MATH1072

Advanced Multivariate Calculus and Ordinary Differential Equations

Lecture pre-reading material

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Contents

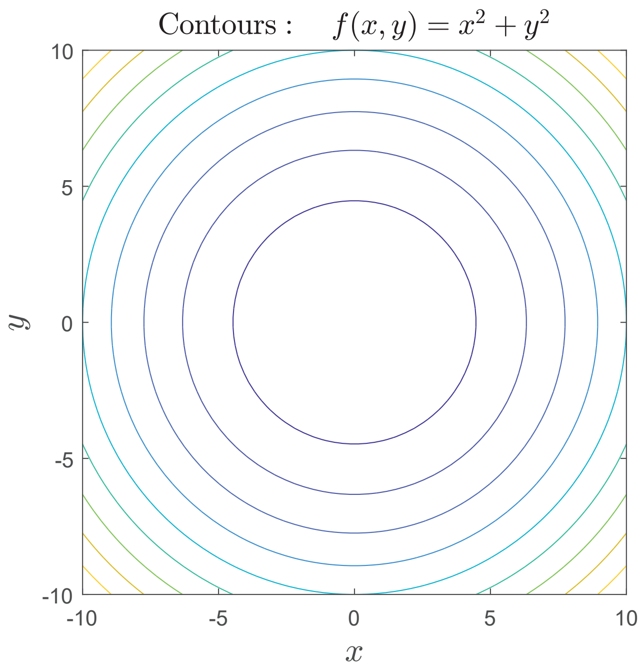
1 PRE-LECTURE READING: Contours and Cross-sections

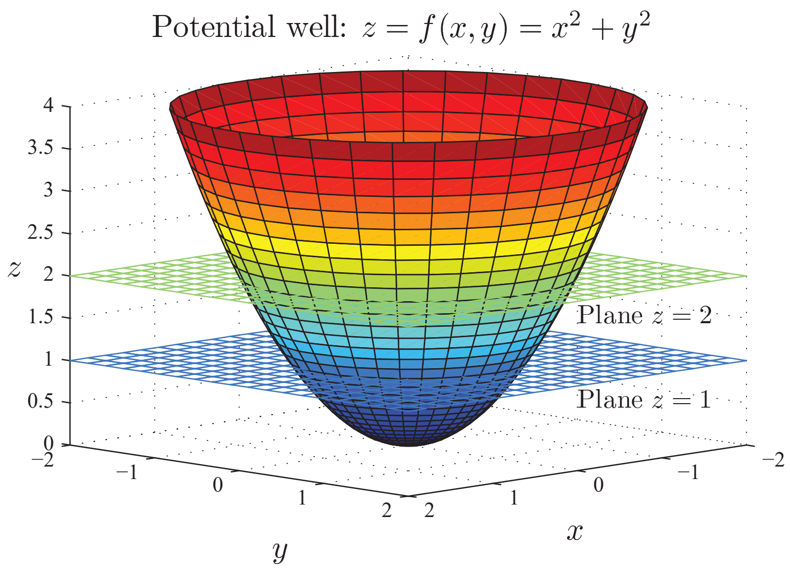
Geographical maps have curves of constant height above sea level, or curves of constant air pressure (isobars), or curves of constant temperature (isothermals). Drawing contours is an effective method of representing a 3-dimensional surface in two dimensions. We now look at functions f of two variables. A contour is a curve corresponding to the equation $z = f(x, y) = C$,

Consider the surface $z = f(x, y) = x^2 + y^2$ sliced by horizontal planes $z = 0$, $z = 1$, $z = 2, \dots$

Plane	Contour	Description
$z = 0$	$x^2 + y^2 = 0$	$x = y = 0$
$z = 1$	$x^2 + y^2 = 1$	Circle, radius 1
$z = 2$	$x^2 + y^2 = 2$	Circle, radius $\sqrt{2}$
$z = 3$	$x^2 + y^2 = 3$	Circle, radius $\sqrt{3}$
$z = 4$	$x^2 + y^2 = 4$	Circle, radius 2

Note that as the radius increases, the contours are more closely spaced.





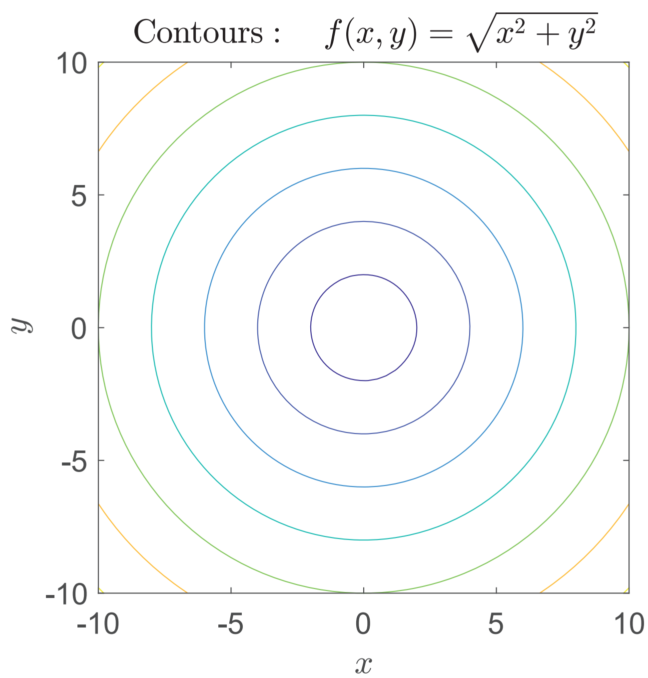
Example: Draw a contour diagram of f given by

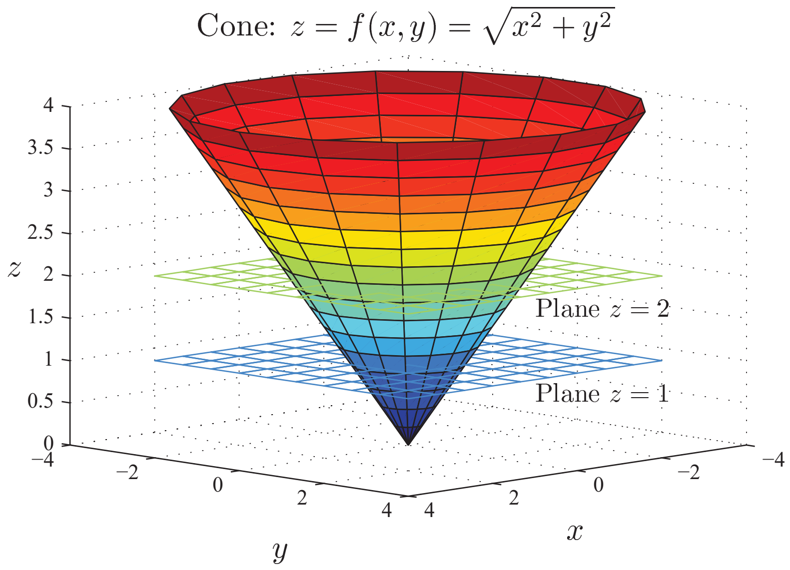
$$f(x, y) = \sqrt{x^2 + y^2}.$$

In MATLAB, the command is

```
ezcontour('sqrt(x^2+y^2)',[-10,10,-10,10]);
```

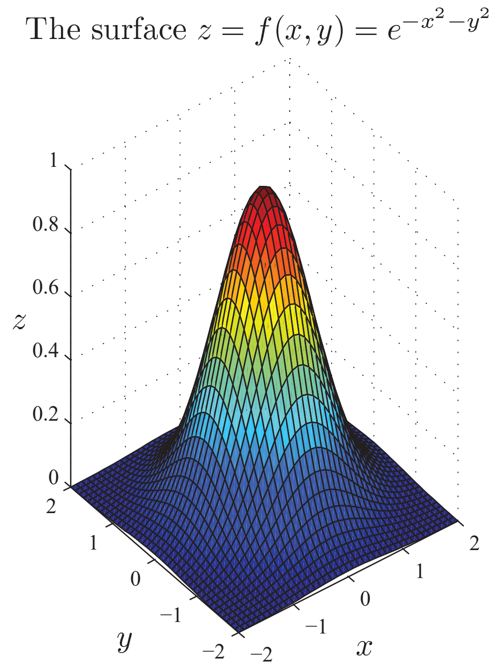
If horizontal planes are equally spaced, say $z = 0, c, 2c, 3c, \ldots,$ it is not hard to visualise the surface from its contour diagram. Spread-out contours mean the surface is quite flat and closely spaced ones imply a steep climb.

