Matematiske metoder (MM 529)

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Tangent space of f at a point $\mathbf{a}$.
In which direction is the largest ascent?
Solution: Find the direction $\mathbf{v}$ in which the directional derivative $D_{\mathbf{v}}f(\mathbf{a})$ is largest.

$$D_{\mathbf{v}}f(\mathbf{a}) = \lim_{h\to 0+} \frac{f(\mathbf{a}+h\mathbf{v_0}) - f(\mathbf{a})}{h} = \mathbf{v_0} \cdot \nabla f(\mathbf{a})$$

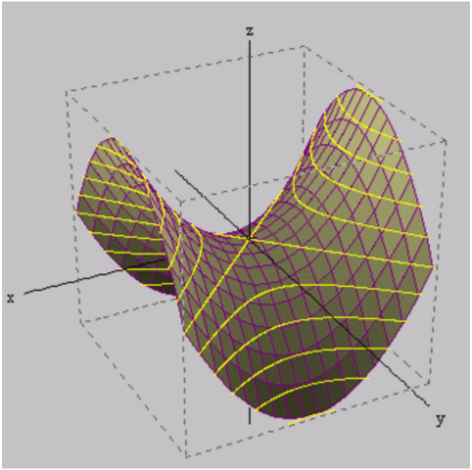
largest, if $\mathbf{v_0} \cdot \nabla f(\mathbf{a})$ is largest. But:

$$\begin{aligned} \mathbf{v_0} \cdot \nabla f(\mathbf{a}) &= |\mathbf{v_0}| \cdot |\nabla f(\mathbf{a})| \cdot \cos\theta \\ &= 1 \cdot |\nabla f(\mathbf{a})| \cdot \cos\theta \end{aligned}$$

largest, if $\theta = 0$ (provided the gradient $\nabla f(\mathbf{a}) \neq (0, 0, \ldots, 0)$).
Conclusion: The gradient points in the direction of the largest ascent in $\mathbf{a}$, and
$|\nabla f(\mathbf{a})|$ is the slope (directional derivative) in this direction.

Example: Saddle surface

$f(x, y) = x^2 - y^2,$
$(a, b) = (2, 1).$
$\nabla f(x, y) = (2x, -2y),$
$\nabla f(2, 1) = (4, -2).$



Largest ascent in direction $(4, -2)$, slope:
$|(4, -2)| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}.$

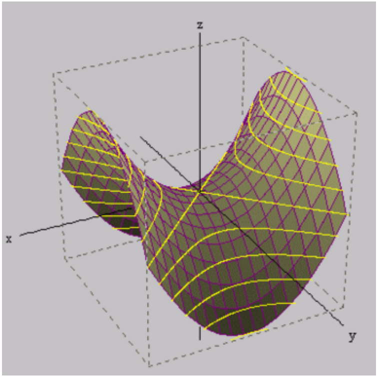
$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. Find local extrema in the interior $D \setminus \partial D$ of D. Extrema on the boundary $\partial D \cap D$ have to be determined separately.

Local maxima and minima

$f(\mathbf{a})$ is a local maximum (minimum, resp.) at point $\mathbf{a}$, if there exists $\delta > 0$ such that $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in B_\delta(\mathbf{a})$ ($f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in B_\delta(\mathbf{a})$, resp.).

If tangent space at $\mathbf{a}$ exists (which is the case if f is continuous and all partial derivatives exist and are continuous in $\mathbf{a}$):
Candidates for local extrema: points $\mathbf{a}$ where $\nabla f(\mathbf{a}) = (0, 0, \ldots, 0)$ (critical points).
Otherwise there are points with larger function value in the direction of $\nabla f(\mathbf{a})$ and points with smaller function value in the direction of $-\nabla f(\mathbf{a})$.
Is a critical point $\mathbf{a}$ (with $\nabla f(\mathbf{a}) = (0, 0, \ldots, 0)$) an extreme point?

