Matematiske metoder (MM 529)

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L'Hospital's rule (limits as $x \to \infty$)

If f, g are both differentiable functions and

$$\frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \quad \text{then} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

L'Hospital's rule (repeated application)

If f, g are both twice differentiable and

$$\frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \quad \text{then} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

If also

$$\frac{\lim_{x\to\infty} f'(x)}{\lim_{x\to\infty} g'(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \quad \text{then} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f''(x)}{g''(x)}.$$

L'Hospital's rule, repeated application

L'Hospital's rule (repeated application)

If f, g are both n times differentiable and

$$\frac{\lim_{x\to\infty} f^{(k)}(x)}{\lim_{x\to\infty} g^{(k)}(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \quad \text{for every } k \text{ with } 0 \le k < n \text{ then}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

Application: $f(x) = b^x$, $b > 1$ (exponential function);

$g(x) = p(x) = \sum_{i=0}^{n} a_i x^i$, $a_n > 0$ (polynomial function)

Determine $\lim_{x\to\infty} \frac{f(x)}{g(x)}$!

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} = \frac{\infty}{\infty}.$$

Application: Polynomial growth versus exponential growth

$f(x) = b^x$, $b > 1$:
$f'(x) = \frac{d}{dx} e^{x \ln b} = (\ln b) e^{x \ln b} = b^x \ln b. \quad f^{(n)}(x) = b^x (\ln b)^n.$

$$g(x) = p(x) = \sum_{i=0}^{n} a_i x^i:$$

$$g'(x) = \sum_{i=0}^{n-1} (i+1) a_{i+1} x^i. \quad g^{(n)}(x) = n! a_n.$$

$$\text{Since } \frac{\lim_{x\to\infty} f^{(k)}(x)}{\lim_{x\to\infty} g^{(k)}(x)} = \frac{\infty}{\infty} \quad \text{for every } k \text{ with } 0 \le k < n$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \lim_{x\to\infty} \frac{b^x (\ln b)^n}{n! a_n} = \frac{\infty}{n! a_n} = \infty.$$

Exponential growth beats polynomial growth!
f grows faster than g as $x \to \infty$ (even if $b = 1.01$ and $a_n = n = 1000!$).
Running time of algorithms (polynomial versus exponential in the input size).

Mean value theorem

If f is continuous on the interval $[a, b]$ and differentiable on the open interval (a, b) then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

There is a value c, where the tangent has the same slope as the straight line through $(a, f(a))$, $(b, f(b))$.

f is continuous (or even differentiable) and $f(10) = 100$. Guess:
What is $f(10.1)$?
My suggestion: $f(10.1) = 100$. Probably not correct but perhaps not too bad.
More information:
f is differentiable, $f(10) = 100$ and $f'(10) = 20$. What is $f(10.1))$?
My suggestion: $f(10.1) = 102$. Probably not correct but perhaps not too bad.
Why? Explanation: 102 is the function value at $x = 10.1$ on the tangent line of the function f at $x = 10$.
If, say, $f(x) = x^2$, second guess much better than first guess, since $(10.1)^2 = 102.01$.
In a certain sense the best guess we can do.

Aim: Approximating a differentiable function f in the vicinity of a point a by a line.

Linear approximation

If f is differentiable in a, then we call the linear function

$$P(x) = f(a) + f'(a)(x - a)$$

the linear approximation of f at a.

Equation of the tangent line at f in a.
Best possible approximation in the sense, that $P(x)$ is the only linear function, where the limit of the error relative to the distance of x and a is zero as $x \to a$.

$$\lim_{x \to a} \frac{f(x) - P(x)}{x - a} = 0.$$

For a function f a value x^* where $f(x^*) = 0$ is called a zero of f.
How to find zeroes of f?
Example: Polynomials.
Easy for degree two: $p(x) = x^2 + px + q$. Zeroes:

$$x^* = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

Quite difficult for degree 3 and 4.
Provably no algorithm to find the zeroes of polynomials of degree 5 and more.
What can we do?
Try to approximate the zeroes, i.e. find a value x very near to the zero x^*.

Aim: approximate the zeroes of a function f, e.g. f high degree polynomial.
Use the idea of linear approximation: Take the zero of the tangent as a point which is (hopefully) closer to the zero.
Create sequence $(a_0, a_1, a_2, \ldots)$ where a_0 is an initial point "close" to the zero and a_{n+1} is the zero of the tangent at f in a_n for $n \geq 1$.

