Matematiske metoder (MM 529)

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Definition Power series

A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

is called a power series about the center c, with the coefficients $a_0, a_1, a_2, \dots$.

Recall: Value of an infinite series is the limit of the partial sums.
Example: Geometric series (all coefficients $a_n = 1$ and center $c = 0$)

$$1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $-1 < x < 1$, since the sequence partial sums (s_n) satisfies
$s_n = \sum_{n=0}^{k} x^n = \dfrac{x^{n+1}-1}{x-1}$ (see exercises to Lecture 3).

Example: Representation of a function f by its Taylor series with center c.

$$\begin{aligned} f(x) &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{k!}(x-c)^3 + \ldots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n. \end{aligned}$$

Coefficients $a_n = \frac{f^{(n)}(x)}{n!}$.
Recall: Identity true for $x = c$ (using the usual agreement $0^0 = 1$).

Power series, radius of convergence

Power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges for $x = c$ against a_0 (usual agreement $0^0 = 1$).
There is always a value $r \in [0, \infty) \cup \{\infty\}$ (radius of convergence), such that the series is convergent for $x \in (c-r, c+r)$, and divergent on $\mathbb{R} \setminus [c-r, c+r]$ (can be convergent or divergent for the boundary values $c-r$ and $c+r$).

Example: Geometric series has radius of convergence $r = 1$,
It converges for $-1 < x < 1$ against $\frac{1}{1-x}$,
diverges to $+\infty$ for $x \geq 1$,
and diverges without a limit for $x \leq -1$.
E.g. for $x = -1$ we get the divergent sequence of partial sums:
$(s_n) = (1, 0, 1, 0, 1, 0, \ldots)$.

Power series, radius of convergence

Power series is a function

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

with domain $(c-r, c+r)$, if $r > 0$ is the radius of convergence.

How to calculate the radius of convergence?
Frequently, the ratio test helps:

Ratio test

For the series $\sum_{n=0}^{\infty} a_n$ suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \gamma.$$

If $\gamma < 1$ then the series is convergent and if $\gamma > 1$ then the series is divergent. If $\gamma = 1$ both is possible.

Ratio test applied to power series

Ratio test

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For the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

consider

$$\gamma = \lim_{n\to\infty} \left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| = |x-c| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

If $0 < \gamma < \infty$, it depends on x whether $\gamma < 1$.

Power series in calculus vs. computer science

Different points of view towards a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

1. In Calculus: method to represent function f in another way.
2. In Discrete Mathematics/Computer Science: method for manipulating sequences $(a_n)_{n\in\mathbb{N}_0}$ (usually with $c=0$, the coefficients being the elements of the sequence).

Common term: Generating function of the sequence (a_n).
Example: $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ generating function of sequence $(1, 1, 1, 1, \dots)$.
Many manipulation methods useful for both, but:
Convergence problems almost exclusively considered from Calculus viewpoint, while radius of convergence $r = 0$ usually does not matter from the Discrete Mathematics viewpoint.

Power series, function vs. sequence

In the sequel $c = 0$. (Taylor series with $c = 0$ are also called Maclaurin series)

Function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

representing the sequence of coefficients $(a_0, a_1, a_2, a_3, \ldots)$.

Power series of $k \cdot f(x)$, $k \in \mathbb{R}$:

$$k \cdot f(x) = k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} k a_n x^n,$$

representing the sequence of coefficients $(ka_0, ka_1, ka_2, ka_3, \ldots)$.

Power series of $x \cdot f(x)$:

$$x \cdot f(x) = x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n,$$

representing the sequence of coefficients $(0, a_0, a_1, a_2, a_3, \ldots)$.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad g(x) = \sum_{n=0}^{\infty} b_n x^n,$$

with radius of convergence r of f and s of g.

Sums and differences

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$

with radius of convergence $\min\{r, s\}$.

Product

$$f(x) \cdot g(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$ with radius of convergence $\min\{r, s\}$.

Algebraic operations, example

Product

$$f(x) \cdot g(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$ with radius of convergence $\min\{r, s\}$.

Example: If $(b_n) = (1, 1, 1, \dots)$ (i.e. $\sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} x^n$ is the geometric series) then

$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \sum_{i=0}^{n} a_i$,

e.g. $\dfrac{1}{(1-x)^2} = \left(\sum_{n=0}^{\infty} x^n\right)^2 = \sum_{n=0}^{\infty} (n+1) x^n$ for $-1 < x < 1$.

What is the power series of the derivative $f'(x)$?

Derivatives

If $\sum_{n=0}^{\infty} a_n x^n$ converges to $f(x)$ on an interval $(-r; r)$, $r > 0$, then f is differentiable on $(-r; r)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

for all $x \in (-r; r)$.

Sequence: $(a_0, a_1, a_2, a_3, \dots)$
Sequence of derivative: $(a_1, 2a_2, 3a_3, 4a_4, \dots)$
Multiply a_n by n and shift it one position to the left ($0a_0 = 0$ vanishes).
Example: Geometric series $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$:

Derivative $f'(x) = \dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$.

Manipulation of power series, example

What is the function h of the power series with sequence of coefficients $(a_n) = (1, 1, 2, 2, 3, 3, 4, 4, \ldots)$?
Looks vaguely similar to the sequence $(1, 2, 3, 4, \ldots)$ corresponding to the function $f(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$.
New trick: substitute x^2 for x, then
$g(x) = f(x^2) = \frac{1}{(1-x^2)^2} = \sum_{k=0}^{\infty}(k+1)x^{2k}$.
Sequence of coefficients $(b_n)_{n\in\mathbb{N}_0}$? Observe that $b_1, b_3, b_5, \ldots = 0$, so $(b_n) = (1, 0, 2, 0, 3, \ldots)$, which gives the correct coefficients at the even positions. For the correct coefficients at the odd positions, shift sequence one to the right, how?
Multiplying by x gives $xg(x) = \frac{x}{(1-x^2)^2} = \sum_{k=0}^{\infty}(k+1)x^{2k+1}$ corresponding to the sequence $(c_n) = (0, 1, 0, 2, 0, 3, \ldots)$.
Therefore $a_n = b_n + c_n$ and

$$h(x) = g(x) + xg(x) = \frac{1+x}{(1-x^2)^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Radius of convergence (apply ratio test): $r = 1$.

What is the power series of the integral $\int_0^x f(t)\,dt$?

Integral of power series

If $\sum_{n=0}^{\infty} a_n x^n$ converges to $f(x)$ on an interval $(-r;r)$, $r>0$, then f is integrable on $(-r,r)$ and for $x \in (-r,r)$

$$\begin{aligned}\int_0^x f(t)\,dt &= \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \\ &= a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \ldots .\end{aligned}$$

Coefficients b_n of $\displaystyle\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{n=0}^{\infty} b_n x^n$:

$b_0 = 0$, $b_n = \frac{a_{n-1}}{n}$, $n \geq 1$, $(b_n) = (0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \ldots)$.

Example: Integral of power series

Determine the power series of $\ln(1+x)$.
Note that $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$.
Start with the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } -1 < x < 1.$$

Substitute $-t$ for x:

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n \quad \text{for } -1 < t < 1.$$

Integrate both sides:

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\, dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \text{ for } -1 < t < 1. \end{aligned}$$