國立交通大學

應用數學系

碩士論文

Markov-Cayley Tree 上拓樸熵存在性之探討

On the Existence of the Topological Entropy on Markov-Cayley Tree

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On the existence of the topological entropy on Markov-Cayley tree

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摘要

有鑑於近年來群與半群上的符號動態系統拓樸受到廣泛關注，而拓樸熵在這類系統上又是研究者們關注的問題，本文探討在稱為馬可夫-凱萊樹的半群之上的符號動態系統與其拓樸熵極限的存在性。透過莖熵極限的存在性，本文不僅給出了拓樸熵極限的存在性的充分條件，也同時展示在這些條件下這兩種熵值相同。同時，透過符號動態系統的圖表示法，本文得以對拓樸熵的收斂性質有更細膩的刻畫。最後，這些本文末也提供了熵計算的演算法與數值結果以輔證論述。

關鍵字：符號動態系統, 半群, 莖熵, 拓樸熵

On the Existence of the Topological Entropy on Markov-Cayley Tree

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ABSTRACT

The study of shift spaces over groups and semigroups has received extensive attention. Among all, the problem of existence in limit of the topological entropy is of interest among researcher. Inspired by the current literature, this thesis investigates the topological entropy of shift spaces over a collection of finitely generated groups and semigroups called Markov-Cayley trees. By virtue of the existence in limit of the stem entropy for Markov-Cayley tree-shifts, this thesis not only unveils several sufficient conditions for the existence in limit of the topological entropy, but demonstrates the coincidence of topological entropy and stem entropy in these cases. Furthermore, with the aid of graph representation on these spaces, a more refined discussion on the convergence of topological entropy is also carried out. These results are further supplemented by algorithms for computation of entropy for the nearest-neighbor Markov-Cayley tree-shifts.

Keywords: shift spaces, semigroups, stem entropy, topological entropy

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吳昱良　謹誌于

應用數學系應用數學碩士班

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Chapter 1

Introduction

In the fields of information theory and computer science, all forms of information are coded into sequences of digits for processing and transmission. With the rapid development of these fields in recent years, the rise of abundant applications naturally stimulates investigations into the study, with various tools invented in the meanwhile. One specific formulation among all is the adoption of the shift space, which, roughly speaking, comprises of a set consisting of labelings over a prescribed structure using a given alphabet, and a shift transformation defined on the set of the labelings. For a better conception of such space, the idea of entropy on shift spaces is proposed as an essential indicator.

Entropy, in general, serves as an indicator of the complexity of the system, and with the aid of entropy, it is easier to identify the potential of shift spaces exhibiting similar chaotic behavior. Claude Shannon first introduces the idea [12, 13] in the modeling of communication, which is later recognized as the founding work of information theory. Lind and Marcus [7] present an extensive study on the entropy of shift spaces over $\mathbb{Z}$, which includes a fundamental classification of shift spaces based on their intrinsic topological properties, as well as the interrelationship between entropy and topological properties.

In fact, the entropy is proved to exist in limit for such spaces and is related to the spectral radius of the adjacency matrix in the case of shifts of finite type and sofic shifts. Aside from well-studied results over $\mathbb{Z}$, researchers are also interested in the entropy of other generalizations and variations of shift spaces. Chow and Mallet-Paret [6, 8] study the lattice dynamical systems over $\mathbb{Z}^d$, in which they first consider a continuous-time dynamical system on a discrete lattice $\mathbb{Z}^2$ depicted by differential equations and then reformulate the dynamics at each lattice point to a discretized version, which is a space consisting of the solutions with no appearance of some patterns. Therein, the authors provide an equivalence condition for the discretized mosaic equilibrium solutions, supplemented by discussions of their stability and the entropy of the space of solutions. The entropies of other generalizations of shift spaces over higher-dimensional $\mathbb{Z}^d$ spaces are also studied in other works. For instance, the book written by Ceccherini-Silberstein and Coornaert [5] considers cellular automata over amenable groups (and hence all shift spaces over $\mathbb{Z}^d$,) and addresses, by exploiting Følner sequence, the existence of the entropy therein.

Besides the aforementioned fruitful achievements, works considering an alternative generalization over semigroups with tree-like Cayley greaphs are also sprouting recently. Shift spaces over free semigroups are first introduced and referred to *tree-shifts* by Aubrun and Béal [1], in which they dig into the topological properties using the language of automata. The entropy of a hom tree-shift, a tree-shift induced by a one-dimensional shift space, is considered in [9] and [10]. Therein, the authors prove the existence of limit of the topological entropy of tree-shifts, accompanied by abundant results both in numerical experiments as well as in theory. In particular, the entropy of a hom tree-shift is shown to be at least as large as those over $\mathbb{Z}$, provided they share a common forbidden set on any path in their Cayley graphs. The authors also relate the topological entropy with the topological properties in the case of the nearest-neighbor tree-shifts, which inspires the partially generalized theorems in the Chapter 3 this thesis. Aside from the tree-shifts, the entropies of shift spaces over regular trees, or free groups, are also studied by Piantadosi [11], in which the golden mean shift is investigated, yet the entropy in the article is defined using limit superior without a discussion of the existence of the limit. In addition to the topological entropy over the semigroups and groups mentioned, a slightly different definition of entropy over a free semigroup is also considered in [2] under a more general setting, and this definition is later used in the exposition of the Fibonacci-Cayley tree-shifts, the shift spaces over a special semigroup named Fibonacci-Cayley tree, carried out by Ban and Chang in [3]. In the article, the authors develop a scheme for entropy computation utilizing the nonlinear recursive system associated with the Fibonacci-Cayley tree-shifts. Despite a lack of existence of entropy in limit in general, it is inferred from the Theorem 3.4 therein that the entropy exists in limit if every symbol is essential and the system is simple. The works naturally raise a question toward the existence of the entropy in limit, yet it is not until recently that a partial answer is given.

In view of the elucidation of the existence of entropy in limit over tree-like structures above, a recent work makes a further step toward such study over a more extensive collection. Based on the formulation in [4], Wu [14] studies the stem entropy over a Markov-Cayley tree, which is essentially a semigroup with relators consisting merely of two generators. In the work, the author exposes the intrinsic tiling structure of mixing Markov-Cayley trees, which is later utilized to prove the existence in limit of the stem entropy, the entropy defined on the substructure rooted at each generator of the tree. Such a result makes a substantial leap in the study, yet currently to our knowledge, the existence in limit of the topological entropy is left unknown.

As an application of the theorems derived in [14], Chapter 2 aims to deal with the existence of topological entropy over Markov-Cayley trees and proves the existence of topological entropy over two particular types of Markov-Cayley trees. For the first type, we show that the limit in entropy exists if a *nearest-neighbor Markov-Cayley tree-shift*, a shift space over a Markov-Cayley tree that is characterized by a single adjacency matrix, has an associated matrix of the relators K containing a full row (**Theorem 2.2.2**). With the theorem, the existence of topological entropy of Fibonacci-Cayley tree-shift and the system with safe symbols follows immediately as an conequence. As for the second type, we prove the existence of entropy of a nearest-neighbor Markov-Cayley tree-shift, which is closely related to nearest-neighbor shifts of finite type over free groups (**Theorem 2.2.5, Theorem 2.2.6, Theorem 2.2.7**).

The third chapter is devoted to the exposition of a partially generalized theorem presented in [10] over the mixing Markov-Cayley. In their work, it is pointed out the entropy of a nearest-neighbor hom tree-shift can be computed by counting the a subset of patterns rather than the set of all admissible patterns provided the adjacency matrix is either irreducible or primitive, while there is no knowing it is also valid for the shift spaces even over the tree-like structures in general. In Chapter 3, we are able to extend the result to any nearest-neighbor Markov-Cayley tree-shift, by introducing the an analogous condition in the strong connectedness of symbols of the space in terms of the graph representation of the shift space. Furthermore, this generalization proves its effectiveness in showing not only the existence of entropy in limit but its equality with the stem entropy (**Theorem 3.2.4**). The techniques used in Theorem 3.2.4 also give rise to an attempt to the existence in limit of the topological entropy in described in Section 3.3. The thesis is then concluded in the last section by a short discussion of the previous sections as well as some problem remained unanswered. As for numerical experiments, the methodology and the data are presented in Appendix A.

Chapter 2

Markov-Cayley Tree-Shifts

This chapter is devoted to the setup for notations and fundamental properties for the upcoming discussions on shift spaces over Markov-Cayley trees.

2.1 Preliminaries

This section covers the general setting of the thesis. Let $\mathcal{T}$ be a finitely generated semigroup with identity element ϵ. We say $\mathcal{T}$ is *Markov-Cayley tree* if there exists $k \times k$ binary matrix K indexed by a finite generating set $\Sigma = \{s_1, s_2, \ldots, s_k\}$ of G such that $\mathcal{T} = \langle \Sigma | R \rangle$, where $R = \{s_i s_j \in \Sigma \times \Sigma : K(s_i, s_j) = 0\}$. That is, every $s_i s_j = \epsilon$ if and only if $K(s_i, s_j) = 0$. With respect to the generating set Σ, every $g \in \mathcal{T}$ is associated with a unique minimal representation $g = g_1 g_2 \cdots g_n$ with $g_i \in \Sigma$ in the sense $K(g_i, g_{i+1}) = 1$ for every $1 \leq i \leq n$. According to this minimal representation, the *length of the* g, written as $|g|$, is defined to be n. This representation is assumed throughout the thesis unless mentioned otherwise. Denote by $\Delta_n^{(g)}$ the n-semiball rooted at $g = g_1 g_2 \cdots g_m \in \mathcal{T}$, i.e., $\Delta_n^{(g)} = \{g\} \cup \{g g_{m+1} \ldots g_{m+l} \in \mathcal{T} : g_j \in \Sigma, l \leq n, K(g_j, g_{j+1}) = 1, \forall 1 \leq j \leq m + l\}$ for all $n \geq 0$. Furthermore, $\Delta_n'^{(s_i)} := \Delta_{n-1}^{(s_i)} \cup \{\epsilon\}$ for all $s_i \in \Sigma$ and $\Delta_n := \cup_{s_i \in \Sigma} \Delta_{n-1}'^{(s_i)}$.

Let $\mathcal{A} = \{1, 2, \cdots, d\}$ be a finite alphabet. A *labeled Markov-Cayley tree* is a function $t : \mathcal{T} \to \mathcal{A}$ for which $t_g := t(g)$ is the label attached to $g \in \mathcal{T}$, and the set $\mathcal{A}^{\mathcal{T}}$ consisting of all labeled Markov-Cayley trees is called the *full d-Markov-Cayley tree-shift*. On $\mathcal{A}^{\mathcal{T}}$, the *shift map* is a semigroup action $\sigma : \mathcal{T} \times \mathcal{A}^{\mathcal{T}} \to \mathcal{A}^{\mathcal{T}}$ such that $(\sigma(g, t))_h = t_{gh}$, for which we also write $\sigma_g t = \sigma(g, t)$. A *pattern* is a function $u : \mathcal{A}^S \to \mathcal{A}$ with $S \subset \mathcal{T}$ a finite subset, for which we call S the *support of* u and denote $s(u) = S$. A pattern u is said to be *accepted* by $t \in X_{\mathbf{A}}$ if there exists $g \in \mathcal{T}$ such that the restriction $(\sigma_g t)|_{s(u)} = u$, and it is *rejected* by t if it is not accepted by t. Let $\mathcal{F}$ be a set of patterns. A space $X = X_{\mathcal{F}} \subseteq \mathcal{A}^{\mathcal{T}}$ is called a *Markov-Cayley tree-shift* if every t by which all patterns in $\mathcal{F}$ are rejected is included in X, and a pattern is said to be *admissible* (by X) if it is accepted by some $t \in X$. A Markov-Cayley tree-shift X is called a *nearest-neighbor* Markov-Cayley tree-shift (NNMCTS) if there exists a k-tuple of $d \times d$ binary matrices $\mathbf{A} = (A_1, A_2, \cdots, A_k)$ such that $X = X_{\mathbf{A}}$, where

$$X_{\mathbf{A}} = \{t \in \mathcal{A}^{\mathcal{T}} : A_i(t_g, t_{gs_i}) = 1, s_i \in \Sigma, |gs_i| = |g| + 1\}.$$

For the case $\mathcal{T}$ is a semigroup, we simply refer to X and $X_{\mathbf{A}}$ as a tree-shift and a nearest-neighbor tree-shift, respectively. Furthermore, due to the frequent appearance later in the thesis, the notations below are also defined. Given $g \in \mathcal{T}$,

1. $p_n^{(g)} = p_n^{(g)}(X_{\mathbf{A}}) := \{u \in \mathcal{A}^{\Delta_n^{(g)}} : u \text{ is admissible by } \in X_{\mathbf{A}}\}$
2. $p_n = p_n(X_{\mathbf{A}}) := p_n^{(\epsilon)}(X_{\mathbf{A}})$
3. $p_{n;a}^{(g)} = p_{n;a}^{(g)}(X_{\mathbf{A}}) := \{u \in \mathcal{A}^{\Delta_n^{(g)}} : u \text{ is admissible by } X_{\mathbf{A}}, u_g = a\}$
4. $p_{n;a} = p_{n;a}(X_{\mathbf{A}}) := p_{n;a}^{(\epsilon)}(X_{\mathbf{A}})$
5. $p_{n;a}^{\prime(g)} = p_{n;a}^{\prime(g)}(X_{\mathbf{A}}) := \{u \in \mathcal{A}^{\Delta^{\prime(g)}} : u \text{ is admissible by } X_{\mathbf{A}}, u_\epsilon = a\}$

With the notations above, we define the entropy of a NNMCTS as follows.

