第3回 数理物理演習会（19 07/15 月）

問 2.3

V : inner product space.

次を示せ.

$$\forall u, v \in V \quad \big|\, \|u\| - \|v\| \,\big| \le \|u - v\|.$$

3角不等式（p.73 (2.11)）より,

$$\|u\| = \|(u-v)+v\| \le \|u-v\| + \|v\|$$

$$\therefore \|u\| - \|v\| \le \|u-v\|$$

$$\|v\| = \|(v-u)+u\| \le \|v-u\| + \|u\|$$

$$\begin{aligned}\therefore \|v\| - \|u\| &\le \|v-u\| \\ &= \|(-1)(u-v)\| \\ &= |-1|\,\|u-v\| \quad ∵ \text{p.68 (2.4)} \\ &= \|u-v\|\end{aligned}$$

よって,

$$\begin{aligned}&\big|\, \|u\| - \|v\| \,\big| \\ &= \max\{\|u\| - \|v\|,\ \|v\| - \|u\|\} \\ &\le \|u-v\|\end{aligned}$$

□.

問. 2. $\{ \cancel{2}(1), \cancel{3}, 5, 6, 7 \}$

問 2.5

$$* \tau_n \overset{?①}{=} 1, \quad * 1 \overset{?②}{=} \varepsilon(\tau_n)\tau_n$$

Def. Hodge star-operator. (p.91〜92)

V : n-dim real met. vec. sp. $\quad h_{\tau_n} =: *$

τ_n : basis of $\Lambda^n V$, $\quad |\langle \tau_n, \tau_n \rangle| = 1$

the n-th exterior power of V

$$\exists_1 * : \Lambda^p V \longrightarrow \Lambda^{n-p} V, \quad p = 0, \cdots, n$$

$$\tau_n \longrightarrow T \longmapsto *T$$

s.t. $T \wedge S = \langle *T, S \rangle \tau_n$, $\quad S \in \Lambda^{n-p} V$

$T \in \Lambda^p V$, $S \in \Lambda^{n-p}V$, $\langle *T, S\rangle \in \mathbb{K} = \Lambda^0 V$, $\tau_n \in \Lambda^n V$

① $* \tau_n \overset{?}{=} 1$

$\tau_n \in \Lambda^n V$ より, $*\tau_n$ を考えると, 定ギで $p = n$ のバアイであるなら,

相手となるテンソルは $S \in \Lambda^{n-n} V = \Lambda^0 V = \mathbb{K}$.

任意に $S \in \mathbb{K}$ をとる.

$$\langle *\tau_n, S \rangle_{\otimes^p V} \tau_n = \tau_n \wedge S \overset{?}{=} S \tau_n \overset{?}{=} \langle 1, S \rangle_{\otimes^p V} \tau_n \qquad (S \in \mathbb{K})$$

$$\therefore * \tau_n = 1.$$

② $* 1 \overset{?}{=} \varepsilon(\tau_n)\tau_n$

$1 \in \Lambda^0 V$ より, $*1$ を考えると, 定ギで $p = 0$ のバアイであるから,

相手となるテンソルとして任意に $w \in \Lambda^{n-0} V = \Lambda^n V$ をとる.

w は τ_n を用いて,

$$w = \lambda \tau_n, \quad \lambda = \varepsilon(\tau_n)\langle \tau_n, w \rangle \quad ∵ \text{p.79 (2.25)}$$

と表すことができ,

$$\langle *1, w \rangle \tau_n = 1 \wedge w \overset{?}{=} w = \langle \varepsilon(\tau_n)\tau_n, w \rangle \tau_n$$

$$\therefore *1 = \varepsilon(\tau_n)\tau_n.$$

□.

∵ p.55 例1.31 ∵ p.53 (1.60)

(2.1)

$$v \wedge v = \frac{v \otimes v - v \otimes v}{\sqrt{2}}$$

$$u_1 \wedge \cdots \wedge u_p = \sqrt{p!}\, A_p(u_1 \otimes \cdots \otimes u_p)$$

$$\sqrt{p!} \times \frac{1}{p!} \sum \mathrm{sgn}(\sigma)\, u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(p)}$$

