Inferential Approaches for Network Analysis:
AMEN for Latent Factor Models

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Abstract

Many network approaches have been developed in descriptive fashion, but attention to inferential approaches to network analysis has been growing. We introduce a new approach that models interdependencies among observations using additive and multiplicative effects (AME). This approach can be applied to binary, ordinal, and continuous network data, and provides a set of tools for inference from longitudinal networks as well. The AME approach is shown a) to be easy to implement; b) interpretable in a general linear model framework; c) computationally straightforward; d) not prone to degeneracy; e) captures 1st, 2nd, and 3rd order network dependencies; and f) notably outperforms multiple regression quadratic assignment procedures, exponential random graph models, and alternative latent space approaches on a variety of metrics both in- and out-of-sample.
Word Count: xxxx

Network analysis provides a way to represent and study "relational data", that is data which defines characteristics between pairs of actors. Data structures that extend beyond the actor level are quite simply the norm in fields such as international relations, where the focus often rests on how countries conflict or cooperate with one another. Yet, the dominant paradigm in international relations for dealing with such data structures is not a network approach but rather a dyadic design, in which an interaction between a pair of countries is considered independent of interactions between any other pair in the system. The implication of this assumption is that when, for example, Vietnam and the United States decide to form a trade agreement, they make this decision independently of what they have done with other countries and what other countries in the international system have done among themselves.[1] A common defense of the dyad-only approach is that many events are only bilateral (Diehl and Wright, 2016), thus alleviating the need for an approach that incorporates interdependencies between observations. The network perspective asserts that even bilateral events and processes take place within a broader system. At a minimum, we do not know whether independence of events and processes characterizes what we observe, thus we should at least examine this assertion.

The potential for interdependence among observations poses a challenge to theoretical as well as statistical modeling since the assumption made by standard approaches used across the social sciences is that observations are, at least, conditionally independent (Snijders, 2011). The consequence of ignoring this assumption has been frequently noted within the political science literature already.[2] Just as relevant is the fact that a wealth of research from other disciplines suggests that carrying the independence assumption into a study with relational data is misguided and most often leads to biased inferences.[3]

A variety of empirical frameworks have been developed to deal with the interdependencies

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[1]There has been plenty of work done on treaty formation that would challenge this claim, e.g., see Manger et al. (2012).

[2]For example, see Beck et al. (1998); Signorino (1999); Aronow et al. (2015).

[3]From Computer Science see: Bonabeau (2002); Brandes and Erlebach (2005). From Economics see: Goyal (2012); Jackson (2014). From Psychology see: Pattison and Wasserman (1999); Kenny et al. (2006). From Statistics and Sociology see: Snijders (1996); Hoff et al. (2002).

inherent in relational data. In this article, we introduce the additive and multiplicative effects model (AME). To illustrate the contrasts between AME, earlier latent variable models, and more traditional approaches such as ERGM we apply each to studying a cross-sectional network measuring collaborations during the policy design of the Swiss CO_2 act. By doing so we are able to show that AME provides a superior goodness of fit to the data than alternative approaches, both in terms of ability to predict linkages and capture network dependencies.

1. Addressing Dependencies in Dyadic Data

Relational, or dyadic, data provide measurements of how pairs of actors relate to one another. The easiest way to organize such data is the directed dyadic design in which the unit of analysis is some set of n actors that have been paired together to form a dataset of z directed dyads. A tabular design such as this for a set of n actors, $\{i, j, k, l\}$ results in $n \times (n-1)$ observations, as shown in Table 1.

Sender	Receiver	Event
i	j	y_{ij}
	k	y_{ik}
$\vdots$	l	y_{il}
j	i	y_{ji}
	k	y_{jk}
$\vdots$	l	y_{jl}
k	i	y_{ki}
	j	y_{kj}
$\vdots$	l	y_{kl}
l	i	y_{li}
	j	y_{lj}
$\vdots$	k	y_{lk}

Table 1: Structure of datasets used in canonical design.

$\longrightarrow$

	i	j	k	l
i	NA	y_{ij}	y_{ik}	y_{il}
j	y_{ji}	NA	y_{jk}	y_{jl}
k	y_{ki}	y_{kj}	NA	y_{kl}
l	y_{li}	y_{lj}	y_{lk}	NA

Table 2: Adjacency matrix representation of data in Table 1. Senders are represented by the rows and receivers by the columns.

When modeling relational data, scholars typically employ a generalized linear model (GLM). An assumption we make when applying this modeling technique is that each of the dyadic observations is conditionally independent.[4] However, this is a strong assumption to make given

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[4]See the online appendix for a longer discussion on the limitations of using GLM to study relational data.

that events between actors in a network are often interdependent. The dependencies that tend to develop in relational data can be more easily understood when we move away from stacking dyads on top of one another and turn instead to adjacency matrices as shown in Table 2. Operationally, this type of data structure is represented as a $n \times n$ matrix, $\mathbf{Y}$, where the diagonals in the matrix are typically undefined. The ij^{th} entry defines the relationship sent from i to j and can be continuous or discrete.[5]

The most common type of dependency that arises in networks are first-order, or nodal dependencies, and these point to the fact that we typically find significant heterogeneity in activity levels across nodes. The implication of this across-node heterogeneity is within-node homogeneity of ties, meaning that values across a row, say $\{y_{ij}, y_{ik}, y_{il}\}$, will be more similar to each other than other values in the adjacency matrix because each of these values has a common sender i. This type of dependency manifests in cases where sender i tends to be more active or less active in the network than other senders. Similarly, while some actors may be more active in sending ties to others in the network, we might also observe that others are more popular targets, this would manifest in observations down a column, $\{y_{ji}, y_{ki}, y_{li}\}$, being more similar. Last, we might also find that actors who are more likely to send ties in a network are also more likely to receive them, meaning that the row and column means of an adjacency matrix may be correlated. Another ubiquitous type of structural interdependency is reciprocity. This is a second-order, or dyadic, dependency relevant only to directed datasets, and asserts that values of y_{ij} and y_{ji} may be statistically dependent. The prevalence of these types of potential interactions within directed dyadic data also complicates the basic assumption of observational independence.

