Principles of Machine Learning: Exercise 1

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Task 1.2.0

- Given the following two rule sets, implement code that solves the least squares problem between X^T and y.
- Afterwards the code should calculate $\hat{y}$ using the calculated weights.

Task 1.2.0

Table: Rule 110

X^T			y
$+1$	$+1$	$+1$	$+1$
$+1$	$+1$	-1	-1
$+1$	-1	$+1$	-1
$+1$	-1	-1	-1
-1	$+1$	$+1$	$+1$
-1	$+1$	-1	-1
-1	-1	$+1$	-1
-1	-1	-1	$+1$

Table: Rule 126

X^T			y
$+1$	$+1$	$+1$	$+1$
$+1$	$+1$	-1	-1
$+1$	-1	$+1$	-1
$+1$	-1	-1	-1
-1	$+1$	$+1$	-1
-1	$+1$	-1	-1
-1	-1	$+1$	-1
-1	-1	-1	$+1$

Task 1.2.1

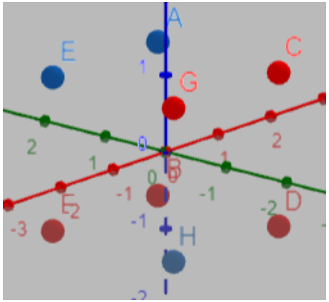
```python
# Solve the Least Squares Problem
w110 = la.lstsq(matXT, y110, rcond=None)

# Calculate yhat
yhat110 = matXT @ w110[0]
```

$$\hat{y}_{110} = \begin{pmatrix} +0.25 \\ -0.25 \\ -0.25 \\ -0.75 \\ +0.75 \\ +0.25 \\ +0.25 \\ -0.25 \end{pmatrix} \qquad \hat{y}_{126} = \begin{pmatrix} +1.57e-16 \\ -1.23e-32 \\ +1.57e-16 \\ -1.23e-32 \\ +1.23e-32 \\ -1.57e-16 \\ +1.23e-32 \\ -1.57e-16 \end{pmatrix}$$

Task 1.2.1

- The first calculation is numerically stable
- Even though the difference between the two rulesets is only one number, the second calculation becomes numerically unstable
- Our data points are the vertices of a cube of side length 2, which are not linearly separable. Therefore no linear model can predict all points correctly.
- We will avoid this issue in the next task by using the kernel trick (embedding our data in a higher dimensional space, in which the points are linearly separable).



Task 1.2.2

Implement a function phi which takes a vector x with n elements and realizes the following transformation: $\varphi : \{+1, -1\}^n \rightarrow \{+1, -1\}^{2^n}$

```python
import itertools as it

def phi(x):
    n = len(x)
    # Generate all possible sets of x
    sets = it.chain.from_iterable(it.combinations(x, r) for r in range(n+1))
    # Multiply each set together to a single value and return these as an array
    return np.array([np.prod(s, dtype=int) for s in sets], dtype=int)
```

Task 1.2.3

Given the previously introduced function phi we compute the matrix $\phi^T = \begin{bmatrix} — & \varphi_0^T & — \\ — & \varphi_1^T & — \\ & \vdots & \\ — & \varphi_7^T & — \end{bmatrix}$

Solve the least squares problem between ϕ^T and y and calculate $\hat{y}$ using the calculated weights.

```
Phi = np.apply_along_axis(phi,1,x.T)

w110 = la.lstsq(Phi, y110 ,rcond=None)[0]
yhat110 = x.T @ w110
```

Task 1.2.3

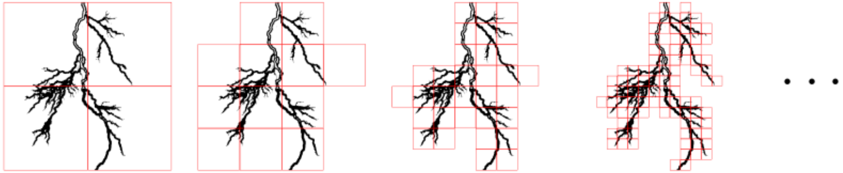
$$\hat{y}_{110} = \begin{pmatrix} +1 \\ -1 \\ -1 \\ -1 \\ +1 \\ -1 \\ -1 \\ +1 \end{pmatrix} \qquad \hat{y}_{126} = \begin{pmatrix} +1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ +1 \end{pmatrix}$$

We now observe that $\hat{y}_{110} = y_{110}$ and $\hat{y}_{126} = y_{126}$. We trained our parameters w with all possible inputs/data to archieve an output as close as possible to our rule so that we get a good output according to our rule for every input. We had to pay the price of computing a high dimensional feature map Phi. Because of this our training aka. The search for the least squares solution is highly complex (in contrast to the computing of the rule once the parameters have been found).

Fractal dimensions

1. Binarize the image
2. Partition the image into 2^l boxes for $l = 1, \ldots, L-2$,
3. Calculate the fractal dimension using linear regression

$$D \cdot \log\left(\frac{1}{s_l}\right) + b = \log(n_l)$$



$s_l = \frac{1}{2}, n_l = 4$ $s_l = \frac{1}{4}, n_l = 12$ $s_l = \frac{1}{8}, n_l = 32$ $s_l = \frac{1}{16}, n_l = 94$

Implementation: Box counting

```python
def box_counting(img):
    w, h = img.shape
    n_ls = []
    # l runs from 1 to 9-2 =7
    for l in range(1, 8):
        n_l, s_l = 0, 1/2**l                      # setup counter and scale
        box_sizeW, box_sizeH = s_l * w, s_l * h   # get box sizes

        for box_w in range(0,(2**l)):             # each l has 2**l boxes
            for box_h in range(0,(2**l)):
                #check if any value in the box is equal 1.
                #If so increment n_l by one
                if (np.any(img[int(box_w * box_sizeW): int((box_w+1) * box_sizeW),\
                               int(box_h * box_sizeH): int((box_h+1) * box_sizeH)]\
                           ==1)):
                    n_l+=1
        n_ls.append(n_l)
    return n_ls
```

Implementation: Calculating the fractal dimension

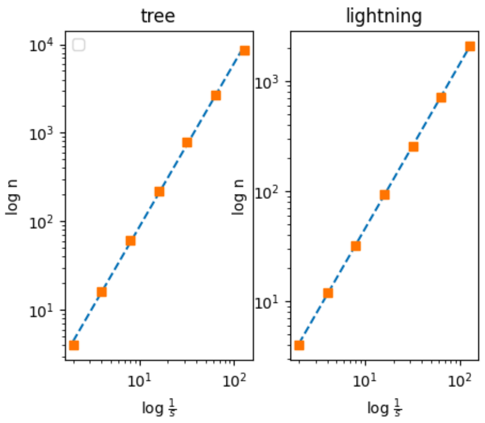
```
def slope(n_l):
    inverted_s_l=[2**l for l in range(1,8)]
    matX=np.vander(np.log(inverted_s_l),2,increasing=True)
    b,D=la.lstsq(matX, np.log(n_l),rcond=None)[0]
    return b,D
```

where np.vander generates the Vandermonde matrix

$$V = \begin{bmatrix} 1 & \log(2^1)^1 \\ 1 & \log(2^2)^1 \\ \vdots & \vdots \\ 1 & \log(2^{L-2})^1 \end{bmatrix}$$

Results

1. Fractal dimension
    - Tree: ≈ 1.846
    - Lighting: ≈ 1.493
2. The fractal dimension of the image “lighting.png” is higher



Learnings

1. Fractal dimensions
2. Fractal dimensions as a least squares problem using box counting
3. Application of least squares to a wider class of problems