Matematiske metoder (MM 529)

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Improper single integrals:

$$\int_a^b f(x)\,dx \quad \text{improper, if}$$

1. $a = -\infty$ or $b = +\infty$ (infinite integration limits), or
2. $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to b^-} f(x) = \pm\infty$ (integrand unbounded at an endpoint).

Similar for double integrals:

$$\iint_D f(x,y)\,dx\,dy \quad \text{improper, if}$$

1. D is unbounded, or
2. $\lim_{(x,y)\to(a,b)} f(x,y) = \pm\infty$ for a point $(a,b) \in D \cup \partial D$ (integrand unbounded approaching a point in D or on its boundary).

If $f(x, y) \geq 0$ on D, and f is integrable, then the improper integral

$$\iint_D f(x, y)\, dx\, dy$$

either exists, i.e.

$$\iint_D f(x, y)\, dx\, dy = a \geq 0,$$

or it diverges to $+\infty$.

Similarly:

If $f(x, y) \leq 0$ on D, and f is integrable, then the improper integral

$$\iint_D f(x, y)\, dx\, dy$$

either exists, i.e.

$$\iint_D f(x, y)\, dx\, dy = a \leq 0,$$

or it diverges to $-\infty$.

How to calculate improper integrals?
Sometimes iteration helps:
Example:

$$\iint_D e^{-x^2}\,dx\,dy, \text{ where}$$

$D = \{(x, y) \mid x \geq 0, y \leq |x|\}$.
D unbounded, function value $f(x, y) = e^{-x^2}$ only depends on x, not on y.
Iteration:

$$\begin{aligned}
\iint_D e^{-x^2}\,dx\,dy &= \int_0^\infty dx \int_{-x}^{x} e^{-x^2}\,dy \\
&= \int_0^\infty e^{-x^2} dx \int_{-x}^{x} 1\,dy \\
&= \int_0^\infty 2xe^{-x^2} dx,
\end{aligned}$$

integral of a function of one variable x.

Improper double integrals, example, contd.

$$\iint_D e^{-x^2}\, dx\, dy, \text{ where}$$

$D = \{(x, y) \mid x \geq 0, y \leq |x|\}$.
Substituting $t = -x^2$, $dt = -2x\, dx$, we obtain

$$\int 2xe^{-x^2} dx = -\int e^t dt = -e^t + C = -e^{-x^2} + C.$$

Therefore

$$\begin{aligned}
\iint_D e^{-x^2}\, dx\, dy &= \int_0^\infty 2xe^{-x^2} dx \\
&= \lim_{t\to\infty} \int_0^t 2xe^{-x^2} dx \\
&= \lim_{t\to\infty} (-e^{-t^2} + e^{-0^2}) = -0 + 1 = 1.
\end{aligned}$$

Improper double integrals, unbounded integrand

Example:

$$\iint_D \frac{1}{(x+y)^2}\,dx\,dy, \text{ where}$$

$D = \{(x,y)\,|\,0 \le x \le 1, 0 \le y \le x^2\}$.
D bounded, but $f(x,y) = \frac{1}{(x+y)^2}$ unbounded, as $(x,y) \to (0,0)$.
Iteration:

$$\begin{aligned}
\iint_D \frac{1}{(x+y)^2}\,dx\,dy &= \int_0^1 dx \int_0^{x^2} \frac{1}{(x+y)^2}\,dy \\
&= \lim_{t\to 0^+} \int_t^1 dx \int_0^{x^2} \frac{1}{(x+y)^2}\,dy \\
&= \lim_{t\to 0^+} \int_t^1 dx \left(-\frac{1}{x+y}\right)\Bigg|_{y=0}^{y=x^2} \\
&= \lim_{t\to 0^+} \int_t^1 \left(\frac{1}{x} - \frac{1}{x^2+x}\right)\,dx = \int_0^1 \frac{1}{x+1}\,dx \\
&= \ln 2 - \ln 1 = \ln 2.
\end{aligned}$$

Example:

$$\iint_D \frac{1}{xy}\,dx\,dy, \text{ where}$$

$D = \{(x,y)\,|\,0 \le x \le 1,\, x \ge y \ge x^2\}$.
D bounded, but $f(x,y) = \frac{1}{xy}$ unbounded, as $(x,y) \to (0,0)$.
Iteration:

$$\begin{aligned}\iint_D \frac{1}{xy}\,dx\,dy &= \int_0^1 \frac{dx}{x} \int_{x^2}^{x} \frac{dy}{y}\\ &= \int_0^1 \frac{1}{x}(\ln x - \ln x^2)\,dx\\ &= -\int_0^1 \frac{\ln x}{x}\,dx.\end{aligned}$$

Substitution $t = \ln x$, $dt = \frac{dx}{x}$, yields

$$\int \frac{\ln x}{x}\,dx = \int t\,dt = \frac{t^2}{2} + C = \frac{\ln^2 x}{2} + C.$$

Improper double integrals, unbounded integrand, example contd.

Example:

$$\iint_D \frac{1}{xy}\, dx\, dy, \text{ where}$$

$D = \{(x, y) \,|\, 0 \le x \le 1,\ x \ge y \ge x^2\}$.

