Matematiske metoder (MM 529)

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Rational function:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P, Q are polynomials.
If $\deg P \geq \deg Q$ then divide P by Q with rest to obtain

$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$

where R, S are polynomials and $\deg R < \deg Q$.

$$\int f(x)\,dx = \int \left(S(x) + \frac{R(x)}{Q(x)} \right) dx = \int S(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx$$

Since $\int S(x)\,dx$ easy to calculate, main objective:
Calculate $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$, where $\deg P < \deg Q$.

Integration of rational functions

Example:

$$f(x) = \frac{x^2}{2x-1}$$

where $\deg x^2 = 2 \geq 1 = \deg(2x-1)$.
Divide x^2 by $2x-1$ to obtain

$$f(x) = \frac{x}{2} + \frac{1}{4} + \frac{1}{4(x-1)}$$

$$\begin{aligned}
\int f(x)\,dx &= \int \frac{x}{2}\,dx + \int \frac{1}{4}\,dx + \int \frac{1}{4(2x-1)}\,dx \\
&= \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{4}\int \frac{1}{2x-1}\,dx
\end{aligned}$$

Substituting $u = 2x-1$, $du = 2dx$, we obtain

$$\int \frac{1}{2x-1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|2x-1| + C.$$

Integration of rational functions, deg Q = 1

Determine

$$f(x) = \frac{A}{ax+b}$$

where $A, a \neq 0$, hence $\deg A = 0 < 1 = \deg(ax+b)$.

$$\int f(x)\,dx = A \int \frac{1}{ax+b}\,dx$$

Substituting $u = ax+b$, $du = adx$, we obtain

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\int \frac{1}{u}\,du = \frac{A}{a}\ln|ax+b| + C.$$

Integration of rational functions, deg Q = 2

$$f(x) = \frac{Ax + B}{ax^2 + bx + c}$$

where $a \neq 0$, hence $\deg(Ax + B) \leq 1 < 2 = \deg(ax^2 + bx + c)$.
First case: $Q(x) = ax^2 + bx + c$ has a zero α where $Q(\alpha) = 0$.
Dividing Q by $(x - \alpha)$ we get:
$Q(x) = ax^2 + bx + c = a(x - \alpha)(x - \beta)$ and if $\alpha \neq \beta$
$f(x) = \frac{1}{a}\left(\frac{A_1}{x-\alpha} + \frac{A_2}{x-\beta}\right)$.Therefore

$$\begin{aligned}
\int f(x)\,dx &= \frac{1}{a}\int\left(\frac{A_1}{x-\alpha} + \frac{A_2}{x-\beta}\right)dx \\
&= \frac{1}{a}\left(\int \frac{A_1}{x-\alpha}\,dx + \int \frac{A_2}{x-\beta}\,dx\right) \\
&= \frac{1}{a}\left(A_1 \ln|x-\alpha| + A_2 \ln|x-\beta|\right)
\end{aligned}$$

How to find A_1, A_2?

Integration of rational functions, deg Q = 2

Given A, B, a, b, c with $a \neq 0$. Determine A_1, A_2 in

$$f(x) = \frac{Ax + B}{ax^2 + bx + c} = \frac{1}{a}\left(\frac{A_1}{x-\alpha} + \frac{A_2}{x-\beta}\right)$$

Multiply by $Q(x) = ax^2 + bx + c = a(x-\alpha)(x-\beta)$:
Set $x = \alpha$ to determine $A_1 = \frac{A\alpha+B}{\alpha-\beta}$ and
set $x = \beta$ to determine $A_2 = \frac{A\beta+B}{\beta-\alpha}$.
Example:

$$f(x) = \frac{x+4}{x^2-5x+6} = \frac{A_1}{x-2} + \frac{A_2}{x-3}.$$

$A = 1,\ B = 4,\ a = 1,\ b = -5,\ c = 6$:
Multiply by $Q(x) = x^2 - 5x + 6 = (x-2)(x-3)$:

$$x + 4 = A_1(x-3) + A_2(x-2).$$

Set $x = 2$: $A_1 = \frac{1\cdot 2+4}{2-3} = -6$ and $x = 3$: $A_2 = \frac{1\cdot 3+4}{3-2} = 7$.
Hence $f(x) = \dfrac{x+4}{x^2-5x+6} = \dfrac{-6}{x-2} + \dfrac{7}{x-3}$.