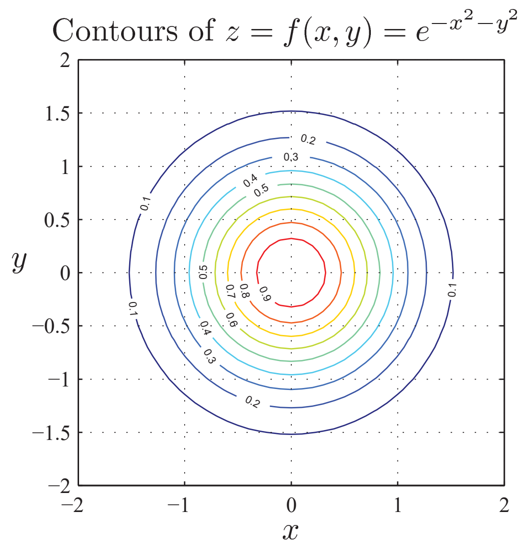




Note that the contours of the last two functions were all circles. Such surfaces have circular symmetry. When x and y only appear as x^2+y^2 in the definition of f, then the graph of f has circular symmetry about the z axis. The height z depends only on the radial distance $r = \sqrt{x^2+y^2}$.

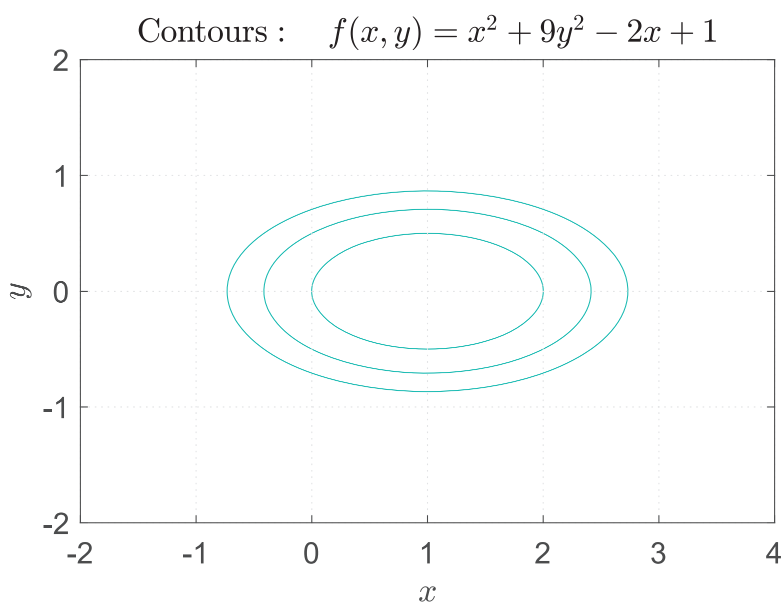
Example: $z = f(x,y) = \mathrm{e}^{-x^2-y^2}$



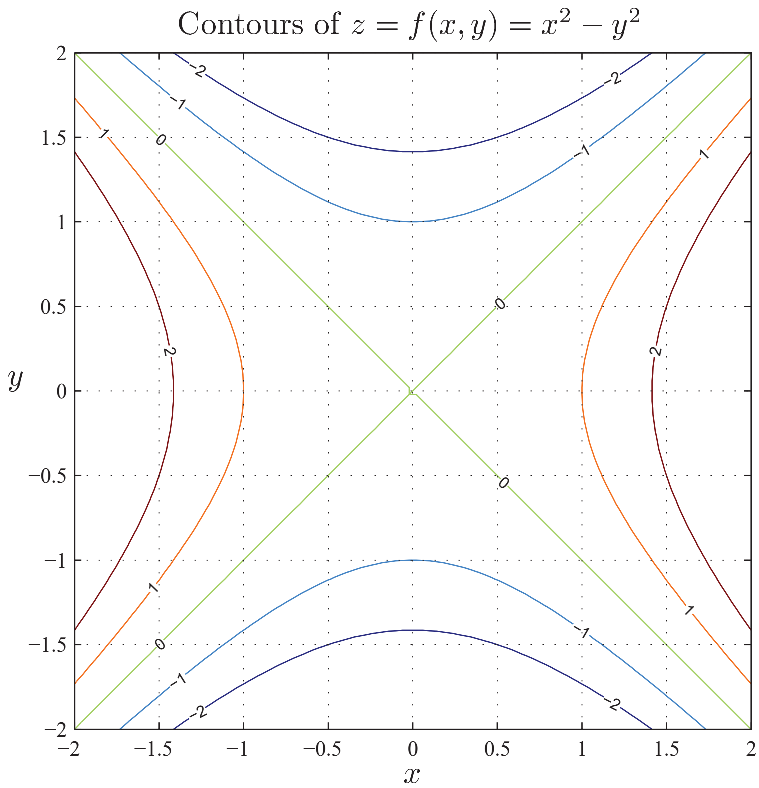


Example: Use the "completing the squares" method (see **Conic Sections**) to determine a contour diagram for $z = f(x, y) = x^2 + 4y^2 - 2x + 1$.

Plane	Contour	Description
$z = 1$	$(x-1)^2 + 4y^2 = 1$	ellipse centre $(1, 0)$ x intercepts: $x = 0$, $x = 2$ intersects line $x = 1$ at $y = \pm 1/2$
$z = 2$	$(x-1)^2 + 4y^2 = 2$	ellipse centre $(1, 0)$ x intercepts: $x = 1 \pm \sqrt{2}$ intersects line $x = 1$ at $y = \pm 1/\sqrt{2}$
$z = 3$	$(x-1)^2 + 4y^2 = 3$	ellipse centre $(1, 0)$ x intercepts $x = 1 \pm \sqrt{3}$ intersects line $x = 1$ at $y = \pm\sqrt{3}/2$



Example: A saddle $z = x^2 - y^2$ has hyperbolic contours.



Note: Contour diagrams of functions whose graphs are planes consist of equidistant parallel lines when the function values are equidistant. Equidistant lines in $\mathbb{R}^2$ are in fact always parallel, unlike $\mathbb{R}^3$.

You can construct the plane itself from its contour diagram provided the contours are labelled. Let the plane be $z = mx + ny + c$. Any point in the plane (x_0, y_0, z_0) gives c:

$$c = z_0 - mx_0 - ny_0.$$

To find m consider moving along the plane in the positive x direction between two contours. Calculate the change in z, Δz, between the two contours, and the change in x, Δx, as you move from one contour line to the next. (Remember move in the x direction only, i.e. in a plane $y = C$).

Then $m = \dfrac{\Delta z}{\Delta x}$.

Similarly take a plane $x = C$ and move in the positive y direction to calculate Δy.

Then $n = \dfrac{\Delta z}{\Delta y}$.

Example: Find the plane given by the following contour diagram:

Plane	Contour
$z = 0$	$y = -2x + 3$
$z = 1$	$y = -2x + 2$

- First note that $\Delta z = 1 - 0 = 1$. Moreover, if $y = 0$ then from $2x + y = 3$ and $2x+y = 2$ we obtain $x = 3/2$ and $x = 1$, respectively. So $\Delta x = 1-3/2 = -1/2$. It follows that $m = \dfrac{\Delta z}{\Delta x} = -2$.
- Similarly, if $x = 0$ then from $2x + y = 3$ and $2x + y = 2$ we obtain $y = 3$ and $y = 2$, respectively. So $\Delta y = 2 - 3 = -1$. It follows that $n = \dfrac{\Delta z}{\Delta y} = -1$.
- So the plane is $z = c - 2x - y$. To find c we see that the point $(1, 1, 0)$ is on the plane $z = 0$ and satisfies $2x + y = 3$. So this point must be on the plane $z = c - 2x - y$. This leads to $c = 3$.

A plane $z = mx + ny + c$ does not have just one slope. It has slope m in the x direction and slope n in the y direction.

To visualise this, imagine starting at $(0,\ 0,\ c)$ and walking in the $(x,\ z)$ plane along the line $z = mx + c$ (with slope m). Or walk in the $(y,\ z)$ plane along the line $z = ny + c$ (with slope n).

Example: Find the plane with slope 6 in the x direction and 4 in the y direction which passes through the point $(1,\ 5,\ 4)$.

$z = mx + ny + c$ with $m = 6$, $n = 4$ and since

$$\begin{aligned} c &= 4 - 6 \times 1 - 4 \times 5 \\ &= -22, \\ \text{we have} \quad z &= x + 4y - 22. \end{aligned}$$

Summary

- You should be able to plot contour diagrams in MATLAB using the **ezcontour** function (and its variants).
- You should be able to recognise circular symmetry in an equation.
- You should be able to match contour diagrams with functions.
- You should be able to sketch simple contour diagrams.

Cross-sections of a surface

A cross-section is the intersection of a surface with a vertical plane such as $y = C$.

Example:

The height z of a vibrating guitar string can be expressed as a function of horizontal distance x, and time t

$$z = f(x,t) = A\sin(\pi x)\cos(2\pi t) \qquad \text{where} \qquad 0 < x < 1.$$

The snapshots where t is constant are cross-sections of the 'surface'.

