「Abs. Alg.」#10 $19^{10}/_{18}$ 金

Thm 3.18 (The 2nd or Diamond iso. thm.)

$$A, B \leq G$$
$$A \leq N_G(B)$$

$$AB \leq G$$
$$B \trianglelefteq AB$$
$$A \cap B \trianglelefteq A$$
$$AB/B \cong A/A\cap B$$

<proof>

$A, B \leq G$, $A \leq N_G(B)$ と Cor 3.13 より. $AB \leq G$.

$\forall a \in H\ \forall b \in H\ ab \in H$

$A \leq N_G(B)$. $B \leq N_G(B)$ ∵1 より. $AB \leq \underbrace{N_G(B)}_{\{g\in G \mid gBg^{-1}=B\}}$

$\therefore \forall ab \in AB\quad ab\,B(ab)^{-1} = B$

$\therefore\ B \trianglelefteq AB$

∵1 $B \leq N_G(B)$

$\forall b \in B,\quad b \in \{g \in G \mid gBg^{-1} = B\}$

i) $bBb^{-1} \subset B$

ii) $bBb^{-1} \supset B$

$\forall b' \in B.\ b' = b(b^{-1}b'b)b^{-1} \in bBb^{-1}$.

$B \trianglelefteq AB$ より, AB/B が定ぎされる.

$$\begin{array}{cccc} \varphi: & A & \longrightarrow & AB/B \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & a & \longmapsto & \varphi(a) := aB \end{array}$$

と定ぎすると.

i) φ: homo. ∵1

ii) φ: surj. ∵2

iii) $\ker\varphi = A \cap B$. ∵3

$$\begin{tikzcd} A \arrow[r, "\text{i) } \varphi:\text{homo.}"] \arrow[d, "\pi"'] & AB/B \\ A/\ker\varphi \arrow[r, "\cong"'] & \varphi(A) \arrow[u, phantom, "\cup \ \| \text{ ii)}"] \end{tikzcd}$$

$A/\ker\varphi \ \| \text{ iii)}\ A\cap B$

i)〜iii) と Thm 3.16 より,

$A\cap B \trianglelefteq A$, $A/A\cap B \cong AB/B$.

∵1 $\varphi(a_1a_2) = a_1a_2B = a_1Ba_2B = \varphi(a_1)\varphi(a_2)$.

∵2 $abB \in AB/B$ を任意にとると. $abB = aB$

$\left(\begin{array}{l} ∵\ \subset\ \forall b_0 \in B\quad abb_0 = a(bb_0) \in aB. \\ \quad\ \supset\ \forall b_0 \in B\quad ab_0 = ab(b^{-1}b_0) \in abB. \end{array} \right)$

より, $\exists a \in A$ s.t. $\varphi(a) = aB = abB$.

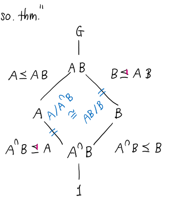
∵3 $\ker\varphi = \{a \in A \mid \varphi(a) = 1B\}$

$= \{a \in A \mid aB = 1B\}$

$= \{a \in A \mid a \in B\}$ ∵ Prop 3.4.

$= A \cap B$. ☒.

☆ "Diamond iso. thm."



Thm 3.19 (3rd iso. thm.)

$$H, K \trianglelefteq G$$
$$H \leq K$$

$$K/H \trianglelefteq G/H$$
$$(G/H)/(K/H) \cong G/K.$$

<proof>

$H \trianglelefteq K$

∵ $\forall k \in K\ kHk^{-1} \subset H$ ∵ $H \trianglelefteq G$

∵ $\forall khk^{-1} \in kHk^{-1}$. $khk^{-1} \in H$

$K/H \trianglelefteq G/H$

∵ $\forall gH \in G/H\quad gH(K/H)(gH)^{-1} \subset (K/H)$

$\rotatebox{90}{$\in$}$ $gH\ kH\ g^{-1}H$ $\|$ $gkg^{-1}H =: k'H \in (K/H)$

$gkg^{-1} \cap K$ ∵ $K \trianglelefteq G$

$(G/H)/(K/H) \cong G/K$

$$\begin{array}{ccccc} ∵ & \varphi: & G/H & \longrightarrow & G/K \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & & gH & \longmapsto & \varphi(gH) := gK \end{array}$$

とおくと.

0) φ: well defined ∵0

i), ii) φ: homo. surj.

iii) $\ker\varphi = K/H$

$$\begin{tikzcd} G/H \arrow[r, "\varphi:\text{homo.}"] \arrow[d, "\pi"'] & G/K \arrow[d, phantom, "\|"] \\ (G/H)/\underbrace{\ker\varphi}_{K/H} \arrow[r, "\cong"'] & \varphi(G/H) \end{tikzcd}$$

0)〜iii) と. 1st iso. thm. より.

∵0 $g_1H = g_2H$ とすると.

$g_1 = g_1 1 \in g_2H \subset g_2K\quad \therefore\ g_1 \in g_2K$

Prop 3.4 より. $g_1K = g_2K$. i.e. $\varphi(g_1H) = \varphi(g_2H)$. ☒.

Thm 3.20 (4th or Lattice iso. thm.)

$$N \trianglelefteq G$$

$\exists \Phi: \{A \leq G \mid N \leq A\} \longrightarrow \{\underset{\overset{\|}{\bar{A}}}{A/N} \leq G/N\}$

: bij.

とくに,

$\forall \bar{H} \leq \bar{G}\ \exists A \leq G$ s.t. $N \leq A\quad \bar{H} = \bar{A}$