Matematiske metoder (MM 529)

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Second order inhomogeneous linear differential equation:

$$y'' + a(x)y' + b(x)y = g(x).$$

Structure of solutions:

All functions y of the form

$$y = y_p + y_h,$$

where y_p is one particular solution of the inhomogeneous equation and $y_h \in Y_H$, the set of solutions of the corresponding homogeneous differential equation

$$y'' + a(x)y' + b(x)y = 0.$$

Problems left:

1. How to find all solutions of a homogeneous LDE.
2. How to find one solution of the inhomogeneous LDE.

Second order linear differential equation:

$$y'' + a(x)y' + b(x)y = g(x).$$

Finding all solutions in general quite complicated.
Easier special case:
Second order linear differential equations with constant coefficients:

$$y'' + ay' + by = g(x), \quad a, b \in \mathbb{R} \text{ (constant coefficients)}.$$

Corresponding homogeneous equation.

$$y'' + ay' + by = 0.$$

Assumption: Solution has the form $y = e^{\lambda x}$ for some λ.
Then λ must be a zero of the polynomial

$$p(\lambda) = \lambda^2 + a\lambda + b.$$

$p(\lambda)$ characteristic polynomial of the homogeneous LDE.

Homogeneous LDE with constant coefficients:

$$y'' + ay' + by = 0.$$

Characteristic polynomial

$$p(\lambda) = \lambda^2 + a\lambda + b.$$

For a real zero λ of p, $y = e^{\lambda x}$ is one solution of corresponding homogeneous LDE. How to find all solutions?
Three cases:

1. p has two different zeroes $\lambda_1 \neq \lambda_2 \in \mathbb{R}$.
2. p has a double zero $\lambda \in \mathbb{R}$.
3. p has two complex zeroes $\lambda, \overline{\lambda} \in \mathbb{C} \setminus \mathbb{R}$.

Homogeneous LDE with constant coefficients:

$$y'' + ay' + by = 0.$$

First case: $p(\lambda) = \lambda^2 + a\lambda + b$ has two different zeroes $\lambda_1 \neq \lambda_2 \in \mathbb{R}$.
Then $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are both solutions,
$y = c_1 y_1 + c_2 y_2$, $c_1, c_2 \in \mathbb{R}$ is the general solution.

Second case: $p(\lambda) = \lambda^2 + a\lambda + b$ has a double zero $\lambda \in \mathbb{R}$.
Then $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$are solutions, and
$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$, $c_1, c_2 \in \mathbb{R}$ is the general solution.

Third case: $p(\lambda) = \lambda^2 + a\lambda + b$ has a zero $\lambda = p + iq \in \mathbb{C} \setminus \mathbb{R}$,
Solutions are real and imaginary part of complex solution function
$y = e^{(p+iq)x} = e^{px} \cos qx + ie^{px} \sin qx$.
Real part: $y_1(x) = e^{px} \cos qx$, imaginary part: $y_2(x) = e^{px} \sin qx$.
General solution: $y = c_1 e^{px} \cos qx + c_2 e^{px} \sin qx$, $c_1, c_2 \in \mathbb{R}$.

Second order inhomogeneous linear differential equation:

$$y'' + a(x)y' + b(x)y = g(x).$$

General solution of the corresponding homogeneous LDE:

$$Y_H = \{c_1 y_1(x) + c_2 y_2(x) \ : \ c_1, c_2 \in \mathbb{R}\},$$

where y_1, y_2 are any two structurally different solutions (non of them is a multiple by a constant of the other).
General solution of the inhomogeneous LDE:

$$Y_I = \{y_p(x) + c_1 y_1(x) + c_2 y_2(x) \ : \ c_1, c_2 \in \mathbb{R}\},$$

where $y_p(x)$ is any solution of the inhomogeneous LDE.
Problem: How to find y_p?

Easiest option (if it works): Guess a solution y_p and verify that it is a solution!

