Matematiske metoder (MM 529)

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A (definite single) integral determines the (signed) area between the function graph and the x-axis.
A double integral determines the (signed) volume between the function graph and the x, y-plane.
A triple integral measures a four-dimensional object between the (three-dimensional) function graph and the x, y, z-space.
B axis-parallel box in $\mathbb{R}^3$.

$$\iiint_B f(x, y, z)\, dx\, dy\, dz$$

thoroughly defined by subdividing B into smaller axis-parallel sub-boxes and forming the Riemann sum over all sub-boxes where the summands are of the form function value at an element of the box times volume of the box. Details omitted here!
Will apply iteration instead, like we did with double integrals.

Let D be a (not too pathological) subset of $\mathbb{R}^3$, and the constant function $f(x, y, z) = 1$. Then

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_D dx\,dy\,dz = \mathrm{Vol}(D).$$

Example (iteration):

$$I = \iiint_B (xy^2 + z^3)\,dx\,dy\,dz,$$

where the box $B = \{(x, y, z)\,|\, x, y, z \geq 0, x \leq a, y \leq b, z \leq c\}$.

$$\begin{aligned} I &= \int_0^c dz \int_0^b dy \int_0^a (xy^2 + z^3)\,dx \\ &= \int_0^c dz \int_0^b dy \left(\frac{1}{2}x^2y^2 + xz^3\right)\Bigg|_{x=0}^{x=a} \\ &= \int_0^c dz \int_0^b \left(\frac{1}{2}a^2y^2 + az^3\right) dy \end{aligned}$$

Example (iteration):

$$I = \iiint_B (xy^2 + z^3)\, dx\, dy\, dz,$$

where the box $B = \{(x, y, z) \,|\, x, y, z \geq 0, x \leq a, y \leq b, z \leq c\}$.

$$\begin{aligned}
I &= \int_0^c dz \int_0^b \left(\frac{1}{2}a^2 y^2 + az^3\right) dy \\
&= \int_0^c dz \left(\frac{1}{6}a^2 y^3 + ayz^3\right)\Bigg|_{y=0}^{y=b} \\
&= \int_0^c \left(\frac{1}{6}a^2 b^3 + abz^3\right) dz \\
&= \left(\frac{1}{6}a^2 b^3 z + \frac{1}{4}abz^4\right)\Bigg|_{z=0}^{z=c} = \frac{1}{6}a^2 b^3 c + \frac{1}{4}abc^4.
\end{aligned}$$

Example (more general domains):

$$I = \iiint_D y \, dx \, dy \, dz,$$

where D is the tetrahedron with vertices $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$.
Observation: only variable y occurs explicitly, therefore by iteration

$$I = \int_0^1 dy \iint_{T(y)} y \, dx \, dz = \int_0^1 y \, dy \iint_{T(y)} dx \, dz,$$

where $\iint_{T(y)} dx \, dz$ is just the area of the triangle $T(y)$, having side lengths $1-y$. Therefore

$$\iint_{T(y)} dx \, dz = \frac{1}{2}(1-y)^2$$

and

$$I = \int_0^1 \frac{1}{2} y (1-y)^2 \, dy = \left(\frac{1}{4} y^2 - \frac{1}{3} y^3 + \frac{1}{8} y^4 \right) \Big|_{y=0}^{y=1} = \frac{1}{24}.$$

Iteration of triple integrals over "nice" solids D can be performed in any order (but the order of iteration may influence whether we can solve the integral).
But: In general, the integration limits depend on the iteration order.
Previous example, D is the tetrahedron with vertices $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$.

$$I = \iiint_D y\,dx\,dy\,dz = \int_0^1 y\,dy \iint_{T(y)} dx\,dz,$$

can be iterated further

$$I = \int_0^1 y\,dy \int_0^{1-y} dx \int_0^{1-y-x} dz,$$

or

$$I = \int_0^1 y\,dy \int_0^{1-y} dz \int_0^{1-y-z} dx.$$

Triple integrals, iteration order

Six possible iteration orders for the tetrahedron D with vertices $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$ and $I = \iiint_D y\,dx\,dy\,dz$.

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} y\,dz,$$

$$I = \int_0^1 dy \int_0^{1-x} dz \int_0^{1-x-z} y\,dy,$$

$$I = \int_0^1 dy \int_0^{1-y} dx \int_0^{1-y-x} y\,dz,$$

$$I = \int_0^1 dy \int_0^{1-y} dz \int_0^{1-y-z} y\,dx,$$

$$I = \int_0^1 dz \int_0^{1-z} dx \int_0^{1-z-x} y\,dy,$$

$$I = \int_0^1 dz \int_0^{1-z} dy \int_0^{1-z-y} y\,dx,$$

Vectors in $\mathbb{R}^n$: A vector $\mathbf{v}$ is an object having a length and a direction (representable by an arrow).
Two vectors with the same length and same direction are considered equal (no matter where they are located in $\mathbb{R}^n$).
Ordered pair (P, Q) of two points in $\mathbb{R}^n$ determines a vector $\vec{PQ} = \mathbf{v}$ whose tail is P and whose head is Q.
The length $|\mathbf{v}|$ (or $||\mathbf{v}||$) of $\mathbf{v}$ is the Euclidean distance between P and Q.
Example:
$\mathbf{v}$ vector from $(1, 1)$ to $(2, 3)$ in $\mathbb{R}^2$.
$\mathbf{v}$ is also the vector from $(2, 3)$ to $(3, 5)$ and from $(0, 0)$ to $(1, 2)$.
The length $|\mathbf{v}| = \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5}$.
Representation of a vector: Coordinates $(a_1, a_2, \ldots .a_n)$ of the head, if the tail is $(0, 0, \ldots, 0)$ (position vector of the point $P = (a_1, a_2, \ldots .a_n)$).

