Matematiske metoder (MM 529)

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Definition: Ordinary differential equation (ODE)

An ordinary differential equation is an equation of the form

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}),$$

where $y = y(x)$ is a function of the variable x.
The highest derivative n occurring in the equation is the order of the ODE.

Counterpart: partial differential equation (PDE) with more than one variable x.
The general solution: all functions $y(x)$ that satisfy the equality.

First order differential equation

$$y'(x) = f(x, y) = f(x, y(x)).$$

The derivative of y at x depends on x and on the function value $y(x)$.

Initial value problem

Solve $y'(x) = f(x, y)$ subject to $y(x_0) = a$.

Geometric interpretation:

$$y'(x) = f(x, y).$$

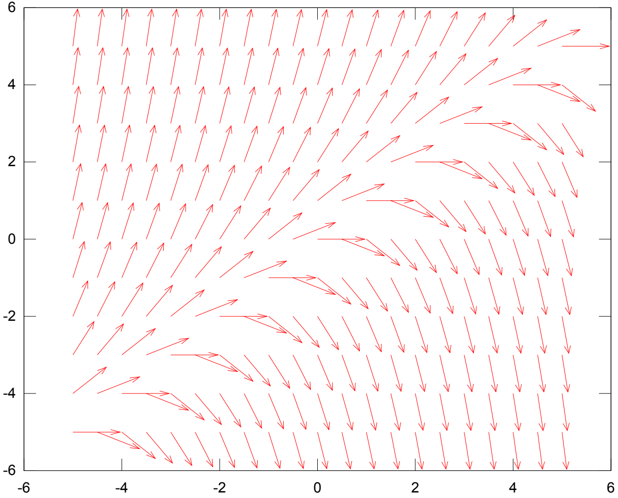
For every point (x, y) of the plane, $f(x, y)$ is the derivative of the function $y(x)$ at x.

Slope field

To every point (x, y) of the plane we assign the slope $f(x, y)$.

Solutions of the ODE: any function whose graph follows the slopes of the slope field.

Slope field of $y' = f(x, y) = y - x$: Solution $y = x + 1$ visible.



Solving first order ODE

General idea: integration.
$y'(x) = f(x, y)$. Therefore

$$y(x) = \int y'(x)\,dx = \int f(x, y)\,dx.$$

Problem: The unknown function $y(x)$ appears on the left and on the right. But:
If y' does not depend on y, i.e. $y'(x) = g(x)$ then the solution is

$$y(x) = \int y'(x)\,dx = \int g(x)\,dx.$$

Example: $y'(t) = e^t + \frac{1}{3}t^2$.
General solution: All functions $y(t)$ that satisfy
$y(t) = \int (e^t + 3t^2)\,dt = \int e^t\,dt + 3\int t^2\,dt = e^t + t^3 + C$.

Separable ODE

A first order differential equation $y' = f(x, y)$ is separable, if $f(x, y) = g(x) \cdot h(y)$.

Examples:

$$y' = \frac{y^2}{x}, \quad y' = (xy)^2, \quad y' = \frac{e^x}{y}, \quad y' = e^{x+y}.$$

Non-separable:

$$y' = y - x, \quad y' = e^{xy}.$$

Separable ODE

$$y'(x) = \frac{dy}{dx} = g(x) \cdot h(y).$$

Separation of variables:

$$\frac{1}{h(y)} y'(x) = \frac{1}{h(y)} \frac{dy}{dx} = g(x).$$

Solution:

$$\int \frac{1}{h(y)} y'(x)\, dx = \int g(x)\, dx,$$

or (a little sloppy):

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx.$$

We integrate the right hand side by x and the left hand side by y (treating the function y like a variable).

