Matematiske metoder (MM529)

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Sequence (of real numbers)

$$a : \mathbb{N}_0 \to \mathbb{R}$$

Written:
$(a(0), a(1), a(2), a(3), \dots) = (a_0, a_1, a_2, a_3, \dots) = (a_n)_{n \in \mathbb{N}_0}$.

Examples:
$(\frac{1}{2^n})_{n \in \mathbb{N}_0} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$
$((-1)^n)_{n \in \mathbb{N}_0} = (1, -1, 1, -1, 1, \dots)$

Convergence and the limit of a sequence

A sequence $(a_n)_{n\in\mathbb{N}_0}$ is convergent to a limit $\gamma \in \mathbb{R}$, if for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}_0$ such that $|\gamma - a_n| < \varepsilon$ for all $n > n_0$.

$$\lim_{n\to\infty} a_n = \gamma \quad \text{or short} \quad (a_n) \to \gamma \quad \text{or} \quad a_n \to \gamma.$$

Otherwise the sequence is divergent.

Infinite limits

Let (a_n) be a divergent sequence. If for every $K \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}_0$ such that for every $n > n_0$

- $a_n > K$ then we say $\lim\limits_{n\to\infty} a_n = \infty$;
- $a_n < K$ then we say $\lim\limits_{n\to\infty} a_n = -\infty$.

Examples: $\lim\limits_{n\to\infty} n = \infty$, $\lim\limits_{n\to\infty} \ln \dfrac{1}{n} = -\infty$.

Limit laws for convergent series

Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then

(1) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(2) $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$;

(3) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, if $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}_0$;

(4) $\lim_{n\to\infty}(a \cdot b_n) = a \cdot b$ for $a \in \mathbb{R}$.

Example: $\lim_{n\to\infty} \left[(1 + \frac{1}{n}) \cdot (1 - \frac{1}{n})\right] = 1 \cdot 1 = 1.$

$$\lim_{n\to\infty} 1 + \frac{1}{n} = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} = 1 + 0 = 1,$$

$$\lim_{n\to\infty} 1 - \frac{1}{n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1.$$

The $\sum$-notation

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n.$$

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + \ldots + 99 + 100 = 5050 \quad \text{(little Gauß)}.$$

$$\sum_{i=0}^{n} x^i = \begin{cases} n+1 & \text{if } x = 1. \\ \frac{x^{n+1}-1}{x-1} & \text{if } x \neq 1, \end{cases}$$

since $(x^n + x^{n-1} + \ldots + x + 1)(x-1) = x^{n+1} - 1$ (see exercises)!

Example

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2.$$

Infinite sum

For a sequence $(a_i)_{i \in \mathbb{N}_0}$

$$\sum_{i=0}^{\infty} a_i = \lim_{n \to \infty} s_n,$$

where $s_n = \sum_{i=0}^{n} a_i$ is the nth partial sum.

Sequence $(s_n)_{n \in \mathbb{N}_0}$ of partial sums in example:
$\left(1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, \ldots\right) = \left(\frac{1}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots\right) = \left(\frac{2^{n+1}-1}{2^n}\right)_{n \in \mathbb{N}_0}$.

$$\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

Sequence $(s_n)_{n\in\mathbb{N}_0}$ of partial sums:

$$\left(1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}, \dots\right) = \left(\frac{2^{n+1}-1}{2^n}\right)_{n\in\mathbb{N}_0} = \left(2-\frac{1}{2^n}\right).$$

Limit of sequence of partial sums:

$$\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2-\frac{1}{2^n}\right) = \lim_{n\to\infty} 2 - \lim_{n\to\infty}\frac{1}{2^n} = 2-0 = 2.$$

Note: $\displaystyle\sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{6} = 1.64\dots$ but $\displaystyle\sum_{i=1}^{\infty}\frac{1}{i} = \infty$ (since $\displaystyle\ln(n+1) \le \sum_{i=1}^{n}\frac{1}{i}$) but $\displaystyle\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \dots = \ln 2.$

Leftsided and rightsided limit

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. The leftsided (rightsided) limit of f at a is

$$\lim_{x \to a^-} f(x) = \gamma \quad \left(\text{or} \quad \lim_{x \to a^+} f(x) = \gamma \right),$$

if for any convergent sequence $a_n \to a$ with $a_n \in D$, $a_n < a$ ($a_n > a$, resp.)

$$\lim_{n \to \infty} f(a_n) = \gamma.$$

Limit of a function

The limit of f as $x \to a$ is $\gamma \in \mathbb{R} \cup \{\infty, -\infty\}$, written

$$\lim_{x \to a} f(x) = \gamma,$$

if

$$\lim_{x \to a^-} f(x) = \gamma = \lim_{x \to a^+} f(x).$$

Alternative definition of one-sided limits

Leftsided and rightsided limit

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. The leftsided (rightsided) limit of f at a is

