Assignment 1.

1. a.

P	Q	P V (~P ∧ Q)	P V Q
T	T	T $\boxed{T}$ F T F T	T $\boxed{T}$ T
T	F	T $\boxed{T}$ F T F	T $\boxed{T}$ F
F	T	F $\boxed{T}$ T F T T	F $\boxed{T}$ T
F	F	F $\boxed{F}$ F F	F $\boxed{F}$ F

b. ~P V ~Q ≡ ~(P ∧ Q)

P	Q	~P V ~Q	~(P ∧ Q)
T	T	F T $\boxed{F}$ F T	$\boxed{F}$ T T T
T	F	T $\boxed{T}$ T F	$\boxed{T}$ T F F
F	T	T F $\boxed{T}$	$\boxed{T}$ F F T
F	F	T F $\boxed{T}$	$\boxed{T}$ F F F

c.

P	Q	~P V P
T		F T $\boxed{T}$ T
F		T F $\boxed{T}$

→ always
→ true.

d.

P	Q	R	P V (Q ∧ R)	(P V Q) ∧ (P V R)
T	T	T	T $\boxed{T}$ T T T	T $\boxed{T}$ T
T	T	F	T $\boxed{T}$ T F F	T $\boxed{T}$ T
T	F	T	T $\boxed{T}$	T $\boxed{T}$ T
T	F	F	T $\boxed{T}$	T $\boxed{T}$ T
F	T	T	F $\boxed{T}$ T T T	F F T $\boxed{T}$ F F T
F	T	F	F $\boxed{F}$ T F F	F T T $\boxed{F}$ F F F
F	F	T	F $\boxed{F}$ F F T	F F F $\boxed{F}$ F T T
F	F	F	F $\boxed{F}$ F F F	F F F $\boxed{F}$

2.

(a)

$|A| = 4$, $|B| = 4$, $A \cup B = \{1, 2, 4, 6, 9, 10\}$ $A \cap B = \{2, 10\}$

$A \setminus B = \{1, 6\}$ $B \setminus A = \{4, 9\}$

(b) $|A| = \infty$ $|B| = \infty$ $A \cup B = \{x \mid x \in \mathbb{N}\}$ $A \cap B = \{x \mid x \text{ is even}\}$

$A \setminus B = \{x \mid x \text{ is odd}\}$ $B \setminus A = \{\ \}$

3.

(a) reflective, antisymmetric, transitive

(b) transitive, antisymmetric

(c) antisymmetric, transitive

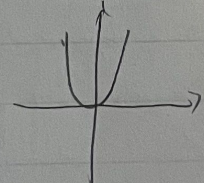
d. reflective, symmetric, transitive.

5. if $g(x)$ and $f(x)$ are bijections, then they are injective and the domain and range are the same, so their composition should also be bijective

4. (a) bijective

(b) injective ($f: \mathbb{Z} \to \mathbb{Z}$, so $2x-3$ is odd. no even number)

(c) none



a y with 2 . X and. there's no range under y=0.

6 (a) base. $n=1$ $\sum_{i=1}^{k} i = 1\cdot(1+1)/2 = 1$.

induction. $n=k$ $\sum_{i=1}^{k} i = k\cdot(k+1)/2$

$n=k+1$ $\sum_{i=1}^{k} k + k+1 = \frac{(k+1)(k+2)}{2}$

$\frac{k\cdot(k+1)}{2} + k+1 = \frac{(k+1)(k+2)}{2}$

$\frac{k\cdot(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

$(k+2)(k+1) = (k+1)(k+2)$ correct.

(b) base: $n=1$. $1^2 = 1\cdot 2\cdot 3/6 = 1$

inductive: $n=k$: $\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$

$n=k+1$ $\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

$\frac{k(k+1)(2k+1)+(k+1)^2}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$

$\cancel{(k+1)}[(2k+1)k + 6(k+1)] = \cancel{(k+1)}(k+2)(2k+3)$

$(2k+1)k + 6k+6 = (k+2)(2k+3)$

$2k^2+k+6k+6 = 2k^2+4k+3k+6$

$\underline{2k^2+7k+6} = \underline{2k^2+7k+6}$

same.

