Abs. Alg #16 19/11/29（金）

Cor 4.5. $|G| = n$
$p = \min\{\, p : \text{prime} \mid p \mid n \,\}$

$\forall H \leq G$ with $|G:H| = p$ $\quad H \trianglelefteq G$.

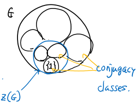
§4.3. Group Acting on Themselves by Conjugation

$$(\cdot) : G \times G \longrightarrow G$$
$$(g, a) \longmapsto g \cdot a := g a g^{-1}$$

Def.
$a, b \in G$: conjugate in G
$:\Leftrightarrow \exists g \in G$ s.t. $b = g a g^{-1}$
(i.e. a, b are in the same orbit of G acting itself by conj.)

この§の $(\cdot)$ のもとでの orbits of G を conjugacy classes of G という.

例.
(2) $|G| > 1$
G does not act transitively.



D_8. $Z(D_8) = \{1, r^2\}$. $r s (r^2)^{-1} = s$.

$\{a\}$: conjugacy class
$\Leftrightarrow \forall g \in G \quad g a g^{-1} = a$
$\Leftrightarrow a \in Z(G)$.

cf. Left Multi.
$G = S_4$, $A = \{1, \cdots, 4\}$
G : transitive.

☆ この§の action を一般化.
$S \subset G$
$g S g^{-1} := \{\, g s g^{-1} \mid s \in S \,\}$

$$(\cdot) : G \times \mathcal{P}(G) \longrightarrow \mathcal{P}(G)$$
$$(g, S) \longmapsto g \cdot S := g S g^{-1}$$

とくに, $S = \{a\}$ のとき, はじめの $(\cdot)$ と同一視.

Def.
$S, T \subset G$: conjugate in G
$:\Leftrightarrow \exists g \in G$ s.t. $T = g S g^{-1}$

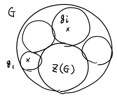
☆ **Prop 4.2 より.**

$$\#\{\, g S g^{-1} \mid g \in G \,\} = \#\{\, T \in \mathcal{P}(G) \mid S \sim T \,\}$$
$$= |G : G_S| \qquad \{\, g \in G \mid \underbrace{g \cdot S}_{g S g^{-1}} = S \,\}$$
$$= |G : N_G(S)|.$$

Prop 4.6.
$\#\{\, g S g^{-1} \mid g \in G \,\} = |G : N_G(S)|$
とくに,
$\#\{\, g s g^{-1} \mid g \in G \,\} = |G : C_G(s)|$. $\{\, g \in G \mid g s g^{-1} = s \,\}$

Thm 4.7 (The Class Equation.)
$|G| < \infty$.

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

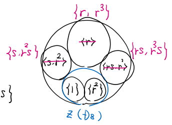


例 (3).

D_8
abelian: $\langle s, r^2 \rangle$, $\langle r \rangle$, $\langle rs, r^2 \rangle$ (index 2)

$\langle s, r^2 \rangle = \{1, r^2, s, r^2 s\}$, $\langle r \rangle = \{1, r, r^2, r^3\}$, $\langle rs, r^2 \rangle = \{1, r^2, rs, r^3 s\}$

$Z(D_8)$



$|D_8 : C_{D_8}(s)|$
$= \quad \{\, g \in D_8 \mid g s g^{-1} = s \,\}$
$g s = s g$

$$|D_8| = |Z(D_8)| + |D_8 : C_{D_8}(s)| + |D_8 : C_{D_8}(r)| + |D_8 : C_{D_8}(rs)|$$
$$= 2 + 2 + 2 + 2$$

Thm 4.8. p : prime.
$|P| = p^{\alpha} \quad (\alpha \geq 1)$.

P has a non-trivial center.
i.e. $Z(P) \neq 1$

<proof>.
Thm 4.7. より.
$|P| = |Z(P)| + \sum |P : C_P(g_i)|$.
仮に $\underbrace{C_P(g_i)}_{\{\, p \in P \mid p g_i p^{-1} = g_i \,\}} = P$ とすると, $g_i \in Z(P)$ となり矛盾.

$\because \forall p \in P \quad p g_i p^{-1} = g_i$. $\; g_i \in Z(P)$.

よって, $C_P(g_i) \neq P$.
$\therefore \; p \mid |P : C_P(g_i)|$
これと, $p \mid |P|$ より,
$p \mid |Z(P)| = |P| - \sum |P : C_P(g_i)|$.
$\therefore \; |Z(P)| \neq 1$.

Cor 4.9. $|P| = p^2$. $\quad p$: prime.

P : abelian

$P \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$

p.125. Conjugacy in S_n.

Prop 4.10. $\sigma, \tau \in S_n$
$\sigma = (a_1\, a_2 \cdots a_{k_1})(b_1\, b_2 \cdots b_{k_2}) \cdots$
$\tau \sigma \tau^{-1}$ has cycle decomp :
$$(\tau(a_1) \cdots \tau(a_{k_1}))(\tau(b_1) \cdots \tau(b_{k_2})) \cdots$$

<proof>
$\sigma(i) = j \;\Rightarrow\; \tau \sigma \tau^{-1}(\tau(i)) = \tau(\sigma(i)) = \tau(j)$
よって, σ の cycle decomp. に i, j がこの順で現れる
なら, $\tau \sigma \tau^{-1}$ の 〃 $\tau(i), \tau(j)$ が 〃

$$i \xrightarrow{\sigma} j \qquad \tau(i) \xrightarrow{\tau \sigma \tau^{-1}} \tau(j)$$

Def.
(1) $\sigma = \underbrace{(\cdot)}_{n_1} \underbrace{(\cdot)}_{n_2} (\cdot\,\cdot) \cdots \underbrace{(\cdots\cdots)}_{n_r}$, $\quad n_1 \leq n_2 \leq \cdots \leq n_r$

$(n_1, \cdots, n_r)$: cycle type of σ.

(2) 非減少自然数列 $\{n_i\}_i$ で $\sum n_i = n$ なものを a partition of n という.