_{-\infty}^{x} f(t)\,dt\,dx\right)$$

$$= \lim_{M\to\infty}\left(\left[\int_{-\infty}^{\infty} f(t)\,dt\right](M-K) - \int_{K}^{M}\left[1-\int_{-\infty}^{\infty} f(t)\,dt\right]dx\right)$$

$$= \int_{K}^{\infty}\left(\int_{x}^{\infty} f(t)\,dt\right)\,dx$$

b. X: Random variable with μ & σ^2 as parameters

$\therefore$ Density f^n; $f(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$,

$-\infty < x < \infty$

$\therefore$ Expected value of $(x-K)^+$

$$E[(x-K)^+] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-K)^+ e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{K}^{\infty} (x-K)\, e^{-(x-\mu)^2/2\sigma^2} dx.$$

let $\frac{x-\mu}{\sigma} = t$

$$\therefore E[(x-K)^+] = \frac{1}{\sqrt{2\pi}} \int_{\frac{(K-\mu)}{\sigma}}^{\infty} (\mu-K+t\sigma)\, e^{-t^2/2}\, dt$$

$$= \frac{\mu-K}{\sqrt{2\pi}} \int_{\frac{(K-\mu)}{\sigma}}^{\infty} e^{-t^2/2} dt + \frac{\sigma}{\sqrt{2\pi}} \int_{\frac{(K-\mu)}{\sigma}}^{\infty} t e^{-t^2/2} dt$$

$$\therefore F_Z\left(\frac{\mu-K}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} \int_{\frac{(K-\mu)}{\sigma}}^{\infty} e^{-t^2/2} dt$$

is deriv^n of C.D.F for std random variance

$$\therefore \int_{(K-\mu)/\sigma}^{\infty} t e^{-t^2/2} \, dt$$

$$= \int t e^{-t^2/2} \, dt$$

$$= \int e^{-\theta} \, dt \qquad \left| \begin{array}{l} \frac{t^2}{2} = \theta \\ t\,dt = d\theta \end{array} \right.$$

$$= -e^{-\theta} + C$$

$$= -e^{-t^2/2} + C$$

$$\therefore \int_{\frac{(K-\mu)}{\sigma}}^{\infty} t e^{-t^2/2} \, dt = \left[-e^{-t^2/2} \right]_{\left(\frac{K-\mu}{\sigma}\right)}^{\infty}$$

$$= 0 + e^{-\frac{(K-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow E\left[(r-K)^+\right] = \frac{\mu-K}{\sqrt{2\pi}} \int_{\frac{(K-\mu)}{\sigma}}^{\infty} e^{-t^2/2} \, dt$$

$$+ \frac{\sigma}{\sqrt{2\pi}} \int_{\frac{(K-\mu)}{\sigma}}^{\infty} t e^{-t^2/2} \, dt$$

$$= (\mu-K) F_Z\left(\frac{\mu-K}{\sigma}\right) + \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(K-\mu)^2}{2\sigma^2}}$$

Q5.

$S(0) = 1500$

$K = 1470$

$r = 0.03$

$\sigma = 0.22$

$T = \frac{44}{252}$

$= 0.175$

$\therefore$ one period binomial model

$\Delta t = T = 0.175$

$u = e^{\sigma\sqrt{\Delta t}} = e^{0.22 \times \sqrt{0.175}}$

$= 1.096$

$\therefore$ it will move up by 9.6%

$d = \frac{1}{u} = 0.912$

ie. will move down by 8.8%

$S(0)$	$\Delta t = T$	Payoff $(S(T) - K)^+$
1500	1644	$1644 - 1470 = 174$
	1368	0

$$\tilde{p} = \frac{e^{r \times \Delta t} - d}{u - d}$$

$$= \frac{e^{0.03 \times 0.175} - 0.912}{1.096 - 0.912}$$

$$\approx 0.506$$

$$\approx 1/2$$

$$\therefore \quad 1 - \tilde{p} = 1/2$$

$\therefore$ Price of European call option

$$C^E(0) = e^{-0.03 \times 0.175} \left[\frac{1}{2} \times 174 + \frac{1}{2} \times 0 \right]$$

$$= ₹\ 86.54$$

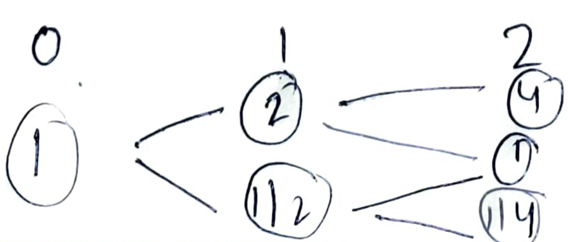
Q6): $S(0) = 1$

$u = 2$

$d = 1/2$

$r = 0$

$N = 2$



$M = \frac{S_0 + S_1 + S_2}{3}$ Payoff $(M - S_N)^+$

$7/3$ $f(S_2(uu)) = 0$

$4/3$ $f(S_2(ud)) = 1/3$

$5/6$ $f(S_2(du)) = 0$

$7/12$ $f(S_2(dd)) = 1/3$

Now $\tilde{p} = \frac{e^{0\times1} - 1/2}{2 - 1/2} = 1/3$

$\therefore \tilde{p} = \frac{1}{3}$

$(1 - \tilde{p}) = \frac{2}{3}$

$P_1(u) = e^{-0}\left[0 \times 1/3 + \frac{2}{3} \times 1/3\right]$

$= 2/9$

$P_1(d) = e^{-0}\left[0 \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3}\right]$

$= \frac{2}{9}$

$\therefore$ Asian Put

$P(0) = e^{-0}\left[\frac{1}{3} \times \frac{2}{9} + \frac{2}{9} \times \frac{2}{3}\right]$

$= \frac{2}{9}$

Hedging strategy (α_i, β_i)

For $n = 0$

$\alpha_1 S_1(u) + \beta_1 B_1 = P_1(u)$

$\alpha_1 S_1(d) + \beta_1 B_1 = P_1(d)$

st $B_n = e^{rn}$.

