Matematiske metoder (MM 529)

Stephan Brandt

Syddansk Universitet, Odense

05. 11. 2013

The definite integral

Let f be a function on $[a, b]$ and $(P_k)_{k\in\mathbb{N}}$ be a sequence of partitions of $[a, b]$. If $\lim\limits_{k\to\infty} ||P_k|| = 0$ then the definite integral of f is

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} S_k,$$

if the limit exists. The value of the limit is independent of the choice of the partitions P_k and the intermediate points c_i.

Existence of the definite integral

If f is piecewise continuous on $[a, b]$ and there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ (f is bounded) then $\displaystyle\int_a^b f(x)\,dx$ exists.

Agreement: $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Rules: $\int_a^b f(x)\,dx = \int_a^t f(x)\,dx + \int_t^b f(x)\,dx$ for any $t \in [a,b]$.
Linearity:

- $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$;
- $\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$ for any $c \in \mathbb{R}$.

Examples:

$$\int_0^2 \lceil x \rceil\,dx = \int_0^1 \lceil x \rceil\,dx + \int_1^2 \lceil x \rceil\,dx = 1 + 2 = 3.$$

$$\int_{-2}^2 -|x|\,dx = -\int_{-2}^2 |x|\,dx = -\int_{-2}^0 |x|\,dx - \int_0^2 |x|\,dx = -2-2 = -4.$$

Problem: How to find definite integrals without computing limits?

Antiderivative

The function F is an antiderivative of f on the interval I if $F'(x) = f(x)$ for every $x \in I$.

Notation: $\int f(x)\,dx$ denotes the set of all functions F with this property (also called the indefinite integral).

Observation: By the differentiation rules, for any antiderivative F of f, $F + C$ is also an antiderivative for any $C \in \mathbb{R}$.

But the converse also holds:

If F and G are antiderivatives of f then

$$0 = F'(x) - G'(x) = \frac{d}{dx}(F(x) - G(x))$$

and the only differentiable functions whose derivatives are zero on the whole interval are constant functions, i.e.

$F(x) - G(x) = C \in \mathbb{R}$.

Common (sloppy) notation: $\int f(x)\,dx = F(x) + C$.

Antiderivative

The function F is an antiderivative of f on the interval I if $F'(x) = f(x)$ for every $x \in I$.

How to find $\displaystyle\int f(x)\,dx$?

First option: Looking it up in a table (for standard functions).
Second option: Knowing a function F with $F'(x) = f(x)$, e.g.

- $\frac{d}{dx}\sin x = \cos x$, hence $\int \cos x \; dx = \sin x + C$;
- $\frac{d}{dx}e^x = e^x$, hence $\int e^x \; dx = e^x + C$;
- $\frac{d}{dx}x^3 = 3x^2$, hence $\int 3x^2 \; dx = x^3 + C$.

Third option: Reversing the differentiation rules. Linearity:

- $\int (f(x) + g(x)) dx = \int f(x)\,dx + \int g(x)\,dx$;
- $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$ for any $c \in \mathbb{R}$.

Reversing product rule and chain rule:later.

What is the relation between definite integrals and antiderivatives (except the common notation)?

Fundamental Theorem of Calculus

If f is continuous on the interval $I = [a, b]$ then

(1) $F(x) = \int_a^x f(u)\,du$ is an antiderivative of f on I, and

(2) for any antiderivative F of f, $\int_a^b f(x)\,dx = F(b) - F(a)$.

Good news, by (2) no need to calculate limits for the definite integral if F is known.
Examples:

$$\ln x = \int_1^x \frac{1}{u}\,du = \int_1^x \frac{du}{u}$$

for $x > 0$, since $\frac{d}{dx}\ln x = \frac{1}{x}$. So $C + \ln x$ are the antiderivatives of $\frac{1}{x}$ and since for $x = 1$, $\int_1^1 \frac{1}{u}\,du = 0 = \ln 1$, therefore $C = 0$.

