Abs. Alg. #13 19/11/08 金

$$\begin{tikzcd}
 & & \langle s, r^2\rangle =: \bar{H} & \\
 & \langle s\rangle \arrow[ur, "\pi"] & \langle r^2 s\rangle \arrow[u, "\pi"] & \langle r^2\rangle \arrow[ul, "\pi", "\times"']
\end{tikzcd}$$

$D_8 \geq H := \langle s\rangle \cup \langle r^2 s\rangle \cup \langle r^2\rangle \geq \langle r^2\rangle$, $\exists 1\ \pi^{-1}$

$\bar{H} = \{1, r^2, s, r^2 s\} =: H/\langle r^2\rangle$, $\exists 1$, $D_8 \geq H \geq \langle r^2\rangle$ ∵ Thm 3.20.

$\pi^{-1}(\bar{H})$

$\langle s\rangle/\langle r^2\rangle = \{1, s\}\{1, r^2\} = \{1, r^2, s, r^2 s\} = \bar{H}$

$D_8 \geq \{1, r^2, s, r^2 s\} \geq \langle r^2\rangle$, $\exists 1$, $\{1, r^2, s, r^2 s\} \not\trianglerighteq \langle r^2\rangle$

$\exists 1\ \{1, r^2, s, r^2 s\}/\langle r^2\rangle = \bar{H}$

★ $\varphi : G/N \longrightarrow H$: homo. を定ぎする方法

$$\begin{tikzcd}
G \arrow[r, "\pi"] \arrow[dr, "\Phi"'] & G/N \arrow[d, "\varphi"] \\
 & H
\end{tikzcd}$$

① $\Phi : G \to H$: homo. を定ぎ.

② $N \leq \ker\Phi$ をチェック.

Def.

$1 = N_0 \trianglelefteq \cdots \trianglelefteq N_k = G$

N_{i+1}/N_i : simple group

a composition series

composition factors of G.

*Jordan 標準形.

Thm 3.22. (Jordan-Hölder)

$G \neq 1$. $|G| < \infty$.

(1) G has a composition series.

(2) The comp. factors in a comp. series are unique:

$$\left.\begin{array}{l} 1 = N_0 \trianglelefteq \cdots \trianglelefteq N_r = G \\ 1 = M_0 \trianglelefteq \cdots \trianglelefteq M_s = G \end{array}\right\} \text{comp. ser.}$$

$\Rightarrow r = s$

$\exists \pi$: permutation of $\{1, \cdots, r\}$ s.t.

$M_{\pi(i)}/M_{\pi(i)-1} \cong N_i/N_{i-1}$.

$1 = N_0 \trianglelefteq \cdots \trianglelefteq N_{i-1} \trianglelefteq N_i \trianglelefteq \cdots \trianglelefteq N_r = G$

N_i/N_{i-1}

$\cong$

M_j/M_{j-1}

$1 = M_0 \trianglelefteq \cdots \trianglelefteq M_{j-1} \trianglelefteq M_j \trianglelefteq \cdots \trianglelefteq M_r = G$

Thm.

18 の simple group の族と,

26 の simple group がより,

任意の finite sim. g. はそのいずれかに iso.

Thm (Feit-Thompson).

G : simp. g. $|G|$: odd. $\Rightarrow \exists p$: prime s.t. $G \cong \mathbb{Z}_p$

☆ Hölder Prog. (2) のイミ.

given $A, B \rightsquigarrow B \cong N$. $G/N \cong A$ 見つける.

例.

$A = B = \mathbb{Z}_2$ のとき. $\mathbb{Z}_2 \cong \mathbb{Z}_2$. $G/\mathbb{Z}_2 \cong \mathbb{Z}_2$

なる G として. $\mathbb{Z}_4$, V_4 が知られてる.

が. $\mathbb{Z}_2, \mathbb{Z}_2$ から $\mathbb{Z}_4, V_4$ を構成するような一般的方法は何か?

Def.

G : solvable

$:\Longleftrightarrow \exists$ a chain of subg. : $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_s = G$.

s.t. G_{i+1}/G_i : abelian.

Thm. $|G| < \infty$.

G : solvable

$\Longleftrightarrow \forall n \mid |G|$ with $(n, \frac{|G|}{n}) = 1$.

$\exists H \leq G$ s.t. $|H| = n$.

* Sylow's thm. の一般化.

☆

$N, G/N$: solvable $\Rightarrow G$: solvable.

<proof>

$G/N =: \bar{G}$

$1 = N_0 \trianglelefteq \cdots \trianglelefteq N_n = N$: N_{i+1}/N_i abelian.

$\bar{1} = \bar{G}_0 \trianglelefteq \cdots \trianglelefteq \bar{G}_m = \bar{G}$: $\bar{G}_{i+1}/\bar{G}_i$ abelian.

Lattice iso. thm. より,

$\exists N \leq G_i \leq G$ s.t. $\bar{G}_i = G_i/N$. , $G_i \trianglelefteq G_{i+1}$

∵ thm. (5) と. $\bar{G}_i \trianglelefteq \bar{G}_{i+1}$

3rd iso. thm. より,

$\bar{G}_{i+1}/\bar{G}_i = (G_{i+1}/N)/(G_i/N)$

$\cong G_{i+1}/G_i$

∵ $N, G_i \trianglelefteq G_{i+1}$

$N \leq G_i$

∴ $1 = N_0 \trianglelefteq \cdots \trianglelefteq N_n = N = G_0 \trianglelefteq \cdots \trianglelefteq G_m = G$

$N \leq G_0$ s.t. □/△ : abelian.

($\psi : G \cong H$, G : abelian

$a, b \mapsto \psi(a), \psi(b)$.

$\psi(a)\psi(b) = \psi(ab) = \psi(ba) = \psi(b)\psi(a)$.

$\Rightarrow H$: abelian)

∴ G : solvable. ∎