2. Intro to Quasicategories

Let us see how quasi-categories can be used to do category theory.

Definition 1. A quasi-category is a simplicial set S such that the following diagram has a lift

$$\begin{tikzcd} \Lambda[n]_i \arrow[r] \arrow[d] & S \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

for all $n \geq 2$ and $0 < i < n$.

Exercise 10. Let S be a *quasi-category*. Using the simplicial set structure in S define:

(1) The set of objects of S
(2) The set of morphisms of S
(3) The set of compositions of S

Moreover, for every morphism f in S define the domain and codomain as objects in S via the simplicial set structure. Finally, for every object define the identity morphism denoted id_x for an object x.

Exercise 11. Let S be a quasicategory. Let f, g be two morphisms in S such that the codomain of f coincides with the domain of g. Prove the composition of f and g exists. We denote any such choice of composition $g \circ f$.

Exercise 12. Let S be a quasicategory and $f, g : x \to y$ two morphisms.

- We say f and g are *left homotopic* if there exists $\sigma \in S_2$ such that
  - $d_0\sigma = f$
  - $d_1\sigma = g$
  - $d_2\sigma = \mathrm{id}x$
- We say f and g are *right homotopic* if there exists $\sigma \in S_2$ such that
  - $d_0\sigma = \mathrm{id}y$
  - $d_1\sigma = g$
  - $d_2\sigma = f$

Prove f and g are left homotopic if and only if they are right homotopic. Hence we can simply say f and g are homotopic and denote it $f \sim g$.

Recap from last time. A morphism $\alpha\colon \Lambda[3]_1 \to S$, where S is a quasi-category, is determined by the values $\sigma_3 = \alpha(012)$, $\sigma_2 = \alpha(013)$ and $\sigma_0 = \alpha(123)$ which satisfy the relations

- $d_0\sigma_3 = d_2\sigma_0$
- $d_2\sigma_3 = d_2\sigma_2$
- $d_0\sigma_2 = d_1\sigma_0$

Similarly, a morphism $\alpha\colon \Lambda[3]_2 \to S$ is determined by the values $\sigma_3 = \alpha(012)$, $\sigma_1 = \alpha(023)$ and $\sigma_0 = \alpha(123)$ which satisfy the relations

- $d_0\sigma_3 = d_2\sigma_0$
- $d_1\sigma_3 = d_2\sigma_1$
- $d_0\sigma_1 = d_0\sigma_0$

Given such a morphism, for example $\alpha\colon \Lambda[3]_1 \to S$, there exists a lift $\beta\colon \Delta[3] \to S$, which in particular gives us a $\sigma_1 = \beta(023) \in S_2$ with $d_0\sigma_1 = d_0\sigma_0$, $d_1\sigma_1 = d_1\sigma_2$ and $d_2\sigma_1 = d_1\sigma_3$.

Solution for Ex. 12. Assume f and g are left homotopic with the given $\sigma \in S_2$. Consider the morphism $\alpha\colon \Lambda[3]_1 \to S$ with $\alpha(013) = \sigma_2 = \sigma$, $\alpha(012) = \sigma_3 = s_0 f$ and $\alpha(123) = \sigma_0 = s_1 f$. We check the relations

- $d_0\sigma_3 = d_0 s_0 f = f = d_2 s_1 f = d_2\sigma_0$
- $d_2\sigma_3 = d_2 s_0 f = s_0 d_1 f = s_0 x = \mathrm{id}_x = d_2\sigma_2$
- $d_0\sigma_2 = f = d_1 s_1 f = d_1\sigma_0$

So α is indeed a morphism of simplicial sets and there is a lift $\beta\colon \Delta[3] \to S$.
Then $\sigma_1 = \beta(023) \in S_2$ has the faces

- $d_0\sigma_1 = d_0\sigma_0 = d_0 s_1 f = \mathrm{id}_y$
- $d_1\sigma_1 = d_1\sigma_2 = d_1\sigma = g$
- $d_2\sigma_1 = d_1\sigma_3 = d_1 s_0 f = f$

which proves that f and g are right homotopic.

Exercise 13. Let S be a quasicategory and x, y two objects. Prove the homotopy relation is an equivalence relation on the set of morphism with domain x and codomain y.

