Matematiske metoder (MM 529)

Stephan Brandt

Syddansk Universitet, Odense

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Linear differential equation (LDE)

A linear differential equation of order n is an ordinary differential equation of the form

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=g(x),$$

where $a_i(x)$ and $g(x)$ are functions (only) depending on x.

Homogeneous/inhomogeneous linear differential equation

A linear differential equation

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=g(x),$$

is homogeneous, if $g(x)=0$ (for every x), otherwise inhomogeneous.

Solution of first order homogeneous linear DEs

First order homogeneous linear differential equation:

$$y' = a(x)y$$

General solution:

$$y = ce^{A(x)},$$

where $c \in \mathbb{R}$ and $A'(x) = a(x)$.

Linearity of differentiation:

$$(f(x) + g(x))' = f'(x) + g'(x),$$

$$(c \cdot f(x))' = c \cdot f'(x).$$

First order inhomogeneous LDE:

$$y' + a(x)y = g(x).$$

Corresponding homogeneous LDE:

$$y' + a(x)y = 0.$$

Let $y_p(x)$ be a solution of the inhomogeneous LDE and $y_h(x), y_H(x)$ be solutions of the corresponding homogeneous LDE. Then $y_h + y_H$ is another solution of the homogeneous LDE and $y_h + y_p$ is another solution of the inhomogeneous LDE.

Inhomogeneous LDE:

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=g(x),$$

Corresponding homogeneous LDE:

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=0,$$

$y_p(x)$ solution of inhomogeneous LDE.
$y_h(x), y_H(x)$ solutions of corresponding homogeneous LDE.
Then $y_h + y_H$ is another solution of the homogeneous LDE and therefore $y_h + y_p$ is another solution of the inhomogeneous LDE.

Inhomogeneous LDE:

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=g(x),$$

Corresponding homogeneous LDE:

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=0,$$

Y_H the set of all solutions of the corresponding homogeneous LDE.
y_p any particular solution of the inhomogeneous LDE.
Then $Y_I=\{y_p+y_h : y_h\in Y_H\}$ is the set of solutions of the inhomogeneous LDE.
Problems left:

1. How to find all solutions of a homogeneous LDE.
2. How to find a particular solution of the inhomogeneous LDE.

Example: $y' = y - x$.
Corresponding homogeneous LDE: $y' = y$.
One particular solution: $y_p = x + 1$ (e.g. guess from the slope field and verify $1 = (x+1)' = (x+1) - x = 1$).
All solutions of corresponding homogeneous LDE:
$Y_H = \{ce^x : c \in \mathbb{R}\}$ (solve separable differential equation).
All solutions of inhomogeneous LDE: $Y_I = \{x + 1 + ce^x : c \in \mathbb{R}\}$.
Initial value problem: $y(0) = 0$: solve $0 = 0 + 1 + ce^0$ for c.
Thus $c = -1$ and $y(x) = x + 1 - e^x$ is the single solution.

First order homogeneous linear differential equation:

$$y' + a(x)y = g(x)$$

General solution of corresponding homogeneous equation:

$$y = ce^{-A(x)},$$

where $c \in \mathbb{R}$ and $A'(x) = a(x)$.

Assumption: A solution of the inhomogeneous equation has the form ("variation of constants")

$$y = k(x)e^{-A(x)}.$$

Then

$$k'(x)e^{-A(x)} - k(x)A'(x)e^{-A(x)} + a(x)k(x)e^{-A(x)} = y' + a(x)y = g(x),$$

and

$$k'(x)e^{-A(x)} = g(x).$$

Solving first order inhomogeneous LDEs

First order homogeneous linear differential equation:

$$y' + a(x)y = g(x).$$

Assumption: A solution of the inhomogeneous equation has the form ("variation of constants")

$$y = k(x)e^{-A(x)}.$$

Then

$$k'(x)e^{-A(x)} = g(x)$$

hence

$$k(x) = \int g(x)e^{A(x)}\,dx.$$

General solution of inhomogeneous equation:

$$y = k(x)e^{-A(x)} + ce^{-A(x)},$$

where $c \in \mathbb{R}$, $A'(x) = a(x)$, and $k(x) = \int g(x)e^{A(x)}\,dx$.

Second order inhomogeneous linear differential equation:

$$y'' + a(x)y' + b(x)y = g(x).$$

Structure of solutions:

All functions y of the form

$$y = y_p + y_h,$$

where y_p is one particular solution of the inhomogeneous equation and $y_h \in Y_H$, the set of solutions of the corresponding homogeneous differential equation

$$y'' + a(x)y' + b(x)y = 0.$$

Problems left:

1. How to find all solutions of a homogeneous LDE.
2. How to find one solution of the inhomogeneous LDE.

Second order linear differential equation:

$$y'' + a(x)y' + b(x)y = g(x).$$

Finding all solutions in general quite complicated.
Easier special case:
Second order linear differential equations with constant coefficients:

$$y'' + ay' + by = g(x), \quad a, b \in \mathbb{R} \text{ (constant coefficients)}.$$

Corresponding homogeneous equation.

$$y'' + ay' + by = 0.$$

Assumption: Solution has the form $y = e^{\lambda x}$ for some λ.
Then $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ and
$0 = \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = (\lambda^2 + a\lambda + b)e^{\lambda x}$.
Since $e^{\lambda x} > 0$ for all x, we have $\lambda^2 + a\lambda + b = 0$.

