Matematiske metoder (MM 529)

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Functions of n variables: $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $z = f(x_1, \ldots, x_n)$.
Difference quotient for variable x_i ($1 \leq i \leq n$):

$$\frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}.$$

Partial derivatives:

$$\begin{aligned} f_{x_i}(x_1, \ldots, x_n) &= \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \\ &= \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}. \end{aligned}$$

Note: For a function $f : \mathbb{R}^n \to \mathbb{R}$ the partial derivative w.r.t. x_i at $(a_1, \ldots, a_n)$ is the derivative at a_i of the function $g : \mathbb{R} \to \mathbb{R}$
$g(x_i) = f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$.

Partial derivatives w.r.t. x_i: Treat all variables except x_i as constants and form the derivative w.r.t. the variable x_i.

Vectors in $\mathbb{R}^n$: A vector $\mathbf{v}$ is an object having a length and a direction (representable by an arrow).

Two vectors with the same length and same direction are considered equal (no matter where they are located in $\mathbb{R}^n$).

Ordered pair (P, Q) of two points in $\mathbb{R}^n$ determines a vector $\vec{PQ} = \mathbf{v}$ whose tail is P and whose head is Q.

The length $|\mathbf{v}|$ (or $||\mathbf{v}||$) of $\mathbf{v}$ is the Euclidean distance between P and Q.

Example:

$\mathbf{v}$ vector from $(1, 1)$ to $(2, 3)$ in $\mathbb{R}^2$.

$\mathbf{v}$ is also the vector from $(2, 3)$ to $(3, 5)$ and from $(0, 0)$ to $(1, 2)$.

The length $|\mathbf{v}| = \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5}$.

Representation of a vector: Coordinates $(a_1, a_2, \ldots .a_n)$ of the head, if the tail is $(0, 0, \ldots, 0)$.

Scalar multiples of $\mathbf{v} = (a_1, a_2, \dots, a_n)$, multiplying $\mathbf{v}$ with the scalar $t \in \mathbb{R}$:

$$t \cdot \mathbf{v} = (t \cdot a_1, t \cdot a_2, \dots, t \cdot a_n)$$

Length $|t \cdot \mathbf{v}| = |t| \cdot |\mathbf{v}|$.
$(-1) \cdot \mathbf{v}$ is the vector with the same length as $\mathbf{v}$ but the opposite direction.
Unit vector: Vector of length 1.
$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{|\mathbf{v}|} \cdot \mathbf{v}$ is the unit vector in the direction of $\mathbf{v}$.
$\mathbf{e_i}$: Unit vector in the direction of the ith coordinate axis.
Addition of vectors $\mathbf{v} = (a_1, a_2, \dots, a_n)$ and $\mathbf{w} = (b_1, b_2, \dots, b_n)$:

$$\mathbf{v} + \mathbf{w} = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n).$$

Subtraction: $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1) \cdot \mathbf{w}$.

Dot product (also called scalar product) of a vector $\mathbf{v} = (a_1, a_2, \ldots, a_n)$ with a vector $\mathbf{w} = (b_1, b_2, \ldots, b_n)$:

$$\mathbf{v} \cdot \mathbf{w} = a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_n \cdot b_n = \sum_{i=1}^{n} a_i \cdot b_i \in \mathbb{R}.$$

Properties:

$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \cdot |\mathbf{w}| \cdot \cos\theta$ where θ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$.

Rules:

$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ (commutative law)

$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive law)

$t \cdot (\mathbf{v} \cdot \mathbf{w}) = (t \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (t \cdot \mathbf{w})$ if $t \in \mathbb{R}$

$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is partially differentiable in the point $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ then the gradient is the tuple (vector)

$$\nabla f(\mathbf{a}) = \left(\frac{\partial}{\partial x_1} f(\mathbf{a}), \frac{\partial}{\partial x_2} f(\mathbf{a}), \ldots, \frac{\partial}{\partial x_n} f(\mathbf{a}) \right) \in \mathbb{R}^n.$$

$n = 2$: Gradient at $(a, b) \in \mathbb{R}^2$ is

$$\nabla f(a, b) = (f_x(a, b), f_y(a, b)).$$

More general: $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ partially differentiable on D then

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y)).$$

Example: $f(x, y) = x^2 - y^2$ partially differentiable on $\mathbb{R}^2$.

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y)) = (2x, -2y).$$

$$\nabla f(-1, 3) = (-2, -6).$$

$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ continuous and partially differentiable w.r.t. x and y and f_x and f_y continuous in $(a, b) \in \mathbb{R}^2$.
Equation of the tangent plane at $(a, b) \in \mathbb{R}^2$:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ partially differentiable w.r.t. x_i and f_{x_i} continuous for all i with $1 \leq i \leq n$ in $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$.
Equation of the tangent space (hyperplane) at $\mathbf{a} \in \mathbb{R}^n$:

$$z = f(\mathbf{a}) + f_{x_1}(\mathbf{a})(x_1 - a_1) + \ldots + f_{x_n}(\mathbf{a})(x_n - a_n)$$

$$z = f(a_1, \ldots, a_n) + \sum_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i).$$

Sum is the dot product of vectors:

$$z = f(a_1, \ldots, a_n) + \nabla f(\mathbf{a}) \cdot (x_1 - a_1, x_2 - a_2 \ldots, x_n - a_n).$$

