Abs. Alg. #14 19/11/15 金

☆ 置換 ~ 互換の積 ~ 符号.

Def. $(p\ q) \in S_n$

A 2-cycle is called a transposition.

Def. $x_1, \cdots, x_n$: ind. variables.

$$\Delta := \prod_{1 \le i \le j \le n} (x_i - x_j)$$

$$\sigma(\Delta) := \prod_{1 \le i \le j \le n} (x_{\sigma(i)} - x_{\sigma(j)}) \quad (= \pm \Delta)$$

$$\varepsilon(\sigma) := \begin{cases} +1 & \text{if } \sigma(\Delta) = +\Delta \\ -1 & \sigma(\Delta) = -\Delta \end{cases}$$

: the sign of σ.

$\varepsilon(\sigma) = +1 \Rightarrow \sigma$: even permu.

$ -1 \Rightarrow \sigma$: odd permu.

Prop 3.23.
$\varepsilon : S_n \longrightarrow \{\pm 1\}$: homo.

Prop 3.24.
a transpostion is an odd permu.
ε is surjective. $n=1$?

Def.

$A_n := \ker \varepsilon$
$= ($the set of even permu.s$)$
: the alternating group of degree n.

☆

$$\begin{tikzcd} S_n \arrow[r, "\varepsilon : \text{homo. surj.}"] \arrow[d] & \{\pm 1\} \\ S_n / \ker \varepsilon \arrow[ur, "\cong"'] & \end{tikzcd}$$

1st iso. thm より

$S_n / A_n \cong \{\pm 1\}$

$|S_n| / |A_n| = 2.$

$|A_n| = \frac{1}{2}(n!)$.

Prop 3.25.
$\varepsilon(\sigma) = -1$
$\Leftrightarrow$ σ を cycle decomposition したときの even長の cycle の個数が odd.

* $\varepsilon(\sigma_1 \sigma_2 \sigma_3 \cdots) = \varepsilon(\sigma_1)\varepsilon(\sigma_2)\cdots$

例. $\sigma = (1\ 2)(3\ 4)(\underbrace{5\ 6\ 7\ 8}_{(5\ 6)(6\ 7)(7\ 8)})$

ε: $-1 \times -1 \times -1 = \varepsilon(\sigma)$.

4.1. Group actions and Permutation Representation

☆ Group action について.

Sylow's thm. で頂点に達する. (p.139).

☆ G : a group acting on $\phi \neq A$.

i.e. $(\cdot) : G \times A \longrightarrow A$: act.

§1.7で見たように, 各 $g \in G$ に対して,

$$\sigma_g : A \longrightarrow A, \quad a \longmapsto \sigma_g(a) := g \cdot a$$

と定めると, σ_g は A の置換で,

$$\varphi : G \longrightarrow S_A : \text{homo.}, \quad g \longmapsto \varphi(g) := \sigma_g$$

: the permutation representation associated to the given action

が定まる.

Def. $(\cdot) : G \times A \longrightarrow A$: act.

(1) $\ker(\cdot) := \{g \in G \mid \forall a \in A\ g \cdot a = a\}$

(2) $G_a := \{g \in G \mid g \cdot a = a\}$
: the stabilizer of a in G.

(3) $\ker(\cdot) = 1 \Rightarrow (\cdot)$: faithful

☆ Note that ... (p.113).

① $\ker(\cdot) = \ker \varphi \trianglelefteq G$

∵ $G \trianglerighteq \ker\varphi = \{g \in G \mid \varphi(g) = \sigma_1\}$
$= \{\ \text{〃} \mid \forall a \in A\ \sigma_g(a) = \sigma_1(a)\}$
$= \{\ \text{〃} \mid \text{〃}\ g \cdot a = a\}$
$= \ker(\cdot)$.

② $\varphi : G \longrightarrow S_A$

$\sigma_g = \sigma_{g'}$
$\Leftrightarrow gN = g'N$ where $\ker\varphi = N$
$\Leftrightarrow g' \in gN = \varphi^{-1}(\sigma_g)$.

③ $(\cdot) : G \times A \longrightarrow A$

$\sim (\bar{\cdot}) : G / \underbrace{\ker\varphi}_{1} \times A \longrightarrow A$: faithful

④ $\ker\varphi = \ker(\cdot)$ $\qquad G_a \le G$
$= \{g \in G \mid \forall a \in A\ g \cdot a = a\}$
$= \{g \in G \mid \forall a \in A\ g \in G_a\}$
$= \bigcap_{a \in A} G_a$
$\trianglelefteq G$.

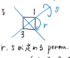
例. (permu. repr., faithful, stab.)

(1) $G = S_n$
$A = \{1, \cdots, n\}$
$(\cdot) : G \times A \longrightarrow A$, $(\sigma, i) \longmapsto \sigma \cdot i := \sigma(i)$

$\underbrace{\varphi}_{\mathrm{id}_{S_n}} : G \longrightarrow S_A$, $\sigma \longmapsto \underbrace{\varphi(\sigma)}_{\sigma} : A \longrightarrow A$, $i \longmapsto \sigma(i)$

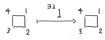
(2) $G = D_8$

$A =$ 

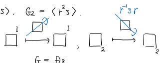
$r, s \in D_8$ を 

と定めると, r, s が定める permu. は,
$\sigma_r = (1\ 2\ 3\ 4)$
$\sigma_s = (\ \ 2\ \ 4)$

$(\cdot) : D_8 \times A \longrightarrow A$ は faithful.

∵ 

$G_1 = \langle s \rangle$, $G_2 = \langle r^2 s \rangle$.

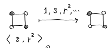
∵ 

(3). $G = D_8$

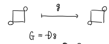
$A =$ 

r, s は(2)と同.
$\sigma_r = (\bar{1}\ \bar{2})$
$\sigma_s = 1$.

$(\cdot) : D_8 \times A \longrightarrow A$ は not faithful.

∵ 

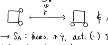
$G_{\bar{1}} = \langle s, r^2 \rangle$

∵ 

(4) $G = D_8$

$A =$ 

D_8 does not act on A.

∵  $\notin A$

☆ $\varphi : G \longrightarrow S_A$: homo. から, act. $(\cdot)$ が定められる :

$$(\cdot) : G \times A \longrightarrow A, \quad (g, a) \longmapsto g \cdot a := \varphi(g)(a).$$

Prop 4.1. G : group
$\phi \neq A$: set.

$$\{(\cdot) : G \times A \to A\} \xrightarrow{\exists \text{ bij.}} \{\varphi : G \to S_A : \text{homo.}\}$$

Def. G : group
$\phi \neq A$: set
$\varphi : G \to S_A$: homo.

φ : a permu. repr. of G.

a given $(\cdot) : G \times A \to A$ affords the φ.
(induces)
いつ?