第 10 章 動態規劃

題號 10.3-2

11.3-2.

Let x_n be the number of crates allocated to store n, $p_n(x_n)$ be the expected profit from allocating x_n to store n and s_n be the number of crates remaining to be allocated to stores $k \geq n$. Then $f_n^*(s_n) = \max_{0 \leq x_n \leq s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)]$. Number of stages: 3

S3	f3*(S3)	X3*
0	0	0
1	4	1
2	9	2
3	13	3
4	18	4
5	20	5

\ X2 \ S2\	f2(S2, X2) 0	1	2	3	4	5	f2*(S2)	X2*
0	0	---	---	---	---	---	0	0
1	4	6	---	---	---	---	6	1
2	9	10	11	---	---	---	11	2
3	13	15	15	15	---	---	15	1,2,3
4	18	19	20	19	19	---	20	2
5	20	24	24	24	23	22	24	1,2,3

\ X1 \ S1\	f1(S1, X1) 0	1	2	3	4	5	f1*(S1)	X1*
5	24	25	24	25	23	21	25	1,3

Optimal solution	X1*	X2*	X3*
1	1	2	2
2	3	2	0

題號 10.3-3

11.3-3.

Let x_n be the number of study days allocated to course n, $p_n(x_n)$ be the number of grade points expected when x_n days are allocated to course n and s_n be the number of study days remaining to be allocated to courses $k \geq n$. Then

$$f_n^*(s_n) = \max_{1 \leq x_n \leq \min(s_n, 4)} [p_n(x_n) + f_{n+1}^*(s_n - x_n)].$$

Number of stages: 4

s_4	$f_4^*(s_4)$	x_4^*
1	6	1
2	7	2
3	9	3
4	9	4

	$f_3(s_3, x_3)$					
s_3	1	2	3	4	$f_3^*(s_3)$	x_3^*
2	8	—	—	—	8	1
3	9	10	—	—	13	2
4	11	11	13	—	13	3
5	11	13	14	14	14	3, 4

	$f_2(s_2, x_2)$					
s_2	1	2	3	4	$f_2^*(s_2)$	x_2^*
3	13	—	—	—	13	1
4	15	13	—	—	15	1
5	18	15	14	—	18	1
6	19	18	16	17	19	1

	$f_1(s_1, x_1)$					
s_1	1	2	3	4	$f_1^*(s_1)$	x_1^*
7	22	23	21	20	23	2

Optimal Solution	x_1^*	x_2^*	x_3^*	x_4^*
1	2	1	3	1

題號 10.3-5

11.3-5.

Let x_n be the number of workers allocated to precinct n, $p_n(x_n)$ be the increase in the number of votes if x_n workers are assigned to precinct n and s_n be the number of workers remaining at stage n. Then

$$f_n^*(s_n) = \max_{0 \le x_n \le s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)].$$

Number of stages: 4

S4	f4*(S4)	X4*
0	0	0
1	6	1
2	11	2
3	14	3
4	15	4
5	17	5
6	18	6

S3 \ X3	f3(S3, X3) 0	1	2	3	4	5	6	f3*(S3)	X3*
0	0	---	---	---	---	---	---	0	0
1	6	5	---	---	---	---	---	6	0
2	11	11	10	---	---	---	---	11	0,1
3	14	16	16	15	---	---	---	16	1,2
4	16	19	21	21	18	---	---	21	2,3
5	17	21	24	26	24	21	---	26	3
6	18	22	26	29	29	27	22	29	3,4

S2 \ X2	f2(S2, X2) 0	1	2	3	4	5	6	f2*(S2)	X2*
0	0	---	---	---	---	---	---	0	0
1	6	7	---	---	---	---	---	7	1
2	11	13	11	---	---	---	---	13	1
3	16	18	17	16	---	---	---	18	1
4	21	23	22	22	18	---	---	23	1
5	26	28	27	27	24	20	---	28	1
6	29	33	32	32	29	26	21	33	1

S1 \ X1	f1(S1, X1) 0	1	2	3	4	5	6	f1*(S1)	X1*
6	33	32	32	33	31	29	24	33	0,3

Optimal solution	X1*	X2*	X3*	X4*
1	0	1	3	2
2	3	1	0	2
3	3	1	1	1

題號 10.3-8

11.3-8.

Let x_n be the number of parallel units of component n that are installed, $p_n(x_n)$ be the probability that the component will function if it contains x_n parallel units, $c_n(x_n)$ be the cost of installing x_n units of component n, s_n be the amount of money remaining in hundreds of dollars. Then

$$f_n^*(s_n) = \max_{x_n=0,\ldots,\min(3,\alpha_{s_n})} [p_n(x_n) f_{n+1}^*(s_n - c_n(x_n))]$$

where $\alpha_{s_n} \equiv \max\{\alpha : c_n(\alpha) \leq s_n, \alpha \text{ integer}\}$.