(1) **Reflexive.** For every morphism f, $\sigma = s_0 f \in S_2$ proves that f is (left) homotopic to f.
(2) **Symmetrical.** Assume f and g are (left) homotopic with witness σ_3. Consider $\sigma_2 = s_0 f$ and $\sigma_0 = s_1 f$. Then
   - $d_0\sigma_3 = f = d_2 s_1 f = d_2\sigma_0$
   - $d_2\sigma_3 = \mathrm{id}_x = s_0 d_1 f = d_2 s_0 f = d_2\sigma_2$
   - $d_0\sigma_2 = d_0 s_0 f = f = d_1 s_1 f = d_1\sigma_0$

   So α is indeed a morphism of simplicial sets and there is a lift $\beta\colon \Delta[3] \to S$.
   Then $\sigma_1 = \beta(023) \in S_2$ has the faces
   - $d_0\sigma_1 = d_0\sigma_0 = d_0 s_1 f = \mathrm{id}_y$
   - $d_1\sigma_1 = d_1\sigma_2 = d_1 s_0 f = f$
   - $d_2\sigma_1 = d_1\sigma_3 = g$

   which proves that g and f are (right) homotopic.
(3) **Transitive.** Assume $f \sim g$ and $g \sim h$, with witnesses σ_2 and σ_0 such that $d_2\sigma_0 = d_2\sigma_2 = id_x$, $d_1\sigma_2 = f$, $d_0\sigma_2 = d_1\sigma_0 = g$ and $d_0\sigma_0 = h$. Also, consider $\sigma_3 = s_0 s_0 x$. They satisfy the relations
   - $d_0\sigma_3 = s_0 x = d_2\sigma_0$
   - $d_2\sigma_3 = s_0 x = d_2\sigma_2$
   - $d_0\sigma_2 = g = d_1\sigma_0$

   so the lift yields a σ_1 with
   - $d_0\sigma_1 = d_0\sigma_0 = h$
   - $d_1\sigma_1 = d_1\sigma_2 = f$
   - $d_2\sigma_1 = d_1\sigma_3 = \mathrm{id}_x$

   so h and f are (left) homotopic.

Exercise 14. Let S be a quasicategory and x, y, z three objects. Let $f : x \to y$ and $g : y \to z$ and let $h, h' : x \to z$ be two possible compositions. Prove $h \sim h'$.

Assume $g \circ f = h$, call the composition σ_3, and $g \circ f = h'$, call the composition σ_2. Consider the morphism $\alpha \colon \Lambda[3]_1 \to S$ with $\alpha(013) = \sigma_2$, $\alpha(012) = \sigma_3$ and $\alpha(123) = \sigma_0 = s_1 g$. We check the relations

- $d_0\sigma_3 = g = d_2 s_1 g = d_2\sigma_0$
- $d_2\sigma_3 = f = d_2\sigma_2$
- $d_0\sigma_2 = g = d_1 s_1 g = d_1\sigma_0$

So α is indeed a morphism of simplicial sets and there is a lift $\beta \colon \Delta[3] \to S$.
Then $\sigma_1 = \beta(023) \in S_2$ has the faces

- $d_0\sigma_1 = d_0\sigma_0 = d_0 s_1 g = \mathrm{id}_z$
- $d_1\sigma_1 = d_1\sigma_2 = h'$
- $d_2\sigma_1 = d_1\sigma_3 = h$

which proves that h and h' are (right) homotopic.

Exercise 15. Let S be quasicategory and let x, y, z, w be four morphisms. Prove for three morphisms $f : x \to y$, $g : y \to z$, $h : z \to w$, there exists a homotopy $h \circ (g \circ f) \sim (h \circ g) \circ f$.

Denote by σ_3 the element in S_2 for the composition $g \circ f$, and by σ_0 the element for the composition $h \circ g$. There also exists an element in S_2 for the composition $(h \circ g) \circ f$, call it σ_2. Consider the morphism $\alpha \colon \Lambda[3]_1 \to S$ with $\alpha(013) = \sigma_2$, $\alpha(012) = \sigma_3$ and $\alpha(123) = \sigma_0$. We check the relations

- $d_0\sigma_3 = g = d_2\sigma_0$
- $d_2\sigma_3 = f = d_2\sigma_2$
- $d_0\sigma_2 = h \circ g = d_1\sigma_0$

So α is indeed a morphism of simplicial sets and there is a lift $\beta \colon \Delta[3] \to S$.
Then $\sigma_1 = \beta(023) \in S_2$ has the faces

- $d_0\sigma_1 = d_0\sigma_0 = h$
- $d_1\sigma_1 = d_1\sigma_2 = (h \circ g) \circ f$
- $d_2\sigma_1 = d_1\sigma_3 = g \circ f$

This proves that $(h \circ g) \circ f$ is a composition of h and $g \circ f$, so by Ex. 14 it is homotopic to $h \circ (g \circ f)$.

It's not homotopic, it "is" a composition.

Exercise 16. Let S be a quasicategory and let $f : x \to y$ be a morphism. Prove there are equivalences $f \sim f \circ \mathrm{id}_x \sim \mathrm{id}_y \circ f$.