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if for any convergent sequence $a_n \to a$ with $a_n \in D$, $a_n < a$ ($a_n > a$, resp.)

$$\lim_{n \to \infty} f(a_n) = \gamma.$$

Hard to verify ("for any convergent sequence"), therefore:

Leftsided and rightsided limit, (ε, δ) formulation

$$\lim_{x \to a^-} f(x) = \gamma \quad \left(\text{or} \quad \lim_{x \to a^+} f(x) = \gamma \right), \quad \gamma \in \mathbb{R},$$

if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\gamma - f(x)| < \varepsilon$ whenever $0 < a - x < \delta$ ($0 < x - a < \delta$, resp.).

Leftsided and rightsided limit, (ε, δ) formulation

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$$\lim_{x \to 0^-} \cos x = 1 = \lim_{x \to 0^+} \cos x = \lim_{x \to 0} \cos x.$$

Leftsided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \quad \left(\text{or} \quad \lim_{x \to a^-} f(x) = -\infty \right),$$

if for any $K \in \mathbb{R}$ there is a $\delta > 0$ such that $f(x) > K$ ($f(x) < K$) whenever $0 < a - x < \delta$ (rightsided infinite limits similar).

Examples: $\lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty; \qquad \lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty.$

Limits at infinity

$$\lim_{x\to\infty} f(x) = \gamma \quad \left(\text{or} \quad \lim_{x\to-\infty} f(x) = \gamma\right), \; \gamma \in \mathbb{R},$$

if for any $\varepsilon > 0$ there is a $x_0 \in \mathbb{R}$ such that $|\gamma - f(x)| < \varepsilon$ whenever $x > x_0$ ($x < x_0$, resp.).

Limits at infinity

$$\lim_{x\to\infty} f(x) = \infty \quad \left(\text{or} \quad \lim_{x\to\infty} f(x) = -\infty\right)), \; \gamma \in \mathbb{R},$$

if for any $K \in \mathbb{R}$ there is a $x_0 \in \mathbb{R}$ such that $f(x) > K$ ($f(x) < K$, resp.) whenever $x > x_0$.

Examples: $\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}$.

$$\lim_{x\to\infty} x^2 = \infty = \lim_{x\to-\infty} x^2.$$

Limit laws for functions (analogues hold for onesided limits)

Suppose that $\lim_{x \to a} f(x) = s$ and $\lim_{x \to a} g(x) = t$. Then

(1) $\lim_{x \to a} (f(x) + g(x)) = s + t$;

(2) $\lim_{x \to a} (f(x) \cdot g(x)) = s \cdot t$;

(3) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{s}{t}$, if $t \neq 0$ and $g(x) \neq 0$ for all $x \in (a - \delta, a + \delta)$ for a $\delta > 0$.

Squezze (or Sandwich) theorem (leftsided limit)

If $f(x) \leq g(x) \leq h(x)$ for all $x \in (x_0 - \delta, x_0)$, $\delta > 0$, and

$$\lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^-} h(x)$$

then

$$\lim_{x \to x_0^-} g(x) = L.$$

Example: $\lim_{x \to 0} g(x)$, where $g(x) = x \sin \frac{1}{x}$.
For $f(x) = x$ and $h(x) = -x$ we have $f(x) \leq g(x) \leq h(x)$ for all $x \in (-1, 0)$. Therefore:

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^-} h(x).$$

A limit theorem

Squezze (or Sandwich) theorem (rightsided limit)

If $f(x) \leq g(x) \leq h(x)$ for all $x \in (x_0, x_0 + \varepsilon)$, $\varepsilon > 0$, and

$$\lim_{x \to x_0^+} f(x) = L = \lim_{x \to x_0^+} h(x)$$

then

$$\lim_{x \to x_0^+} g(x) = L.$$

Example: $\lim_{x \to 0} g(x)$, where $g(x) = x \sin \frac{1}{x}$.
For $f(x) = x$ and $h(x) = -x$ we have $h(x) \leq g(x) \leq f(x)$ for all $x \in (0, 1)$. Therefore:

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} f(x) = 0.$$

Altogether: $\displaystyle \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = \lim_{x \to 0} g(x) = 0.$

Continuous function

A function f is continuous in x_0, if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

f is continuous on (a, b), if it is continuous for all x_0 with $a < x_0 < b$.

f is continuous in x_0 if the leftsided limit, the rightsided limit and the function value are equal,

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

Examples: $f(x) = \ln x$ is continuous on $(0, \infty)$.

$$g(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0. \\ 0 & \text{if } x = 0, \end{cases} \qquad \text{is not continuous in } x_0 = 0.$$

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$$h_1(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3. \\ 6 & \text{if } x = 3, \end{cases} \qquad \text{is continuous in } (-\infty, \infty) = \mathbb{R}.$$

$$h_2(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3. \\ 7 & \text{if } x = 3, \end{cases} \qquad \text{is not continuous in } x_0 = 3.$$

Continuous function

A function f is continuous in x_0, if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

f is continuous on (a, b), if it is continuous for all x_0 with $a < x_0 < b$.

