第 10 章 動態規劃

題號 10.3-2

11.3-2.

Let x_n be the number of crates allocated to store n, $p_n(x_n)$ be the expected profit from allocating x_n to store n and s_n be the number of crates remaining to be allocated to stores $k \geq n$. Then $f_n^*(s_n) = \max_{0 \leq x_n \leq s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)]$. Number of stages: 3

S3	f3*(S3)	X3*
0	0	0
1	4	1
2	9	2
3	13	3
4	18	4
5	20	5

S2 \ X2	f2(S2, X2): 0	1	2	3	4	5	f2*(S2)	X2*
0	0	---	---	---	---	---	0	0
1	4	6	---	---	---	---	6	1
2	9	10	11	---	---	---	11	2
3	13	15	15	15	---	---	15	1,2,3
4	18	19	20	19	19	---	20	2
5	20	24	24	24	23	22	24	1,2,3

S1 \ X1	f1(S1, X1): 0	1	2	3	4	5	f1*(S1)	X1*
5	24	25	24	25	23	21	25	1,3

Optimal solution	X1*	X2*	X3*
1	1	2	2
2	3	2	0

題號 10.3-3

11.3-3.

Let x_n be the number of study days allocated to course n, $p_n(x_n)$ be the number of grade points expected when x_n days are allocated to course n and s_n be the number of study days remaining to be allocated to courses $k \geq n$. Then

$$f_n^*(s_n) = \max_{1 \leq x_n \leq \min(s_n, 4)} [p_n(x_n) + f_{n+1}^*(s_n - x_n)].$$

Number of stages: 4

s_4	$f_4^*(s_4)$	x_4^*
1	6	1
2	7	2
3	9	3
4	9	4

	$f_3(s_3, x_3)$					
s_3	1	2	3	4	$f_3^*(s_3)$	x_3^*
2	8	—	—	—	8	1
3	9	10	—	—	13	2
4	11	11	13	—	13	3
5	11	13	14	14	14	3, 4

	$f_2(s_2, x_2)$					
s_2	1	2	3	4	$f_2^*(s_2)$	x_2^*
3	13	—	—	—	13	1
4	15	13	—	—	15	1
5	18	15	14	—	18	1
6	19	18	16	17	19	1

	$f_1(s_1, x_1)$					
s_1	1	2	3	4	$f_1^*(s_1)$	x_1^*
7	22	23	21	20	23	2

Optimal Solution	x_1^*	x_2^*	x_3^*	x_4^*
1	2	1	3	1

題號 10.3-8

11.3-8.

Let x_n be the number of parallel units of component n that are installed, $p_n(x_n)$ be the probability that the component will function if it contains x_n parallel units, $c_n(x_n)$ be the cost of installing x_n units of component n, s_n be the amount of money remaining in hundreds of dollars. Then

$$f_n^*(s_n) = \max_{x_n=0,\ldots,\min(3,\alpha_{s_n})} [p_n(x_n) f_{n+1}^*(s_n - c_n(x_n))]$$

where $\alpha_{s_n} \equiv \max\{\alpha : c_n(\alpha) \le s_n, \alpha \text{ integer}\}$.

s_4	$f_4^*(s_4)$	x_4^*
$0, 1$	0	0
2	0.5	1
3	0.7	2
$4 \le s_4 \le 10$	0.9	3

$f_3(s_3, x_3) = \mathrm{P}_3(x_3) f_4^*(s_3 - c_3(x_3))$

	$f_3(s_3, x_3)$					
s_3	0	1	2	3	$f_3^*(s_3)$	x_3^*
0	0	—	—	—	0	0
$1, 2$	0	0	—	—	0	$0, 1$
3	0	0.35	0	—	0.35	1
4	0	0.49	0	0	0.49	1
5	0	0.63	0.40	0	0.63	1
6	0	0.63	0.56	0.45	0.63	1
7	0	0.63	0.72	0.63	0.72	2
$8 \le s_3 \le 10$	0	0.63	0.72	0.81	0.81	3

$f_2(s_2, x_2) = \mathrm{P}_2(x_2) f_3^*(s_2 - c_2(x_2))$

	$f_2(s_2, x_2)$					
s_2	0	1	2	3	$f_2^*(s_2)$	x_2^*
$0, 1$	0	—	—	—	0	0
$2, 3$	0	0	—	—	0	$0, 1$
4	0	0	0	—	0	$0, 1, 2$
5	0	0.210	0	0	0.210	1
6	0	0.294	0	0	0.294	1
7	0	0.378	0.245	0	0.378	1
8	0	0.378	0.343	0.280	0.378	1
9	0	0.432	0.441	0.392	0.441	2
10	0	0.486	0.441	0.504	0.504	3

$f_1(s_1, x_1) = \mathrm{P}_1(x_1) f_2^*(s_1 - c_1(x_1))$

	$f_1(s_1, x_1)$					
s_1	0	1	2	3	$f_1^*(s_1)$	x_1^*
10	0	0.22	0.227	0.302	0.302	3

The optimal solution is $x_1^* = 3$, $x_2^* = 1$, $x_3^* = 1$ and $x_4^* = 3$, yielding a system reliability of 0.3024.

