Notas

December 8, 2025

Lista de tareas pendientes

Contents

1 Question

Saul: Hola Why is the correlation hole not affected by the precense of a high degree of symmetries in Ref. [1]?

2 Relevant literature

In this section, I will list paper's I believe we should be aware of. These works either provide valuable insights for exploring our ideas or are significant contributions that merit citation in any future publication.

- Ref [2]: decomposition of the spectral form factor into its contributions coming from kth order level spacings.
- Ref [3]: derivation of the distribution $P^{(k)}(s)$ of kth order level spacings $s^{(k)} = E_{i+k} - E_i$. They numerically test their analytical approximation in the XXZ spin chain model with random fields (we have it available in `QMB.wl`). Fig. 4(middle) makes an analysis very similar to that of Figs. **??** and **??**.
- Ref. [4]: exhaustive analysis of kth order spacings in superposed spectra with comparison to spacing ratios, mostly for the COE and GOE ensembles, and also tested in phsyical systems: the intermediate map and the quantum kicked top (QKT)[1].
- Ref. [5]: derivation of an analytical approximation for the distribution of kth order spacing ratios. They present numerical evidence for the following approximation:

$$P^{(k)}(r,\beta) = P(r,\beta'), \quad \beta' = \frac{k(k+1)}{2}\beta + (k-1), \tag{1}$$

eq:tekur_higherorder_ansatz

  where $P(r,\beta')$ corresponds to the distribution found by Atas et al. [6] and $\beta = 1,2,4$ correspond to the orthogonal, unitary and symplectic circular or Gaussian ensembles, respectively. JA: Important: for large k, I think the distribution becomes Gaussian, as it was noted for the kth order spacings [3]. Therefore, Eq. (1) should be used up until some k

3 Mostly ideas

3.1 Survival probability in a system with the simplest symmetric structure

The survival probability is defined as

$$S_p(t) = |\langle\psi_0|\psi(t)\rangle|^2, \tag{2}$$

Consider that $H|\phi_n\rangle = E_n|\phi_n\rangle$, then the survival probability S_p yields

$$S_p(t) = \left|\sum_k |\langle\psi_0|\phi_k\rangle|^2 e^{-iE_k t}\right|^2. \tag{3}$$

Let us consider a symmetry of H under an arbitrary operator Π such that $H = H_1 \oplus H_2$—for instance, you may consider Π to be the reflection operator. Then, we can relabel the energies E_n such that $\{E_n\}_{n=1}^{p}$ and $\{E_n\}_{n=p+1}^{N}$ are the spectra of H_1 and H_2, respectively. N denotes the dimension of H, p the dimension of H_1 and $N-p$ is the dimension of H_2. After this relabeling, $S_p(t)$ takes the form:

$$\begin{aligned} S_p(t) &= \left|\sum_{k=1}^{p} |\langle\psi_0|\phi_k\rangle|^2 e^{-iE_k t} + \sum_{l=p+1}^{N} |\langle\psi_0|\phi_l\rangle|^2 e^{-iE_l t}\right|^2 &(4)\\ &= \left|\sum_{k=1}^{p} |\langle\psi_0|\phi_k\rangle|^2 e^{-iE_k t}\right|^2 + \left|\sum_{l=p+1}^{N} |\langle\psi_0|\phi_l\rangle|^2 e^{-iE_l t}\right|^2 &(5)\\ &+ 2\,\mathrm{Re}\left\{\sum_{k=1}^{p+1}\sum_{l=p+1}^{N} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_l\rangle|^2 e^{-i(E_k-E_l)t}\right\}. &(6) \end{aligned}$$

[1] We don't have any numerics for the QKT, but Miguel does

The first two terms are recognized as the survival probability in the two symmetric subspaces. The third term God knows what the hell it is, but it is the term explaining why the correlation whole survives (or not).

3.2 Survival probability as function of level spacings

Leer esta sección para discutir con JA si vale la pena seguir explorando esta idea o darle muerte

Let us rewrite the survival probability as follows:

$$
\begin{aligned}
S_p(t) &= \left| \sum_k |\langle\psi_0|\phi_k\rangle|^2 e^{-iE_k t} \right|^2 && (7)\\
&= \sum_{k,l} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_l\rangle|^2 e^{-i(E_k-E_l)t} && (8)\\
&= \sum_{k=1}^{N} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_k\rangle|^2 + 2\,\mathrm{Re}\left[\sum_{k=1}^{N-1} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_{k+1}\rangle|^2 e^{-i(E_k-E_{k+1})t} \right.\\
&\quad + \sum_{k=1}^{N-2} |\langle\psi_0|\phi_k\rangle|^2 |\langle\psi_0|\phi_{k+2}\rangle|^2 e^{-i(E_k-E_{k+2})t} + \dots\\
&\quad \left. + |\langle\psi_0|\phi_N\rangle|^4 e^{-i(E_1-E_N)t} \right]. && (9)
\end{aligned}
$$

eq:Sp:spacings

In this form, it is explicit that the survival probability can be written as a sum of the contributions given by higher order spacings. From Ref. [7] we learned not only that RMT's predictions can be verified even a non-desymmetrized system, but also that higher order spacing ratios exhibit correlations that match RMT's predictions. Very recently, a work on the distribution of higher order spacings for the circular gaussian ensembles appeared on the arXiv [4] JA: I think these results we can use them to study the behavior of each term in Eq. (9). My guess is that the most relevant contributions to $S_p(t)$ come from the higher order spacing correlations.

Let us consider two identical spectra $\{E_1^{(1)}, E_2^{(1)}, E_3^{(1)}\}$ and $\{E_1^{(2)}, E_2^{(2)}, E_3^{(2)}\}$ (that is, $E_i^{(1)} = E_i^{(2)}$). The whole ordered spectrum will be $\{E_1^{(1)}, E_1^{(2)}, E_2^{(1)}, E_2^{(2)}, E_3^{(1)}, E_3^{(2)}\}$. Then, compute the spacings:

$$
\begin{aligned}
s_1^{(1)} &= E_1^{(2)} - E_1^{(1)} = 0 && (10)\\
s_1^{(2)} &= E_2^{(1)} - E_1^{(1)} && (11)\\
s_1^{(3)} &= E_2^{(2)} - E_1^{(1)} && (12)\\
s_1^{(4)} &= E_3^{(1)} - E_1^{(1)} && (13)\\
s_1^{(5)} &= E_3^{(2)} - E_1^{(1)} && (14)
\end{aligned}
$$

The spacings of first order $s_n^{(1)}$ follow a Poissonian distribution, possibly a zero-inflated Poisson distribution as half the spacings will become zero. The second-order spacings become the first-order spacings of both subspaces JA: I think the PDF here will be the convolution of two Wigner surmises. The third-order spacings become a mix between first- and second-order spacings JA: maybe the convolution of three Wigner surmises?. The fourth-order spacings will become second-order spacings from both subspaces JA: once again, the convolution of four Wigner surmises?. The fifth-order spacings become a mix between second- and third-order spacings JA: you know the drill by now, the convolution of five Wigner surmises?