Formula

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}, \quad \text{if } f'(a_n) \neq 0.$$

Approach works if a_0 is close enough to the zero x^*, precise conditions are complicated.
Example: Find the largest zero of $f(x) = x^2 - 2$.
f is differentiable on $\mathbb{R}$ and $f'(x) = 2x$.
Since $f(1) = -1$ and $f(x) \geq 2$ for $x \geq 2$ and f is continuous, the largest zero is between 1 and 2 (Intermediate value theorem).
Choose starting point close to the zero, e.g.: $a_0 = \frac{3}{2}$.

Example: Find the largest zero of $f(x) = x^2 - 2$.
f is differentiable on $\mathbb{R}$ and $f'(x) = 2x$.
Largest zero of f is between 1 and 2.
Choose starting point close to the zero, e.g.: $a_0 = \frac{3}{2}$.

Formula

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}, \quad \text{if } f'(a_n) \neq 0..$$

in our example

$$a_{n+1} = a_n - \frac{a_n^2 - 2}{2a_n} = \frac{a_n}{2} + \frac{1}{a_n}.$$

$a_0 = \frac{3}{2}$, $a_1 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$, $a_2 = \frac{17}{24} + \frac{12}{17} = \frac{577}{408} = 1.4142156\ldots$,
already very close to $x^* = \sqrt{2} = 1.4142135\ldots$.
If f is 'well-behaved' then a_{n+1} has about two more correct decimal digits than a_n close to x^* (quadratic convergence).

(after Brook Taylor 1685–1731)
Can we do better than using linear approximation?
Yes, we can (provided that our function has higher order derivatives)!

The Taylor polynomial of f at a

Let f be k-times differentiable on an interval I and $a \in I$. Then the k-th order Taylor polynomial of f at a is

$$\begin{aligned} P_k(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \\ &\quad \dots + \frac{f^{(k-1)}(a)}{(k-1)!}(x-a)^{k-1} + \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

First order Taylor polynomial of f at a:

$$P_1(x) = f(a) + f'(a)(x-a),$$

equation of the tangent of f at a (linear approximation).

Taylor polynomial (examples)

Slightly easier to write:

k-th order Taylor polynomial at $a = 0$

$$\begin{aligned} P_k(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots \\ &\quad \dots + \frac{f^{(k-1)}(0)}{(k-1)!}x^{k-1} + \frac{f^{(k)}(0)}{k!}x^k. \end{aligned}$$

Example: Third order Taylor polynomial of $f(x) = e^x$ at $a = 0$.

Obs.: $f^{(k)}(x) = e^x$ for every k, therefore $f^{(k)}(0) = 1$ for every k.

$$P_3(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

In general:

$$P_k(x) = 1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{k!}x^k = \sum_{i=0}^{k} \frac{1}{i!}x^i.$$

Taylor polynomial (examples)

k-th order Taylor polynomial at $a = 0$

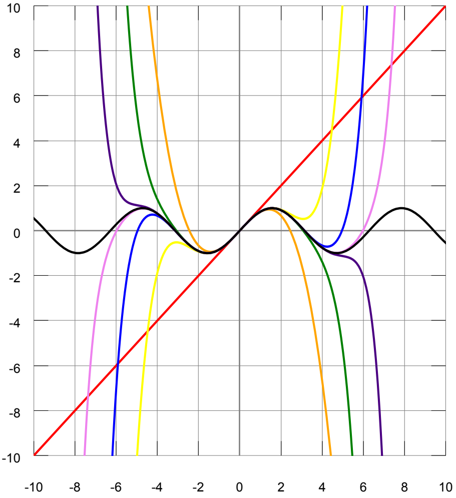
$$\begin{aligned} P_k(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots \\ &\quad \dots + \frac{f^{(k-1)}(0)}{(k-1)!}x^{k-1} + \frac{f^{(k)}(0)}{k!}x^k. \end{aligned}$$

Example: Sixth order Taylor polynomial of $f(x) = \sin x$ at $a = 0$.

$$f^{(n)}(0) = \begin{cases} 0 = \sin 0 & \text{if } n = 4k, \\ 1 = \cos 0 & \text{if } n = 4k+1, \\ 0 = -\sin 0 & \text{if } n = 4k+2, \\ -1 = -\cos 0 & \text{if } n = 4k+3, \end{cases}$$

$$Q_6(x) = \frac{1}{1!}x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 = x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$$

Example: Taylor approximation of $f(x) = \sin x$



In colours: The kth order Taylor polynomials $Q_k(x)$ of $f(x) = \sin x$ at $a = 0$ for $k = 1, 3, 5, \ldots 15$.

