# Matematiske metoder (MM 529)

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Some integrals not expressable in standard functions (or you just do not know a method to solve the integral).
Example: Sine integral

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

not expressable in standard functions.
Idea: Replace function by its power series and manipulate the power series instead.
Example:

$$E(x) = \int_0^x e^{-t^2}\, dt.$$

Calculate $E(1)$ up to an error of less than $\frac{1}{1000}$.
Start with the Taylor series of $e^x$, $c = 0$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

for every $x \in \mathbb{R}$.

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for every $x \in \mathbb{R}$.
Substitute $-t^2$ for $x$:

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = 1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$$

for every $t \in \mathbb{R}$.

Example:

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Calculate $E(1)$ up to an error of less than $\frac{1}{1000}$.

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$$

for every $t \in \mathbb{R}$.

$$\begin{aligned}\int_0^x e^{-t^2}\, dt &= \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}\, dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n \frac{t^{2n}}{n!}\, dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}\end{aligned}$$

for every $x \in \mathbb{R}$.

Example:

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Calculate $E(1)$ up to an error of less than $\frac{1}{1000}$.

$$E(x) = \int_0^x e^{-t^2}\,dt \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$

for every $t \in \mathbb{R}$.

$$\begin{aligned} E(1) = \int_0^1 e^{-t^2}\,dt \quad &= \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \\ &= \quad 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} + \gamma \end{aligned}$$

where $0 \geq \gamma \geq \frac{-1}{5!\cdot 11} = \frac{-1}{1320}$ by the error criterion for alternating series. We obtain $0.7465 < E(1) < 0,7475$.

(First) binomial formula: $(x+y)^2 = x^2 + 2xy + y^2$.
What is $(x+y)^n$ ?

$$\begin{aligned} (x+y)^3 &= (x+y)\cdot(x+y)\cdot(x+y) \\ &= x^3 + x^2y + xyx + xy^2 + yx^2 + yxy + y^2x + y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3. \end{aligned}$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Each summand of $(x+y)^n$ has the form $x^k y^{n-k}$, $0 \le k \le n$.
Denote by $\binom{n}{k}$ the number of summands of the form $x^k y^{n-k}$, then $\binom{n}{0} = 1 = \binom{n}{n}$. Terms $\binom{n}{k}$ are called binomial coefficients.

## Binomial Theorem

For every $n \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

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Why Binomial Theorem?

Lat. *nomen*: name.

*Bi-nomial*: two names: two variables (i.e. $x$ and $y$).

Calculating binomial coefficients $\binom{n}{k}$:

Substitute variable $y$ by $a \in \mathbb{R}$ and calculate Taylor series of $f(x) = (x+a)^n$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Calculating binomial coefficients $\binom{n}{k}$:
Substitute variable $y$ by $a \in \mathbb{R}$ and calculate Taylor series of $f(x) = (x+a)^n$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

$f^{(0)}(x) = (x+a)^n$.
$f^{(1)}(x) = n \cdot (x+a)^{n-1}$.
$f^{(2)}(x) = n(n-1) \cdot (x+a)^{n-2}$.
$f^{(k)}(x) = n(n-1)\cdots(n-k+1) \cdot (x+a)^{n-k}$, for $k \leq n$
$f^{(n)}(x) = n!$,
$f^{(k)}(x) = 0$, for $k > n$. Therefore

$$f(x) = (x+a)^n = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k,$$

where $f^{(k)}(0) = n(n-1)\cdots(n-k+1) \cdot a^{n-k}$ for $k \leq n$.

# Binomial Theorem, binomial coefficients

$$f(x) = (x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$$

Calculating binomial coefficients $\binom{n}{k}$:
$f^{(k)}(x) = n(n-1)\cdots(n-k+1)\cdot(x+a)^{n-k}$, for $k \leq n$,
$f^{(k)}(x) = 0$, for $k > n$. Therefore
$f^{(k)}(0) = n(n-1)\cdots(n-k+1)\cdot a^{n-k}$ for $k \leq n$, and

$$(x+a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k a^{n-k}.$$

Binomial coefficients

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

for $n \geq k \geq 0$.