$\to u_1 \otimes u_2 \otimes u_3$ $\quad \to u_1 \wedge u_2 \wedge u_3$

$u_1 \wedge u_2 \wedge u_3 \quad$ 例 3

$$\frac{\sqrt{3!}}{3!}\big[u_1 \otimes u_2 \otimes u_3 + \mathrm{sgn}(\sigma) u_2 \otimes u_1 \otimes u_3 + \mathrm{sgn}(\sigma) u_3 \otimes u_1 \otimes u_2 + \mathrm{sgn}(\sigma) u_1 \otimes u_3 \otimes u_2 + \mathrm{sgn}(\sigma) u_2 \otimes u_3 \otimes u_1 + \mathrm{sgn}(\sigma) u_3 \otimes u_2 \otimes u_1\big]$$

σ

$$g(u_1 \otimes \cdots \otimes u_p,\ v_1 \otimes \cdots \otimes v_p)$$

$$= \langle u_1 \wedge \cdots \wedge u_p,\ v_1 \wedge \cdots \wedge v_p \rangle$$

$$\langle \sqrt{p!} A_p(u_1 \otimes \cdots \otimes u_p),\ \sqrt{p!} A_p(v_1 \otimes \cdots \otimes v_p) \rangle$$

$$\Big\langle \frac{1}{\sqrt{p!}} \sum \mathrm{sgn}(\sigma)(u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(p)}),\ \frac{1}{\sqrt{p!}} \sum \mathrm{sgn}(\tau)(v_{\tau(1)} \otimes \cdots \otimes v_{\tau(p)}) \Big\rangle$$

$g(\alpha u, \beta v) = \langle \alpha u, \beta v \rangle = \alpha\beta \langle u, v \rangle$ となるので

$$\frac{1}{p!} \sum\sum \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \langle u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(p)},\ v_{\tau(1)} \otimes \cdots \otimes v_{\tau(p)} \rangle$$

$= g(\cdots, \cdots)$

p.85 (2.36)

$$g(e_{i_1} \otimes \cdots \otimes e_{i_p},\ e_{j_1} \otimes \cdots \otimes e_{j_p}) = g(e_{i_1}, e_{j_1})\, g(e_{i_2}, e_{j_2}) \cdots g(e_{i_p}, e_{j_p})$$ とする

$$\frac{1}{p!} \sum \mathrm{sgn}(\sigma) \sum \mathrm{sgn}(\tau)\, g(u_{\sigma(1)}, v_{\tau(1)}) \cdots g(u_{\sigma(p)}, v_{\tau(p)})$$

とおくと

$\tau = \tau\sigma^{-1}\sigma$ (5)

同じ

$$g(u_1, v_{\tau\sigma^{-1}(1)}) \cdots g(u_p, v_{\tau\sigma^{-1}(p)})$$

$$g(u_{\sigma(1)}, v_{\tau\sigma^{-1}\sigma(1)})\, g(u_{\sigma(2)}, v_{\tau\sigma^{-1}\sigma(2)})$$

$$\mathrm{sgn}(\tau') = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma^{-1}) = \mathrm{sgn}(\tau)\cdot\mathrm{sgn}(\sigma)$$

$\tau' = \tau\sigma^{-1}$

$$= \sum \mathrm{sgn}(\tau')\, g(u_1, v_{\tau'(1)}) \cdots g(u_p, v_{\tau'(p)})$$

$$= \det(g(u_i, v_j))$$

$$\begin{bmatrix} u_1 v_1 & u_1 v_2 & u_3 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}$$

行列式の計算と同じ

#

Σ → last 1 $\qquad$ 積 → last 2.

$$\frac{1}{p!} \sum_{\sigma \in S_p} \sum_{\tau \in S_p} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\, g(u_{\sigma(1)}, v_{\tau(1)}) \cdots g(u_{\sigma(p)}, v_{\tau(p)})$$

（σ と τ の対応を保ちながら, 次のように総和の順序を入れ替える.

$$\sigma \rightsquigarrow \sigma\sigma^{-1} = \underset{\text{何も置換しない.}}{1}$$

$$\tau \rightsquigarrow \tau\sigma^{-1} =: \tau'$$ ）

$$= \frac{1}{p!} \underbrace{\sum_{1 \in S_p}}_{p!\text{コの足しあげ.}} \sum_{\tau' \in S_p} \underbrace{\mathrm{sgn}(1)}_{+1} \mathrm{sgn}(\tau')\, g(u_1, v_{\tau'(1)}) \cdots g(u_p, v_{\tau'(p)})$$

$$= \frac{1}{\cancel{p!}} \cancel{p!} \sum_{\tau' \in S_p} \mathrm{sgn}(\tau')\, g(u_1, v_{\tau'(1)}) \cdots g(u_p, v_{\tau'(p)})$$

$$= \det\big(g(u_i, u_j)\big)_{i,j = 1, \cdots, p}$$

□.