Scalar multiples of $\mathbf{v} = (a_1, a_2, \ldots, a_n)$, multiplying $\mathbf{v}$ with the scalar $t \in \mathbb{R}$:

$$t \cdot \mathbf{v} = (t \cdot a_1, t \cdot a_2, \ldots, t \cdot a_n)$$

Length $|t \cdot \mathbf{v}| = |t| \cdot |\mathbf{v}|$.
$(-1) \cdot \mathbf{v}$ is the vector with the same length as $\mathbf{v}$ but the opposite direction.
Zero vector: $\mathbf{0} = (0, 0, \ldots, 0) = 0 \cdot \mathbf{v}$ for any vector $\mathbf{v}$.
Unit vector: Vector of length 1.
$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{|\mathbf{v}|} \cdot \mathbf{v}$ is the unit vector in the direction of $\mathbf{v}$.
$\mathbf{e_i}$: Unit vector in the direction of the ith coordinate axis.
$\mathbf{e_i} = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is the i-th coordinate.
Addition of vectors $\mathbf{v} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{w} = (b_1, b_2, \ldots, b_n)$:

$$\mathbf{v} + \mathbf{w} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

Subtraction: $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1) \cdot \mathbf{w}$.

Dot product (also called scalar product) of a vector $\mathbf{v} = (a_1, a_2, \ldots, a_n)$ with a vector $\mathbf{w} = (b_1, b_2, \ldots, b_n)$:

$$\mathbf{v} \bullet \mathbf{w} = a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_n \cdot b_n = \sum_{i=1}^{n} a_i \cdot b_i \in \mathbb{R}.$$

(Change of notation compared to lecture 7!)

Properties:

$\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}| \cdot |\mathbf{w}| \cdot \cos\theta$ where θ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$. (I.e. $\mathbf{v} \bullet \mathbf{w} = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$, or $\mathbf{w} = \mathbf{0}$, or $\theta = \frac{\pi}{2} = 90^o$).

Rules:

$\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$ (commutative law)

$\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$ (distributive law)

$t \cdot (\mathbf{v} \bullet \mathbf{w}) = (t \cdot \mathbf{v}) \bullet \mathbf{w} = \mathbf{v} \bullet (t \cdot \mathbf{w})$ if $t \in \mathbb{R}$

$\mathbf{v} \bullet \mathbf{v} = |\mathbf{v}|^2$.

Vector projection of $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u}_{\mathbf{v}} = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|^2} \cdot \mathbf{v}.$$

In $\mathbb{R}^2$:
Unit vectors in direction of the coordinate axes:
$\mathbf{i} = \mathbf{e_1} = (1, 0)$,
$\mathbf{j} = \mathbf{e_2} = (0, 1)$.
In $\mathbb{R}^3$:
Unit vectors in direction of the coordinate axes:
$\mathbf{i} = \mathbf{e_1} = (1, 0, 0)$,
$\mathbf{j} = \mathbf{e_2} = (0, 1, 0)$.
$\mathbf{k} = \mathbf{e_3} = (0, 0, 1)$.
Representation of the vector
$(a_1, a_2, a_3) = a_1 \cdot \mathbf{i} + a_2 \cdot \mathbf{j} + a_3 \cdot \mathbf{k}$

The cross product in $\mathbb{R}^3$

The cross product $\mathbf{u} \times \mathbf{v}$ of two vectors $\mathbf{u}, \mathbf{v} \in R^3$ is the unique vector in $\mathbb{R}^3$ satisfying the following three properties::

(i) $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{v} = 0$,

(ii) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

(iii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a right hand triad.

Geometric interpretation:
$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \sin\theta$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.
Calculating the cross product of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ with $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$:

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.$$

Cross products of unit vectors:
$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.
$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$.
$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Cross product

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.$$

Example: $(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (-2\mathbf{j} + 5\mathbf{k}) =$
$= (1 \cdot 5 - (-2) \cdot (-3))\mathbf{i} + ((-3) \cdot 0 - 5 \cdot 2)\mathbf{j} + (2 \cdot (-2) - 1 \cdot 0)\mathbf{k}$
$= -\mathbf{i} - 10\mathbf{j} - 4\mathbf{k}.$

Properties of the cross product $\times$

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, and $t \in \mathbb{R}$ then

(i) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$,

(ii) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$,

(iii) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$,

(iv) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,

(v) $(t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}) = t(\mathbf{u} \times \mathbf{v})$.

(vi) $(\mathbf{u} \times \mathbf{v}) = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$, or $\mathbf{v} = \mathbf{0}$, or both vectors are parallel.