Supporting Information for "Inferential Approaches for Network Analysis: AMEN for Latent Factor Models"[☆]

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Additive and Multiplicative Effects Gibbs Sampler

To estimate, the effects of our exogenous variables and latent attributes we utilize a Bayesian probit model in which we sample from the posterior distribution of the full conditionals until convergence. Specifically, given observed data **Y** and **X** – where **X** is a design array that includes our sender, receiver, and dyadic covariates – we estimate our network of binary ties using a probit framework where: $y_{ij,t} = 1(\theta_{ij,t} > 0)$ and $\theta_{ij,t} = \boldsymbol{\beta}^\top \mathbf{X}_{ij,t} + a_i + b_j + \mathbf{u}_i^\top \mathbf{D}\mathbf{v}_j + \epsilon_{ij}$. The derivation of the full conditionals is described in detail in Hoff (2005) and Hoff (2008), thus here we only outline the Markov chain Monte Carlo (MCMC) algorithm for the AME model that we utilize in this paper.

- Given initial values of $\{\boldsymbol{\beta}, \mathbf{a}, \mathbf{b}, \mathbf{U}, \mathbf{V}, \Sigma_{ab}, \rho,$ and $\sigma_\epsilon^2\}$, the algorithm proceeds as follows:
  - sample $\boldsymbol{\theta} \mid \boldsymbol{\beta}, \mathbf{X}, \boldsymbol{\theta}, \mathbf{a}, \mathbf{b}, \mathbf{U}, \mathbf{V}, \Sigma_{ab}, \rho,$ and σ_ϵ^2 (Normal)
  - sample $\boldsymbol{\beta} \mid \mathbf{X}, \boldsymbol{\theta}, \mathbf{a}, \mathbf{b}, \mathbf{U}, \mathbf{V}, \Sigma_{ab}, \rho,$ and σ_ϵ^2 (Normal)
  - sample $\mathbf{a}, \mathbf{b} \mid \boldsymbol{\beta}, \mathbf{X}, \boldsymbol{\theta}, \mathbf{U}, \mathbf{V}, \Sigma_{ab}, \rho,$ and σ_ϵ^2 (Normal)
  - sample $\Sigma_{ab} \mid \boldsymbol{\beta}, \mathbf{X}, \boldsymbol{\theta}, \mathbf{a}, \mathbf{b}, \mathbf{U}, \mathbf{V}, \rho,$ and σ_ϵ^2 (Inverse-Wishart)
  - update ρ using a Metropolis-Hastings step with proposal $p^*|p \sim$ truncated normal$_{[-1,1]}(\rho, \sigma_\epsilon^2)$
  - sample $\sigma_\epsilon^2 \mid \boldsymbol{\beta}, \mathbf{X}, \boldsymbol{\theta}, \mathbf{a}, \mathbf{b}, \mathbf{U}, \mathbf{V}, \Sigma_{ab},$ and ρ (Inverse-Gamma)
  - For each $k \in K$:
    * Sample $\mathbf{U}_{[,k]} \mid \boldsymbol{\beta}, \mathbf{X}, \boldsymbol{\theta}, \mathbf{a}, \mathbf{b}, \mathbf{U}_{[,-k]}, \mathbf{V}, \Sigma_{ab}, \rho,$ and σ_ϵ^2 (Normal)
    * Sample $\mathbf{V}_{[,k]} \mid \boldsymbol{\beta}, \mathbf{X}, \boldsymbol{\theta}, \mathbf{a}, \mathbf{b}, \mathbf{U}, \mathbf{V}_{[,-k]}, \Sigma_{ab}, \rho,$ and σ_ϵ^2 (Normal)
    * Sample $\mathbf{D}_{[k,k]} \mid \boldsymbol{\beta}, \mathbf{X}, \boldsymbol{\theta}, \mathbf{a}, \mathbf{b}, \mathbf{U}, \mathbf{V}, \Sigma_{ab}, \rho,$ and σ_ϵ^2 (Normal)[1]

[1]Subsequent to estimation, **D** matrix is absorbed into the calculation for **V** as we iterate through K.

