Survival probability

October 14, 2025

Contents

Todo list

1 Question

Why is the correlation hole not affected by the precense of a high degree of symmetries in Ref. [1]?

2 Mostly ideas

2.1 Survival probability in a system with the simplest symmetric structure

The survival probability is defined as

$$S_p(t) = |\langle\psi_0|\psi(t)\rangle|^2, \tag{1}$$

Consider that $H|\phi_n\rangle = E_n|\phi_n\rangle$, then the survival probability S_p yields

$$S_p(t) = \left|\sum_k |\langle\psi_0|\phi_k\rangle|^2 e^{-iE_k t}\right|^2. \tag{2}$$

Let us consider a symmetry of H under an arbitrary operator Π such that $H = H_1 \oplus H_2$—for instance, you may consider Π to be the reflection operator. Then, we can relabel the energies E_n such that $\{E_n\}_{n=1}^{p}$ and $\{E_n\}_{n=p+1}^{N}$ are the spectra of H_1 and H_2, respectively. N denotes the dimension of H, p the dimension of H_1 and $N-p$ is the dimension of H_2. After this relabeling, $S_p(t)$ takes the form:

$$\begin{aligned}
S_p(t) &= \left|\sum_{k=1}^{p} |\langle\psi_0|\phi_k\rangle|^2 e^{-iE_k t} + \sum_{l=p+1}^{N} |\langle\psi_0|\phi_l\rangle|^2 e^{-iE_l t}\right|^2 &(3)\\
&= \left|\sum_{k=1}^{p} |\langle\psi_0|\phi_k\rangle|^2 e^{-iE_k t}\right|^2 + \left|\sum_{l=p+1}^{N} |\langle\psi_0|\phi_l\rangle|^2 e^{-iE_l t}\right|^2 &(4)\\
&\quad + 2\,\mathrm{Re}\left\{\sum_{k=1}^{p+1}\sum_{l=p+1}^{N} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_l\rangle|^2 e^{-i(E_k-E_l)t}\right\}. &(5)
\end{aligned}$$

The first two terms are recognized as the survival probability in the two symmetric subspaces. The third term God knows what the hell it is, but it is the term explaining why the correlation whole survives (or not).

2.2 Survival probability as function of level spacings

Let us rewrite the survival probability as follows:

$$\begin{aligned}
S_p(t) &= \left|\sum_k |\langle\psi_0|\phi_k\rangle|^2 e^{-iE_k t}\right|^2 &(6)\\
&= \sum_{k,l} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_l\rangle|^2 e^{-i(E_k-E_l)t} &(7)\\
&= \sum_{k=1}^{N} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_k\rangle|^2 + 2\,\mathrm{Re}\left[\sum_{k=1}^{N-1} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_{k+1}\rangle|^2 e^{-i(E_k-E_{k+1})t}\right.\\
&\quad + \sum_{k=1}^{N-2} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_{k+2}\rangle|^2 e^{-i(E_k-E_{k+2})t} + \ldots\\
&\quad \left. + |\langle\psi_0|\phi_N\rangle|^4 e^{-i(E_1-E_N)t}\right]. &(8)
\end{aligned}$$

In this form, it is explicit that the survival probability can be written as a sum of the contributions given by higher order spacings. From Ref. [2] we learned not only that RMT's predictions can be verified even a non-desymmetrized system, but also that higher order spacing ratios exhibit correlations that match RMT's predictions. Very recently, a work on the distribution of higher order spacings for the circular gaussian ensembles appeared on the arXiv [3] JA: I think these results we can use them to study the behavior of each term in Eq. (8). My guess is that the most relevant contributions to $S_p(t)$ come from the higher order spacing correlations.

Let us consider two identical spectra $\{E_1^{(1)}, E_2^{(1)}, E_3^{(1)}\}$ and $\{E_1^{(2)}, E_2^{(2)}, E_3^{(2)}\}$ (that is, $E_i^{(1)} = E_i^{(2)}$). The whole ordered spectrum will be $\{E_1^{(1)}, E_1^{(2)}, E_2^{(1)}, E_2^{(2)}, E_3^{(1)}, E_3^{(2)}\}$. Then, compute the spacings:

$$s_1^{(1)} = E_1^{(2)} - E_1^{(1)} = 0 \tag{9}$$
$$s_1^{(2)} = E_2^{(1)} - E_1^{(1)} \tag{10}$$
$$s_1^{(3)} = E_2^{(2)} - E_1^{(1)} \tag{11}$$
$$s_1^{(4)} = E_3^{(1)} - E_1^{(1)} \tag{12}$$
$$s_1^{(5)} = E_3^{(2)} - E_1^{(1)} \tag{13}$$