JA: I guess the convolution of many Wigner surmises should converge to something...? Hence, after some order the PDF should be almost the same. Something to ask Deepseek or chatGPT

JA: If all of this is, at the very least, a good approximation, we still have to take into account the coefficients $|\langle\psi_0|\phi_i\rangle|^2 |\langle\psi_0|\phi_{i+k}\rangle|^2$. Therefore, we may consider the case where all coefficients are equal, case in which I'm almost sure the $S_p(t)$ becomes the spectral form factor. If it is indeed the case we should expect a dip-ramp-plateau behavior, the ramp being the hallmark of long-range correlations of a spectrum's chaotic system.

3.3 Spectral form factor (SFF)

The SFF can be decomposed into the following sums:

$$K(t) = \frac{1}{N^2}\overline{\sum_{k,l} e^{-i(E_k - E_l)t}} \tag{15}$$

eq:sff

$$= 1 + \frac{2}{N^2}\,\mathrm{Re}\left[\overline{\sum_{k=1}^{N-1} e^{-i(E_{k+1}E_k)t}} + \overline{\sum_{k=1}^{N-2} e^{-i(E_{k+2}-E_k)t}} + \ldots + \overline{e^{-i(E_N - E_1)t}}\right]. \tag{16}$$

eq:sff:spacings

We try to move forward with the following approximation. We assume the spacings $s_k^{(1)} = E_{k+1} - E_k$ are independent and identically distributed (i.i.d), thus:

$$\overline{\sum_{k=1}^{N-1} e^{-i(E_{k+1}E_k)t}} \approx (N-1)\,\overline{e^{-is^{(1)}t}} = \int_0^\infty e^{-ist} P(s)\,\mathrm{d}s\,, \tag{17}$$

eq:sff:spacings1

where $P(s)$ is either a Poisson or a Wigner-Dyson distribution, depending on the case,

$$\int_0^\infty e^{-ist} P_{\mathrm{WD}}(s)\,\mathrm{d}s = (N-1)\left[1 - te^{-t^2/\pi}\left(\mathrm{erfi}\left(\frac{t}{\sqrt{\pi}}\right) + i\right)\right] \tag{18}$$

eq:sff:spacings1:WDapprox

$$\int_0^\infty e^{-ist} P_P(s)\,\mathrm{d}s = \frac{N-1}{1+it}. \tag{19}$$

eq:sff:spacings1:Papprox

JA: In Ref. [8] they study the distribution of higher order level spacings.

A checking was performed for the proposition made in Eq. (18):

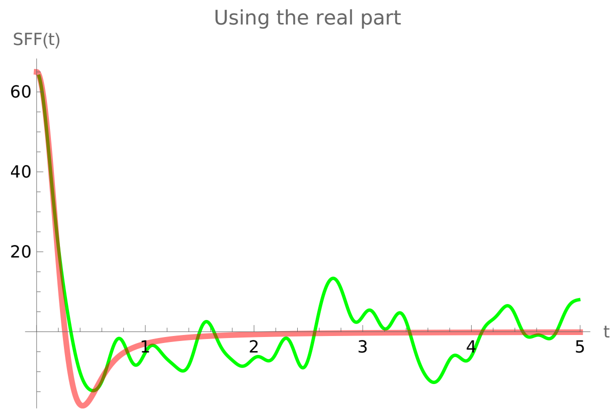

Figure 1: Comparing if the left side of Eq. (17) is well described by Eq. (18) when considering a single symmetric subspace.

fig:ApproxWDRe

Hacer la gráfica que compruebe/refute si se cumple la Eq. (19) al considerar (1) un subespacio en un régimen integrable y (2) considerando todo el espectro

Pensar si puede ser interesante seguir explorando este tipo de aproximaciones usando los resultados de la Ref. [3]

JA: Yo creo que puede ser interesante porque sigue testeando si existe o no un “breakdown” de las correlaciones de largo alcance para algún orden k de vecinos

3.4 Interesting things about SFF

There is a direct need in averanging the SFF computed, here a comparison between two methods will be shown, based on the data obtained from the BHH model. A time window averanging is performed on the data and a sector averanging aswell (averanging the SFF at each time with each symmetric sector of the hamiltonian).

The conclusion is that in a general context, a time averanging is best by far in comparison to a sector averanging, mainly due to the fact that a hamiltonian might only have 2 symmetric sectors. It is remarkable that applying both a sector averanging and a time window averanging proves to be the most precise approach.

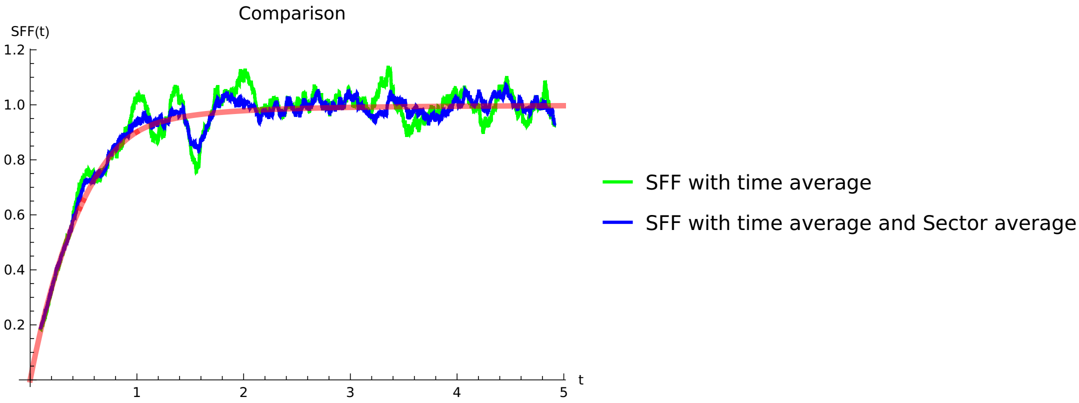

Figure 2: Comparison between using just time averanging and time averanging with sector averanging (red line is the GOE prediction). fig:CompSectAndTime

Using the GOE, numerical calculations have been performed in order to observe a characteristic behavior that is presented when SFF is considered. Two approaches where inspected. First, a high order matrix (dimension $\approx 3,000$) that belonged to GOE was generated, with this a SFF calculation was performed using time averanging.

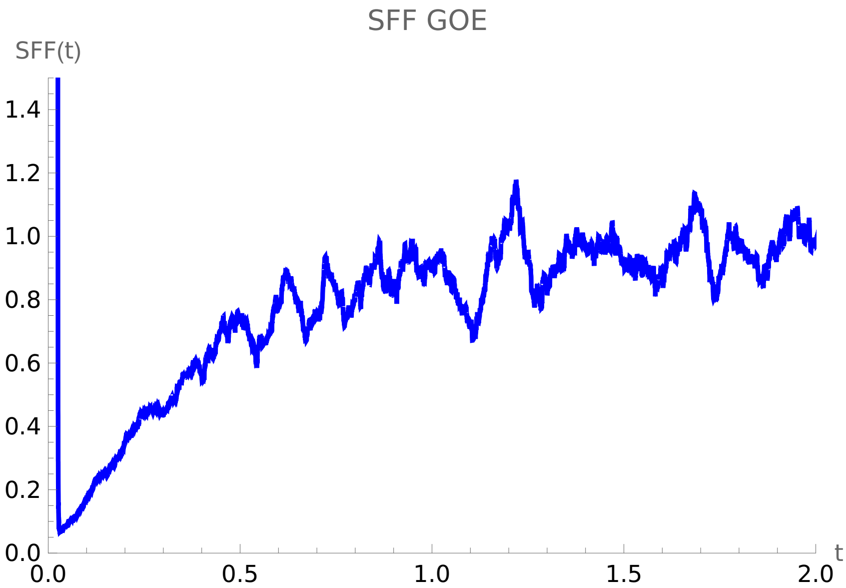

Figure 3: GOE SFF behavior using time averanging and a GOE matrix of dimension 3000. fig:TAvergGOE

Remarkably, using a different approach, where a list of GOE matrices was generated, then a sector averanging with a time averanging was performed, proving that applying both methodologies proves to be useful (valid only with a decent amount of statistics.)