Ingold & Fischer Model Specification and Expected Effects

Variable	Description	Expected Effect
Conflicting policy preferences		
Business v. NGO	Binary, dyadic covariate that equals one when one actor is from the business sector and the other an NGO.	$-$
Opposition/alliance	Binary, dyadic covariate that equals one when i, sender, perceives j, receiver, as having similar policy objectives regarding climate change.	$+$
Preference dissimilarity	Transformation of four core beliefs into a Manhattan distance matrix, smaller the distance the closer the beliefs of i and j.	$-$
Transaction costs		
Joint forum participation	Binary, dyadic covariate that equals one when i and j belong to the same policy forum.	$+$
Influence		
Influence attribution	Binary, dyadic covariate that equals one when i considers j to be influential.	$+$
Alter's influence in-degree	Number of actors that mention i as being influential, this is a measure of reputational power.	$+$
Influence absolute diff.	Absolute difference in reputational power between i and j.	$-$
Alter = Government Actor	Binary, nodal covariate that equals one when j is a state actor.	$+$
Functional requirements		
Ego = Environment NGO	Binary, nodal covariate that equals one when i is an NGO.	$+$
Same actor type	Binary, dyadic covariate that equals when i and j are the same actor type.	$+$
Endogenous dependencies: ERGM Specific Parameters		
Mutuality	Captures concept of reciprocity, if i indicates they collaborated with j then j likely collaborates with i.	$+$
Outdegree popularity	Captures idea that actors sending more ties will be more popular targets themselves for collaboration.	$+$
Twopaths	Counts the number of two-paths in the network, two-path is an instance where i is connected to j, j to k, but i is not connected to k.	$-$
GWIdegree (2.0)	Takes into account how many ties a node sends in the network, used to capture network structures that result from some highly active nodes.	$+$
GWESP (1.0)	Counts the number of shared partners for each pair and sums across.	$+$
GWOdegree (0.5)	Takes into account how many ties a node receives in the network, used to capture networks structures that result from some highly popular nodes.	$+$

Table 1: Summary of variables to be included in model specification.

AME Model Convergence

Trace plot for AME model presented in paper.

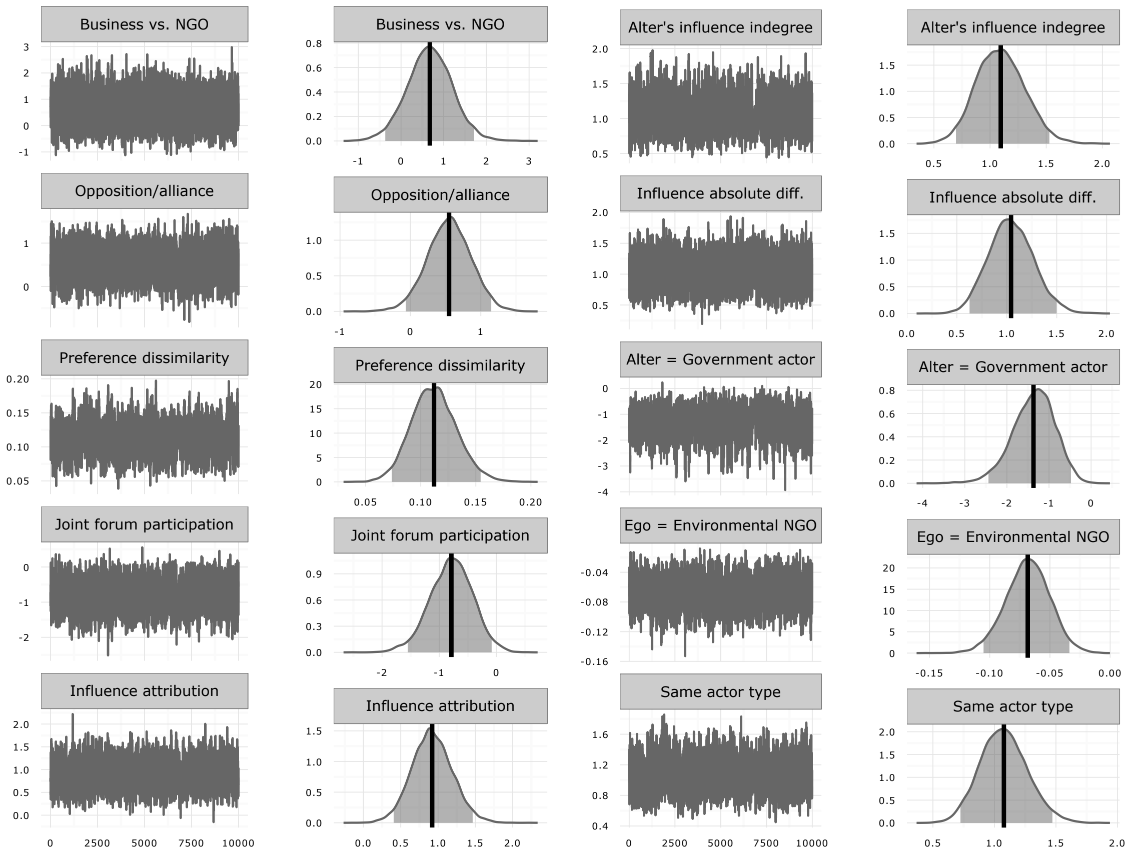

Figure 1: Trace plot for AME model presented in paper. In this model, we utilize the SRM to account for first and second-order dependence. To account for third order dependencies we use the latent factor approach with $K = 2$.

