Additive and Multiplicative Latent Factor Models for Network Inference

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Motivation

Relational data consists of

- a set of units or nodes
- a set of measurements, y_{ij}, specific to pairs of nodes (i, j)

Sender	Receiver	Event
i	j	y_{ij}
$\vdots$	k	y_{ik}
	l	y_{il}
j	i	y_{ji}
$\vdots$	k	y_{jk}
	l	y_{jl}
k	i	y_{ki}
$\vdots$	j	y_{kj}
	l	y_{kl}
l	i	y_{li}
$\vdots$	j	y_{lj}
	k	y_{lk}

$\longrightarrow$

	i	j	k	l
i	NA	y_{ij}	y_{ik}	y_{il}
j	y_{ji}	NA	y_{jk}	y_{jl}
k	y_{ki}	y_{kj}	NA	y_{kl}
l	y_{li}	y_{lj}	y_{lk}	NA

Relational data assumptions

GLM: $y_{ij} \sim \beta^T X_{ij} + e_{ij}$

Networks typically show evidence against independence of $e_{ij} : i \neq j$

Not accounting for dependence can lead to:

- biased effects estimation
- uncalibrated confidence intervals
- poor predictive performance
- inaccurate description of network phenomena

We've been hearing this concern for decades now:

Thompson & Walker (1982)	Beck et al. (1998)	Snijders (2011)
Frank & Strauss (1986)	Signorino (1999)	Erikson et al. (2014)
Kenny (1996)	Li & Loken (2002)	Aronow et al. (2015)
Krackhardt (1998)	Hoff & Ward (2004)	Athey et al. (2016)

Outline

- Nodal and dyadic dependencies in networks
    - Can model using the “A” in AME
- Third order dependencies
    - Can model using the “M” in AME
- Application

What network phenomena? Sender heterogeneity

Values across a row, say $\{y_{ij}, y_{ik}, y_{il}\}$, may be more similar to each other than other values in the adjacency matrix because each of these values has a common sender i

	i	j	k	l
i	NA	y_{ij}	y_{ik}	y_{il}
j	y_{ji}	NA	y_{jk}	y_{jl}
k	y_{ki}	y_{kj}	NA	y_{kl}
l	y_{li}	y_{lj}	y_{lk}	NA

What network phenomena? Receiver heterogeneity

Values across a column, say $\{y_{ji}, y_{ki}, y_{li}\}$, may be more similar to each other than other values in the adjacency matrix because each of these values has a common receiver i

	i	j	k	l
i	NA	y_{ij}	y_{ik}	y_{il}
j	y_{ji}	NA	y_{jk}	y_{jl}
k	y_{ki}	y_{kj}	NA	y_{kl}
l	y_{li}	y_{lj}	y_{lk}	NA

What network phenomena? Sender-Receiver Covariance

Actors who are more likely to send ties in a network may also be more likely to receive them

	i	j	k	l
i	NA	y_{ij}	y_{ik}	y_{il}
j	y_{ji}	NA	y_{jk}	y_{jl}
k	y_{ki}	y_{kj}	NA	y_{kl}
l	y_{li}	y_{lj}	y_{lk}	NA

What network phenomena? Reciprocity

Values of y_{ij} and y_{ji} may be statistically dependent

	i	j	k	l
i	NA	y_{ij}	y_{ik}	y_{il}
j	y_{ji}	NA	y_{jk}	y_{jl}
k	y_{ki}	y_{kj}	NA	y_{kl}
l	y_{li}	y_{lj}	y_{lk}	NA

Social Relations Model (The “A” in AME)

We use this model to form the additive effects portion of AME

$$
\begin{aligned}
y_{ij} &= \mu + e_{ij} \\
e_{ij} &= a_i + b_j + \epsilon_{ij} \\
\{(a_1, b_1), \ldots, (a_n, b_n)\} &\sim N(0, \Sigma_{ab}) \\
\{(\epsilon_{ij}, \epsilon_{ji}) : \; i \neq j\} &\sim N(0, \Sigma_{\epsilon}), \text{ where}
\end{aligned}
$$

$$
\Sigma_{ab} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} \qquad \Sigma_{\epsilon} = \sigma_{\epsilon}^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}
$$

- μ baseline measure of network activity
- e_{ij} residual variation that we will use the SRM to decompose