Responder por qué una matriz GOE grande es estadísticamente lo mismo a tomar muchas matrices GOE de dimensión pequeña

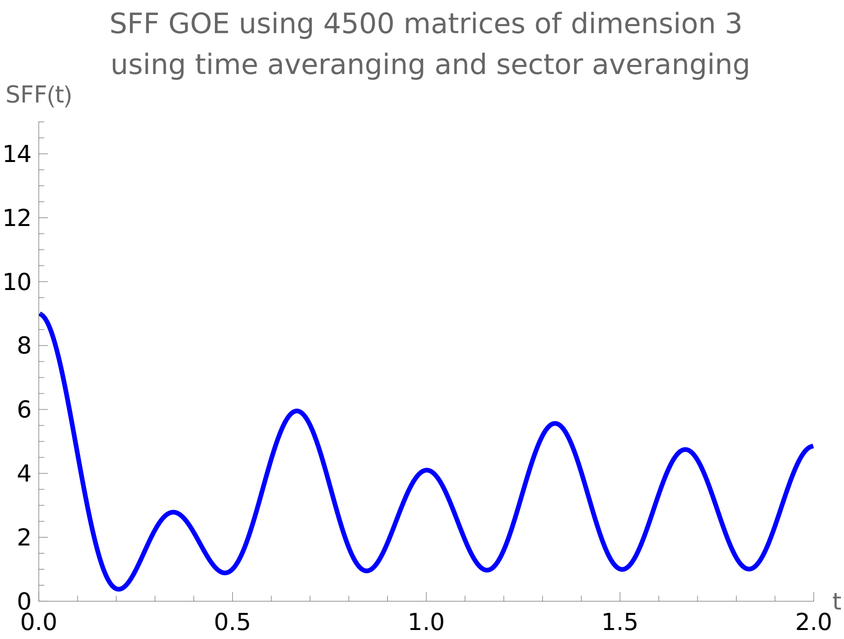

Figure 4: GOE SFF behavior using time averanging and sector averanging of a list of 4500 GOE matrices of dimension 3. JA: Vimos que con matrices COE hasta con matrices de 2×2 se puede construir el SFF

fig:SFFGOETS3

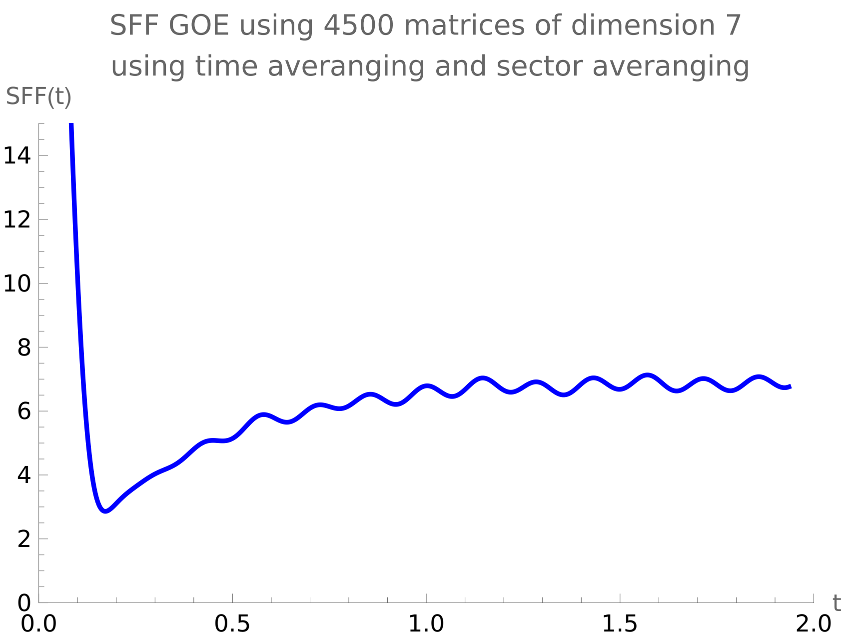

Figure 5: GOE SFF behavior using time averanging and sector averanging of a list of 4500 GOE matrices of dimension 7

fig:SFFGOETS7

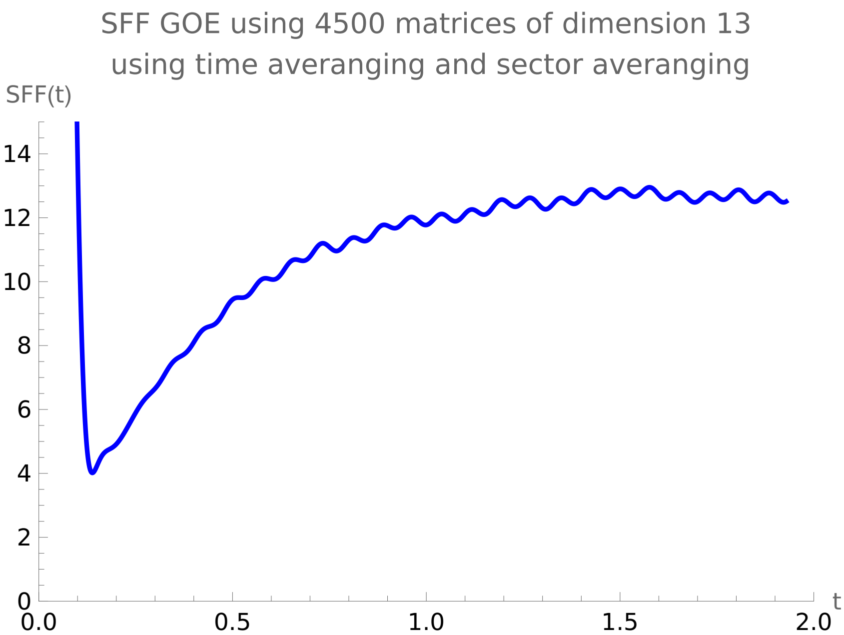

Figure 6: GOE SFF behavior using time averanging and sector averanging of a list of 4500 GOE matrices of dimension 13 fig:SFFGOETS13

