Matematiske metoder (MM 529)

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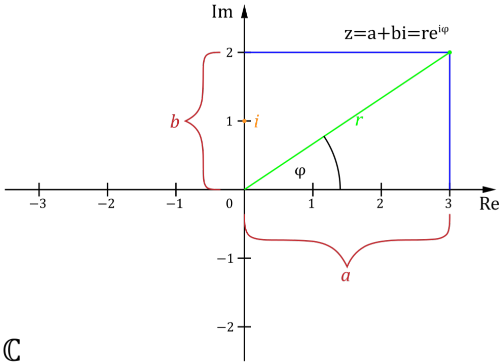
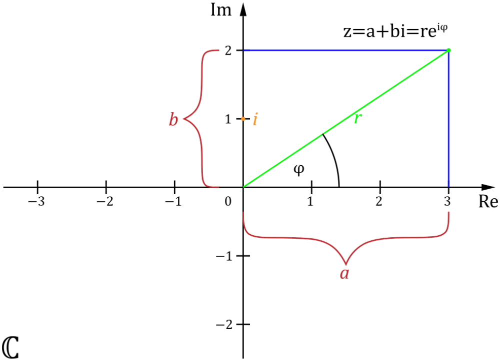
Set of complex numbers $\mathbb{C} = \{a + i \cdot b \mid a, b \in \mathbb{R}\}$.
i: imaginary unit.
$z = a + ib$: $a = \mathrm{Re}(z)$ real part, $b = \mathrm{Im}(z)$ imaginary part.
Conjugate complex number of $z = a + ib$: $\bar{z} = a - ib$,
i.e. the complex number with $\mathrm{Re}(\bar{z}) = \mathrm{Re}(z)$ and $\mathrm{Im}(\bar{z}) = -\mathrm{Im}(z)$.
Representation in the complex plane: Real axis for the real part, imaginary axis for the imaginary part.

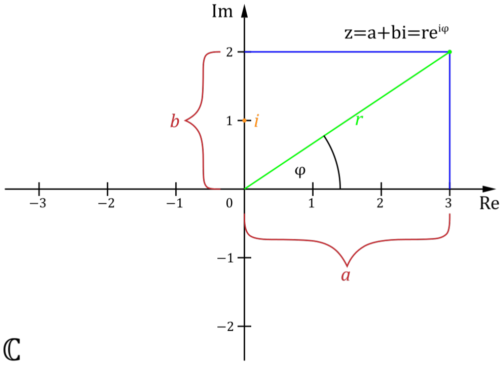


Complex plane



Each complex number z is a point in the complex plane represented by a vector.
Uniquely described by pair (a, b), where $a =$ Re(z) and $b =$Im(z), or
by pair (r, φ) of length r and direction (angle) φ of the vector.

Polar coordinates



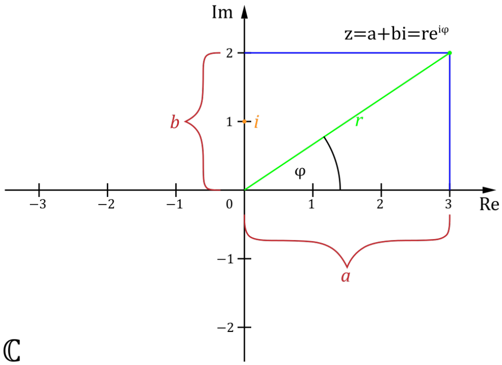
Uniquely described by pair (r, φ) of length r and angle φ.

Polar coordinates:

Length: $r = |z| = \sqrt{a^2 + b^2}$ absolute value (or modulus) of z.

Angle $\varphi = \arg(z)$ with the positive real axis (argument of z).

Argument not unique, $\varphi + 2k\pi$, $k \in \mathbb{Z}$ further arguments (usual agreement: any angle φ is an argument of $z = 0$).



Uniquely described by pair (r, φ) of length r and angle φ.

Polar coordinates:

Length: $r = |z| = \sqrt{a^2 + b^2}$ absolute value (or modulus) of z.