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6. base case: $n=1$ $\quad 1^3 = 1^2 \cdot 2^2/4 = 1$

induction $n=k$ $\quad \sum_{i=1}^{k} i^3 = k^2(k+1)^2/4$

$n=k+1$ $\quad \sum_{i=1}^{k} i^3 + (k+1)^3 = (k+1)^2(k+2)^2/4$

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{4(k+1)^3 + k^2(k+1)^2}{4}$$

$$= \frac{(k+1)^2 \cdot 4(k+1) + k^2(k+1)^2}{4}$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2 \cdot (k+2)^2}{4} \Rightarrow \text{Right} = \frac{(k+1)^2(k+2)^2}{4}$$

7. Adjacency matrix

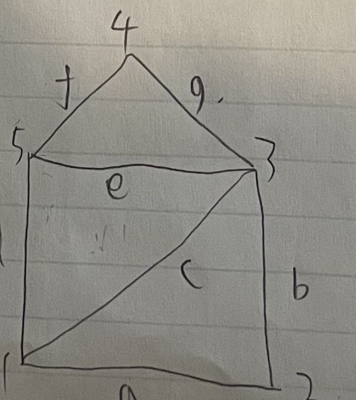
	1	2	3	4	5
1	0	1	1	0	1
2	1	0	1	0	0
3	1	1	0	1	1
4	0	0	1	0	1
5	1	0	1	1	0

adjacent list

1. (2, 3, 5)
2. (1, 3)
3. (1, 2, 5)
4. (3, 5)
5. (1, 3, 4)

edge list

(1,2), (1,3), (1,5)
(2,3), (3,4), (3,5)
(4,5)



Incidence Matrix

	a	b	c	d	e	f	g
1	1	0	1	1	0	0	0
2	1	1	0	0	0	0	0
3	0	1	1	0	1	0	1
4	0	0	0	0	0	1	1
5	0	0	0	1	1	1	0

8. There are $E = n(n-1)/2$ edges.

prove: base case $n=1$ $\qquad E(1) = 0$

Inductive step: $n=k$ $\qquad E(k) = k(k-1)/2$

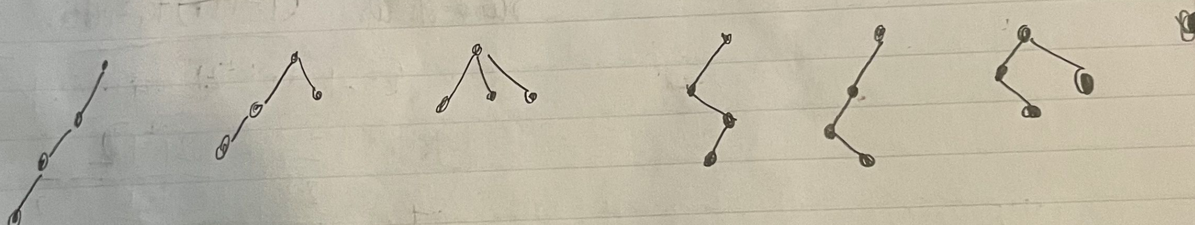
$n=k+1$ $\qquad E(k+1) = (k+1)(k+1-1)/2 = \dfrac{(k+1)k}{2}$

Left $= \dfrac{k(k-1)}{2} + k$

$= \dfrac{k(k-1)+2k}{2}$

$= \dfrac{(k+2-1)k}{2}$

$= \dfrac{(k+1)k}{2}$ = Right

9. 

10. base case $|V|=1$ $\qquad |E| = 1-1=0$

induction step $|V|=k$

$|E| = k-1$

$|V| = k+1$

$|E| = k+1-1 = k$

Left: $k-1 + 1 = k$ = Right

when $|V|=k$ $\qquad$ one more edges added with one more vertices o Algorithms (Spring 2023)

11. 0-9 10 digits
$10 \times 10 \times 10 = 1000$

12.
1 1
2 2 3
3 4 5 6
4 7 8 9 10
⋮
n n numbers.

number in i th, let x be the last number in i-1 th level
$\sum_{k=1}^{i} x+k$

the number in the last row is $\frac{i(i-1)}{2}$, so the first number in ith is $\frac{i(i-1)}{2}+1 = \frac{i^2-i+2}{2}$

Sum in ith: $\frac{(i^2-i+2)+(i^2-i+2i)}{2} \cdot \frac{i}{2}$

$= \frac{2i^2+2}{2} \cdot \frac{i}{2}$

$= \frac{i^3+i}{2}$

13.
(a) 4

(b) 1-10 have straight flush
$4 \times 10 = 40$
$40-4=36$ (not include Royal)

(c) $4 \times C_{13}^{5} = 4 \times \frac{13!}{5!(13-5)!} = 5148$

(d) $13 \times C_{12}^{3} \times C_{4}^{2} = 698240$

$f(x) = 2x$

14. base, $x=1$ $2x=2$ even.