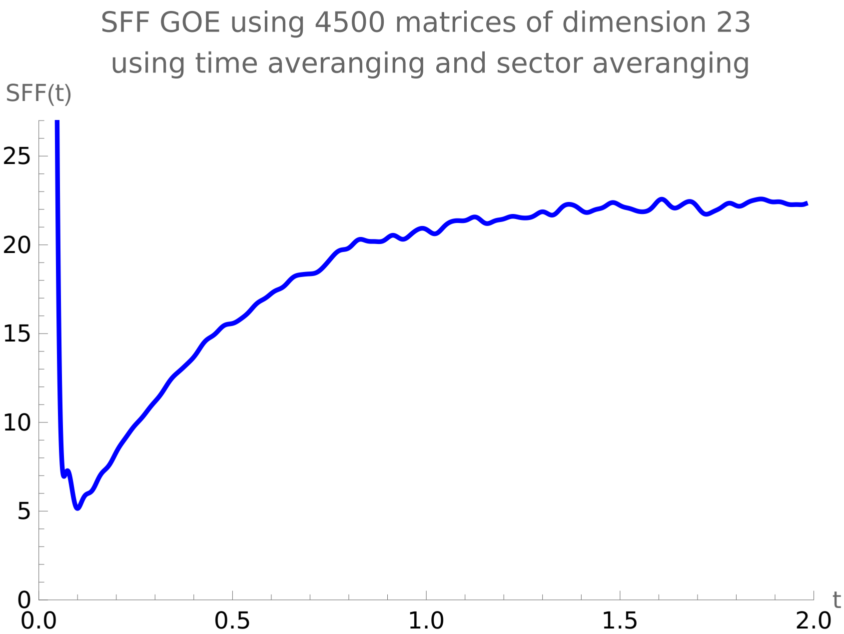

Figure 7: GOE SFF behavior using time averanging and sector averanging of a list of 4500 GOE matrices of dimension 23 fig:SFFGOETS23

An observation: It is to be noted that using a high dimension GOE matrix proves to be the most direct way of approaching statistical results, an effort must be made on relying in a method that involves a list of low dimensional GOE matrices, due to the fact that a high dimensional matrix might prove a difficutly considering the diagonalization that must be carried eventually.

4 Mean Spacings Ratio

The following generalization was used:

$$r_n^k = \frac{\min(s_n^k, s_{n-1}^k)}{\max(s_n^k, s_{n-1}^k)}. \tag{20}$$

Employing only a first order corrleation yields a characteristic value of $\langle r \rangle_{GOE} = 0.5307$ and $\langle r \rangle_{Poisson} = 0.38629$. Employing several BH realizations, with different J/U values, a region of values was found to be in accordance, at least for fist order correlations, with the GOE.

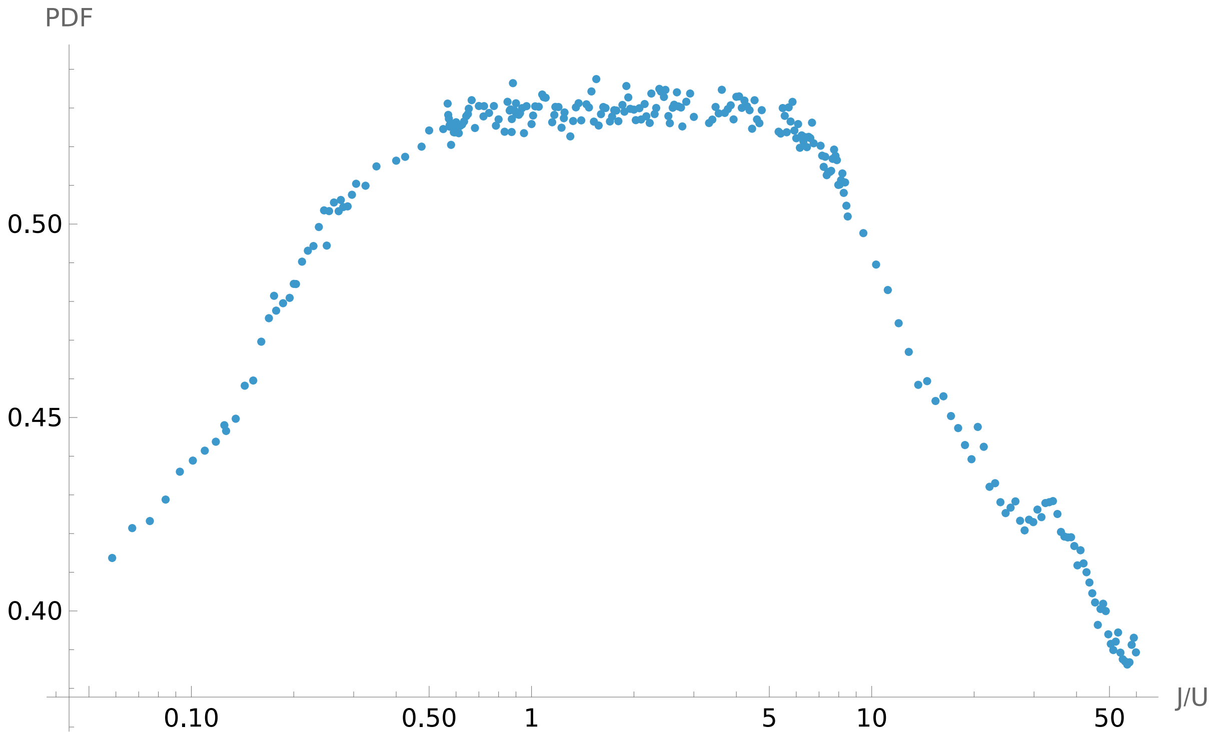

Figure 8: Mean spacing ratio of first correlation of several BHH hamiltonians with differnt J/U values. Notice that from 0.5 to 10, the mean value is similar to the GOE prediction. Using a BHH of 9 sites and 9 bosons.

fig:Map

Responder a la siguiente pregunta: será que la gráfica para el promedio de cocientes de orden $k > 1$ da cualitativamente lo mismo a la Fig. 8? Obviamente, $k > 1$ pero considerablemente más pequeño que la dimensión del subespacio

Considering the mean spacing ratio of k order proves to be a useful quantity to observe short range correlation, long range correlations are difficult to check, and numerical calculation have been performed in order to show that, as k grows, the mean spacing ratio value of the GOE approaches the Poisson mean spacing ratio of k order, therefore, it motivates to think that, above certain k, there is no way of distinguishing between GOE or Poisson statistics. The following figure shows the idea, numerical calculation where performed for the GOE and for Poisson.

Even though the mean spacing ratio of first order is unfolding independent, it does not guarantee that higher order ratios are not sensible to unfolding.

Tarea para JA: explicación con ecuaciones de porqué este es el caso

Therefore, numerical calculation where performed to conclude that **the mean spacing ratio of k order for $k > 1$ is sensitive to unfolding and therefore is a necessary procedure when considering higher order correlations.**

The following figures show the behavior of $\langle r_n^k \rangle$ for all k values ranging from k=1 to k=Total amount of eigenvalues of several BH realizations using differnt J/U. the Poisson and GOE characteristic curves are given as reference.