Definition 2.1.1. Suppose $\mathcal{A}$ is a finite alphabet, and $X_{\mathbf{A}} \subseteq \mathcal{A}^{\mathcal{T}}$ is an NNMCTS.

1. The *topological entropy of* $X_{\mathbf{A}}$ is defined to be
$$h = h(X_{\mathbf{A}}) := \limsup_{n\to\infty} \frac{\log p_n(X_{\mathbf{A}})}{|\Delta_n|}. \tag{2.1}$$
2. The *stem entropy rooted at* g is defined as
$$h^{(g)} = h^{(g)}(X_{\mathbf{A}}) := \limsup_{n\to\infty} \frac{\log p_n^{(g)}(X_{\mathbf{A}})}{|\Delta_n^{(g)}|}. \tag{2.2}$$

In [14], it is proved that $h^{(s_i)}$ are equal for all $s_i \in \Sigma$ if K is a primitive matrix, i.e. there exists $N \in \mathbb{N}$ such that $K^N(s_i, s_j) > 0$ for all $s_i, s_j \in \Sigma$. Later in the thesis, it is surprising to see that the stem entropy are actually coincident with the topological entropy in many cases.

2.2 Existence of Topological Entropy in Limit

This section is devoted to the development of existence of entropy in limit. In Wu's thesis [14], the stem entropy of Markov-Cayley tree-shifts is proved to exist in the following theorem:

Theorem 2.2.1. *Suppose that $\mathcal{T}$ is a Markov-Cayley tree with $K \in \{0,1\}^{k\times k}$ a primitive matrix indexed by a generating set Σ, that $\mathcal{A}$ is a finite alphabet, and that $X_{\mathbf{A}} \subseteq \mathcal{A}^{\mathcal{T}}$ is an NNMCTS. Then, the limit $\lim_{n\to\infty} \frac{\log p_n^{(s)}(X_{\mathbf{A}})}{|\Delta_n^{(s)}|}$ exists for all $s \in \Sigma$. In addition, $h^{(s_i)}(X_{\mathbf{A}}) = h^{(s_j)}(X_{\mathbf{A}})$ for every $s_i, s_j \in \Sigma$.*

On top of the theorem, this section further addresses the existence of limit in limit of the topological entropy over two different types of Markov-Cayley trees, which are, in essence, generalizations of Markov-Cayley tree-shifts over a free group and over the Fibonacci-Cayley tree [3], respectively.

2.2.1 Topological Entropy over Markov-Cayley Trees with a Full Row

This subsection is devoted to the existence of limit in the topological entropy of the NNMCTS $X_{\mathbf{A}}$ defined on a Markov-Cayley tree $\mathcal{T}$ with $\sum_{j=1}^{k} K(s_i, s_j) = k$ for some $s_i \in \Sigma$, which is shown in the following theorem.

Theorem 2.2.2. *Suppose that $\mathcal{T}$ is a Markov-Cayley tree with $K \in \{0,1\}^{k\times k}$ a primitive matrix such that $\sum_{j=1}^{k} K(s_i, s_j) = k$ for some $s_i \in \Sigma$, that $\mathcal{A}$ is a finite alphabet, and that $X_{\mathbf{A}} \subseteq \mathcal{A}^{\mathcal{T}}$ is an NNMCTS. Then, the limit $\lim_{n\to\infty} \frac{\log p_n(X_{\mathbf{A}})}{|\Delta_n|}$ exists and equals $h^{(s_1)}(X_{\mathbf{A}})$.*

Proof. Note that every n-block u can be uniquely expressed as a $(k+1)$-tuple

$$(u_\epsilon, u|_{\Delta_{n-1}^{(s_1)}}, u|_{\Delta_{n-1}^{(s_2)}}, \cdots, u|_{\Delta_{n-1}^{(s_k)}})$$

and thus $p_n \leq |\mathcal{A}| \cdot \prod_{j=1}^{k} p_{n-1}^{(s_j)}$. As a consequence,

$$\limsup_{n\to\infty} \frac{\log p_n}{|\Delta_n|} \leq \limsup_{n\to\infty} \frac{|\mathcal{A}|}{|\Delta_n|} + \sum_{j=1}^{k} \frac{\log p_{n-1}^{(s_j)}}{|\Delta_{n-1}^{(s_j)}|} \frac{|\Delta_{n-1}^{(s_j)}|}{|\Delta_n|} = h^{(s_i)} = h^{(s_1)}$$

holds by applying Theorem 2.2.1. On the other hand, $p_n^{(s_i)} \leq p_n$ holds naturally, which further implies

$$\liminf_{n\to\infty} \frac{\log p_n}{|\Delta_n|} \geq \liminf_{n\to\infty} \frac{\log p_n^{(s_i)}}{|\Delta_n|} = \liminf_{n\to\infty} \frac{\log p_n^{(s_i)}}{|\Delta_n^{(s_i)}|} = h^{(s_i)} = h^{(s_1)}.$$

The proof is finished by combining the inequalities above. □

Theorem 2.2.2 directly leads to the following corollary.

Corollary 2.2.3. *Suppose $X_{\mathbf{A}}$ is an NNMCTS over a Markov-Cayley tree $K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then, the limit $\lim_{n\to\infty} \frac{\log p_n(X_{\mathbf{A}})}{|\Delta_n|}$ exists and equals $h^{(s_1)}(X_{\mathbf{A}})$.*

Proof. It is a straightforward application of Theorem 2.2.2. □

For such an NNMCTS in Corollary 2.2.3, $\mathcal{T}$ is also referred to as a Fibonacci-Cayley tree, and $X_{\mathbf{A}}$ is referred to as a Fibonacci-Cayley tree-shift [3].

Example 2.2.4. Suppose X_{A_1,A_2} is an NNMCTS over a Fibonacci-Cayley tree. Algorithm 2 is implemented to numerically evaluates the topological entropy and the stem entropy when A_1, A_2 are either

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and the results is presented in Table 2.

2.2.2 Topological Entropy over Free Groups

It follows from Section 2.1 that a free group $\mathcal{T}$ of rank k is inherently a Markov-Cayley tree with a generating set

$$\Sigma = \{s_1, s_2, \cdots, s_k, s_1^{-1}, s_2^{-1}, \cdots, s_k^{-1}\}$$

and with $K \in \{0,1\}^{2k\times 2k}$ a primitive matrix defined as

$$K(s,s') = \begin{cases} 1, & \text{if } s' \neq s^{-1}; \\ 0, & \text{if } s' = s^{-1}. \end{cases}$$

Due to the requirement $s_i s_i^{-1} = \epsilon$, an NNMCTS of this type has the form $X_{\mathbf{A},\mathbf{A}^T}$, where $\mathbf{A} = (A_1, A_2, \cdots, A_k)$ and $\mathbf{A}^T = (A_1^T, A_2^T, \cdots, A_k^T)$. The topological entropy of an NNMCTS over a free group is first considered by Piantadosi in [11], in which such a space is referred to as a nearest-neighbor shift of finite type, and the entropy is studied under the definition (2.2) with no mention of the existence of the limit. The following theorem, on the other hand, shows such a limit does exist if one assumes $A_1 = A_1^T = A_2 = A_2^T = \cdots = A_k = A_k^T$.

Theorem 2.2.5. *Suppose $X_{\mathbf{A}}$ is an NNMCTS and K is a primitive matrix. If $\sum_{j=1}^k K(s_i, s_j) = \sum_{j=1}^k K(s_{i'}, s_j)$ for every $1 \leq i, i' \leq k$ and $A_1 = A_2 = \cdots = A_k = A$, then the limit $\lim_{n\to\infty} \frac{\log p_n(X_{\mathbf{A}})}{|\Delta_n|}$ exists and equals $h^{(s_1)}(X_{\mathbf{A}})$.*

In fact, this theorem has several applications. In particular, when $\mathcal{T}$ is a free semigroup, this leads to the existence of limit of topological entropy shown in [9]. For our purposes of application to free groups, it is seen that not only the limit in the entropy exists, but also $h(X_{\mathbf{A},\mathbf{A}^T}) = h^{(s)}(X_{\mathbf{A},\mathbf{A}^T})$ for every $s \in \Sigma$. Some other variations are also considered, each of which states a similar result as Theorem 2.2.5 exclusively over free groups, with slightly different constraints either on the matrices of rules $\mathbf{A}$ or on the number of symbols in the alphabet $\mathcal{A}$.

Theorem 2.2.6. *Suppose $X_{\mathbf{A},\mathbf{A}^T}$ is an NNMCTS over the free group rank k, and $A_1 = A_2 = \cdots = A_k = A$. The limit $\lim_{n\to\infty} \frac{\log p_n(X_{\mathbf{A},\mathbf{A}^T})}{|\Delta_n|}$ exists and equals $h^{(s_1)}(X_{\mathbf{A},\mathbf{A}^T})$.*

Theorem 2.2.7. *Suppose $X_{\mathbf{A},\mathbf{A}^T}$ is an NNMCTS over the free group rank k, and the dimension of matrices A_i is no more than $(2k-1)$. Then, the limit $\lim_{n\to\infty} \frac{\log p_n(X_{\mathbf{A},\mathbf{A}^T})}{|\Delta_n|}$ exists and equals $h^{(s_1)}(X_{\mathbf{A},\mathbf{A}^T})$.*

However, before we dig into the the detailed elucidation, two prerequisite lemmas need to be proved. The first one shows that $h \leq h^{(s)}$ for all $s \in \Sigma$ in any NNMCTS over free groups in general, which not only provides an upper bound for topological entropy estimation but sets up the foundation stone of the discussion of the existence in limit of topological entropy.

Lemma 2.2.8. *Suppose* $X_{\mathbf{A}}$ *is an NNMCTS and* K *is a primitive matrix. Then,* $h^{(s_i)}(X_{\mathbf{A}}) \geq h(X_{\mathbf{A}})$ *for every* $s_i \in \Sigma$.

Proof. Suppose $\Sigma = \{s_1, s_2, \dots, s_k\}$. Note that a pattern u with support Δ_n can be uniquely expressed as a $(k+1)$-tuple

$$\left(u_\epsilon, u\big|_{s_1\Delta^{(s_1)}_{n-1}}, u\big|_{s_2\Delta^{(s_2)}_{n-1}}, \cdots, u\big|_{s_l\Delta^{(s_k)}_{n-1}}\right).$$

Using the identity proved above one derives

$$p_n \leq |\mathcal{A}| \cdot \prod_{j=1}^{k} p^{(s_j)}_{n-1} p^{(s_j^{-1})}_{n-1}.$$

As a consequence,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log p_n}{|\Delta_n|} &\leq \limsup_{n\to\infty} \frac{|\mathcal{A}|}{|\Delta_n|} + \sum_{j=1}^{k} \frac{\log p^{(s_j)}_{n-1}}{|\Delta^{(s_j)}_{n-1}|} \frac{|\Delta^{(s_j)}_{n-1}|}{|\Delta_n|} \\
&= \sum_{j=1}^{k} h^{(s_i)} \cdot \left(\lim_{n\to\infty} \frac{|\Delta^{(s_1)}_{n-1}|}{|\Delta_n|}\right) \\
&= h^{(s_i)}
\end{aligned}$$

holds by applying Theorem 2.2.1. ∎

The second lemma is related to the proof of Theorem 2.2.5

Lemma 2.2.9 (Jensen-Minkowski inequality)**.** *Let* $\{x_i\}_{i=1}^N$ *be a finite set of non-negative real numbers and* $p \geq 1$. *Then,* $N^{p-1}\sum_{i=1}^N x_i^p \geq (\sum_{i=1}^N x_i)^p \geq \sum_{i=1}^N x_i^p$

Proof. Note that since $x \mapsto x^p$ is a convex function,

$$N^{p-1}\sum_{i=1}^{N} x_i^p = N^p \sum_{i=1}^{N} \frac{1}{N} x_i^p \geq N^p(\sum_{i=1}^{N} \frac{1}{N} x_i)^p = (\sum_{i=1}^{N} x_i)^p,$$

and thus the first inequality holds for all $N \in \mathbb{N}$ and all $p \geq 1$.