Linear approximation for functions $f : \mathbb{R} \to \mathbb{R}$: Approximate differentiable function f near $a \in \mathbb{R}$ by tangent line through point $(a, f(a))$.
Linear approximation: $f(x) \approx P(x) = f(a) + f'(a)(x - a)$ for x near a.
Linear approximation of functions $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$: Approximate $f(\mathbf{x})$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ near $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in D$ by the corresponding point on the tangent space of f in $\mathbf{a}$.
Equation of the tangent plane at $(a, b) \in D$:

$$z = L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Example: $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$, $(a, b) = (2, 1)$.
Partial derivatives $f_x(x, y) = 2x$ and $f_y(x, y) = -2y$,
Tangent plane in $(2, 1)$ consists of all points (x, y, z) satisfying the equation

$$\begin{aligned} z = L(x, y) &= f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) \\ &= 4 - 1 + 4(x - 2) - 2(y - 1) = 4x - 2y - 3. \end{aligned}$$

Linear approximation (example)

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Linear approximation of f at $(2.2, 0.9)$ near $(2,1)$:

$$4.03 = 4.84 - 0.81 = f(2.2, 0.9) \approx L(2.2, 0.9) = 8.8 - 1.8 - 3 = 4.$$

The general case:
$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ partially differentiable w.r.t. x_i and f_{x_i} continuous for all i with $1 \leq i \leq n$ in $(a_1, \ldots, a_n) \in D$ (D open subset of $\mathbb{R}^n$).
Linear approximation of f near $\mathbf{a} = (a_1, \ldots, a_n)$:
$f(x_1, \ldots, x_n) \approx L(x_1, \ldots, x_n)$, where

$$\begin{aligned} L(x_1, \ldots, x_n) &= f(\mathbf{a}) + f_{x_1}(\mathbf{a})(x_1 - a_1) + \ldots + f_{x_n}(\mathbf{a})(x_n - a_n) \\ &= f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (x_1 - a_1, x_2 - a_2 \ldots, x_n - a_n). \end{aligned}$$

Unit vector in the direction of $\mathbf{v} \neq (0,0,\ldots,0)$:
$\mathbf{v_0} = \frac{1}{|\mathbf{v}|} \cdot \mathbf{v}$.
Directional derivative in the direction of $\mathbf{v} \neq (0,0,\ldots,0)$ at $\mathbf{a} = (a_1,\ldots,a_n) \in \mathbb{R}^n$:

$$D_{\mathbf{v}} f(\mathbf{a}) = \lim_{h \to 0+} \frac{f(\mathbf{a} + h\mathbf{v_0}) - f(\mathbf{a})}{h},$$

if the limit exists.
Case $n = 2$: $\mathbf{a} = (a,b) \in \mathbb{R}^2$, $\mathbf{v_0} = (u,w) \in \mathbb{R}^2$.
Directional derivative in direction of $\mathbf{v}$ and $\mathbf{a}$:

$$D_{\mathbf{v}} f(a,b) = \lim_{h \to 0+} \frac{f(a+hu, b+hw) - f(a,b)}{h}.$$

Case $n = 2$: $\mathbf{a} = (a, b) \in \mathbb{R}^2$, $\mathbf{v_0} = (u, w) \in \mathbb{R}^2$.
Directional derivative in direction of $\mathbf{v}$ at $\mathbf{a}$:

$$D_{\mathbf{v}} f(a, b) = \lim_{h \to 0+} \frac{f(a + hu, b + hw) - f(a, b)}{h}.$$

If $\mathbf{v_0} = \mathbf{e_1} = (1, 0)$ (i.e. $\mathbf{v}$ parallel to the x-axis) then

$$D_{\mathbf{v}} f(a, b) = \lim_{h \to 0+} \frac{f(a + h, b) - f(a, b)}{h} = f_x(a, b).$$

If $\mathbf{v_0} = \mathbf{e_2} = (0, 1)$ (i.e. $\mathbf{v}$ parallel to the y-axis) then

$$D_{\mathbf{v}} f(a, b) = \lim_{h \to 0+} \frac{f(a, b + h) - f(a, b)}{h} = f_y(a, b).$$

In general (directional derivatives in direction of a coordinate axis):

$$D_{\mathbf{e_i}} f(\mathbf{a}) = \lim_{h \to 0+} \frac{f(\mathbf{a} + h\mathbf{e_i}) - f(\mathbf{a})}{h} = f_{x_i}(\mathbf{a}).$$

The directional derivative in $\mathbf{a}$ need not exists if all partial (first order) derivatives exist (the function need not be continuous in $\mathbf{a}$).
If $D \subseteq \mathbb{R}^n$ is open, and all (first order) partial derivatives of $f : D \to \mathbb{R}$ exist and are continuous, then the directional derivative exists for every vector $\mathbf{v} \neq (0, \dots, 0)$.
In that case

$$D_{\mathbf{v}} f(\mathbf{a}) = \lim_{h \to 0+} \frac{f(\mathbf{a} + h\mathbf{v_0}) - f(\mathbf{a})}{h} = \mathbf{v_0} \cdot \nabla f(\mathbf{a}).$$

Example: $f(x, y) = x^2 - y^2$, partial derivatives $f_x(x, y) = 2x$, $f_y(x, y) = -2y$ are continuous. Therefore:

$$D_{(1,1)} f(2, 1) = \mathbf{v_0} \cdot \nabla f(\mathbf{a}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \cdot (4, -2) = \sqrt{2}.$$

Indicates the slope of f in $(2, 1)$ in direction of the angle bisector.