Considering only J/U values located in the chaos region:

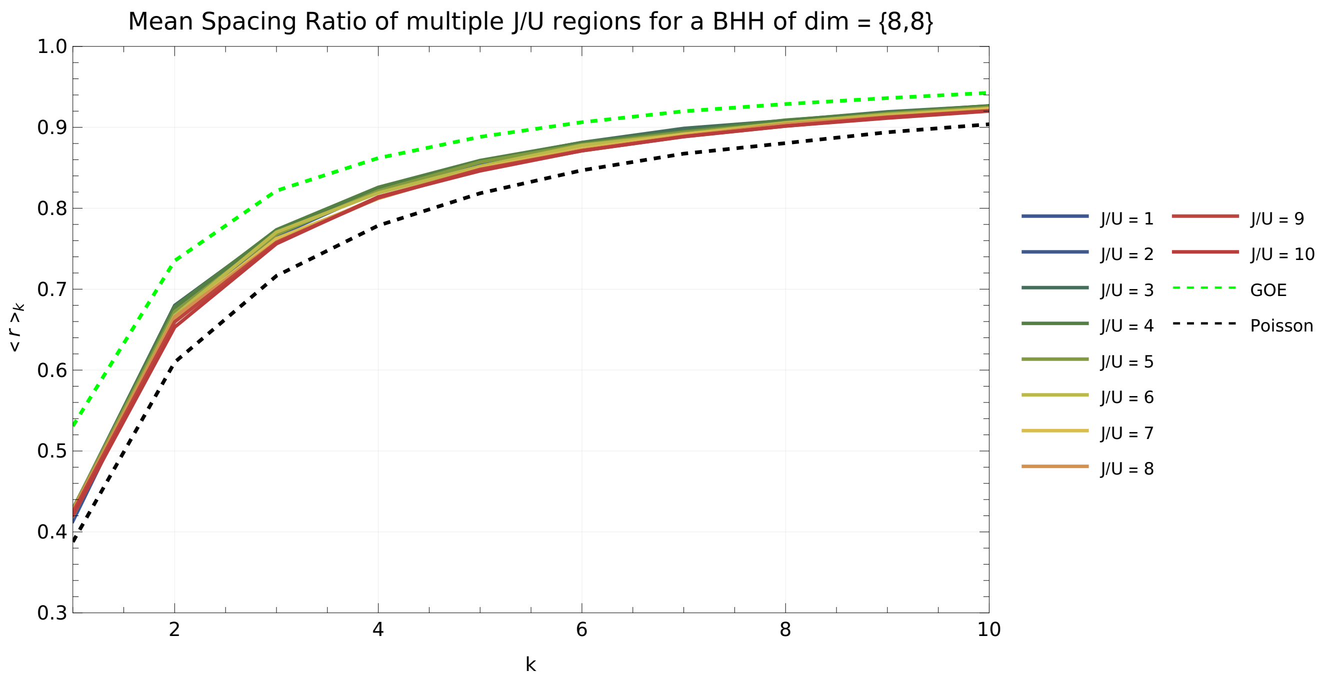

Figure 9: Mean spacing ratio for low k of several BHH hamiltonians of dim 8,8 with different J/U values located in the chaos regime.
fig:BHMeanKChaosRegionLowK

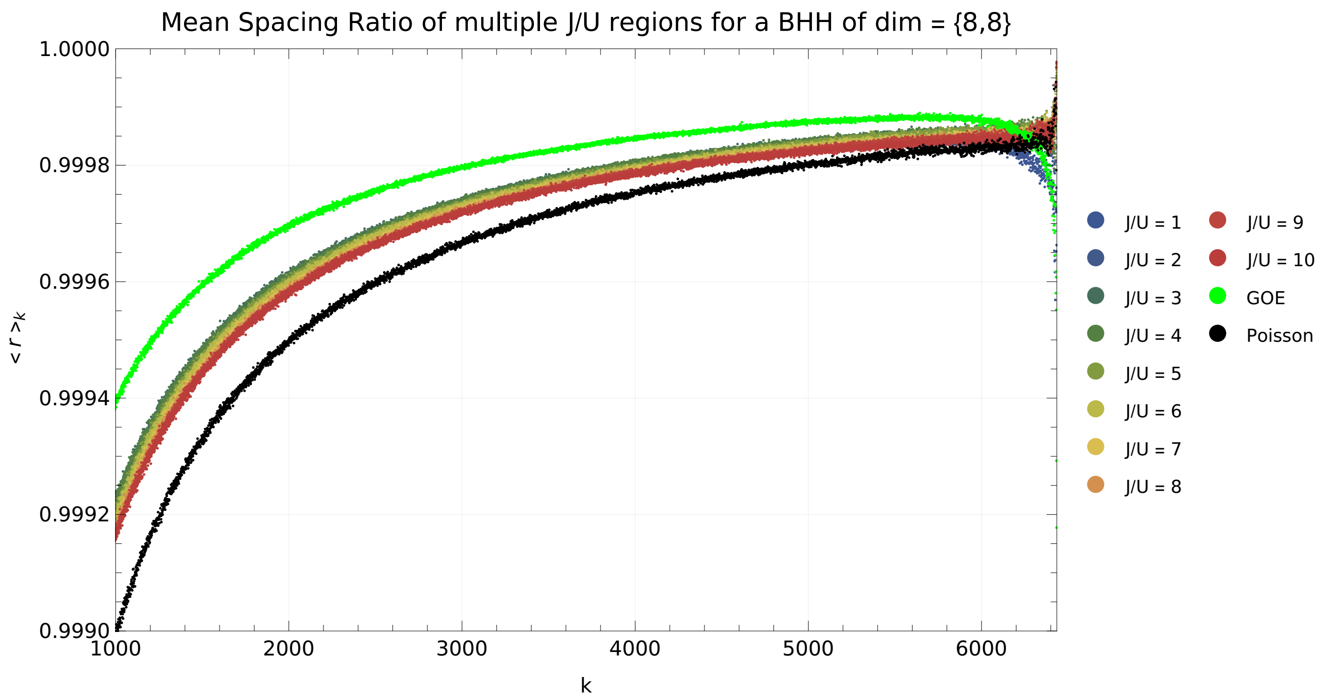

Figure 10: Mean spacing ratio as k grows of several BHH hamiltonians of dim 8,8 with different J/U values located in the chaos regime.
fig:BHMeanKChaosRegionMiddleK