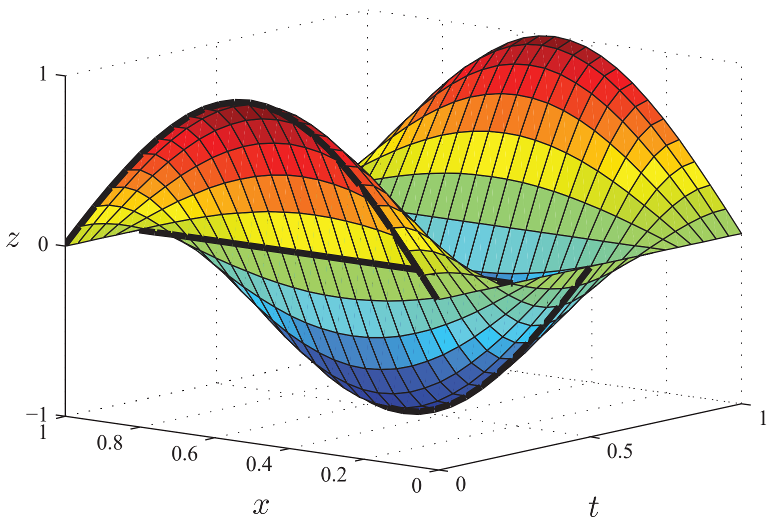
Varying time we find

$$\begin{aligned} t = 0: &\qquad z = A\sin(\pi x) \\ t = \frac{1}{8}: &\qquad z = \frac{1}{2}\sqrt{2}A\sin(\pi x) \\ t = \frac{1}{4}: &\qquad z = 0 \\ t = \frac{3}{8}: &\qquad z = -\frac{1}{2}\sqrt{2}A\sin(\pi x) \\ t = \frac{1}{2}: &\qquad z = -A\sin(\pi x). \end{aligned}$$

These represent sine curves, with amplitudes between 0 and A.

We can also consider the cross-sections in x. For instance $x = \frac{1}{2}$ (at the top of the sine wave), then $z = A\cos(2\pi t)$ which equals the amplitude of the sine wave.

Vibration of a guitar string: $z = \sin(\pi x)\cos(2\pi t)$



MATLAB can be used to make a movie of the 2-dimensional surface by plotting cross-sections at different t values in sequence. The sequence of plots can be stored in a vector and played as a movie using the following code:

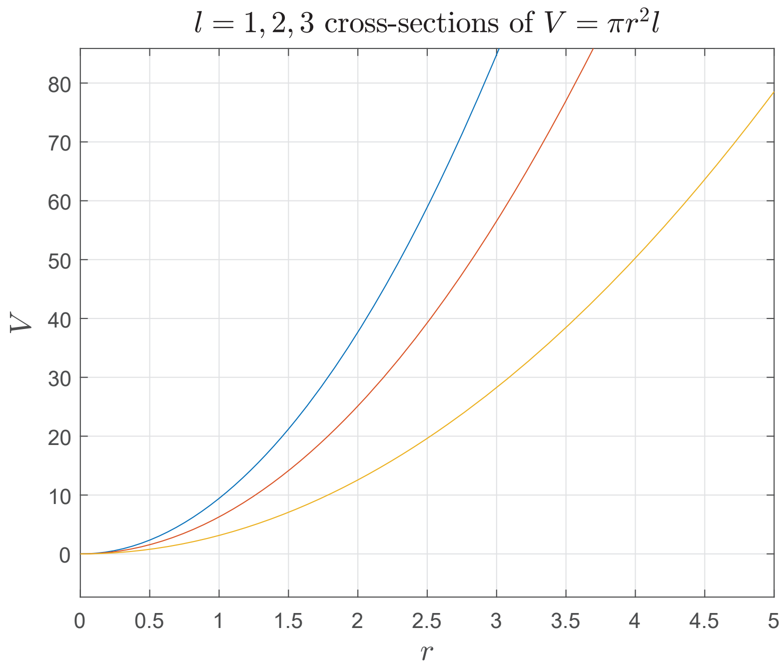
```
x=(0:0.25:1);
for j=1:100
   t=j/25;
   z=sin(pi*x)*cos(2*pi*t);
   plot(x,z);axis([0,1,-1,1]);
   M(j)=getframe;
end
```

Note: **ezplot** cannot be used to do this because MATLAB gets confused about which of t, x is a variable and which is a number.

Consider the volume of a cylinder as a function of two variables: $V(r,\ell) = \pi r^2 \ell$.

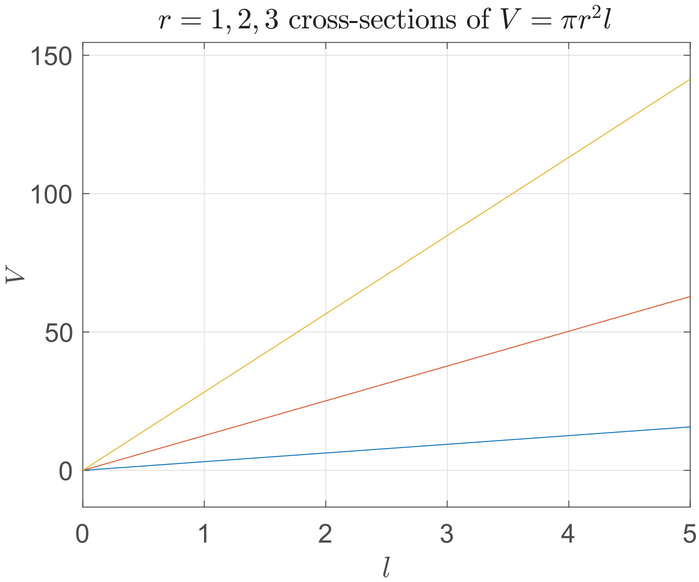
Using MATLAB, the **ezplot** function is the simplest graphing aid:

```
ezplot('pi*r^2*1',[0,5])
hold on
ezplot('pi*r^2*2',[0,5])
ezplot('pi*r^2*0.5',[0,5])
```

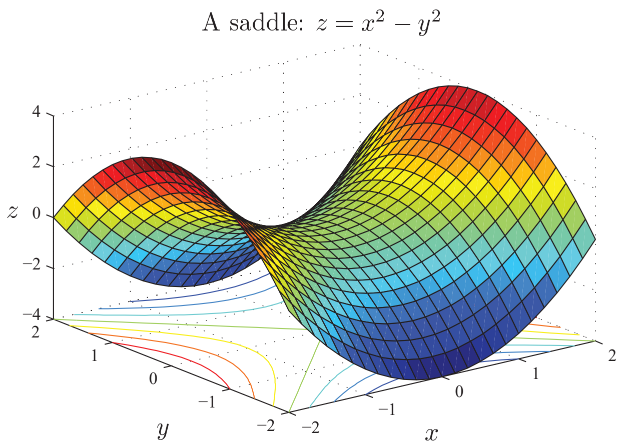


Alternatively, we could keep r fixed and plot V as a function of ℓ:

```
ezplot('pi*1^2*l',[0,5])
hold on
ezplot('pi*0.5^2*l',[0,5])
ezplot('pi*2^2*l',[0,5])
```



Example: The cross-sections of a saddle $z = x^2 - y^2$ are parabolas. For $y = y_0$ they point up: $z = x^2 - (y_0)^2$; and for $x = x_0$ they point down: $z = -y^2 + x_0^2$.



Summary

- You should be able to construct cross-sections of multivariate functions.
- Cross-sections are 2-dimensional graphs.
- Animation of cross-sections is another way to visualise multivariate functions.

2 PRE-LECTURE READING: Lines and Planes

2.1 Scalar equation for a plane

The equation of a plane in $\mathbb{R}^3$ can be expressed using scalars or vectors. In this first section we will discuss scalar equations.

Note that in $\mathbb{R}^3$, we adopt the convention to take the z-axis pointing upwards, and the (x, y)-plane to be horizontal.

Horizontal planes

The x- and y-axes lie in the horizontal plane $z = 0$. All other horizontal planes are parallel to $z = 0$, and are given by the equation $z = c$.

Vertical planes

A vertical plane has the form $ax + by = d$; it depends on x and y only and z does not appear. If you are not told that this is an equation of a plane, or, equivalently, an equation in $\mathbb{R}^3$, then you cannot distinguish it from the equation of a line in $\mathbb{R}^2$.

Consider the plane $x + y = 1$. First imagine the line $x + y = 1$ in the xy-plane. Then the plane $x + y = 1$ in $\mathbb{R}^3$ contains this line, and is parallel to the z-axis.

Arbitrary planes

The general equation of a plane in $\mathbb{R}^3$ is given by

$$ax + by + cz = d$$

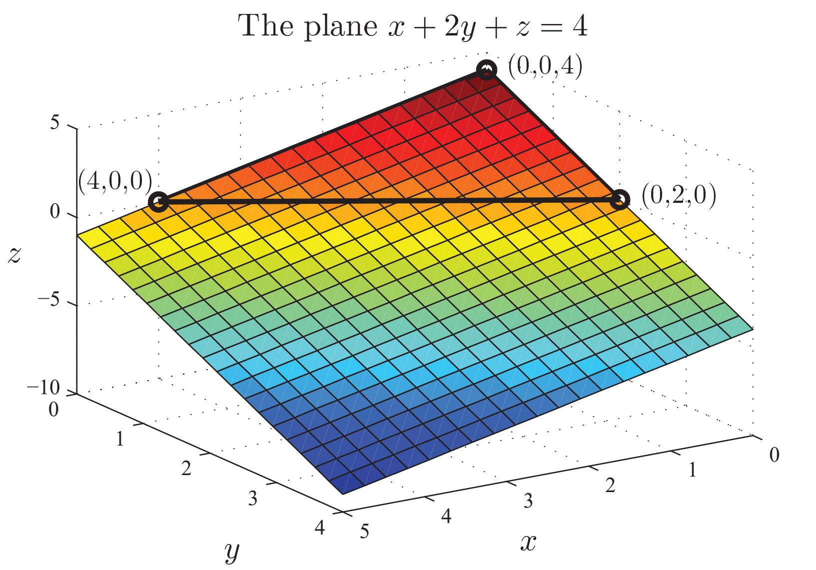
with a, b, c, d fixed real numbers. If the plane is not vertical, i.e., $c \neq 0$, this equation can be rearranged so that z is expressed as a function of x and y:

$$z = F(x,\, y) = -(a/c)x - (b/c)y + (d/c) = mx + ny + z_0.$$

To plot the plane in MATLAB, simply plot $F(x, y)$ using **ezsurf**.