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$$

- row/sender effect (a_i) & column/receiver effect (b_j)

- Modeled jointly to account for correlation in how active an actor is in sending and receiving ties

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$$

- σ_a^2 and σ_b^2 capture heterogeneity in the row and column means
- σ_{ab} describes the linear relationship between these two effects (i.e., whether actors who send [receive] a lot of ties also receive [send] a lot of ties)

Social Relations Model (The "A" in AME)

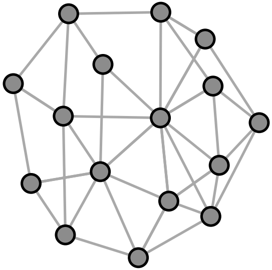
$$
\begin{aligned}
y_{ij} &= \mu + e_{ij} \\
e_{ij} &= a_i + b_j + {\color{red}\epsilon_{ij}} \\
\{(a_1, b_1), \ldots, (a_n, b_n)\} &\sim N(0, \Sigma_{ab}) \\
{\color{red}\{(\epsilon_{ij}, \epsilon_{ji}) : \ i \neq j\}} &\sim N(0, {\color{red}\Sigma_\epsilon}), \ {\color{red}\text{where}}
\end{aligned}
$$

$$
\Sigma_{ab} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} \qquad {\color{red}\Sigma_\epsilon = \sigma_\epsilon^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}}
$$

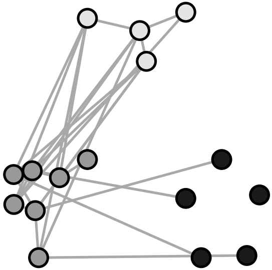
- ϵ_{ij} captures the within dyad effect
- Second-order dependencies are described by σ_ϵ^2
- Reciprocity, aka within dyad correlation, represented by ρ

Third Order Dependencies

HOMOPHILY



STOCHASTIC EQUIVALENCE



To account for these patterns we can build on what we have so far and find an expression for γ:

$$y_{ij} \approx \beta^T X_{ij} + a_i + b_j + \gamma(u_i, v_j)$$

Latent Factor Model: The “M” in AME

Each node i has an unknown latent factor

$$\mathbf{u}_i, \mathbf{v}_j \in \mathbb{R}^k \quad i, j \in \{1, \ldots, n\}$$

The probability of a tie from i to j depends on their latent factors

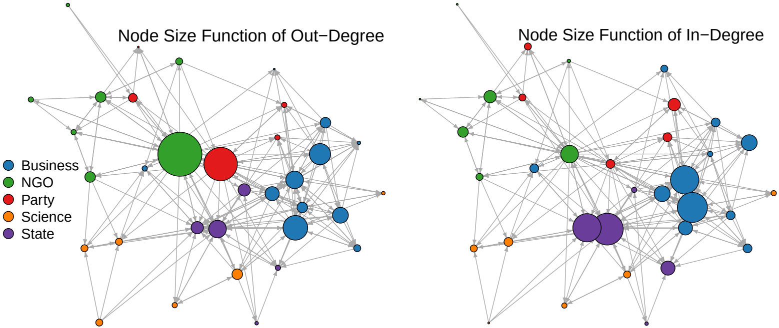
$$\begin{aligned}\gamma(\mathbf{u}_i, \mathbf{v}_j) &= \mathbf{u}_i^T D \mathbf{v}_j \\ &= \sum_{k \in K} d_k u_{ik} v_{jk}\end{aligned}$$

