Abs. Alg. #18 19/12/13 金

p135.

Def. $H \le G$

H : characteristic in G, H char G

$:\Leftrightarrow \forall \sigma \in \mathrm{Aut}(G)\ \ \sigma(H) = H$

☆ (1) H char $G \Rightarrow H \trianglelefteq G$

(2) $\exists^1 H \le G$ with $|H| = n \Rightarrow H$ char G

(3) K char H, $H \trianglelefteq G \Rightarrow K \trianglelefteq G$

Prop 4.16
$\mathrm{Aut}(\mathbb{Z}_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times \ (= \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} \mid (a,n) = 1\})$.

<proof> 略.

例. $|G| = pq$, $p<q$, prime. $|G| = 3 \cdot 5$

$p \nmid (q-1) \Rightarrow G$: abelian

☆ G : cyclic.

Prop 4.17.

(1) p : odd prime.

$\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_{p-1}$

$\mathrm{Aut}(\mathbb{Z}_{p^n}) = \mathbb{Z}_{p^{n-1}(p-1)}$ cf. Cor 9.20 (p.314).

(2) $n \ge 3$

$\mathrm{Aut}(\mathbb{Z}_{2^n}) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$ cf. Cor 9.20

(3) p : prime.

$(V, +)$: abelian with $\forall v \in V\ \ pv = 0$?

$|V| = p^n \Rightarrow V$: n-dim. vec. sp. over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

$\mathrm{Aut}(V) \cong GL(V) \cong GL_n(\mathbb{F}_p)$

cf. §10.2, §11.1.

(4) $n \ne 6$ $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n) \cong S_n$ (Exe 18)

$n = 6$ $|\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)| = 2$. (Exe 19, Exe 10 §6.3)

(5) $\mathrm{Aut}(D_8) \cong D_8$

$\mathrm{Aut}(Q_8) \cong S_4$ (Exe 4,5 Exe 9 §6.3).

☆ (3) の V を the elementary abelian g. of of p^n という.

例. $P \trianglelefteq G$

$3^2 = |P|$, $|G| = 3^2 \cdot 5$

G : abelian.

$\mathrm{Aut}(G) := \{ f : G \to G : \text{iso.} \}$

bij, homo.

§ 4.5 Sylow's Theorem.

Def. p : prime, $H \le G$

(1) G : p-group

$:\Leftrightarrow \exists \alpha \ge 0$ s.t $|G| = p^\alpha$

左 $:\Leftrightarrow$ 右

$G = V_4 = \{1, a, b, c\}$

$|G| = 4 = 2^2$. V_4 : 2-group.

H : p-subgroup

$:\Leftrightarrow H$: p-group.

(2) $|G| = p^\alpha m$, $p \nmid m$

H : Sylow p-subgroup of G

$:\Leftrightarrow H$: p-subgroup, $|H| = p^\alpha$,

(3) $\mathrm{Syl}_p(G) := \{ H \le G \mid H : \text{Sylow } p\text{-subg. of } G \}$

$n_p(G) := \# \mathrm{Syl}_p(G)$.

Thm 4.18 (Sylow's Theorem).

$|G| = p^\alpha m$, $p \nmid m$.

(1) $\mathrm{Syl}_p(G) \ne \emptyset$

(2) $P \in \mathrm{Syl}_p(G)$, $Q \le G$: p-subgroup.

$\Rightarrow \exists g \in G$ s.t. $Q \le gPg^{-1}$

とくに,

$Q \in \mathrm{Syl}_p(G) \Rightarrow P \overset{\text{conj.}}{\sim} Q$

i.e. $\exists g \in G$ s.t. $Q = gPg^{-1}$.

(3) $n_p \equiv 1 \pmod p$

さらに

$\forall P \in \mathrm{Syl}_p(G)$ $n_p = |G : N_G(P)|$

$n_p \mid m$.

Lem 4.19 $P \in \mathrm{Syl}_p(G)$

$Q \le G$: p-subgroup.

$Q \cap N_G(P) = Q \cap P$.

<proof>

$N_G(P) \cap Q =: H$

($\supset$)

$P \le N_G(P)$ より, $P \cap Q \le N_G(P) \cap Q \ (= H)$.

($\subset$)

$H = N_G(P) \cap Q \le Q$ だから, $H \le P$ を示せば十分.

$H = N_G(P) \cap Q \le N_G(P)$ より, Cor 3.15 を用いて,

$PH \le G$.

Prop 3.13 より,

$|PH| = |P||H| / |P \cap H|$

商に現れた数はいずれも p のべき乗だから,

PH : p-group.

$P \le PH$ より, $p^\alpha \mid |PH| \mid p^\alpha m$

よって, $|PH| = p^\alpha = |P|$.

$\therefore P = PH$, $H \le P$ ∎.