Finding a solution y_p of

$$y'' + a(x)y' + b(x)y = g(x).$$

Another option: Variation of constants.
Known: General solution of the corresponding homogeneous LDE

$$y = c_1 y_1(x) + c_2 y_2(x),\ c_1, c_2 \in \mathbb{R}.$$

Assumption: a solution y_p has a similar form, where the constants c_1, c_2 are replaced by functions $c_1(x), c_2(x)$ depending on the variable x

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x).$$

Forming the derivative using the product rule:

$$y_p' = c_1'(x)y_1(x) + c_1(x)y_1'(x) + c_2'(x)y_2(x) + c_2(x)y_2'(x).$$

Second derivative? Too complicated to be helpful.

Solving second order inhomogeneous LDEs, variation of constants

Finding a solution y_p of

$$y'' + a(x)y' + b(x)y = g(x)$$

by variation of constants.
Assumption: a solution y_p has the form

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x).$$

First derivative:

$$y_p' = c_1'(x)y_1(x) + c_1(x)y_1'(x) + c_2'(x)y_2(x) + c_2(x)y_2'(x).$$

Additional assumption $c_1'(x)y(x) + c_2'(x)y(x) = 0$ (fine as long as we still get a solution, we only need one!)
Second derivative (with additional assumption):

$$y_p'' = c_1'(x)y_1'(x) + c_1(x)y_1''(x) + c_2'(x)y_2'(x) + c_2(x)y_2''(x).$$

Solving second order inhomogeneous LDEs, variation of constants

Finding a solution y_p of

$$y'' + a(x)y' + b(x)y = g(x)$$

by variation of constants.
Assumptions: a solution y_p has the form

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$$

and $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$.
Derivatives:

$$y_p' = c_1(x)y_1'(x) + c_2(x)y_2'(x).$$

$$y_p'' = c_1'(x)y_1'(x) + c_1(x)y_1''(x) + c_2'(x)y_2'(x) + c_2(x)y_2''(x).$$

Insertion into LDE:

$$\begin{aligned} g(x) &= c_1'(x)y_1'(x) + c_1(x)y_1''(x) + c_2'(x)y_2'(x) + c_2(x)y_2''(x) \\ &+ a(x)(c_1(x)y_1'(x) + c_2(x)y_2'(x)) \\ &+ b(x)(c_1(x)y_1(x) + c_2(x)y_2(x)) \end{aligned}$$

Finding a solution y_p of

$$y'' + a(x)y' + b(x)y = g(x)$$

by variation of constants.
Assumptions: a solution y_p has the form

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$$

and $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$.
Inserting y_p and its derivatives into LDE:

$$\begin{aligned}
g(x) &= c_1'(x)y_1'(x) + c_1(x)y_1''(x) + c_2'(x)y_2'(x) + c_2(x)y_2''(x) \\
&+ a(x)(c_1(x)y_1'(x) + c_2(x)y_2'(x)) \\
&+ b(x)(c_1(x)y_1(x) + c_2(x)y_2(x)) = \\
&= c_1'(x)y_1'(x) + c_2'(x)y_2'(x) \\
&+ c_1(x)(y_1''(x) + a(x)y_1'(x) + b(x)y_1(x)) \\
&+ c_2(x)(y_2''(x) + a(x)y_2'(x) + b(x)y_2(x)) = \\
&= c_1'(x)y_1'(x) + c_2'(x)y_2'(x).
\end{aligned}$$

Solving second order inhomogeneous LDEs, variation of constants

Finding a solution y_p of

$$y'' + a(x)y' + b(x)y = g(x))$$

by variation of constants.
Assumptions: a solution y_p has the form

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$$

and $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$ and insertion of y_p and its derivatives into LDE yields system of linear equations:

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= g(x) \end{aligned}$$

in the indeterminates $c_1'(x)$ and $c_2'(x)$.
Final aim: Calculate $c_1(x)$ and $c_2(x)$.