The easiest way to sketch the plane by hand is to use the triangle method: If all of $a, b, c \neq 0$ the plane $ax + by + cz = d$ intercepts each axis at precisely one point. These three points make up a triangle which fixes the plane.

Example: The plane $x + 2y + z = 4$ intersects the x-axis at $x = 4$, the y-axis at $y = 2$ and the z-axis at $z = 4$.

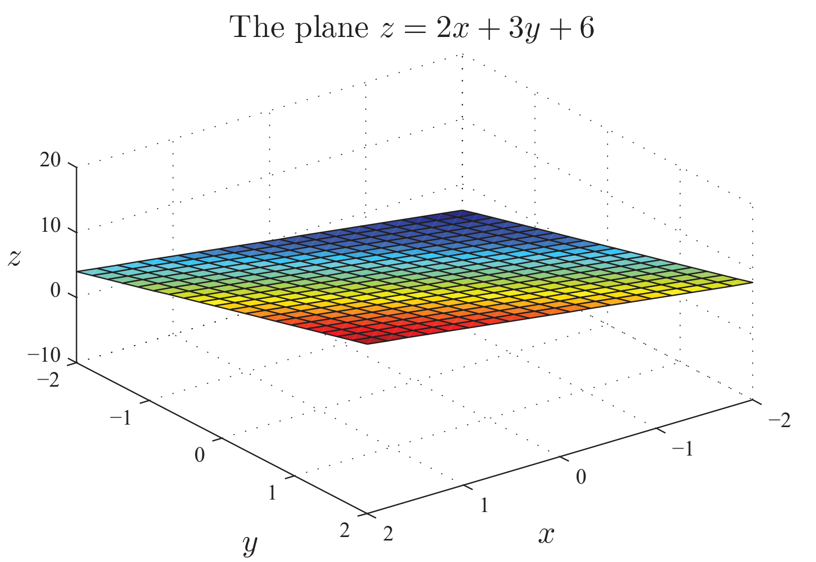


The triangle method is based on the simple fact that *any* three points that lie in a plane uniquely determine this plane *provided these three points do not lie on a single straight line*.

Example: Consider the plane $2x + 3y - z + 6 = 0$.

In MATLAB, the graphing command is simply:

```
ezsurf('2*x+3*y+6',[-2,2,-2,2])
```



Important remark: It is customary to say *the* equation of a plane, even though it is not unique. Multiplying the equation of a plane by a nonzero constant gives another equation for the same plane. For example, $x - 2y + 3z = 4$ and $-2x + 4y - 6z = -8$ are equations of the same plane.

Example: Find the equation of the plane through $(0, 0, 5)$, $(1, 3, 2)$ and $(0, 1, 1)$.

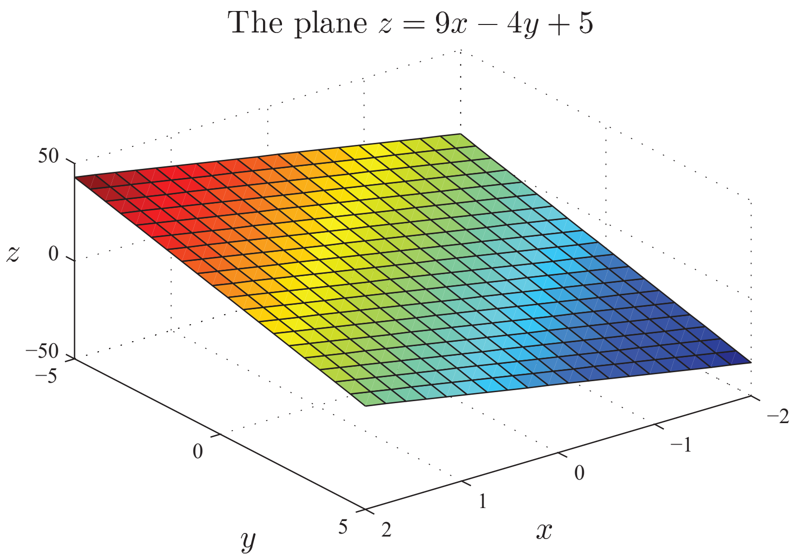
Let $z = ax + by + d$.

The first point gives $d = 5$.

The 2nd point gives $2 = a + 3b + 5$ and the 3rd point gives $1 = b + 5$.

Solving simultaneously gives $a = 9$ and $b = -4$.

So the plane is $z = 9x - 4y + 5$.

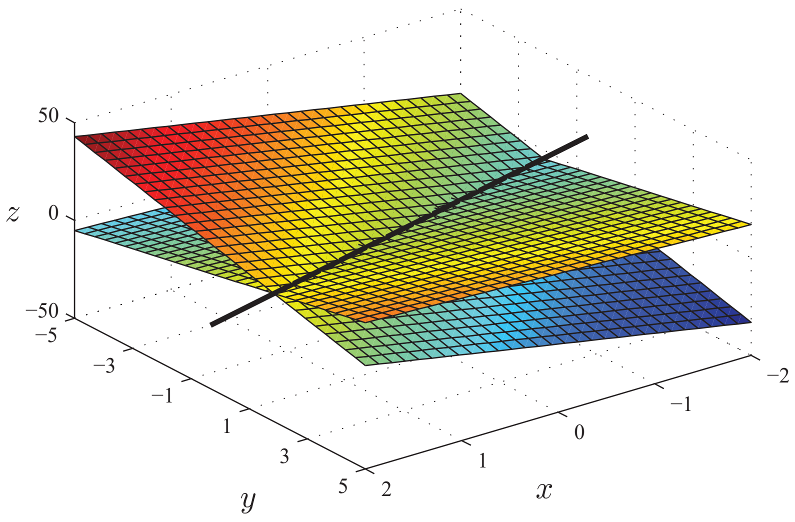


The MATLAB command to produce a plot of this plane would be:

```
ezsurf('9*x-4*y+5',[-2,2,-5,5])
```

The following picture shows the intersection of the planes $2x + 3y - z + 6 = 0$ and $z = 9x - 4y + 5$. Note that the intersection is a line.

The intersection of $z = 9x - 4y + 5$ and $z = 2x + 3y + 6$



Remember that to produce two or more plots on the same figure use the MATLAB command:

```
hold on;
```

2.2 Vector equation of a plane I

Revision of vectors

Pythagoras gives the distance d between the points (x_0, y_0) and (x_1, y_1) as

$$d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

In three dimensions, the distance d between (x_0, y_0, z_0) and (x_1, y_1, z_1) is

$$d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

The norm of a vector is its length. In MATLAB, you can directly call the norm of a vector:

```
v=[1 2 3]
v =
     1     2     3
norm(v)
ans =
    3.7417
```

Recall that the dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

It is also true that

$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos\theta,$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\|\boldsymbol{a}\|$ is the norm of $\boldsymbol{a}$:

$$\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}\,.$$

If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal.

The projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is "the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$":

$$\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \|\boldsymbol{b}\| \cos\theta \times \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|^2}\,\boldsymbol{a}.$$

Recall that the cross product $\boldsymbol{a} \times \boldsymbol{b}$ of two vectors in $\mathbb{R}^3$ is a vector orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$ given by

$$\boldsymbol{a} \times \boldsymbol{b} = (a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

This is best memorised using a 3 by 3 determinant.

The cross product points in the direction given by the right-hand-rule.

It is also true that

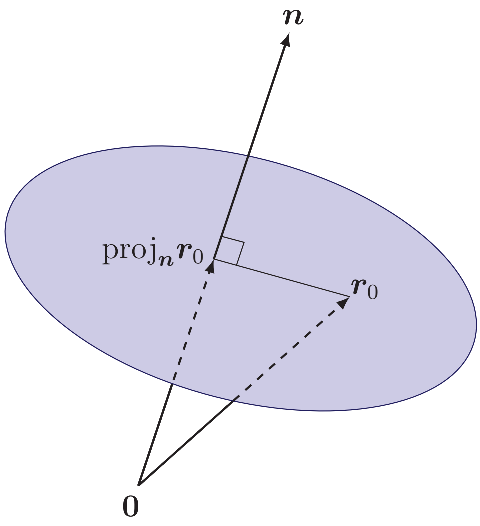
$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \sin\theta.$$

This means that $\|\boldsymbol{a} \times \boldsymbol{b}\|$ gives the area of the parallelogram spanned by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Vector equation of a plane I

We have seen in Section 2.1 that any three points in $\mathbb{R}^3$ which do not lie on a straight line determine a plane. There is another geometric way to view a plane, which gives rise to the vector equation of a plane.