Angle $\varphi = \arg(z)$ with the positive real axis (argument of z).

$z = r(\cos\varphi + i\sin\varphi)$, i.e. $\mathrm{Re}(z) = r\cos\varphi$ and $\mathrm{Im}(z) = r\sin\varphi$.

Algebraic representation by real part a and imaginary part b:
$z = a + ib$.
Polar coordinates: absolute value r and angle φ:
$z = r(\cos\varphi + i\sin\varphi)$.
Exponential representation: $z = re^{i\varphi}$.
(in fact, exponential function on the imaginary axis satisfies
$e^{i\varphi} = \cos\varphi + i\sin\varphi$, $\varphi \in \mathbb{R}$. Periodic with period 2π.)
Changing representations:
Given (r, φ) then
$a = r\cos\varphi$, $b = r\sin\varphi$.
Given (a, b) then $r = \sqrt{a^2 + b^2}$, $\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{b}{a}$ if $a \neq 0$.
Therefore

$$\varphi = \begin{cases} \arctan\frac{b}{a} & \text{if } a > 0, \\ \arctan\frac{b}{a} + \pi & \text{if } a < 0, \\ \frac{\pi}{2} & \text{if } a = 0, b > 0, \\ -\frac{\pi}{2} & \text{if } a = 0, b < 0, \end{cases}$$

is an argument of the form $-\frac{\pi}{2} \leq \varphi < \frac{3\pi}{2}$.

Arithmetic operations, algebraic representation:

Addition/Subtraction

$$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d).$$

Multiplication

$$(a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc).$$

Division

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2},$$

if $c + id \neq 0$.

Powers? Example:

$$(a + ib)^3 = a^3 + 3a^2 ib + 3a(ib)^2 + (ib)^3 = a^3 - 3ab^2 + i(3a^2 b - b^3).$$

Polar coordinates, exponential representation:
$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$, $w = s(\cos\theta + i\sin\theta) = se^{i\theta}$.

Addition/Subtraction

Rewrite z and w in algebraic representation!

Multiplication

$$\begin{aligned} z \cdot w &= rs(\cos(\varphi+\theta) + i\sin(\varphi+\theta)) \\ &= rse^{i(\varphi+\theta)} = |z||w|e^{i(\arg(z)+\arg(w))}. \end{aligned}$$

Division

$$\begin{aligned} \frac{z}{w} &= \frac{r}{s}(\cos(\varphi-\theta) + i\sin(\varphi-\theta)) \\ &= \frac{r}{s}e^{i(\varphi-\theta)} = \frac{|z|}{|w|}e^{i(\arg(z)-\arg(w))}, w \neq 0. \end{aligned}$$

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$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$, $w = s(\cos\theta + i\sin\theta) = se^{i\theta}$.

Multiplication

$$\begin{aligned} z \cdot w &= rs(\cos(\varphi+\theta) + i\sin(\varphi+\theta)) \\ &= rse^{i(\varphi+\theta)} = |z||w|e^{i(\arg(z)+\arg(w))}. \end{aligned}$$

Multiply absolute values, add angles.

Division

$$\begin{aligned} \frac{z}{w} &= \frac{r}{s}(\cos(\varphi-\theta) + i\sin(\varphi-\theta)) \\ &= \frac{r}{s}e^{i(\varphi-\theta)} = \frac{|z|}{|w|}e^{i(\arg(z)-\arg(w))}, w \neq 0. \end{aligned}$$

Form quotient of absolute values, substract angles.

Polar coordinates, exponential representation:
$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$.

Powers of complex numbers

nth power of z.

$$z^n = r^n(\cos n\varphi + i\sin n\varphi) = r^n e^{in\varphi} = |z|^n e^{in\arg(z)}, n \in \mathbb{Z}.$$

Take nth power of absolute value, multiply angle by n.
Example:

$$(1+i)^{10} = (\sqrt{2}e^{i\pi/4})^{10} = (\sqrt{2})^{10} e^{i\cdot 10\cdot\pi/4} = 2^5 e^{i5\pi/2} = 32i.$$

Roots: Square roots of 4: ± 2, all solutions of $x^2 = 4$, all zeroes of the polynomial $x^2 - 4$.
In $\mathbb{R}$: Polynomial of degree $n \geq 0$ has at most n zeroes.