The equivalences are given by $s_0 f$ and $s_1 f \in S_2$:
$d_0 s_0 f = f$, $d_1 s_0 f = f$ and $d_2 s_0 f = id_x$, so f is a composition $f \circ \mathrm{id}_x$.
$d_0 s_1 f = \mathrm{id}_y$, $d_1 s_1 f = f$ and $d_2 s_1 f = f$, so f is a composition $\mathrm{id}_y \circ f$.

Exercise 17. Let S be a quasicategory. Let $\mathrm{Ho}S$ be the graph with

$$\mathrm{Obj}_{\mathrm{Ho}S} = S_0$$

and $\mathrm{Hom}_{\mathrm{Ho}S}(x, y)$ given by the equivalence classes of morphism with domain x and codomain y with respect to the homotopy relation. Prove this gives us the data of a category, called the *homotopy category*.

to check: well-definedness of composition −¿ Ex. 14, associativity −¿ Ex. 15, identity −¿ Ex. 16

Exercise 18. Let $\mathcal{C}$ be a category. Take for granted that $N\mathcal{C}$ is quasi-category. What are the objects and morphisms in the quasi-category $N\mathcal{C}$? Moreover, compute $\mathrm{Ho}N\mathcal{C}$.

The objects in $N\mathcal{C}$ are by definition the elements of $N\mathcal{C}_0$, i.e. the objects of $\mathcal{C}$. The morphisms in $N\mathcal{C}$ are the elements of $N\mathcal{C}_1$, i.e. the morphisms of $\mathcal{C}$. The face maps send a morphism $f\colon x \to y$ to $d_1 f =$ the source of $f \ = x$, and $d_0 f =$ the target of $f \ = y$.
In Ho$N\mathcal{C}$, the morphisms between some x and y are $\mathrm{Hom}_{\mathrm{Ho}N\mathcal{C}}(x,y) = \{[f] \mid f \in N\mathcal{C}_1, d_1 f = x, d_0 f = y\} = \mathrm{Hom}_{\mathcal{C}}(x,y)$. The equivalence classes each contain only one element because in $\mathcal{C}$, we have $f \circ \mathrm{id}_x = f = \mathrm{id}_y \circ f$.

Exercise 19. Let K be a Kan complex. Prove K is a quasi-category. What are the objects and morphisms in the quasi-category K?

Exercise 20. Let T be a topological space and let $S(T)$ be the *singular Kan complex*. Describe the objects and morphisms of the category Ho$S(T)$.

The objects of Ho$S(T)$ are the points in T, and the morphisms between two points x, y are $\mathrm{Hom}_{\mathrm{Ho}S(T)}(x,y) = \text{paths } \gamma\colon [0,1] \to T \mid \gamma(0) = x,\ \gamma(1) = y/\sim$ (i.e. a morphism between x and y is a homotopy class of paths with endpoints x and y)
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Definition 2. Let S be a quasi-category. A morphism is an equivalence if it is an isomorphism in HoS.

Exercise 21. Let $f : y \to x$ be a morphism. Let Sq be the pushout of the following diagram

$$\Delta[2] \xleftarrow{<12>} \Delta[1] \xrightarrow{<01>} \Delta[2]$$

Prove f is an equivalence if and only if there exists a diagram $H : \mathrm{Sq} \to S$ of the following form

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}x"] \arrow[d] & x \arrow[d] \\ y \arrow[ur, "f"] \arrow[r, "\mathrm{id}y"'] & y \end{tikzcd}$$

By definition, f is an equivalence iff there exist morphisms $g, g'\colon x \to y$ such that $f \circ g = \mathrm{id}_x$ and $g' \circ f = \mathrm{id}_y$, that is, iff there are elements $\sigma, \sigma' \in S_2$ with $d_0\sigma = f,\ d_1\sigma = \mathrm{id}_x,\ d_1\sigma' = \mathrm{id}_y,\ d_2\sigma' = f$. This is the same as the information that there exist morphisms of simplicial sets $\alpha,\ \alpha'\colon \Delta[2] \to S$ such that $\alpha(12) = d_0\alpha(012) = f,\ \alpha(02) = d_1\alpha(012) = \mathrm{id}_x,\ \alpha'(01) = d_2\alpha(012) = f,\ \alpha'(02) = d_1\alpha(012) = \mathrm{id}_y$. $(*)$

When the Square Sq is defined as this pushout $\begin{tikzcd} \Delta[1] \arrow[r] \arrow[d] & \Delta[2] \arrow[d] \\ \Delta[2] \arrow[r] & \mathrm{Sq} \end{tikzcd}$ the vertical $\Delta[2] \to \mathrm{Sq}$ morphism is the inclusion of the upper left triangle and the horizontal one is the inclusion of the lower right triangle.
If there exists a diagram $H\colon \mathrm{Sq} \to S$ of the given form, then composition with these inclusions yields diagrams $\Delta[2] \to S$

$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d] & x \\ y \arrow[ur, "f"] & \end{tikzcd}$. and $\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[d] \\ y \arrow[r, "\mathrm{id}_y"'] & y \end{tikzcd}$ respectively. These satisfy the $(*)$ conditions.