The second inequality is proved by induction on N. It is clear that when $N = 1$ the inequality is actually an equality. For the case $N = 2$, it is sufficient to prove the case

$x_2 \geq x_1 > 0$, since the case $x_1 = 0$ or $x_2 = 0$ falls back to the case $N = 0$. Observe that when $x_2 = x_1 > 0$,

$$(x_1 + x_2)^p - (x_1^p + x_2^p) = (2^p - 2)x_1^p \geq 0,$$

and that

$$\frac{\partial}{\partial x_2}[(x_1 + x_2)^p - (x_1^p + x_2^p)] = p((x_1 + x_2)^{p-1} - x_2^{p-1}) \geq 0.$$

These yield the desired result. Now suppose the induction hypothesis holds for $N = n$. We show that it also holds for $N = n + 1$. Indeed, by induction hypothesis,

$$(\sum_{i=1}^{N} x_i)^p \geq x_N^p + (\sum_{i=1}^{N-1} x_i)^p \geq \sum_{i=1}^{N} x_i^p.$$

The lemma is then proved to hold for all N by induction. □

Proof of Theorem 2.2.5. Since $m = \sum_{j=1}^{k} K(s_i, s_j) = \sum_{j=1}^{k} K(s_{i'}, s_j)$ for every $1 \leq i, i' \leq k$ and $A_1 = A_2 = \cdots = A_k = A$, it follows immediately that $p_{n;a}^{\prime(s_i)} = p_{n;a}^{\prime(s_j)}$ for every $s_i, s_j \in \Sigma$, for which we simply denote $p'_{n;a}$ in the rest of the proof.

Note that since $\frac{k}{m} \geq 1$, the following inequality holds for every $s \in \Sigma$ by Lemma 2.2.9:

$$\begin{aligned}
(p_n^{(s)})^{\frac{k}{m}} &= (\sum_{a=1}^{d} (p'_{n;a})^m)^{\frac{k}{m}} = (d \sum_{a=1}^{d} \frac{1}{d} \cdot (p'_{n;a})^m)^{\frac{k}{m}} \\
&\leq d^{\frac{k-m}{m}} \sum_{j=1}^{d} (p'_{n;a})^k \\
&= d^{\frac{k-m}{m}} p_n.
\end{aligned}$$

On the other hand, it can also be deduced by applying Lemma 2.2.9 that

$$(p_n^{(s)})^{\frac{k}{m}} = (\sum_{a=1}^{d} (p'_{n;a})^m)^{\frac{k}{m}} \geq \sum_{j=1}^{d} (p'_{n;a})^k = p_n.$$

The inequalities above yield that $(p_n^{(s)})^{\frac{k}{m}} \geq p_n \geq |\mathcal{A}|^{\frac{m-k}{m}}(p_n^{(g)})^{\frac{k}{m}}$ and thus

$$
\begin{aligned}
&\frac{\log p_n^{(s)}}{|\Delta_n^{(s)}|}\frac{\frac{k}{m}|\Delta_n^{(s)}|}{|\Delta_n|} + \frac{m-k}{m}\frac{\log|\mathcal{A}|}{|\Delta_n|} = \frac{\log\left(|\mathcal{A}|^{\frac{m-k}{m}}(p_n^{(g)})^{\frac{k}{m}}\right)}{|\Delta_n|} \\
\leq &\frac{\log p_n}{|\Delta_n|} \\
\leq &\frac{\log p_n^{(s)}}{|\Delta_n^{(s)}|}\frac{\frac{k}{m}|\Delta_n^{(s)}|}{|\Delta_n|} = \frac{\log\left((p_n^{(s)})^{\frac{k}{m}}\right)}{|\Delta_n|}.
\end{aligned}
$$

Since $\lim_{n\to\infty}\frac{\log p_n^{(s)}}{|\Delta_n^{(s)}|}$ is proved to be $h^{(s_1)}$ for all $s \in \Sigma$ in Theorem 2.2.1 and since $\lim_{n\to\infty}\frac{\frac{k}{m}|\Delta_n^{(s)}|}{|\Delta_n|} = 1$, the proof is finished. □

Proof of Theorem 2.2.6. Suppose $\Sigma = \{s_1, s_2, \ldots, s_k, s_1^{-1}, s_2^{-1}, \ldots, s_k^{-1}\}$. Due to the structure of the free group, it can be proved by inspection that $p_{n;a}^{\prime(s_i)} = p_{n;a}^{\prime(s_j)}$ and that $p_{n;a}^{\prime(s_i^{-1})} = p_{n;a}^{\prime(s_j^{-1})}$ for all s_i, s_j. For simplicity, we write $p'_{n;a} = p_{n;a}^{\prime(s_i)} = p_{n;a}^{\prime(s_j)}$, $p''_{n;a} = p_{n;a}^{\prime(s_i^{-1})} = p_{n;a}^{\prime(s_j^{-1})}$, and $|\Delta_n^{\prime(s_i)}| = |\Delta_n^{\prime(s_i^{-1})}| = |\Delta'_n|$ in the rest of the proof.

First, we claim that

$$
\limsup_{n\to\infty}\frac{\log p'_{n;a}}{|\Delta'_n|} = \limsup_{n\to\infty}\frac{\log p''_{n;a}}{|\Delta'_n|} \leq h^{(s_1)} \tag{2.3}
$$

To this end, Lemma 2.2.8 implies that

$$
\begin{aligned}
&\frac{\log p'_{n;a}}{|\Delta'_n|} \\
= &\frac{\log|\{u \in \mathcal{A}^{\Delta_n^{\prime(s_1)}} : u \text{ is accepted by some } t \in X_{\mathbf{A},\mathbf{A}^T}, u_\epsilon = a\}|}{|\Delta_{n-1}^{(s_1)}| + 1} \\
\leq &\frac{\log|\{u \in \mathcal{A}^{\Delta_{n-1}^{(s_1)}} : u \text{ is accepted by some } t \in X_{\mathbf{A},\mathbf{A}^T}\}|}{|\Delta_{n-1}^{(s_1)}| + 1} = \frac{\log p_{n-1}^{(s_1)}}{|\Delta_{n-1}^{(s_1)}| + 1},
\end{aligned}
$$

since $\Delta_n^{\prime(s_1)} = \Delta_n^{\prime(s_1)} \cup \{\epsilon\}$. The inequality is then proved by taking limit superior of both sides.

Now we claim that $\lim_{n\to\infty}\frac{\log p_n}{|\Delta_n|}$ exists and equals $h^{(s_1)}$. Since it follows from Lemma 2.2.8 that $\limsup_{n\to\infty}\frac{\log p_n}{|\Delta_n|} \leq h^{(s_1)}$, it is left to show that $\liminf_{n\to\infty}\frac{\log p_n}{|\Delta_n|} \geq h^{(s_1)}$. Since $p_n^{(s_1)} = \sum_{a\in\mathcal{A}}(p'_{n;a})^k \cdot (p''_{n;a})^{k-1}$, there exists $a_n \in \mathcal{A}$ for each n such that $(p'_{n;a_n})^k \cdot (p''_{n;a_n})^{k-1} \geq$

$\frac{p_n^{(s_1)}}{|\mathcal{A}|}$. Hence, by applying (2.3), for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$p'_{n;a_n}, p''_{n;a_n} < e^{(h^{(s_1)}+\epsilon)|\Delta'_n|},$$

whenever $n \geq N$. Furthermore, from Theorem 2.2.1, one may assume that for $n \geq N$,

$$(p'_{n;a_n})^k \cdot (p''_{n;a_n})^{k-1} \geq \frac{p_n^{(s_1)}}{|\mathcal{A}|} > \frac{1}{|\mathcal{A}|} e^{(h^{(s_1)}-\epsilon)|\Delta_n^{(s_1)}|}.$$

The inequalities above indicates that

$$\begin{aligned}
p''_{n;a_n} &= \frac{(p'_{n;a_n})^k \cdot (p''_{n;a_n})^{k-1}}{(p'_{n;a_n})^k \cdot (p''_{n;a_n})^{k-2}} \\
&\geq \frac{\frac{1}{|\mathcal{A}|} e^{(h^{(s_1)}-\epsilon)|\Delta_n^{(s_1)}|}}{e^{(h^{(s_1)}+\epsilon)|\Delta'_n|(2k-2)}} \\
&= \frac{1}{|\mathcal{A}|} e^{(h^{(s_1)}-\epsilon)[(2k-1)|\Delta'_n|-(2k-2)]-(h^{(s_1)}+\epsilon)|\Delta'_n|(2k-2)} \\
&\geq \frac{1}{|\mathcal{A}|} e^{-(2k-2)(h^{(s_1)}-\epsilon)} e^{(h^{(s_1)}-(4k-3)\epsilon)|\Delta'_n|}.
\end{aligned}$$

Hence,

$$\begin{aligned}
p_{n;a_n} &\geq (p'_{n;a_n})^k \cdot (p''_{n;a_n})^{k-1} \cdot p''_{n;a_n} \\
&\geq \frac{1}{|\mathcal{A}|^2} e^{(h^{(s_1)}-\epsilon)|\Delta_n^{(s_1)}|} \cdot e^{-(2k-2)(h^{(s_1)}-\epsilon)} e^{(h^{(s_1)}-(4k-3)\epsilon)|\Delta'_n|} \\
&\geq \frac{1}{|\mathcal{A}|^2} e^{(h^{(s_1)}-(4k-3)\epsilon)|\Delta_n^{(s_1)}|} \cdot e^{-(2k-2)(h^{(s_1)}-\epsilon)} e^{(h^{(s_1)}-(4k-3)\epsilon)|\Delta'_n|} \\
&= \frac{1}{|\mathcal{A}|^2} e^{(h^{(s_1)}-(4k-3)\epsilon)(|\Delta_n^{(s_1)}|+|\Delta'_n|)} \cdot e^{-(2k-2)(h^{(s_1)}-\epsilon)} \\
&= \frac{1}{|\mathcal{A}|^2} e^{(h^{(s_1)}-(4k-3)\epsilon)(|\Delta_n|+1)} \cdot e^{-(2k-2)(h^{(s_1)}-\epsilon)}.
\end{aligned}$$

By taking limit inferior from both sides, one derives

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\log p_n}{|\Delta_n|} \\
\geq &\liminf_{n\to\infty} \frac{\log p_{n;a}}{|\Delta_n|} \\
\geq &\liminf_{n\to\infty} -\frac{2\log|\mathcal{A}|}{|\Delta_n|} + (h^{(s_1)} - (4k-3)\epsilon)\frac{|\Delta_n|+1}{|\Delta_n|} - \frac{(2k-2)(h^{(s_1)}-\epsilon)}{|\Delta_n|} \\
\geq &h^{(s_1)} - (4k-3)\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the proof is complete. $\square$

Proof of Theorem 2.2.7. For simplicity, we write $|\Delta'^{(s_i)}| = |\Delta'^{(s_i^{-1})}| = |\Delta'_n|$ in the rest of the proof.