1.1. *Social Relations Model: Additive Part of AME*

The relevance of modeling first- and second-order dependencies has long been recognized within some social sciences particularly in psychology. Warner et al. (1979) developed the social

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[5]Relations between actors in a network setting at times does not involve senders and receivers. Networks such as these are referred to as undirected and all the relations between actors are symmetric, meaning $y_{ij} = y_{ji}$.

relations model (SRM), a type of ANOVA decomposition technique, that facilitates this under-taking. The SRM is of particular note as it provides the error structure for the additive effects component of the AME framework that we introduce here. The goal of the SRM is to decompose the variance of observations in an adjacency matrix in terms of heterogeneity across row means (out-degree), heterogeneity along column means (in-degree), correlation between row and column means, and correlations within dyads. Wong (1982) and Li and Loken (2002) provide a random effects representation of the SRM:

$$y_{ij} = \mu + e_{ij}$$

$$e_{ij} = a_i + b_j + \epsilon_{ij}$$

$$\{(a_1, b_1), \ldots, (a_n, b_n)\} \overset{\text{iid}}{\sim} N(0, \Sigma_{ab})$$

$$\{(\epsilon_{ij}, \epsilon_{ji}) : \ i \neq j\} \overset{\text{iid}}{\sim} N(0, \Sigma_\epsilon), \text{ where}$$

$$\Sigma_{ab} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} \qquad \Sigma_\epsilon = \sigma_\epsilon^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

(1)

The basic idea here is simple, μ provides a baseline measure of the density or sparsity of a network, and e_{ij} represents residual variation. The residual variation decomposes into parts: a row/sender effect (a_i), a column/receiver effect (b_j), and a within-dyad effect (ϵ_{ij}). The row and column effects are modeled jointly to account for correlation in how active an actor is in sending and receiving ties. Heterogeneity in the row and column means is captured by σ_a^2 and σ_b^2, respectively, and σ_{ab} describes the linear relationship between these two effects (i.e., whether actors who send [receive] a lot of ties also receive [send] a lot of ties). Beyond these first-order dependencies, second-order dependencies are described by σ_ϵ^2 and a within dyad correlation, or reciprocity, parameter ρ.

The SRM covariance structure described in Equation 1 can be incorporated into the systematic component of a GLM framework to produce the social relations regression model (SRRM): $\boldsymbol{\beta}^\top \mathbf{X}_{ij} + a_i + b_j + \epsilon_{ij}$, where $\boldsymbol{\beta}^\top \mathbf{X}_{ij}$ accommodates the inclusion of dyadic, sender, and re-

ceiver covariates (Hoff, 2005). The SRRM approach incorporates row, column, and within-dyad dependence in way that is widely used and understood by applied researchers: a regression framework and additive random effects to accommodate variances and covariances often seen in relational data.[6]

1.2. Latent Factor Model: Multiplicative Part of AME

Missing from the framework provided by the SRM is an accounting of third-order dependence patterns that can arise in relational data. The ubiquity of third-order effects in relational datasets arises from the presence of some set of shared attributes between nodes that affects their probability of interacting with one another.[7]

For example, one finding from the gravity model of trade is that neighboring countries are more likely to trade with one another; in this case, the shared attribute is simply geographic proximity. A finding common in the political economy literature is that democracies are more likely to form trade agreements with one another, and the shared attribute here is a country's political system. Both geographic proximity and a country's political system are examples of homophily, which captures the idea that the relationships between actors with similar characteristics in a network are likely to be stronger than nodes with different characteristics.[8]

A binary network where actors tend to form ties with others based on some set of shared characteristics often leads to a network graph with a high number of "transitive triads" in which sets of actors $\{i, j, k\}$ are each linked to one another. The left-most plot in Figure 1 provides a representation of a network that exhibits this type of pattern. The relevant implication of this when it comes to conducting statistical inference is that–unless we are able to specify the

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[6]Furthermore, this handles a diversity of outcome distributions. In the case of binary data this can be done by utilizing a latent variable representation of a probit regression model.

[7]Another reason why we may see the emergence of third-order effects is the "sociology" explanation: that individuals want to close triads because this is putatively a more stable or preferable social situation (Wasserman and Faust, 1994).

