Matematiske metoder (MM529)

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Polynomial

A (real) polynomial is a function $p : \mathbb{R} \to \mathbb{R}$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where the coefficients $a_i \in \mathbb{R}$ for $i \in \{0, 1, \ldots, n-1, n\}$ and $a_n \neq 0$.
The number a_n is called the leading coefficient.
The degree $n = \deg p$ of p is the index of the leading coefficient.

The zero-polynomial $p(x) = 0$ is the function mapping every $x \in \mathbb{R}$ to 0.
Its degree is defined to be $-\infty$.

Example: Multiplication of two polynomials

$$(x^3 - 3x^2 + 3x + 1)(2x^2 + x + 4) = 2x^5 - 5x^4 + 7x^3 - 7x^2 + 13x + 4.$$

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$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0.$$

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Product of polynomials

$$p(x) \cdot q(x) = c_{n+m} x^{n+m} + c_{n+m-1} x^{n+m-1} + \ldots + c_1 x + c_0,$$

where $c_k = a_0 b_k + a_1 b_{k-1} + \ldots + a_{k-1} b_1 + a_k b_0$ (telescope sum).

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Degree $\deg(p \cdot q) = \deg p + \deg q$.

Rational function

A rational function is a function of the form

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Famous rational function

$$\begin{aligned} f : \mathbb{R} \setminus \{0\} &\to \mathbb{R} \\ f(x) &= \frac{1}{x}. \end{aligned}$$

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Law of radioactive decay: $W(t) = W_0 e^{-\lambda t}$.
Decay rate λ only depending on the isotope.

Properties of functions

A function $f : A \to B$ is called

- injective, if every $b \in B$ has at most one preimage,
- surjective, if every $b \in B$ has at least one preimage,
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Interpretation in the graph

A function $f : \mathbb{R} \to \mathbb{R}$ is

- injective, if every horizontal line intersects the graph in at most one point,
- surjective, if every horizontal line intersects the graph in at least one point,
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Injective, surjective, and bijective functions

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Note: Every injective function with the codomain restricted to the range is bijective.

Inverse function

For a bijective function $f : A \to B$, $f(x) = y$, there is a unique function $f^{-1} : B \to A$ with

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What is the inverse of f^{-1}? $(f^{-1})^{-1} = f$.

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Most common bases:
2 (computer sci.), 10 (economics), $e = 2.718\ldots$ (natural sci.).

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Application: Logarithmic scale

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(e.g. 100 decibel is a thousend times louder than 70 decibel).

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$$\tan : A \to \mathbb{R}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}},$$

where $A = \mathbb{R} \setminus \{(2k+1)\frac{\pi}{2} \mid k \in \mathbb{Z}\}$.

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Sums of angles:

$$\sin(\theta+\varphi) = \sin\theta\ \cos\varphi + \sin\varphi\ \cos\theta$$

$$\cos(\theta+\varphi) = \cos\theta\ \cos\varphi - \sin\theta\ \sin\varphi$$

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Inverse functions

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right];$$

$$\arccos : [-1, 1] \to [0, \pi];$$

$$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

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Convergence and the limit of a sequence

A sequence $(a_n)_{n\in\mathbb{N}_0}$ is convergent to a limit $\gamma \in \mathbb{R}$ if for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}_0$ such that $|\gamma - a_n| < \varepsilon$ for all $n > n_0$.

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Let (a_n) be a divergent sequence. If for every $K \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}_0$ such that for every $n > n_0$

- $a_n > K$ then we say $\lim_{n\to\infty} a_n = \infty$;
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Examples: $\lim\limits_{n\to\infty} n = \infty$, $\lim\limits_{n\to\infty} \ln \frac{1}{n} = -\infty$.

$$(a_0, a_1, a_2, a_3, a_4, \ldots) = (0, 0.3, 0.33, 0.333, 0.3333, \ldots) \to \tfrac{1}{3}.$$

Examples of limits

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If $\varepsilon \geq 10^{-k} = \frac{1}{10^k}$ take $n_0 = k + 1$.

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Fibonacci sequence:
$(F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, \ldots) = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$:

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$((-1)^n)_{n\in\mathbb{N}_0} = (1, -1, 1, -1, 1, \ldots)$ is not convergent (take $\varepsilon = \frac{1}{2}$) and has no infinite limit (take $K = 0$).

Limit laws for convergent series

Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then

(1) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(2) $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$;

(3) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, if $b_n \neq 0$ for all $n \in \mathbb{N}_0$;

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