By applying the same argument as in Theorem 2.2.6 (2.3), one obtains that

$$\limsup_{n\to\infty} \frac{\log p'^{(s)}_{n;a}}{|\Delta'^{(s)}_n|} \le h^{(s_1)} \tag{2.4}$$

for every $s \in \Sigma$. Furthermore, since it follows from Lemma 2.2.8 that $\limsup_{n\to\infty} \frac{\log p_n}{|\Delta_n|} \le h^{(s_1)}$, it is left to show that $\liminf_{n\to\infty} \frac{\log p_n}{|\Delta_n|} \ge h^{(s_1)}$. Since $p_n^{(s)} = \sum_{a\in\mathcal{A}} \prod_{w\neq s^{-1}} p'^{(w)}_{n;a}$, for every $s \in \Sigma$ and $n \ge 0$ there exists $a_{n;s} \in \mathcal{A}$ such that $\prod_{w\neq s^{-1}} p'^{(w)}_{n;a_{n;s}} \ge \frac{p_n^{(s)}}{|\mathcal{A}|}$. Hence, by applying Theorem 2.2.1 and (2.4), for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $s \in \Sigma$, $w \in \Sigma \setminus \{s^{-1}\}$ and all $n \ge N$

$$p'^{(w)}_{n;a_{n;s}} < e^{(h^{(s_1)}+\epsilon)|\Delta'_n|}, \tag{2.5}$$

and that

$$\prod_{w\neq s^{-1}} p'^{(w)}_{n;a_{n;s}} > \frac{1}{|\mathcal{A}|} e^{(h^{(s_1)}-\epsilon)|\Delta^{(s)}_n|}. \tag{2.6}$$

At this moment, It is noteworthy that the restriction imposed on the dimension of A_i leads to the coincidence of some $a_{n;z} = a_{n;z'}$ $(z \neq z')$ such that $K(z', z^{-1}) = 1$ by the pigeonhole principle. As a result,

$$\begin{aligned} p_{n;a_{n;z}} &= p^{(z)}_{n;a_{n;z}} \cdot p'^{(z^{-1})}_{n;a_{n;z'}} \\ &= \left(\prod_{w\neq z^{-1}} p'^{(w)}_{n;a_{n;z}}\right) \cdot p'^{(z^{-1})}_{n;a_{n;z'}}, \end{aligned} \tag{2.7}$$

To see this, note that for every admissible patterns $u \in \mathcal{A}^{\Delta_n}$ with $u_\epsilon = a_{n;z} = a_{n;z'}$, it can be uniquely expressed as a 2-tuple $(u|_{\Delta_n^{(z)}}, u|_{\Delta_n'^{(z^{-1})}})$, where $u|_{\Delta_n^{(z)}}, u|_{\Delta_n'^{(z^{-1})}}$ are admissible patterns and $(u|_{\Delta_n^{(z)}})_\epsilon = a_{n;z}$ and $(u|_{\Delta_n'^{(z')}})_\epsilon = a_{n;z'}$. Due to the fact that u and $(u|_{\Delta_n^{(z)}}, u|_{\Delta_n'^{(z^{-1})}})$ actually defines a one-one correspondence, the equality of (2.7) follows as a consequence.

Now apply (2.5)(2.6)(2.7), it follows that

$$
\begin{aligned}
p'^{(z^{-1})}_{n;a_{n;z}} = p'^{(z^{-1})}_{n;a_{n;z'}} &= \frac{\prod_{w\neq z'^{-1}} p'^{(w)}_{n;a_{n;z'}}}{\prod_{w\neq z'^{-1}, w\neq z^{-1}} p'^{(w)}_{n;a_{n;z'}}} \\
&\geq \frac{1}{|\mathcal{A}|} e^{(h^{(s_1)}-\epsilon)|\Delta_n^{(z)}|-(h^{(s_1)}+\epsilon)|\Delta'_n|(2k-2)} \\
&= \frac{1}{|\mathcal{A}|} e^{(h^{(s_1)}-\epsilon)[(2k-1)|\Delta'_n|-(2k-2)]-(h^{(s_1)}+\epsilon)|\Delta'_n|(2k-2)} \\
&\geq \frac{1}{|\mathcal{A}|} e^{-(2k-2)(h^{(s_1)}-\epsilon)} e^{(h^{(s_1)}-(4k-3)\epsilon)|\Delta'_n|}.
\end{aligned}
$$

By combining the results above,

$$
\begin{aligned}
p_n \geq p_{n;a_{n;z}} &= \Big(\prod_{w\neq z^{-1}} p'^{(w)}_{n;a_{n;z}}\Big) \cdot p'^{(z^{-1})}_{n;a_{n;z}} \\
&\geq \frac{1}{|\mathcal{A}|^2} e^{(h^{(s_1)}-\epsilon)|\Delta_n^{(z)}|} \cdot e^{-(2k-2)(h^{(s_1)}-\epsilon)} e^{(h^{(s_1)}-(4k-3)\epsilon)|\Delta'_n|} \\
&\geq \frac{1}{|\mathcal{A}|^2} e^{(h^{(s_1)}-(4k-3)\epsilon)|\Delta_n^{(z)}|} \cdot e^{-(2k-2)(h^{(s_1)}-\epsilon)} e^{(h^{(s_1)}-(4k-3)\epsilon)|\Delta'_n|} \\
&= \frac{1}{|\mathcal{A}|^2} e^{(h^{(s_1)}-(4k-3)\epsilon)(|\Delta_n^{(z)}|+|\Delta'_n|)} \cdot e^{-(2k-2)(h^{(s_1)}-\epsilon)} \\
&= \frac{1}{|\mathcal{A}|^2} e^{(h^{(s_1)}-(4k-3)\epsilon)(|\Delta_n|+1)} \cdot e^{-(2k-2)(h^{(s_1)}-\epsilon)}
\end{aligned}
$$

Hence, one obtains $\liminf_{n\to\infty} \frac{\log p_n}{|\Delta_n|} \geq h^{(s_1)}$, which finishes the proof. □

Example 2.2.10. Suppose $X_{\mathbf{A},\mathbf{A}^T}$ is an NNMCTS over a free group of rank k, where

$$
\mathbf{A} = (A, A, \cdots, A) \text{ and } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
$$

Then, the limit of $\lim_{n\to\infty} \frac{\log p_n}{\Delta_n}$ exists and equals $h^{(s_1)}(X_{\mathbf{A},\mathbf{A}^T})$ by either Theorem 2.2.5, Theorem 2.2.6 or Theorem 2.2.7. In particular, since A is a symmetric matrix, $h(X_{\mathbf{A},\mathbf{A}^T}) = h^{(s_1)}(X_{\mathbf{A},\mathbf{A}^T}) = h(X_{(A,A,\dots,A)})$, where $X_{(A,A,\dots,A)}$ is a nearest-neighbor tree-shift over the semigroup of rank $(2k-1)$. For the case $k = 2$, it is numerically evaluated by Algorithm 1 and by (A.3) that $h(X_{\mathbf{A},\mathbf{A}^T}) \in (0.2332621211030, 0.2332621211030 + 8 \cdot 10^{-12})$. More experimental results of NNMCTS over free group of rank 2 are provided in Table 1.

It is remarkable that for every NNMCTS $X_{\mathbf{A}}$ satisfying the assumption of Theorem 2.2.5, one can define an nearest-neighbor tree-shift of the form $X_{(A,A,\dots,A)}$ as in 2.2.10

while preserving the topological entropy. By virtue of Algorithm 1, the computational complexity is alleviated compared to Algorithm 2.

Chapter 3

Graph Representation and Existence in Limit of Topological Entropy

The purpose of this chapter is two-fold. On one hand, Section 3.1 and Section 3.2 carry out the partial generalization of the proposition of Petersen and Salama [9, Proposition 3.1], which manifests the connection between topological properties of a nearest-neighbor hom tree-shift, a nearest-neighbor tree-shift of the form $X_{(A,A,\ldots,A)}$, and the convergence of its topological entropy. On the other, techniques developed in Section 3.2 are adapted in Section 3.3 in an attempt to answer the question of existence of limit in the topological entropy for general NNMCTS's.

3.1 Topological Properties of One-Sided Nearest-Neighbor Shift Spaces

Let $\mathbb{Z}_+$ denote the set of non-negative integers, and $A \in \{0,1\}^{k\times k}$ be a $k \times k$ binary matrix with each row a non-zero row. A *one-sided nearest-neighbor shift of finite type* $X_A \subseteq \mathcal{A}^{\mathbb{Z}_+}$ is a degenerate NNMCTS defined on a Markov-Cayley tree generated by Σ with $K = [1]$. Since each row of A is non-zero, every *locally admissible word by* A, $a_0a_1\cdots a_{n-1} \in \mathcal{A}^n$ satisfying $A(a_i, a_{i+1}) = 1$ for every $0 \le i < n$, admits a labeling $x \in X_A$ satisfying $x|_{[0,n)} = a_0a_1\cdots a_{n-1}$. That is, every locally admissible word by A is an admissible word by X_A. In the following, we recall several fundamental properties of such a shift space.

Suppose $X_A \subseteq \mathcal{A}^{\mathbb{Z}_+}$ is a one-sided nearest-neighbor shift of finite type. A directed graph $G_A = (V_A, E_A)$ is called the *graph representation of X_A* if

$$V_A = \mathcal{A} \text{ and } E_A = \{(a,b) \in V_A \times V_A : A(a,b) = 1\}$$

It is not hard to see from the definition that a one-sided nearest-neighbor shift of finite type can also be defined in this manner. That is, if $G = (V, E)$ is a directed graph, one can define a one-sided nearest-neighbor shift of finite type $X_G \subseteq \mathcal{A}_G^{\mathbb{Z}_+}$ by the adjacency matrix A_G as follows:

$$\mathcal{A}_G = V \text{ and } A_G(a,b) = \begin{cases} 1 & \text{if } (a,b) \in V; \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the two definitions are equivalent in the sense that $X_G = X_{A_G}$ for every directed graph G and that $X_A = X_{G_A}$ for every binary matrix A.

In the past literature, the classical classification of a one-sided nearest-neighbor shift of finite type consists of two important categories defined as follows, which play crucial roles in our later discussion on entropy. Suppose X_A is a one-sided nearest-neighbor shift of finite type. X_A is said to be *transitive* if for all $x, y \in X_A$ and finite intervals $[0, n), [0, m) \subset \mathbb{Z}_+$, there exists $N \in \mathbb{N}$ and $z \in X_A$ such that $z|_{[0,n)} = x|_{[0,n)}$ and that $z|_{[n+N,n+N+m)} = y|_{[0,m)}$. X_A is said to be *mixing* if for all finite intervals $[0, n), [0, m) \subset \mathbb{Z}_+$, there exists $N \in \mathbb{N}$ such that for all $l \geq N$ there exists $z \in X_A$ satisfying $z|_{[0,n)} = x|_{[0,n)}$ and that $z|_{[n+l,n+l+m)} = y|_{[0,m)}$. A directed graph $G = (V, E)$ is said to be *strongly connected* if every $a, b \in V$, there exists a walk of length N starts at a and terminates at b, which we denotes $a \xrightarrow{N}\!\!\!\!\rightarrow b$. It follows immediately from the definition that X_A is transitive if X_A is mixing. Other fundamental properties are provided and proved in the following.

Proposition 3.1.1. *Suppose $X_A \subseteq \mathcal{A}^{\mathbb{Z}_+}$ is a one-sided nearest-neighbor shift of finite type. The following are equivalent.*

1. *X_A is transitive.*

2. *For all $a, b \in \mathcal{A}$ there exists $N \in \mathbb{N}$ and $x \in X_A$ such that $x_0 = a$ and $x_N = b$.*

3. *A is an irreducible matrix, i.e. there exists $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} A^n(a,b) > 0$ for all $a, b \in \mathcal{A}$.*

4. *The graph representation $G = (V, E)$ of X_A is strongly connected.*

Proof. We first demonstrate the equivalence between (1) and (2), and then their equivalence with the rest.

(1) $\Leftrightarrow$ (2). The necessity follows immediately, so it is left to show the sufficiency. Suppose $x, y \in X_A$ and finite intervals $[0, n), [0, m) \subset \mathbb{Z}_+$ are given. By the hypothesis of the proposition, there exist $N \in \mathbb{N}$ and $z \in X_A$ such that $z_0 = x_{n-1}$ and that $z_N = y_0$. Then, the labeling $\bar{z} = x|_{[0,n)} z|_{[1,N-1]} y|_{[0,\infty)}$ lies in X_A, $\bar{z}|_{[0,n)} = x|_{[0,n)}$, and $\bar{z}|_{[n+N,n+N+m)} = y|_{[0,m)}$. The proof is then finished.

(2) $\Leftrightarrow$ (3). Note that $A^n(a, b)$ is the number of all locally admissible $(n + 1)$-words starting with a and ending with b. Since each row of A is non-zero, every locally admissible word is an admissible word in X_A and hence the equivalence follows.

(2) $\Leftrightarrow$ (4). Note that (2) is equivalent to that for all $a, b \in \mathcal{A}$ there exists an locally admissible word $aa_1 \cdots a_{n-1}b$ by A. The proof is finished by noting that a locally admissible word $aa_2 \cdots a_{n-1}b$ also denotes a path in G and vice versa. □

Proposition 3.1.2. *Suppose $X_A \subseteq \mathcal{A}^{\mathbb{Z}_+}$ is a one-sided nearest-neighbor shift of finite type. The following are equivalent.*

1. *X_A is mixing.*

2. *There exists $N \in \mathbb{N}$ such that for all $a, b \in \mathcal{A}$ and $n \geq N$ there exists $x \in X_A$ satisfying $x_0 = a$ and that $x_n = b$.*

3. *A is an irreducible matrix, i.e. there exists $N \in \mathbb{N}$ such that $A^N(a, b) > 0$ for all $a, b \in \mathcal{A}$.*

4. *The graph representation $G = (V, E)$ of X_A is strongly connected and there exist $a \in V$ and $N \in \mathbb{N}$ such that for all $n \geq N$, G admits a walk $a \xrightarrow{n} a$.*

Proof. We first demonstrate the equivalence between (1) and (2), and then their equivalence with (3) and (4)

(1) $\Leftrightarrow$ (2). The necessity follows immediately, so it is left to show the sufficiency. Suppose $[0, m_1), [0, m_2) \subset \mathbb{Z}_+$ are given. By the hypothesis of the proposition, there exist $N \in \mathbb{N}$ such that for all $n \geq N$ and $z \in X_A$ such that $z_0 = x_{m_1-1}$ and that $z_n = y_0$. Then, the labeling $\bar{z} = x|_{[0,m_1)} z_{[1,n-1]} y|_{[0,\infty)}$ lies in X_A, $\bar{z}|_{[0,m_1)} = x|_{[0,m_1)}$, and $\bar{z}|_{[m_1+n,m_1+n+m_2)} = y|_{[0,m_2)}$. Since $n \geq N$ is arbitrary, the proof is then finished.