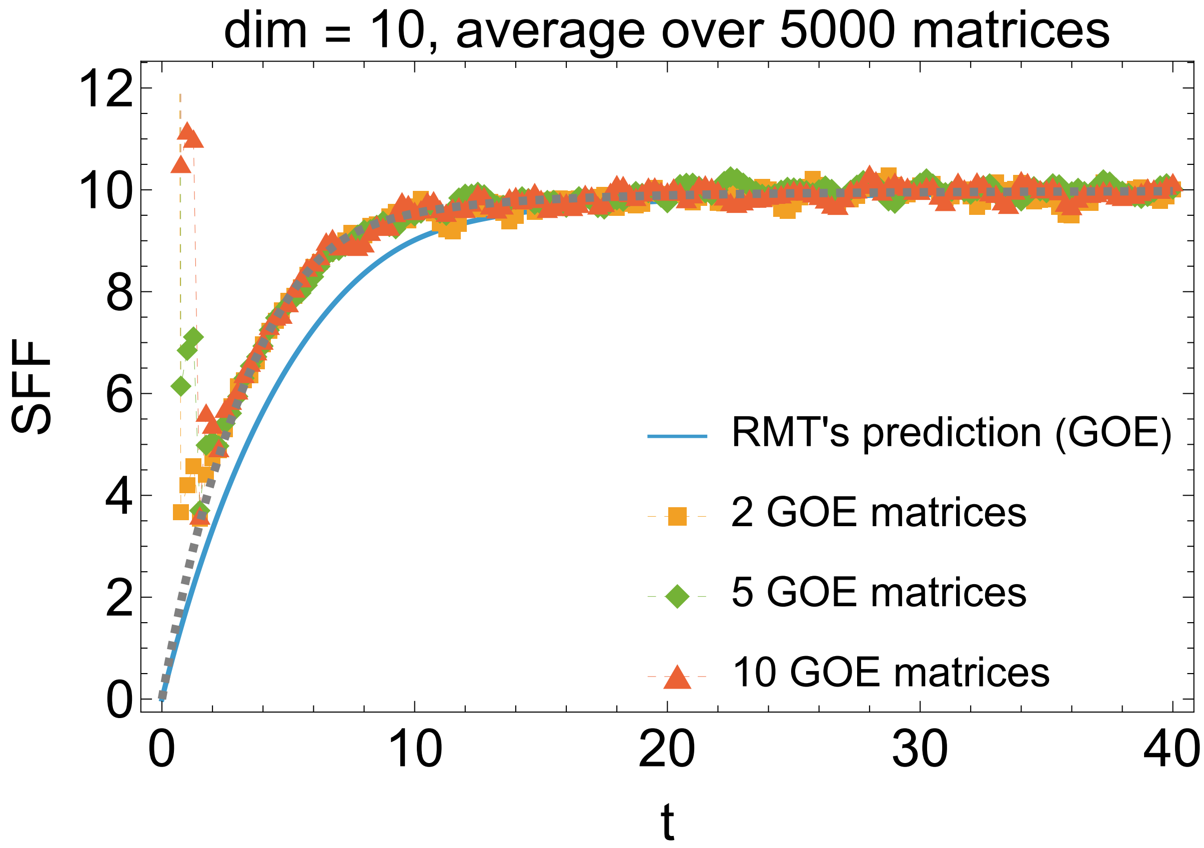

Figure 11: 5,000 samples of k matrices of dimension 10 drawn from the GOE. JA: Será que esto es genérico?

Considering low J/U values located in the transition phase between chaos and integrability:

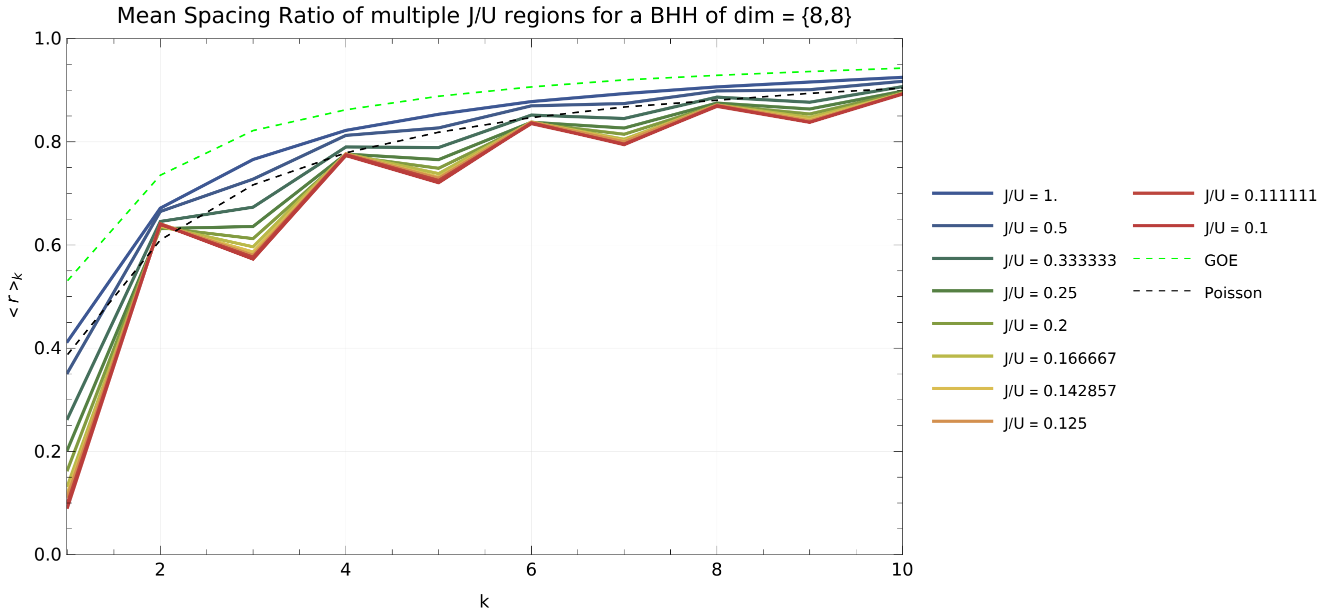

Figure 12: Mean spacing ratio for low k of several BHH hamiltonians of dim 8,8 with different J/U values located in transition phase. JA: Será que si vemos el SFF parcial de las correlaciones de segundo orden para $J/U = 0.1$ veríamos una curva que se ajusta más a la aproximación de WD o de Poisson propuesta en la Eq. (17)? fig:BHMeanKNearChaosAndPoissonRegionLowK

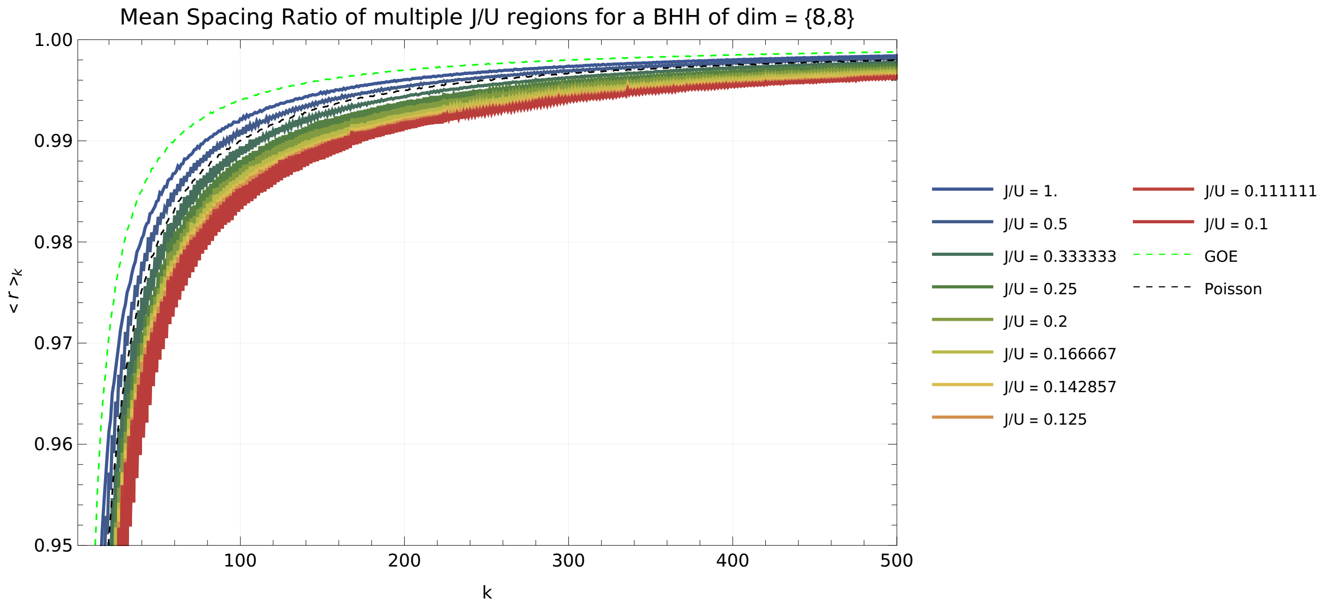

Figure 13: Mean spacing ratio for middle values of k of several BHH hamiltonians of dim 8,8 with different J/U values located in transition phase. fig:BHMeanKNearChaosAndPoissonRegionMiddleK