$f(x, y) = x^2 - y^2$.
Critical points:
$\nabla f(x, y) = (2x, -2y) = (0, 0)$,
if and only if $(x, y) = (0, 0)$.



Is $(0, 0)$ an extreme point?
No! For $g(x) = f(x, 0)$ and $h(y) = f(0, y)$, $g''(x) = f_{xx}(x, 0) = 2$ and $h''(y) = f_{yy}(0, y) = -2$, so g is convex and h is concave.
In $(x, y) = (0, 0)$, g has a minimum and h has a maximum (f has a saddle point).

Finding maxima and minima

$f : D \to \mathbb{R}$. Find local extrema in the interior $D \setminus \partial D$ of $D \subseteq \mathbb{R}^n$, if f is continuous and partially differentiable with continuous partial derivatives on D.
Answer only for the case $n = 2$ (for general n more mathematics is needed: determinants of matrices).
$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ two times partially differentiable and f and all first and second order partial derivatives are continuous (in particular, Schwarz' Theorem gives that $f_{yx} = f_{xy}$).
If $f_{xx}(\mathbf{a})f_{yy}(\mathbf{a}) - (f_{xy}(\mathbf{a}))^2$

- > 0, then $\mathbf{a}$ is an extreme point,
- < 0, then $\mathbf{a}$ is a saddle point.

If $\mathbf{a}$ is an extreme point and $f_{xx}(\mathbf{a})$

- > 0, then $f(\mathbf{a})$ is a local minimum,
- < 0, then $f(\mathbf{a})$ is a local maximum.

Example

$f(x, y) = x^3 + y^3 - 3xy$ satisfies
$f_x(x, y) = 3x^2 - 3y$ and
$f_y(x, y) = 3y^2 - 3x$.
$f_x = 0$ if and only if $x^2 = y$,
$f_y = 0$ if and only if $y^2 = x$.
$\nabla f(x, y) = (0, 0)$ if and only if $(x, y) = (0, 0)$ or $(x, y) = (1, 1)$ (critical points).
Are they extreme points or saddle points?
$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = f_{yx} = -3$.
$f_{xx}(0, 0)f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = 0 - 9 < 0$
therefore $(0, 0)$ saddle point.
$f_{xx}(1, 1)f_{yy}(1, 1) - (f_{xy}(1, 1))^2 = 36 - 9 > 0$
therefore $(1, 1)$ extreme point.
$f_{xx}(1, 1) = 6 > 0$ therefore $f(1, 1) = -1$ is a local minimum.

Two seemingly unrelated topics:

Definite integral

Measure the area between the graph of a function f and the x-axis. If $f(x)$ is negative on an interval, then the contribution of this interval to the area is negative.
Result: a real number.

Notation: For $f : [a, b] \to \mathbb{R}$ the area is $A = \int_a^b f(x)\,dx$.

Antiderivative

Given a function f, for which function(s) F does the following relation hold: $F' = f$?
Result: a function F, called an antiderivative of f.

Notation: $\int f(x)\,dx$ denotes the set of all functions F with this property.

Mean value theorem of differentiation

If f is continuous on the interval $[a, b]$ and differentiable on the open interval (a, b) then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

There is a value c, where the tangent line has the same slope as the straight line through $(a, f(a))$, $(b, f(b))$.

Mean value theorem for the area

If f is continuous on the open interval (a, b) then there is a $c \in (a, b)$ such that the area under f on $[a, b]$ is

$$A = f(c)(b - a).$$

Piecewise continuous function

A function $f : [a, b] \to \mathbb{R}$ is piecewise continuous if there are only finitely many points in $[a, b]$ where f is not continuous.

Partition of $[a, b]$

A set P of points $\{x_0, x_1, \dots, x_n\}$ with
$a = x_0 < x_1 < x_2 < \dots < x_n = b$ is called a partition of $[a, b]$.
The width of $[x_{i-1}, x_i]$ is denoted $\Delta x_i = x_i - x_{i-1}$.
The norm of the partition is $||P|| = \max\limits_{1 \le i \le n} \Delta x_i$.