Solution: Separation of variables, example

Separable ODE

$$y'(x) = \frac{dy}{dx} = x \cdot y.$$

Separation of variables:

$$\frac{1}{y}\frac{dy}{dx} = x.$$

One solution is the constant function $y(x) = 0$. Further solutions:

$$\int \frac{1}{y}\,dy = \int x\,dx.$$

$$\ln|y| = \frac{1}{2}x^2 + C, \quad C \in \mathbb{R}.$$

$$|y| = e^{x^2/2} \cdot e^C = \widetilde{C}e^{x^2/2}, \quad \widetilde{C} > 0, \text{ therefore}$$

$$y = \pm\widetilde{C}e^{x^2/2}, \quad \widetilde{C} > 0.$$

All solutions (together with $y(x) = 0$ for $c = 0$):
$y = ce^{x^2/2}, \quad c \in \mathbb{R}.$

Solution: Separation of variables, example

Separable ODE

$$y'(x) = \frac{dy}{dx} = x \cdot y.$$

All solutions:

$$y = ce^{x^2/2}, \quad c \in \mathbb{R}.$$

Initial value problem (IVP): $y(1) = 1$.
Single solution of IVP:

$$1 = y(1) = ce^{1/2}, \quad \text{thus } c = \frac{1}{\sqrt{e}} = 0.606\ldots$$

and the solution is

$$y(x) = e^{(x^2-1)/2}.$$

Indeed, $y(1) = 1$ and the chain rule gives

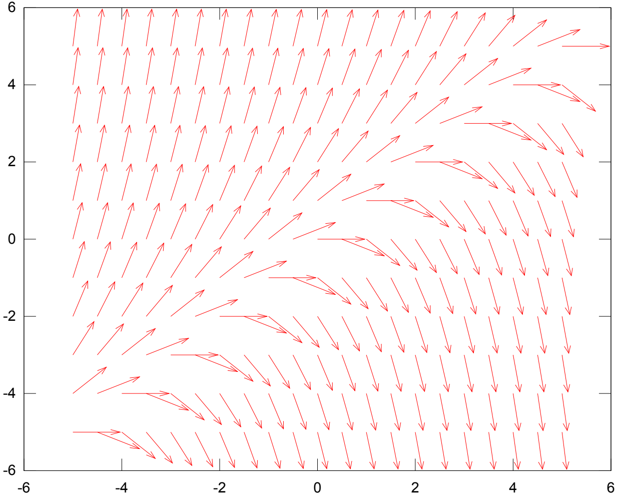
$$y'(x) = \frac{1}{2} 2xe^{(x^2-1)/2} = x \cdot y(x).$$

Problem: First order differential equation with initial value condition:

$$y' = f(x, y), \text{ and } y(x_0) = y_0.$$

Estimate $y(x)$ (e.g., because you cannot calculate the function $y(x)$ explicitly).
Idea: Follow the slope field.
Euler method: Split interval $[x_0, x]$ into n sub-intervals of equal length $h = \frac{|x - x_0|}{n}$.
Recursively, calculate
$y_1 = y_0 + hf(x_0, y_0)$
$y_2 = y_1 + hf(x_0 + h, y_1)$
$y_3 = y_2 + hf(x_0 + 2h, y_2)$
$\vdots$
$y_n = y_{n-1} + hf(x_0 + (n-1)h, y_{n-1})$.
Hope: $y_n \approx y(x)$ for the solution $y(x)$ of the IVP.

Slope field of $y' = f(x, y) = y - x$.



Example: First order differential equation with initial value condition:

$$y' = f(x, y) = y - x, \text{ and } y(0) = 0.$$

Correct function:
$y(x) = x + 1 - e^x$. E.g. $y(2) = 3 - e^2 = -4.389\ldots$.
Estimate $y(2)$ with Eulers method: Split interval $[0, 2]$ into 2 sub-intervals of equal length $h = \frac{|2-0|}{2} = 1$.
Calculate
$y_1 = y_0 + hf(x_0, y_0) = 0 + 1(0 - 0) = 0$,
$y_2 = y_1 + hf(x_0 + h, y_1) = 0 + 1(0 - 1) = -1$,
far away from the correct value of $y(2)$.
Improvement: Halved interval length $h = \frac{1}{2}$.
$y_1 = y_0 + hf(x_0, y_0) = 0 + \frac{1}{2}(0 - 0) = 0$
$y_2 = y_1 + hf(x_0 + h, y_1) = 0 + \frac{1}{2}(0 - \frac{1}{2}) = -\frac{1}{4}$,
$y_3 = y_2 + hf(x_0 + 2h, y_2) = -\frac{1}{4} + \frac{1}{2}(-\frac{1}{4} - 1) = -\frac{1}{4} - \frac{5}{8} = -\frac{7}{8}$
$y_4 = y_3 + hf(x_0 + 3h, y_3) = -\frac{7}{8} + \frac{1}{2}(-\frac{7}{8} - \frac{3}{2}) = -\frac{7}{8} - \frac{19}{16} = -\frac{33}{16}$,
better, but still far away.