s_4	$f_4^*(s_4)$	x_4^*
$0, 1$	0	0
2	0.5	1
3	0.7	2
$4 \leq s_4 \leq 10$	0.9	3

$f_3(s_3, x_3) = \mathrm{P}_3(x_3) f_4^*(s_3 - c_3(x_3))$

	$f_3(s_3, x_3)$					
s_3	0	1	2	3	$f_3^*(s_3)$	x_3^*
0	0	—	—	—	0	0
$1, 2$	0	0	—	—	0	$0, 1$
3	0	0.35	0	—	0.35	1
4	0	0.49	0	0	0.49	1
5	0	0.63	0.40	0	0.63	1
6	0	0.63	0.56	0.45	0.63	1
7	0	0.63	0.72	0.63	0.72	2
$8 \leq s_3 \leq 10$	0	0.63	0.72	0.81	0.81	3

$f_2(s_2, x_2) = \mathrm{P}_2(x_2) f_3^*(s_2 - c_2(x_2))$

	$f_2(s_2, x_2)$					
s_2	0	1	2	3	$f_2^*(s_2)$	x_2^*
$0, 1$	0	—	—	—	0	0
$2, 3$	0	0	—	—	0	$0, 1$
4	0	0	0	—	0	$0, 1, 2$
5	0	0.210	0	0	0.210	1
6	0	0.294	0	0	0.294	1
7	0	0.378	0.245	0	0.378	1
8	0	0.378	0.343	0.280	0.378	1
9	0	0.432	0.441	0.392	0.441	2
10	0	0.486	0.441	0.504	0.504	3

$f_1(s_1, x_1) = \mathrm{P}_1(x_1) f_2^*(s_1 - c_1(x_1))$

	$f_1(s_1, x_1)$					
s_1	0	1	2	3	$f_1^*(s_1)$	x_1^*
10	0	0.22	0.227	0.302	0.302	3

The optimal solution is $x_1^* = 3$, $x_2^* = 1$, $x_3^* = 1$ and $x_4^* = 3$, yielding a system reliability of 0.3024.

題號 10.4-3

11.4-3.

$$f_n(1,x_n) = K(x_n) + x_n + \left(\frac{1}{3}\right)^{x_n} f_{n+1}^*(1) + \left[1 - \left(\frac{1}{3}\right)^{x_n}\right] f_{n+1}^*(0)$$
$$= K(x_n) + x_n + \left(\frac{1}{3}\right)^{x_n} f_{n+1}^*(1)$$

since $f_n^*(0) = 0$ for every n. $f_3^*(1) = 16$, $f_3^*(0) = 0$ and $K(x_n) = 0$ if $x_n = 0$, $K(x_n) = 3$ if $x_n > 0$.

	$f_2(s_2, x_2)$						
s_2	0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	–	–	–	–	0	0
1	16	9.33	6.78	6.59	7.20	6.59	3

	$f_1(s_1, x_1)$						
s_1	0	1	2	3	4	$f_1^*(s_1)$	x_1^*
1	6.59	6.20	5.73	6.24	7.08	5.73	2

The optimal policy is to produce two in the first run and to produce three in the second run if none of the items produced in the first run is acceptable. The minimum expected cost is $573.

題號 10.4-4

11.4-4.

$$f_n^*(s_n) = \max_{x_n \geq 0} \left\{ \tfrac{1}{3} f_{n+1}^*(s_n - x_n) + \tfrac{2}{3} f_{n+1}^*(s_n + x_n) \right\},$$

with $f_6^*(s_6) = 0$ for $s_6 < 5$ and $f_6^*(s_6) = 1$ for $s_6 \geq 5$.

s_5	$f_5^*(s_5)$	x_5^*
0	0	0
1	0	0
2	0	0
3	2/3	$x_5^* \geq 2$
4	2/3	$x_5^* \geq 1$
$s_5 \geq 5$	1	$x_5^* \leq s_5 - 5$

	$f_4(s_4, x_4)$						
s_4	0	1	2	3	4	$f_4^*(s_4)$	x_4^*
0	0	–	–	–	–	0	0
1	0	0	–	–	–	0	0
2	0	4/9	4/9	–	–	4/9	1, 2
3	2/3	4/9	2/3	2/3	–	2/3	0.2, 3
4	2/3	8/9	2/3	2/3	2/3	8/9	1
$s_4 \geq 5$	1	–	–	–	–	1	$x_4^* \leq s_4 - 5$

	$f_3(s_3, x_3)$						
s_3	0	1	2	3	4	$f_3^*(s_3)$	x_3^*
0	0	–	–	–	–	0	0
1	0	8/27	–	–	–	8/27	1
2	4/9	4/9	16/27	–	–	16/27	2
3	2/3	20/27	2/3	2/3	–	20/27	1
4	8/9	8/9	22/27	2/3	2/3	~~22/27~~ $\frac{8}{9}$	0, 1
$s_3 \geq 5$	1	–	–	–	–	1	$x_3^* \leq s_3 - 5$

	$f_2(s_2, x_2)$						
s_2	0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	–	–	–	–	0	0
1	8/27	32/81	–	–	–	32/81	1
2	16/27	48/81	48/81	–	–	48/81	0, 1, 2
3	20/27	64/81	62/81	2/3	–	64/81	1
4	24/27	74/81	70/81	62/81	2/3	74/81	1
$s_2 \geq 5$	1	–	–	–	–	1	$x_2^* \leq s_2 - 5$

	$f_1(s_1, x_1)$				
s_1	0	1	2	$f_1^*(s_1)$	x_1^*
2	48/81	160/243	~~124/243~~ $\frac{148}{243}$	160/243	1

The probability of winning the bet using the policy given above is $160/243 = 0.658$.