[8]Homophily can be used to explain the emergence of patterns such as transitivity ("a friend of a friend is a friend") and balance ("an enemy of a friend is an enemy"). See Shalizi and Thomas (2011) for a more detailed discussion on the concept of homophily.

list of exogenous variable that may explain this prevalence of triads–the probability of j and k forming a tie is not independent of the ties that already exist between those actors and i.
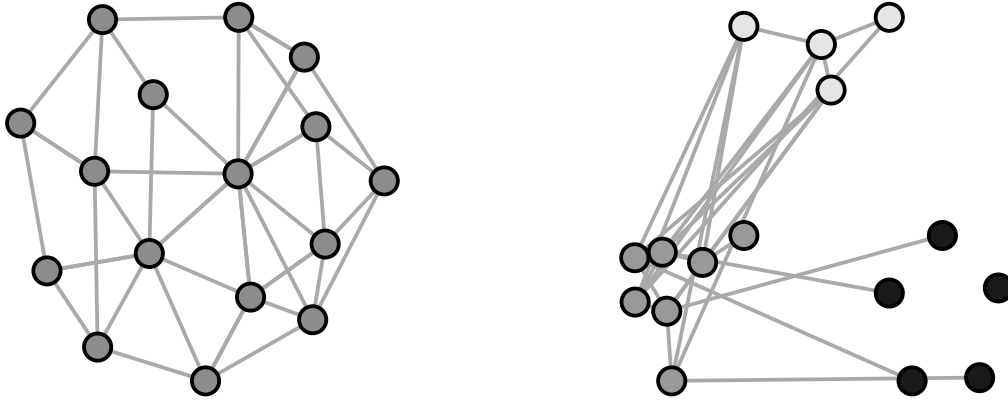
Figure 1: Graph on the left is a representation of an undirected network that exhibits a high degree of homophily, while on the right we show an undirected network that exhibits stochastic equivalence.

Another third-order dependence pattern that cannot be accounted for in the additive effects framework is stochastic equivalence. A pair of actors ij are stochastically equivalent if the probability of i relating to, and being related to, by every other actor is the same as the probability for j. This refers to the idea that there will be groups of nodes in a network with similar relational patterns. The occurrence of a dependence pattern such as this is not uncommon in the social science applications. Manger et al. (2012) posit and estimate a stochastic equivalence structure to explain the formation of preferential trade agreements (PTAs). Specifically, they suggest that PTA formation is related to differences in per capita income levels between countries. Countries falling into high, middle, and low income per capita levels will have patterns of PTA formation that are determined by the groups into which they fall. Such a structure is represented in the right-most panel of Figure 1, here the lightly shaded group of nodes at the top can represent high-income countries, nodes on the bottom-left middle-income, and the darkest shade of nodes low-income countries. The behavior of actors in a network can at times be governed by group level dynamics, and failing to account for such dynamics leaves potentially important parts of the data generating process ignored.

To account for third-order dependence patterns within the context of the SRRM we turn to

latent variable models, which have become a popular approach for modeling relational data in fields as diverse as biology to computer science to the social sciences. These models assumes that relationships between nodes are mediated by a small number (K) of node-specific unobserved latent variables. One reason for their increased usage is that they enable researchers to capture and visualize third-order dependencies in a way that other approaches are not able to replicate. Additionally, the conditional independence assumption facilitates the testing of a variety of nodal and dyadic level theories, and provides a range of computational advantages.

A number of latent variable approaches have been developed to represent third-order dependencies in relational data, we focus on two here: the latent distance model and the latent factor model. For the sake of exposition, we consider the case where relations are symmetric to describe the differences between these approaches. Both of these approaches can be incorporated into an undirected version of the framework that we have been constructing through the inclusion of an additional term to the model for y_{ij}, $\alpha(u_i, u_j)$, that captures latent third-order characteristics of a network, where u_i and u_j are node-specific latent variables. General definitions for how $\alpha(u_i, u_j)$ is defined for these latent variable models are shown in Equations 2. One other point of note about these approaches is that researchers have to specify a value for K.

Latent distance model

$$\alpha(\mathbf{u}_i, \mathbf{u}_j) = -|\mathbf{u}_i - \mathbf{u}_j|$$

$$\mathbf{u}_i \in \mathbb{R}^K,\ i \in \{1, \dots, n\}$$

Latent factor model   (2)

$$\alpha(\mathbf{u}_i, \mathbf{u}_j) = \mathbf{u}_i^\top \Lambda \mathbf{u}_j$$

$$\mathbf{u}_i \in \mathbb{R}^K,\ i \in \{1, \dots, n\}$$

$$\Lambda \text{ a } K \times K \text{ diagonal matrix}$$

The latent distance model was developed by Hoff et al. (2002) to capture homophily. In this

approach, each node i has some unknown latent position in K dimensional space, $\mathbf{u}_i \in \mathbb{R}^K$, and the probability of a tie between a pair ij is a function of the negative Euclidean distance between them: $-|\mathbf{u}_i - \mathbf{u}_j|$. Hoff et al. (2002) show that because latent distances for a triple of actors obey the triangle inequality, this formulation models the tendencies toward homophily commonly found in social networks. This approach has been operationalized in the **latentnet** package developed by Krivitsky and Handcock (2015). However, this approach also comes with an important shortcoming: it confounds stochastic equivalence and homophily. Consider two nodes i and j that are proximate to one another in K dimensional Euclidean space, this suggests not only that $|\mathbf{u}_i - \mathbf{u}_j|$ is small but also that $|\mathbf{u}_i - \mathbf{u}_l| \approx |\mathbf{u}_j - \mathbf{u}_l|$, the result being that nodes i and j will by construction assumed to possess the same relational patterns with other actors such as l (i.e., that they are stochastically equivalent). Thus latent distance models confound strong ties with stochastic equivalence. This approach cannot adequately model data with many ties between nodes that have different network roles.

An early iteration of the latent factor approach was presented in Hoff (2005) and introduced to political science by Hoff and Ward (2004), but the revised approach is motivated by an eigenvalue decomposition of a network.[9] The motivation for this alternative framework stems from the fact that many real networks exhibit varying degrees of stochastic equivalence and homophily. In these situations, using either the latent distance or class model would end up representing only a part of the network structure. In the latent factor model, each actor has an unobserved vector of characteristics, $\mathbf{u}_i = \{u_{i,1}, \ldots, u_{i,K}\}$, which describe their behavior as an actor in the network. The probability of a tie from i to j depends on the extent to which $\mathbf{u}_i$ and $\mathbf{u}_j$ are "similar" (i.e., point in the same direction) and on whether the entries of Λ are greater or less than zero.

