Matematiske metoder (MM 529)

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Differentiation rules and standard derivatives:
Tables, e.g. inside the cover of the textbook!

Derivative of f^{-1}

If $g = f^{-1} : B \to A$ is the inverse function of $f : A \to B$, then

$$g'(x) = \frac{1}{f'(g(x))},$$

whenever $f'(g(x)) \neq 0$.

Chain rule:

$$1 = \frac{d}{dx}x = (f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

thus $g'(x) = \frac{1}{f'(g(x))}$.

Derivative of f^{-1}

If $g = f^{-1} : B \to A$ is the inverse function of $f : A \to B$, then

$$g'(x) = \frac{1}{f'(g(x))},$$

whenever $f'(g(x)) \neq 0$.

Examples:

- $g(x) = \ln x$ inverse of $f(x) = e^x$:

$$g'(x) = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

- $g(x) = \arctan x$ inverse of $f(x) = \tan x$:

$$g'(x) = \frac{1}{\tan'(\arctan x)} = \frac{1}{1 + \tan^2(\arctan x)} = \frac{1}{1 + x^2}.$$

If the derivative f' is differentiable, we can build the derivative $\frac{d}{dx}f'(x) = f''(x)$. In general:

Second, third, ... derivative of a function

(Recursive definition) Set $f^{(0)} = f$. If $f^{(n)}$ is differentiable then

$$f^{(n+1)} = \frac{d}{dx}f^{(n)}.$$

Therefore, $f^{(1)} = f'$, $f^{(2)} = f''$, $f^{(3)} = f'''$, etc.

n times differentiable

We call f n times differentiable, if $f^{(n)}$ exists.

Example: $f(x) = 5x^4$: $f' = 20x^3$, $f'' = 60x^2$, $f''' = 120x$, $f^{(4)} = 120$, $f^{(5)} = f^{(6)} = f^{(7)} = \ldots = 0$.
f ist n times differentiable, for every n.

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0. \\ -x^2 & \text{if } x < 0, \end{cases} \qquad \text{is differentiable in } x_0 = 0.$$

(Check the left– and rightsided limits of the difference quotient or the derivatives of x^2 and $-x^2$ at $x = 0$.)
But: $g'(x) = |2x|$ is not differentiable in $x = 0$.
Therefore: g is only once but not two times differentiable in $x_0 = 0$.

$h(x) = \sin x$: $h'(x) = \cos x$, $h''(x) = -\sin x$, $h'''(x) = -\cos x$, $h^{(4)}(x) = \sin x$, etc.
In general:

$$h^{(n)}(x) = \begin{cases} \sin x & \text{if } n = 4k, \\ \cos x & \text{if } n = 4k+1, \\ -\sin x & \text{if } n = 4k+2, \\ -\cos x & \text{if } n = 4k+3, \end{cases}$$

$k \in \mathbb{N}_0$. Therefore, $\sin x$ is n times differentiable for every n.

Ascending/descending function

A function f is ascending on an interval $I \subseteq \mathbb{R}$, if $f(x) < f(y)$ for all $x, y \in I$ with $x < y$.
f is descending on I, if $f(x) > f(y)$ for all $x, y \in I$ with $x < y$.

Example: $f(x) = x^2$ is descending on $I = (-\infty, 0)$, and ascending on $I = (0, \infty)$.

(First) derivatives and ascending/descending functions

Let f be differentiable on $I \subseteq \mathbb{R}$.

- If $f'(x) > 0$ for all $x \in I$ then f is ascending on I.
- If $f'(x) < 0$ for all $x \in I$ then f is descending on I.

Example continued: $f'(x) = 2x < 0$ on $I = (-\infty, 0)$, and $f'(x) = 2x > 0$ on $I = (0, \infty)$.

Convex/concave function

A function f is convex on an interval $I \subseteq \mathbb{R}$, if for any two points $(x, f(x)), (y, f(y))$, $x, y \in I$, the graph is below the straight line joining the two points.
f is concave on I, if for any two points $(x, f(x)), (y, f(y))$, $x, y \in I$, the graph is above the straight line joining the two points.

Condition for convexity for a continuous function f:

$$f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2} \quad \text{for all pairs } x, y \in I,\ x < y.$$

Example:
$f(x) = x^3$ is concave on $I = (-\infty, 0)$, and convex on $I = (0, \infty)$.
Analogy with convex/concave lenses: look from below.
Terminology in the textbook:
convex: concave up,
concave: concave down.

Convex/concave function

A function f is convex on an interval $I \subseteq \mathbb{R}$, if for any two points $(x, f(x)), (y, f(y))$, $x, y \in I$, the graph is below the straight line joining the two points.
f is concave on I, if for any two points $(x, f(x)), (y, f(y))$, $x, y \in I$, the graph is above the straight line joining the two points.

Example: $f(x) = x^3$ is concave on $I = (-\infty, 0)$, and convex on $I = (0, \infty)$.

Second derivatives and convex/concave functions

Let f be two times differentiable on $I \subseteq \mathbb{R}$.

- If $f''(x) > 0$ for all $x \in I$ then f is convex on I.
- If $f''(x) < 0$ for all $x \in I$ then f is concave on I.

Example continued: $f''(x) = 6x < 0$ on $I = (-\infty, 0)$, and $f''(x) = 6x > 0$ on $I = (0, \infty)$.

Local maximum and minimum

$f(x)$ is a local maximum of f on an interval I, if there is a $\delta > 0$ such that $f(y) \leq f(x)$ for all $y \in (x - \delta, x + \delta) \cap I$.
$f(x)$ is a local minimum of f on an interval I, if there is a $\delta > 0$ such that $f(y) \geq f(x)$ for all $y \in (x - \delta, x + \delta) \cap I$.