Multiplicative Effects Visualization

When it comes to estimating higher-order effects, ERGM is able to provide explicit estimates of a variety of higher-order parameters, however, this comes with the caveat that these are the "right" set of endogenous dependencies. The AME approach, as shown in Equation **??**, estimates network dependencies by examining patterns left over after taking into account the observed covariates. For the sake of space, we focus on examining the third-order dependencies left over after accounting for the observed covariates and network covariance structure modeled by the SRM. A visualization of remaining third-order dependencies is shown in Figure 2.

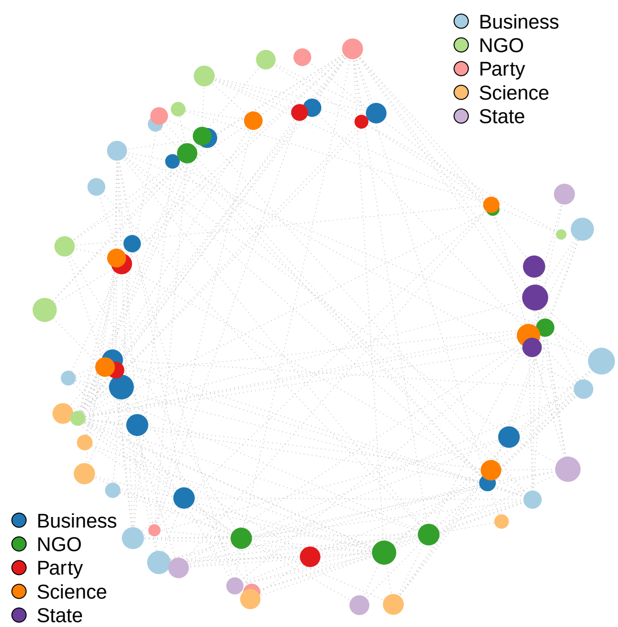

Figure 2: Circle plot of estimated latent factors.

In Figure 2, the directions of $\hat{u}_i$'s and $\hat{v}_i$'s are noted in lighter and darker shades, respectively, of an actor's type.[2] The size of actors is a function of the magnitude of the vectors, and dashed lines between actors indicate greater than expected levels of

[2]For example, actors from industry and business are assigned a color of blue and the direction of $\hat{u}_i$ for these actors is shown in light blue and $\hat{v}_i$ in dark blue

collaboration based on the regression term and additive effects. In the case of the application dataset that we are using here organization names have been anonymized and no additional covariate information is available. However, if we were to observe nodes sharing certain attributes clustering together in this circle plot that would mean such an attribute could be an important factor in helping us to understand collaborations among actors in this network. Given how actors of different types are distributed in almost a random fashion in this plot, we can at least be sure that it is unlikely other third-order patterns can be picked up by that factor.

Other Network Goodness of Fit Tests

Below we show a standard set of statistics upon which comparisons are usually conducted:[3]

Variable	**Description**
Dyad-wise shared partners	Number of dyads in the network with exactly i shared partners.
Edge-wise shared partners	Similar to above except this counts the number of dyads with the same number of edges.
Geodesic distances	The proportion of pairs of nodes whose shortest connecting path is of length k, for $k = 1, 2, \ldots$. Also, pairs of nodes that are not connected are classified as $k = \infty$.
Incoming k-star	Propensities for individuals to have connections with multiple network partners.
Indegree	Proportion of nodes with the same value of the attribute as the receiving node.
Outdegree	Proportion of nodes with the same value of the attribute as the sending node.

Table 2: Description of a set of standard statistics used to assess whether a model captures network dependencies.

We simulate 1,000 networks from the LSM, ERGM, and AME model and compare how well they align with the observed network in terms of the statistics described in Table 2. The results are shown in Figure 3. Values for the observed network are indicated by a gray bar and average values from the simulated networks for the AME, ERGM, and LSM are represented by a diamond, triangle, and square, respectively. The densely shaded interval around each point represents the 95% interval from the simulations and the taller, less dense the 90% interval.[4] Looking across the panels in Figure 3 it is clear that there is little difference between the ERGM and AME models in terms of how well they capture network dependencies. The LSM model, however, does perform somewhat worse in comparison here as well. Particularly, when it comes to assessing the number of edge-wise shared partners and in terms of capturing the indegree and outdegree distributions of the collaboration network.

[3] See Morris et al. (2008) for details on each of these parameters. If one was to examine goodness of fit in the **ergm** package these parameters would be calculated by default.

[4] Calculation for the incoming k-star statistic is not currently supported by the **latentnet** package.

