Matematiske metoder (MM 529)

Stephan Brandt

Syddansk Universitet, Odense

18. 02. 2014

Functions of two variables: $f(x, y)$,
defined on a domain $D \subseteq \mathbb{R}^2$.
Special case: D is an axis parallel rectangle, i.e.
$D = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$.
Can form the integral

$$g(x) = \int_c^d f(x, y)\, dy$$

for all $x \in \mathbb{R}$ with $a \leq x \leq b$, provided that f is integrable.
Treat x like a constant.
Similarly, can form the integral

$$h(y) = \int_a^b f(x, y)\, dx$$

for all $y \in \mathbb{R}$ with $c \leq x \leq d$, provided that f is integrable.
Treat y like a constant.

Integrals of functions with more than one variable, rectangular domains

$f(x, y)$, defined on an axis parallel rectangle $D \subseteq \mathbb{R}^2$.

$$g(x) = \int_c^d f(x, y)\, dy, \quad h(y) = \int_a^b f(x, y)\, dx$$

Consider

$$\int_a^b g(x)\, dx = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

and

$$\int_c^d h(y)\, dy = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Example: $f(x, y) = k$ for a constant $k \in \mathbb{R}$. Then

$$g(x) = \int_c^d k\, dy = k(d - c), \quad h(y) = \int_a^b k\, dx = k(b - a), \text{ and}$$

$$\int_a^b g(x)\, dx = \int_c^d h(y)\, dy = k(b - a)(d - c)$$

$f(x, y)$, defined on an axis parallel rectangle $D \subseteq \mathbb{R}^2$.
Example: $f(x, y) = k$ for a constant $k \in \mathbb{R}$. Then

$$g(x) = \int_c^d k\,dy = k(d - c), \quad h(y) = \int_a^b k\,dx = k(b - a)$$

and integration of $g(x)$ and $h(y)$ gives

$$\int_a^b \left(\int_c^d k\,dy\right) dx = k(b - a)(d - c) = \int_c^d \left(\int_a^b k\,dx\right) dy,$$

volume of the solid of height k over D.
Here integration order does not matter (in some later cases it does matter). Shorthand:

$$\int_a^b \int_c^d k\,dy\,dx = k(b-a)(d-c) = \int_c^d \int_a^b k\,dx\,dy = \iint_D k\,dx\,dy,$$

$f(x, y)$, defined on an axis parallel rectangle $D \subseteq \mathbb{R}^2$.
$f(x, y)$ any (suitable) function.
Aim: measuring the volume of the solid over D with top surface $z = f(x, y)$.
Approach: Riemann sums.

Riemann sum, function of one variable

Let f be a function on $[a, b]$ and P_k be a partition of $[a, b]$ and $c_i \in [x_{i-1}, x_i]$. Then for $\Delta x_i = x_i - x_{i-1}$ the Riemann sum is

$$S_k = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

(Depends on f, the partition P_k and the choice of the c_i.)

Riemann sum, function of one variable

Let f be a function on $[a, b]$ and P_k be a partition of $[a, b]$ into intervals of length $\Delta x_i = x_i - x_{i-1}$ and $c_i \in [x_{i-1}, x_i]$. Then the Riemann sum is

$$S_k = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

(Depends on f, the partition P_k and the choice of the c_i.)

The definite integral, function of one variable

Let f be a function on $[a, b]$ and $(P_k)_{k\in\mathbb{N}}$ be a sequence of partitions of $[a, b]$. If $\lim_{k\to\infty} ||P_k|| = 0$ then the definite integral of f is

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} S_k,$$

if the limit exists. The value of the limit is independent of the choice of the partitions P_k and the intermediate points c_i.

$f(x, y)$, defined on an axis parallel rectangle $D \subseteq \mathbb{R}^2$.
$f(x, y)$ any (suitable) function.
Aim: measuring the volume of the solid over D with top surface $z = f(x, y)$.
Approach: Riemann sums.
Partition P_k of D by rectangular grid into grid cells
$R_{ij} = \{(x, y) \mid x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}$
of area $A_{ij} = \Delta x_i \cdot \Delta y_j = (x_i - x_{i-1}) \cdot (y_j - y_{j-1})$.
Diameter of a grid cell R_{ij}:
$\mathrm{diam}(R_{ij}) = \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2} = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$.
For every grid cell R_{ij}, chose any point $(x_{ij}^*, y_{ij}^*) \in R_{ij}$.
Volume of the box of height $f(x_{ij}^*, y_{ij}^*)$ over R_{ij}:
$V_{ij} = f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta x_j$.

Partition P_k of rectangle D by rectangular grid into grid cells
$R_{ij} = \{(x, y) \,|\, x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}$
of area $A_{ij} = \Delta x_i \cdot \Delta y_j = (x_i - x_{i-1}) \cdot (y_j - y_{j-1})$.
Diameter of a grid cell R_{ij}:
$\mathrm{diam}(R_{ij}) = \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2} = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$.
For every grid cell R_{ij}, chose any point $(x^*_{ij}, y^*_{ij}) \in R_{ij}$.
Volume of the box of height $f(x^*_{ij}, y^*_{ij})$ over R_{ij}:
$V_{ij} = f(x^*_{ij}, y^*_{ij})\Delta x_i \Delta x_j$

Riemann sum,functions of two variables

Let f be a function on D and P_k be a partition of D into rectangles R_{ij} and $(x^*_{ij}, y^*_{ij}) \in R_{ij}$. Then the Riemann sum is

$$S_k = \sum_{i=1}^{n} \sum_{j=1}^{m} V_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x^*_{ij}, y^*_{ij})\Delta x_i \Delta y_j.$$

(Depends on f, the partition P_k and the choice of the (x^*_{ij}, y^*_{ij}).)