Iteration:

$$\iint_D \frac{1}{xy}\, dx\, dy \quad = \quad -\int_0^1 \frac{\ln x}{x}\, dx.$$

Substitution $t = \ln x$, $dt = \frac{dx}{x}$, yields

$$\int \frac{\ln x}{x}\, dx = \int t\, dt = \frac{t^2}{2} + C = \frac{\ln^2 x}{2} + C.$$

Therefore

$$\iint_D \frac{1}{xy}\, dx\, dy = -\lim_{t \to 0^+} \int_t^1 \frac{\ln x}{x}\, dx = -\lim_{t \to 0^+} (\ln^2 1 - \ln^2 t) = +\infty.$$

Mean value theorem, single integrals:

Mean value theorem for the area

If f is continuous on the open interval (a, b) then there is a $c \in (a, b)$ such that the area under f on $[a, b]$ is

$$A = \int_a^b f(x)\,dx = f(c)(b-a).$$

Double integrals determine a volume.
A set $D \subseteq \mathbb{R}^2$ is connected, if any two points in D are joined by a continuous curve in D.

Mean value theorem for the volume, double integrals

If f is continuous on the closed, bounded and connected set $D \subseteq \mathbb{R}^2$ then there is an $(a, b) \in D$, such that the volume under f on D is

$$V = \iint_D f(x, y)\,dx\,dy = f(a, b) \cdot \mathrm{area}(D).$$

Mean value theorem for the volume, double integrals

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Average value of f over D

The average (or mean) value of the integrable function f over the domain $D \subseteq \mathbb{R}^2$ is

$$\overline{f} = \frac{1}{\mathrm{area}(D)} \iint_D f(x, y)\, dx\, dy.$$

Example: The average value of x over a domain D with area A is

$$\overline{x} = \frac{1}{A} \iint_D x\, dx\, dy.$$

Polar coordinates

Every point $P \in \mathbb{R}^2$ can be represented in
Cartesian coordinates: $P = (x, y)$, or in
Polar coordinates: $P = (r, \theta)$,
r distance from the origin $(0, 0)$, θ angle with the x-axis.
(Other coordinate systems used as well).
Relations between cartesian and polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta;$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Integration of functions f over domain D in polar coordinates, why?

1. f representable in an easier way in polar coordinates (e.g. f only depends on r), or
2. D representable in an easier way in polar coordinates (e.g. D is a sector of a circle)

Riemann sums in cartesian coordinates:
Partition P_k of D by rectangular grid into grid cells
$R_{ij} = \{(x, y) \,|\, x_{i-1} \leq x \leq x_i,\, y_{j-1} \leq y \leq y_j\}$
of area $A_{ij} = \Delta x_i \cdot \Delta y_j = (x_i - x_{i-1}) \cdot (y_j - y_{j-1})$.
Riemann sums in polar coordinates:
Partition P_k of D by polar grid into grid cells
$R_{ij} = \{(r, \theta) \,|\, r_{i-1} \leq r \leq r_i,\, \theta_{j-1} \leq \theta \leq \theta_j\}$
of area

$$\begin{aligned} A_{ij} &= (r_i^2 - r_{i-1}^2)\frac{(\theta_i - \theta_{i-1})}{2} \\ &= \frac{r_i + r_{i-1}}{2}\Delta r \Delta\theta \\ &\approx r_i \Delta r \Delta\theta, \end{aligned}$$

as $r_i - r_{i-1} \to 0$.

$$dx\, dy = dA = r\, dr\, d\theta.$$

Example:
Calculate the volume V under the graph of $f(x, y) = 1 - x^2 - y^2$ over the unit circle:

$$V = \iint_D (1 - x^2 - y^2)\, dx\, dy,$$

where $D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$.
Here the domain D is a cycle and the function f only depends on $r^2 = x^2 + y^2$. Changing to polar coordinates

$$V = \iint_D (1 - x^2 - y^2)\, dx\, dy = \iint_D (1 - r^2) r\, dr\, d\theta.$$

Iteration yields

$$\begin{aligned} V &= \iint_D (1 - r^2) r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 (1 - r^2) r\, dr \\ &= \int_0^{2\pi} d\theta \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_{r=0}^{r=1} = \frac{1}{4} \int_0^{2\pi} d\theta = \frac{\pi}{2}. \end{aligned}$$

Area of a polar region

What is the area A of the region D between the rays $\theta = \alpha$, $\theta = \beta$, and the graph of $r = f(\theta)$, $\alpha \le \theta \le \beta$?

$$\begin{aligned} A &= \iint_D dx\, dy = \iint_D r\, dr\, d\theta \\ &= \int_\alpha^\beta d\theta \int_0^{f(\theta)} r\, dr = \frac{1}{2}\int_\alpha^\beta (f(\theta))^2 d\theta. \end{aligned}$$

Example: Detour via double integrals. Determine the improper integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

The improper integral converges, and, obviously, $I = \int_{-\infty}^{\infty} e^{-y^2}\, dy$, so

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy \\ &= \iint_{\mathbb{R}^2} e^{-r^2} r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r\, dr \\ &= \int_0^{2\pi} d\theta \lim_{R\to\infty} \left(-\frac{1}{2} e^{-r^2}\right)\Big|_0^R = \frac{1}{2}\int_0^{2\pi} d\theta = \pi. \end{aligned}$$

Since $I \geq 0$ we obtain $I = \sqrt{\pi}$.