(2) $\Leftrightarrow$ (3). Note that $A^n(a,b)$ is the number of all locally admissible $(n+1)$-words starting with a and terminating with b. Since each row of A is non-zero, every locally admissible word is an admissible word in X_A, and hence the equivalence follows.

(2) $\Leftrightarrow$ (4). The necessity follows from definition and Proposition 3.1.1 that if X_A is mixing, then X_A is transitive by definition and thus G is strongly connected. As for sufficiency, it follows from hypothesis of (2) that there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there is a locally admissible word $aa_1 \cdots a_{n-1}a$ by A. That is, $aa_1 \cdots a_{n-1}a$ is a n-walk in G. As for necessity, we assume $a, b \in \mathcal{A} = V$ are given. Then, by hypothesis of (4), there exists $c \in \mathcal{A} = V$ and $M \in \mathbb{N}$ such that for every $n \geq M$, there is a walk $c \xrightarrow{n} c$. In addition, since G is strongly connected, there exists walks $a \xrightarrow{M_{a,c}} c$ and $c \xrightarrow{M_{c,b}} b$. These yields a walk $a \xrightarrow{M_{a,c}+n+M_{c,b}} b$ for every $n \geq N$. Since $\mathcal{A}$ is a finite set and $n \geq M$ is arbitrary, the proof is finished by noting $N = M + \max_a M_{a,c} + \max_b M_{c,b}$. $\square$

3.2 Topological Properties of Nearest-Neighbor Markov-Cayley Tree-Shifts

The purpose of this section is to partially generalize the following proposition of Petersen and Salama [9].

Proposition 3.2.1 (Petersen-Salama [9]). *Suppose $X_{\mathbf{A}}$ is a nearest-neighbor tree-shift with $\mathbf{A} = (A, A, \cdots, A)$. Then, the following statements hold.*

- *If A is irreducible, then $h(X_{\mathbf{A}}) = \limsup_{n\to\infty} \frac{\log p_{n;a}(X_{\mathbf{A}})}{|\Delta_n|}$ for all $a \in \mathcal{A}$.*
- *If A is primitive, then $h(X_{\mathbf{A}}) = \lim_{n\to\infty} \frac{\log p_{n;a}(X_{\mathbf{A}})}{|\Delta_n|}$ for all $a \in \mathcal{A}$.*

To establish an analogy of the proposition, it is a prerequisite to generalize the idea of the graph representation of a one-sided nearest-neighbor shift of finite type, as discussed in Proposition 3.1.2, to an analogous graph representation of $X_{\mathbf{A}}$.

Definition 3.2.2. Suppose $X_{\mathbf{A}}$ is an NNMCTS. A *graph $G = (V, E)$ associated with $X_{\mathbf{A}}$* is a directed graph with the vertex set and with the edge set

$$V = \mathcal{A} \times \Sigma \quad \text{and} \quad E = \{((a, s_i), (b, s_j)) \in V \times V : K(s_i, s_j) = 1, A_j(a, b) = 1\},$$

respectively.

1. G is said to be *strongly connected* if for every $(a, s_i), (b, s_j) \in V$ there is a N-walk from (a, s_i) to (b, s_j) in G (denoted by $(a, s_i) \xrightarrow{N}\!\!\!\!\rightarrow (b, s_j)$) for some N depending on (a, s_i) and (b, s_j).

2. A vertex $(a, s_i) \in V$ is called a *pivot* if there exist $s_j \in \Sigma$ and an integer $N \in \mathbb{N}$ such that every $(b, s_j) \in V$ admits a walk $(a, s_i) \xrightarrow{N}\!\!\!\!\rightarrow (b, s_j)$.

The definition forms the foundation stone of the generalization of Proposition 3.1.1 and 3.1.2. In fact, it is not hard to see the strong connectedness of the graph in the above definition agrees with Proposition 3.1.1 (4). In addition, the definition of the pivot mimics the requirement for a shift space to admit periodic labelings as indicated in Proposition 3.1.2 (4). These generalizations are summarized in the following proposition.

Proposition 3.2.3. *Suppose $X_{\mathbf{A}}$ is an NNMCTS with $\mathbf{A} = (A, A, \ldots, A)$ a k-tuple of matrices, K is a primitive matrix, and $G = (V, E)$ is the directed graph associated with $X_{\mathbf{A}}$. Then,*

1. *G is strongly connected if and only if the adjacency matrix associated with X_G is irreducible. In addition, it is also equivalent to A being irreducible.*

2. *G is strongly connected and contains a pivot if and only if the adjacency matrix associated with X_G is primitive. In addition, it is also equivalent to A being primitive.*

Proof. (1) It is proved in Proposition 3.1.1 that G is strongly connected if and only if the adjacency matrix A_G associated with G is irreducible. It remains to demonstrate their equivalence with A being irreducible. It is not hard to see that A is irreducible if G is strongly connected, since for $(a, s_i), (b, s_i) \in V$, there exists a walk

$$(a, s_i)(a_1, s_{i_1})(a_2, s_{i_2}) \cdots (a_{n-1}, s_{i_{n-1}})(b, s_i)$$

and thus $aa_1 \cdots a_{n-1}b$ is a word admissible by A. We now show the converse, i.e., for $(a, s_i), (b, s_j) \in V$, there exists a walk $(a, s_i) \xrightarrow{M}\!\!\!\!\rightarrow (b, s_j)$. Since K is a primitive matrix, there exists, by Proposition 3.1.2 (2), N such that for every $n \in \mathbb{N}$ and $s_i, s_j \in \Sigma$, there exists an locally admissible $(n+1)$-word $s_i s_{i_1} s_{i_2} \cdots s_{i_{n-1}} s_j$ by K. On the other hand, since A is irreducible, for every $a, b \in \mathcal{A}$ there exists an integer $M \geq N$ and an $(M+1)$-word $aa_1a_2 \cdots a_{M-1}b$ admissible by A. This results in a walk $(a, s_i)(a_1, s_{i_1}) \cdots (a_{M-1}, s_{i_{M-1}})(b, s_j)$ in G. This completes the proof.

(2) First of all, we show that A is primitive if the adjacency matrix A_G associated with G is primitive. Indeed, since A_G is primitive, there exists N such that for all $(a, s_i), (b, s_j)$ and $n \geq N$, there exists an walk

$$(a, s_i)(a_1, s_{i_1})(a_2, s_{i_2}) \cdots (a_{n-1}, s_{i_{n-1}})(b, s_i)$$

in G by Proposition 3.1.2. This naturally yields an admissible $(n+1)$-word $aa_1a_2 \cdots a_{n-1}b$ by X_A which starts at a and terminates at b.

Secondly, we show that G is strongly connected and contains a pivot provided A is primitive. To this end, we show every $(a, s_i) \in V$ is a pivot of G. Since K is a primitive matrix, there exists an integer N_1 such that for every $s_j \in \Sigma$ and every $n \geq N_1$, there exists, by Proposition 3.1.2, an $(n+1)$-word admissible by K which starts from s_i and terminates at s_j. On the other hand, since A is primitive, there exists $N_2 \geq N_1$ such that for every $b \in \mathcal{A}$ there exists a word $a_0a_1a_2 \cdots a_{N_2-1}a_{N_2}$ with $a_0 = a$, $a_{N_2} = b$ and $A(a_k, a_{k+1})$ for every $k = 0, 1, \cdots, N_2 - 1$. This implies for all $n \geq N_2$ there is a walk $(a_1, s_{i_1})(a_2, s_{i_2}) \cdots (a_{n-1}, s_{i_{n-1}})(a_n, s_{i_n})$ in G with $a_1 = a$, $a_n = b$, $s_{i_1} = s_i$ and $s_{i_n} = s_j$. This finishes the proof of our claim. Note since every (a, s_i) is a pivot, it follows from Proposition 3.1.2 that G is strongly.

Finally, it remains to show that if G is strongly connected and contains a pivot, then A_G is primitive. By Proposition 3.1.2, it is also equivalent to show that G is strongly connected and there exists $(a, s_i) \in V$ and $N \in \mathbb{N}$ such that every $n \geq N$ admits a walk $(a, s_i) \xrightarrow{n}\mathrel{\mkern-14mu}\rightarrow (a, s_i)$. Since strong connectedness follows immediately, it is left to show the latter. Suppose (a, s_i) is a pivot such that there exist $s_j \in \Sigma$ and walks $(a, s_i) \xrightarrow{N}\mathrel{\mkern-14mu}\rightarrow (b_k, s_j)$ for every $b_k \in \mathcal{A}$ as follows:

$$\begin{gathered}
(a, s_i)(a_{1,1}, s_{l_{1,1}}) \cdots (a_{1,N-1}, s_{l_{1,N-1}})(a, s_j), \\
(a, s_i)(a_{2,1}, s_{l_{2,1}}) \cdots (a_{2,N-1}, s_{l_{2,N-1}})(b_2, s_j), \\
\vdots \\
(a, s_i)(a_{d,1}, s_{l_{d,1}}) \cdots (a_{d,N-1}, s_{l_{d,N-1}})(b_d, s_j).
\end{gathered}$$

Hence, the following are admissible words by A:

$$aa_{1,1}\cdots a_{1,N-1}a,$$
$$aa_{2,1}\cdots a_{2,N-1}b_2,$$
$$\vdots$$
$$aa_{d,1}\cdots a_{d,N-1}b_d.$$

From these, we are able to construct a $(n+1)$-word of length for all $n \geq N$ with both starting and ending symbol a. For instance, for the case $n = N+2$, we may observe if $a_{1,N-2} = b_k$ for some $1 \leq k \leq d$, then $aa_{k,1}\cdots a_{k,N-1}b_k a_{1,N-1}a$ is an admissible word by A. This process can be done for $N+1 \leq n \leq 2N+1$, and further extension process for $n > 2N+1$ is done by a proper concatenation with the prefix $aa_{1,2}\cdots a_{1,N}a$. Now since K is a primitive matrix, we can also prove that for every $s_i \in \Sigma$ and any sufficiently large $n \in \mathbb{N}$ there is an $(n+1)$-word admissible by K which both starts and terminates at s_i. Combining these two facts, we are able to construct a walk $(a, s_i) \overset{n}{\twoheadrightarrow} (a, s_i)$ for all sufficiently large n, and the proof is completed. $\square$

With the generalization of the mixing property in hand, we are at a position to present the existence of limit in the topological entropy.