Fundamental Theorem of Calculus

If f is continuous on the interval $[a, b]$ then

(1) $F(x) = \displaystyle\int_a^x f(u)\,du$ is an antiderivative of f, and

(2) for any antiderivative F of f, $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

Examples:

$$\int_0^\pi \sin x\,dx = -\cos\pi - -\cos 0 = 1 + 1 = 2.$$

$$\int_{-\pi}^{\pi} (3e^x - \cos x)\,dx = 3\int_{-\pi}^{\pi} e^x\,dx - \int_{-\pi}^{\pi} \cos x\,dx = 3(e^\pi - e^{-\pi}).$$

Fundamental Theorem of Calculus, "counterexample"

Fundamental Theorem of Calculus

If f is continuous on the interval $I = [a, b]$ then

(1) $F(x) = \int_a^x f(u)\,du$ is an antiderivative of f on I, and

(2) for any antiderivative F of f, $\int_a^b f(x)\,dx = F(b) - F(a)$.

Example: $F(x) = -\frac{1}{x}$ is an antiderivative of $f(x) = \frac{1}{x^2}$. So

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = F(1) - F(-1) = -1 - 1 = -2?$$

But: The graph of f shows that the area is positive! What went wrong? f is unbounded near $x = 0$ (and not continuous).
Piecewise continuous functions can be split up at discontinuities: $\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$.
How to handle unboundedness, next time!

... that invert the product rule and the chain rule:

Product rule

If f and g are differentiable functions then

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

Chain rule

If f and g are functions such that f is differentiable on the range of g, then the derivative of the composition is

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Aim: Inversion of the product rule.

Product rule

If f and g are differentiable functions then

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides (finding an antiderivative):

$$\begin{aligned} u(x)v(x) + C &= \int (u(x)v(x))'\,dx = \int (u'(x)v(x) + u(x)v'(x))\,dx \\ &= \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx \end{aligned}$$

Therefore:

Rule for integration by parts

If u and v are differentiable functions then

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

Rule for integration by parts

If u and v are differentiable functions then

$$\int uv'\,dx = uv - \int u'v\,dx.$$

$$\text{Example:} \quad \int x\cos x\,dx = ?$$

Choose u and v appropriately to solve the right hand side:
If $u(x) = x$ and $v'(x) = \cos x$ then
the derivative $u'(x) = 1$ and an antiderivative is $v(x) = \sin x$.
Therefore:

$$\begin{aligned} \int x\cos x\,dx &= x\sin x - \int 1\cdot\sin x\,dx \\ &= x\sin x + \cos x + C. \end{aligned}$$

Rule for integration by parts

If u and v are differentiable functions then

$$\int uv'\,dx = uv - \int u'v\,dx.$$

$$\text{Example:} \quad \int \ln x\,dx = ?$$

Where is the second factor?
$u(x) = \ln x,\ v'(x) = 1$
$u'(x) = \frac{1}{x},\ v(x) = x.$
Therefore:

$$\begin{aligned} \int \ln x\,dx &= x\ln x - \int \frac{1}{x}\cdot x\,dx \\ &= x\ln x - x + C = x(-1+\ln x) + C, \end{aligned}$$

for $x \in (0,\infty)$.

Rule for integration by parts

If u and v are differentiable functions then

$$\int uv'\,dx = uv - \int u'v\,dx.$$

$$\text{Example:} \quad \int \cos^2 x\,dx = ?$$

$u(x) = \cos x,\ v'(x) = \cos x$
$u'(x) = -\sin x,\ v(x) = \sin x.$

$$\begin{aligned}
\int \cos^2 x\,dx &= \cos x \sin x - \int -\sin^2 x\,dx \\
&= \cos x \sin x + \int (1 - \cos^2 x)\,dx \\
&= \cos x \sin x + \int 1\,dx - \int \cos^2 x\,dx,
\end{aligned}$$

Integration by parts, examples

$u(x) = \cos x, \; v'(x) = \cos x$
$u'(x) = -\sin x, \; v(x) = \sin x.$

$$\begin{aligned}
\int \cos^2 x \, dx &= \cos x \sin x - \int -\sin^2 x \, dx \\
&= \cos x \sin x + \int (1 - \cos^2 x) \, dx \\
&= \cos x \sin x + \int 1 \, dx - \int \cos^2 x \, dx,
\end{aligned}$$

Therefore

$$2 \int \cos^2 x \, dx = \cos x \sin x + x + C, \text{ and}$$

$$\int \cos^2 x \, dx = \frac{\cos x \sin x + x + C}{2}.$$

Next aim: Inversion of the chain rule.