Let P be a plane and $\boldsymbol{n}$ be a vector perpendicular to P. Such a vector is called a normal to P. Let $\boldsymbol{r}_0$ be a vector from the origin to a point $\boldsymbol{r}_0$ in P and $\mathrm{proj}_{\boldsymbol{n}}\boldsymbol{r}_0$ its projection onto $\boldsymbol{n}$. Then the length of this projection vector, i.e., $\|\mathrm{proj}_{\boldsymbol{n}}\boldsymbol{r}_0\|$ is the distance of the plane to the origin.



If we had taken another vector, say $\boldsymbol{r}$, from the origin to a point $\boldsymbol{r}$ in P then

$$\text{proj}_{\boldsymbol{n}}\boldsymbol{r} = \text{proj}_{\boldsymbol{n}}\boldsymbol{r}_0.$$

In other words, the plane P is given by the collection of *all* points r whose corresponding vectors $\boldsymbol{r}$ have the same orthogonal projection onto $\boldsymbol{n}$ as $\boldsymbol{r}_0$. Hence

$$\boxed{\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0.}$$

This is the vector equation for a plane.

The vector and scalar equations are in fact the same, as shown next.

We start with the vector equation, where $\boldsymbol{n} = (a, b, c)$ is a normal, $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ and $\boldsymbol{r} = (x, y, z)$. Then

$$\begin{gathered}
\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0 \\
(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0 \\
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \\
ax + by + cz = d
\end{gathered}$$

where $d = \boldsymbol{n} \cdot \boldsymbol{r}_0 = ax_0 + by_0 + cz_0$.

Important remark: Given a plane $ax + by + cz = d$ we now have a geometric interpretation of the vector (a, b, c): it is a normal to the plane. Obviously the normal is not unique since any vector of the form $k(a, b, c)$, where k is a nonzero scalar, is also a normal.

Our understanding of normal vectors allows us to compute angles between planes, defined as the angle between their respective normal vectors.

Important remark: By convention, the angle between two planes cannot exceed $\pi/2$.

2.3 Equations for a line

There are three common ways to represent a line:

(a) the vector representation;

(b) the parametric representation;

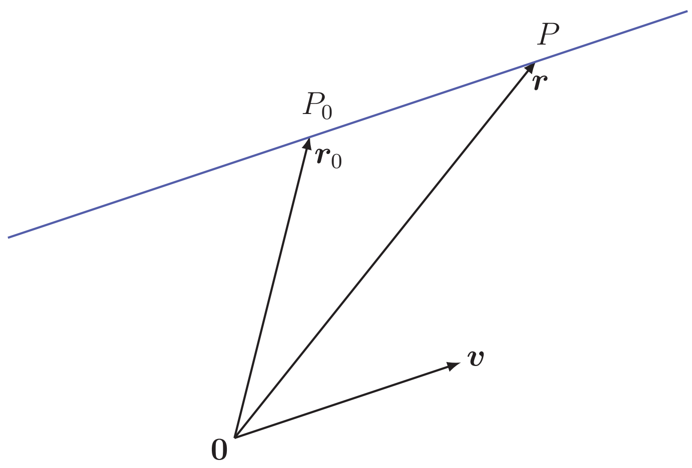
(c) the symmetric equations, obtained by eliminating parameters in (b).

The representation that is best depends on the particular problem at hand.

(a) Vector representation

Let $P_0 = (x_0, y_0, z_0)$ be a point on the line L with corresponding position vector $\boldsymbol{r}_0$ and let $\boldsymbol{v}$ be a vector parallel to L, known as a direction vector. For an arbitrary point $P = (x, y, z)$ lying on L we have

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0).$$



Since $\overrightarrow{P_0P}$, is parallel to $\boldsymbol{v}$, we have $\overrightarrow{P_0P} = \lambda\boldsymbol{v}$ for some scalar λ. Hence

$$\boldsymbol{r} = \boldsymbol{r}_0 + \overrightarrow{P_0P} = \boldsymbol{r}_0 + \lambda\boldsymbol{v}.$$

The equation

$$\boldsymbol{r} = \boldsymbol{r}_0 + \lambda\boldsymbol{v}, \qquad \lambda \in \mathbb{R}$$

is know as the vector equation of the line L.

Important remark: As with *the* equation of a plane, *the* vector equation of a line is not unique. One can choose *any* point P_0 on the line as “starting point”and one can multiply the vector $\boldsymbol{v}$ by any nonzero constant.

(b) Parametric representation

The parametric representation of a line is a scalar representation of the vector equation $\boldsymbol{r} = \boldsymbol{r}_0 + \lambda \boldsymbol{v}$. Writing each vector in component form,

$$\begin{aligned} \boldsymbol{r} &= (x, y, z) \\ \boldsymbol{r}_0 &= (x_0, y_0, z_0) \\ \boldsymbol{v} &= (a, b, c) \end{aligned}$$

gives

$$(x, y, z) = (x_0, y_0, z_0) + \lambda(a, b, c).$$

Matching components results in three scalar equations

$$\begin{cases} x = x_0 + a\lambda \\ y = y_0 + b\lambda \\ z = z_0 + c\lambda \end{cases} \qquad (*)$$

known as the parametric equations of a line.

(c) Symmetric equations

The parameter λ can be eliminated from the parametric equations of a line. For example, by eliminating λ from each of the three equations $x = 5 + \lambda$, $y = 1 + 4\lambda$, $z = 3 - 2\lambda$ for the line L, we obtain

$$\lambda = x - 5 = \frac{y-1}{4} = \frac{z-3}{-2}.$$

The equations

$$x - 5 = \frac{y-1}{4} = \frac{z-3}{-2}$$

are known as the symmetric equations of L.

What these equation really are is a set of two non-identical, non-parallel planes

$$x - 5 = \frac{y-1}{4} \quad \text{and} \quad x - 5 = \frac{z-3}{-2},$$

which, as we know, must intersect to give a line.

The general form of the symmetric equations of the line $(*)$ from the previous page is given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

but some care is needed when one (or two) of a, b or c is equal to zero. For example, if $a = 0$ the above needs to be replaced by

$$x = x_0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

and so on.

To plot a line in three-dimensional space, use the parametric form. The MATLAB command is **ezplot3**. As input it takes three functions of a parameter, and the range of the parameter.

Example: Plot the line

$$\begin{cases} x = 100 + 200t \\ y = 200 + 300t \\ z = 300 - 10t \end{cases}$$

for t between -100 and 100. The MATLAB command would be

```
ezplot3('100+200*t','200+300*t','300-10*t',[-100,100])
```

Remember, you can grab and rotate the resulting axes with these three-dimensional plots.

2.4 Parallel, skew and orthogonal lines.

Two lines are parallel if, when written as $\boldsymbol{r} = \boldsymbol{r}_0 + \lambda \boldsymbol{v}$, and $\boldsymbol{s} = \boldsymbol{s}_0 + \mu \boldsymbol{u}$, the direction vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are linear multiples of each other, i.e., $\boldsymbol{u} = k\boldsymbol{v}$ for some nonzero scalar k.

Two straight lines in three dimensional space rarely intersect. Non parallel, non-intersecting lines are called skew lines.

Example: Check that the lines L_1 and L_2 given by

$$L_1 : \begin{cases} x = 1 + \lambda \\ y = -2 + 3\lambda \\ z = 4 - \lambda \end{cases} \qquad L_2 : \begin{cases} x = 2\mu \\ y = 3 + \mu \\ z = -3 + 4\mu \end{cases}$$

for $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, are skew lines.

Two lines are orthogonal if their direction vectors are orthogonal. An easy test for orthogonality is to take the dot product of the direction vectors; if and only if this gives zero are the lines orthogonal.

Example: Check that the lines L_1 and L_2 given by

$$\begin{aligned} L_1: &\quad \frac{x-1}{2} = y - 3 = \frac{z+5}{4}, \\ L_2: &\quad 2 - x = \frac{5-y}{2} = z - 5. \end{aligned}$$

are orthogonal.

Further theory

Unlike lines, planes can not be skew in 3 dimensions. Given two planes, the only possibilities are that they intersect, or that they are parallel. If two planes are parallel, then their normal vectors must be parallel. If two (distinct) planes intersect, then they intersect in a line.

2.5 Vector equation of a plane II

Previously we discussed the scalar equation of a plane, which is an equation of the form $ax + by + cz = d$, and the vector equation of a plane, which is an equation of the form $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0$.