$$D \text{ is a } K \times K \text{ diagonal matrix}$$

Can account for both stochastic equivalence and homophily

Swiss Climate Change Application

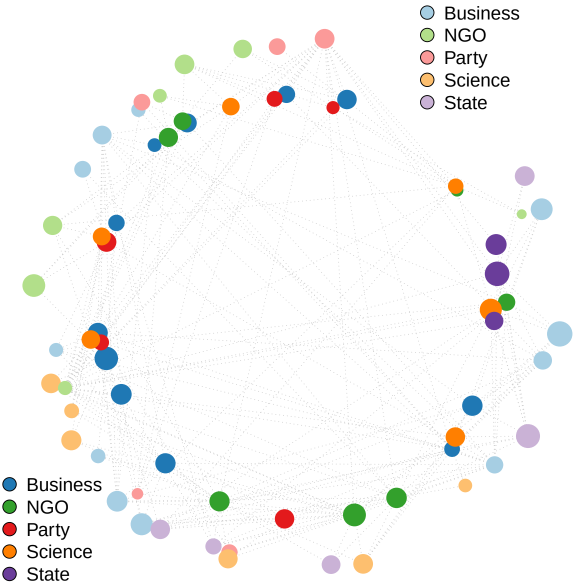
Cross-sectional network measuring whether an actor indicated that they collaborated with another during the policy design of the Swiss CO_2 act (Ingold 2008)



Parameter Estimates

	Expected Effect	Logit	MRQAP	LSM	ERGM	AME
Conflicting policy preferences						
Business vs. NGO	−	-0.86	-0.87*	-1.37*	-1.11*	-1.37*
Opposition/alliance	+	1.21*	1.14*	0.00	1.22*	1.08*
Preference dissimilarity	−	-0.07	-0.60	-1.76*	-0.44	-0.79*
Transaction costs						
Joint forum participation	+	0.88*	0.75*	1.51*	0.90*	0.92*
Influence						
Influence attribution	+	1.20*	1.29*	0.08	1.00*	1.09*
Alter's influence indegree	+	0.10*	0.11*	0.01	0.21*	0.11*
Influence absolute diff.	−	-0.03*	-0.06*	0.04	-0.05*	-0.07*
Alter = Government actor	+	0.63*	0.68	-0.46	1.04*	0.55
Functional requirements						
Ego = Environmental NGO	+	0.88*	0.99	-0.60	0.79*	0.67
Same actor type	+	0.74*	1.12*	1.17*	0.99*	1.04*

Latent Factor Visualization

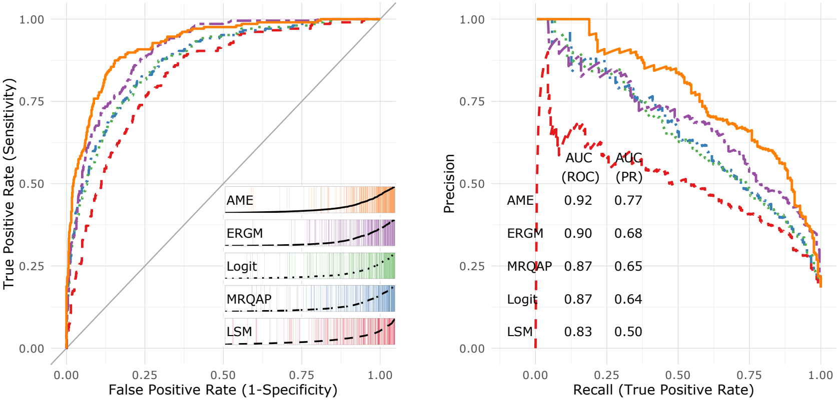


Out of Sample Performance Assessment

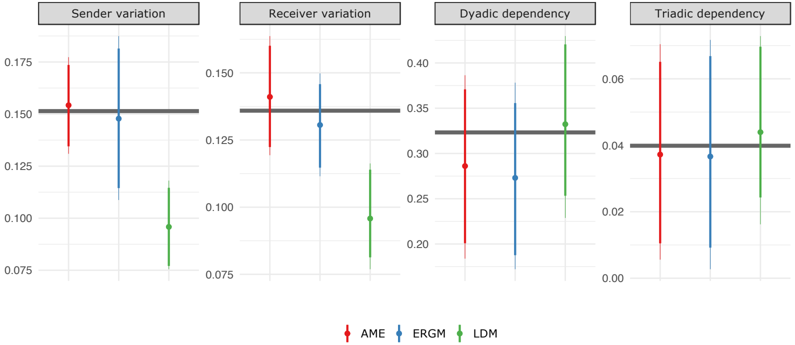
- Randomly divide the $n \times (n-1)$ data points into S sets of roughly equal size, letting s_{ij} be the set to which pair $\{ij\}$ is assigned.
- For each $s \in \{1, \ldots, S\}$:
  - Obtain estimates of the model parameters conditional on $\{y_{ij} : s_{ij} \neq s\}$, the data on pairs not in set s.
  - For pairs $\{kl\}$ in set s, let $\hat{y}_{kl} = E[y_{kl} | \{y_{ij} : s_{ij} \neq s\}]$, the predicted value of y_{kl} obtained using data not in set s.