$$f(x) = \frac{x+4}{x^2-5x+6} = \frac{-6}{x-2} + \frac{7}{x-3}.$$

Therefore

$$\begin{aligned} \int f(x)\,dx &= \int \frac{-6}{x-2}\,dx + \int \frac{7}{x-3}\,dx \\ &= -6\ln|x-2| + 7\ln|x-3| + C. \end{aligned}$$

If $\alpha = \beta$ then

$$f(x) = \frac{Ax+B}{ax^2+bx+c} = \frac{1}{a}\left(\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2}\right)$$

Again: Multiply by $Q(x)$:

$$Ax + B = A_1(x-\alpha) + A_2.$$

Set $x = \alpha$ to determine A_2, then determine A_1.

Example: $f(x) = \frac{2x+1}{(x-2)^2} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2}$.
Multiply by $Q(x)$:

$$2x + 1 = A_1(x-2) + A_2.$$

Set $x = 2$ to determine $A_2 = 2 \cdot 2 + 1 = 5$, then determine A_1, e.g. by setting $x = 3$:
$A_1 = 2 \cdot 3 + 1 - A_2 = 2$. Therefore:

$$f(x) = \frac{2x+1}{(x-2)^2} = \frac{2}{x-2} + \frac{5}{(x-2)^2}.$$

Second case: $Q(x)$ has no zeroes. No simplification possible.
In general: Rational function can be written as a sum of terms of the form $\frac{A}{(x-\alpha)^k}$ and $\frac{Ax+B}{(x^2+ax+b)^k}$.
In particular: If $Q(x)$ has degree n and n different real zeroes $\alpha_1, \ldots, \alpha_n$ then

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{A_i}{x - \alpha_i}.$$

Aim: Measure the area

$$\int_a^b f(x)\,dx, \quad \text{if}$$

1. $a = -\infty$ or $b = +\infty$ (infinite integration limits), or
2. $\lim\limits_{x\to a^+} f(x) = \pm\infty$ or $\lim\limits_{x\to b^-} f(x) = \pm\infty$
   (integrand unbounded at an endpoint).

We required that f is bounded on $[a, b]$.

Approach: take limits! Infinite integration limits:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx \quad \text{for any } b \in \mathbb{R}.$$

Improper integrals

Aim: Measure the area

$$\int_a^b f(x)\,dx, \quad \text{if}$$

the integrand f is unbounded at an endpoint:

$$\lim_{x\to b^-} f(x) = \pm\infty \quad \text{then} \quad \int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{then} \quad \int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

Rule: Never integrate over points, where f is unbounded.
Split the integral at points, where f is unbounded.
If f unbounded at $c \in [a, b]$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

if both integrals finite (otherwise the integral is divergent).

Improper integrals, examples

$$\int_0^\infty e^{-x}\,dx = ?$$

With substitution:

$$\int e^{-x}\,dx = -e^{-x} + C.$$

Therefore

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-b} + e^{-0} = 0 + 1 = 1$$

Finite area!

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b x^{-1}\,dx = \lim_{b\to\infty} \ln|b| - \ln|1| = \lim_{b\to\infty} \ln b = +\infty.$$

Integral divergent, infinite area!

Numerical integration, motivation

Some functions not integrable, in the sense that they cannot be represented in terms of standard functions (while derivatives of standard functions are again standard functions).
Example: the sine integral

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

is (provably) not expressable via an antiderivative in standard functions.

Data from practical applications behave like an unknown function.
How can we estimate integrals?
Answer: Riemann sums.
Mean value theorem for integrals:

$$\int_a^b f(x)\, dx = f(c)(b-a) \quad \text{for a suitable } c \in (a,b).$$

Numerical integration, approximation by rectangles

Mean value theorem for integrals:

$$\int_a^b f(x)\,dx = f(c)(b-a) \quad \text{for a suitable } c \in (a,b).$$

Where would you expect c to be?
Rather in the middle of the interval (a,b).
Approach: Split the interval $[a,b]$ into n sub-intervals $[x_i, x_{i+1}]$ of equal length $h = \frac{b-a}{n}$, $(a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b)$.
Estimate

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx f\left(\frac{x_i + x_{i+1}}{2}\right) h.$$

Therefore:

Midpoint rule

$$\int_a^b f(x)\,dx \approx h \cdot \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right).$$

Numerical integration, approximation by trapezoids

Midpoint rule:
Split interval $[a, b]$ into n sub-intervals $[x_i, x_{i+1}]$ of equal length $h = \frac{b-a}{n}$, $(a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b)$.

$$\int_a^b f(x)\,dx \approx h \cdot \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right).$$

Two problems:

1. convergence not quick enough, and
2. halving the sub-interval length requires a new set of function values.

