Why Unfolding is not necessary for ratio of consecutive level spacings

1 Unfolding: Removing System-Specific Density

In quantum systems, the density of states often exhibits both a smooth background and superimposed fluctuations. To analyze the underlying statistical properties of the energy levels, it is essential to separate these components. This process is known as unfolding. Let us discuss how to perform the unfolding of a spectrum.

We begin defining the cumulative level count function $N(E)$, also known as staircase function:

$$N(E) = \sum_i \Theta(E - E_i). \tag{1}$$

One then needs to find a smooth function $\overline{N}(E)$ that approximates $N(E)$, by using techniques such as polynomial fitting or spectral averaging. The goal is to capture the average trend of energy levels distribution, filtering out the individual deviations. Then, the unfolded levels, denoted as ϵ_i, are defined as:

$$\epsilon_i = \overline{N}(E_i) \tag{2}$$

This transformation ensures that the unfolded levels have a uniform average density. Consequently, the spacings between consecutive unfolded levels, $\tilde{s}_i$, are given by:

$$\tilde{s}_i = \epsilon_{i+1} - \epsilon_i \tag{3}$$

An important property of the unfolded spacings is that their average, $\langle \tilde{s}_i \rangle$, is equal to 1 throughout the entire spectrum. This normalization is absolutely necessary for the comparison of the spacing distribution with the predictions of RMT.

2 The Ratio Statistics: Automatic Scale Invariance

Consider three consecutive levels: E_{i-1}, E_i, E_{i+1}. The ratio of consecutive spacings is defined as:

$$r_i = \frac{\min(s_i, s_{i-1})}{\max(s_i, s_{i-1})} \tag{4}$$

where $s_i = E_{i+1} - E_i$ are the level spacings. This definition, which is not the only one, ensures $0 \leq r_i \leq 1$. Now let's examine what happens under the unfolding transformation.

The unfolded spacings are:

$$\tilde{s}_i = \overline{N}(E_{i+1}) - \overline{N}(E_i) \tag{5}$$

For sufficiently localized triplets, the density of states should be approximately constant we can approximate using the mean value theorem:

$$\tilde{s}_i \approx \rho(E_i) \cdot s_i, \quad \tilde{s}_{i-1} \approx \rho(E_i) \cdot s_{i-1} \tag{6}$$

where $\rho(E) = \frac{\mathrm{d}\overline{N}}{\mathrm{d}E}$ is the local density of states. Now compute the ratio of unfolded spacings:

$$\tilde{r}_i = \frac{\min(\tilde{s}_i, \tilde{s}_{i-1})}{\max(\tilde{s}_i, \tilde{s}_{i-1})} = \frac{\min(\rho(E_i)s_i, \rho(E_i)s_{i-1})}{\max(\rho(E_i)s_i, \rho(E_i)s_{i-1})} = \frac{\min(s_i, s_{i-1})}{\max(s_i, s_{i-1})} = r_i \tag{7}$$

Therefore, we have proved that unfolding is not needed for computing the ratio of consecutive level spacings, since ratios are "invariant" under unfolding.