Conversely, let $\alpha,\ \alpha'\colon \Delta[2] \to S$ be morphisms of simplicial sets that satisfy $(*)$. Then these define a cocone of the diagram $\Delta[2] \xleftarrow{<12>} \Delta[1] \xrightarrow{<01>} \Delta[2]$ (in dimension 1, we have $01 \overset{<12>}{\mapsto} 12 \overset{\alpha}{\mapsto} f$ and $01 \overset{<01>}{\mapsto} 01 \overset{\alpha'}{\mapsto} f$).
Since Sq is defined as the pushout of that diagram, there is a morphism $\mathrm{Sq} \to S$ which, by definition of α and α', looks exactly as needed.

Exercise 22. Let S be a Kan complex. Prove that every morphism in $\mathrm{Ho}S$ is an isomorphism. i.e every morphism in S is an equivalence.

Use Exercise 21. Let f be a morphism in S with $d_0 f = x,\ d_1 f = y$.
Consider the morphisms of simplicial sets $\beta\colon \Lambda[2]_2 \to S$, given by the diagram
$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] & x \\ y \arrow[ur, "f"] & \end{tikzcd}$$
and $\beta'\colon \Lambda[2]_0 \to S$ given by the diagram
$$\begin{tikzcd} & x \\ y \arrow[ur, "f"] \arrow[r, "\mathrm{id}_y"'] & y \end{tikzcd}.$$
Since S is a Kan complex, β and β' have lifts α and $\alpha'\colon \Delta[2] \to S$ respectively.
These α and α' are now a cocone for the diagram $\Delta[2] \xleftarrow{<12>} \Delta[1] \xrightarrow{<01>} \Delta[2]$,

since the compositions are $\alpha\circ <12> = \alpha'\circ <01> =$ the diagram $y \xrightarrow{f} x$.
So there is a morphism $\mathrm{Sq} \to S$ such that the upper left triangle is given by α' and the lower right triangle by α.
(I now notice that the existence of α and α' was already enough)

From this point on we will take for granted that a quasi-category in which every morphism is an equivalence is precisely a Kan complex (this is a surprisingly difficult result to prove).

3. The Quasi-Category of Spaces

Let's focus on one specific quasi-category: the quasi-category of spaces. What is a space?

Exercise 23. Denote by $|\Delta^n|$ the *topological n-simplex* given as a specific subspace of $\mathbb{R}^{n+1}$. Let $S : Top \to s\mathcal{S}et$ be the functor that takes a topological space T to the simplicial set $S(T)$ given by $S(T)_n = \mathrm{Hom}(|\Delta^n|, T)$. Prove that $S(T)$ is a Kan complex.

To prove: every morphism in $S(T)$ is an equivalence. Let $\gamma\colon |\Delta^1| \to T$ be a morphism in $S(T)$ with $\gamma(0,1) = x,\ \gamma(1,0) = y$. Consider the morphism $\bar\gamma\colon |\Delta^1| \to T$ defined by $\bar\gamma(s,t) = \gamma(t,s)$. We can now define $\sigma\colon |\Delta^2| \to T$ by $\sigma(x_0,x_1,x_2) = \gamma(1-x_1,x_1)$. The faces of this are:
$d_0\sigma(x_0,x_1) = \sigma(0,x_0,x_1) = \gamma(1-x_0,x_0) = \gamma(x_1,x_0) = \bar\gamma(x_0,x_1)$
$d_1\sigma(x_0,x_1) = \sigma(x_0,0,x_1) = \gamma(1-0,0) = \gamma(1,0) = y = s_0 y(x_0,x_1)$
$d_2\sigma(x_0,x_1) = \sigma(x_0,x_1,0) = \gamma(1-x_1,x_1) = \gamma(x_0,x_1)$
So the composition $\bar\gamma\gamma$ is homotopic to the constant path at $\gamma(1,0)$.

Similarly, $\sigma'(x_0,x_1,x_2) = \gamma(x_1, 1-x_1)$ is the homotopy between $\gamma\bar\gamma$ and the constant path at $\gamma(0,1)$.