Better: Approximation by trapezoids:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \left(\frac{f(x_i) + f(x_{i+1})}{2}\right) h.$$

(Arithmetic mean of the function values, not of the arguments)

Numerical integration, trapezoid rule

Approximation by trapezoids:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \left(\frac{f(x_i)+f(x_{i+1})}{2}\right) h.$$

Trapezoid rule

Split interval $[a,b]$ into n sub-intervals $[x_i, x_{i+1}]$ of equal length $h = \frac{b-a}{n}$, $(a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b)$.

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{2}\sum_{i=0}^{n-1}(f(x_i)+f(x_{i+1})) \\ &= \frac{h}{2}(f(x_0)+2f(x_1)+\ldots+2f(x_{n-1})+f(x_n))\end{aligned}$$

Numerical integration, trapezoid rule, example

Trapezoid rule

$$\int_a^b f(x)\,dx \approx \frac{h}{2}(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n))$$

Example:

$$\int_0^4 x^3\,dx = \frac{1}{4}4^4 - \frac{1}{4}0^4 = 4^3 = 64.$$

Trapezoid rule with 4 sub-intervals of equal length 1:

$$\int_0^4 x^3\,dx \approx \frac{1}{2}(0^3 + 2\cdot 1^3 + 2\cdot 2^3 + 2\cdot 3^3 + 4^3) = 68.$$

Your boss wants a better estimate? No problem:

Numerical integration, trapezoid rule, halving the interval length

Trapezoid sum

n intervals of length $h = \frac{b-a}{n}$:

$$T_n = \frac{h}{2}(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)).$$

Halving the intervals: $2n$ intervals of length $\frac{h}{2} = \frac{b-a}{2n}$:

$$T_{2n} = \frac{h}{4}(f(y_0) + 2f(y_1) + \ldots + 2f(y_{2n-1}) + f(y_{2n})),$$

where $y_{2i} = x_i$. Therefore

Trapezoid sum, halved interval length

$$\begin{aligned} T_{2n} &= \frac{1}{2}T_n + \frac{h}{4}(2f(y_1) + 2f(y_3) + \ldots + 2f(y_{2n-1})) \\ &= \frac{1}{2}T_n + \frac{h}{2}(f(y_1) + f(y_3) + \ldots + f(y_{2n-1})) \end{aligned}$$

Numerical integration, trapezoid rule, halving the interval length, example

Trapezoid sum, halved interval length

$$
\begin{aligned}
T_{2n} &= \frac{1}{2}T_n + \frac{h}{4}\left(2f(y_1) + 2f(y_3) + \ldots + 2f(y_{2n-1})\right) \\
&= \frac{1}{2}T_n + \frac{h}{2}\left(f(y_1) + f(y_3) + \ldots + f(y_{2n-1})\right)
\end{aligned}
$$

Example with four intervals of length 1:

$$\int_0^4 x^3\,dx \approx T_4 = \frac{1}{2}(0^3 + 2\cdot 1^3 + 2\cdot 2^3 + 2\cdot 3^3 + 4^3) = 68.$$

Your boss wants a better estimate! Eight intervals of length $\frac{1}{2}$:

$$
\begin{aligned}
T_8 &= \frac{1}{2}T_4 + \frac{1}{2}\left(\left(\frac{1}{2}\right)^3 + \left(\frac{3}{2}\right)^3 + \left(\frac{5}{2}\right)^3 + \left(\frac{7}{2}\right)^3\right) \\
&= \frac{68}{2} + \frac{1}{16}(1^3 + 3^3 + 5^3 + 7^3) = 34 + 31 = 65
\end{aligned}
$$