Mean value theorem for the area

If f is continuous on the open interval (a, b) then there is a $c \in (a, b)$ such that the area under f is $A = f(c)(b - a)$.

Area under f

If f is piecewise continuous on $[a, b]$ and P is a partition of $[a, b]$ containing all points where f is discontinuous then there are $c_i \in (x_{i-1}, x_i)$ such that the area under f is

$$A = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

Problem: How to find the correct values c_i?
Solution: Not necessary, take any $c_i \in [x_{i-1}, x_i]$ and keep refining the partition and take the limit.

Riemann sum

Let f be a function on $[a, b]$ and P_k be a partition of $[a, b]$ and $c_i \in [x_{i-1}, x_i]$. Then the Riemann sum is

$$S_k = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

(Depends on f, the partition P_k and the choice of the c_i.)

Example: $f(x) = x^2$, $[a, b] = [0, 2]$, $P_1 = \{0, 1, 2\}$, $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{2}$:

$$S_1 = \sum_{i=1}^{2} f(c_i)\Delta x_i = \frac{1}{4} \cdot 1 + \frac{9}{4} \cdot 1 = \frac{5}{2} = 2.5.$$

Refine the partition: $P_2 = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$, $c_i = \frac{x_i + x_{i-1}}{2}$:

$$S_2 = \sum_{i=1}^{4} f(c_i)\Delta x_i = \frac{1}{2}\left(\frac{1}{16} + \frac{9}{16} + \frac{25}{16} + \frac{49}{16}\right) = \frac{21}{8} = 2.625.$$

The definite integral

Let f be a function on $[a, b]$ and $(P_k)_{k\in\mathbb{N}}$ be a sequence of partitions of $[a, b]$. If $\lim_{k\to\infty} ||P_k|| = 0$ then the definite integral of f is

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} S_k,$$

if the limit exists. The value of the limit is independent of the choice of the partitions P_k and the intermediate points c_i.

Example: $f(x) = \lceil x \rceil$ on $[a, b] = [0, 2]$. Take $P_1 = \{0, 1, 2\}$ and $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{2}$. Then

$$S_1 = \sum_{i=1}^{2} f(c_i)\Delta x_i = 1 \cdot 1 + 2 \cdot 1 = 3.$$

Example: $f(x) = \lceil x \rceil$ on $[a, b] = [0, 2]$. Take $P_1 = \{0, 1, 2\}$ and $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{2}$. Then

$$S_1 = \sum_{i=1}^{2} f(c_i)\Delta x_i = 1 \cdot 1 + 2 \cdot 1 = 3.$$

Different choice $c_1 = 0$, $c_2 = 1$ (left end of the intervals):

$$S_1 = \sum_{i=1}^{2} f(c_i)\Delta x_i = 0 \cdot 1 + 1 \cdot 1 = 1.$$

Refining the partition with $||P_k|| \to 0$, the contribution of the (≤ 3) intervals containing $x = 0$ and $x = 1$ becomes arbitrarily small.
Precisely: $3 - 3||P_k|| \leq S_k \leq 3 + 3||P_k||$. Therefore

$$\int_0^2 f(x)\,dx = \lim_{k\to\infty} S_k = 3.$$

The definite integral

Let f be a function on $[a, b]$ and $(P_k)_{k\in\mathbb{N}}$ be a sequence of partitions of $[a, b]$. If $\lim\limits_{k\to\infty} ||P_k|| = 0$ then the definite integral is

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} S_k,$$

if the limit exists.

Existence of the definite integral

If f is piecewise continuous on $[a, b]$ and there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ (f is bounded) then $\displaystyle\int_a^b f(x)\,dx$ exists.

Agreement: $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.