As a result of this exercise, we can think of a Kan complex as a way to define a "space". We will take the following fact for granted. There exists a quasi-category $N\mathcal{S}$:

(1) $N\mathcal{S}_0$ is the set of Kan complexes K.
(2) $N\mathcal{S}_1$ is the of morphisms of simplicial sets $f : K \to L$
(3) $N\mathcal{S}_2$ is the set where an element is given by a tuple (X, Y, Z, f, g, h, H), where X, Y, Z are Kan complexes, $f : X \to Y$, $g : Y \to Z$, $h : X \to Z$ are morphisms of simplicial sets and $H : \Delta[1] \times X \to Z$ is a morphism of simplicial set with $H(-, 0) = g \circ f$ and $H(-, 1) = h$.

To better understand how H works as a natural transformation in a case where $f, g\colon X \to Y$ and $H\colon \Delta[1] \times X \to Y$ with $H(0, -) = f,\ H(1, -) = g$, let $x \in X_1$. Then $H_0(0, d_1x) = f_0(d_1x)$ and $H_0(1, d_1x) = g_0(d_1x)$. Also, $H_1(00, x) = f_1(x)$ and $H_1(11, x) = g_1(x)$. The interesting data is $H_1(01, -)$: there are

- $H_1(01, s_0d_1x)$ which has source $f_0(d_1x)$ and target $g_0(d_1x)$,
- $H_1(01, x)$ which has source $f_0(d_1x)$ and target $g_0(d_0x)$,
- and $H_1(01, s_0d_0x)$ which has source $f_0(d_0x)$ and target $g_0(d_0x)$.

All this can be written in a diagram
$$\begin{tikzcd} f_0(d_1x) \arrow[r, "f_1x"] \arrow[d, "H_1(01.s_0d_1x)"] \arrow[dr, "H_1(01.x)"] & f_0(d_0x) \arrow[d, "H_1(01.s_0d_0x)"] \\ g_0(d_1x) \arrow[r, "g_1x"] & g_0(d_0x) \end{tikzcd}$$
This diagram is actually commutative in the sense that the upper right and lower left triangle are elements in Y_2, given by

- $H_2(011, s_0x) = \sigma$ is the lower left triangle
  - $d_0\sigma = H_1(11, x) = g_1x$
  - $d_1\sigma = H_1(01, x)$
  - $d_2\sigma = H_1(01, \mathrm{id}_{d_1x})$
- $H_2(001, s_1x) = \tau$ is the upper right triangle
  - $d_0\tau = H_1(01, \mathrm{id}_{d_0x})$
  - $d_1\tau = H_1(01, x)$
  - $d_2\tau = H_1(00, x) = f_1x$

Exercise 24. Use the explicit description above to verify that $N\mathcal{S}$ satisfy the lifting condition against the inner horn inclusion $\Lambda[2]_1 \to \Delta[2]$.

To show: Given a morphism of simplicial sets $\alpha\colon \Lambda[2]_1 \to N\mathcal{S}$, there is a lift $\Delta[2] \to N\mathcal{S}$.
The morphism α consists of a choice of three Kan complexes $0 \mapsto X,\ 1 \mapsto Y,\ 2 \mapsto Z$ and two morphisms of Kan complexes $01 \mapsto (f\colon X \to Y),\ 12 \mapsto (g\colon Y \to Z)$.
What we now need for the lift is a morphism of simplicial sets $h\colon X \to Z$ (for $02 \mapsto h$) and an element of $N\mathcal{S}_2$ of the form (X, Y, Z, f, g, h, H) where only H is still to be defined (for $012 \mapsto (X, Y, Z, f, g, h, H)$).
Set $h := g \circ f$, and $H\colon \Delta[1] \times X \to Z$ shall be 'the natural transformation from $g \circ f$ to $g \circ f$', i.e. $H_n(i_0...i_n, -) = g \circ f\colon X_n \to Z_n$ in every dimension n.

Exercise 25. Use the explicit description given to describe the homotopy category $\mathrm{Ho}N\mathcal{S}$.

The objects of Ho$N\mathcal{S}$ are the Kan complexes.
For two objects $K,\ L$, the morphisms in Ho$N\mathcal{S}$ between K and L are the equivalence classes of morphisms of Kan complexes $f\colon K \to L$, where two morphisms $f,\ f'\colon K \to L$ are equivalent iff there is a 'natural transformation' between $f = f \circ \mathrm{id}_K$ and f' (iff there is a 'natural transformation' between $f = \mathrm{id}_L \circ f$ and f').

Exercise 26. Show that $N\mathcal{S}$ is not a Kan complex.

