Matematiske metoder (MM 529)

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(with similar methods one can also calculate the area of surfaces, the volume of a solid, if the functions of the boundary are differentiable)

Approximating the length of a curve by the length of a polygonal line, with straight line segments between consecutive points on the curve.

Curve in $\mathbb{R}^2$ with initial point (a, b) and terminal point (c, d):

Sequence of consecutive points

$(a, b) = (x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) = (c, d)$.

Length of the straight line segment between (x_{i-1}, y_{i-1}) and (x_i, y_i): $\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$.

Length L_n of the polygonal line between (a, b) and (c, d):

$$L_n = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}.$$

Function $f : D \subseteq \mathbb{R} \to \mathbb{R}$. Estimate the length of the graph of f between $(a, f(a))$ and $(b, f(b))$, $a, b \in D$.
Length L_n of the polygonal line with intermediate points $(a, f(a)) = (x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n)) = (b, f(b))$ on the graph, $a = x_0 < x_1 < \ldots < x_n = b$ partition of $[a, b]$:

$$L_n = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

If f is differentiable:

Mean value theorem of differentiation

If f is continuous on the interval $[a, b]$ and differentiable on the open interval (a, b) then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$. Calculate the length of the graph of f between $(a, f(a))$ and $(b, f(b))$, $a, b \in D$.

Mean value theorem

If f is continuous on the interval $[a, b]$ and differentiable on the open interval (a, b) then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

With $\Delta x_i = x_i - x_{i-1}$ there must be $c_i \in (x_{i-1}, x_i)$, such that

$$\begin{aligned}
L_n &= \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\
&= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (f(x_i) - f(x_{i-1}))^2} \\
&= \sum_{i=1}^{n} \sqrt{1 + \frac{(f(x_i) - f(x_{i-1}))^2}{(\Delta x_i)^2}}\Delta x_i = \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2}\Delta x_i
\end{aligned}$$

Differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$. Calculate the length of the graph of f between $(a, f(a))$ and $(b, f(b))$, $a, b \in D$.
Length of polygonal line for partition P_n:

$$L_n = \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2} \Delta x_i.$$

Riemann sum for the function $g(x) = \sqrt{1 + (f'(x))^2}$, partition P_n, and selection of intermediate points c_i.
Sequence of partitions P_n, $n \to \infty$ with $\lim_{n\to\infty} ||P_n|| = 0$

$$\lim_{n\to\infty} L_n = \int_a^b g(x)\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

length of the graph of the differentiable function $f(x)$ between $x = a$ and $x = b$.

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Example: Length of the graph of $f(x) = x^2$ between 0 and t

$$\int_0^t \sqrt{1 + (f'(x))^2}\,dx = \int_0^t \sqrt{1 + 4x^2}\,dx = \int_0^t 2\sqrt{\frac{1}{4} + x^2}\,dx.$$

Antiderivative of $2\sqrt{\frac{1}{4} + x^2}$ is (according to tables) is $x\sqrt{\frac{1}{4} + x^2} + \frac{1}{4}\ln|x + \sqrt{\frac{1}{4} + x^2}| + C$, therefore

$$\int_0^t 2\sqrt{\frac{1}{4} + x^2}\,dx = t\sqrt{\frac{1}{4} + t^2} + \frac{1}{4}\ln(2t + \sqrt{1 + 4t^2}).$$

Example: Length of the graph of $f(x) = x^2$ between 0 and t:

$$\int_0^t 2\sqrt{\frac{1}{4} + x^2}\, dx = t\sqrt{\frac{1}{4} + t^2} + \frac{1}{4}\ln(2t + \sqrt{1 + 4t^2}).$$

For $t = 1$ the length is $\frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(2 + \sqrt{5}) = 1.4789...$, between
$\sqrt{2} = 1.4142...$ (length of the straight line between $(0, 0)$ and $(1, 1)$), and
$\pi/2 = 1.5707...$ (length of the circular arc of radius 1 between $(0, 0)$ and $(1, 1)$.

Infinite series

$$\sum_{k=N}^{\infty} a_k = \lim_{n\to\infty} S_n,$$

where $S_n = \sum_{k=N}^{n} a_k$, sequence of partial sums.

Series may be convergent or divergent.

Criteria for convergence and divergence:

Comparison test

Let (a_k) and (b_k) be sequences with $0 \le a_k \le b_k$ for all $k \ge N$.

1. If $\sum_{k=N}^{\infty} b_k = \beta \in \mathbb{R}$ then $\sum_{k=N}^{\infty} a_k = \gamma \in \mathbb{R}$ with $\gamma \le \beta$.
2. If $\sum_{k=N}^{\infty} a_k = \infty$ then $\sum_{k=N}^{\infty} b_k = \infty$.

Necessary condition for a series to be convergent: $\lim_{k\to\infty} a_k = 0$.