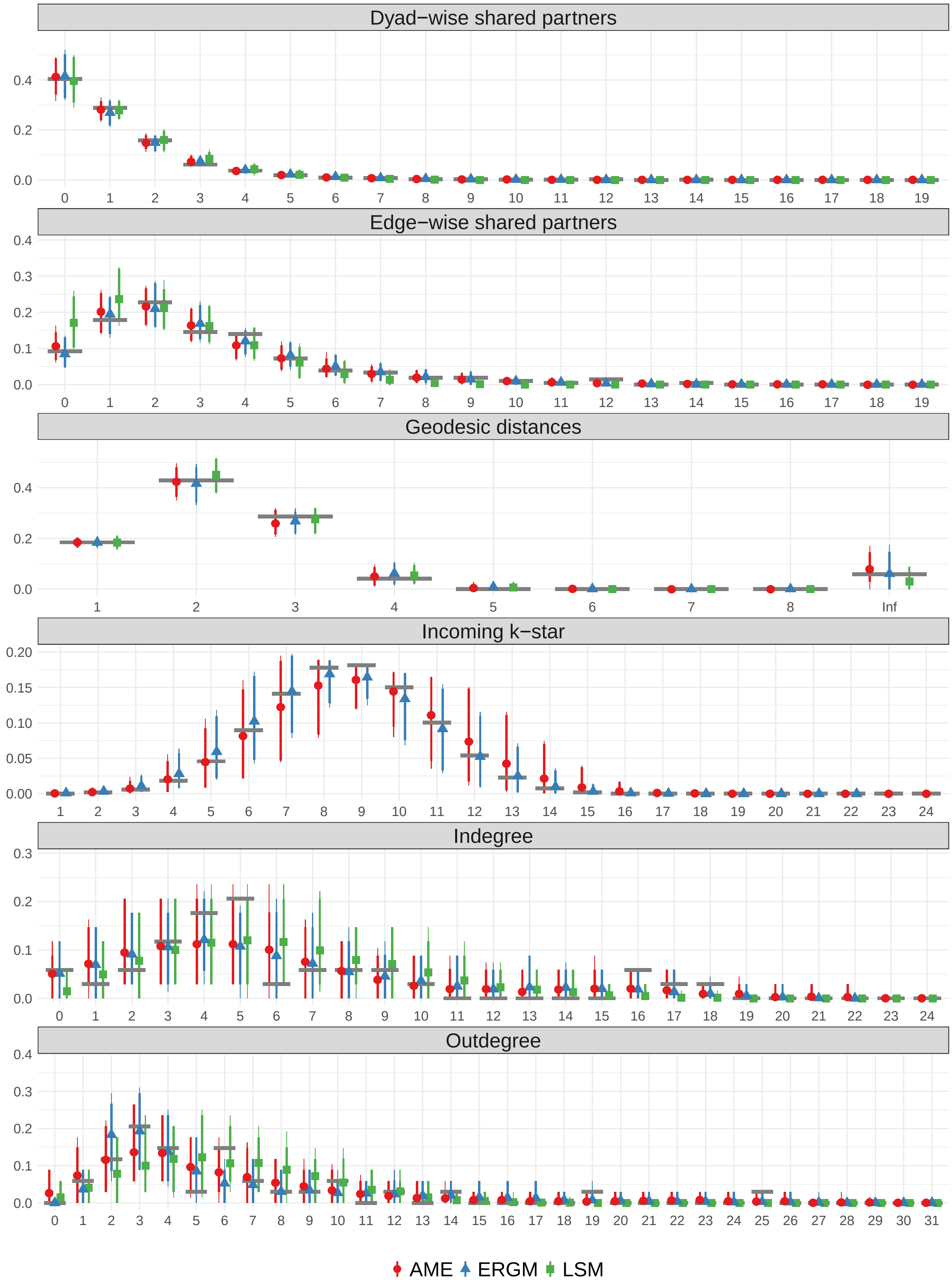

Figure 3: Goodness of fit statistics to assess how well the LSM, ERGM, and AME approaches account for network dependencies.

Comparison with other AME Parameterizations

Here we provide a comparison of the AME model we present in the paper that uses $K = 2$ for multiplicative effects and show how results change when we use $K = \{1, 3, 4\}$. Trace plots for $K = \{1, 3, 4\}$ are available upon request.