(a) induction $x=k$ $f(x)=2k$ even

$x=k+1$? $f(x) = 2(k+1) = 2k+2$ by definition. even number plus even number equals to even number.

left: $2k+2\cdot 1 = 2k+2$

(b) if $2k$ is not even, then it will have remainders when divide by 2. but $2x/2 = x \in N$.

15. (1) $x \geq 0$, $y \geq 0$

$|x+y| = |x|+|y|$

(2) $x \leq 0$ $y \geq 0$ $|x| \leq |y|$

$|x+y| < |x|+|y|$

(3) $x \leq 0$. $y \geq 0$ $|x| \geq |y|$

$|x+y| < |x|+|y|$

(4) $|x| < 0$ $y > 0$ $|x| < |y|$

$|x+y| < |x|+|y|$

16. Loop invariants after i th iteration, min is the minimum number in $a[0 \cdots i]$

(a) base case. $i=1$, min $= a[1]$

induction step $i=k$, min $=$

for $i=k+1$, if $a[k+1] <$ min, min will be $a[k+1]$

so min is minimum in $a[1 \cdots k+1]$ after $k+1$ loops

Correctness. From the invariance, after the loop end, min will be the smallest number in a

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val = 4 i=3 1 7 2

4 6 2 3 6 4 8 7 val : 8 2 3

8 4 6 7

Completeness. the program will end after i equals to the length of the array as i increases every time when a ~~loop~~ ends iteration

(b) invariant: after i th iteration ~~a~~ i-1 th iteration. a[1 ... i] is sorted from lagest to smallest

base case: i=2, a[1 ... 2]: a[1 .. 2] is sorted

induction. suppose i=K. after ith iteration i[1 ... K] is sorted

when i=k+1. number of a[1 .. k] is already sorted - numbers. larger than a[k+1] will be shifted to a[j ... k], numbers larger than a[k+1] will be place to a[1 ... j-1] and a[k+1] will be placed to a[j]. So a[1 ... k] is sorted

Correctness after i = len(a), from the ~~code~~ invariant, a[1 .. len(a)] is sorted.

Completeness: the out loop will ends after i ~~len(a)~~ i = len(a)

17.(a)

$C_n = C_{n-1} + 4$

$= C_{n-2} + 4 + 4$

$= C_{n-3} + 4 + 4 + 4$

$= C_0 + 4n$

$= 1 + 4n$

(b)

$d_n = 3 \cdot d_{n-1}$

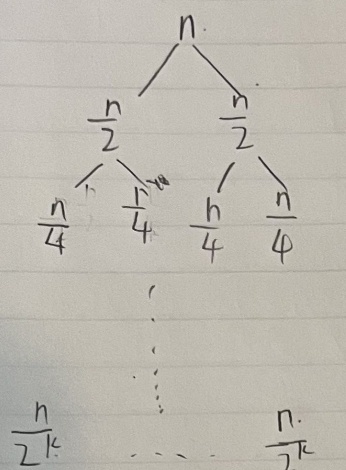
$= 3 \cdot d_{n-2} \cdot 3$

$= 3^3 d_{n-3}$

$= 3^n d_0$

$= 4 \cdot 3^n$

$T(\frac{n}{2}) = 1$



n

$n +$

$n +$

$\vdots +$

n

$$\frac{n}{2^k} = 1$$

$$k = \log_2 n$$

$$\log_2 n \cdot n + n = n\log n + n$$

$$T_n \leq n\log n + n$$

(d).

$$f(n+1) - f(n) = \sum_1^n (i \cdot f(i)) - \sum_1^{n-1} (i \cdot f(i))$$

$$f(n+1) - f(n) = n \cdot f(n)$$

$$f(n+1) = (n+1)\, f(n)$$

$$f(n) = n\, f(n-1)$$

$$= n \cdot (n-1) \cdot f(n-2)$$

$$= n\,(n-1)(n-2)\, f(n-3)$$

$$= \cdots n \cdot (n-1)(n-2) \cdots f(1)$$

$$= n!$$