Theorem 3.2.4. *Suppose $X_{\mathbf{A}}$ is an NNMCTS and K is a primitive matrix. Let $G = (V, E)$ be the directed graph associated with $X_{\mathbf{A}}$. Then, the limit $\lim_{n\to\infty} \frac{\log p_n(X_{\mathbf{A}})}{|\Delta_n|}$ exists and equals $h^{(s_1)}(X_{\mathbf{A}})$ if G admits a pivot and G is strongly connected.*

Proof. First, we show that $\liminf_{n\to\infty} \frac{\log p_{n;a}^{(s_i)}}{|\Delta_n^{(s_i)}|} = \liminf_{n\to\infty} \frac{\log p_{n;b}^{(s_j)}}{|\Delta_n^{(s_j)}|}$ for every $s_i, s_j \in \Sigma$ and $a, b \in \mathcal{A}$. Suppose

$$\liminf_{n\to\infty} \frac{\log p_{n;a}^{(s_i)}}{|\Delta_n^{(s_i)}|} = \min\left\{\liminf_{n\to\infty} \frac{\log p_{n;c}^{(s_l)}}{|\Delta_n^{(s_l)}|} : s_l \in \Sigma, c \in \mathcal{A}\right\} =: \underline{h}$$

and

$$\liminf_{n\to\infty} \frac{\log p_{n;b}^{(s_j)}}{|\Delta_n^{(s_j)}|} = \max\left\{\liminf_{n\to\infty} \frac{\log p_{n;c}^{(s_l)}}{|\Delta_n^{(s_l)}|} : s_l \in \Sigma, c \in \mathcal{A}\right\} =: \overline{h}.$$

We show that $\overline{h} = \underline{h}$. To begin with, we endow an order on Σ so that the set $\{g_i\}_{i=1}^M = \{g \in \mathcal{T} : |g| = N\}$ is endowed with the lexicographic order. With this order, we introduce the notation

$$p^{(s_i)}_{N;a;b_1,\cdots,b_M} := |\{u \in \mathcal{A}^{\Delta^{(s_i)}_N} : u \text{ is accepted by } t \in X_{\mathbf{A}}, u_{g_i} = b_i, \forall 1 \le i \le M\}|. \tag{3.1}$$

Since G is strongly connected, there is an admissible walk $(a, s_i) \overset{N}{\twoheadrightarrow} (b, s_j)$ in G. As a consequence, there exists $g_l = g_l^{(0)} g_l^{(1)} \dots g_l^{(N-1)} s_j \in \{g \in \mathcal{T} : |g| = N\}$ with $g_l^{(0)}, \dots, g_l^{(N-1)}, s_j \in \Sigma$ so that

$$p^{(s_i)}_{N;a;b_1,\cdots,b_{l-1},b,b_{l+1},\cdots,b_M} \ge 1.$$

In this case, (b, s_j) is said to appear in $b_1, \cdots, b_{l-1}, b, b_{l+1}, \cdots, b_M$. Therefore,

$$\begin{aligned} p^{(s_i)}_{N+n} &= \sum_{b_1,\cdots,b_M} p^{(s_i)}_{N;a;b_1,\cdots,b_M} \cdot p^{(s_{l_1})}_{n;b_1} p^{(s_{l_2})}_{n;b_2} \cdots p^{(s_{l_M})}_{n;b_M} \\ &\ge p^{(s_i)}_{N;a;b_1,\cdots,b,\cdots,b_M} p^{(s_{l_1})}_{n;b_1} \cdot p^{(s_{l_2})}_{n;b_2} \cdots p^{(s_j)}_{n;b} \cdots p^{(s_{l_M})}_{n;b_M} \\ &\ge p^{(s_{l_1})}_{n;b_1} \cdot p^{(s_{l_2})}_{n;b_2} \cdots p^{(s_j)}_{n;b} \cdots p^{(s_{l_M})}_{n;b_M}. \end{aligned} \tag{3.2}$$

Hence, it yields

$$\liminf_{n\to\infty} \frac{\log p^{(s_i)}_{N+n}}{|\Delta^{(s_i)}_{N+n}|} \ge \liminf_{n\to\infty} \frac{\log p^{(s_{l_1})}_{n;b_1}}{|\Delta^{(s_{l_1})}_n|} \frac{|\Delta^{(s_{l_1})}_n|}{|\Delta^{(s_i)}_{N+n}|} + \cdots + \frac{\log p^{(s_j)}_{n;b}}{|\Delta^{(s_j)}_n|} \frac{|\Delta^{(s_j)}_n|}{|\Delta^{(s_i)}_{N+n}|} + \cdots + \frac{\log p^{(s_{l_M})}_{n;b_M}}{|\Delta^{(s_{l_M})}_n|} \frac{|\Delta^{(s_{l_M})}_n|}{|\Delta^{(s_i)}_{N+n}|}.$$

Note that $\lim_{n\to\infty} \frac{|\Delta^{(s_l)}_n|}{|\Delta^{(s_i)}_{N+n}|}$ is positive for every $s_l \in \Sigma$ and

$$\lim_{n\to\infty} \frac{|\Delta^{(s_{l_1})}_n| + \cdots + |\Delta^{(s_{l_M})}_n|}{|\Delta^{(s_i)}_{N+n}|} = \lim_{n\to\infty} \frac{|\Delta^{(s_{l_1})}_n| + \cdots + |\Delta^{(s_{l_M})}_n|}{|\Delta^{(s_i)}_{N-1}| + |\Delta^{(s_{l_1})}_n| + \cdots + |\Delta^{(s_{l_M})}_n|} = 1.$$

It follows that $\underline{h} \ge \overline{h}$.

Next, we show that $\liminf_{n\to\infty} \frac{\log p^{(s_i)}_{n;a}}{|\Delta^{(s_i)}|} = \limsup_{n\to\infty} \frac{\log p^{(s_i)}_{n;a}}{|\Delta^{(s_i)}_n|}$ for every $s_i \in \Sigma$ and $a \in \mathcal{A}$. Suppose (a, s_i) is a pivot in G. Then, for every for all $c \in \mathcal{A}$, there is an admissible walk $(a, s_i) \overset{N}{\twoheadrightarrow} (c, s_j)$ in G. As a consequence, there exists $g_l = g_l^{(0)} g_l^{(1)} \dots g_l^{(N-1)} s_j \in \{g \in \mathcal{T} : |g| = N\}$ with $g_l^{(0)}, \dots, g_l^{(N-1)}, s_j \in \Sigma$ so that $p^{(s_i)}_{N;a;b_1,\cdots,b_{l-1},c,b_{l+1},\cdots,b_M} \ge 1$. On the other hand, it follows from the claim above that for every $\epsilon > 0$, there exists N' such that for every $n \ge N'$,$s_l \in \Sigma$ and $c \in \mathcal{A}$,

$$\frac{\log p^{(s_l)}_{n;c}}{|\Delta^{(s_l)}_n|} \ge \overline{h} - \epsilon. \tag{3.3}$$

Hence,

$$\begin{aligned}
p_{N+n;a}^{(s_i)} &= \sum_{b_1,\cdots,b_M} p_{N;a;b_1,\cdots,b_M}^{(s_i)} \cdot p_{n;b_1}^{(s_{l_1})} p_{n;b_2}^{(s_{l_2})} \cdots p_{n;b_M}^{(s_{l_M})} \\
&\geq \sum_{b_1,\cdots,b_M} \frac{1}{|\mathcal{A}|} \sum_{\substack{c\in\mathcal{A}: \\ (c,s_j) \text{ appears in } b_1,\cdots,b_M}} p_{N;a;b_1,\cdots,b_M}^{(s_i)} \cdot p_{n;b_1}^{(s_{l_1})} p_{n;b_2}^{(s_{l_2})} \cdots p_{n;b_M}^{(s_{l_M})},
\end{aligned}$$

for every product in the first line is counted no more than $|\mathcal{A}|$ times in the second sum of the second line. From (3.3), one may further derive

$$\begin{aligned}
p_{N+n;a}^{(s_i)} &\geq \frac{1}{|\mathcal{A}|} \sum_{c} \sum_{\substack{b_1,\cdots,b_M: \\ (c,s_j) \text{ appears in } b_1,\cdots,b_M}} p_{N;a;b_1,\cdots,b_M}^{(s_i)} \cdot p_{n;b_1}^{(s_{l_1})} p_{n;b_2}^{(s_{l_2})} \cdots p_{n;b_M}^{(s_{l_M})} \\
&\geq \frac{1}{|\mathcal{A}|} \sum_{c} 1 \cdot p_{n;c}^{(s_j)} e^{(\overline{h}-\epsilon)(-|\Delta_n^{(s_j)}|+\sum_{s_{l_m}} |\Delta_n^{(s_{l_m})}|)} \\
&= \frac{1}{|\mathcal{A}|} p_n^{(s_j)} e^{(\overline{h}-\epsilon)(-|\Delta_n^{(s_j)}|+\sum_{s_{l_m}} |\Delta_n^{(s_{l_m})}|)}.
\end{aligned}$$

It is inferred from the above inequality that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\log p_{N+n;a}^{(s_i)}}{|\Delta_{N+n}^{(s_i)}|} \\
&\geq \lim_{n\to\infty} \frac{\log p_n^{(s_j)}}{|\Delta_n^{(s_j)}|} \lim_{n\to\infty} \frac{|\Delta_n^{(s_j)}|}{|\Delta_{N+n}^{(s_i)}|} + (\overline{h}-\epsilon) \lim_{n\to\infty} \frac{-|\Delta_n^{(s_j)}| + \sum_{s_{l_m}} |\Delta_n^{(s_{l_m})}|}{|\Delta_{N+n}^{(s_i)}|} \\
&= h^{(s_j)} \cdot \lim_{n\to\infty} \frac{|\Delta_n^{(s_j)}|}{|\Delta_{N+n}^{(s_i)}|} + (\overline{h}-\epsilon) \cdot \lim_{n\to\infty} \frac{-|\Delta_n^{(s_j)}| + \sum_{s_{l_m}} |\Delta_n^{(s_{l_m})}|}{|\Delta_{N+n}^{(s_i)}|}.
\end{aligned}$$

It follows as a result that $h^{(s_j)} = \overline{h} = h^{(s_1)}$, since $\lim_{n\to\infty} \frac{\sum_{s_{l_m}} |\Delta_n^{(s_{l_m})}|}{|\Delta_{N+n}^{(s_i)}|} = 1$, $\lim_{n\to\infty} \frac{|\Delta_n^{(s_j)}|}{|\Delta_{N+n}^{(s_i)}|} > 0$ and righthand side of the inequality is a convex combination of $h^{(s_j)}$ and $\overline{h} - \epsilon$.

We are now ready to prove the proposition. Since it follows from the same argument in Proposition 2.2.2 that $\limsup_{n\to\infty} \frac{\log p_n}{|\Delta_n|} \leq h^{(s_1)}$, it is left to show that $\liminf_{n\to\infty} \frac{\log p_n}{|\Delta_n|} \geq h^{(s_1)}$. For every $a \in \mathcal{A}$, there exist $b_1, b_2, \ldots b_M \in \mathcal{A}$ such that $p_{1+n;a} \geq \prod_{l=1}^{M} p_{n;b_l}^{(s_l)} > 0$ for all $n \in \mathbb{N}$. It can be deduced from above that

$$\liminf_{n\to\infty} \frac{\log p_n}{|\Delta_n|} \geq \liminf_{n\to\infty} \frac{\log p_{n;a}}{|\Delta_n|} = \liminf_{n\to\infty} \frac{\log p_{1+n;a}}{|\Delta_{1+n}|} \geq \liminf_{n\to\infty} \sum_{l=1}^{k} \frac{\log p_{n;b_l}^{(s_l)}}{|\Delta_n^{(s_l)}|} \frac{|\Delta_n^{(s_l)}|}{|\Delta_{1+n}|} = h^{(s_1)}.$$

The proof is then finished. □

Corollary 3.2.5. *Suppose $X_{\mathbf{A}}$ is an NNMCTS and K is a primitive matrix. If $A_1 = \cdots = A_k = A$ is primitive, then the limit $\lim_{n\to\infty} \frac{\log p_n(X_{\mathbf{A}})}{|\Delta_n|}$ exists and equals $h^{(s_1)}(X_{\mathbf{A}})$.*

Proof. This follows immediately from Proposition 3.2.3 and Theorem 3.2.4. □

Finally, it is noteworthy that the assumption in Theorem 3.2.4 is finitely checkable in general.

Proposition 3.2.6. *Suppose $X_{\mathbf{A}}$ is an NNMCTS and K is a primitive matrix with $\mathbf{A}$ and K known. Suppose $G = (V, E)$ is the directed graph associated with $X_{\mathbf{A}}$. It is finitely checkable whether G admits a pivot and whether G is strongly connected.*

Proof. Since G is strongly connected if and only if the adjacency matrix A_G associated with G is irreducible, it is clearly finitely checkable. To see the admittance of pivot is also finitely checkable, we define the matrix B_n for all $n \in \mathbb{Z}_+$ as follows:

$$B_n((a,s_i),(b,s_j)) = \begin{cases} 1 & \text{if } A_G^n((a,s_i),(b,s_j)) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is then clear that G admits a pivot if and only if there exist $s_i, s_j \in \Sigma$, $a \in \mathcal{A}$, and $n \in \mathbb{Z}_+$ such that $B_n((a,s_i),(b,s_j)) = 1$ for all $b \in \mathcal{A}$. Since $|\{B_n\}_{n\geq 0}| \leq 2^{|V|^2}$ and B_n is eventually periodic, there exist $0 \leq N_1 \leq N_2 \leq 2^{|V|^2}$ such that $B_{N_1+n} = B_{N_2+n}$ for all $n \geq 0$. This completes the proof. □

3.3 An Attempt toward the Existence of Topological Entropy

This section presents an attempt toward the existence of topological entropy in general by exploiting the composition of colors on the boundary of all n-blocks. To be more specific, it is inspired by (3.2) that $p_{N+n}^{(s_i)}$ or p_{N+n} can be expressed as a linear combination of products of the form $\prod_{(a,s_i)} (p_{n;a}^{\prime(s_i)})^{\mathbf{v}_{(a,s_i)}}$, which leads to the following definition.

Definition 3.3.1. Suppose $X_{\mathbf{A}}$ is an NNMCTS.