	AME (k=1)	AME (k=2)	AME (k=3)	AME (k=4)
Intercept/Edges	-3.08	-3.39	-3.72	-3.93
	[-3.91; -2.30]	[-4.38; -2.50]	[-4.84; -2.73]	[-5.12; -2.87]
Conflicting policy preferences				
Business vs. NGO	-1.28	-1.37	-1.48	-1.51
	[-2.20; -0.47]	[-2.44; -0.47]	[-2.63; -0.49]	[-2.69; -0.47]
Opposition/alliance	0.95	1.08	1.19	1.28
	[0.64; 1.27]	[0.72; 1.47]	[0.80; 1.64]	[0.86; 1.77]
Preference dissimilarity	-0.65	-0.79	-0.89	-0.95
	[-1.30; -0.03]	[-1.55; -0.08]	[-1.71; -0.12]	[-1.80; -0.14]
Transaction costs				
Joint forum participation	0.84	0.92	1.01	1.06
	[0.38; 1.31]	[0.40; 1.47]	[0.44; 1.62]	[0.43; 1.72]
Influence				
Influence attribution	1.00	1.09	1.21	1.28
	[0.63; 1.39]	[0.69; 1.53]	[0.75; 1.71]	[0.80; 1.84]
Alter's influence indegree	0.10	0.11	0.12	0.13
	[0.07; 0.14]	[0.07; 0.15]	[0.08; 0.17]	[0.09; 0.18]
Influence absolute diff.	-0.06	-0.07	-0.07	-0.08
	[-0.10; -0.03]	[-0.11; -0.03]	[-0.12; -0.04]	[-0.12; -0.04]
Alter = Government actor	0.52	0.55	0.60	0.64
	[-0.04; 1.07]	[-0.07; 1.15]	[-0.07; 1.27]	[-0.07; 1.35]
Functional requirements				
Ego = Environmental NGO	0.61	0.67	0.76	0.80
	[-0.31; 1.56]	[-0.38; 1.71]	[-0.38; 1.90]	[-0.40; 2.04]
Same actor type	0.97	1.04	1.11	1.17
	[0.60; 1.35]	[0.63; 1.50]	[0.64; 1.59]	[0.68; 1.68]

Table 3: 95% posterior credible intervals are provided in brackets.

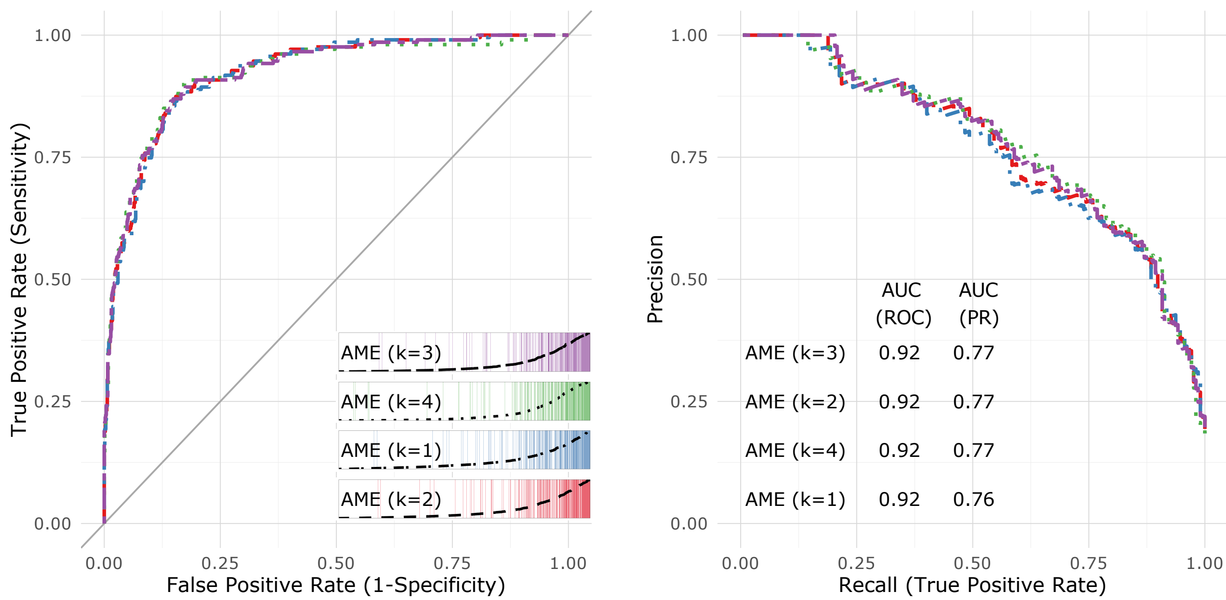

Figure 4: Assessments of out-of-sample predictive performance using ROC curves, separation plots, and PR curves. AUC statistics are provided as well for both curves.