This procedure generates a sociomatrix of out-of-sample predictions of the observed data

Performance Comparison



Network Dependencies



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Thanks.

Hope of Exponential Random Graph Models

- Early 1970s development of pseudolikelihood estimation: Ove Frank (1971); Julian Besag (1972) proposed using a logistic regression with network characteristics as covariates. Birth of ERGM.

- $P(x) = exp(\Theta^T s(x) - \psi(\theta))$, where x is an adjacency matrix that is a graph, $s(\theta)$ are some set of sufficient statistics for the graph, and $\psi(\theta)$ is a normalizing constant, often set to be $\log \sum_x e^{\theta^T zx)}$. This is often estimated via pseudolikelihood, simply by regressing $x \sim \text{logit}(z(x))$.

- Maximum Likelihood is a better approach withRobbins and Monro, or the importance sampling approach of Geyer & Thompson. More advances with Bayesian approaches are available now with MCMC (Koskinen, Robins & Pattison).

- In the 1990s, networks became more widely recognized as and the ERGM approach was employed to estimate models in a variety of network domains. Needle sharing communities, HIV infections, for example.

There is a problem with ERGM.

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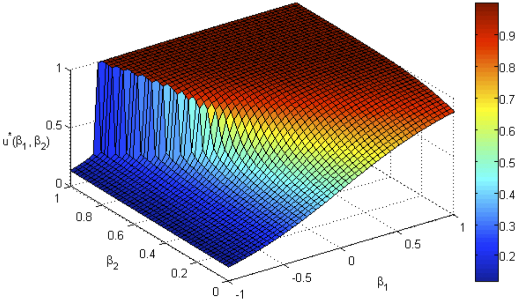
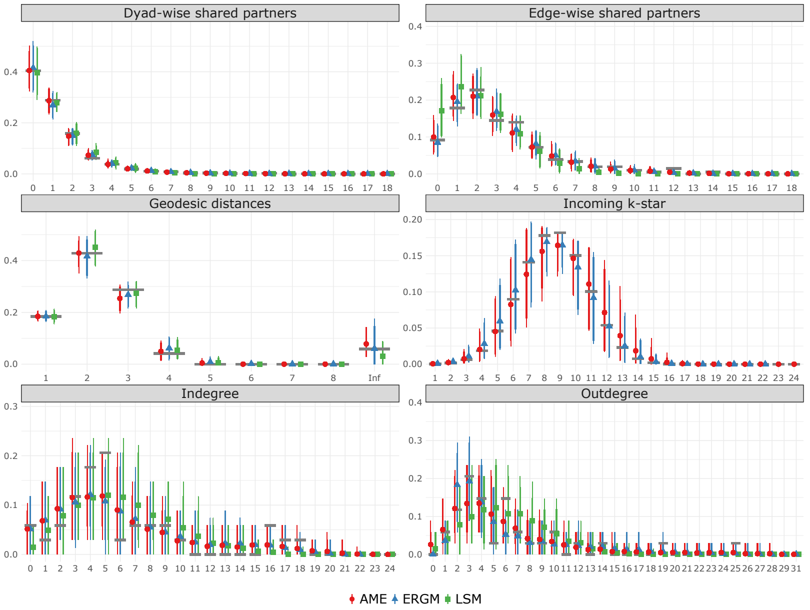
It is a bug, not a feature.

•Probabilistic ERGM models place almost all of the probability on networks that are either nearly empty (degenerate) with no linkages or nearly saturated with all nodes being interconnected.