More specifically, the similarity in the latent factors, $\mathbf{u}_i \approx \mathbf{u}_j$, corresponds to how stochastically equivalent a pair of actors are and the eigenvalue determines whether the network ex-

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[9] An important difference in the earlier approaches such as the GBME compared to the model that we present here is that Λ was taken to be the identity matrix thus stochastic equivalence could not be characterized.

hibits positive or negative homophily. For example, say that that we estimate a rank-one latent factor model (i.e., $K = 1$), in this case $\mathbf{u}_i$ is represented by a scalar $u_{i,1}$, similarly, $\mathbf{u}_j = u_{j,1}$, and Λ will have just one diagonal element λ. The average effect this will have on y_{ij} is simply $\lambda \times u_i \times u_j$, where a positive value of $\lambda > 0$ indicates homophily and $\lambda < 0$ anti-homophily. Hoff (2008) shows that such a model can represent both homophily and stochastic equivalence, and that the alternative latent variable approaches can be represented as a latent factor model but not vice versa. In the directed version of this approach, we use the singular value decomposition,[10] here actors in the network have a vector of latent characteristics to describe their behavior as a sender, denoted by $\mathbf{u}$, and as a receiver, $\mathbf{v}$: $\mathbf{u}_i, \mathbf{v}_j \in \mathbb{R}^K$ (Hoff, 2009). These again can alter the probability, or in the continuous case value, of an interaction between ij additively: $\mathbf{u}_i^\top \mathbf{D} \mathbf{v}_j$, where $\mathbf{D}$ is an $K \times K$ diagonal matrix.

Both the latent distance and factor models are "conditional independence models" in that they assume that ties are conditionally independent given all of the observed predictors and unknown node-specific parameters: $p(Y|X, U) = \prod_{i<j} p(y_{i,j}|x_{i,j}, u_i, u_j)$. Typical parametric models of this form relate $y_{i,j}$ to $(x_{i,j}, u_i, u_j)$ via some sort of link function:

$$p(y_{i,j}|x_{i,j}, u_i, u_j) = f(y_{i,j} : \eta_{i,j})$$
$$\eta_{i,j} = \beta^\top x_{i,j} + \alpha(\mathbf{u}_i, \mathbf{u}_j).$$

The structure of $\alpha(\mathbf{u}_i, \mathbf{u}_j)$ can result in very different interpretations for any estimates of the regression coefficients β. For example, suppose the latent effects $\{u_1, \ldots, u_n\}$ are near zero on average (if not, their mean can be absorbed into an intercept parameter and row and column additive effects). Under the latent factor model, the average value of $\alpha(\mathbf{u}_i, \mathbf{u}_j) = \mathbf{u}_i^\top \Lambda \mathbf{u}_j$ will be near zero and so we have

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[10]The singular value decomposition is a model based analogue to the eigenvalue decomposition for directed networks.

$$\eta_{i,j} = \beta^\top x_{i,j} + \mathbf{u}_i^\top \Lambda \mathbf{u}_j$$

$$\bar{\eta} \approx \beta^\top \bar{x},$$

and so the values of β can be interpreted as yielding the "average" value of $\eta_{i,j}$. On the other hand, under the distance model

$$\eta_{i,j} = \beta^\top x_{i,j} - |\mathbf{u}_i - \mathbf{u}_j|$$

$$\bar{\eta} \approx \beta^\top \bar{x} - \overline{|\mathbf{u}_i - \mathbf{u}_j|} < \beta^\top \bar{x}.$$

In this case, $\beta^\top \bar{x}$ does not represent an "average" value of the predictor $\eta_{i,j}$, it represents a maximal value as if all actors were zero distance from each other in the latent social space. For example, consider the simplest case of a normally distributed network outcome with an identity link. In this case,

$$y_{i,j} = \beta^\top x_{i,j} + \alpha(\mathbf{u}_i, \mathbf{u}_j) + \epsilon_{i,j}$$

$$\bar{y} \approx \beta^\top \bar{x} + \overline{\alpha(\mathbf{u}_i, \mathbf{u}_j)}$$

$$=\approx \beta^\top \bar{x}.$$

Under the distance model, $\bar{y} \approx \beta^\top \bar{x} + \overline{|\mathbf{u}_i - \mathbf{u}_j|} < \beta^\top \bar{x}$, and so we no longer can interpret β as representing the linear relationship between y and x. Instead, it may be thought of as describing some sort of average hypothetical "maximal" relationship between $y_{i,j}$ and $x_{i,j}$.

