Abs Alg リモート輪講#4 [2020.5.8]

§2.2. Centralizer and Normalizer, Stabilizers and Kernels

Def G: group
$g, a \in G$

$C_G(A) := \{ g \in G \mid {}^\forall a \in A \ \ gag^{-1} = a \}$

: the centralizer of A in G

例 $C_G(A) \leq G$

例 略.

$C_G(\{a\}) = : C_G(a)$

Def

$Z(G) := C_G(G)$

: the center of G

Def

$N_G(A) := \{ g \in G \mid \underset{\underset{\{ gag^{-1} \mid a \in A \}}{\shortparallel}}{gAg^{-1}} = A \}$

: the normalizer of A in G

例 $N_G(A) \leq G$

① $1 \in N_G(A)$

② $1A1^{-1} = \{ 1a1^{-1} \mid a \in A \} = A$

③ $x, y \in N_G(A) \Rightarrow xy^{-1} \in N_G(A)$.

① $(xy^{-1})A(xy^{-1})^{-1} \overset{?}{=} A$

$xy^{-1} \underset{\underset{A}{\shortparallel}}{A} yx^{-1} = xAx^{-1} = A$.

$\{ y^{-1}ay \mid a \in A \} = yAy^{-1}$

$= \{ y^{-1}(yay^{-1})y \mid a \in A \}$

$= A$

例 $C_G(A) \leq N_G(A) \leq G$

演 略

Def G: a group acting on a set S
$s \in S$

$G_s := \{ g \in G \mid g \cdot s = s \}$

: the stabilizer of s in G

(G作用S, sを固定するG全体の集合)

$\ker_G(S) := \{ g \in G \mid {}^\forall s \in S \ \ g \cdot s = s \}$

: the kernel of the action of G on S

例(1) $G = D_8$, $S = \{1, 2, 3, 4\}$.

$G_2 = \{1, 2\}$

$\ker_{D_8}(S) = \{1\}$

例(2) $G = D_8$, $S = \{\{1, 3\}, \{2, 4\}\}$

$G_{\{1,3\}} = \{1, r^2, s, sr^2\}$

$G_{\{2,4\}} = \quad 〃$

$\ker_{D_8}(S) = \quad 〃$

★ $G_s \leq G$

★ $\ker_G(S) \leq G$

① $1 \in \ker_G(S)$

① action の def より

closed under inverse: $\forall g \in \ker_G(S) \Rightarrow g^{-1} \in \ker_G(S)$.

① ${}^\forall s \in S \quad g \cdot s = s \Rightarrow (g^{-1}) \cdot s$

$= g^{-1} \cdot (g \cdot s)$ ① action の

$= g^{-1} \cdot s$ ② $g \in \ker_G(S)$

closed under product.

① $x, y \in \ker_G(S) \Rightarrow {}^\forall s \in S \ \ xy \cdot s$

$(xy) \cdot s = x \cdot (y \cdot s) = x \cdot s = s \Rightarrow xy \in \ker_G(S)$.

★ centralizer, normalizer, center は、

stabilizer and kernel の action を用いて説明できる

(1) $\underset{\underset{(g, A)}{\shortparallel}}{G \times \mathcal{P}(G)} \longrightarrow \mathcal{P}(G)$

$(g, A) \longmapsto g \cdot A := \underset{\underset{gAg^{-1}}{\shortparallel}}{gAg^{-1}}$

とする.

$N_G(A) = \underset{\underset{\leq G}{\shortparallel}}{G_A} = \{ g \in G \mid \underset{\underset{gAg^{-1}}{\shortparallel}}{g \cdot A} = A \}$

$\leq G$

(2)

$N_G(A) \times A \longrightarrow A$

$(g, a) \longmapsto g \cdot a := gag^{-1}$

とする.

$C_G(A) = \underset{\underset{\leq G}{\shortparallel}}{\ker_{N_G(A)}(A)} = \{ g \in \underset{\underset{\leq G}{\shortparallel}}{N_G(A)} \mid {}^\forall a \in A \ \ \underset{\underset{gag^{-1}}{\shortparallel}}{g \cdot a} = a \}$

$\leq N_G(A)$

$C_G(A) \leq N_G(A)$. $N_G(A) \leq G$ $\quad \therefore$ transitive に $\leq$. $C_G(A) \leq G$

(3) $Z(G) = \ker_G(G) = \{ g \in G \mid {}^\forall x \in G \ \ \underset{\underset{xgx^{-1}}{\shortparallel}}{g \cdot x} = x \}$

§2.3. Cyclic groups and Cyclic subgroups

例 $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ の subgroup の全 order は?

Def H: group

H: cyclic

$\overset{\text{def}}{\Longleftrightarrow} {}^\exists x \in H \ \ {}^\forall x \in H = \langle x^k \rangle \in \mathbb{Z}$.

write

$H = \langle x \rangle$.

Ex (1) $G = \mathbb{Z}_{24}$

$H = \langle 4 \rangle$

$x^k \ (1 \leq k \leq 6 \ \ 0 \leq k \leq 6) \in \mathbb{Z}/24\mathbb{Z}$

${}^\exists x = x^{k+1} = (x^k)x^1 = 1^k x^1 = x^k$

$H = \{ \overset{\text{初項}}{x^0 = 1}, x^1, \cdots, x^{\overset{\text{末項}}{5}} \}$ に. $|H| = 6$.

Ex (2) $H = \langle 2, 3 \rangle$

$|H| = |\mathbb{Z}| = \infty$

Prop 2.2.

$H = \langle x \rangle \Rightarrow |H| = |x|$

① 略.

Prop 2.3 G: group
$x \in G$
$m, n \in \mathbb{Z}$

$x^m = x^n = 1 \Rightarrow x^{(m,n)} = 1$

ex. $x^m = 1 \Rightarrow |x| \mid m$

① $x^m = x^n = 1$ より. Euclid互除法より.

${}^\exists x, y \in \mathbb{Z}$ $x, y \in \quad (m, n) = mx + ny$

$\therefore x^{(m,n)} = x^{mx+ny} = (x^m)^x (x^n)^y = 1^x 1^y = 1$

「ex に ji 代入 $x \in \mathbb{Z}$ $x^{(m,n)} = 1$」 $\Rightarrow |x| = k$ とする.

1) $(m, n) \quad |x| \mid m$

2) $m \neq n$ の ?

$x^m = 1, x^n = 1$ を $\mathbb{Z}$ 上で考えると, $x^{(m,n)} = 1$.

3.で示. $0 < (m, k) \leq k$ $\quad$ 「k より小」

$(m, k) < k$ より矛盾.

① $|x| = k$ より, k の x の生成元 $\{x, x^2, \cdots, x^k = 1\}$ を $x^m = 1$ となるように m.

$\therefore \quad |x| = k = (m, k) \mid m$.

Thm 2.4

Any two cyclic groups of the same order are isomorphic

(1) $|\langle x \rangle| = n < \infty \Rightarrow \langle x \rangle \cong \langle y \rangle$

(2) $|\langle x \rangle| = \infty \Rightarrow \langle x \rangle \cong \mathbb{Z}$

proof >

(1) $\varphi : \langle x \rangle \longrightarrow \langle y \rangle$

$\quad\quad\quad