Thm 3.20 $N \trianglelefteq G$

$\pi : \{ A \leq G \mid N \leq A \} \longrightarrow \{ A/N \leq G/N \}$

s.t. $\pi :$ bij.

と $\langle$ に, surj. i.e.

$\forall \overline{H} \leq \overline{G} \;\; \exists A \leq G$ s.t. $N \trianglelefteq A, \overline{H} = \overline{A}$

$\pi :$ bij. は次の指標をもつ. $N \leq {}^{\forall} A \leq G$, $N \leq {}^{\forall} B \leq G$,

(1) $A \leq B \iff \overline{A} \leq \overline{B}$

(2) $A \leq B \implies |B : A| = |\overline{B} : \overline{A}|$

(3) $\overline{\langle A, B \rangle} = \langle \overline{A}, \overline{B} \rangle$ $\langle A, B \rangle = \langle A \cup B \rangle$

(4) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

(5) $A \trianglelefteq G \iff \overline{A} \trianglelefteq \overline{G}$

\<proof\>

$\pi : \{ A \leq G \mid N \leq A \} \longrightarrow \{ \overline{A} \leq \overline{G} \}$

$A \longmapsto \pi(A) := \overline{A}$

$(= A/N)$.

と定め, $\pi :$ bij. を示す.

$\pi :$ inj.

① $\overline{A} = \overline{B}$ とし. $a \in A$ とする. $aN \in \overline{A} = \overline{B}$ より. $(= B/N)$
$(= \{ bN \mid b \in B \})$

$\exists b \in B$ s.t. $aN = bN$,

$\therefore b^{-1} aN = 1N$ $\therefore b^{-1} a \in N \leq B$ $\therefore a \in bB = B$.

$\therefore A \subset B$. 同様に $B \subset A$ $\therefore A = B$.

$\pi :$ surj.

② $\overline{A} \leq \overline{G}$ とする.

$\pi : G \longrightarrow \overline{G}$

$g \longmapsto \pi(g) := gN =: \overline{g}$

を考える.

$\pi^{-1}(\overline{A}) = \{ g \in G \mid \pi(g) \in \overline{A} \}$

と考えると,

i) $\pi^{-1}(\overline{A}) \leq G$

① $\pi(1) = \overline{1} \in \overline{A}$ より, $1 \in \pi^{-1}(\overline{A}) \neq \phi$.

$g, h \in \pi^{-1}(A)$ とすると,

$\pi(gh^{-1}) = gh^{-1} N = gN \cdot h^{-1} N$

$= gN (hN)^{-1} = \overline{g}(\overline{h})^{-1} \in \overline{A}$

$\therefore gh^{-1} \in \pi^{-1}(\overline{A})$.

ii) $N \leq \pi^{-1}(\overline{A})$.

① $\forall n \in N$ $\pi(n) = nN = N = \overline{1} \in \overline{A}$

$\therefore n \in \pi^{-1}(\overline{A})$.

i) ii) より, $\overline{A} = \pi(B)$ となる $B = \pi^{-1}(\overline{A})$ を見つけた.

\<proof\> (3)

$\overline{\langle A, B \rangle} = \langle \overline{A}, \overline{B} \rangle$

i.e. $\langle A \cup B \rangle / N = \{ c_1^{\delta_1} \cdots c_n^{\delta_n} \mid c_i \in A \cup B, \delta_i \in \mathbb{Z}, c_i \neq c_{i+1}, n \in \mathbb{Z}^+ \} / N$

$\langle A/N \cup B/N \rangle = \{ \overline{c_1}^{\delta_1} \cdots \overline{c_n}^{\delta_n} \mid \overline{c_i} \in \overline{A} \cup \overline{B}, \delta_i \in \mathbb{Z}, \overline{c_i} \neq \overline{c_{i+1}}, m \in \mathbb{Z}^+ \}$

$(\subset)$

$\forall \overline{cN} \in \overline{\langle A, B \rangle}$

$\exists n \in \mathbb{Z}^+$ $\exists \{ c_i \}_{i=1}^{n} \subset A \cup B$ $\exists \{ \delta_i \}_{i=1}^{n} \subset \mathbb{Z}$ s.t.

$c = c_1^{\delta_1} \cdots c_n^{\delta_n}$, $c_i \neq c_{i+1}$

$\therefore cN = c_1^{\delta_1} \cdots c_n^{\delta_n} N = (c_1 N)^{\delta_1} \cdots (c_n N)^{\delta_n}$

$=: \overline{c_1}^{\delta_1} \cdots \overline{c_n}^{\delta_n} \in \langle \overline{A}, \overline{B} \rangle$

$(\overline{c_i} \in \overline{A} \text{ or } \overline{c_i} \in \overline{B} \text{ i.e. } \overline{c_i} \in \overline{A} \cup \overline{B})$.

$(\supset)$

$\forall \overline{c} \in \langle \overline{A}, \overline{B} \rangle$

$\exists m \in \mathbb{Z}^+$ $\exists \{ \overline{c_i} \}_{i=1}^{m} \subset \overline{A} \cup \overline{B}$ $\exists \{ \delta_i \}_{i=1}^{m} \subset \mathbb{Z}$ s.t.

$\overline{c} = \overline{c_1}^{\delta_1} \cdots \overline{c_m}^{\delta_m}$, $\overline{c_i} \neq \overline{c_{i+1}}$

$\therefore \overline{c} = (c_1 N)^{\delta_1} \cdots (c_m N)^{\delta_m}$

$= c_1^{\delta_1} \cdots c_m^{\delta_m} N \in \overline{\langle A, B \rangle}$. $\square$

例1   (cf. p79 (5))

$\varphi : Q_8 \longrightarrow \mathcal{V}_4$: homo. surj.

$\pm 1 \longmapsto 1$

$\pm i \longmapsto a$

$\pm j \longmapsto b$

$\pm k \longmapsto c$

quaternion q.   Klein-4-g.

$\Psi(A/N) := \varphi(A)$.

$\{ A \leq Q_8 \mid N \leq A \} \xrightarrow{\pi : bij} \{ A/N \leq Q_8/N \} \xrightarrow{\Psi (bij)} \{ B \leq \mathcal{V}_4 \}$

Q_8	$\longmapsto$ Q_8/N	$\longmapsto$ $\Psi(Q_8/N) = \varphi(Q_8) = \mathcal{V}_4$
$\langle i \rangle$	$\langle i \rangle / N$	$\Psi(\langle i \rangle/N) = \varphi(\langle i \rangle) = \varphi(\{i, -1, -i, 1\})$
$\langle j \rangle$		$= \{1, a\} = \langle a \rangle$
$\langle k \rangle$		
$\langle -1 \rangle$	$\langle -1 \rangle / N$	$\Psi(\langle -1 \rangle/N) = \varphi(\langle -1 \rangle) = 1$.

例2. (cf. p84 (3))

$\langle s \rangle / \langle r^2 \rangle = \langle s, r^2 \rangle$

$\langle r^2 \rangle \trianglelefteq \mathcal{D}_8 = \{ 1, r, r^2, r^3, s, rs, r^2s, r^3s \}$

$*$ $\langle r^2 \rangle$ でわると.
$\langle r^2 \rangle$ が1のポジション
になって.
Lattice はちぢむ

$*$ $|G| = 4 \implies G \cong \mathbb{Z}_4$ or $G \cong \mathcal{V}_4$

$\langle x \rangle$ where $|x| = 4$

これより, $\mathcal{D}_8 / \langle r^2 \rangle \cong \mathcal{D}_4$.