Riemann sum, functions of two variables

Let f be a function on D and P_k be a partition of D into rectangles R_{ij} and $(x_{ij}^*, y_{ij}^*) \in R_{ij}$. Then the Riemann sum is

$$S_k = \sum_{i=1}^{n} \sum_{j=1}^{m} V_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j .$$

(Depends on f, the partition P_k and the choice of the (x_{ij}^*, y_{ij}^*).)

Sums the volumes of the boxes over all the grid cells.
The norm of a partition P_k is $||P_k|| = \max_{1 \leq i \leq n, 1 \leq j \leq m} \mathrm{diam}(R_{ij})$.

D axis parallel rectangle,
$||P_k|| = \max_{1 \le i \le n, 1 \le j \le m} \operatorname{diam}(R_{ij})$.

The double integral over rectangular domains

Let f be a function on D and $(P_k)_{k \in \mathbb{N}}$ be a sequence of partitions of D. If $\lim_{k \to \infty} ||P_k|| = 0$ then the double integral of f is

$$\iint_D f(x, y)\, dx\, dy = \lim_{k \to \infty} S_k,$$

if the limit exists and is independent of the choice of the partitions P_k and the intermediate points (x_{ij}^*, y_{ij}^*).
In this case, we call f (Riemann-)integrable over D.

Note: the textbook uses $dA = dx\, dy$, to emphasize that we integrate over an area.

Given: function f defined on $D \subseteq \mathbb{R}^2$ where $D \subseteq R$ for an axis parallel rectangle R.
Define

$$g(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D, \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

The double integral over general bounded domains

If the double integral of g over R exists then we call f integrable over $D \subseteq R$ and

$$\iint_D f(x,y)\,dx\,dy = \iint_R g(x,y)\,dx\,dy.$$

Example: $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$ unit circle, contained in $R = \{(x,y) \mid -1 \leq x, y \leq 1\}$.

$$\iint_D 1\,dx\,dy = \iint_R g(x,y)\,dx\,dy = \pi,$$

the area of the unit circle ($g(x,y) = 1$ if $(x,y) \in D$, otherwise 0).

Properties of double integrals

Area of the domain D:

$$\iint_D 1\, dx\, dy,$$

the volume of the cylinder of height 1 over D.

Linearity:

$$\iint_D (af(x,y) + bg(x,y))\, dx\, dy =$$

$$= a \iint_D f(x,y)\, dx\, dy + b \iint_D g(x,y)\, dx\, dy,$$

for $a, b \in \mathbb{R}$ and on D integrable functions f and g.

Preservation of inequalities:
If $f(x, y) \leq g(x, y)$ on D, then

$$\iint_D f(x, y)\, dx\, dy \leq \iint_D g(x, y)\, dx\, dy.$$

Triangle inequality:

$$\left| \iint_D f(x, y)\, dx\, dy \right| \leq \iint_D |f(x, y))|\, dx\, dy.$$

Additivity of domains:
If $D_1, D_2, \ldots, D_n$ are non-overlapping domains such that $D = D_1 \cup D_2 \cup \ldots \cup D_n$. Then

$$\iint_D f(x, y)\, dx\, dy = \sum_{k=1}^{n} \iint_{D_k} f(x, y)\, dx\, dy.$$

How to calculate double integrals (without calculating Riemann sums)?

x-simple and y-simple domains

A domain is called y-simple, if it is bounded by two vertical lines $x = a$ and $x = b$ and two continuous graphs of functions $y = c(x)$ and $y = d(x)$.
A domain is called x-simple, if it is bounded by two horizontal lines $y = c$ and $y = d$ and two continuous graphs of functions $x = a(y)$ and $x = b(y)$.

Example: Any rectangle is x-simple and y-simple. Circles are both as well.

Iterated integral

If D is a y-simple domain given by $a \leq x \leq b$ and $c(x) \leq y \leq d(x)$, then

$$\iint_D f(x,y)\,dx\,dy = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)\,dy.$$

If D is a x-simple domain given by $c \leq y \leq d$ and $a(y) \leq x \leq b(y)$, then

$$\iint_D f(x,y)\,dx\,dy = \int_c^d dy \int_{a(y)}^{b(y)} f(x,y)\,dx.$$

The triangle $T = \{(x, y) \,|\, 0 \leq x \leq 1, 0 \leq y \leq x\}$ is y-simple. Therefore, for $f(x, y) = xy$,

$$\begin{aligned} \iint_D xy\,dx\,dy &= \int_0^1 dx \int_0^x xy\,dy. \\ &= \int_0^1 dx \left(\frac{xy^2}{2}\right)\bigg|_{y=0}^{y=x} \\ &= \int_0^1 \frac{x^3}{2}\,dx = \frac{1}{8}. \end{aligned}$$