There is a second type of vector equation of a plane, which is similar in form to the vector equation of a line. One fixes a point P_0 on the plane with corresponding position vector $\boldsymbol{r}_0$, but now not one, but two (non-parallel) vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ parallel to the plane are required to fully determine the plane:

$$\boldsymbol{r} = \boldsymbol{r}_0 + \lambda\boldsymbol{u} + \mu\boldsymbol{v}, \qquad \lambda, \mu \in \mathbb{R}.$$

Important remark: In some sense the above vector equation of a plane is even less unique than the vector equation of a line. Again one can take $\boldsymbol{r}_0$ to be any point on the plane and one can multiply both $\boldsymbol{u}$ and $\boldsymbol{v}$ by nonzero constants. But unlike a line, we can also replace $\boldsymbol{u}$ and $\boldsymbol{v}$ by any other pair of (independent) vectors parallel to the plane. For example, $(x, y, z) = (1, 0, 1) + \lambda(0, 1, 2) + \mu(3, 1, -1)$ and $(x, y, z) = (4, 1, 0) + \lambda(-1, 0, 1) + \mu(-3, 1, 5)$ are different vector representations of the same plane! So do not panic if in your exam the person next to you writes down a very different-looking equation; they still might have the right answer...

2.6 Distance from a point to a plane

A common problem that arises in applications is to find the distance from a point to a plane, where, by distance, we always mean minimum distance.

If $P = (x_1, y_1, z_1)$ is a point, with corresponding position vector $\boldsymbol{p}$, and Π is a plane (with normal $\boldsymbol{n} = (a, b, c)$) given by $ax + by + cz = d$, then the formula for the distance between P and Π is

$$D = \frac{|\boldsymbol{n} \cdot \boldsymbol{p} - d|}{\|\boldsymbol{n}\|} = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Important remark: It is best not to try to memorise such a formula because, oddly enough, under exam pressure small details often subtly change for the worse. Also, it depends on one's choice of representation of a plane. For example, sometimes a plane is expressed as $ax + by + cz + d = 0$ in which case the numerator contains $+d$ instead of $-d$. Best is to *understand* the actual derivation given below.

One derivation of the above distance formula uses orthogonal projections. Below we present an alternative method, which is not quite as slick, but which is very easy to carry out for explicit examples, even under exam conditions.

First we construct the line L through the point P and orthogonal to the plane Π:

$$L : \quad \boldsymbol{r} = \boldsymbol{p} + \lambda \boldsymbol{n}, \qquad \lambda \in \mathbb{R}.$$

Next we determine the point, say Q, where L intersects Π. That is, we substitute the equation for L into the equation $\boldsymbol{r} \cdot \boldsymbol{n} = d$ for Π:

$$(\boldsymbol{p} + \lambda \boldsymbol{n}) \cdot \boldsymbol{n} = d.$$

Solving for λ yields

$$\lambda = \frac{d - \boldsymbol{p} \cdot \boldsymbol{n}}{\|\boldsymbol{n}\|^2},$$

so that the point Q, with position vector $\boldsymbol{q}$, is given by

$$\boldsymbol{q} = \boldsymbol{p} + \frac{d - \boldsymbol{n} \cdot \boldsymbol{p}}{\|\boldsymbol{n}\|^2} \boldsymbol{n}.$$

The distance between P and Π is now the distance between P and Q. But

$$\boldsymbol{p} - \boldsymbol{q} = \frac{\boldsymbol{n} \cdot \boldsymbol{p} - d}{\|\boldsymbol{n}\|^2} \boldsymbol{n}$$

so that

$$D^2 = \|\boldsymbol{p} - \boldsymbol{q}\|^2 = \frac{(\boldsymbol{n} \cdot \boldsymbol{p} - d)^2}{\|\boldsymbol{n}\|^4} \boldsymbol{n} \cdot \boldsymbol{n} = \frac{(\boldsymbol{n} \cdot \boldsymbol{p} - d)^2}{\|\boldsymbol{n}\|^2}.$$

The formula now follows by taking the square root on both sides *and remembering that a distance can never be negative*, whereas $\boldsymbol{n} \cdot \boldsymbol{p} - d$ can.

3 PRE-LECTURE READING: Conic Sections

The goal of this section is for you to become an expert in recognising equations of parabolas, circles, ellipses and hyperbolas . The key technique to this is "completing the square".

3.1 Parabolas

Arranging a quadratic $y = ax^2 + bx + c$ into the form

$$y = a(x - h)^2 + k,$$

is called completing the square. In this form $x = h$ is identified as the axis of symmetry of the parabola. The value $y = k$ is the minimum when $a > 0$, and the maximum when $a < 0$.

Completing the square makes use of the identity

$$x^2 + 2xh + h^2 = (x + h)^2.$$

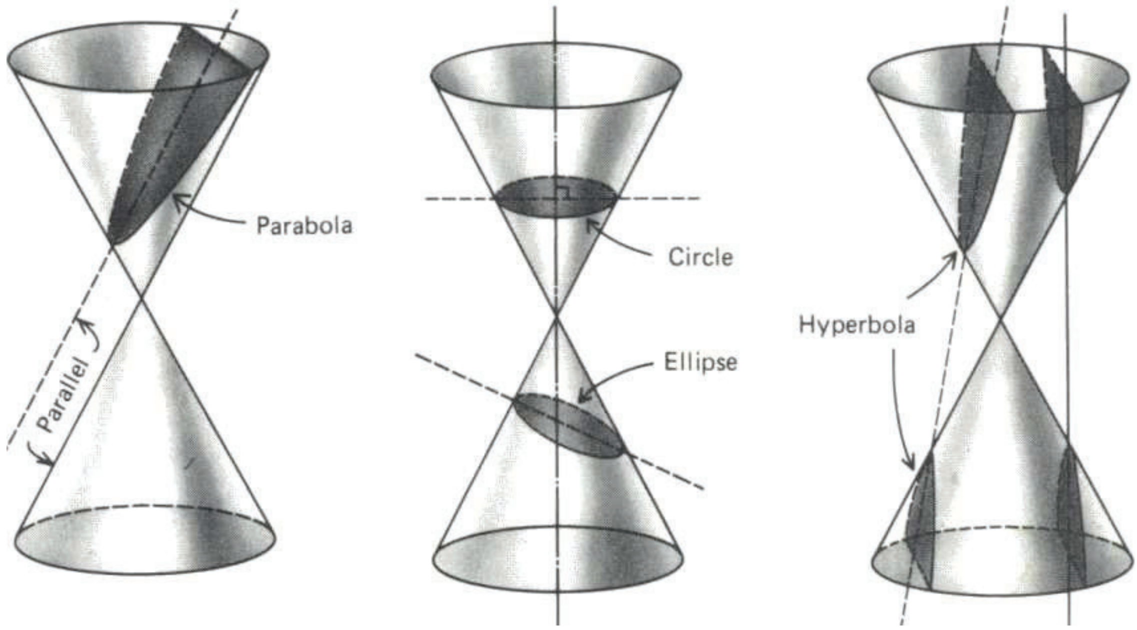
Consider the following example

$$\begin{aligned} y &= -3x^2 - 6x + 1 \\ &= -3(x^2 + 2x) + 1 \\ &= -3(x^2 + 2x + 1 - 1) + 1 \\ &= -3(x + 1)^2 + 4. \end{aligned}$$

Thus we can conclude that the parabola $y = -3x^2 - 6x + 1$ has a maximum value of 4, occurring at $x = -1$.

Parabolas, circles, ellipses and hyperbolas are all examples of conic sections; they all arise by intersecting a double cone such as $z^2 = x^2 + y^2$, with a plane.

Slice an upright cone horizontally and you get a circle. Tip your knife a bit and you will get an ellipse. Keep tipping and you will get a parabola. Continue tipping and you will cut out a hyperbola.



Formally then, the general equation for a conic is given by the formula,

$$(l - ex - dy)^2 = x^2 + y^2,$$

with a focus at the origin.

3.2 Circles

A circle centered at the origin $(0,0)$ with radius r is given by the equation

$$x^2 + y^2 = r^2.$$

A circle with radius r and center (h,k) corresponds to

$$(x-h)^2 + (y-k)^2 = r^2.$$

For a circle center $(2, -3)$, with radius 2, we have $(x-2)^2 + (y+3)^2 = 4$, which can also be represented parametrically:

$$(x-2) = 2\cos\theta, \quad (y+3) = 2\sin\theta, \qquad \text{or} \qquad x = 2 + 2\cos\theta, \quad y = -3 + 2\sin\theta.$$

Given an expression such as $x^2 + 10x + y^2 - 4y + 20 = 0$, we can determine whether it is a circle by completing the square:

$$\begin{aligned} x^2 + 10x + y^2 - 4y + 20 &= 0 \\ x^2 + 10x + 25 - 25 + y^2 - 4y + 4 - 4 + 20 &= 0 \\ (x+5)^2 + (y-2)^2 &= 9 \end{aligned}$$

so $x^2 + 10x + y^2 - 4y + 20 = 0$ represents a circle of radius 3, centred at $(-5, 2)$.

3.3 Ellipses

In standard form, an ellipse centered at the origin is given by the equation

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.}$$

An ellipse may also be defined as the set of points in a plane the sum of whose distances from two fixed focal points F_1 and F_2 is a constant. If $a \geq b$, then the foci are at $(\pm c, 0)$ where $c^2 = a^2 - b^2$. These lie on the major axis (x-axis here). The ellipse intersects the x axis at the vertices $(\pm a, 0)$, and the y-axis at the vertices $(0, \pm b)$.