Results with Euler method (in this particular case) unsatisfactory.
Two possible solutions:

1. work with much smaller interval length h (difficult by hand but not for a computer)
2. improve the method.

Idea of improved Euler method (Heun's method):
Take the slope at the initial point and at the potential endpoint, and take the average of both to find the next point:
See exercises!

Linear differential equation (LDE)

A linear differential equation of order n is an ordinary differential equation of the form

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=g(x),$$

where $a_i(x)$ and $g(x)$ are functions (only) depending on x.

Examples:

$$y'-y=-x; \quad y''-2y'+y=0; \quad y^{(3)}-e^x y'=\sin x.$$

Counterexamples (not linear):

$$y'\cdot y=0; \quad y'-y^2=e^x; \quad y'=e^y.$$

Homogeneous/inhomogeneous linear differential equation

A linear differential equation

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+a_{n-2}(x)y^{(n-2)}+\ldots+a_1(x)y'+a_0(x)y=g(x),$$

is homogeneous, if $g(x)=0$ (for every x), otherwise inhomogeneous.

Examples (homogeneous):

$$y''-2y'+y=0; \quad y'=y$$

Examples (inhomogeneous):

$$y''-2y'+y=\sin x; \quad y'=y-x.$$

Solution of first order homogeneous linear DEs

First order homogeneous linear differential equation:

$$y' = a(x)y$$

is separable. One solution: $y(x) = 0$. Further solutions with separation of variables:

$$\int \frac{1}{y}\,dy = \int a(x)\,dx, \text{ therefore}$$

$$\ln|y| = A(x) + C,$$

where C is an arbitrary constant in $\mathbb{R}$ and $A(x)$ is an antiderivative of $a(x)$ (i.e. $A'(x) = a(x)$). Exponentiating both sides and tracking the constants (see last lecture), we get the general solution

$$y = ce^{A(x)}, \text{ where } c \text{ is an arbitrary constant in } \mathbb{R}.$$

First order homogeneous linear differential equation:

$$y' = a(x)y$$

General solution:

$$y = ce^{A(x)},$$

where $c \in \mathbb{R}$ and $A'(x) = a(x)$.

Example: Radioactive decay

$$y'(t) = -\lambda y(t).$$

Here $a(t) = -\lambda$, therefore $A(t) = -\lambda t$ is an antiderivative, and $y(t) = c \cdot e^{-\lambda t}$ is the general solution.
Starting with an initial amount $y_0 = y(0)$ at time $t = 0$:
$c = y_0$ yields the single solution of the initial value problem.

Linearity of differentiation:

$$(f(x)+g(x))' = f'(x)+g'(x),$$
$$(c\cdot f(x))' = c\cdot f'(x).$$

First order inhomogeneous LDE:

$$y' + a(x)y = g(x).$$

Corresponding homogeneous LDE:

$$y' + a(x)y = 0.$$

Let $y_p(x)$ be a solution of the inhomogeneous LDE and $y_h(x), y_H(x)$ be solutions of the corresponding homogeneous LDE. Then

$$(y_h+y_H)'+a(x)(y_h+y_H) = y_h'+a(x)y_h+y_H'+a(x)y_H = 0+0 = 0,$$

therefore $y_h + y_H$ is another solution of the corresponding homogeneous LDE (similarly $c\cdot y_h$ is a solution for any $c\in\mathbb{R}$).

First order inhomogeneous LDE:

$$y' + a(x)y = g(x).$$

Corresponding homogeneous LDE:

$$y' + a(x)y = 0.$$

$y_p(x)$ solution of inhomogeneous LDE.
$y_h(x), y_H(x)$ solutions of corresponding homogeneous LDE.
Then $y_h + y_H$ is another solution of the homogeneous LDE and

$$(y_h+y_p)'+a(x)(y_h+y_p) = y_h'+a(x)y_h+y_p'+a(x)y_p = 0+g(x) = g(x),$$

therefore $y_h + y_p$ is another solution of the inhomogeneous LDE.