It suffices to show the existence of a morphism in $N\mathcal{S}$ that is not an equivalence. Consider the morphism $t\colon X := S(\{0,1\}) \to S(\{0\}) =: 0$ (where $\{0,1\}$ has the discrete topology, so in each dimension $S(\{0,1\})_n$ is a two-point set and $S(\{0\})_n$ a one-point set).
If t was an equivalence, there would have to be a morphism $f\colon 0 \to X$ and a 'natural transformation' $H\colon \Delta[1] \times X \to X$ from $f \circ t$ to id_X. Necessarily, in each dimension $f \circ t(0) = f \circ t(1)$, let w.l.o.g. $f_0 \circ t_0 = \mathrm{const}_0$, then because of the degeneracy maps, $f \circ t$ has to be const_0 in every dimension. Then $H_1(01, 11) \in X_1$ would have to be an element with source $d_1 H_1(01, 11) = \mathrm{const}_0(d_1 11) = 0$ and target $d_0 H_1(01, 11) = \mathrm{id}(d_1 11) = 1$. But there is no such element in $X_1 = \{00, 11\}$

So, $N\mathcal{S}$ is not a Kan complex itself. What can we do about that?

Exercise 27. Let S be a quasi-category and $f : x \to y, g : y \to z$ two equivalences. Prove that $gf : x \to z$ is an equivalence

Since f and g are equivalences, there are $e, e'\colon y \to x$ and $h, h'\colon z \to y$ such that $e \circ f = \mathrm{id}_x,\ f \circ e' = \mathrm{id}_y = h \circ g,\ g \circ h' = \mathrm{id}_z$. Then $e \circ h$ and $e' \circ h'$ are inverses to $g \circ f$, i.e. $eh \circ gf = \mathrm{id}_x$ and $gf \circ e'h' = \mathrm{id}_z$. This holds because of the associativity proved in Exercise 15.

Let S be a quasi-category. Define $S^{\simeq}$ as follows:

(1) $(S^{\simeq})_0 = S_0$
(2) $(S^{\simeq})_1 = \{f :\in S_1 | f \text{ is an equivalence } \} \subseteq S_1$
(3) $(S^{\simeq})_n = \{\sigma \in S_n | \forall \alpha : [1] \to [n] (\alpha^*(\sigma) \in (S^{\simeq})_1)\} \subseteq S_n$

Exercise 28. Prove that $S^{\simeq}$ is also a quasi-category.

To Do:

- $S^{\simeq}$ is a simplicial set, check that faces and degeneracies are in $(S^{\simeq})_{n-1}$ or $(S^{\simeq})_{n+1}$ resp.
  Let $\sigma \in (S^{\simeq})_n$, then $d_j \sigma \in S_{n-1}$. Let $\alpha\colon [1] \to [n-1]$. The face map d_j actually arises from a map $\delta_j\colon [n-1] \to [n]$ in the simplex category, and $\alpha^*(d_j(\sigma)) = (\delta_j \circ \alpha)^*(\sigma)$. This $\delta_j \circ \alpha$ is now a morphism $[1] \to [n]$ in the simplex category. Since σ lies in $(S^{\simeq})_n$, $(\delta_j \circ \alpha)^*(\sigma)$ is an equivalence.
  Similar for the degeneracy maps:
- $S^{\simeq}$ is a quasi-category, check that the filling of an inner horn is in $(S^{\simeq})_{n+1}$

Exercise 29. Prove that $S^{\simeq}$ is also a Kan complex. Conclude that $N\mathcal{S}^{\simeq}$ is a Kan complex.

We take for granted that a quasi-category in which every morphism is an equivalence is precisely a Kan complex - this is given by definition.

When $S^{\simeq}$ is a Kan complex for every quasi-category S, then in particular, $(N\mathcal{S})^{\simeq}$ is one.
So, (up to size issues) $N\mathcal{S}^{\simeq}$ is an object in $N\mathcal{S}$. That's crazy!

4. Colimits in Sets vs. Spaces

Exercise 30. Compute the following colimits in the category of sets:

- $* \overset{0}{\underset{1}{\rightrightarrows}} \{0,1\}$
- $* \rightrightarrows *$
- $* \longleftarrow \{0,1\} \longrightarrow \{0,1,2\}$
- $* \longleftarrow * \coprod * \longrightarrow *$

- $* \overset{0}{\underset{1}{\rightrightarrows}} \{0,1\} \longrightarrow \{0,1\}/(0 \sim 1) \cong *$
- $* \rightrightarrows * \longrightarrow *$
- $$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \{0,1,2\} \arrow[d] \\ * \arrow[r] & \{0,1,2,*\}/(0 \sim * \sim 1) \cong \{0,2\} \end{tikzcd}$$
- $$\begin{tikzcd} * \coprod * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \coprod */(* \sim *) \cong * \end{tikzcd}$$

Exercise 31. Use the CW description to compute the same colimit in the ∞-category of spaces.