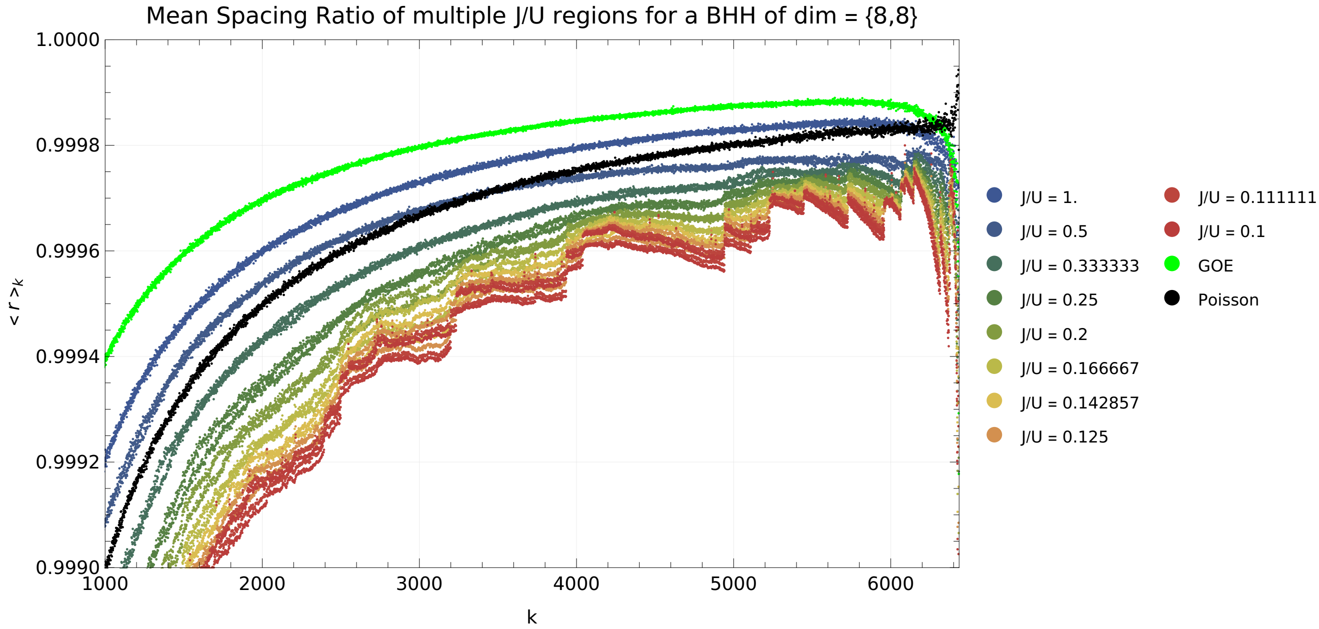

Figure 14: Mean spacing ratio for high values of k of several BHH hamiltonians of dim 8,8 with different J/U values located in transition phase. fig:BHMeanKNearChaosAndPoissonRegionHighK

Notice how the resulting curves for configurations J/U lying in the transition region deviate from the Poisson characteristic curve as k grows. Implying that higher order correlations are not adequately described either by Poisson and GOE.

5 Bose Hubbard and SFF

Two Bose Hubbard configurations where tested, one with periodic boundary conditions and the other with open boundary conditions. For both configurations, multiple J/U realizations where made.

For the moment, a stricking difference is noted in the scaling among both configurations.

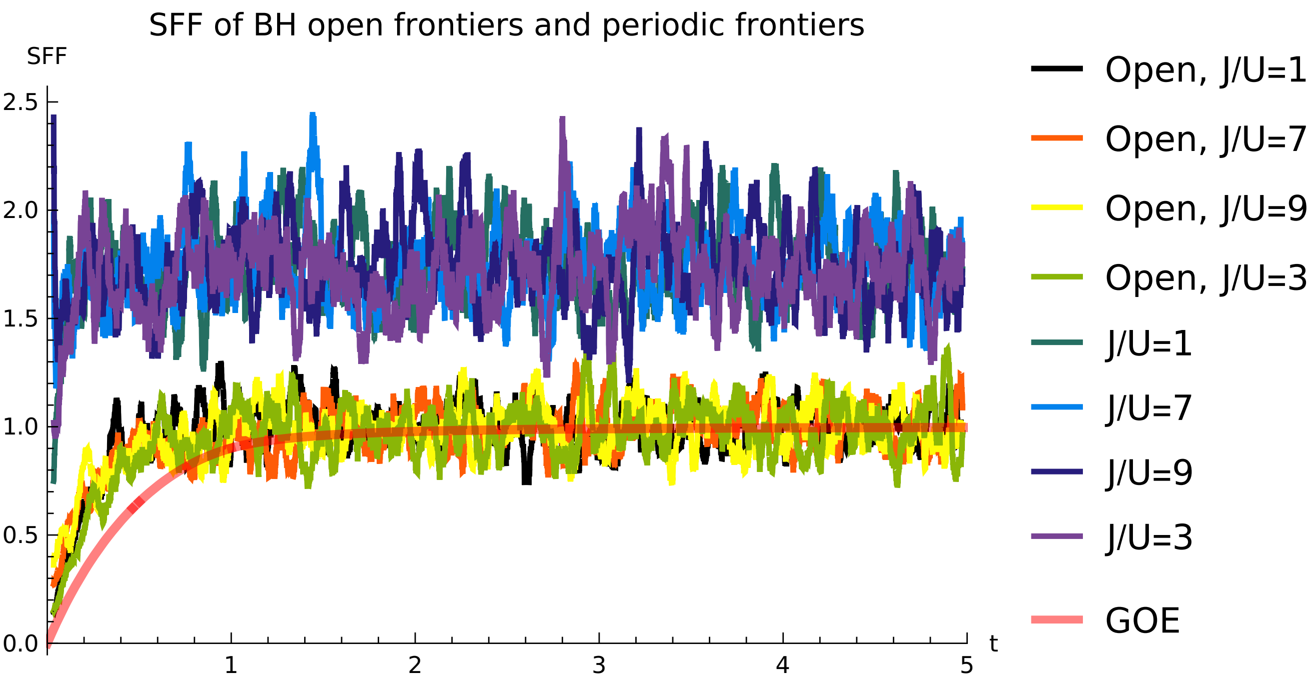

Figure 15: SFF of both configurations, periodic and open boundary conditions, for multiple J/U realizations. fig:SFFPeriodicOpen

It is to be noted that, considering periodic boundary conditions, the J/U values that are regarded as chaos in the open frontier conditions, are not in general in accordance with what is seen with the SFF results for periodic conditions. In particular, the following figure shows that some J/U values are linked with integrability behavior.

