Matematiske metoder (MM529)

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Polynomial

A (real) polynomial is a function $p: \mathbb{R} \to \mathbb{R}$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where the coefficients $a_i \in \mathbb{R}$ for $i \in \{0, 1, ..., n-1, n\}$ and $a_n \neq 0$.

The number a_n is called the leading coefficient.

The degree $n = \deg p$ of p is the index of the leading coefficient.

The zero-polynomial p(x) = 0 is the function mapping every $x \in \mathbb{R}$ to 0.

Its degree is defined to be $-\infty$.

Multiplication of polynomials

Example: Multiplication of two polynomials

$$(x^3 - 3x^2 + 3x + 1)(2x^2 + x + 4) = 2x^5 - 5x^4 + 7x^3 - 7x^2 + 13x + 4.$$

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Two polynomials:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0.$$

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Product of polynomials

$$p(x) \cdot q(x) = c_{n+m}x^{n+m} + c_{n+m-1}x^{n+m-1} + \ldots + c_1x + c_0,$$

where $c_k = a_0 b_k + a_1 b_{k-1} + \ldots + a_{k-1} b_1 + a_k b_0$ (telescope sum).

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Degree $\deg(p \cdot q) = \deg p + \deg q$.



Rational functions

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Famous rational function

$$f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$$

 $f(x) = \frac{1}{x}$

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Examples of exponential functions

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Law of radioactive decay: $W(t) = W_0 e^{-\lambda t}$.

Decay rate λ only depending on the isotope.

A function $f: A \rightarrow B$ is called

- injective, if every $b \in B$ has at most one preimage,
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Interpretation in the graph

A function $f: \mathbb{R} \to \mathbb{R}$ is

- injective, if every horizontal line intersects the graph in at most one point,
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Note: Every injective function with the codomain restricted to the range is bijective.

Inverse function

For a bijective function $f: A \to B$, f(x) = y, there is a unique function $f^{-1}: B \to A$ with $f^{-1}(y) = x$, if and only if f(x) = y.

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 $f: [0, \infty) \to [0, \infty)$, $f(x) = x^2$ is bijective. Inverse?
 $f^{-1}: [0, \infty) \to [0, \infty)$, $f^{-1}(x) = \sqrt{x}$.

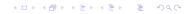
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What is the inverse of f^{-1} ? $(f^{-1})^{-1} = f$.



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Most common bases:

2 (computer sci.), 10 (economics), e = 2.718... (natural sci.).



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One bel more represents a noise whose amplitude is ten times bigger.

(e.g. 100 decibel is a thousend times louder than 70 decibel).

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long side hypothenuse,

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 $\tan : A \to \mathbb{R}, \quad \tan \theta = \frac{y}{x},$

where $A = \mathbb{R} \setminus \{(2k+1)\frac{\pi}{2} \mid k \in \mathbb{Z}\}$ (the cases where x = 0).

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Sums of angles:

$$\sin(\theta + \varphi) = \sin\theta \, \cos\varphi + \sin\varphi \, \cos\theta$$

$$\cos(\theta + \varphi) = \cos\theta \, \cos\varphi - \sin\theta \, \sin\varphi$$

Inverses of trigonometric functions

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Inverse functions

$$\begin{split} & \text{arcsin}: [-1,1] \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right]; \\ & \text{arccos}: [-1,1] \to [0,\pi]; \\ & \text{arctan}: \mathbb{R} \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right). \end{split}$$

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 $\big(0,1,1,2,3,5,8,13,21,\ldots\big)$

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Sequence (of real numbers)

$$a:\mathbb{N}_0\to\mathbb{R}$$

Written:

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 $((-1)^n)_{n\in\mathbb{N}_0} = (1, -1, 1, -1, 1, \ldots)$

Limits

Convergence and the limit of a sequence

A sequence $(a_n)_{n\in\mathbb{N}_0}$ is convergent to a limit $\gamma\in\mathbb{R}$ if for every $\varepsilon>0$ there is an $n_0\in\mathbb{N}_0$ such that $|\gamma-a_n|<\varepsilon$ for all $n>n_0$.

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Let (a_n) be a divergent sequence. If for every $K \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}_0$ such that for every $n > n_0$

- $a_n > K$ then we say $\lim_{n \to \infty} a_n = \infty$;
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Examples:
$$\lim_{n\to\infty} n = \infty$$
, $\lim_{n\to\infty} \ln \frac{1}{n} = -\infty$.



$$(a_0,a_1,a_2,a_3,a_4,\ldots) = (0,0.3,0.33,0.333,0.3333,\ldots) \to \tfrac{1}{3}.$$

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 $((-1)^n)_{n\in\mathbb{N}_0}=(1,-1,1,-1,1,\ldots)$ is not convergent (take $\varepsilon=\frac{1}{2}$) and has no infinite limit (take K=0).

- $(1) \lim_{n\to\infty} (a_n+b_n)=a+b;$
- (2) $\lim_{n\to\infty}(a_n\cdot b_n)=a\cdot b;$
- (3) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, if $b_n \neq 0$ for all $n \in \mathbb{N}_0$;
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