Fundamental Theorem of Algebra

Every (real or complex) polynomial of degree $n \geq 0$ has exactly n zeroes in $\mathbb{C}$ (counting multiplicities).

Polar coordinates, exponential representation:
$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$.
Calculate all nth roots of z, $n \in N$.
If w is an nth root of z, then $w^n = z$, therefore
$|w| = \sqrt[n]{|z|}$ and $n\arg(w) = \arg(z) = \varphi + 2k\pi$, $k \in \mathbb{Z}$.
Example: All 5th roots of $1 = 1 + i \cdot 0 = 1e^{i \cdot 0} \in \mathbb{C}$.
$w^5 = 1$, therefore
$|w| = 1$ and $5\arg(w) = \arg(1) = 0 + 2k\pi$, $k \in \mathbb{Z}$,
$\arg(w) = \frac{2k\pi}{5}$, $k \in \mathbb{Z}$ has the five different solutions for $0 \leq k \leq 4$:
$\arg(w) = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$, since $\frac{10\pi}{5} = 2\pi = 0 + 2\pi$,
$\frac{12\pi}{5} = \frac{2\pi}{5} + 2\pi$.
Geometric interpretation: The 5th (nth) roots of one are the vertices of a regular 5-gon (n-gon) inscribed in the unit circle in the complex plane.

Polar coordinates, exponential representation:
$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$.
Calculate all nth roots of z, $n \in N$.
If w is an nth root of z, then $w^n = z$, therefore
$|w| = \sqrt[n]{|z|}$ and $n\arg(w) = \arg(z) = \varphi + 2k\pi,\ k \in \mathbb{Z}$.
Therefore $\arg(w) = \frac{\varphi}{n} + 2\pi\frac{k}{n}$ with the n different solutions for $0 \le k \le n-1$:

$$\arg(w) = \frac{\varphi}{n}, \frac{\varphi}{n} + \frac{2\pi}{n}, \frac{\varphi}{n} + \frac{4\pi}{n}, \ldots, \frac{\varphi}{n} + \frac{2(n-1)\pi}{n}.$$

Geometric interpretation: The nth roots of $z \ne 0$ are the vertices of a regular n-gon inscribed in the circle of radius $\sqrt[n]{|z|}$ around the origin of the complex plane.
Example: Third roots w of $z = 1 + i = \sqrt{2}e^{i\pi/4}$ satisfy
$|w| = \sqrt[3]{\sqrt{2}} = \sqrt[6]{2}$ and $3\arg w = \frac{\pi}{4}$, with the three different solutions $\arg(w) = \frac{\pi}{12} + k\frac{2\pi}{3},\ k = 0, 1, 2$. Therefore

$$w = \sqrt[6]{2}e^{i\pi/12}, \sqrt[6]{2}e^{i9\pi/12}, \sqrt[6]{2}e^{i17\pi/12}$$

are the three different third roots of z.

$$\text{Is} \quad \sum_{n=k}^{\infty} a_i \quad \text{convergent, where all } a_n > 0?$$

Two more criteria:

Root test

Suppose

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n} = c.$$

If $c < 1$ then the series is convergent and if $c > 1$ then the series diverges to infinity. If $c = 1$ both is possible.

Ratio test

Suppose

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = c.$$

If $c < 1$ then the series is convergent and if $c > 1$ then the series diverges to infinity. If $c = 1$ both is possible.

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$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n} = c.$$

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Example: $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent, since $\lim_{n\to\infty} (a_n)^{1/n} = \frac{1}{2} < 1$.