5. Weak Descent

Exercise 32. Let $\{X_i\}_I$ be an I indexed collection of sets. Show that the collection of pullback functors

$$\mathcal{S}\mathrm{et}_{/\coprod X_i} \to \prod_i \mathcal{S}\mathrm{et}_{/X_i}$$

is an isomorphism of categories.

reminder: pullback functor. Let $\mathcal{C}$ be a category that has finite limits. Then there is the contravariant functor $\mathcal{C} \to Cat$
On objects, it sends $A \in \mathrm{Obj}(\mathcal{C})$ to the category $\mathcal{C}/_A$. On morphisms, it takes $f\colon A \to B$ and makes it a (covariant) functor $f^*\colon \mathcal{C}/_B \to \mathcal{C}/_A$ that does:
On objects $X \xrightarrow{h} B$ is sent to $X \times_B A \xrightarrow{f^*(h)} A$.
On morphisms, it's nothing surprising: $X_1 \xrightarrow{g}$ becomes $X_1 \times_B A \xrightarrow{(g,\mathrm{id})} X_2 \times_B A$
To prove that "pullback" is a functor, check that for every $A \xrightarrow{f} B \xrightarrow{g} C$ holds $f^* \circ g^* = (g \circ f)^*\colon \mathcal{C}/C \to \mathcal{C}/_A$. The former sends $(Y \to C)$ to $(Y \times_C B) \times_B A$. Use the "pasting law for pullbacks" from ncatlab.

For arbitrary products:
Let $(X_i)_{i\in I}$ be a family of objects of $\mathcal{S}\mathrm{et}$. Consider the functor

$$(\iota^*_{X_i})_{i\in I}\colon \mathcal{S}\mathrm{et}/_{\coprod X_i} \to \prod \mathcal{S}\mathrm{et}/_{X_i}$$

. To prove: this functor is actually an iso.
What does it do concretely? Let $h\colon Z \to \coprod X_i$ be an object of $\mathcal{S}\mathrm{et}/_{\coprod X_i}$.
Diagram

$$\begin{tikzcd} h^{-1}(X_j) \arrow[r] \arrow[d, "\iota_i^* h = h|_{h^{-1}(X_j)}"] & Z \arrow[d, "h"] \\ X_j \arrow[r, "\iota_j"] & \coprod X_i \end{tikzcd}$$

The functor $(\iota_i^*)_{i\in I}\colon \mathcal{S}\mathrm{et}/_{\coprod X_i} \to \prod \mathcal{S}\mathrm{et}/_{X_i}$ sends an object $h\colon Z \to \coprod X_i$ in $\mathcal{S}\mathrm{et}/_{\coprod X_i}$ to $(\iota_i^* h\colon h^{-1}(X_i) \to X_i)_{i\in I}$.
The inverse functor $\prod \mathcal{S}\mathrm{et}/_{X_i} \to \mathcal{S}\mathrm{et}/_{\coprod X_i}$ that we'll define will be called $\coprod_{i\in I} -$ because it does the following: Given a collection $(Y_i \xrightarrow{f_i} X_i)_{i\in I}$ in $\prod \mathcal{S}\mathrm{et}/_{X_i}$, send it to $\coprod f_i\colon \coprod Y_i \to \coprod X_i$ which is f_j on each Y_j. It has been checked that this is compatible with composition of morphisms in $\mathcal{S}\mathrm{et}/_{\coprod X_i}$.
See that the composition of these functors in both directions is the identity:
Start with $h\colon Z \to \coprod X_i$, then $\coprod h^{-1}(X_i) = Z$ and the morphism that comes back is exactly h. Start with $(f_i\colon Y_i \to X_i)_{i\in I}$, then we first get $\coprod f_i\colon \coprod Y_i \to \coprod X_i$ and then for each $j \in I$, $(\coprod f_i)^{-1}(X_j)$ is exactly Y_j and $\iota_j^*(\coprod f_i) = f_j$.
They are inverse.

Exercise 33. Let $\mathcal{C}$ be a small category, show the last result also holds for the category $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$, using the fact that colimits and pullbacks are computed pointwise.

Exercise 34. Let $\mathcal{C}$ be a category with finite limits and colimits. Fix a diagram $B \xleftarrow{g} A \xrightarrow{f} C$ in $\mathcal{C}$. Define a functor

$$\mathcal{C}_{/B\coprod_A C} \to \mathcal{C}_{/B} \times_{\mathcal{C}_{/A}} \mathcal{C}_{/C}$$

For abbreviation $P := B\coprod_A C$.
The objects of the category $\mathcal{C}/_B \times_{\mathcal{C}/_A} \mathcal{C}/_C$ are pairs $((h_1\colon X \to B), (h_2\colon Y \to C))$ with the property that $g^*(h_1) = f^*(h_2)$, in particular $X \times_B A = Y \times_C A$.