The equation of an ellipse centred at the point (h, k) is simply

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

3.4 Hyperbolas

A hyperbola centered at the origin, with asymptotes $y = \pm bx/a$, is given by the equation

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \qquad \text{or by} \qquad \boxed{-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.}$$

A hyperbola may also be defined as the set of points in a plane, the difference of whose distances from two fixed focal points F_1 and F_2 is a constant. For such a hyperbola, the foci are at $(\pm c, 0)$ and $(0, \pm c)$ respectively, where $c^2 = a^2 + b^2$. The equation of a hyperbola centred at the point (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

or

$$-\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Another kind of hyperbola is the very familiar $y = 1/x$. This can also be expressed as $xy = 1$, and is an example of a hyperbola with axes of symmetry which do not align with the x- and y-axes. These more general cases with cross-terms are studied in MATH2001/2901.

3.5 Quiz

Can you match each equation with one of the graphs?

(a) $x^2 - y^2 - 1 = 0$

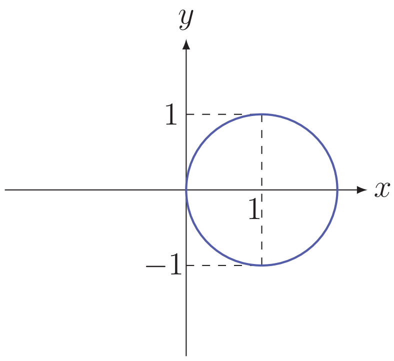
(b) $x + y^2 - 1 = 0$

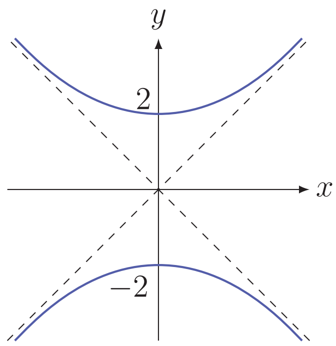
(c) $x^2 - 2x + y^2 = 0$

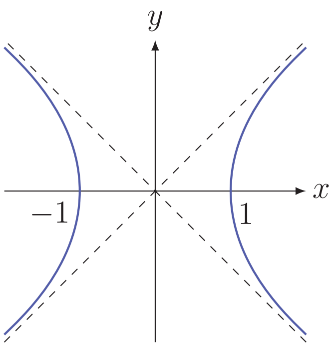
(d) $x^2 + 2x - y + 1 = 0.$

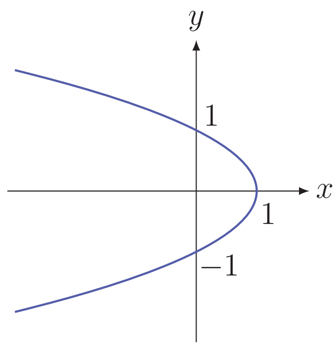
(e) $x^2 - y^2 + 4 = 0.$

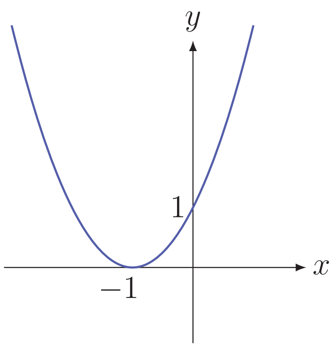
(f) $4x^2 - 8x + y^2 = 0.$

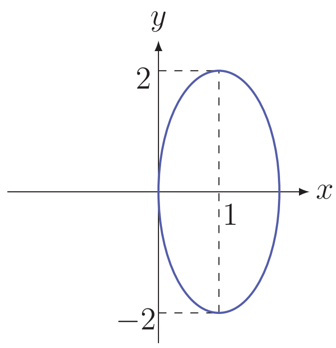












Double-check each answer by putting each equation in standard form.

4 PRE-LECTURE READING: Newton's Laws of Motion.

4.1 Force and acceleration

Consider a particle of mass m which moves in a straight line. Let $x(t)$ be the distance of the particle from the origin at time t.

- Velocity: $v(t) = \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$.
- Acceleration: $a(t) = \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x}$

The motion of this particle is determined by **Newton's Second Law**;

$$m\ddot{x} = F,$$

where F is the net force exerted on the particle.

This is called the **equation of motion.**

Figure 1: 1-D motion

4.2 Gravity

Consider a particle at height y above the Earth's surface. According to **Newton's Law of Gravitation**, the force exerted on the particle is;

$$F = \frac{-GMm}{r^2},$$

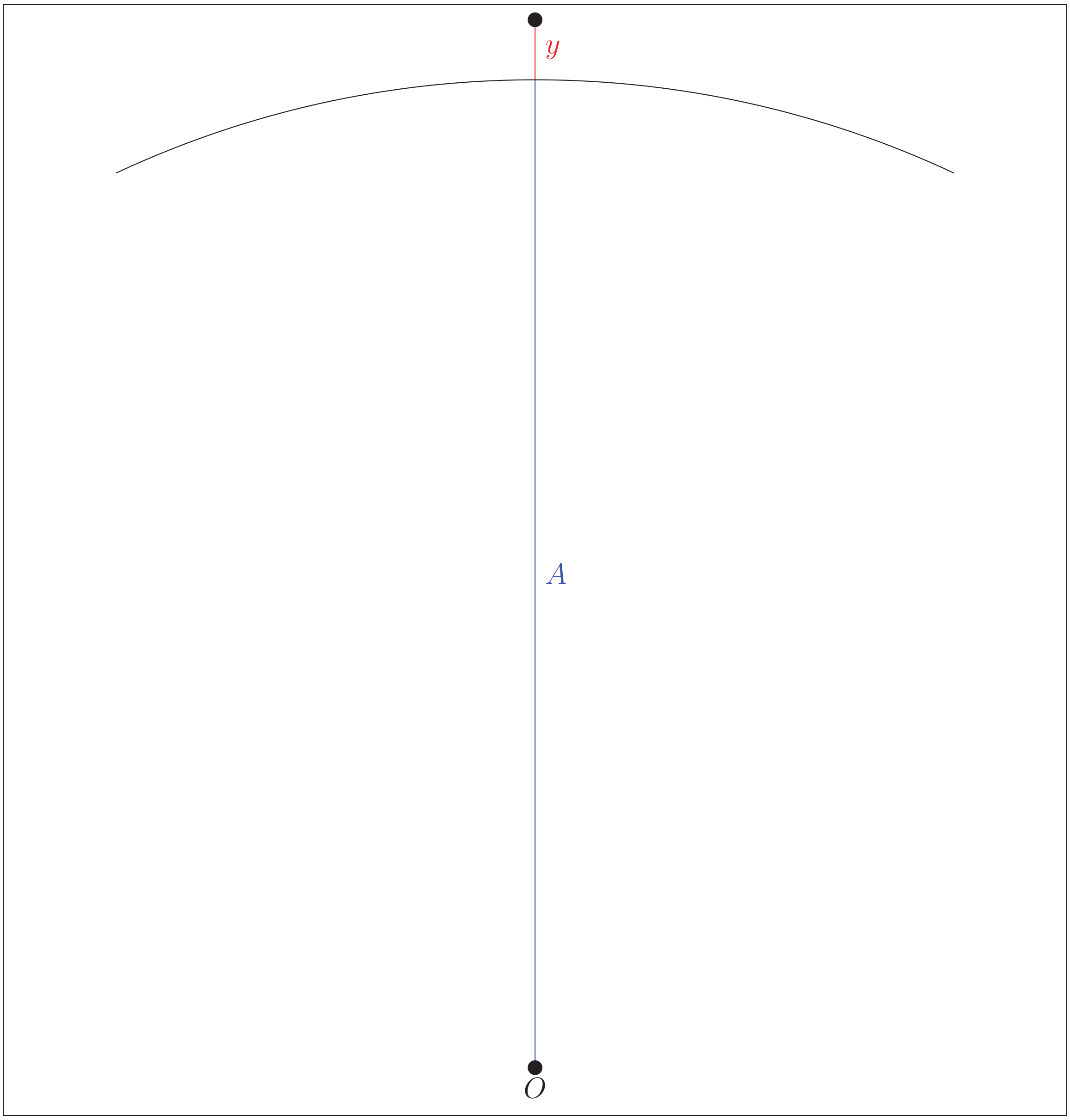


Figure 2: Gravitational force

where

$$
\begin{aligned}
M &= \text{mass of the Earth } (5.9 \times 10^{24}\,\text{kg}) \\
A &= \text{radius of the Earth } (6.4 \times 10^{6}\,\text{m}) \\
G &= \text{universal gravitational constant } (6.7 \times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}) \\
r &= y + A \text{ (i.e. the distance from the particle to the centre of the Earth.)}
\end{aligned}
$$

If $y \ll A$ then F can be expressed in the form $F = -mg$.

Using $r = y + A$, we have

$$F = -\frac{GMm}{(y+A)^2} \approx -\frac{GMm}{A^2}.$$

(As an aside, informally, "$a \ll b$" means that a is negligible compared to b.) We write this approximation as

$$F = -mg,$$

where $g = \dfrac{GM}{A^2}$ is *acceleration due to gravity near the surface of the earth.* The numerical value is approximately $g = 9.8\,\mathrm{ms}^{-2}$.