Finding a solution y_p of

$$y'' + a(x)y' + b(x)y = g(x).$$

Assumptions: a solution y_p has the form

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$$

and $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$ then

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= g(x) \end{aligned}$$

First line implies $c_1'(x) = -\frac{c_2'(x)y_2(x)}{y_1(x)}$ and $c_2'(x) = -\frac{c_1'(x)y_1(x)}{y_2(x)}$.
Substitute into second line:

$$\begin{aligned} c_1'(x)y_1'(x) - \frac{c_1'(x)y_1(x)}{y_2(x)}y_2'(x) &= g(x) \\ -\frac{c_2'(x)y_2(x)}{y_1(x)}y_1'(x) + c_2'(x)y_2'(x) &= g(x) \end{aligned}$$

Finding a solution y_p of

$$y'' + a(x)y' + b(x)y = g(x).$$

Assumptions: a solution y_p has the form

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$$

and $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$ then

$$\begin{aligned} c_1'(x)y_1'(x) - \frac{c_1'(x)y_1(x)y_2'(x)}{y_2(x)} &= g(x), \\ -\frac{c_2'(x)y_2(x)y_1'(x)}{y_1(x)} + c_2'(x)y_2'(x) &= g(x), \end{aligned}$$

implying

$$\begin{aligned} c_1'(x) &= \frac{-g(x)y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}, \\ c_2'(x) &= \frac{g(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}. \end{aligned}$$

Solving second order inhomogeneous LDEs, variation of constants

Finding a solution y_p of

$$y'' + a(x)y' + b(x)y = g(x).$$

Assumptions: a solution y_p has the form

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$$

and $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$ then

$$\begin{aligned} c_1'(x) &= \frac{-g(x)y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} \\ c_2'(x) &= \frac{g(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}. \end{aligned}$$

and therefore

$$\begin{aligned} c_1(x) &= \int c_1'(x)\,dx = \int \frac{-g(x)y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}\,dx \\ c_2(x) &= \int c_2'(x)\,dx = \int \frac{g(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}\,dx. \end{aligned}$$

Solving second order inhomogeneous LDEs, variation of constants, example

General solution of

$$y'' + y' - 2y = e^x.$$

Characteristic polynomial of the corresponding homogeneous LDE $y'' + y' - 2y = 0$:

$$p(\lambda) = \lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1)$$

has the two real zeroes $\lambda_1 = -2$ and $\lambda_2 = 1$.
General solution of the homogeneous LDE:

$$y_h = c_1 e^{-2x} + c_2 e^x,\ c_1, c_2 \in \mathbb{R}.$$

Assumptions: a solution y_p has the form

$$y_p = c_1(x)e^{-2x} + c_2(x)e^x$$

and $c_1'(x)e^{-2x} + c_2'(x)e^x = 0.$

Solving second order inhomogeneous LDEs, variation of constants, example

General solution of
$$y'' + y' - 2y = e^x.$$
General solution of the homogeneous LDE:
$$y_h = c_1 e^{-2x} + c_2 e^x,\ c_1, c_2 \in \mathbb{R}.$$
Assumptions: a solution y_p has the form
$$y_p = c_1(x)e^{-2x} + c_2(x)e^x$$
and $c_1'(x)e^{-2x} + c_2'(x)e^x = 0$ then
$$\begin{aligned} c_1'(x) &= \frac{-g(x)y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} = \frac{-e^x e^x}{e^{-2x}e^x + 2e^{-2x}e^x} = -\frac{1}{3}e^{3x} \\ c_2'(x) &= \frac{g(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} = \frac{e^x e^{-2x}}{e^{-2x}e^x + 2e^{-2x}e^x} = \frac{1}{3}. \end{aligned}$$
thus
$$c_1(x) = \int c_1'(x)\,dx = -\frac{1}{9}e^{3x} + C \text{ and } c_2(x) = \int c_2'(x)\,dx = \frac{1}{3}x + C.$$