Previous examples: Third order Taylor polynomial of $f(x) = e^x$:

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

$P_3(1) = 2.666\ldots \approx 2.718\ldots = f(1) = e^1$.
Sixth order Taylor polynomial of $f(x) = \sin x$.

$$Q_6(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$$

$Q_6(1) = 0.84166\ldots \approx 0.84147\ldots = f(1) = \sin 1$.
By chance? No!
$P_k(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k$.

Taylor's Theorem

If f is k times differentiable, then
$f(x) = P_k(x) + R_k(x)$, where the error term
$R_k(x) = h_k(x)(x-a)^k$ and $\lim\limits_{x \to a} h_k(x) = 0$.

$P_k(x)$ is the only polynomial of degree k with this property.

$$P_k(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Taylor's Theorem

If f is k times differentiable, then
$f(x) = P_k(x) + R_k(x)$, where the error term
$R_k(x) = h_k(x)(x-a)^k$ and $\lim_{x \to a} h_k(x) = 0$.

$P_k(x)$ is the unique best approximation by a degree k polynomial of $f(x)$ near a.
Closer analysis of the error term $R_k(x)$:

Lagrange remainder theorem

If f is $k+1$ times differentiable and $f^{(k+1)}$ continuous, then
$f(x) = P_k(x) + R_k(x)$, where the remainder

$$R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$$

for a point c in the interval between x and a.

$$P_k(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Lagrange remainder theorem

If f is $k+1$ times differentiable and $f^{(k+1)}$ continuous, then $f(x) = P_k(x) + R_k(x)$, where the remainder

$$R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$$

for a point c in the interval between x and a.

Example: Third order Taylor polynomial of $f(x) = e^x$ at $a = 0$.

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

Error we make by using $P_3(x)$ instead of $f(x)$:

$$R_3(x) = \frac{e^c}{4!}x^4$$

for a point c in the interval between x and 0.

Example: Third order Taylor polynomial of $f(x) = e^x$ at $a = 0$.

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

Error we make by using $P_3(x)$ instead of $f(x)$:

$$R_3(x) = f(x) - P_3(x) = \frac{e^c}{4!}x^4$$

for a point c in the interval between x and 0.
Example $x = 1$: Error

$$R_3(1) = \frac{e^c}{4!}1^4 = \frac{e^c}{4!}$$

for a point c in the interval between 0 and 1. Therefore
$0.041\ldots = \frac{1}{4!} \leq R_3(1) \leq \frac{e}{4!} = 0.113\ldots$.
Actual error:
$R_3(1) = f(1) - P_3(1) = 2.718\ldots - 2.666\ldots = 0.051\ldots$.

Example: Sixth order Taylor polynomial of $f(x) = \sin x$ at $a = 0$.

$$Q_6(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$$

Error we make by using $Q_6(x)$ instead of $f(x)$:

$$R_6(x) = f(x) - Q_6(x) = \frac{f^{(7)}(c)}{7!}(x-a)^7$$

for a value c in the interval between x and a.
Example $x = 1$: Error

$$R_6(1) = \frac{-\cos c}{7!}1^7 = \frac{-\cos c}{7!}$$

for a value c in the interval between 0 and 1. Therefore
$-0.000198\ldots = \frac{-1}{7!} \leq R_6(1) \leq \frac{-\cos 1}{7!} = -0.000107\ldots$.
Actual error:
$R_6(1) = f(1) - P_6(1) = 0.84147\ldots - 0.84166\ldots = -0.000195\ldots$.

f differentiable arbitrarily many times. Then we can replace Taylor polynomial of f at a by an infinite series:

Taylor series

$$\begin{aligned} T(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \\ &= \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i \end{aligned}$$

Remember: Infinite series is limit of partial sums.
For which x will it converge? Sure for $x = a$. There is always an $r \in [0, \infty) \cup \{\infty\}$, such that $T(x)$ convergent for $x \in (a-r, a+r)$, and divergent on $\mathbb{R} \setminus [a-r, a+r]$ (radius of convergence).

Example: $T(x) = 1 + x + x^2 + x^3 + \dots = \sum_{i=0}^{\infty} x^i$ (geometric series)

Taylor series of $f(x) = \frac{1}{1-x}$ at $a = 0$ converges only on $(-1, 1)$, $r = 1$.

f differentiable arbitrarily many times.

Taylor series

$$\begin{aligned} T(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{k!}(x-a)^3 + \dots \\ &= \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i. \end{aligned}$$

In our two examples $T(x) = f(x)$ for every $x \in \mathbb{R}$ $(r = \infty)$:

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots = \sum_{i=0}^{\infty} \frac{1}{i!}x^i \quad \text{for every } x \in \mathbb{R}.$$

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!}x^{2i+1} \quad \text{for every } x \in \mathbb{R}.$$