Global maximum and minimum

$f(x)$ is a global maximum of f on I, if $f(y) \leq f(x)$ for all $y \in I$.
$f(x)$ is a global minimum of f on I, if $f(y) \geq f(x)$ for all $y \in I$.

The points x are called extreme points.

Inflection points

A point x is an inflection point of a function f, if the function changes from convex to concave or from concave to convex at x.

That means: there is a $\delta > 0$ such that f is convex on $(x - \delta, x)$ and concave on $(x, x + \delta)$ or vice versa.

Necessary condition for extreme points

If f is a differentiable function on an open interval I and $x \in I$ is an extreme point then $f'(x) = 0$.

Candidates for extreme points of f on a closed interval $[a, b]$:

- points with $f'(x) = 0$,
- points where f is not differentiable,
- the boundary points a, b.

Sufficient condition for local maxima/minima

If f is two times differentiable on I and $f'(x) = 0$ then $f(x)$ is a

- local maximum, if $f''(x) < 0$,
- local minimum, if $f''(x) > 0$.

Maximum: tangent horizontal and function concave in x.
Minimum: tangent horizontal and function convex in x.

Necessary condition for inflection points

If f is a two times differentiable function on an open interval I and $x \in I$ is an inflection point then $f''(x) = 0$.

Sufficient condition for inflection points

Let f be three times differentiable on I. If $f''(x) = 0$ and $f'''(x) \neq 0$ then x is an inflection point.

Change:
concave/convex: $f'''(x) > 0$.
convex/concave: $f'''(x) < 0$.

General rule

Let f be $n \geq 2$ times differentiable on I and for $x \in I$,
$f'(x) = f''(x) = \ldots = f^{(n-1)}(x) = 0$ and $f^{(n)}(x) \neq 0$.
Then x is an

- extreme point, if n is even,
- inflection point, if n is odd.

Geometric behaviour (examples)

General rule

Let f be $n \geq 2$ times differentiable on I and for $x \in I$,
$f'(x) = f''(x) = \ldots = f^{(n-1)}(x) = 0$ and $f^{(n)}(x) \neq 0$.
Then x is an

- extreme point, if n is even,
- inflection point, if n is odd.

Example: $f(x) = x^5$.
$f'(x) = 5x^4$. We have $f'(0) = 0$, therefore $x = 0$ candidate for an extreme point.
$f''(x) = 20x^3$. We have $f''(0) = 0$, therefore $x = 0$ candidate for an inflection point.
$f'''(x) = 60x^2$, $f^{(4)}(x) = 120x$, $f^{(5)}(x) = 120$. Therefore, $f'(0) = \ldots = f^{(4)}(0) = 0$ and $f^{(5)}(0) = 120 \neq 0$ and $n = 5$ is odd, so x is an inflection point.

Limit laws for functions

Suppose that $\lim_{x\to a} f(x) = s$ and $\lim_{x\to a} g(x) = t$. Then

(1) $\lim_{x\to a}(f(x) + g(x)) = s + t$;

(2) $\lim_{x\to a}(f(x) \cdot g(x)) = s \cdot t$;

(3) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{s}{t}$, if $t \neq 0$,

(4) $\lim_{x\to a}(f(x))^{g(x)} = s^t$, if $s > 0$.

What happens if some of the limits are $\pm\infty$ or 0?
Some expressions give rise to a limit. Assume that $b \in \mathbb{R}$, $b > 0$:

$$b + \infty = \infty,\ b - \infty = -\infty,\ b \cdot \infty = \infty,\ (-b) \cdot \infty = -\infty,$$

$$(-b) \cdot (-\infty) = \infty,\ \infty \cdot \infty = \infty,\ \infty \cdot (-\infty) = -\infty,\ \frac{b}{\pm\infty} = 0.$$

If $\lim_{x \to a} f(x) = b > 0$ and $\lim_{x \to a} g(x) = 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \infty & \text{if } g(x) > 0, \\ -\infty & \text{if } g(x) < 0, \end{cases}$$

where $x \in (a - \delta, a + \delta)$ for a $\delta > 0$.
Example:

$$\lim_{x \to 0} \frac{e^x}{x^2} = \infty.$$

Indeterminate expressions obtained by taking limits

$$0 \cdot \infty, \ \frac{\pm\infty}{\pm\infty}, \ \frac{0}{0}, \ 0^0.$$

(also L'Hôpital)

L'Hospital's rule

If f, g are both differentiable functions and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example:

$$h(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3, \\ 6 & \text{if } x = 3. \end{cases}$$

Complicated representation of $h(x) = x + 3$.

L'Hospital's rule (example)

L'Hospital's rule

If f, g are both differentiable functions and

$$\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example: For $f(x) = x^2 - 9$ and $g(x) = x - 3$ and

$$h(x) = \begin{cases} \frac{f(x)}{g(x)} & \text{if } x \neq 3, \\ 6 & \text{if } x = 3, \end{cases} \quad \text{is continuous in } (-\infty, \infty) = \mathbb{R},$$

because

$$\frac{\lim_{x\to 3} f(x)}{\lim_{x\to 3} g(x)} = \frac{0}{0} \text{ and } \lim_{x\to 3} \frac{f(x)}{g(x)} = \lim_{x\to 3} \frac{f'(x)}{g'(x)} = \lim_{x\to 3} \frac{2x}{1} = 6.$$

L'Hospital's rule (further example)

L'Hospital's rule

If f, g are both differentiable functions and

$$\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example: For $f(x) = e^x - 1$ and $g(x) = x$ and

$$h(x) = \frac{f(x)}{g(x)} = \frac{e^x - 1}{x} \text{ difference quotient of } e^x \text{ at } x_0 = 0.$$

$$\frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)} = \frac{0}{0} \text{ and } \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{e^x}{1} = 1.$$