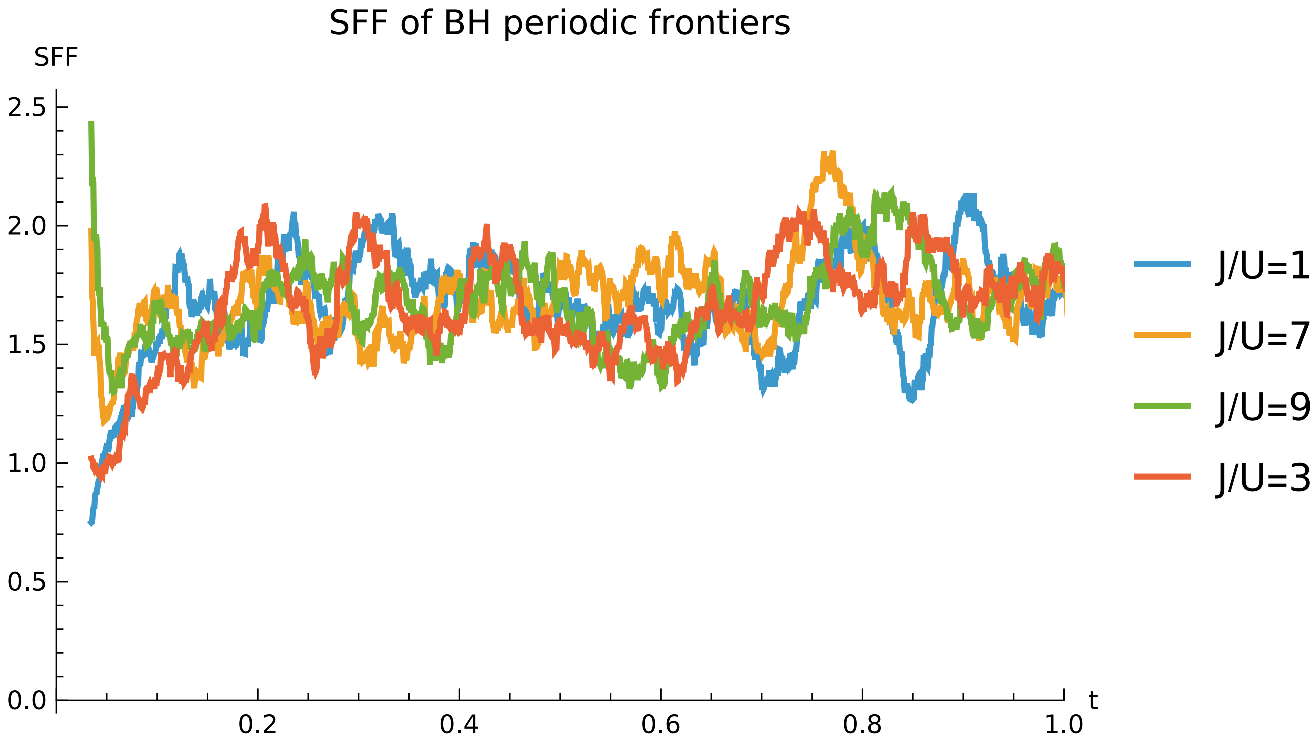

Figure 16: SFF of periodic boundary conditions, for multiple J/U realizations. fig:SFFPeriodic

A Bose Hubbard Model for bosons:

$$\hat{H} = -\frac{J}{2}\sum_{\langle i,j\rangle}\left(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_j^\dagger \hat{a}_i\right) + \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i - 1). \tag{21}$$

$\hat{a}_i^\dagger, \hat{a}_j$ are ladder operators and $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$, J and U are related to the kinetics and interactions among the bosons, respectively.

B Average of $\sum_k e^{-is_k t}$ for Wigner-Surmise Spacings

We consider the nearest-neighbor spacings s_k distributed according to the **Wigner surmise**:

$$p(s) = \frac{\pi s}{2} e^{-\pi s^2/4}, \quad s \geq 0.$$

We wish to compute:

$$\left\langle \sum_k e^{-is_k t} \right\rangle.$$

Assumption

Assuming the spacings are independent and identically distributed (i.i.d.) with the Wigner surmise distribution:

$$\left\langle \sum_k e^{-is_k t} \right\rangle \approx (N-1)\left\langle e^{-ist} \right\rangle.$$

Characteristic Function

The characteristic function is:

$$\phi(t) = \int_0^\infty e^{-ist}\, p(s)\, \mathrm{d}s = \int_0^\infty e^{-ist} \cdot \frac{\pi s}{2} e^{-\pi s^2/4} \mathrm{d}s.$$

This integral evaluates to:

$$\phi(t) = 1 - te^{-t^2/\pi} \left(\operatorname{erfi}\left(\frac{t}{\sqrt{\pi}}\right) + i \right),$$

where $\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{u^2} \mathrm{d}u$ is the imaginary error function.

Final Result

$$\boxed{\left\langle \sum_k e^{-is_k t} \right\rangle \approx (N-1) \left[1 - te^{-t^2/\pi} \left(\operatorname{erfi}\left(\frac{t}{\sqrt{\pi}}\right) + i \right) \right]}$$

Remarks

- This result assumes i.i.d. spacings (a common approximation using the Wigner surmise)
- In full RMT (GOE), spacings are not independent, but this captures key features
- For $t = 0$: $\phi(0) = 1$ as expected
- For small t: $\phi(t) \approx 1 - it - \frac{2}{\pi} t^2 + \cdots$

C df

For a quantum system with Poissonian level statistics, the nearest-neighbor spacings s_k are independent and identically distributed with the exponential distribution:

$$p(s) = e^{-s}, \quad s \geq 0,$$

where the mean spacing is normalized to unity.

We wish to compute the average:

$$\left\langle \sum_k e^{-is_k t} \right\rangle.$$

Assuming the spacings are independent, this becomes:

$$\left\langle \sum_k e^{-is_k t} \right\rangle = (N-1) \left\langle e^{-ist} \right\rangle,$$

where the characteristic function is:

$$\phi(t) = \left\langle e^{-ist} \right\rangle = \int_0^\infty e^{-ist} e^{-s}\, \mathrm{d}s = \int_0^\infty e^{-s(1+it)}\, \mathrm{d}s\,.$$

Evaluating this integral gives:

$$\phi(t) = \frac{1}{1+it}.$$

Therefore, the final result is:

$$\boxed{\frac{N-1}{1+it}}$$

D A possibly useful integral

$$\left\langle e^{-ist} \right\rangle = \frac{1}{2} e^{-\frac{1}{2}t(\sigma^2 t + 2i)} \left[1 + \operatorname{erf}\left(\frac{1 - i\sigma^2 t}{\sqrt{2}\sigma} \right) \right] \tag{22}$$

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