If all $a_k \geq 0$ for all $k \geq N$ then the series $\sum_{k=N}^{\infty} a_k$ geometrically is the area under the graph of a staircase function.
Assume that the elements of the sequence are function values of the form $a_k = f(k)$ and f is non-increasing:
Geometric comparison gives:

Integral test

$$\int_N^{\infty} f(x)\,dx \leq \sum_{n=N}^{\infty} f(n) \leq f(N) + \int_N^{\infty} f(x)\,dx.$$

The series is convergent, if and only if the (improper) integral is convergent.

Example: $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent since for $f(x) = \frac{1}{x^2}$ and an antiderivative $F(x) = -\frac{1}{x}$, $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{x\to\infty} F(x) - F(1) = 1$.

Sets of numbers:

1. Natural numbers $\mathbb{N}$: Addition Subtraction: $1 - 3 \notin \mathbb{N}$.
2. Integers $\mathbb{Z}$: Subtraction Division: $1/3 \notin \mathbb{Z}$.
3. Rational numbers $\mathbb{Q}$: Division Limits of sequences:
   $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \notin \mathbb{Q}$.
4. Rational numbers $\mathbb{R}$: Limits of convergent sequences
   Roots of polynomials: No solution of $x^2 + 1 = 0$ in $\mathbb{R}$ (in other words $\sqrt{-1} \notin \mathbb{R}$).

We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
Define i to be a solution of $x^2 + 1 = 0$. Then $-i = (-1) \cdot i$ is another solution, because

$$(-i)^2 + 1 = (-1)^2 \cdot i^2 + 1 = 1 \cdot (-1) + 1 = 0.$$

Set of complex numbers $\mathbb{C} = \{a + i \cdot b \mid a, b \in \mathbb{R}\}$.

Set of complex numbers $\mathbb{C} = \{a + i \cdot b \mid a, b \in \mathbb{R}\}$.
i: imaginary unit.
$z = a + ib$: $a = \mathrm{Re}(z)$ real part, $b = \mathrm{Im}(z)$ imaginary part.
Conjugate complex number of $z = a + ib$: $\bar{z} = a - ib$,
i.e. the complex number with $\mathrm{Re}(\bar{z}) = \mathrm{Re}(z)$ and $\mathrm{Im}(\bar{z}) = -\mathrm{Im}(z)$.
We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$,
because $\mathbb{R}$ is the set of complex numbers with imaginary part
$b = 0$.
Complex means complicated?
Arithmetic operations with complex numbers:
Treat i like a variable, except $i^2 = -1$!
Example:
$1 + 3i^3 = 1 + 3i \cdot i^2 = 1 + 3i \cdot (-1) = 1 - 3i = 1 + i \cdot (-3)$,
complex number with real part 1 and imaginary part -3.

Arithmetic of complex numbers

Set of complex numbers $\mathbb{C} = \{a + i \cdot b \mid a, b \in \mathbb{R}\}$.
Arithmetic operations with complex numbers:
Treat i like a variable, except $i^2 = -1$!

Addition/Subtraction

$$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d).$$

Multiplication

$$(a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc).$$

Division

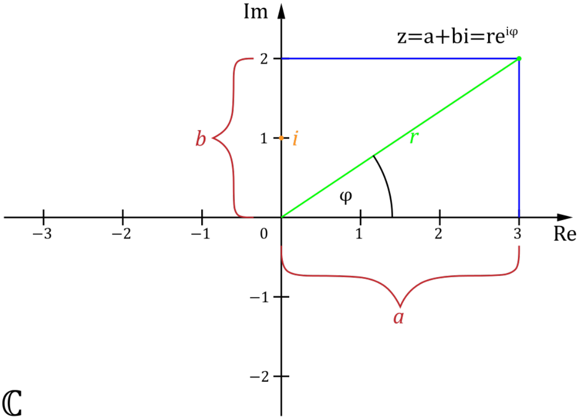
$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2},$$

if $c + id \neq 0$. Real part: $\dfrac{ac + bd}{c^2 + d^2}$. Imaginary part: $\dfrac{bc - ad}{c^2 + d^2}$.

Complex plane

Associate complex number $a + ib$ with the point (a, b).

Like cartesian plane, except x-axis now real axis and y-axis now imaginary axis.



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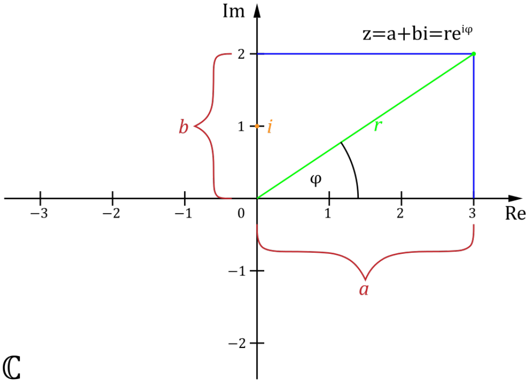

Second interpretation: Associate $a + ib \in \mathbb{C}$ with vector (a, b) in the complex plane.

Geometric interpretation of addition of complex numbers: Vector addition in the complex plane.