『Abs. Alg.』輪講#6 19.09.20 金

ch3. Quotient Groups and Homomorphisms.

★ Gの quotient の subgroup の lattice は，Gの lattice の top と関係.

Gの subgroup の lattice は，Gの 〃 bottom 〃

★ quotient groups を調べることは 本質的に homo. を調べることと同値，

例.
$$\begin{array}{ccc} \varphi : \mathbb{Z} & \longrightarrow & Z_n = \langle x \rangle \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a & \longmapsto & \varphi(a) := x^a \end{array}$$

☑ φ : homo.

∵ $\varphi(a+b) = x^{a+b} = x^a x^b = \varphi(a)\varphi(b)$.

☑ φ : surj.

☑ the fiber of φ over x^a is...

$$\begin{aligned} \varphi^{-1}(x^a) &= \{ m \in \mathbb{Z} \mid \varphi(m) = x^a \} \\ &= \{ m \in \mathbb{Z} \mid m \equiv a \pmod{n} \} \\ &= \bar{a} \end{aligned}$$

$$\begin{array}{c|ccccccc} & 0 & 1 & \cdots & a & \cdots & n-1 \\ \mathbb{Z} & \pm n & 1 \pm n & \cdots & a \pm n & \cdots & (n-1) \pm n \\ & \pm 2n & 1 \pm 2n & \cdots & a \pm 2n & \cdots & (n-1) \pm 2n \\ & \vdots & \vdots & & \vdots & & \vdots \\ & \downarrow \varphi & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\ \hline Z_n & x^0 & x^1 & & x^a & & x^{n-1} \end{array}$$

$$\mathbb{Z}/n\mathbb{Z} \cong Z_n$$

Def.

$\varphi : G \longrightarrow H$: homo.

$\ker \varphi := \{ g \in G \mid \varphi(g) = 1 \}$

Prop 3.1

G, H : groups.

$\varphi : G \longrightarrow H$: homo.

(1) $\varphi(1_G) = 1_H$

(2) $\varphi(g^{-1}) = \varphi(g)^{-1}$

(3) $\varphi(g^n) = \varphi(g)^n \quad (\forall n \in \mathbb{Z})$.

(4) $\ker \varphi \leq G$

(5) $\mathrm{im}(\varphi) \leq H$

<proof>

(3) $n \in \mathbb{Z}^-$ での成立を示す.

$n = 0, -1$ のとき，(1), (2) より，成立.

$n = -k$ のとき成立を仮定すると，$n = -(k+1)$ でも成立.

∵ $\varphi(g^{-(k+1)}) = \varphi(g^{-k} g^{-1}) = \varphi(g^{-k})\varphi(g^{-1})$ ∵ homo.

$= \underbrace{\varphi(g)^{-k}}_{∵\text{仮定}} \underbrace{\varphi(g)^{-1}}_{∵(2)}$

$= \varphi(g)^{-(k+1)}$.

帰納法より，$n \in \mathbb{Z}^-$ で成立. ☑

Def.

$\varphi : G \to H$: homo.
$K := \ker \varphi$

$G/K := \{ \varphi^{-1}(a) \mid a \in H \}$

ただし，演算は次のとおり.

$X, Y \in G/K$, $X = \varphi^{-1}(a)$, $Y = \varphi^{-1}(b)$

$XY := \varphi^{-1}(ab)$

★ 本当に group をなすか？

$\varphi^{-1}(1) = \ker \varphi \in G/K$.

$X, Y \in G/K$, ($X = \varphi^{-1}(a)$, $Y = \varphi^{-1}(b)$, $a, b \in H$) とする.

☑ K は identity.
∵ $XK = \varphi^{-1}(a1) = \varphi^{-1}(a) = X$.

☑ $Y^{-1} = \varphi^{-1}(b^{-1})$
∵ $YY^{-1} = \varphi^{-1}(bb^{-1}) = K$.

$XY^{-1} = \varphi^{-1}(ab^{-1})$, $ab^{-1} \in H$

∴ $XY^{-1} \in G/K$.

★ この演算は well-defined？

$X, Y \in G/K$, $X = \varphi^{-1}(a)$, $Y = \varphi^{-1}(b)$,

$XY = \varphi^{-1}(ab)$.

とする.

$X = \varphi^{-1}(a')$, $Y = \varphi^{-1}(b')$

$\overset{?}{\Rightarrow}$ $\varphi^{-1}(a'b') = \varphi^{-1}(ab)$

<proof>

$\varphi^{-1}(a) = X = \varphi^{-1}(a')$

と

$a \neq a' \Rightarrow \varphi^{-1}(a) \neq \varphi^{-1}(a')$

の対偶を用いて

$a = a'$.

同様に

$b = b'$.

∴ $\varphi^{-1}(a'b') = \varphi^{-1}(ab)$