Matematiske metoder (MM 529)

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Function $f(x, y) = x^2 - y^2$ of two variables.

Domain: $D = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.
In general: function of n variables
$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$,
$(x_1, x_2, \ldots, x_n) \mapsto z$.

Functions of two variables: $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, $z = f(x, y)$.

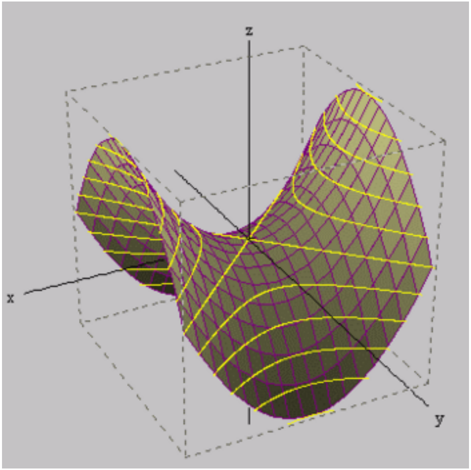
$$\operatorname{graph} f = \{(x, y, z) \in \mathbb{R}^3 \ : \ z = f(x, y)\}.$$

Functions of n variables $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $z = f(x_1, x_2, \dots, x_n)$.

$$\operatorname{graph} f = \{(x_1, x_2, \dots, x_n, z) \in \mathbb{R}^{n+1} \ : \ z = f(x_1, x_2, \dots, x_n)\}.$$

For $n \geq 3$ hard to visualize in our 3-dimensional world.

Example: Saddle surface



Graph of the function $f(x, y) = x^2 - y^2$.

Level sets/level curves

Sets of elements of the domain D with the same function value.

$$L(k) = \{(x_1, \ldots, x_n) \in D \ : \ f(x_1, \ldots, x_n) = k\}.$$

Examples:
Level curves in a geographic map of constant height k ($n = 2$).
Helps visualizing the behaviour of functions $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$.

Example: Level curves

$n = 2$: Given two points $(x, y), (a, b) \in \mathbb{R}^2$.
The (Euclidean) distance of (x, y) and (a, b) equals $\sqrt{(x-a)^2 + (y-b)^2}$.
Theorem of Pythagoras.
Arbitrary n: Given two points $(x_1, x_2, \ldots, x_n), (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$.
The (Euclidean) distance of $(x_1, x_2, \ldots, x_n)$ and $(a_1, a_2, \ldots, a_n)$ equals $\sqrt{(x_1-a_1)^2 + (x_2-a_2)^2 + \ldots + (x_n-a_n)^2}$.
Open disc of radius δ around the point $(a, b) \in \mathbb{R}^2$:

$$B_\delta(a, b) = \{(x, y) \in \mathbb{R}^2 \;:\; \sqrt{(x-a)^2 + (y-b)^2} < \delta\}.$$

Open ball of radius δ around the point $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$:

$B_\delta(a_1, \ldots, a_n) =$

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \;:\; \sqrt{(x_1-a_1)^2 + \ldots + (x_n-a_n)^2} < \delta\}.$$

Open and closed sets, boundary

Open/closed intervals in $\mathbb{R}$:
$(a, b) = \{x \in \mathbb{R} \ : \ a < x < b\}$ open, boundary points a, b do not belong to (a, b).
$[a, b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}$ closed, boundary points a, b belong to (a, b).
$[a, b) = \{x \in \mathbb{R} \ : \ a \leq x < b\}$ and $(a, b]$ neither open nor closed.

Boundary points

Set $D \subseteq \mathbb{R}^2$.
$(a, b) \in \mathbb{R}^2$ boundary point of D, if for every $\delta > 0$
$B_\delta(a, b) \cap D \neq \emptyset$ and $B_\delta(a, b) \cap (\mathbb{R}^2 \setminus D) \neq \emptyset$.

∂D: set of boundary points of D.
$D \setminus \partial D$: set of inner points of D.

Open/closed sets

$D \subseteq \mathbb{R}^2$ is open if $\partial D \cap D = \emptyset$.
$D \subseteq \mathbb{R}^2$ is closed if $\partial D \subseteq D$.

Boundary points

Set $D \subseteq \mathbb{R}^n$.
$(a_1, \ldots, a_n) \in \mathbb{R}^n$ boundary point of D, if for every $\delta > 0$
$B_\delta(a_1, \ldots, a_n) \cap D \neq \emptyset$ and $B_\delta(a_1, \ldots, a_n) \cap (\mathbb{R}^n \setminus D) \neq \emptyset$.