So $2\alpha_1 + \beta_1 = 2/9$ ——(1)

$\frac{\alpha_1}{2} + \beta_1 = \frac{2}{9}$ ——(2)

(1) − (2)

$\Rightarrow 2\alpha_1 - \frac{\alpha_1}{2} = 0$

$\alpha_1 = 0$

& $\beta_1 = 2/9$

For $n = 1$

$S_1 = 2$

$\alpha_2 S_2(uu) + \beta_2 B_2 = P_2(uu)$

$\alpha_2 S_2(ud) + \beta_2 B_2 = P_2(ud)$

$\Rightarrow 4\alpha_2 + \beta_2 = 0$

$\alpha_2 + \beta_2 = 1/3$

$\alpha_2 = -1/9, \quad \beta_2 = 4/9$

For $3_1 = 1/2$

$$\alpha_3 S_2(du) + \beta_3 B_2 = P_2(du)$$

$$\alpha_3 S_2(dd) + \beta_3 B_2 = P_2(dd)$$

$$\Rightarrow \quad \alpha_3 + \beta_3 = 0$$

$$\frac{\alpha_3}{4} + \beta_3 = 1/3$$

$$\Rightarrow \quad \alpha_3 = -4/9$$

$$\beta_3 = 4/9$$

⇒ Given,

$\mu = 0.0165$

$\sigma = 0.0730$

Suppose price ratio of n^{th} week to that of $(n-1)^{th}$ week is $X_{n,n-1} \sim$ Lognormal $(0.0165, 0.0730)$

They are iid random variable.

a) When price increases over each of the next two weeks

$X_{01} > 1$, $X_{12} > 1$, i.e.

Probability that the price increases over each of the next two weeks

$$P(X_{01} > 1) P(X_{12} > 1) = P(X_{01} > 1)^2$$

$$= P(\ln X_{01} > 0)^2$$

$$= P\left(\frac{\ln X_{01} - \mu}{\sigma} > \frac{-0.0165}{0.073}\right)^2$$

$$= P(Z > -0.23)^2$$

$$= (0.59)^2$$

$$= 0.3481.$$

b) Probability that the price at the end of two weeks is larger than it is today

i.e. $X_{01} X_{12} > 1$

we know probability of iid RV's of two lognormal variables is lognormal

$\therefore X_{01} X_{12} \sim \text{lognormal}(\mu' = 2 \times 0.0165, \sigma' = \sqrt{(0.073)^2 + (0.073)^2}$

$\sim \text{lognormal}(0.033, 0.103)$

$P(X_{01} X_{12} > 1) = P(\ln X_{01} X_{12} > 0)$

$= P\left(\frac{\ln X_{01} X_{12} - 0.033}{0.103} > -0.32\right)$

$= P(z > -0.32) = 0.6255$

Given,

$S_0 = 69,$

$K = 70$

$r = 0.05$

$\sigma = 0.35$

$T = 0.5$

$$\therefore d_1 = \frac{\ln(69/70) + \left(0.05 + \frac{(0.35)^2}{2}\right) \times 0.5}{0.35\sqrt{0.5}}$$

$$= 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5}$$

$$= -0.0809$$

$\therefore$ The European put

$$= 70e^{-0.05 \times 0.5} N(0.0809) - 69N(-0.1666)$$

$$= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338$$

$$= 6.40$$

$S(0) = 110$

$K = 105$

$r = 0.08$

$T = \frac{1}{4} = 0.25\,y$

$\sigma = 0.2$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(S_0/K\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right\}$$

$$= \frac{\ln\left(\frac{110}{105}\right) + \left(0.08 \times \frac{0.2^2}{2}\right) \times 0.25}{0.2\sqrt{0.25}}$$

$$= \frac{0.0715}{0.1}$$

$$= 0.715$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$= 0.715 - 0.2 \times \sqrt{0.25}$$

$$= 0.615$$

$\therefore$ Price of call option

$$C^E(0) = S(0)\,N(d_1) - e^{-rT} K N(d_2)$$

$$= 110 \times N(0.715) - e^{-0.02} \times 105 \times N(0.615)$$

$$= 110 \times 0.76115 - e^{-0.02} \times 105 \times 0.72907$$

$$= 8.689$$

II^{nd}

Given $\sigma = 0.2$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$= e^{0.2 \times \sqrt{0.25}}$$

$$= e^{0.1}$$

$$= 1.105$$

$$d = \frac{1}{u} = 0.905$$

$$\begin{tikzcd}
 & 121.55 \\
110 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 99.55
\end{tikzcd}$$

$$c_1(u) = 121.55 - 105$$

$$= 16.55$$

$$c_1(d) = 0$$

$$\therefore \Delta = \frac{c_1(u) - c_1(d)}{(u-d)\,[S(0)]}$$

$$= \frac{16.55 - 0}{(1.105 - 0.905) \times 110}$$

$$= \frac{16.55}{22}$$

$$= 0.752$$

By Put Call Parity

European Put

$p^E(0) = c(0) - S(0) + e^{-rT} K$

$= 8.689 - 110 + e^{-0.08 \times 0.25} \times 105$

$= 111.6098 - 110$

$= 1.6098$

≈ 1.61