Thus the latent factor model provides two important benefits over the distance framework. First, we are able to capture a wider assortment of dependence patterns that arise in relational data, and, second, parameter interpretation is more straightforward. The AME approach considers the regression model shown in Equation 3:

$$y_{ij} = g(\theta_{ij})$$

$$\theta_{ij} = \boldsymbol{\beta}^\top \mathbf{X}_{ij} + e_{ij}$$

$$e_{ij} = a_i + b_j + \epsilon_{ij} + \alpha(\mathbf{u}_i, \mathbf{v}_j) \text{ , where} \tag{3}$$

$$\alpha(\mathbf{u}_i, \mathbf{v}_j) = \mathbf{u}_i^\top \mathbf{D} \mathbf{v}_j = \sum_{k \in K} d_k u_{ik} v_{jk}$$

Using this framework, we are able to model the dyadic observations as conditionally inde-pendent given $\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ depends on the the unobserved random effects, $\mathbf{e}$. $\mathbf{e}$ is then modeled to account for the potential first, second, and third-order dependencies that we have discussed. As described in Equation 1, $a_i + b_j + \epsilon_{ij}$, are the additive random effects in this framework and account for sender, receiver, and within-dyad dependence. The multiplicative effects, $\mathbf{u}_i^\top \mathbf{D} \mathbf{v}_j$, are used to capture higher-order dependence patterns that are left over in $\boldsymbol{\theta}$ after accounting for any known covariate information. A Bayesian procedure in which parameters are iteratively updated using a Gibbs sampler is available in the **amen** package to estimate this type of gen-eralized linear mixed effects model from continuous, binary, ordinal, and other relational data types.[11]

2. Empirical Comparison

Their application utilizes a cross-sectional network measuring whether an actor indicated that they collaborated with another during the policy design of the Swiss CO_2 act (Ingold, 2008).[12] The Swiss government proposed this act in 1995 with the goal of undertaking a 10%

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[11]The set of parameters that are estimated in the model from the observed data, $\{\mathbf{Y}, \mathbf{X}\}$, are: latent Gaussian variables ($\boldsymbol{\theta}$); nodal and/or dyadic regression coefficients ($\boldsymbol{\beta}$); additive nodal random effects ($\{(a_i, b_i)\} \in \{i = 1, \ldots, n\}$); network covariance ($\Sigma_{ab}, \Sigma_\epsilon$); multiplicative effects ($\mathbf{U}, \mathbf{V},$ and $\mathbf{D}$).

[12]This is a directed relational matrix as an actor i can indicate that they collaborated with j but j may not have stated that they collaborated with i.

reduction in CO_2 emissions by 2012. The act was accepted in the Swiss Parliament in 2000 and implemented in 2008. Ingold (2008), and subsequent work by Ingold and Fischer (2014), sought to determine what drives collaboration among actors trying to affect climate change policy. The set of actors included in this network are those that were identified by experts as holding an important position in Swiss climate policy.[13] In total, Ingold (2008) identifies 34 relevant actors: five state actors, eleven industry and business representatives, seven environmental NGOs and civil society organizations, five political parties, and six scientific institutions and consultants.

To understand what factors may play a role in shaping collaboration in this relational data structure a modeling approach is necessary. Cranmer et al. (2016) follow Ingold and Fischer (2014) in developing a model specification. We do not review the specification in detail here, instead we just provide a summary of the variables to be included and the theoretical expectations of their effects in Table 3.

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[13]For further details on the methodology utilized in choosing the set of actors see Ingold (2008); Ingold and Fischer (2014).

Variable	Description	Expected Effect
Conflicting policy preferences		
Business v. NGO	Binary, dyadic covariate that equals one when one actor is from the business sector and the other an NGO.	−
Opposition/alliance	Binary, dyadic covariate that equals one when i, sender, perceives j, receiver, as having similar policy objectives regarding climate change.	+
Preference dissimilarity	Transformation of four core beliefs into a Manhattan distance matrix, smaller the distance the closer the beliefs of i and j.	−
Transaction costs		
Joint forum participation	Binary, dyadic covariate that equals one when i and j belong to the same policy forum.	+
Influence		
Influence attribution	Binary, dyadic covariate that equals one when i considers j to be influential.	+
Alter's influence in-degree	Number of actors that mention i as being influential, this is a measure of reputational power.	+
Influence absolute diff.	Absolute difference in reputational power between i and j.	−
Alter = Government Actor	Binary, nodal covariate that equals one when j is a state actor.	+
Functional requirements		
Ego = Environment NGO	Binary, nodal covariate that equals one when i is an NGO.	+
Same actor type	Binary, dyadic covariate that equals when i and j are the same actor type.	+
Endogenous dependencies: ERGM Specific Parameters		
Mutuality	Captures concept of reciprocity, if i indicates they collaborated with j then j likely collaborates with i.	+
Outdegree popularity	Captures idea that actors sending more ties will be more popular targets themselves for collaboration.	+
Twopaths	Counts the number of two-paths in the network, two-path is an instance where i is connected to j, j to k, but i is not connected to k.	−
GWIdegree (2.0)	Takes into account how many ties a node sends in the network, used to capture network structures that result from some highly active nodes.	+
GWESP (1.0)	Counts the number of shared partners for each pair and sums across.	+
GWOdegree (0.5)	Takes into account how many ties a node receives in the network, used to capture networks structures that result from some highly popular nodes.	+

Table 3: Summary of variables to be included in model specification.

2.1. Parameter Estimates

Using the specification described in Table 3 we compare five different modeling approaches. The first four approaches chosen here, as in Cranmer et al. (2016), are a logistic regression model, MRQAP, ERGM, and a latent space model (LSM) in which third-order dependencies are accounted for via a two-dimensional Euclidean distance metric.[14] Parameter estimates for these four approaches are shown in Table 4.

The fifth column shows the results from using the additive and multiplicative effects model (AME), in which we account for nodal and dyadic heterogeneity using the SRM and third-order effects represented by a latent factor approach in which we set $K = 2$.[15] Cranmer et al. (2016)

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[14]For a detailed discussion on the MRQAP see Dekker et al. (2007).