The spacings of first order $s_n^{(1)}$ follow a Poissonian distribution, possibly a zero-inflated Poisson distribution as half the spacings will become zero. The second-order spacings become the first-order spacings of both subspaces JA: I think the PDF here will be the convolution of two Wigner surmises. The third-order spacings become a mix between first- and second-order spacings JA: maybe the convolution of three Wigner surmises?. The fourth-order spacings will become second-order spacings from both subspaces JA: once again, the convolution of four Wigner surmises?. The fifth-order spacings become a mix between second- and third-order spacings JA: you know the drill by now, the convolution of five Wigner surmises?

JA: I guess the convolution of many Wigner surmises should converge to something...? Hence, after some order the PDF should be almost the same. Something to ask Deepseek or chatGPT

JA: If all of this is, at the very least, a good approximation, we still have to take into account the coefficients $|\langle\psi_0|\phi_i\rangle|^2|\langle\psi_0|\phi_{i+k}\rangle|^2$. Therefore, we may consider the case where all coefficients are equal, case in which I'm almost sure the $S_p(t)$ becomes the spectral form factor. If it is indeed the case we should expect a dip-ramp-plateau behavior, the ramp being the hallmark of long-range correlations of a spectrum's chaotic system.

2.3 Spectral form factor (SFF)

The SFF can be decomposed into the following sums:

$$K(t) = \frac{1}{N^2}\overline{\sum_{k,l} e^{-i(E_k - E_l)t}} \tag{14}$$
$$= 1 + \frac{2}{N^2}\,\mathrm{Re}\left[\overline{\sum_{k=1}^{N-1} e^{-i(E_{k+1}E_k)t}} + \overline{\sum_{k=1}^{N-2} e^{-i(E_{k+2}-E_k)t}} + \ldots + \overline{e^{-i(E_N - E_1)t}}\right]. \tag{15}$$

SN: As a sanity check, check numerically in the BH model that Eqs. (14) and (15) give the same SFF curve, computing separately term by term of Eq. (15), and then adding them up

We try to move forward with the following approximation. We assume the spacings $s_k^{(1)} = E_{k+1} - E_k$ are independent and identically distributed (i.i.d), thus:

$$\overline{\sum_{k=1}^{N-1} e^{-i(E_{k+1}E_k)t}} \approx (N-1)\,\overline{e^{-is^{(1)}t}} = \int_0^\infty e^{-ist}P(s)\,\mathrm{d}s\,, \tag{16}$$

where $P(s)$ is either a Poisson or a Wigner-Dyson distribution, depending on the case,

$$\int_0^\infty e^{-ist}P_{\mathrm{WD}}(s)\,\mathrm{d}s = (N-1)\left[1 - te^{-t^2/\pi}\left(\mathrm{erfi}\left(\frac{t}{\sqrt{\pi}}\right) + i\right)\right] \tag{17}$$
$$\int_0^\infty e^{-ist}P_P(s)\,\mathrm{d}s = \frac{N-1}{1+it}. \tag{18}$$

SN: Check numerically if the left side of Eq. (16) is well described by either Eq. (17) when a single symmetric subspace is considered, and by Eq. (18) when the whole spectrum is considered (we have the intuition that the whole spectrum should exhibit Poisson statistics at first order spacings or ratios)

JA: In Ref. [4] they study the distribution of higher order level spacings.

A Average of $\sum_k e^{-is_k t}$ for Wigner-Surmise Spacings

We consider the nearest-neighbor spacings s_k distributed according to the **Wigner surmise**:

$$p(s) = \frac{\pi s}{2} e^{-\pi s^2/4}, \quad s \geq 0.$$

We wish to compute:

$$\left\langle \sum_k e^{-is_k t} \right\rangle.$$

Assumption

Assuming the spacings are independent and identically distributed (i.i.d.) with the Wigner surmise distribution:

$$\left\langle \sum_k e^{-is_k t} \right\rangle \approx (N-1) \left\langle e^{-ist} \right\rangle.$$

Characteristic Function

The characteristic function is:

$$\phi(t) = \int_0^\infty e^{-ist} p(s)\, \mathrm{d}s = \int_0^\infty e^{-ist} \cdot \frac{\pi s}{2} e^{-\pi s^2/4} \mathrm{d}s.$$

This integral evaluates to:

$$\phi(t) = 1 - t e^{-t^2/\pi} \left(\operatorname{erfi}\left(\frac{t}{\sqrt{\pi}} \right) + i \right),$$

where $\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{u^2} \mathrm{d}u$ is the imaginary error function.