題號 10.4-1

11.4-1.

Let s_n be the current fortune of the player, A be the event to have \$100 at the end and X_n be the amount bet at the nth match.

$$f_3^*(s_3) = \max_{0 \leq x_3 \leq s_3} \{\mathrm{P}\{A|s_3\}\}$$

$0 \leq s_3 < 50,\ f_3^*(s_3) = 0.$

$$50 \leq s_3 < 100,\ f_3^*(s_3) = \begin{cases} 0 & \text{if } x_3^* \neq 100 - s_3 \\ 1/2 & \text{if } x_3^* = 100 - s_3 \end{cases}$$

$$s_3 = 100,\ f_3^*(s_3) = \begin{cases} 0 & \text{if } x_3^* > 0 \\ 1 & \text{if } x_3^* = 0 \end{cases}$$

$$s_3 > 100,\ f_3^*(s_3) = \begin{cases} 0 & \text{if } x_3^* \neq s_3 - 100 \\ 1/2 & \text{if } x_3^* = s_3 - 100 \end{cases}$$

s_3	$f_3^*(s_3)$	x_3^*
$0 \leq s_3 < 50$	$\mathbf{0}$	$0 \leq x_3^* \leq 50$
$50 \leq s_3 < 100$	$\mathbf{1/2}$	$100 - s_3$
$s_3 = 100$	$\mathbf{1}$	0
$100 < s_3$	$\mathbf{1/2}$	$s_3 - 100$

$$f_2^*(s_2) = \max_{0 \leq x_2 \leq s_2} \left[\tfrac{1}{2} f_3^*(s_2 - x_2) + \tfrac{1}{2} f_3^*(s_2 + x_2)\right]$$

s_2	$f_2^*(s_2)$	x_2^*
$0 \leq s_2 < 25$	$\mathbf{0}$	$0 \leq x_2 \leq s_2$
$25 \leq s_2 < 50$	0	$0 \leq x_2 \leq 50 - s_2$
	$\mathbf{1/4}$	$50 - s_2 \leq x_2 \leq s_2$
$s_2 = 50$	$1/4$	$0 \leq x_2 < 50$
	$\mathbf{1/2}$	$x_2 = 50$
$50 < s_2 < 75$	$\mathbf{1/2}$	$0 \leq x_2 < s_2 - 50$
	$1/4$	$s_2 - 50 < x_2 < 100 - s_2$
	$\mathbf{1/2}$	$x_2 = 100 - s_2$
	$1/4$	$100 - s_2 < x_2 \leq s_2$
$s_2 = 75$	$1/2$	$0 \leq x_2 < 25$
	$\mathbf{3/4}$	$x_2 = 25$
	$1/4$	$25 \leq x_2 \leq 75$
$75 < s_2 < 100$	$1/2$	$0 \leq x_2 < 100 - s_2$
	$\mathbf{3/4}$	$x_2 = 100 - s_2$
	$1/2$	$100 - s_2 < x_2 \leq s_2 - 50$
	$1/4$	$s_2 - 50 < x_2 \leq s_2$
$s_2 = 100$	$\mathbf{1}$	$x_2 = 0$
	$1/2$	$0 < x_2 \leq 50$
	$1/4$	$50 \leq x_2 \leq 100$
$100 < s_2$	$1/2$	$0 \leq x_2 \leq s_2 - 100$
	$\mathbf{3/4}$	$x_2 = s_2 - 100$
	$1/2$	$s_2 - 100 < x_2 \leq s_2 - 50$
	$1/4$	$s_2 - 50 < x_2 \leq s_2$

The entries in bold represent the maximum value in each case.