[15]In the online appendix, we show that the parameter estimates presented here for the AME model remain very

provide a lengthy discussion of the differences between the first four modelling approaches that we will not repeat here. More relevant for us are how parameter estimates from AME relate to other approaches. The first point to note is that, in general, the parameter estimates returned by the AME are similar to those of MRQAP and ERGM but quite different from the LSM. For example, while the LSM returns a result for the `Opposition/alliance` variable that diverges from MRQAP and ERGM, the AME returns a result that is not only similar to those approaches but in line with the theoretical expectations of Ingold and Fischer (2014). Similar discrepancies between LSM and other approaches appear for parameters such as `Influence attribution` and `Alter's influence degree`. Each of these discrepancies are resolved when using AME. In part, this is a function of how the LSM approach as operationalized in the **latentnet** package can confound the effects of covariates with the latent space metric.[16]

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similar no matter the K chosen.

[16]In the online appendix, we show that these differences persist even when incorporating sender and receiver random effects or when switching to a bilinear approach to handle third-order dependencies.

	Logit	MRQAP	LSM	ERGM	AME
Intercept/Edges	-4.44*	-4.24*	0.94*	-12.17*	-3.39*
	(0.34)		[0.09; 1.82]	(1.40)	[-4.38; -2.50]
Conflicting policy preferences					
Business vs. NGO	-0.86	-0.87*	-1.37*	-1.11*	-1.37*
	(0.46)		[-2.42; -0.41]	(0.51)	[-2.44; -0.47]
Opposition/alliance	1.21*	1.14*	0.00	1.22*	1.08*
	(0.20)		[-0.40; 0.39]	(0.20)	[0.72; 1.47]
Preference dissimilarity	-0.07	-0.60	-1.76*	-0.44	-0.79*
	(0.37)		[-2.62; -0.90]	(0.39)	[-1.55; -0.08]
Transaction costs					
Joint forum participation	0.88*	0.75*	1.51*	0.90*	0.92*
	(0.27)		[0.86; 2.17]	(0.28)	[0.40; 1.47]
Influence					
Influence attribution	1.20*	1.29*	0.08	1.00*	1.09*
	(0.22)		[-0.40; 0.55]	(0.21)	[0.69; 1.53]
Alter's influence indegree	0.10*	0.11*	0.01	0.21*	0.11*
	(0.02)		[-0.03; 0.04]	(0.04)	[0.07; 0.15]
Influence absolute diff.	-0.03*	-0.06*	0.04	-0.05*	-0.07*
	(0.02)		[-0.01; 0.09]	(0.01)	[-0.11; -0.03]
Alter = Government actor	0.63*	0.68	-0.46	1.04*	0.55
	(0.25)		[-1.08; 0.14]	(0.34)	[-0.07; 1.15]
Functional requirements					
Ego = Environmental NGO	0.88*	0.99	-0.60	0.79*	0.67
	(0.26)		[-1.32; 0.09]	(0.17)	[-0.38; 1.71]
Same actor type	0.74*	1.12*	1.17*	0.99*	1.04*
	(0.22)		[0.63; 1.71]	(0.23)	[0.63; 1.50]
Endogenous dependencies					
Mutuality	1.22*	1.00*		0.81*	0.39
	(0.21)			(0.25)	[-0.12; 0.96]
Outdegree popularity				0.95*	
				(0.09)	
Twopaths				-0.04*	
				(0.02)	
GWIdegree (2.0)				3.42*	
				(1.47)	
GWESP (1.0)				0.58*	
				(0.16)	
GWOdegree (0.5)				8.42*	
				(2.11)	

Table 4: * $p < 0.05$. Logistic regression and ERGM results are shown with standard errors in parentheses. MRQAP provides no standard errors. LSM and AME are shown with 95% posterior credible intervals provided in brackets.

There are also notable differences between the parameter estimates that result from the MRQAP, ERGM, and the AME. Using the AME we find evidence that `Preference dissimilarity` is associated with a reduced probability of collaboration between a pair of actors, which is in line with the theoretical expectations stated earlier. Additionally, the AME and MRQAP results differ from ERGM for the nodal effects related to whether a receiver of a collaboration is a government actor, `Alter=Government actor`, and whether the sender is an environmental NGO, `Ego=Environmental NGO`.

2.2. Tie Formation Prediction

How do these approaches fit the data out-of-sample? We utilize a cross-validation procedure to assess the out-of-sample performance for each of the models presented in Table 4 as follows:

- Randomly divide the $n \times (n-1)$ data points into S sets of roughly equal size, letting s_{ij} be the set to which pair $\{ij\}$ is assigned.

- For each $s \in \{1, \ldots, S\}$:

  - Obtain estimates of the model parameters conditional on $\{y_{ij} : s_{ij} \neq s\}$, the data on pairs not in set s.

  - For pairs $\{kl\}$ in set s, let $\hat{y}_{kl} = E[y_{kl}|\{y_{ij} : s_{ij} \neq s\}]$, the predicted value of y_{kl} obtained using data not in set s.