Chain rule

If F and g are functions such that F is differentiable on the range of g with derivative $F' = f$, then

$$F'(g(x)) = f(g(x))g'(x).$$

Integrating both sides (finding an antiderivative):

$$\begin{aligned} \int f(g(x))g'(x)\,dx &= \int F'(g(x))\,dx \\ &= F(g(x)) + C \end{aligned}$$

Therefore:

Substitution rule

If F and g are differentiable functions and $F' = f$ then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Substitution rule (definite integrals)

If F and g are differentiable functions and $F' = f$ then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

For the definite integral we obtain

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(t)\,dt.$$

Two versions, how to apply the substitution rule:

1. recognize the structure $f(g(x))g'(x)$ of the integrand, and
2. substitute a suitable function $g(t)$ for x in the integrand $f(x)$ of $\displaystyle\int f(x)\,dx$.

Substitution rule

If F and g are differentiable functions and $F' = f$ then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Recognize the structure:
Wanted:

$$\int f(g(x))g'(x)\,dx.$$

Substitute $t = g(x)$ then $\frac{dt}{dx} = \frac{dg(x)}{dx} = g'(x)$, therefore $dt = g'(x)dx$. We obtain:

$$\int f(g(x))g'(x)\,dx = \int f(t)\,dt = F(t) + C.$$

Back substitution $g(x) = t$:

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Substitution rule

If F and g are differentiable functions and $F' = f$ then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

$$\int \frac{(\ln x)^2}{x}\,dx = ?$$

Substitute $t = g(x) = \ln x$ then $dt = g'(x)dx = \frac{dx}{x}$. We obtain:

$$\int \frac{(\ln x)^2}{x}\,dx = \int t^2\,dt = \frac{1}{3}t^3 + C.$$

Back substitution $\ln x = t$:

$$\int \frac{(\ln x)^2}{x}\,dx = \frac{1}{3}(\ln x)^3 + C.$$

Substitution rule (definite integrals)

If F and g are differentiable functions on $[a, b]$ and $F' = f$ then

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(t)\,dt.$$

$$\int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$$

Substitute $t = \cos x$ then $dt = g'(x)dx = -\sin x\,dx$. We obtain:

$$\int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{t}\,dt = -\ln|t|\,dt, \quad \text{and}$$

$$\int_0^{\frac{\pi}{4}} \tan x\,dx = -\int_{\cos 0}^{\cos\frac{\pi}{4}} \frac{1}{t}\,dt = -\ln|\cos\frac{\pi}{4}| + \ln|\cos 0| = -\ln\cos\frac{\pi}{4}.$$

No back substitution needed!

Substitution rule

If F and g are differentiable functions and $F' = f$ then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Substitute a (new) function for x:
Wanted:

$$F(x) + C = \int f(x)\,dx.$$

Substitute $g(t) = x$ then $dx = g'(t)\,dt$, the function g being injective. We obtain:

$$H(t) + C = \int f(g(t))g'(t)\,dt = F(g(t)) + C.$$

Back substitution $g^{-1}(x) = t$:

$$F(x) + C = F(g(g^{-1}(x))) + C = H(g^{-1}(x)) + C.$$

Substitution rule, substituting a function, example

$$\int \sqrt{1-x^2}\,dx = ? \quad \text{for } |x| \le 1.$$

Substitution $x = g(t) = \sin t$ (injective on $[\frac{-\pi}{2}, \frac{\pi}{2}]$ with range $[-1, 1]$) then $dx = g'(t)dt = \cos t\, dt$. We obtain:

$$\begin{aligned} F(\sin t) &= \int \sqrt{1-\sin^2 t}\cos t\, dt = \int \cos^2 t\, dt \\ &= \frac{1}{2}(t + \sin t \cos t + C) \quad \text{(see above).} \end{aligned}$$

Back substitution $t = \arcsin x$:

$$\begin{aligned} F(x) &= \frac{1}{2}(\arcsin x + x\cos(\arcsin x) + C) \\ &= \frac{1}{2}(\arcsin x + x\sqrt{1-x^2} + C), \end{aligned}$$

using $\cos x = \sqrt{1-\sin^2 x}$! Integration sometimes needs creativity!