Further examples:

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \qquad \text{is not continuous in } x_0 = 0.$$

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \qquad \text{is continuous in } x_0 = 0.$$

By the sandwich theorem,

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = g(0) = 0.$$

Intermediate value theorem:

Let f be a continuous function on the closed interval $[a, b]$.
If $L \in (f(a), f(b))$ (or $L \in (f(b), f(a))$ if $f(b) < f(a)$)
then there is a c in the open interval (a, b) such that $f(c) = L$.

Every value between $f(a)$ and $f(b)$ is attained by the function on the interval (a, b).

Difference quotient

The difference quotient of f for x and x_0 is the quotient

$$\frac{f(x) - f(x_0)}{x - x_0}$$

of the difference of the function values $f(x) - f(x_0)$ and the difference $x - x_0$.

Alternative representation for $x = x_0 + h$:

$$\frac{f(x_0 + h) - f(x_0)}{h}.$$

Geometric interpretation: The difference quotient is the slope of the secant line through the points $(x, f(x))$ and $(x_0, f(x_0))$.

Derivative

The derivative of f in x_0 is the limit of the difference quotients

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

A function f' is called the derivative of f on the interval (a, b), if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

for every $x_0 \in (a, b)$. The function f is called differentiable in x_0 (on (a, b), resp.) if the limit exists.

Geometric interpretation: The value $f'(x_0) \in \mathbb{R}$ is the slope of the tangent line at the graph of f in the point $(x_0, f(x_0))$.
Further notation of derivatives: $\frac{df}{dx} = \frac{d}{dx} f(x)$ (Leibniz notation)
Physics: functions $y(t)$: $\dot{y} = \dot{y}(t)$ (Newton notation).

Examples of derivatives

Derivative

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Linear functions: $f(x) = ax + b$.

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{ax + b - (ax_0 + b)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{a(x - x_0)}{x - x_0} = a. \end{aligned}$$

Quadratic functions: $f(x) = ax^2 + bx + c$.

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{ax^2 + bx + c - (ax_0^2 + bx_0 + c)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{a(x^2 - x_0^2) + b(x - x_0)}{x - x_0} = \lim_{x \to x_0} a(x + x_0) + b \\ &= 2ax_0 + b. \end{aligned}$$

Example:

$$f(x) = |x|.$$

is continuous in $x = 0$:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0.$$

But:

$$-1 = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} \neq \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 1,$$

therefore f is not differentiable in $x_0 = 0$.

Necessary condition to be differentiable

If f is differentiable in x_0 then it must be continuous in x_0.
The converse does not hold (see example).

Derivative of the sum

$$\begin{aligned}
\frac{d}{dx}(f(x_0)+g(x_0)) &= \lim_{x\to x_0} \frac{f(x)+g(x)-(f(x_0)+g(x_0))}{x-x_0} \\
&= \lim_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0} \right) \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \\
&= \frac{d}{dx} f(x_0) + \frac{d}{dx} g(x_0).
\end{aligned}$$

Rules for derivatives

$$(f(x)+g(x))' = f'(x) + g'(x).$$

Derivative of the product

$$\begin{aligned}
\frac{d}{dx}(f(x)\cdot g(x)) &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h)-f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0}\left(\frac{g(x+h)(f(x+h)-f(x))}{h}+\frac{f(x)(g(x+h)-g(x))}{h}\right) \\
&= \lim_{h\to 0} g(x+h)\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h} \\
&= g(x)\frac{df}{dx}(x)+f(x)\frac{dg}{dx}(x).
\end{aligned}$$

Rules for derivatives

$$(f(x)+g(x))' = f'(x)+g'(x),$$

$$(f(x)\cdot g(x))' = f'(x)g(x)+f(x)g'(x).$$

Rules for derivatives

If $f(x)$ and $g(x)$ are differentiable functions in x, then

$$(f(x) \pm g(x))' = f'(x) \pm g'(x), \quad \text{(Sums and differences)}$$

$$(f(x) \cdot g(x))' = f(x)g'(x) + f'(x)g(x), \quad \text{(Product rule)}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}, \quad \text{if } g(x) \neq 0. \text{ (Quotient rule)}$$

Chain rule

If f and g are functions such that f is differentiable on the range of g, then the derivative of the composition is

$$(f \circ g)'(x) = f'(g(x))g'(x).$$