The procedure summarized in the steps above generates a sociomatrix of out-of-sample predictions of the observed data. Each entry $\hat{y}_{ij}$ is a predicted value obtained from using a subset of the data that does not include y_{ij}. In this application we set S to 45 which corresponds to randomly excluding approximately 2% of the data from the estimation. Such a low number of observations were excluded in every fold because excluding any more observations would cause the ERGM specification to result in a degenerate model that empirically can not be fit. This highlights the computational difficulties associated with ERGMs in the presence of even

small levels of missingness. Not only do you have to have the exactly correct model specification, you also have to have virtually all of the data on the true network.
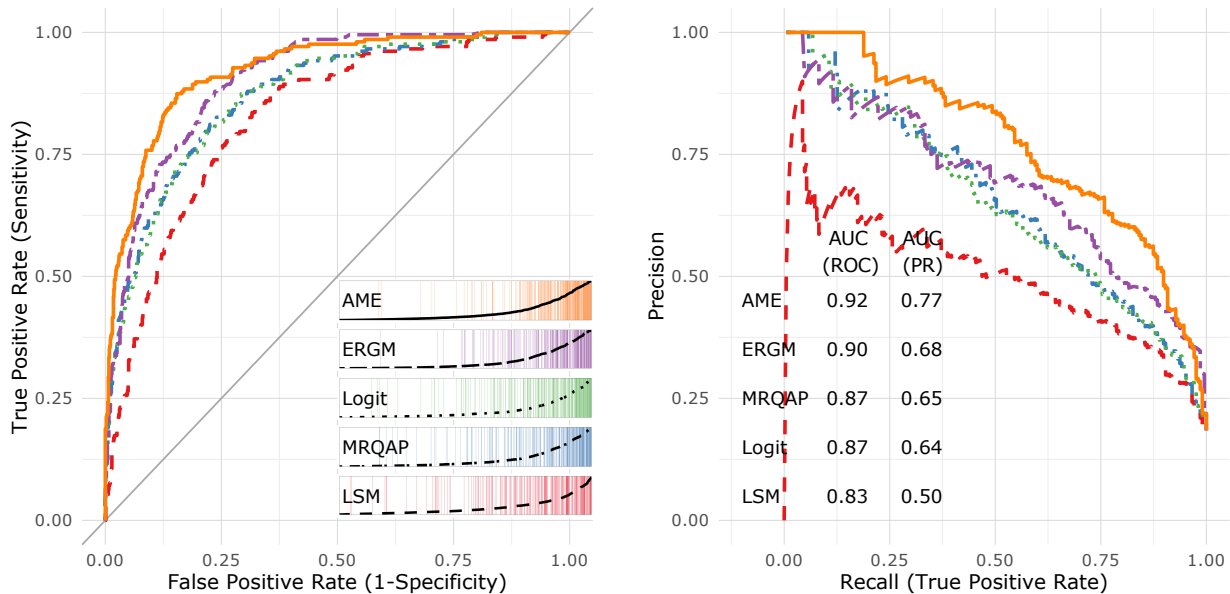
Figure 2: Assessments of out-of-sample predictive performance using ROC curves, separation plots, and PR curves. AUC statistics are provided as well for both curves.

Using the set of out-of-sample predictions we generate from the cross-validation procedure, we provide a series of tests to assess model fit. First, is a diagnostic that is common in the political science literature. The left-most plot in Figure 2 compares the five approaches in terms of their ability to predict the out-of-sample occurrence of collaboration based on Receiver Operating Characteristic (ROC) curves. ROC curves provide a comparison of the trade-off between the True Positive Rate (TPR), sensitivity, False Positive Rate (FPR), 1-specificity, for each model. On this diagnostic, the AME model performs best closely followed by ERGM. The MRQAP and Logit approaches perform similarly, and the LSM approach lags notably behind the other specifications.[17]

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[17]In the online appendix, we provide additional comparisons between our AME approach and various parameterizations of the LSM, in each case we find that the AME approach provides far superior results in terms of out-of-sample predictive performance. However, the LSM approach does begin to perform notably better when incorporating sender and receiver random effects. We also compare performance when using varying values of K

A more intuitive visualization of the differences between these modeling approaches can be gleaned through examining the separation plots included on the right-bottom edge of the ROC plot. This visualization tool plots each of the observations, in this case actor pairs, in the dataset according to their predicted value from left (low values) to right (high values). Models with a good fit should have all network links, here these are colored by the modeling approach, towards the right of the plot. Using this type of visualization we can again see that the AME and ERGM models performs better than the alternatives.

The last diagnostic we highlight to assess predictive performance are precision-recall (PR) curves. In both ROC and PR space we utilize the TPR, also referred to as recall–though in the former it is plotted on the y-axis and the latter the x-axis. The difference, however, is that in ROC space we utilize the FPR, while in PR space we use precision. FPR measures the fraction of negative examples that are misclassified as positive, while precision measures the fraction of examples classified as positive that are truly positive. PR curves are useful in situations where correctly predicting events is more interesting than simply predicting non-events (Davis and Goadrich, 2006). This is especially relevant in the context of studying many relational datasets in political science such as conflict, because events in such data are extremely sparse and it is relatively easy to correctly predict non-events. In the case of our application dataset, the vast majority of dyads, 80%, do not have a network linkage, which points to the relevance of assessing performance using the PR curves as we do in the right-most plot of Figure 2. We can see that the relative-ordering of the models remains similar but the differences in how well they perform become much more stark. Here we find that the AME approach performs notably better in actually predicting network linkages than each of the alternatives. Area under the curve (AUC) statistics are provided in Figure 2 and these also highlight AME's superior out-of-sample performance.

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for the AME model, we find that increasing K to 3 or 4 does not improve out-of-sample model fit. Typically setting $K = 2$ works well for most applied cases.