Solving second order inhomogeneous LDEs, variation of constants, example

General solution of

$$y'' + y' - 2y = e^x.$$

General solution of the homogeneous LDE:

$$y_h = c_1 e^{-2x} + c_2 e^x,\ c_1, c_2 \in \mathbb{R}.$$

Assumption: a solution y_p has the form

$$y_p = c_1(x)e^{-2x} + c_2(x)e^x$$

then

$$c_1(x) = -\frac{1}{9}e^{3x} + C \text{ and } c_2(x) = \frac{1}{3}x + C,$$

and

$$y_p = -\frac{1}{9}e^{3x}e^{-2x} + \frac{1}{3}xe^x = \frac{1}{3}(x - \frac{1}{3})e^x.$$

General solution:
$y_p + Y_H = \frac{1}{3}(x - \frac{1}{3})e^x + c_1 e^{-2x} + c_2 e^x$, $c_1, c_2 \in \mathbb{R}$.

Inhomogeneous LDE:

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=g(x),$$

Corresponding homogeneous LDE:

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=0,$$

Y_H the set of all solutions of the corresponding homogeneous LDE.
y_p any particular solution of the inhomogeneous LDE.
Then $Y_I=\{y_p+y_h : y_h\in Y_H\}$ is the set of solutions of the inhomogeneous LDE.
Problems left:

1. How to find all solutions of a homogeneous LDE.
   In case of constant coefficients, all solutions can be read off the zeroes of characteristic polynomial.
2. How to find a particular solution of the inhomogeneous LDE.
   e.g. use variation of constants.

Alternating series

A sequence (a_n) is called alternating if $a_n \cdot a_{n+1} < 0$ for all $n \geq k$.

The infinite sum $\sum_{n=k}^{\infty} a_n$ over an alterating sequence is an alternating series.

I.e., consecutive summands have different signs (where 0 counts for both signs).

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating series.

Convergence of alternating series under much weaker circumstances than in the general case:

Alternating series test

If $(|a_n|)$ is monotonously decreasing (i.e. $|a_{n+1}| \leq |a_n|$) and $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=k}^{\infty} a_n$ is convergent.

Revisiting alternating series

Alternating series

If $a_n \cdot a_{n+1} < 0$ for all $n \geq k$ then the infinite sum $\sum_{n=k}^{\infty} a_n$ is an alternating series.

Bounding the error of the n-th partial sum $s_n = \sum_{i=k}^{n} a_i$ for series passing the alternating series test.

Error estimate for alternating series

If $(|a_n|)$ is monotonously decreasing (i.e. $|a_{n+1}| \leq |a_n|$) and $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=k}^{\infty} a_n = s \in \mathbb{R}$ and the error of the n-th partial sum $s - s_n$ has the same sign as a_{n+1} and satisfies

$$|s - s_n| \leq |s_{n+1} - s_n| = |a_{n+1}|.$$

Error estimate for alternating series

If $a_n \cdot a_{n+1} < 0$ for all $n \geq k$ and $(|a_n|)$ is monotonously decreasing (i.e. $|a_{n+1}| \leq |a_n|$) and $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=k}^{\infty} a_n = s \in \mathbb{R}$ and the error of the n-th partial sum $s - s_n$ has the same sign as a_{n+1} and satisfies

$$|s - s_n| \leq |s_{n+1} - s_n| = |a_{n+1}|.$$

Example: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = s$ converges by the alternating series test.

The 6th partial sum is

$s_6 = \sum_{k=1}^{6} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} = \frac{37}{60}$, the error

$s - s_n$ has a positive sign and is at nost $\frac{1}{7}$.

Therefore $0.616... \approx \frac{37}{60} \leq s \leq \frac{37}{60} + \frac{1}{7} \approx 0.759...$

If needed, calculate better approximates by summing more terms.

But what is the sum?