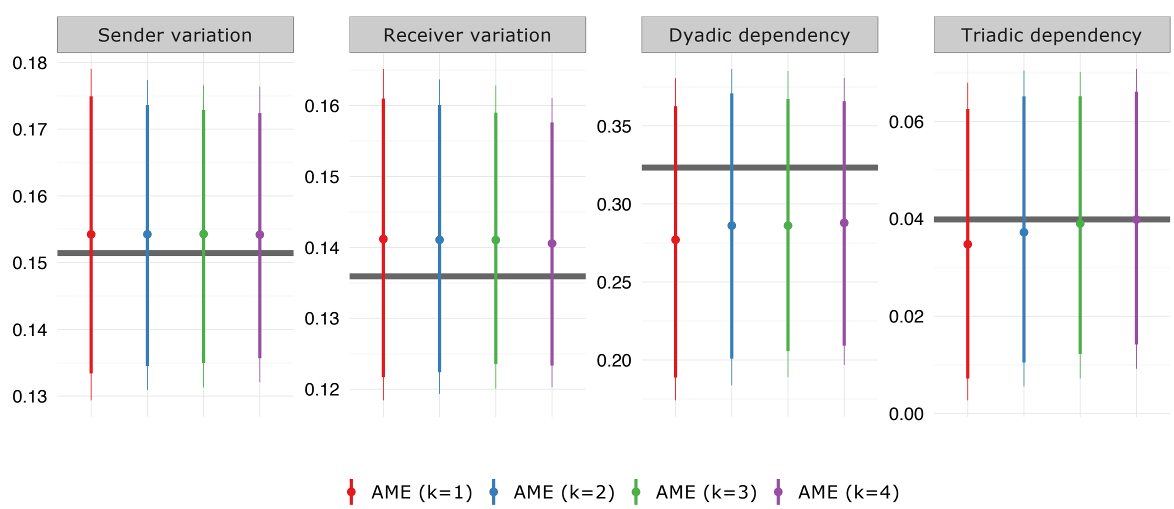

Figure 5: Network goodness of fit summary using **amen**.

Comparison of amen & latentnet R Packages

Here we provide a comparison of the AME model we present in the paper with a variety of parameterizations from the **latentnet** package. The number of dimensions in the latent space in each of these cases is set to 2. LSM (SR) represents a model in which random sender and receiver effects are included.

	LSM	LSM (SR)	AME
Intercept/Edges	0.94	0.60	-3.39
	[0.09; 1.82]	[-1.10; 2.37]	[-4.38; -2.50]
Conflicting policy preferences			
Business vs. NGO	-1.37	-3.07	-1.37
	[-2.42; -0.41]	[-4.77; -1.56]	[-2.44; -0.47]
Opposition/alliance	0.00	0.31	1.08
	[-0.40; 0.39]	[-0.24; 0.86]	[0.72; 1.47]
Preference dissimilarity	-1.76	-1.88	-0.79
	[-2.62; -0.90]	[-3.07; -0.68]	[-1.55; -0.08]
Transaction costs			
Joint forum participation	1.51	1.56	0.92
	[0.86; 2.17]	[0.69; 2.41]	[0.40; 1.47]
Influence			
Influence attribution	0.08	0.30	1.09
	[-0.40; 0.55]	[-0.37; 0.96]	[0.69; 1.53]
Alter's influence indegree	0.01	0.06	0.11
	[-0.03; 0.04]	[-0.03; 0.14]	[0.07; 0.15]
Influence absolute diff.	0.04	-0.08	-0.07
	[-0.01; 0.09]	[-0.14; -0.02]	[-0.11; -0.03]
Alter = Government actor	-0.46	-0.11	0.55
	[-1.08; 0.14]	[-1.91; 1.76]	[-0.07; 1.15]
Functional requirements			
Ego = Environmental NGO	-0.60	-1.69	0.67
	[-1.32; 0.09]	[-3.74; 0.23]	[-0.38; 1.71]
Same actor type	1.17	1.82	1.04
	[0.63; 1.71]	[1.10; 2.54]	[0.63; 1.50]

Table 4: 95% posterior credible intervals are provided in brackets.