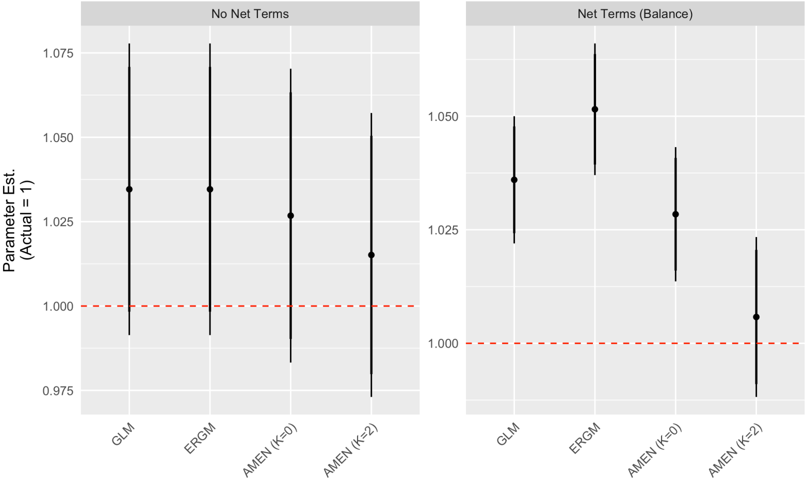
•The likelihood surface contains steep or discontinuous gradients that render it impossible to solve numerically (or analytically). Even (especially) for very small networks this is problematic.



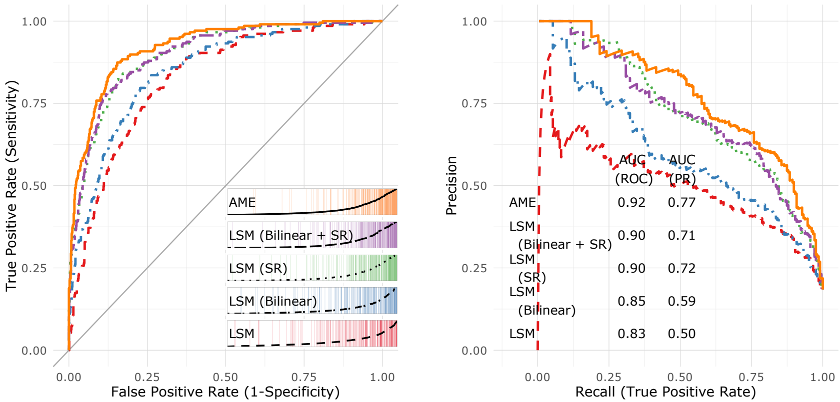
Standard Network Dependence Measures



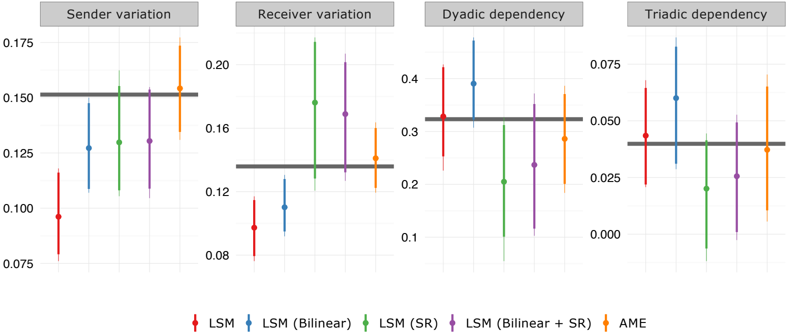
Simulation Comparison



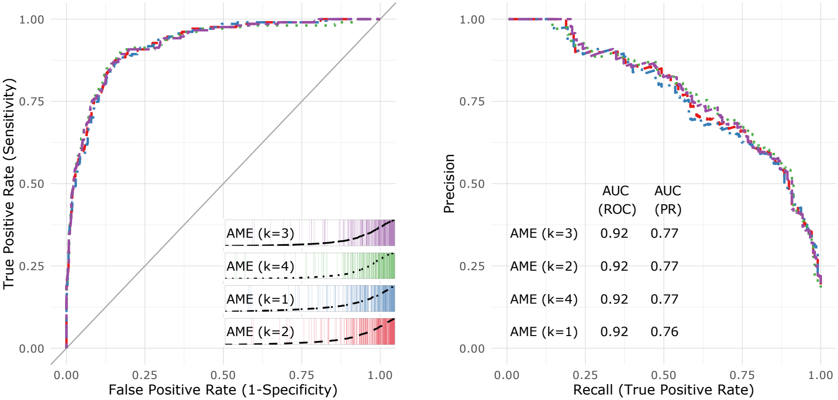
AMEN v LSM Performance



AMEN versus LSM Net Dependence



AMEN varying K Performance



AMEN varying K Net Dependence