Define a functor $\mathcal{C}/_{B\coprod_A C} \to \mathcal{C}/_B \times_{\mathcal{C}/_A} \mathcal{C}/_C$.
On objects: Let $h\colon W \to P$ in $\mathcal{C}/_P$. The functor (ι_B^*, ι_C^*) yields a pair of objects $((\iota_B^* h\colon W \times_P B \to B), (\iota_C^* h\colon W \times_P C \to C))$ in $\mathcal{C}/_B \times \mathcal{C}/_C$. This pair actually lies in $\mathcal{C}/_B \times_{\mathcal{C}/_A} \mathcal{C}/_C$, since $g^*(\iota_B^*(h)) = (\iota_B \circ g)^*(h) = (\iota_C \circ f)^*(h) = f^*(\iota_C^*(h))$.
On morphisms, also apply the functor pair (ι_B^*, ι_C^*). I think $\mathcal{C}/_B \times_{\mathcal{C}/_A} \mathcal{C}/_C$ is a full subcategory of $\mathcal{C}/_B \times \mathcal{C}/_C$, so nothing could go wrong (look at this again later?)

Exercise 35. Give an example in $\mathcal{S}\mathrm{et}$ that demonstrates that this is not an equivalence.

In fact the situation is much worse than one might expect.

Exercise 36. Let $\mathcal{C}$ be a category such that

$$\mathcal{C}_{/B\coprod_A C} \to \mathcal{C}_{/B} \times_{\mathcal{C}_{/A}} \mathcal{C}_{/C}$$

is an isomorphism for all diagrams $B \leftarrow A \to C$. Prove that for any two objects X, Y, $\mathrm{Hom}(X, Y)$ is either empty or the point.

We can in fact further prove that it is always the point, which means the category is very trivial.

6. Locally Cartesian Closed Categories

Exercise 37. Let $\mathcal{C}$ be a finitely complete category. Prove the following are equivalent.

(1) For every object X, $\mathcal{C}_{/X}$ is Cartesian closed.
(2) For every morphism $f : X \to Y$, the pullback functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ has a right adjoint.
(3) For every object X, $\mathcal{C}_{/X}$ has a global sections. (A finitely complete category $\mathcal{D}$ has global sections if for every object Y in $\mathcal{D}$, the product map $- \times Y : \mathcal{D} \to \mathcal{D}_{/Y}$ has a right adjoint).

$1 \Rightarrow 2$ Consider a morphism $f\colon X \to Y$; by (1), the category $\mathcal{C}/_Y$ is Cartesian closed. This means in particular that there is a functor $-^X\colon \mathcal{C}/_Y \to \mathcal{C}/_Y$ such that for all objects $A \to Y$, $Z \to Z$ there is the natural isomorphism

$$\mathrm{Hom}_{/Y}(A \times_Y X, Z) \xleftrightarrow{\sim} \mathrm{Hom}_{/Y}(A, Z^X)$$

Now define the functor $\bar{f}\colon \mathcal{C}/_X \to \mathcal{C}/_Y$ as follows:
An object $Z \to X$ will first be sent to the composition $Z \to X \to Y$ by $f_!$, this is then an object in $\mathcal{C}/_Y$. Applying $-^X$ then yields an object $Z^X \to Y$ in $\mathcal{C}/_Y$. Claim: for every $A \to X$ and $Z \to Y$,

$$\mathrm{Hom}_{/X}(A \times_Y X, Z) \xleftrightarrow{\sim} \mathrm{Hom}_{/Y}(A, Z^X)$$

The "$\rightarrow$" direction is given by the iso from $- \times_X Y \dashv -^X$, since every morphism over X is also a morphism over Y.
The other direction of that same iso gives us back a morphism over Y but why would this automatically be a morphism over X? I'm not sure if it is

$2 \Rightarrow 3$ (Note: maybe use Exercise 5b?)

$3 \Rightarrow 1$ Global sections for every $\mathcal{C}/_X$ means: For every object X in $\mathcal{C}$ and every object $Y \to X$ in $\mathcal{C}/_X$, the product functor $- \times_X Y\colon \mathcal{C}/_X \to (\mathcal{C}/_X)/_Y$ (what?) has a right adjoint

Hence all three can be taken as the definition of a locally Cartesian closed category.

Exercise 38. Let $\mathcal{C}$ be a small category. Prove that $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ is locally Cartesian closed.

Exercise 39. Let $s\mathcal{S}\mathrm{et}$ be the category of simplicial sets. For given morphisms $f : X \to Y$ and $p : Z \to X$ give a description of the simplicial set $f_* p$ by explicitly describing the simplicial set $f_* Z$.