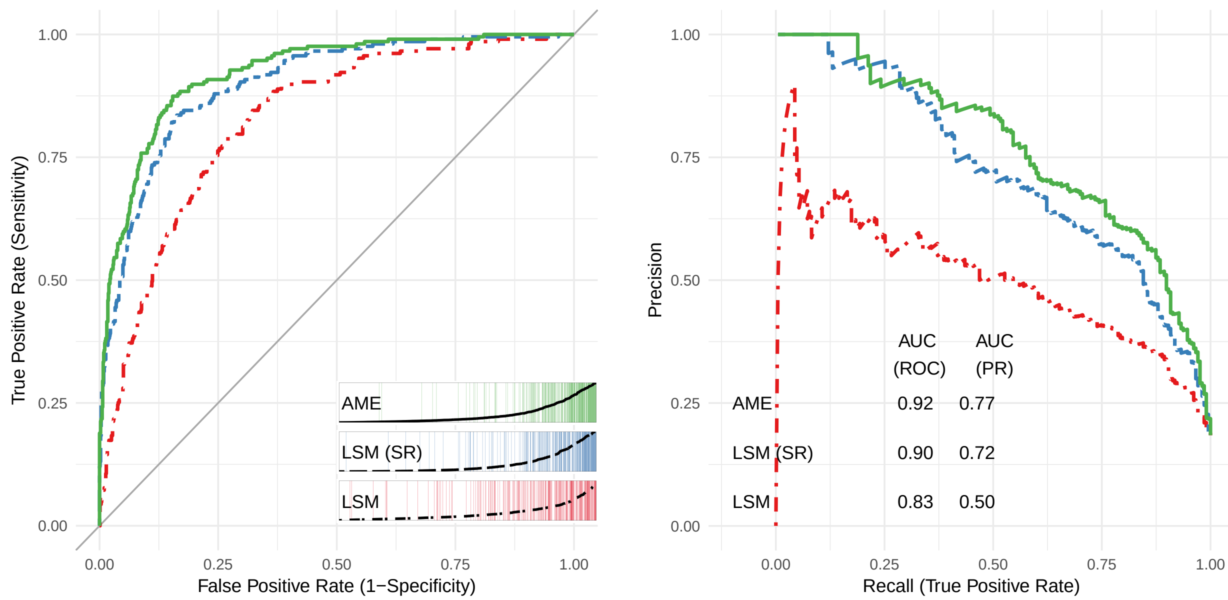

Figure 6: Assessments of out-of-sample predictive performance using ROC curves, separation plots, and PR curves. AUC statistics are provided as well for both curves.

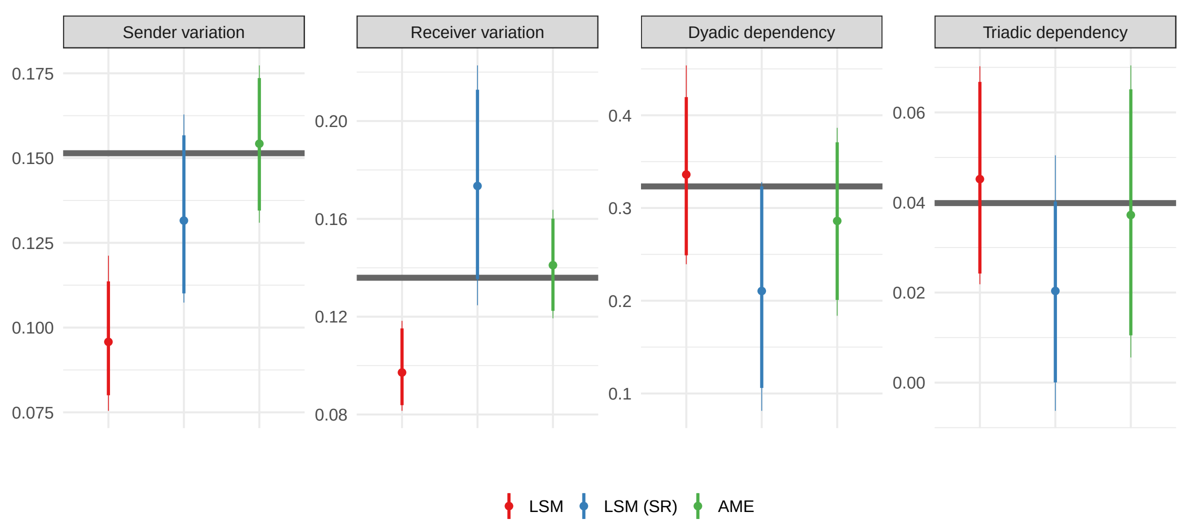

Figure 7: Network goodness of fit summary using **amen**.

Simulation Based Comparison of ***amen*** *&* ***latentnet***

We construct a simulation study to examine differences in the ability of LSM and AME to capture network dependencies.

