Abs. Alg. #15 19/11/22 (金)

Prop 4.2. G acts on A.

$a \sim b$
$:\Leftrightarrow \exists g \in G$ s.t. $a = g \cdot b$

(1). $(\sim)$: an equivalence rel.

(2). $\#\{b \in A \mid a \sim b\} = |G : G_a|$ $\quad |\{gG_a\}|$

<proof> (2).

$$\begin{aligned}
&\{b \in A \mid a \sim b\} \\
&= \{b \in A \mid \exists g \in G \text{ s.t. } a = g \cdot b\} \\
&= \{b \in A \mid \exists \tilde{g} \in G \text{ s.t. } b = \tilde{g} \cdot a\} \\
&= \{\tilde{g} \cdot a \in A \mid \tilde{g} \in G\} \\
&= \{g \cdot a \mid g \in G\} =: \mathcal{C}_a
\end{aligned}$$

そこで $\{g \in G \mid g \cdot a = a\}$

$$\begin{array}{ccc}
\psi : \mathcal{C}_a & \longrightarrow & \{\text{left coset of } G_a \text{ in } G\} =: (G : G_a). \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
b = g \cdot a & \longmapsto & \psi(g \cdot a) := gG_a
\end{array}$$

とおき、ψ : bij を示す。

★ Prop 4.2 より、

G acts on A $\rightsquigarrow$ partition

Def. G acts on $\emptyset \neq A$.

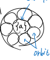
(1) $\{g \cdot a \mid g \in G\}$: the orbit of G containing a.

(2) (action) : transitive
$:\Leftrightarrow$ there's only one orbit.
i.e. $\forall a, b \in G$ $a \sim b$.

例. G acts on A.

(1) G acts trivially : $\forall a \in A$ $g \cdot a = a$

(action) : transitive $\Leftrightarrow |A| = 1$.



(2) $G = S_n$, $A = \{1, \cdots, n\}$

$\{\sigma \cdot i \mid \sigma \in S_n\} = A$ $\quad \therefore$ transitive.
$g \cdot a$

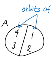
Note : $|G : G_i| = |\mathcal{C}_i| = |A| = n$.

(3) $H \leq G$ acts on A.

G : trans. $\not\Rightarrow$ H : trans.

例. $H = \langle (1\,2), (3\,4) \rangle \leq S_4 =: G$

G trans.ly acts on $A := \{1, 2, 3, 4\}$

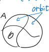


$(\cdots)(\cdot\cdot) \Leftarrow$
$(\cdot\cdot)(\cdot\cdot)$

★ $\forall \sigma \in S_n$ has the unique cycle decomp. :

$A = \{1, \cdots, n\}$
$G = \langle \sigma \rangle$.

S_n acts on A, $G \leq S_n$. G acts on A.



$x \in \mathcal{O}$ をとる。Prop 4.2 より、

$\#\underbrace{\{y \in A \mid x \sim y\}}_{\mathcal{O}} = |G : G_x|$

$$\begin{array}{ccc}
\exists \psi : \mathcal{O} & \longrightarrow & (G : G_x) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
y = \sigma^i x & \longmapsto & \psi(\sigma^i x) := \sigma^i G_x
\end{array}$$

G : cyclic より、$G_x \trianglelefteq G$, G / G_x は order d の cyclic g.

ただし d は (cf. p.84 例(2))
$\sigma^d \in G_x$ なる最小の自然数。

$d = |G : G_x| = |\mathcal{O}|$.

より、

$(G : G_x) = \{1G_x, \sigma G_x, \cdots, \sigma^{d-1} G_x\}$

ψ : bij. より、

$\mathcal{O} = \{x, \sigma x, \cdots, \sigma^{d-1} x\}$

σ が x を回している。

つまり、σ は d-cycle として $\mathcal{O}$ に act.

$\underbrace{(\cdots)}_{d}$

他の $\mathcal{O}'$ についても同様 (σ は d'-cycle として $\mathcal{O}'$ に act.)

存在性 O.K.

唯一性. uniq.

$\sigma \rightsquigarrow \langle \sigma \rangle = G$ の orbits $\qquad (A)(B)$
$\qquad (a\,b\,c)(b\,c\,a)$.

$G = \langle \sigma \rangle$ の orbits $=:$ $\sigma \in S_n$ の orbits とみなすと、

σ の orbits $=$ σ を cycle decomp. したときの cycles.