Homogeneous LDE with constant coefficients:

$$y'' + ay' + by = 0.$$

Assumption: Solution has the form $y = e^{\lambda x}$ for some λ.
Then λ must be a zero of the polynomial

$$p(\lambda) = \lambda^2 + a\lambda + b = 0.$$

$p(\lambda)$ characteristic polynomial of the homogeneous LDE.
If $\lambda \in \mathbb{R}$ is a zero of the characteristic polynomial, then $y = e^{\lambda x}$ is a solution of the homogeneous LDE, since

$$y'' + ay' + by = (\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Homogeneous LDE with constant coefficients:

$$y'' + ay' + by = 0.$$

Characteristic polynomial

$$p(\lambda) = \lambda^2 + a\lambda + b.$$

For a real zero λ of p, $y = e^{\lambda x}$ is one solution of corresponding homogeneous LDE. How to find all solutions?
Three cases:

1. p has two different zeroes $\lambda_1 \neq \lambda_2 \in \mathbb{R}$.
2. p has a double zero $\lambda \in \mathbb{R}$.
3. p has two complex zeroes $\lambda, \overline{\lambda} \in \mathbb{C} \setminus \mathbb{R}$.

Homogeneous LDE with constant coefficients:

$$y'' + ay' + by = 0.$$

First case: $p(\lambda) = \lambda^2 + a\lambda + b$ has two different zeroes $\lambda_1 \neq \lambda_2 \in \mathbb{R}$.
Then $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are both solutions.
By linearity, $y = c_1 y_1 + c_2 y_2$ is a solution for any $c_1, c_2 \in \mathbb{R}$, and all solutions can be written in this form.
Example:

$$y'' + y' - 6y = 0.$$

Characteristic polynomial: $p(\lambda) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$ has zeroes $\lambda_1 = -3$ and $\lambda_2 = 2$.
General solution: $y = c_1 e^{-3x} + c_2 e^{2x}$, $c_1, c_2 \in \mathbb{R}$.

Homogeneous LDE with constant coefficients:

$$y'' + ay' + by = 0.$$

Second case: $p(\lambda) = \lambda^2 + a\lambda + b$ has a double zero $\lambda \in \mathbb{R}$.
Then $y_1 = e^{\lambda x}$ is a solution, and $y = c_1 y_1$, $c_1 \in \mathbb{R}$ are further solutions. One more structurally different solution needed:
$y_2 = xe^{\lambda x}$ is a solution, since $y_2' = (\lambda x + 1)e^{\lambda x}$ and
$y_2'' = (\lambda^2 x + 2\lambda)e^{\lambda x}$.
$y_2'' + ay_2' + by_2 = e^{\lambda x}[(\lambda^2 + a\lambda + b)x + 2\lambda + a] = 0.$
General solution $y = c_1 e^{\lambda x} + c_2 xe^{\lambda x}$, $c_1, c_2 \in \mathbb{R}$.
Example:

$$y'' + 2y' + y = 0.$$

Characteristic polynomial: $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$ has the double zero $\lambda = -1$.
General solution: $y = c_1 e^{-x} + c_2 xe^{-x}$, $c_1, c_2 \in \mathbb{R}$.

Homogeneous LDE with constant coefficients:

$$y'' + ay' + by = 0.$$

Third case: $p(\lambda) = \lambda^2 + a\lambda + b$ has a zero $\lambda = p + iq \in \mathbb{C} \setminus \mathbb{R}$, then the conjugate complex number $\overline{\lambda}$ is another zero.
Then $y = e^{\lambda x}$ is a complex solution function, but we need two structurally different real solution functions:
Approach: Split up the function into its real and its imaginary part:
$y = e^{\lambda x} = y_1 + iy_2$, where $y_1(x)$ and $y_2(x)$ are real functions.
We know:
$0 = (e^{\lambda x})'' + a(e^{\lambda x})' + be^{\lambda x} = y_1'' + ay_1' + by_1 + i(y_2'' + ay_2' + by_2)$,
therefore y_1 and y_2 are both (real) solutions of the differential equation.
Finding real and complex part of $y = e^{(p+iq)x}$:
$e^{(p+iq)x} = e^{px} \cdot e^{iqx} = e^{px}(\cos qx + i \sin qx) =$
$e^{px} \cos qx + ie^{px} \sin qx$.
Real part: $y_1(x) = e^{px} \cos qx$, imaginary part $y_2(x) e^{px} \sin qx$.
General solution $y = c_1 e^{px} \cos qx + c_2 e^{px} \sin qx,\ c_1, c_2 \in \mathbb{R}$.

Homogeneous LDE with constant coefficients:

$$y'' + ay' + by = 0.$$

Third case: $p(\lambda) = \lambda^2 + a\lambda + b$ has a zero $\lambda = p + iq \in \mathbb{C} \setminus \mathbb{R}$,
Finding real and imaginary part of complex solution function $y = e^{(p+iq)x}$:
$e^{(p+iq)x} = e^{px} \cdot e^{iqx} = e^{px}(\cos qx + i \sin qx) =$
$e^{px} \cos qx + ie^{px} \sin qx$.
Real part: $y_1(x) = e^{px} \cos qx$, imaginary part $y_2(x) = e^{px} \sin qx$.
General solution $y = c_1 e^{px} \cos qx + c_2 e^{px} \sin qx$, $c_1, c_2 \in \mathbb{R}$.
Example:

$$y'' + y = 0.$$

Characteristic polynomial: $p(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$.
Complex solution function: $y = e^{ix} = \cos x + i \sin x$.
General solution: $y = c_1 \cos x + c_2 \sin x$, $c_1, c_2 \in \mathbb{R}$.