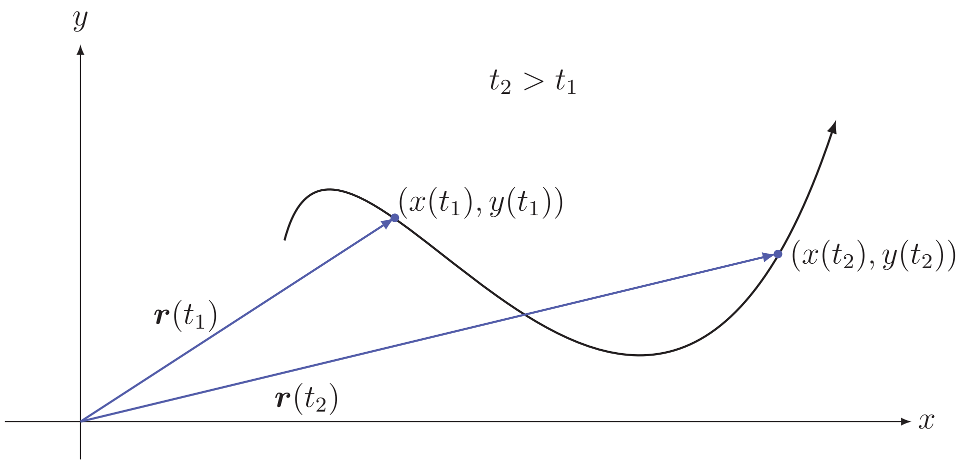
4.3 Position vectors, velocity and acceleration

When we are working in dimensions greater than one, parametric equations are used to describe position, velocity and acceleration.

The position vector

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

traces out the path given by the parametrisation $(x(t), y(t))$.



Similarly in three dimensions, the position vector

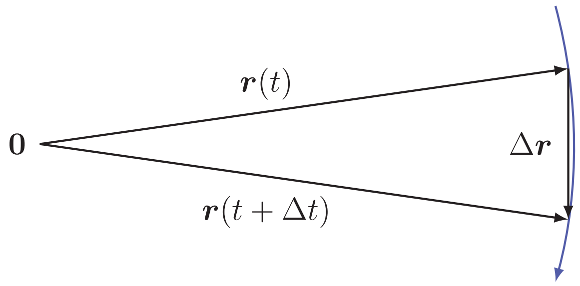
$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

traces out the path given by the parametrisation $(x(t), y(t), z(t))$.

The vector tangent to the path of motion, with magnitude equal to the speed is the velocity vector. If $\boldsymbol{r}(t)$ is the position vector, then

$$\Delta\boldsymbol{r} = \boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)$$

is approximately tangent to the curve traced out by $\boldsymbol{r}(t)$. The approximation gets better as $\Delta t \to 0$.



The velocity vector $\boldsymbol{v}$ is given by

$$\begin{aligned}\boldsymbol{v}(t) &= \lim_{\Delta t\to 0}\frac{\boldsymbol{r}(t+\Delta t)-\boldsymbol{r}(t)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{r}}{\Delta t}\\ &= \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}.\end{aligned}$$

In component-form, if $\boldsymbol{r}(t)=x(t)\boldsymbol{i}+y(t)\boldsymbol{j}+z(t)\boldsymbol{k}$, then the velocity is

$$\boldsymbol{v}(t)=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}t}\boldsymbol{i}+\frac{\mathrm{d}y}{\mathrm{d}t}\boldsymbol{j}+\frac{\mathrm{d}z}{\mathrm{d}t}\boldsymbol{k}.$$

Speed is the magnitude of velocity:

$$\text{speed } = \|\boldsymbol{v}(t)\| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}.$$

Note that if an object moves with constant velocity, this means that it has constant speed and direction, i.e., it travels in a straight line.

The acceleration vector is defined as

$$\boldsymbol{a}(t)=\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\boldsymbol{i}+\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\boldsymbol{j}+\frac{\mathrm{d}^2z}{\mathrm{d}t^2}\boldsymbol{k}.$$

Example: The velocity of an object is $\boldsymbol{v}=2\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}$ and at $t=0$ the object passes through $(-2,1,0)$. Find the position vector.

Solution: The velocity is constant

$$\begin{aligned}\boldsymbol{r}(t) &= \int \boldsymbol{v}\ \mathrm{d}t\\ &= t\boldsymbol{v}+\boldsymbol{r}_0\\ &= 2(t-1)\boldsymbol{i}+(3t+1)\boldsymbol{j}+4t\boldsymbol{k}\end{aligned}$$

Example: Find the velocity vector of a car moving along the helical path (e.g., the Indooroopilly Shopping Centre carpark)

$$x = \cos 5t, \quad y = \sin 5t, \quad z = t.$$

Solution:

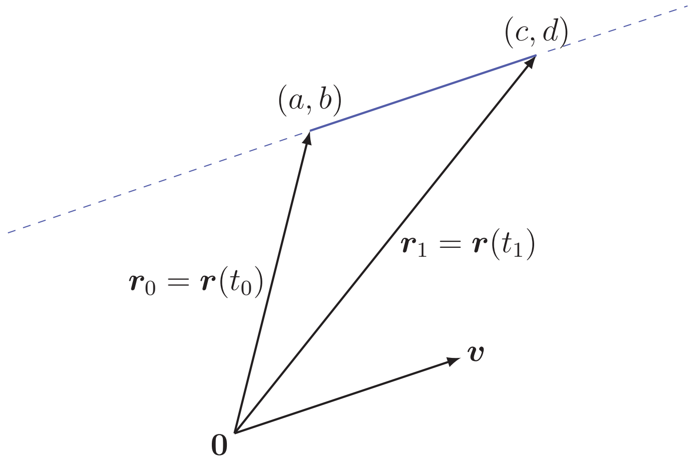
$$\boldsymbol{r}(t) = \cos 5t\boldsymbol{i} + \sin 5t\boldsymbol{j} + \frac{t}{2}\boldsymbol{k}$$
$$\boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = -5\sin 5t\boldsymbol{i} + 5\cos 5t\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}.$$

Example: Say we want to parametrise motion, with constant velocity, along the straight line between the points (a, b) and (c, d). We require that (a, b) corresponds to parameter value $t = t_0$, and (c, d) corresponds to $t = t_1$, with $t_1 = t_0 + \Delta t$ and $\Delta t > 0$.

Let $\boldsymbol{r}_0$ be the position vector for (a, b) and $\boldsymbol{r}_1$ the position vector for (c, d). Then

$$\boldsymbol{v} = \boldsymbol{r}_1 - \boldsymbol{r}_0$$

is a direction vector for the line that goes through (a, b) and (c, d).



The position vector for points on the line through $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ is

$$\boldsymbol{r} = \boldsymbol{r}_0 + \lambda\boldsymbol{v} = (1-\lambda)\boldsymbol{r}_0 + \lambda\boldsymbol{r}_1, \qquad \lambda \in \mathbb{R}.$$

So all we need to do to parametrise the portion of the line from $\boldsymbol{r}_0$ to $\boldsymbol{r}_1$ is choose a relationship between λ and t that gives $\lambda = 0$ when $t = t_0$ and $\lambda = 1$ when $t = t_1$. The linear relation between λ and t that does this is

$$\lambda = \frac{t - t_0}{\Delta t}.$$

A parametrisation for the line segment is therefore

$$\boxed{\boldsymbol{r}(t) = \boldsymbol{r}_0 + \frac{t - t_0}{\Delta t}\boldsymbol{v}, \qquad t_0 \le t \le t_1.}$$

4.4 Example - stone on a string

Imagine you spin a stone on a string above your head so that it moves in a circle of radius 1m at a height of 2m above the ground at a constant speed with period π seconds. If the string breaks, find the position, velocity, and acceleration vectors of the stone after the string breaks.

Before the string breaks

$$\boldsymbol{r}(t) = \cos 2t\boldsymbol{i} + \sin 2t\boldsymbol{j} + 2\boldsymbol{k}$$
$$\boldsymbol{v}(t) = -2\sin 2t\boldsymbol{i} + 2\cos 2t\boldsymbol{j},$$

so that $\boldsymbol{r}(0) = \boldsymbol{i} + 2\boldsymbol{k}$ and $\boldsymbol{v}(0) = 2\boldsymbol{j}$.

After the string breaks the stone moves under gravity

$$m\boldsymbol{a}(t) = -mg\boldsymbol{k} \quad \Rightarrow \quad \boldsymbol{a}(t) = -g\boldsymbol{k} \quad \Rightarrow \quad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -g\boldsymbol{k}.$$

Hence

$$\boldsymbol{v} = -gt\boldsymbol{k} + \boldsymbol{v}(0) = 2\boldsymbol{j} - gt\boldsymbol{k}.$$

Since $\dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}$ this in turn implies that

$$\boldsymbol{r}(t) = 2t\boldsymbol{j} - \frac{1}{2}gt^2\boldsymbol{k} + \boldsymbol{r}(0)$$

so that

$$\boldsymbol{r}(t) = \boldsymbol{i} + 2t\boldsymbol{j} + \left(2 - \frac{1}{2}gt^2\right)\boldsymbol{k}.$$