∂D: set of boundary points of D.
$D \setminus \partial D$: set of inner points of D.

Open/closed sets

$D \subseteq \mathbb{R}^n$ is open if $\partial D \cap D = \emptyset$.
$D \subseteq \mathbb{R}^n$ is closed if $\partial D \subseteq D$.

Example: $D_1 = \{(x, y) : x, y \leq 1, x^2 + y^2 \geq 1\}$ and
$D_2 = \{(x, y) : x, y < 1, x^2 + y^2 > 1\}$ both have the same boundary
$\partial D_1 = \partial D_2 = \{(x, y) : x = 1 \text{ or } y = 1\} \cup \{(x, y) : x^2 + y^2 = 1\}$,
D_1 is closed (since $\partial D_1 \subseteq D_1$),
D_2 is open (since $\partial D_2 \cap D_2 = \emptyset$).

$D \subseteq \mathbb{R}^2$ open subset. Function $f : D \to \mathbb{R}$:

Limit: (ε, δ)-definition

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \in \mathbb{R},$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|L - f(x,y)| < \varepsilon, \text{ whenever } (x,y) \in B_\delta(a,b).$$

Continuity

f is continuous in $(a,b) \in D$, if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

f is continuous on D, if f is continuous in every point $(a,b) \in D$.

$D \subseteq \mathbb{R}^n$ open subset. Function $f : D \to \mathbb{R}$:

Limit: (ε, δ)-definition

$$\lim_{(x_1,\ldots,x_n)\to(a_1,\ldots,a_n)} f(x,y) = L \in \mathbb{R},$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|L - f(x_1,\ldots,x_n)| < \varepsilon, \text{ whenever } (x_1,\ldots,x_n) \in B_\delta(a_1,\ldots,a_n).$$

Continuity

f is continuous in $(a_1,\ldots,a_n) \in D$, if

$$\lim_{(x_1,\ldots,x_n)\to(a_1,\ldots,a_n)} f(x_1,\ldots,x_n) = f(a_1,\ldots,a_n).$$

f is continuous on D, if f is continuous in every point $(a_1,\ldots,a_n) \in D$.

$f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$.

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0,$$

since for every given $\varepsilon > 0$ choosing $\delta = \sqrt{\varepsilon}$ all elements of $B_\delta(0, 0) = \{(x, y) : \sqrt{x^2 + y^2} < \delta\}$ satisfy $-\delta < x, y < \delta$.
Therefore $0 \leq x^2, y^2 < \delta^2 \leq \varepsilon$ and $-\varepsilon < f(x, y) = x^2 - y^2 < \varepsilon$ for all $(x, y) \in B_\delta(0, 0)$,
hence $|f(x, y) - 0| < \varepsilon$.

Since $f(0, 0) = 0^2 - 0^2 = 0$,

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0),$$

therefore f is continuous in the point $(0, 0)$.

Partial derivatives

Functions of two variables: $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, $z = f(x, y)$.
Difference quotient for variable x at the point (a, b):

$$\frac{f(a+h, b) - f(a, b)}{h}.$$

Difference quotient for variable y at the point (a, b):

$$\frac{f(a, b+h) - f(a, b)}{h}.$$

Partial derivatives of f at the point $(a, b) \in D$:

$$f_x(a, b) = \frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h},$$

$$f_y(a, b) = \frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h},$$

if the limit exists. (Result: a real number).

If the limit exists for all $(a, b) \in D$:
Partial derivative with respect to x (y, resp.):

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Result: functions $f_x : D \to \mathbb{R}$ and $f_y : D \to \mathbb{R}$.
∂ spoken "dee" or "del' emphasizes partial derivatives'.
(Attention! Textbook: $f_1 = f_x$ and $f_2 = f_y$.)
Example: $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$ (saddle surface).
$f_x(x, y) = 2x$, $f_y(x, y) = -2y$.

Functions of n variables: $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $z = f(x_1, \ldots, x_n)$.
Difference quotient for variable x_i ($1 \leq i \leq n$):

$$\frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}.$$

Partial derivatives:

$$\begin{aligned} f_{x_i}(x_1, \ldots, x_n) &= \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \\ = & \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}. \end{aligned}$$

Note: For a function $f : \mathbb{R}^n \to \mathbb{R}$ the partial derivative w.r.t. x_i at $(a_1, \ldots, a_n)$ is the derivative at a_i of the function $g : \mathbb{R} \to \mathbb{R}$
$g(x_i) = f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$.