Final Result

$$\boxed{\left\langle \sum_k e^{-is_k t} \right\rangle \approx (N-1) \left[1 - t e^{-t^2/\pi} \left(\operatorname{erfi}\left(\frac{t}{\sqrt{\pi}} \right) + i \right) \right]}$$

Remarks

- This result assumes i.i.d. spacings (a common approximation using the Wigner surmise)
- In full RMT (GOE), spacings are not independent, but this captures key features
- For $t = 0$: $\phi(0) = 1$ as expected
- For small t: $\phi(t) \approx 1 - it - \frac{2}{\pi} t^2 + \cdots$

B df

For a quantum system with Poissonian level statistics, the nearest-neighbor spacings s_k are independent and identically distributed with the exponential distribution:

$$p(s) = e^{-s}, \quad s \geq 0,$$

where the mean spacing is normalized to unity.

We wish to compute the average:

$$\left\langle \sum_k e^{-is_k t} \right\rangle.$$

Assuming the spacings are independent, this becomes:

$$\left\langle \sum_k e^{-is_k t} \right\rangle = (N-1)\left\langle e^{-ist}\right\rangle ,$$

where the characteristic function is:

$$\phi(t) = \left\langle e^{-ist}\right\rangle = \int_0^\infty e^{-ist} e^{-s}\, \mathrm{d}s = \int_0^\infty e^{-s(1+it)}\, \mathrm{d}s\,.$$

Evaluating this integral gives:

$$\phi(t) = \frac{1}{1+it}.$$

Therefore, the final result is:

$$\boxed{\frac{N-1}{1+it}}$$

C A possibly useful integral

$$\left\langle e^{-ist}\right\rangle = \frac{1}{2} e^{-\frac{1}{2}t(\sigma^2 t+2i)}\left[1+\operatorname{erf}\left(\frac{1-i\sigma^2 t}{\sqrt{2}\sigma}\right)\right] \tag{19}$$

D Poster

D.1 Main goal

The primary objective is to develop an intuitive understanding of the correlations associated with higher-order spacings in the Bose-Hubbard (BH) model, specifically within the parameter regime characterized by the transition from Poisson to Wigner-Dyson statistics suggested by the short-range correlation indicators we have previously examined, namely the mean level spacing ratio $\langle r\rangle$ and the Kullback-Leibler (KL) divergence. More precisely, I aim to test the hypothesis that long-range correlations emerge gradually, akin to the gradual breakdown of tori in classical chaotic systems as the chaotic parameter is increased.

D.2 What to do

Compare the mean spacing ratios of order k of a GOE matrix (sufficiently big) with that obtained from the BH model. That is, plot k in the horizontal axis, and $\langle r^{(k)}\rangle$ in the vertical axis. I would expect that if the system is chaotic the curve of GOE and of that of the BH model are the same. But in the transition between integrable and chaotic, maybe we will observe deviations from the GOE?

References

[1] p. l. u. family=Cruz given=Javier, S. Lerma-Hernández, and J. G. Hirsch, "Quantum chaos in a system with high degree of symmetries", Physical Review E **102**, 032208 (2020).

[2] S. H. Tekur and M. S. Santhanam, "Symmetry deduction from spectral fluctuations in complex quantum systems", Physical Review Research **2**, 032063 (2020).

[3] S. Rout and U. T. Bhosale, *Higher-order spacings in the superposed spectra of random matrices with comparison to spacing ratios and application to complex systems*, (Oct. 1, 2025) arXiv:2510.00503 [physics], http://arxiv.org/abs/2510.00503.

[4] D. Engel, J. Main, and G. Wunner, "Higher-order energy level spacing distributions in the transition region between regularity and chaos", Journal of Physics A: Mathematical and General **31**, 6965 (1998).