## Binomial Theorem, binomial coefficients

$$(x+a)^n = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k a^{n-k}.$$

Substituting variable $y$ for $a \in \mathbb{R}$:

### Binomial Theorem

For every $n \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

$$\text{where } \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

## Binomial Series

$$(x+a)^n = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k a^{n-k}.$$

Substituting 1 for $a$:

### Binomial Series for integers

For every $n \in \mathbb{N}$

$$(1+x)^n = (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k,$$

$$\text{where } \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} \text{ if } k \leq n,$$

$$\text{and } \binom{n}{k} = 0 \text{ if } k > n.$$

## Binomial Series for integers

For every $n \in \mathbb{N}$

$$(1+x)^n = (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k,$$

$$\text{where } \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Observe: $\binom{n}{k} = 0$ if $k > n$, since the enumerator has a factor $= 0$.
Power series of the sequence of coefficients
$(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots) = (\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n-1}, \binom{n}{n}, 0, 0, \dots).$

# Binomial Recursion

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \text{ for } n, k \in \mathbb{N}_0.$$

## Binomial recursion

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \text{ if } n, k \in \mathbb{N}.$$

Proof:

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)\cdots(n-k)}{k!} + \frac{(n-1)\cdots(n-k+1)}{(k-1)!} \\
&= \left(\frac{n-k}{k} + 1\right) \frac{(n-1)\cdots(n-k+1)}{(k-1)!} \\
&= \frac{n}{k} \cdot \frac{(n-1)\cdots(n-k+1)}{(k-1)!} \\
&= \binom{n}{k}. \quad \square
\end{aligned}$$

## Binomial recursion

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \text{ if } n, k \in \mathbb{N}.$$

Pascal's Triangle for $\binom{n}{k}$:

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 2 | 1 | 0 | 0 | 0 | 0 |
| 3 | 1 | 3 | 3 | 1 | 0 | 0 | 0 |
| 4 | 1 | 4 | 6 | 4 | 1 | 0 | 0 |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | 0 |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

## Binomial Series for integers

For every $n \in \mathbb{N}$

$$(1+x)^n = (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k,$$

$$\text{where } \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Summation over binomial coefficients in a row (set $x = 1$):

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{\infty} \binom{n}{k}.$$

Alternating sum over binomial coefficients in a row (set $x = -1$):

$$0 = (1-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k}.$$

# Binomial Series for real exponents

## Binomial Series for integers

For every $n \in \mathbb{N}$

$$(1+x)^n = (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k,$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ for every $x \in \mathbb{R}$.

What is $\sqrt{(1+x)} = (1+x)^{1/2}$ ?

## Binomial Series

For every $r \in \mathbb{R}$

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k,$$

where $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$, if $-1 < x < 1$.

# Binomial Series for real exponents

## Binomial Series

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$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k,$$

where $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$, if $-1 < x < 1$.

For $|x| < 1$,

$$\begin{aligned}
(1+x)^{1/2} &= \sum_{k=0}^{\infty} \binom{1/2}{k} x^k \\
&= \binom{1/2}{0} x^0 + \binom{1/2}{1} x^1 + \binom{1/2}{2} x^2 + \binom{1/2}{3} x^3 + \dots \\
&= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2} x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!} x^3 + \dots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots
\end{aligned}$$

## Binomial Series

For every $z \in \mathbb{C}$

$$(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k,$$

where $\binom{z}{k} = \frac{z(z-1)\cdots(z-k+1)}{k!}$, if $-1 < x < 1$.

Example:

$$\binom{i}{2} = \frac{i(i-1)}{2} = \frac{1}{2}(-1-i),$$

the complex number whose real and imaginary parts are both $-\frac{1}{2}$. Complex numbers $z = a + ib$ in the exponent of a function:

$$\begin{aligned}(1+x)^z &= (e^{\ln(1+x)})^z = e^{(a+ib)\cdot\ln(1+x)} = e^{a\ln(1+x)} \cdot e^{ib\ln(1+x)} \\ &= (1+x)^a \cdot \left[\cos(b\ln(1+x)) + i\sin(b\ln(1+x))\right],\end{aligned}$$

by Euler's formula $e^{ix} = \cos x + i \sin x$.