§4.2 G. Acting on Themselves by L. Multip.

$$\begin{array}{ccc}
(\cdot) : G \times G & \longrightarrow & G \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
(g, a) & \longmapsto & g \cdot a := ga.
\end{array}$$

$|G| = n$.
$G = \{g_1, \cdots, g_n\}$

* 有限
$\rightsquigarrow$ 添字
$\rightsquigarrow$ $A \sim G$ とみなす。
$\quad \| $
$\{1, \cdots, n\}$

通常の perm. repr. $\tilde{\varphi}$ を考えると、

$$\begin{array}{ccc}
\tilde{\varphi} : G & \longrightarrow & S_G \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
g & \longmapsto & \tilde{\varphi}(g) := \tilde{\sigma}_g : G \longrightarrow G
\end{array}$$

$\qquad g_i \longmapsto \tilde{\sigma}_g(g_i) := g \cdot g_i = gg_i$

そこで $\{1, \cdots, n\} =: A$ として、

$$\begin{array}{ccc}
\varphi : G & \longrightarrow & S_A \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
g & \longmapsto & \varphi(g) := \sigma_g : A \longrightarrow A
\end{array}$$

$\qquad i \longmapsto \sigma_g(i) := j$

where
$\tilde{\sigma}_g(g_i) = g_j$

を $(\cdot)$ の perm. repr. と思う。

例. $G = V_4 = \{1, a, b, c\} =: \{g_1, g_2, g_3, g_4\}$

$\sigma_a : \{1, \cdots, 4\} \longrightarrow \{1, \cdots, 4\}$

$$\begin{array}{ccc}
1 & & 2 \\
2 & & 1 \\
3 & \longmapsto & \sigma_a(3) = \boxed{4} \\
4 & & 3
\end{array}$$

$\tilde{\sigma}_a(g_3) = a \cdot g_3 = ab = c = g_4$

$\therefore \sigma_a = (1\ 2)(3\ 4)$.

★ この §の act. は trans. (1), faithful (3) で
$\forall a \in G$ $G_a = 1$. (2)
(それぞれ Thm 4.3 の (1)(3)(2) の特別なバタく)

★ この §の act. の一般化.

$H \leq G$
$A = (G : H)$

$$\begin{array}{ccc}
(\cdot) : G \times A & \longrightarrow & A \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
(g, aH) & \longmapsto & g \cdot aH := gaH.
\end{array}$$

とくに、$H = 1$ のとき、

$A = \{\{a\} \mid a \in G\} \sim \{a \mid a \in G\} = G$

と同一視すると、$(\cdot)$ は §のはじめに見たもの。

一般化した $(\cdot)$ の perm. repr. について。

$|G : H| = m$
$A = (G : H) = \{a_1 H, \cdots, a_m H\}$

$\sigma_g(i) = j$
where
$\tilde{\sigma}_g(a_i H) = a_j H$.

例.

$H = \langle s \rangle \leq D_8 = G$.

$(G : H) = \{\underset{a_1}{1}H, rH, r^2H, \underset{a_4}{r^3}H\}$

$\sigma_s : \{1, \cdots, 4\} \longrightarrow \{1, \cdots, 4\}$

$$\begin{array}{ccc}
1 & & 1 \\
2 & \longmapsto & \sigma_s(2) = \boxed{4} \\
3 & & 3 \\
4 & & 2
\end{array}$$

$$\begin{aligned}
\tilde{\sigma}_s(a_2) &= s \cdot a_2 H \\
&= s a_2 H \\
&= s r H \\
&= sr\{1, s\} = r^3\{s, 1\} \\
&= r^3 H = a_4 H.
\end{aligned}$$

$\therefore \sigma_s = (2\ 4)$

Thm 4.3. $H \leq G$
$A = (G : H)$

$$\begin{array}{ccc}
(\cdot) : G \times A & \longrightarrow & A \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
(g, aH) & \longmapsto & g \cdot aH := gaH.
\end{array}$$

さっきの φ.

π_H : perm. repr. afforded by $(\cdot)$.

(1) G acts transitively on A.

(2) $G_{1H} = H$.

(3) $\ker \pi_H = \bigcap_{x \in G} xHx^{-1}$

: the largest one of $\{J \trianglelefteq G \mid J \leq H\}$

Cor 4.4 (Cayley's thm.).

Every group is iso. to a subg. of some sym. group.

とくに、

$|G| = n \Rightarrow \exists H \leq S_n$ s.t. $G \cong H$.

<proof>.

Thm 4.3 で $H = 1$ とする。

この§での $(\cdot) : G \times G \to G$ を考える。

$\ker \pi_1 = \bigcap x 1 x^{-1} = 1$.

1st iso. thm. より、

$$\begin{tikzcd}
G \arrow[r, "\pi_1 : \text{homo.}"] \arrow[d, "\pi"'] & S_G \\
G/\ker\pi_1 \arrow[r, "\cong"] & \pi_1(G) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$G \cong G/\ker\pi_1 \cong \pi_1(G) \leq S_G$

$G \cong \bigcirc \leq S_G$

$\square$.