1. For the product $\prod_{(a,s_i)} (p_{n;a}^{\prime(s_i)})^{\mathbf{v}_{(a,s_i)}}$ appearing in the calculation of entropy with

$\mathbf{v}_{(a,s_i)} \in \mathbb{Z}_+$, we denote by a vector $\mathbf{v} \in \mathbb{Z}_+^{|\mathcal{A}|\times|S_k|}$. Note that

$$W := \left\{ \sum_{\mathbf{v}\in\mathbb{Z}_+^{|\mathcal{A}|\times|S_k|}} r_{\mathbf{v}} \cdot \mathbf{v} : r_{\mathbf{v}} \in \mathbb{Z}, r_{\mathbf{v}} \neq 0 \text{ for finitely many } \mathbf{v} \in \mathbb{Z}_+^{|\mathcal{A}|\times|S_k|} \right\}$$

is a free vector space over $\mathbb{Z}_+^{|\mathcal{A}|\times|S_k|}$.

2. Define the *fundamental form homomorphism* $F : W \to W$ as

$$(F(\mathbf{v}))_{(a,s_i)} = \begin{cases} 1, & \text{if } \mathbf{v}_{(a,s_i)} > 0 \\ 0, & \text{if } \mathbf{v}_{(a,s_i)} = 0 \end{cases}.$$

3. Let $\mathbf{v} \in W$ and $F^* : W \to W$ be defined as

$$F^*\left(\sum_{\mathbf{v}} r_{\mathbf{v}} \cdot \mathbf{v}\right) = \sum_{\mathbf{v}} r'_{\mathbf{v}} \cdot F^*(\mathbf{v}),$$

where

$$r'_{\mathbf{v}} = \begin{cases} 1, & \text{if } r_{\mathbf{v}} > 0 \\ 0, & \text{if } r_{\mathbf{v}} = 0 \end{cases}.$$

$F^*(\mathbf{v})$ is called the *simplified fundamental form* of $\mathbf{v} \in W$.

4. Define the shift homomorphism $\sigma : W \to W$ by

$$\sigma(\mathbf{v}) = \sigma\left(\prod_{(a,s_i)} p'^{(s_i)\,\mathbf{v}_{(a,s_i)}}_{n;a}\right) = \prod_{(a,s_i)} \left(\sum_{b} \prod_{j:K(i,j)=1} A_i(a,b) p'^{(s_j)}_{b;n}\right)^{\mathbf{v}_{(a,s_i)}}$$

5. Suppose $x, y \in W$. We denote $x \succeq y$ if every term $\mathbf{v}$ appearing in $F^*(x)$ admits a term $\mathbf{w}$ appearing in $F^*(y)$ satisfying $\mathbf{v}_{(a,s_i)} \geq \mathbf{w}_{(a,s_i)}$ for every $a \in \mathcal{A}$ and every $s_i \in S_k$.

Remark 3.3.2. The purpose of the shift homomorphism is as follows. Let $n \in \mathbb{N}$ be fixed, and let $p^{(s_i)}_{N+n}, p^{(s_i)}_{N+1+n}$ be expressed as linear combinations $x, y \in W$ of products of the form $\prod_{(a,s_i)} (p'^{(s_i)}_{n;a})^{\mathbf{v}_{(a,s_i)}}$ as in (3.2). Then, $\sigma(p^{(s_i)}_{N+n}) = p^{(s_i)}_{N+n+1}$ as a number, which follows as a consequence of the process that extends every admissible pattern $u \in \mathcal{A}^{\Delta^{(s_i)}_N}$ to a admissible pattern $v \in \mathcal{A}^{\Delta^{(s_i)}_{N+1}}$ with $v|_{s(u)} = u$.

Proposition 3.3.3. *Suppose $X_{\mathbf{A}}$ is an NNMCTS and K is a primitive matrix. If there exist $N_1, N_2 \in \mathbb{N}$ and $s_i \in S_k$ such that $\sigma^{N_1}(p_n^{(s_i)}(X_{\mathbf{A}})) \succeq \sigma^{N_2}(p_n(X_{\mathbf{A}}))$, then $\lim_{n\to\infty} \frac{\log p_n(X_{\mathbf{A}})}{|\Delta_n|}$ exists and equals $h^{(s_1)}(X_{\mathbf{A}})$.*

Proof. As in Proposition 3.2.4, let $\{g_i\}_{i=1}^M = \{g \in \mathcal{T} : |g| = N_1\}$ be endowed with the lexicographic order and let $p_{N;a;b_1,\cdots,b_M}^{(s_i)}$ as in (3.1). Denote $x = \sigma^{N_1}(p_n^{(s_i)})$ and $y = \sigma^{N_2}(p_n)$. Since $x \succeq y$, every term $\mathbf{v}$ appearing in $F^*(x)$ admits a term $\phi(\mathbf{v})$ appearing in $F^*(y)$ satisfying $\mathbf{v}_{(a,s_i)} \geq \phi(\mathbf{v})_{(a,s_i)}$ for every $a \in \mathcal{A}$ and every $s_i \in S_k$. In this proof, we denote $[n, \mathbf{v}]$ for every $\mathbf{v} = \prod_{(a,s_i)} (p_{n;a}'^{(s_i)})^{\mathbf{v}_{(a,s_i)}} \in W$ for an emphasis on the size of the block.

Note that since $\lim_{n\to\infty} \frac{\log p_{N_1+n}^{(s_i)}}{|\Delta_{N_1+n}^{(s_i)}|} = h^{(s_1)}$ and the number of terms in x is constant for all n, there exists by the pigeonhole principle that

$$[n, \mathbf{v}_n] = (p_{n;a_1}'^{(s_{l_1})})^{\mathbf{v}_{(a_1,s_{l_1})}} \cdot \cdots \cdot (p_{n;a_M}'^{(s_{l_M})})^{\mathbf{v}_{(a_M,s_{l_M})}}$$

appearing in x such that $\lim_{n\to\infty} \frac{\log[n,\mathbf{v}_n]}{|\Delta_n^{(s_i)}|} = \lim_{n\to\infty} \frac{\log p_{N_1+n}^{(s_i)}}{|\Delta_{N_1+n}^{(s_i)}|} = h^{(s_1)}$. Hence, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{\log[n,\mathbf{v}_n]}{|\Delta_n^{(s_i)}|} > h^{(s_1)} - \epsilon$ for all $n \geq N$, and that $\frac{\log p_{n;b}'^{(s_j)}}{|\Delta_n^{(s_j)}|} < h^{(s_1)} + \epsilon$ for every $b \in \mathcal{A}$ and $s_j \in \Sigma$. These indicate that for every $(p_{n;a_m}'^{(s_{l_m})})^{\mathbf{v}_{(a_m,s_{l_m})}}$ in $[n, \mathbf{v}_n]$,

$$\begin{aligned}
&\frac{\log p_{n;a_m}'^{(s_{l_m})}}{|\bar{\Delta}_n^{(s_i)}|} \frac{([n,\mathbf{v}_n])_{(a_m,s_{l_m})}}{M} \\
\geq &\frac{\log[n,\mathbf{v}_n]}{|\bar{\Delta}_n^{(s_i)}|} - \sum_{j\neq m} \frac{\log p_{n;a_j}'^{(s_{l_j})}}{|\bar{\Delta}_n^{(s_i)}|} \frac{([n,\mathbf{v}_n])_{(a_j,s_{l_j})}}{M} \\
\geq &(h^{(s_1)} - \epsilon) - (h^{(s_1)} + \epsilon) \cdot \frac{M - ([n,\mathbf{v}_n])_{(a_m,s_{l_m})}}{M}
\end{aligned}$$

and thus

$$\frac{\log p_{n;a_m}'^{(s_{l_m})}}{|\bar{\Delta}_n^{(s_j)}|} \geq h^{(s_1)} - 2M\epsilon.$$

Now observe that

$$\frac{\log p_n}{|\Delta_n|} \geq \frac{\phi([n,\mathbf{v}_n])}{|\Delta_n|} \geq h^{(s_1)} - 2M\epsilon,$$

for all $n \geq N$. The proof is thus finished. □

Remark 3.3.4. Even though Proposition 3.3.3 seem to provide a criterion for the limit in the topological entropy to exist, there is no knowing whether this criterion is finitely checkable.

Chapter 4

Conclusion and Discussion

This thesis aims at investigating the existence of limit of topological entropy defined on Markov-Cayley tree-shifts. Based on Theorem 2.2.1 proved in [14], we are empowered to show herein that three types of nearest-neighbor Markov-Cayley tree-shifts have the topological entropy exist in limit, and that the topological entropy of these types coincides with their stem entropy. To this end, the methods adopted in this thesis fall in the two genres discussed in Chapter 2 and Chapter 3, respectively.

In Chapter 2, two types of Markov-Cayley tree-shifts are studied by a straightforward estimation of the topological entropy, in which we exploit either the structure of the Markov-Cayley tree (Theorem 2.2.2), the symmetry of the rules (Theorem 2.2.5, 2.2.6) or the number of symbols in the alphabet (Theorem 2.2.7). In Chapter 3, we partially generalize the idea of mixing property on Markov tree-shifts to that on Markov-Cayley tree-shifts. With the property, we are enabled to partially generalize [9, Proposition 3.1] and prove the coincidence of the stem entropy and the topological entropy, and techniques developed for this purpose is adapted to prove the existence of topological entropy in limit under special circumstances.

Despite the attempt of this thesis, the problem of the existence of topological entropy in limit is far from being solved. The following problems naturally arise and are still left unanswered.

1. Suppose $X_{\mathbf{A}}$ is a Markov-Cayley tree-shift over a Markov-Cayley tree generated by Σ, where K is primitive. Does the topological entropy of $X_{\mathbf{A}}$ exist in limit? Does the topological entropy coincide with the stem entropy in general?

2. It is shown in Theorem 3.2.4 that for every $a, b \in \mathcal{A}$ and every $s_i, s_j \in \Sigma$,

$$\liminf_{n\to\infty} \frac{\log p_{n;a}^{(s_i)}}{|\Delta_n^{(s_i)}|} = \liminf_{n\to\infty} \frac{\log p_{n;b}^{(s_j)}}{|\Delta_n^{(s_j)}|}$$

if the graph representation of $X_{\mathbf{A}}$ is strongly connected. Does the equality also hold for limit inferior, as pointed out in [9, Proposition 3.1]?

Chapter 5

Reference

[1] N. Aubrun and M. P. Béal, *Decidability of conjugacy of tree-shifts of finite type*, Lect. Notes Comput. Sci. (including Subser. Lect. Notes Artif. Intell. Lect. Notes Bioinformatics), vol. 5555 LNCS, 2009, pp. 132–143.

[2] J.-C. Ban and C.-H. Chang, *Tree-shifts: the entropy of tree-shifts of finite type*, Nonlinearity **30** (2017), no. 7, 2785–2804.

[3] ———, *Complexity of neural networks on Fibonacci-Cayley tree*, J. Algebr. Comb. Discret. Struct. Appl. **6** (2019), no. 2, 105–122.

[4] J.-C. Ban, C.-H. Chang, and Y.-H. Huang, *Complexity of shift spaces on semigroups*, J. Algebr. Comb. (2020).

[5] T. Ceccherini-Silberstein and M. Coornaert, *Cellular Automata and Groups*, Springer Monographs in Mathematics, Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[6] S.-N. Chow and J. Mallet-Paret, *Pattern Formation and Spatial Chaos in Lattice Dynamical Systems—Part I*, IEEE Trans. Circuits Syst. I Fundam. Theory Appl. **42** (1995), no. 10, 746–751.

[7] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, nov 1995.

[8] J. Mallet-Paret and S.-N. Chow, *Pattern Formation and Spatial Chaos in Lattice Dynamical Systems—Part II*, IEEE Trans. Circuits Syst. I Fundam. Theory Appl. **42** (1995), no. 10, 752–756.

[9] K. Petersen and I. Salama, *Tree shift topological entropy*, Theor. Comput. Sci. **743** (2018), 64–71.

[10] ———, *Entropy on regular trees*, Discret. Contin. Dyn. Syst. Ser. A **40** (2020), no. 7, 4453–4477.

[11] S. T. Piantadosi, *Symbolic dynamics on free groups*, Discret. Contin. Dyn. Syst. **20** (2008), no. 3, 725–738.

[12] C. E. Shannon, *A Mathematical Theory of Communication*, Bell Syst. Tech. J. **27** (1948), no. 3, 379–423.

[13] ———, *A Mathematical Theory of Communication*, Bell Syst. Tech. J. **27** (1948), no. 4, 623–656.

[14] Y.-Y. Wu, *On the existence of the stem entropy on Markov-Cayley tree*, Mater thesis (2021), National Central University, Taoyuan City, Taiwan.