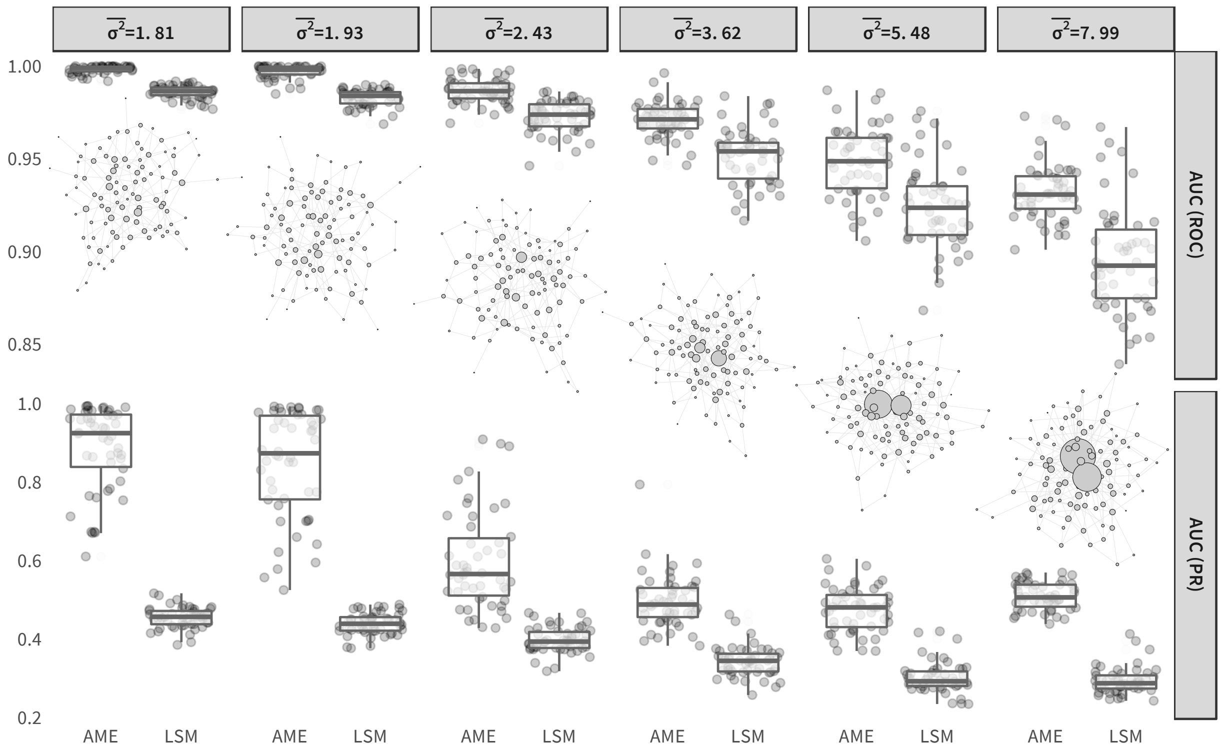

Figure 8: Predictive performance of AME vs LSM for networks with varying degree distributions.

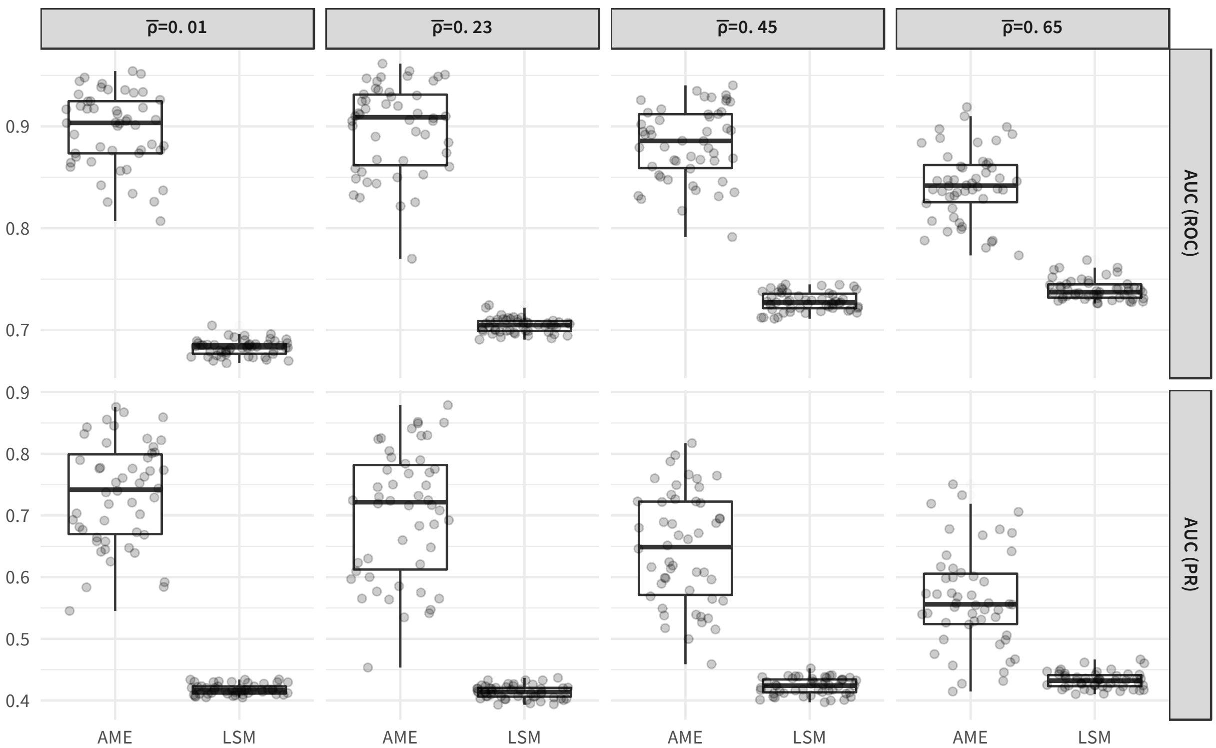

Figure 9: Predictive performance of AME vs LSM for networks with varying levels of reciprocity.

Hoff, P. D., 2005. Bilinear mixed-effects models for dyadic data. Journal of the American Statistical Association 100 (4690), 286–295.

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