2.3. Capturing Network Attributes

For network data it is also important to assess whether a model adequately captures the network parameters of the dependent variable (Hunter et al., 2008). To do this one can compare the observed network with a set of networks simulated from the estimated models. We restrict our focus to the three approaches–LSM, ERGM, and AME–that explicitly seek to model network interdependencies. We simulate 1,000 networks from the three models and compare how well they align with the observed network in terms of four network statistics: (1) the empirical standard deviation of the row means (i.e., heterogeneity of nodes in terms of the ties they send); (2) the empirical standard deviation of the column means (i.e., heterogeneity of nodes in terms of the ties they receive); (3) the empirical within-dyad correlation (i.e., measure of reciprocity in the network); and (4) a normalized measure of triadic dependence. A comparison of the LSM, ERGM, and AME models among these four statistics is shown in Figure 3.[18]
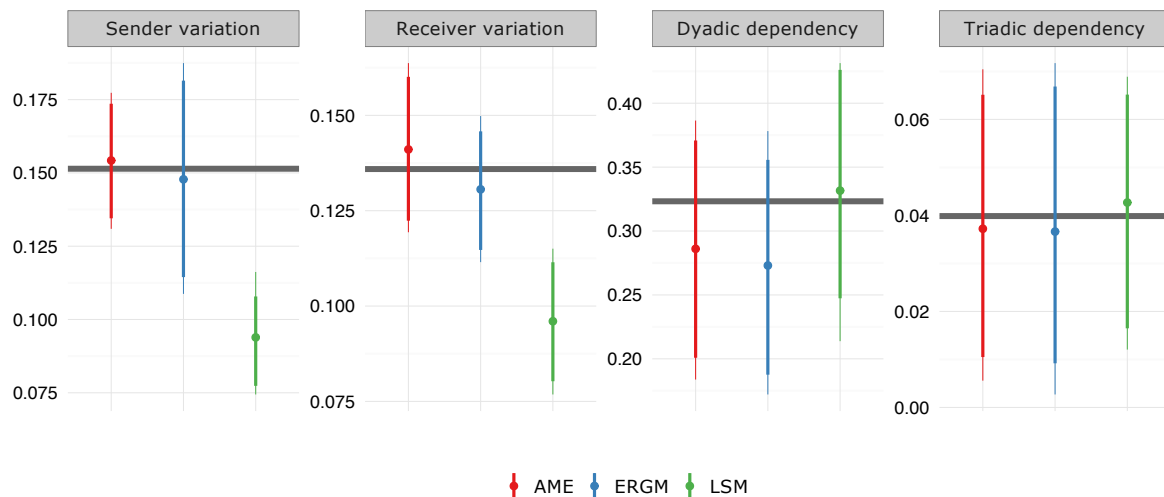
Figure 3: Network goodness of fit summary using **amen**.

Here it becomes quickly apparent that the LSM model fails to capture how active and pop-

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[18]Scholars in the networks field usually test for more specific dependencies in order to ascertain whether a particular endogenous covariate needs to be added or modified. We perform this same performance exericise on a more specific set of statistics and include the results in the online appendix. There we also find that the AME does as well as ERGM, and that the LSM model lags behind.

ular actors are in the Swiss climate change mitigation network.[19] The AME and ERGM specifications again both tend to do equally well.[20] If when running this diagnostic, we found that the AME model did not adequately represent the observed network this would indicate that we might want to increase K to better account for network interdependencies. No changes to the model specification as described by the exogenous covariates a researcher has chosen would be necessary. If the ERGM results did not align with the diagnostic presented in Figure 3, then this would indicate that an incorrect set of endogenous dependencies have been specified. Failing to identify (or find) the right specification will leave the researcher with the problems we introduced earlier.

3. Conclusion

The AME approach to estimation and inference in network data provides a number of benefits over alternative approaches. Specifically, it provides a modeling framework for dyadic data that is based on familiar statistical tools such as linear regression, GLM, random effects, and factor models. We have an understanding of how each of these tools work, they are numerically more stable than ERGM approaches, and more general than alternative latent variable models such as the latent distance or class frameworks. Further the estimation procedure utilized in AME avoids confounding the effects of nodal and dyadic covariates with actor positions in the latent space as the latent distance variable does. For researchers in international relations and more broadly across political science this is of primary interest, as many studies that employ relational data still have conceptualizations that are monadic or dyadic in nature. Additionally, through the application dataset utilized herein we show that the AME approach outperforms both ERGM and latent distance models in out-of-sample prediction, and also is better able to capture network dependencies than the latent distance model.

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[19] Interestingly, even after incorporating random sender and receiver effects into the LSM framework this problem is not completely resolved, see the online appendix for details.

[20] Not surprisingly, if we increase K in the AME approach we are able to better account for triadic dependencies, see the online appendix for details.

More broadly, relational data structures are composed of actors that are part of a system. It is unlikely that this system can be viewed simply as a collection of isolated actors or pairs of actors. The assumption that dependencies between observations occur can at the very least be examined. Failure to take into account interdependencies leads to biased parameter estimates and poor fitting models. By using standard diagnostics such as shown in Figure 3, one can easily assess whether an assumption of independence is reasonable. We stress this point because a common misunderstanding that seems to have emerged within the political science literature relying on dyadic data is that a network based approach is only necessary if one has theoretical explanations that extend beyond the dyadic. This is not at all the case and findings that continue to employ a dyadic design may misrepresent the effects of the very variables that they are interested in. The AME approach that we have detailed here provides a statistically familiar way for scholars to account for unobserved network structures in relational data. Additionally, through this approach we can visualize these dependencies in order to better learn about the network patterns that remain in the event of interest after having accounted for observed covariates.

When compared to other network based approaches such as ERGM, AME is easier to specify and utilize. It is also more straightforward to interpret since it does not require interpretation of unusual features such as *three-stars* which fall outside of the normal language for discussing social science. Further, the **amen** package facilitates the modeling of longitudinal network data. In sum, excuses for continuing to treat relational data as conditionally independent are no longer valid.

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