Partial derivatives w.r.t. x_i: Treat all variables except x_i as constants and form the derivative w.r.t. the variable x_i.

Tangent planes

$f : D \subseteq \mathbb{R} \to \mathbb{R}$ differentiable in $a \in \mathbb{R}$.
Equation of the tangent line at $a \in \mathbb{R}$:

$$y = f(a) + f'(a)(x - a).$$

$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ partially differentiable w.r.t. x and y and f_x and f_y continuous in $(a, b) \in D$ (D open subset of $\mathbb{R}^2$).
Equation of the tangent plane at $(a, b) \in D$:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ partially differentiable w.r.t. x_i and f_{x_i} continuous for all i with $1 \leq i \leq n$ in $(a_1, \ldots, a_n) \in D$ (D open subset of $\mathbb{R}^n$).
Equation of the tangent space (hyperplane) at $(a_1, \ldots, a_n) \in D$:

$$z = f(a_1, \ldots, a_n) + f_{x_1}(a_1, \ldots, a_n)(x_1 - a_1) + \ldots + f_{x_n}(a_1, \ldots, a_n)(x_n - a_n)$$

$$z = f(a_1, \ldots, a_n) + \sum_{i=1}^{n} f_{x_i}(a_1, \ldots, a_n)(x_i - a_i).$$

Tangent planes (Example)

$f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 - y^2$.
Determine the tangent plane at f in $(2,1)$.
$f_x(x,y) = 2x$ and $f_y(x,y) = -2y$ are both continuous in $(x,y) = (2,1)$ (actually on the whole set $\mathbb{R}^2$).
Therefore the tangent plane consists of all points (x,y,z) satisfying the equation

$$\begin{aligned} z &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 4 - 1 + 4(x-2) - 2(y-1) = 4x - 2y - 3. \end{aligned}$$

$f_x : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ partially differentiable w.r.t. x and y in $(a, b) \in \mathbb{R}^2$.
Second order partial derivatives:

$$f_{xx}(a,b) = (f_x)_x(a,b) = \frac{\partial f_x}{\partial x}(a,b) = \frac{\partial^2 f}{\partial x^2}(a,b).$$

$$f_{xy}(a,b) = (f_x)_y(a,b) = \frac{\partial f_x}{\partial y}(a,b) = \frac{\partial^2 f}{\partial x \partial y}(a,b).$$

$f_y : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ partially differentiable w.r.t. x and y in $(a, b) \in \mathbb{R}^2$.
Second order partial derivatives:

$$f_{yx}(a,b) = (f_y)_x(a,b) = \frac{\partial f_y}{\partial x}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b).$$

$$f_{yy}(a,b) = (f_y)_y(a,b) = \frac{\partial f_y}{\partial y}(a,b) = \frac{\partial^2 f}{\partial y^2}(a,b).$$

Higher-order partial derivatives

$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ two times partially differentiable w.r.t. x and y in $(a, b) \in \mathbb{R}^2$. Second order partial derivatives:

$$f_{xx}(a, b) = (f_x)_x(a, b) = \frac{\partial f_x}{\partial x}(a, b) = \frac{\partial^2 f}{\partial x^2}(a, b).$$

$$f_{xy}(a, b) = (f_x)_y(a, b) = \frac{\partial f_x}{\partial y}(a, b) = \frac{\partial^2 f}{\partial x \partial y}(a, b).$$

$$f_{yx}(a, b) = (f_y)_x(a, b) = \frac{\partial f_y}{\partial x}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

$$f_{yy}(a, b) = (f_y)_y(a, b) = \frac{\partial f_y}{\partial y}(a, b) = \frac{\partial^2 f}{\partial y^2}(a, b).$$

Example: $f(x, y) = x^2 - y^2$.
$f_{xx}(x, y) = 2$
$f_{xy}(x, y) = 0$
$f_{yx}(x, y) = 0$
$f_{yy}(x, y) = -2$.

Exchanging the order of partial derivatives

In many cases $f_{xy} = f_{yx}$. By chance?

Theorem of Schwarz

If D is an open set, $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ two times partially differentiable and the partial derivatives f_{xy} and f_{yx} are both continuous then

$$f_{yx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x, y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x, y)\right) = f_{xy}(x, y).$$