Appendix A

Computation of Stem Entropy

A.1 Computation of Stem Entropy

In this section, we provide the pseudo codes for computation of topological entropy of nearest-neighbor tree-shift and stem entropy of NNMCTS, which are shown in Algorithm 1 and Algorithm 2, respectively. For the the discussion of the algorithms, we denote by $\odot$ the entrywise product of vectors.

Remark A.1.1. The idea behind Algorithm 1 is given as follows. Following the notation in Section 2.1, we denote by $\mathbf{p}_n$ the number of admissible patterns as

$$\mathbf{p}_n = (p_{n;1}, p_{n;2}, \cdots, p_{n;k})^T.$$

Then, it is shown (see for example [2]) that the growth rate of is governed by the system of recursive equations

$$\begin{cases} \mathbf{p}_n = f(\mathbf{p}_{n-1}) := (A_1 \mathbf{p}_{n-1}) \odot \cdots \odot (A_k \mathbf{p}_{n-1}) \\ \mathbf{p}_0 = [1, 1, \cdots, 1]^T. \end{cases}$$

However, due to the size required to limited resources of computer, computing entropy directly from the above formula is impractical, which prompts us to consider a normalized system defined as follows. Let $\{r_n > 0 : n \geq 0\}$ be a given sequence of positive real numbers.

$$\begin{cases} \bar{\mathbf{p}}_n = g(\bar{\mathbf{p}}_{n-1}) := \frac{f(\bar{\mathbf{p}}_{n-1})}{r_n}; \\ \bar{\mathbf{p}}_0 = [1/r_0, 1/r_0, \cdots, 1/r_0]^T. \end{cases}$$

It is noteworthy that the following equality holds:

$$\begin{aligned} \mathbf{p}_n &= f^n(\mathbf{p}_0) \\ &= g^n(\bar{\mathbf{p}}_0) \cdot r(0)^{k^n} r(1)^{k^{n-1}} \cdots r(n)^{k^0} \\ &= \bar{\mathbf{p}}_n \cdot r(0)^{k^n} r(1)^{k^{n-1}} \cdots r(i)^{k^0}. \end{aligned}$$

If r_n is chosen to be the maximal element in $\mathbf{p}_n$ in Algorithm 1, the maximal element in $\bar{\mathbf{p}}(i)$ is 1 and thus

$$t_n = \log \max_a p_{n;a} = k^n \log r_0 + k^{n-1} \log r_1 + \cdots + k^0 \log r_n. \tag{A.1}$$

In fact, if r_n is defined as in the algorithm, then r_n is a rational number and

$$h(X_{\mathbf{A}}) = \lim_{n\to\infty} \frac{\log \max_a p_{n;a}}{k^{n+1}/k-1} = \sum_{i=0}^{\infty} \log r_i \cdot \frac{k-1}{k^{i+1}}. \tag{A.2}$$

In particular, if $X_{(A,A,\cdots,A)}$ is a nearest-neighbor tree-shift with A an essential matrix, i.e., for every $b \in \mathcal{A}$ there exists $a \in \mathcal{A}$ satisfying $A(a,b) = 1$, then $|\mathcal{A}| \geq r_n \geq 1$ for all $n \geq 0$ and

$$\sum_{i=0}^{N} \log r_i \cdot \frac{k-1}{k^{i+1}} \leq h(X_{\mathbf{A}}) \leq \sum_{i=0}^{N} \log r_n \cdot \frac{k-1}{k^{i+1}} + \sum_{i=N+1}^{\infty} k \log|\mathcal{A}| \cdot \frac{k-1}{k^{i+1}}. \tag{A.3}$$

Remark A.1.2. As an generalization of Algorithm 1, the number of blocks of an NNM-CTS must satisfy the following recursive system in general.

$$\begin{cases} \mathbf{p}_n^{(s_j)} = (A_1 \mathbf{p}_{n-1}^{(s_1)})^{K(s_j,s_1)} \odot \cdots \odot (A_k \mathbf{p}_{n-1}^{(s_k)})^{K(s_j,s_k)} \\ \mathbf{p}_0^{(s_j)} = [1, 1, \cdots, 1]^T. \end{cases}$$

In the same manner, given any positive sequence of $\{r_n^{(s_j)} : n \geq 0, s_j \in \Sigma_K\}$, one may define the normalized system as

$$\begin{cases} \bar{\mathbf{p}}_n^{(s_j)} = (A_1\bar{\mathbf{p}}_{n-1}^{(s_1)})^{K(s_j,s_1)} \odot \cdots \odot (A_k\bar{\mathbf{p}}_{n-1}^{(s_k)})^{K(s_j,s_k)}/r_n^{(s_j)} \\ \bar{\mathbf{p}}_0^{(s_j)} = \left[1/r_0^{(s_j)}, 1/r_0^{(s_j)}, \cdots, 1/r_0^{(s_j)}\right]^T. \end{cases}$$

By writing $\mathbf{f} = (f_1, f_2, \cdots, f_k)$ and $\mathbf{g} = (g_1, g_2, \cdots, g_k)$ the map $\mathbf{p}_{n-1} \overset{\mathbf{f}}{\mapsto} \mathbf{p}_n$ and the map $\bar{\mathbf{p}}_{n-1} \overset{\mathbf{g}}{\mapsto} \bar{\mathbf{p}}_n$ respectively,

$$\mathbf{p}_n^{(s_j)} = f_j(\mathbf{f}^{n-1}(\mathbf{p}_0^{(s_1)}, \mathbf{p}_0^{(s_2)}, \cdots, \mathbf{p}_0^{(s_k)})) \tag{A.4}$$

$$= g_j\left(\exp(t_{n-1}^{(s_1)}) \cdot \bar{\mathbf{p}}_{n-1}^{(s_1)}, \exp(t_{n-1}^{(s_2)}) \cdot \bar{\mathbf{p}}_{n-1}^{(s_2)}, \cdots, \exp(t_{n-1}^{(s_k)}) \cdot \bar{\mathbf{p}}_{n-1}^{(s_k)}\right) \cdot r_n^{(s_j)} \tag{A.5}$$

$$= \bar{\mathbf{p}}_n^{(s_j)} \cdot \exp(t_n^{(s_j)}), \tag{A.6}$$

where $\max \bar{\mathbf{p}}_n^{(s_j)} = 1$ and

$$t_n^{(s_j)} = \log \max_a p_{n;a}^{(s_j)}. \tag{A.7}$$

Hence,

$$\frac{t_n^{(s_j)}}{|\Delta_n^{(s_j)}|} = \frac{\log \max_a p_{n;a}^{(s_j)}}{|\Delta_n^{(s_j)}|},$$

which tends to the stem entropy of $X_{\mathbf{A}}$.

A.2 Algorithms

input : $\mathbf{A} = (A_1, A_2, \cdots, A_k)$: k binary matrices of dimension k.
$iter$: maximum of iterations in execution
ϵ: threshold for convergence
output: h: approximation of entropy, where $h_n := \frac{\log \max_a p_{n;a}}{|\Delta_n|}$.

Function `normalized_tree_entropy(`$\mathbf{A}$,$iter$,ϵ`)`
  $\bar{\mathbf{p}}_0 = [\bar{p}_{0;1}, \bar{p}_{0;2}, \cdots, \bar{p}_{0;k}]^T = [1, 1, \cdots, 1]^T$;
  $r_0 \leftarrow 1$;
  $t_0 \leftarrow \log r_0$;
  $h_0 \leftarrow s_0/|\Delta_0|$;
  for $n \in \{1, 2, \cdots, iter - 1\}$ **do**
    $\bar{\mathbf{p}}_n = [\bar{p}_{n;1}, \bar{p}_{n;2}, \cdots, \bar{p}_{n;k}]^T \leftarrow (A_1 \bar{\mathbf{p}}_{n-1}) \odot \cdots \odot (A_k \bar{\mathbf{p}}_{n-1})$;
    $r_n \leftarrow \max_a \bar{p}_{n;a}$;
    $\bar{\mathbf{p}}_n \leftarrow \bar{\mathbf{p}}_n / r_n$;
    $t_n \leftarrow k \cdot t_{n-1} + \log r_n$;
    $h_n = t_n/|\Delta_n|$;
    if $|h_n - h_{n-1}| < h_{n-1} \cdot \epsilon$ *or* $h_n < \epsilon$ **then**
      break;
    end
  end
  return h
end

Figure 1: Topological Entropy of nearest-neighbor hom tree-shift

input : K: binary matrix of dimension k.
$\mathbf{A} = (A_1, A_2, \cdots, A_k)$: k binary matrices of dimension k.
$iter$: maximum of iterations in execution
ϵ: threshold for convergence

output: $h^{(s_j)}$: approximation of entropy, where $h_n^{(s_j)} := \frac{\log \max_a p_{n;a}^{(s_j)}}{|\Delta_n|}$.

Function `normalized_mctree_entropy(`$\mathbf{A}$,$iter$,ϵ`)`

 for $j \in \{1, 2, \cdots, k\}$ **do**

  $\bar{\mathbf{p}}_0^{(s_j)} = \left[\bar{p}_{0;1}^{(s_j)}, \bar{p}_{0;2}^{(s_j)}, \cdots, \bar{p}_{0;k}^{(s_j)}\right]^T \leftarrow [1, 1, \cdots, 1]^T$;

  $r_0^{(s_j)} \leftarrow 1$;

  $t_0^{(s_j)} = \log r_0^{(s_j)}$;

  $h_0^{(s_j)} = s_0^{(s_j)} / |\Delta_0^{(s_j)}|$;

 end

 for $n \in \{1, 2, \cdots, iter - 1\}$ **do**

  for $j \in \{1, 2, \cdots, k\}$ **do**

   $\bar{\mathbf{p}}_n^{(s_j)} = \left[\bar{p}_{n;1}^{(s_j)}, \bar{p}_{n;2}^{(s_j)}, \cdots, \bar{p}_{n;k}^{(s_j)}\right]^T \leftarrow$
   $(A_1 \mathbf{p}_{n-1}^{(s_1)})^{K(s_j, s_1)} \odot \cdots \odot (A_k \mathbf{p}_{n-1}^{(s_k)})^{K(s_j, s_k)}$;

   $r_n^{(s_j)} \leftarrow \max_a \bar{p}_{n;a}^{(s_j)}$;

   $\bar{\mathbf{p}}_n^{(s_j)} = \left[\bar{p}_{n;1}^{(s_j)}, \bar{p}_{n;2}^{(s_j)}, \cdots, \bar{p}_{n;k}^{(s_j)}\right]^T \leftarrow \bar{\mathbf{p}}_n^{(s_j)} / r_n^{(s_j)}$;

   $t_n^{(s_j)} = K(s_j, :) \cdot \left[t_{n-1}^{(s_1)}, \cdots, t_{n-1}^{(s_k)}\right] + \log r_n^{(s_j)}$;

   $h_n^{(s_j)} = t_n^{(s_j)} / |\Delta_n^{(s_j)}|$;

  end

  if $\sum_{j=1}^{k} |h_n^{(s_j)} - h_{n-1}^{(s_j)}| < \sum_{j=1}^{k} h_{n-1}^{(s_j)} \cdot \epsilon$ *or* $h_n^{(s_j)} < \epsilon$ **then**

   break;

  end

 end

 return $\left(h^{(s_1)}; h^{(s_k)}; \cdots; h^{(s_k)}\right)$

end

Figure 2: Stem entropy of nearest-neighbor Markov-Cayley tree-shift

A.3 Tables

The experiments in the following are conducted with mpmath library of python with the following configuration: dps=5000,$\epsilon = 10^{-50}$.

A_1	A_2	stem entropy	topological entropy	iteration
$[0,1;1,1]$	$[1,1;1,0]$	0.1261881372008	0.1261881372008	37
$[1,1;1,0]$	$[1,1;1,0]$	0.2332621211030	0.2332621211030	34
$[0,1,0;1,0,1;0,1,0]$	$[0,1,1;1,0,0;0,1,1]$	0.1681464340595	0.1681464340595	36

Table 1: Numerical experiments on the stem entropy of X_{A_1,A_2,A_1^T,A_2^T} over the free group (where log is computed with base 10).

A_1	A_2	stem entropy	topological entropy	iteration
$[1,1;1,0]$	$[1,1;1,0]$	0.2178219813166	0.2178219813166	82
$[0,1;1,1]$	$[0,1;1,1]$	0.2178219813166	0.2178219813166	82
$[0,1;1,1]$	$[1,1;1,0]$	0.1267559612313	0.1267559612313	73
$[1,1;1,0]$	$[0,1;1,1]$	0.1267559612313	0.1267559612313	73

Table 2: Numerical experiments on the stem entropy of X_{A_1,A_2} over Fibonacci-Cayley tree generate by Σ_K, where $K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (where log is computed with base 10).