$$f_1^*(75) = \max_{0 \leq x_1 \leq 75} \left[\tfrac{1}{2} f_2^*(75 - x_1) + \tfrac{1}{2} f_2^*(75 + x_1)\right]$$

$$f_1(75, x_1) = \begin{cases} 3/4 & \text{if } x_1 = 0 \\ 5/8 & \text{if } 0 < x_1 < 25 \\ 3/4 & \text{if } x_1 = 25 \\ 1/2 & \text{if } 25 < x_1 \leq 50 \\ 3/8 & \text{if } 50 < x_1 \leq 75 \end{cases}$$

s_1	$f_1^*(s_1)$	x_1^*
75	3/4	0 or 25

Policy	x_1	won 1st bet	lost 1st bet	won 2nd bet	lost 2nd bet
1	0	25	25	0	50
2	25	0	50	0	0

題號 10.4-3

11.4-3.

$$f_n(1, x_n) = K(x_n) + x_n + \left(\frac{1}{3}\right)^{x_n} f^*_{n+1}(1) + \left[1 - \left(\frac{1}{3}\right)^{x_n}\right] f^*_{n+1}(0)$$
$$= K(x_n) + x_n + \left(\frac{1}{3}\right)^{x_n} f^*_{n+1}(1)$$

since $f^*_n(0) = 0$ for every n. $f^*_3(1) = 16$, $f^*_3(0) = 0$ and $K(x_n) = 0$ if $x_n = 0$, $K(x_n) = 3$ if $x_n > 0$.

	$f_2(s_2, x_2)$						
s_2	0	1	2	3	4	$f^*_2(s_2)$	x^*_2
0	0	–	–	–	–	0	0
1	16	9.33	6.78	6.59	7.20	6.59	3

	$f_1(s_1, x_1)$						
s_1	0	1	2	3	4	$f^*_1(s_1)$	x^*_1
1	6.59	6.20	5.73	6.24	7.08	5.73	2

The optimal policy is to produce two in the first run and to produce three in the second run if none of the items produced in the first run is acceptable. The minimum expected cost is $573.

題號 10.4-4

11.4-4.

$$f_n^*(s_n) = \max_{x_n \geq 0}\left\{\tfrac{1}{3} f_{n+1}^*(s_n - x_n) + \tfrac{2}{3} f_{n+1}^*(s_n + x_n)\right\},$$

with $f_6^*(s_6) = 0$ for $s_6 < 5$ and $f_6^*(s_6) = 1$ for $s_6 \geq 5$.

s_5	$f_5^*(s_5)$	x_5^*
0	0	0
1	0	0
2	0	0
3	$2/3$	$x_5^* \geq 2$
4	$2/3$	$x_5^* \geq 1$
$s_5 \geq 5$	1	$x_5^* \leq s_5 - 5$

	$f_4(s_4, x_4)$						
s_4	0	1	2	3	4	$f_4^*(s_4)$	x_4^*
0	0	–	–	–	–	0	0
1	0	0	–	–	–	0	0
2	0	$4/9$	$4/9$	–	–	$4/9$	$1, 2$
3	$2/3$	$4/9$	$2/3$	$2/3$	–	$2/3$	$0.2, 3$
4	$2/3$	$8/9$	$2/3$	$2/3$	$2/3$	$8/9$	1
$s_4 \geq 5$	1	–	–	–	–	1	$x_4^* \leq s_4 - 5$

	$f_3(s_3, x_3)$						
s_3	0	1	2	3	4	$f_3^*(s_3)$	x_3^*
0	0	–	–	–	–	0	0
1	0	$8/27$	–	–	–	$8/27$	1
2	$4/9$	$4/9$	$16/27$	–	–	$16/27$	2
3	$2/3$	$20/27$	$2/3$	$2/3$	–	$20/27$	1
4	$8/9$	$8/9$	$22/27$	$2/3$	$2/3$	$22/27$	$0, 1$
$s_3 \geq 5$	1	–	–	–	–	1	$x_3^* \leq s_3 - 5$

	$f_2(s_2, x_2)$						
s_2	0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	–	–	–	–	0	0
1	$8/27$	$32/81$	–	–	–	$32/81$	1
2	$16/27$	$48/81$	$48/81$	–	–	$48/81$	$0, 1, 2$
3	$20/27$	$64/81$	$62/81$	$2/3$	–	$64/81$	1
4	$24/27$	$74/81$	$70/81$	$62/81$	$2/3$	$74/81$	1
$s_2 \geq 5$	1	–	–	–	–	1	$x_2^* \leq s_2 - 5$

	$f_1(s_1, x_1)$				
s_1	0	1	2	$f_1^*(s_1)$	x_1^*
2	$48/81$	$160/243$	$124/243$	$160/243$	1

The probability of winning the bet using the policy given above is $160/243 = 0.658$.