Ratio test

Suppose

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = c.$$

If $c < 1$ then the series is convergent and if $c > 1$ then the series diverges to infinity. If $c = 1$ both is possible.

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If $c < 1$ then the series is convergent and if $c > 1$ then the series diverges to infinity. If $c = 1$ both is possible.

Which one to choose? Depends mainly on the structure of the summands (even though the ratio test is a tiny little bit stronger).

Ratio test

Suppose

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = c.$$

If $c < 1$ then the series is convergent and if $c > 1$ then the series diverges to infinity. If $c = 1$ both is possible.

Examples: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ satifies $\displaystyle c = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ satifies $\displaystyle c = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(1 - \frac{2n+1}{(n+1)^2}\right) = 1.$

The first series diverges while the second converges.

Absolute and conditional convergence

$\sum_{n=k}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

If $\sum_{n=k}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent then the first series is called conditionally convergent.

Examples: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent,

while $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

Theorem

Every absolutely convergent series is convergent.

Absolute and conditional convergence

$\sum_{n=k}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

If $\sum_{n=k}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent then the first series is called conditionally convergent.

Reordering summands:

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is (conditionally) convergent, but first summing the even index summands and then the odd index summands gives

$$\begin{aligned}
\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots-\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\right) &= \sum_{n=1}^{\infty}\frac{1}{2n}-\sum_{n=1}^{\infty}\frac{1}{2n-1} \\
&= \infty-\infty=?
\end{aligned}$$

Absolute and conditional convergence

$\sum_{n=k}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

If $\sum_{n=k}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent then the first series is called conditionally convergent.

Reordering Theorem

If $\sum_{n=1}^{\infty} a_n = s \in \mathbb{R}$ is absolutely convergent, then for any reordering (including the signs) of the summands the sum converges to s.

Alternating series

A sequence (a_n) is called alternating if $a_n \cdot a_{n+1} < 0$ for all $n \geq k$.

The infinite sum $\sum_{n=k}^{\infty} a_n$ over an alterating sequence is an alternating series.

I.e., consecutive summands have different signs (where 0 counts for both signs).

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating series.

Convergence of alternating series under much weaker circumstances than in the general case:

Alternating series test

If $(|a_n|)$ is monotonously decreasing (i.e. $|a_{n+1}| \leq |a_n|$) and $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ is convergent.

Definition: Ordinary differential equation (ODE)

An ordinary differential equation is an equation of the form

$$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}),$$

where $y = y(x)$ is a function of the variable x.
The highest derivative n occurring in the equation is the order of the ODE.

Counterpart: partial differential equation (PDE) with more than one variable x.
The general solution: all functions $y(x)$ that satisfy the equality.
Examples:

$$y'' = \frac{y^2}{x}; \quad y' = y - x; \quad y''' = 2y'' - y' + y + e^x.$$

First order differential equation

$$y'(x) = f(x, y) = f(x, y(x)).$$

The derivative of y at x depends on x and on the function value $y(x)$.

Example: $y' = y - x$.

Initial value problem

Solve $y'(x) = f(x, y)$ subject to $y(x_0) = a$.

Example: Solve $y' = y - x$ subject to $y(0) = 1$.
Solution: $y(x) = x + 1$, because $y(0) = 0 + 1 = 1$ and $1 = y'(x) = x + 1 - x = 1$.

General initial value problem

Solve $y^{(n)}(x) = f(x, y, y', y'', \ldots, y^{(n-1)})$ subject to $y(x_0) = a_0$, $y'(x_0) = a_1$, $\ldots, y^{(n-1)}(x_0) = a_{n-1}$.

Geometric interpretation of first order ODE

$$y'(x) = f(x, y).$$

For every point (x, y) of the plane, $f(x, y)$ is the derivative of the function $y(x)$ at x.

Slope field

To every point (x, y) of the plane we assign the slope $f(x, y)$.

Solutions of the ODE: any function whose graph follows the slopes of the slope field.

Slope field of $y' = f(x, y) = y - x$: Solution $y = x + 1$ visible.